Rend. Mat. Appl. (7).
Volume 43 (2022), 37 – 59

RENDICONTI DI MATEMATICA
E DELLE SUE APPLICAZIONI

On the asymptotic behavior and approximate solution of a varicella zoster model using the modified differential transform method

Oluwatayo M. Ogunmiloro, Michael O. Oke and Precious O. Ojakovo

Abstract. *This article proposes a mathematical model describing the evolution and transmission of Varicella Zoster Virus (VZV) disease among classes of human individuals. The model is formulated to accommodate parameters and variables describing direct and indirect forms of transmission, re-activation of infectious shingles as well as the treatment and vaccination of susceptible births and immigrants. The model is analyzed and found to be positive, bounded and well posed. The controlled basic reproduction number R_{vzv}, obtained using the next generation matrix operator reveals that vaccination is effective as a control in creating a level herd immunity. Linearizing the model around the VZV - free equilibrium shows that the model is locally and globally asymptotically stable when R_{vzv} is less than unity. The approximate solution of the model system equations is obtained using the modified differential transform which involves the Differential Tranform Method (DTM) and the Laplace - Pade posttreatment technique (LP). The hybrid LPDTM technique is employed to enlarge the domain of convergence of the approximate solutions of the model. The model solutions using LPDTM is compared with the Fehlberg fourth order Runge - Kutta (RK45) via the Maple computational software to show the efficiency and convergence of the two methods through simulations. Further simulations carried out on the model reveal that timely vaccination and treatment are effective strategies in curtailing the spread of VZV infection in human and environmental host population.*

1. Introduction

Infectious diseases pose a serious challenge to human existence, but the development of safe vaccines, drugs and medical equipments has led to the reduction of the prevalence of some of the infectious diseases. Chickenpox, also known as Varicella Zoster Virus (VZV), is exclusively a human virus that belongs to the $\alpha_o - herpes$ virus family. VZV is present worldwide and it is highly infectious where primary infection leads to acute varicella or chicken pox, usually from exposure through direct contact with skin or lesions, or indirect contact through airborne spread from respiratory droplets or infected environmental sources like towels, toiletries, etc [6]. After initial infection, VZV establishes lifelong latency in cranial nerves and can reactivate later as infectious Herpes Zoster (HZ) or shingles. Varicella

2020 Mathematics Subject Classification: 92B05, 93D05.
Keywords: Varicella Zoster Virus (VZV), Basic Reproduction Number R_{vzv}, Stability Analysis, Laplace-Pade Differential Transform Method (LPDTM).

vaccination is recommended for outbreak control, while a pregnant woman who has symptoms of varicella also needs to be vaccinated to prevent Foetal Varicella Syndrome (FVS), which causes birth defect such as congenital varicella syndrome. This can also cause shingles in the baby during the first $1-2$ years of life. Symptoms of VZV includes onset of slight fever, tiredness and weakness, followed by an itching blister-like rash which resolves within $7-10$ days [20]. In Africa, mortality is low and morbidity is high, this has placed an endemic burden on the African society. However, this disease occurs mostly in children, pregnant women and adults. Mathematical models are good predictive tools in analyzing the spread and control of epidemics. Several works have been done in using deterministic models to analyze the transmission of VZV. Edward, Kuznetsov and Mirau [1], examined the modeling and stability analysis for VZV with vaccination. They investigated the local and global asymptotic behavior of the model and bifurcation analysis to prove that the basic reproduction number of the model is supercritical, while sensitivity analysis of the reproduction number is performed to suggest control strategies to policy health makers. Garnett and Greenfal [2], formulated a mathematical model to describe the epidemiology of VZV infections. A steady state age distribution of Zoster cases predicted by the model are compared to the observed distribution using data involving VZV transmission. The results obtained showed that the likelihood of reactivation increases with age. Corberian - Vallet *et al.* [3], used a discrete time Bayesian stochastic compartmental model approach to study the transmission of VZV in Valencia, Spain. The Bayesian analysis allows the computation of the posterior distribution of the model parameter and posterior predictive distribution of VZV incidence, which enables point forecast and prediction interval. Gommel, Jaros and Luu [7], formulated and discussed the dynamics of VZV with United States as a case study. Korostil, Wood and Regan [4], investigated the impact of periodicity of VZV in the presence of immune boosting and chemical reinfection with VZV. Furthermore, Tang *et al.* [5], investigated the control strategies of VZV in China using mathematical modeling approach where a school based vaccination intervention scenario is compared with a baseline (no intervention) scenario. Moreover, a mathematical approach to characterize the transmission dynamics of the VZV in United States, these results reveal that two doses of vaccination and 90 percent coverage of vaccination allow the zoster virus to decrease for 27 years in the host community.

Several semi-analytical methods like DTM, Homotopy Perturbation Method (HPM), Homotopy Analysis Method (HAM), Variational Iteration Method (VIM) etc., are powerful techniques employed in approximating nonlinear and linear problems in engineering and physics. These analytical methods have been used to solve various kind of models, see [8, 9] and [13 - 15, 17], while the use of Lyapunov techniques to prove asymptotic stability of models can also be seen in [10 - 12], which proved useful to this work. The semi-analytical method of interest in this work is the DTM. This method is efficient with less computational effort in solving linear and nonlinear differential equations, which does not depend on perturbation parameter like other analytical methods. One of the demerits of DTM is that the series solution does not portray the actual behavior of the problem but results to

a good approximation to the actual solution in a very small domain. Due to this drawback, a hybrid modified DTM is considered, for improving the DTMs truncated series solutions convergence rate by combining the DTM, Laplace transform and Pade approximant method. The resulting series solutions obtained by DTM, even if they have large number of terms, may converge in a restricted domain. Therefore, the domain of convergence of the truncated power series by DTM expands by the Laplace - Pade. The Laplace transform is applied to the convergent series obtained by DTM and later form its pade approximant, where the transformed series converts to a meromorphic function. Previous works considered only the impact of vaccination as well as fitting prevalence data on VZV transmission to models describing VZV transmission in human host population to show that vaccine is efficacious in boosting immunity against the disease. Articles, where the modified DTM is used can be seen in [16, 18, 19, 21].

In view of the cited works, we consider extensively a theoretical model based on ordinary differential equations subdivided into seven compartments of human sub-populations and environmental sources. The model incorporates the direct human to human and indirect human to environment forms of transmission using the mass action and saturated nonlinear incidence function, while variables describing the impact of vaccination on susceptible births and immigrants, treatment of VZV infected individuals as well as the reactivation of infectious shingles after recovery for some recovered human individuals are incorporated and studied. Furthermore, the LPDTM method is employed to obtain the approximate solution of the model in comparison with RK45 via Maple computational software to show the efficiency and convergence of the two methods with low error approximation. It is to the author's knowledge that this work has not been considered. This work is partitioned into sections. Section 2 describes the model derivation and basic properties of the model, while the computation of the VZV - free equilibrium solution and R_{vzv} are being carried out. Section 3 involve the analysis of the local and global asymptotic behavior of the VZV - free equilibrium solutions, while Section 4 involves using the modified DTM to obtain the approximate solutions of the model system equations. Section 5 discusses the numerical simulations using LPDTM in comparison with RK45 via.

2. Mathematical Model Derivation

In this section, a mathematical model is derived describing the transmission of VZV in the total human and environmental host population. The total human host population $N(t)$ is sub-divided into population of susceptible individuals $S_h(t)$, Vaccinated individuals $V_h(t)$; Exposed individuals $E_h(t)$; Infected individuals $I_h(t)$; Treated individuals $T_h(t)$ and Recovered individuals $R_h(t)$, such that $N(t) = S_h(t) + V_h(t) + E_h(t) + I_h(t) + T_h(t) + R_h(t)$. Also, the total environmental host population is considered and denoted by $E_v(t)$, where $t > 0$. The population of susceptible human individuals is generated by the rate at which newborns and immigrants who are recruited and vaccinated, denoted by ϕHN and $\rho\lambda$ respectively. Also the terms $(1-\phi)HN$ and $(1-\rho)\lambda$ denote the fractions of unvaccinated

newborns and immigrants respectively. The population of susceptible humans is further reduced by the quantities $\frac{\beta_1 c_1 S_h I_h}{N}$ and $\frac{\beta_2 c_2 S_h E_v}{A+E_v}$ which follows a mass action incidence and nonlinear incidence function respectively, where β_1 and β_2 denote the rate at which susceptible individuals become infectious by coming directly or indirectly in contact with infected individuals and infected environmental sources respectively. Also c_1 and c_2 denote the per capita contact rate associated to direct and indirect transmission, while μ denote the natural mortality rate associated to all classes of human population, A denote the concentration of varicella zoster virus in environmental sources. The exposed population is increased by the quantities $\frac{\beta_1 c_1 S_h I_h}{N}$ and $\frac{\beta_2 c_2 S_h E_v}{A+E_v}$ and reduced by quantity $(\mu+\delta)E_h$, where δ is the progression rate from exposed to infected population. In the infected population, η_1 is denotes the treatment rate while the rate of recovery through treatment is given by η_2. Individuals who have recovered, receive a life long immunity denoted by the quantity $K_o R$, and the reactivation of the virus in some fractions of recovered human individuals, known as shingles is denoted by $(1-\alpha)K_o R$, where $\alpha K_o R$ is the rate at which some fractions of recovered individuals develop infectious shingles. Also, the virus increases in the environmental sources through human infectious contribution to the environment denoted by σ, and there is a natural death of the virus denoted by μ_v. Furthermore, the rate at which vaccination losses its potency overtime, is denoted by τ. The assumptions guiding the model formulation are listed below.

- Birth and death rates are constant.
- The population is homogeneously mixed.
- The vaccinations of immigrants and newborns are considered.
- There is natural death of the virus in the environment.
- Infectious human individuals contribute to the infection of the environment.

The model derived after the assumptions and descriptions of variables is given by

$$\begin{cases} \frac{dS_h}{dt} = (1-\phi)HN + (1-\rho)\lambda - \frac{\beta_1 c_1 S_h I_h}{N} - \frac{\beta_2 c_2 S_h E_v}{A+E_v} - \mu S_h + \tau V_h, \\ \frac{dV_h}{dt} = \rho\lambda + \phi HN - (\mu+\tau)V_h, \\ \frac{dE_h}{dt} = \frac{\beta_1 c_1 S_h I_h}{N} + \frac{\beta_2 c_2 S_h E_v}{A+E_v} - (\mu+\delta)E_h, \\ \frac{dI_h}{dt} = \delta E_h - (\eta_1+\mu)I_h + \alpha K_o R_h, \\ \frac{dT_h}{dt} = \eta_1 I_h - (\eta_2+\mu)T_h, \\ \frac{dR_h}{dt} = \eta_2 T_h + (1-\alpha)K_o R_h - \mu R_h, \\ \frac{dE_v}{dt} = \sigma I_h - \mu_v E_v. \end{cases} \tag{2.1}$$

Subject to the initial conditions $S_h(0) \geq 0$, $V_h(0) \geq 0$, $E_h(0) \geq 0$, $I_h(0) \geq 0$, $T_h(0) \geq 0$, $R_h(0) \geq 0$, $E_v(0) \geq 0$.

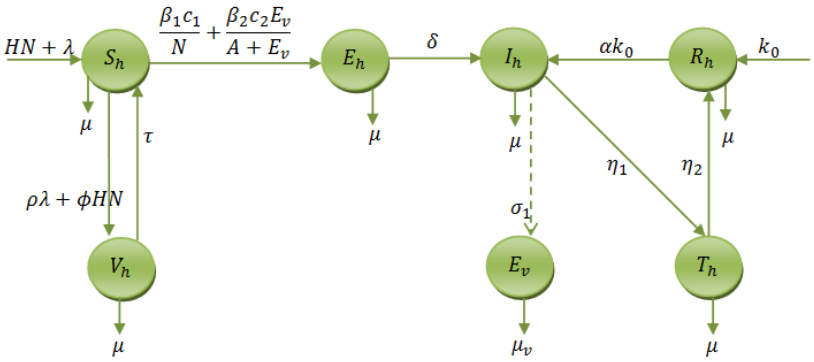


Figure 1. Block diagram describing the VZV infectious interactions among compartments of human individuals

Table 1. Parameter Descriptions

Descriptions	Parameters	Values	Sources
Vaccinated newborn rate	ϕ	0.0052/per day	[1]
Birth rate	H	0.0352/per day	[1]
Immigrants influx rate	λ	0.00356/per day	[1]
Vaccinated immigrants rate	ρ	0.0121/per day	[4]
Direct contact rate	β_1	0.0325/per day	[4]
Indirect contact rate	β_2	0.0241/per day	[4]
Per capita direct contact rate	c_1	0.11/per day	[4]
Per capita indirect contact rate	c_2	0.019/per day	[4]
Natural mortality rate	μ	0.0096/per day	[1]
Progression rate	η_1	0.52/per day	[2]
Concentration of virus rate	A	0.05/ per day	[6]
Lifelong immunity rate	K_o	0.46/per day	[4]
Progression rate	η_2	0.14/ per day	[4]
Natural death rate of the virus	μ_v	0.014/per day	[6]
Vaccination waning rate	τ	0.00033/per day	[2]
Human infectious contribution rate	σ	0.009/per day	[4]
Development of shingles rate	α	0.00002/per day	[4]

3. Basic Analysis of the Model

3.1. Positivity and Boundedness of the Model Solutions

For the model system (2.1) to be epidemiologically reasonable in the sense of VZV transmission, it is pertinent to show that all the solutions of the model system (2.1) and its non-negative initial conditions remain non-negative at time $t > 0$.

Theorem 3.1. *Let the initial conditions* $S_h(0) \geq 0$, $V_h(0) \geq 0$, $E_h(0) \geq 0$, $I_h(0) \geq 0$, $T_h(0) \geq 0$, $R_h(0) \geq 0$, $E_v(0) \geq 0$.

Then, the solutions $(S_h, V_h, E_h, I_h, T_h, R_h, E_v)$ *of model (2.1) are non-negative for* $t > 0$, *with* $N(t) = S_h(t) + V_h(t) + E_h(t) + I_h(t) + T_h(t) + R_h(t) + E_v(t)$ *and*

$$\Omega_1 = \left\{ (S_h, V_h, E_h, I_h, T_h, R_h) \in R_+^7 : S_h, V_h, E_h, I_h, T_h, R_h \leq \frac{HN + \lambda}{\mu} \right\}$$

.

Proof. Let

$$t_0 = \sup \left\{ t > 0 \;\middle|\; \begin{array}{c} S_h(t) > 0, V_h(t) > 0, E_h(t) > 0, I_h(t) > 0, \\ T_h(t) > 0, R_h(t) > 0, E_v(t) > 0 \end{array} \right\}.$$

Since $S_h(0) > 0, V_h(0) > 0, E_h(0) > 0, I_h(0) > 0, T_h(0) > 0, R_h(0) > 0, E_v(0) > 0$ then $t_0 > 0$, if $t_0 < \infty$, then $S_h, V_h, E_h, I_h, T_h, R_h, E_v$ is equal to zero at t_0. From the first equation of model system (1), given by

$$\frac{dS_h}{dt} = (1-\phi)HN + (1-\rho)\lambda - \frac{\beta_1 c_1 S_h I_h}{N} - \frac{\beta_2 c_2 S_h E_v}{A + E_v} - \mu S_h + \tau V_h, \tag{3.1}$$

then,

$$\begin{cases} \frac{d}{dt} S_h(t) exp \left\{ \left[\frac{\beta_1 c_1 S_h I_h}{N} + \frac{\beta_2 c_2 S_h E_v}{A+E_v} - \mu \right] \right\} = (1-\phi)HN + (1-\rho)\lambda \\ +\tau V_h \; exp \Big[\frac{\beta_1 c_1 I_h}{N} - \frac{\beta_2 c_2 E_v}{A+E_v} - \mu \Big] t. \end{cases} \tag{3.2}$$

Hence,

$$\begin{cases} S_h(t_0) exp \Big[\frac{\beta_1 c_1 I_h}{N} + \frac{\beta_2 c_2 E_v}{A+E_v} - \mu \Big] - S_h(0) = \\ \int_0^{t_0} (1-\phi)HN + (1-\rho)\lambda + \tau V_h \; \exp \Big[- \Big(\frac{\beta_1 c_1 I_h}{N} + \frac{\beta_2 c_2 E_v}{A+E_v} - \mu \Big) P \Big] dP \geq 0. \end{cases} \tag{3.3}$$

The same procedure in (3.1) - (3.3) can be shown for $S_h > 0, V_h > 0, E_h > 0, I_h > 0, T_h > 0, R_h > 0, E_v > 0$ for time $t > 0$, which implies that the solutions of (2.1) are positive.

In addition, adding the total human host population in model system equations in (2.1) in the absence of VZV infections yields

$$\frac{dN(t)}{dt} = HN - \mu N \leq HN - \mu N. \tag{3.4}$$

Simplifying (3.4) yields

$$0 \leq \lim_{t \to \infty} inf N(t) \leq \lim_{t \to \infty} sup N(t) \leq \frac{HN + \lambda}{\mu}. \tag{3.5}$$

The invariant region of (2.1) is the feasible domain where the solution of (2.1) is contained, given by,

$$\Omega_1 = \left\{ (S_h, V_h, E_h, I_h, T_h, R_h) \in R_+^7 : S_h, V_h, E_h, I_h, T_h, R_h \leq \frac{HN + \lambda}{\mu} \right\}. \quad (3.6)$$

Hence, (3.6) is well-posed and reasonable in the sense of VZV dynamics, so that model (2.1) solution starts and remain in Ω_1.

3.2. Computation of VZV-Free Equilibrium Solution and Basic Reproduction Number R_{vzv}

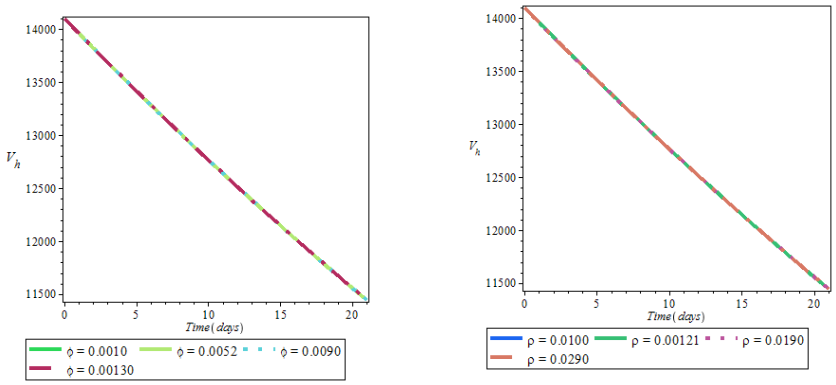
The equilibrium solutions of model system (2.1) is obtained by fixing the right hand side of model system (2.1) to zero. The time independent solutions in the absence of VZV infection is given by,

$$\begin{aligned} W^0 &= (S_h^o, V_h^o, E_h^o, I_h^o, T_h^o, R_h^o, E_v^o) \\ &= \Big(\frac{\tau(\rho\lambda + \phi HN) - (1-\phi)HN - (1-\rho)\lambda}{\mu(\mu+\tau)}, \frac{\rho\lambda + \phi HN}{(\mu+\tau)}, 0, 0, 0, 0, 0 \Big). \end{aligned} \quad (3.7)$$

The basic reproduction number threshold R_{vzv} of model system (2.1) is the rate of average secondary cases of VZV generated when a primary VZV infected individual is introduced into a large susceptible human population during his or her infection period. The next generation matrix approach [1, 7], is used to obtain the value of R_{vzv}. Therefore, the R_{vzv} of the model system (2.1) is given by,

$$R_{vzv} = \frac{\beta_1\beta_2 c_1((1-\phi)HN + (1-\rho)\lambda)\eta_1\alpha K_o}{N\mu(\alpha K(\mu(\mu+\eta_1+\eta_2)) - K_o(\mu^2 + \mu(\eta_1+\eta_2) + \eta_1+\eta_2) + \mu^3 + \mu^2(\eta_1+\eta_2) + \mu\eta_1\eta_2)} \quad (3.8)$$

The R_{vzv} obtained in (3.8) is called the vaccination controlled basic reproduction number, where the susceptible births and influx of immigrants are vaccinated, such that $\phi > 1 - \frac{1}{R_{vzv}}$ and $\rho > 1 - \frac{1}{R_{vzv}}$ leads to a herd immunity level. If $R_{vzv} < 1$, VZV is minimized in the host community and if $R_{vzv} > 1$, VZV infections continues in the population and becomes endemic.



(a) (b)

Figure 2. The impact of the variation of vaccination parameters on the population of vaccinated newborns and immigrants.

3.3. Local and Global Asymptotic Behavior of VZV-free Equilibrium Solution of the Model

Theorem 3.2. *The VZV-free equilibrium (3.7) of model system (2.1) solution is locally asymptotically stable when* $R_{vzv} < 1$

.

Proof. Linearizing the model around the VZV - free equilibrium solutions in (3.7), yields the Jacobian given by

$$J_c = \begin{pmatrix} -\mu & \tau & 0 & -\frac{\beta_1 c_1 S_h}{N} & 0 & 0 & 0 \\ 0 & -(\mu+\tau) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\mu+\delta) & \frac{\beta_1 c_1 S_h}{N} & 0 & 0 & 0 \\ 0 & 0 & \delta & -(\eta_1+\mu) & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_1 & -(\eta_2+\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta_2 & -\mu+(1-\alpha)K_o & 0 \\ 0 & 0 & 0 & \sigma & 0 & 0 & -\mu_v \end{pmatrix} \tag{3.9}$$

It is observed that the real part of (3.9) is negative. The eigenvalues of (3.9) are given by $-\mu, -\mu_v, -(\mu+\tau), -(\mu+\delta)$ which reduces (3.9) to another 3×3 matrix given by

$$\begin{pmatrix} -q_1 & 0 & 0 \\ \eta_1 & -q_2 & 0 \\ 0 & \eta_2 & -q_3 \end{pmatrix}, \tag{3.10}$$

where $q_1 = -(\eta_1+\mu), q_2 = -(\eta_2+\mu)$ and $q_3 = -\mu+(1-\alpha)K_o$. The characteristic equations of (3.10) yields

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3, \tag{3.11}$$

where

$$\begin{cases} A_1 = (q_1 + q_2 - q_3), \\ A_2 = (q_2q_3 + q_1q_3 - q_1q_3), \\ A_3 = q_3q_2q_1(1 - R_{vzv}). \end{cases} \tag{3.12}$$

By the use of the Routh - Hurwitz conditions [10 - 12], $A_i > 0$ for $i = 1, 2, 3$ and $A_1A_2 - A_3 > 0$ could be verified easily. Also, $(1 - R_{vzv}) > 0 \Leftrightarrow R_{vzv} < 1$. Thus, the VZV - free equilibrium (3.7) of model system (2.1) is locally asymptotically stable.

Theorem 3.3. *The VZV - free equilibrium (3.7) of model system (2.1) is globally asymptotically stable when* $R_{vzv} < 1$.

Proof. Let $\frac{dX_1}{dt} = F_1(X_1,\ Z_1)$, $\frac{dZ_1}{dt} = G_1(X_1,\ Z_1)$, where $X_1 = (S_h, V_h, T_h, R_h)$ denotes the population without VZV infections, i.e., $X \in R^{+4}$ and $Z_1 = (E_h, I_h, E_v)$ denote the compartment with VZV infections. The VZV - free equilibrium solution is denoted by $W^0 = (X_1^*, Z_1^*) = (X_1^*, 0)$, where

$$X_1^* = \Big(\frac{(1-\phi)HN + (1-\rho)\lambda}{\mu}, \frac{\rho\lambda + \phi HN}{(\mu+\tau)}\Big),$$

then the following two conditions are established.

- $X_1^* = F_1(X_1, 0), X_1^*$ is globally asymptotically stable
- $G_1(X_1, Z_1) = LZ_1 - G_1(X_1,\ Z_1) \geq 0$ for all $(X_1, Z_1) \in \Omega_1$, which is the domain of relevance of model system (2.1).

Therefore,

$$F_1(X_1, 0) = \begin{pmatrix} (1-\phi)HN + (1-\rho)\lambda - \mu S_h + \tau V_h \\ \rho\lambda + \phi HN - (\mu+\tau)V_h - (\eta_2+\mu)T_h \\ \eta_2 T_h + (1-\alpha)K_o R_h - \mu R_h \end{pmatrix} \tag{3.13}$$

and

$$G_1(X_1, Z_1) = \begin{pmatrix} \frac{\beta_1 c_1 S_h I_h}{N} + \frac{\beta_2 c_2 S_h E_v}{(A+E_v)} - (\mu+\delta)E_h \\ \delta E_h - (\eta_1+\mu)I_h + \alpha K_o R_h \\ \sigma I_h - \mu_v E_v \end{pmatrix}. \tag{3.14}$$

In the absence VZV infections, $G_1(X_1, 0) = 0$. Also, let

$$L = \begin{pmatrix} -\mu & \tau & 0 & -\frac{\beta_1 c_1 S_h^o}{N} & 0 & 0 & 0 \\ 0 & -(\mu+\tau) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\mu+\delta) & -\frac{\beta_1 c_1 S_h^o}{N} & 0 & 0 & 0 \\ 0 & 0 & \delta & -(\eta_1+\mu) & 0 & \alpha K_o & 0 \\ 0 & 0 & 0 & \eta_1 & -(\eta_2+\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta_2 & -(\mu+(1-\alpha)K_o & 0 \\ 0 & 0 & 0 & \sigma & 0 & 0 & -\mu_v \end{pmatrix}, \tag{3.15}$$

then $G_1(X_1,\ Z_1)$ can be re-written as $G_1(X_1,\ Z_1) = LZ_1 - \hat{G}_1(X_1,\ Z_1)$, where

$$\hat{G}_1(X_1,\ Z_1) = \begin{pmatrix} \frac{\beta_1 c_1 (S_h - S_h^o)(I_h - I_h^o)}{N\mu} + \frac{\beta_2 c_2 (S_h - S_h^o)(E_v - E_v^o)}{(A + (E_v - E_v^o)} - (\mu + \delta)(E_h - E_h^o) \\ \delta(E_h - E_h^o) - (\eta_1 + \mu)(I_h - I_h^o) + \alpha K_o (R_h - R_h^o) \\ \sigma(I_h - I_h^o) - \mu_v (E_v - E_v^o) \end{pmatrix}. \tag{3.16}$$

Conclusively, $X_1^* = \left(\frac{(1-\phi)HN + (1-\rho)\lambda}{\mu}, \frac{\rho\lambda + \phi HN}{(\mu+\tau)}\right)$ is a global asymptotic stable equilibrium point of (3.7), where the solution given by $S_h = \frac{(1-\phi)HN + (1-\rho)\lambda}{N\mu} + \left(S_h(0) - \frac{(1-\phi)HN + (1-\rho)\lambda}{N\mu}\right)e^{-\mu t}$ converges to X_1^* as $t \to \infty$. This implies the global convergence of (3.16) in Ω_1. From (3.15) and (3.16), the two conditions are satisfied, that is, $G_1(X_1,\ 0)$ and $G_1(X_1,\ Z_1) = LZ_1 - \hat{G}_1(X_1,\ Z_1) \geq 0$. Thus the model system (2.1) at the VZV - free equilibrium (3.7) is globally asymptotically stable.

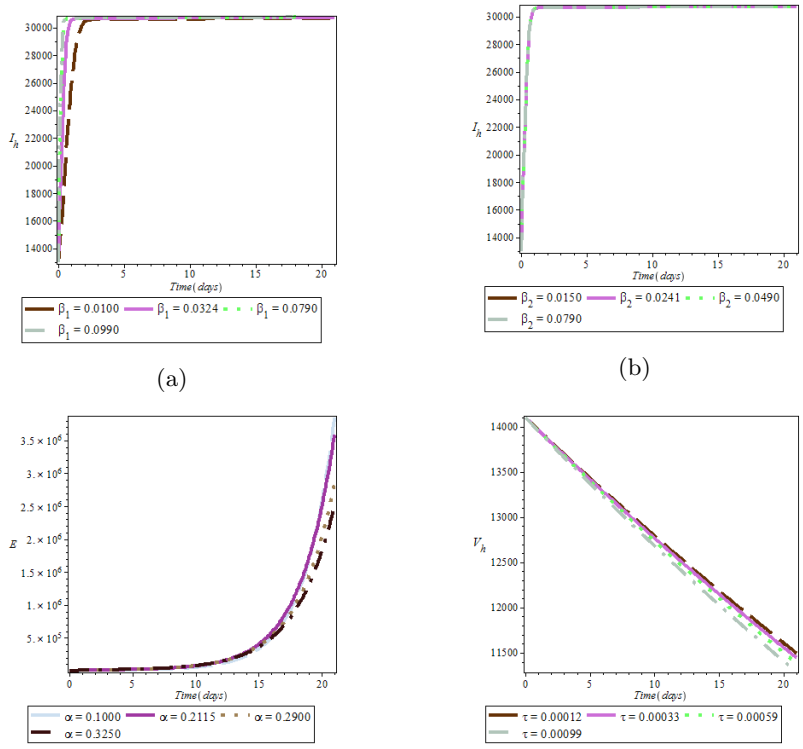


Figure 3. The impact of the variations of parameter values of contact rate, human infectious contribution to the environment and loss of immunity

4. Method of Solution

Here, a modified differential transform method (LPDTM) is employed to obtain the approximate solutions of the model system equations (2.1). The DTM is used to obtain the truncated series solution, while the LP is applied to enlarge the convergence domain of the truncated series solution. The use of this method is highlighted step by step.

4.1. Differential Transform Method (DTM)

Defintion 4.1. . If a function $u(t)$ is differentiable with respect to t, then

$$U(k) = \frac{1}{k!}\Big[\frac{d^k u(t)}{dt^k}\Big]_{t=0}. \tag{4.1}$$

Defintion 4.2. . The differential inverse of set $\{U(k)\}_{k=0}^{n}$ is given by

$$u(t) = \sum_{k=0}^{\infty} U(k)t^k. \tag{4.2}$$

Substituting (4.1) into (4.2) yields

$$u(t) = \sum_{k=0}^{\infty} \frac{1}{k!}\Big[\frac{d^k u(t)}{dt^k}\Big]_{t=0} t^k. \tag{4.3}$$

From (4.1) to (4.3), the concept of DTM is derived. Further examples on application of DTM to models and the derivation of the basic properties involving DTM can be seen in [13, 14, 16, 17].

Table 2. Some Basic Operations of DTM.

Original Function	Transformed Function
$\alpha_1 u(t) \pm \alpha_2 v(t)$	$\alpha_1 U(t) \pm \alpha_2 V(t)$
$u(t)v(t)$	$\sum_{l=0}^{k} U(k)V(k-r)$
$\frac{d}{dt}[u(t)]$	$(k+1)U(k+1)$
t^n	$\delta(k-n) = 1$, when $k = n$ and 0, when k$\neq$ n
$t^n u(t)$	$U(k-n)$
$e^{\lambda t}$	$\frac{\lambda^k}{k!}$
$\sin(\omega t + \alpha_1)$	$\frac{\omega^k}{k!}\sin\left(\frac{\pi k}{2} + \alpha_1\right)$
$\cos(\omega t + \alpha_1)$	$\frac{\omega^k}{k!}\cos\left(\frac{\pi k}{2} + \alpha_1\right)$

4.2. Pade Approximant

Given an analytical function $u(t)$ with Mclaurin series expansion

$$u(t) = \sum_{n=0}^{\infty} u_n t^n, \quad 0 \le t \le T. \tag{4.4}$$

The Pade approximant to $u(t)$ of order $[q_1,\ q_2]$, which is represented by $\left[\frac{q_1}{q_2}\right]_u(t)$, so that

$$\left[\frac{q_1}{q_2}\right]_u(t) = \frac{r_0 + r_1 t + \cdots + r_{q_1} t^{q_1}}{1 + s_1 t + \cdots + s_{q_2} t^{q_2}}, \tag{4.5}$$

where $s_1 = 1$ and the numerator and denominator possess no common factors. The numerator and denominator in (4.5) are developed so that $u(t)$ and $\left[\frac{q_1}{q_2}\right]_u(t)$ and their derivatives agree at $t = 0$ up to $q_1 + q_2$, that is,

$$u(t) - \left[\frac{q_1}{q_2}\right]_u(t) = 0(t^{q_1+q_2+1}). \tag{4.6}$$

From (4.6), we obtain the following algebraic system given by

$$\begin{aligned}
&u_{q_1} s_1 + \cdots + u_{q_1-q_2+1} s_n = -u_{q_1+1} \\
&u_{q_1+1} s_1 + \cdots + u_{q_1-q_2+2} s_n = -u_{q_1+2} \\
&. \\
&. \\
&. \\
&u_{q_1+q_2-1} s_1 + \cdots + u_{q_1} s_n = -uq_1 + q_2
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
&r_o = u_o \\
&r_1 = u_1 + u_o s_1 \\
&. \\
&. \\
&. \\
&r_{q_1} = u_n + u_{q_1-1} s_1 + \cdots + u_o s_{q_1}
\end{aligned} \tag{4.8}$$

From (4.7), we compute for all the coefficients s_n, $1 \leq n \leq q_1$. Then, the coefficients r_n can be determined, so that $0 \leq n \leq q_2$. From (4.8), note that for a fixed value of $q_1 + q_2 + 1$, the error in (4.6) is smallest when the numerator and denominator of (4.5) possess the same degree or when the numerator possesses a degree one higher than the denominator.

4.3. Laplace - Pade Posttreatment Technique

Laplace - Pade posttreatment technique is used to widen the domain of convergence of solutions or to obtain exact solutions. The procedure governing the Laplace - Pade technique is outlined below.

- Apply DTM to the given model system (2.1).
- Perform desirable number of iterations and obtain the solutions in power series form.

- The Laplace transform is applied to the power series solutions obtained using DTM.
- Next, s is substituted by $\frac{1}{z}$ in the equation.
- The transformed series obtained is converted into a meromorphic function by forming its Pade approximant of order $\frac{G}{H}$, where G and H are arbitrarily chosen, but they should be smaller than the order of the power series. In this process, the Pade approximant extends the domain of the truncated series solution to obtain better accuracy and convergence.
- Then, z is substituted by $1/s$.
- Finally, by using the inverse Laplace transformation, the exact or approximate solution is obtained.

Therefore, applying DTM to the formulated model system (2.1) yields following equations

$$
\begin{aligned}
S_h(k+1) &= \frac{1}{(k+1)}\Big((1-\phi)HN + (1-\epsilon)\lambda - \frac{\beta_1 c_1}{N}\Big(\sum_{l=0}^{k} S_h(k) I_h(k-l)\Big)\\
&- \beta_2 c_2\Big(\sum_{l=0}^{k} S_h(k)\frac{E_v(k-l)}{A+E_v(k)}\Big) - \mu S_h(k) + \tau V_h(k)\Big),\\
V_h(k+1) &= \frac{1}{(k+1)}(\rho\lambda + \phi HN - (\mu+\tau)V_h(k)),\\
E_h(k+1) &= \frac{1}{(k+1)}\Big(\frac{\beta_1 c_1}{N}(\sum_{l=0}^{k} S_h(k) I_h(k-l)) +\\
&\beta_2 c_2\Big(\sum_{l=0}^{k} S_h(k)\frac{E_v(k-l)}{A+E_v(k)}\Big) - (\mu+\delta)E_h(k)\Big),\\
I_h(k+1) &= \frac{1}{(k+1)}(\delta E_h(k) - (\eta_1+\mu)I_h(k) + \alpha K_o R_h(k)),\\
T_h(k+1) &= \frac{1}{(k+1)}(\eta_1 I_h(k) - (\eta_2+\mu)T_h(k)),\\
R_h(k+1) &= \frac{1}{(k+1)}(\eta_2 T_h(k) + (1-\alpha)K_o R_h(k) - \mu R_h(k)),\\
E_v(k+1) &= \frac{1}{(k+1)}(\sigma I_h(k) - \mu_v E_v(k)).
\end{aligned}
\tag{4.9}
$$

Using the following assumed initial conditions $S_h = 0.1750, V_h = 0.950, E_h = 0.1000, I_h = 0.1200, T_h = 0.850, R_h = 0.1500, E_v = 0.1000$ and the parameter values given in Table 1, with the aid of Maple computational software, yields

the tenth order series solution approximations for each sub-equations in (2.1) as follows;

$$S_h(t) = \sum_{k=0}^{10} S_h(k)t^k = 0.1750 + 0.000620103333t + 0.002202535436t^2 + 0.001443985993t^3 + 0.001091823929t^4 + 0.0008767035118t^5 + 0.0007322445605t^6 + 0.000628592248t^7 + 0.0005506172475t^8 + 0.0004898381148t^9 + 0.0004411353066t^{10}, \tag{4.10}$$

$$V_h(t) = \sum_{k=0}^{10} V_h(k)t^k = 0.950 - 0.009424616400t + 0.00005123502040t^2 + 0.000002791612082t^3 + 0.000002213969823t^4 + 0.000001772323056t^5 + 0.000001477666805t^6 + 0.000001266989538t^7 + 0.000001108877349t^8 + 9.858432053\ 10^{-7}t^9 + 8.873810577\ 10^{-7}t^{10}, \tag{4.11}$$

$$E_h(t) = \sum_{k=0}^{10} E_h(k)t^k = 0.1000 - 0.01994310333t + 0.002216145340t^2 - 0.0001363775366t^3 + 0.00002075219132t^4 + 0.000007082305302t^5 0.000005091756240t^6 + 0.000003671273394t^7 + 0.000002776977319t^8 + 0.000002173005673t^9 + 0.000001746403645t^{10}, \tag{4.12}$$

$$I_h(t) = \sum_{k=0}^{10} I_h(k)t^k = 0.1200 - 0.0278085000t + 0.01362001802t^2 - 0.001410535693t^3 + 0.0002507779532t^4 - 0.00001896508281t^5 + 0.000002193728463t^6 + 1.263436314\ 10^{-8}t^7 + 9.670781112\ 10^{-8}t^8 + 5.962821041\ 10^{-8}t^9 + 4.280278913\ 10^{-8}t^{10}, \tag{4.13}$$

$$T_h(t) = \sum_{k=0}^{10} T_h(k)t^k = 0.850 - 0.0.0647600t - 0.002386162000t^2 + 0.002479793068t^3 - 0.0002761139008t^4 + 0.00003434223506t^5 - 0.000002499906905t^6 + 2.163892677\ 10^{-7}t^7 - 3.225245702\ 10^{-9}t^8 + 5.641173171\ 10^{-10}t^9 + 3.016274990\ 10^{-9}t^{10}, \tag{4.14}$$

$$R_h(t) = \sum_{k=0}^{10} R_h(k)t^k = 0.1500 + 0.1719665000t + 0.02582834541t^2 + 0.002928728123t^3 + 0.0003453335540t^4 + 0.0001665695703t^5 + 0.00001781608500t^6 + 3.987383010\ 10^{-8}t^7 + 5.546793204\ 10^{-9}t^8 + 1.674548611\ 10^{-10}t^9 + 8.488942299\ 10^{-11}t^{10}, \tag{4.15}$$

$$E_v(t) = \sum_{k=0}^{10} E_v(k)t^k = 0.1000 - 0.0003200t - 0.0001228982500t^2$$
$$+ 0.00004143357923t^3 - 0.000003318722838t^4 + 4.606927398\ 10^{-7}t^5 \quad (4.16)$$
$$+ 2.95225739510^{-8}t^6 + 2.879553173\ 10^{-9}t^7 + 9.174440485\ 10^{-12}t^8$$
$$+ 9.669353977\ 10^{-11}t^9 + 5.353001841\ 10^{-11}t^{10}$$

The approximate series solutions obtained from (4.10) - (4.16) may have restricted domains of convergence. The accuracy is enhanced by applying the Laplace - Pade posttreatment technique earlier described. On applying the Laplace transformations to (4.10) - (4.16), yields the following;

$$L[S_h(t)] = \frac{1}{s^{11}}(1.000000000\ 10^{-12}(1.750000000\ 10^{11}s^{10} + 6.20103333\ 10^8 s^9$$
$$+ 4.405070872\ 10^9 s^8 + 8.663915958\ 10^9 s^7 + 2.620377430\ 10^{11}s^6$$
$$+ 1.052044214\ 10^{11}s^5 + 5.272160836\ 10^{11}s^4 + 3.168104933\ 10^{12}s^3$$
$$+ 2.220088741\ 10^{13}s^2 + 1.777524551\ 10^{14}s + 1.600791801\ 10^{15})), \quad (4.17)$$

$$L[V_h(t)] = \frac{1}{s^{11}}(4.000000000\ 10^{-15}(2.375000000\ 10^{14}s^{10} - 2.356154100\ 10^1 2s^9$$
$$+ 2.561751020\ 10^{10}s^8 + 4.187418123\ 10^9 s^7 + 1.328381894\ 10^{10}s^6$$
$$+ 5.316969168\ 10^{10}s^5 + 2.659800249\ 10^{11}s^4 + 1.596406818\ 10^{12}s^3$$
$$+ 1.117748368\ 10^{13}s^2 + 8.943569558\ 10^{13}s + 8.050320955\ 10^{14})), \quad (4.18)$$

$$L[E_h(t)] = \frac{1}{s^{11}}(8.000000000\ 10^{-14}(1.250000000\ 10^{12}s^{10} - 2.492887916\ 10^{11}s^9$$
$$+ 5.540363350\ 10^{10}s^8 - 1.022831524\ 10^{10}s^7 + 6.225657396\ 10^9 s^6$$
$$+ 1.062345795\ 10^{10}s^5 + 4.582580616\ 10^{10}s^4 + 2.312902238\ 10^{11}s^3$$
$$+ 1.399596569\ 10^{12}s^2 + 9.856753733\ 10^{12}s + 7.921686934\ 10^{13})), \quad (4.19)$$

$$L[I_h(t)] = \frac{1}{s^{11}}(1.600000000\ 10^{-15}(7.500000000\ 10^{13}s^{10} - 1.738031250\ 10^{13}s^9$$
$$+ 1.702502252\ 10^{13}s^8 - 5.28950849\ 10^{12}s^7 + 3.761669298\ 10^{12}s^6$$
$$- 1.422381211\ 10^{12}s^5 + 9.871778084\ 10^{11}s^4 + 3.979824389\ 10^{10}s^3$$
$$+ 2.437036840\ 10^{12}s^2 + 1.35236812\ 10^{13}s + 9.707672575\ 10^{13})), \quad (4.20)$$

$$L[T_h(t)] = \frac{1}{s^{11}}(3.200000000\ 10^{-16}(2.656250000\ 10^{15}s^{10} - 2.023750000\ 10^{14}s^9$$
$$- 1.491351250\ 10^{13}s^8 + 4.649612002\ 10^{13}s^7 - 2.070854256\ 10^{13}s^6$$
$$+ 1.287833815\ 10^{13}s^5 - 5.624790536\ 10^{12}s^4 + 3.408130966\ 10^{12}s^3$$
$$- 4.063809585\ 10^{11}s^2 + 6.397090376\ 10^{12}s + 3.420455839\ 10^{13})), \quad (4.21)$$

$$L[R_h(t)] = \frac{1}{s^{11}}(6.400000000\ 10^{-17}(2.343750000\ 10^{15}s^{10} + 2.686976562\ 10^{15}s^9 + 8.071357941\ 10^{14}s^8 + 2.74568261510^{14}s^7 + 1.29500082810^{14}s^6 + 3.123179443\ 10^{13}s^5 + 2.004309562\ 10^{13}s^4 + 3.140064120\ 10^{12}s^3 + 3.494479719\ 10^{12}s^2 + 9.494690624\ 10^{11}s + 4.813230284\ 10^{12})), \tag{4.22}$$

$$L[E_v(t)] = \frac{1}{s^{11}}(3.200000000\ 10^{-18}(3.125000000\ 10^{16}s^{10} - 1.000000000\ 10^{14}s^9 - 7.681140625\ 10^{13}s^8 + 7.768796106\ 10^{13}s^7 - 2.489042128\ 10^{13}s^6 + 1.72759774\ 10^{13}s^5 - 6.642579139\ 10^{12}s^4 + 4.535296247\ 10^{12}s^3 + 1.155979501\ 10^{11}s^2 + 1.096504741\ 10^{13}s + 6.070304088\ 10^{13})), \tag{4.23}$$

Substituting $s = \frac{1}{z}$ in (4.17) to (4.23) yields the following;

$$L[S_h(t)] = 0.1750000000z + 0.0006201033330z^2 + 0.004405070872z^3 + 0.008663915958z^4 + 0.02620377430z^5 + 0.1052044214z^6 + 0.5272160838z^7 + 3.168104933z^8 + 22.20088741z^9 + 177.7524551z^{10} + 1600.791801z^{11}, \tag{4.24}$$

$$L[V_h(t)] = 0.9500000000z - 0.009424616400z^2 + 0.0001024700408z^3 + 0.00001674967249z^4 + 0.00005313527576z^5 + 0.0002126787667z^6 + 0.001063920100z^7 + 0.006385627272z^8 + 0.04470993472z^9 + 0.3577427823z^{10} + 3.220128382z^{11}, \tag{4.25}$$

$$L[E_h(t)] = 0.1000000000z - 0.01994310333z^2 + 0.004432290680z^3 + 0.0008182652192z^4 + 0.0004980525917z^5 + 0.0008498766360z^6 + 0.003666064493z^7 + 0.01850321790z^8 + 0.1119677255z^9 + 0.7885402986z^10 + 6.337349547z^{11}, \tag{4.26}$$

$$L[I_h(t)] = 0.1553227612z^{11} + 0.02163788499z^10 + 0.003899258944z^9 + 0.00006367719022z^8 + 0.001579484493z^7 - 0.002275809938z^6 + 0.006018670877z^5 - 0.008463214158z^4 + 0.02724003603z^3 - 0.02780850000z^2 + 0.1200000000z, \tag{4.27}$$

$$L[T_h(t)] = 0.8500000000z - 0.06476000000z^2 + 0.004772324000z^3 + 0.01487875841z^4 - 0.006626733619z^5 + 0.004121068208z^6 - 0.001799932972z^7 + 0.001090601909z^8 - 0.0001300419067z^9 + 0.002047068920z^{10} + 0.01094545868z^{11}, \tag{4.28}$$

$$
\begin{aligned}
L[R_h(t)] &= 0.1500000000z - 0.1719665000z^2 + 0.05165669082z^3 \\
&+ 0.01757236874z^4 + 0.008288005299z^5 + 0.001998834844z^6 + \\
&0.001282758120z^7 + 0.0002009641037z^8 + 0.0002236467020z^9 \\
&+ 0.00006076601999z^{10} + 0.000308046382z^{11},
\end{aligned}
\tag{4.29}
$$

$$
\begin{aligned}
L[E_v(t)] &= 0.0001942497308z^{11} + 0.00003508815171z^{10} + 3.699134403\ 10^{-7}z^9 \\
&+ 0.00001451294799z^8 - 0.00002125625324z^7 + 0.0000528312877z^6 \\
&- 0.00007964934810z^5 + 0.0002486014754z^4 - 0.000245796500z^3 \\
&- 0.0003200000000z^2 + 0.1000000000z.
\end{aligned}
\tag{4.30}
$$

Again, substituting $z = \dfrac{1}{s}$, we compute the $\left[\frac{4}{4}\right]$ Pade approximant of (4.24) - (4.30) to obtain the following

$$
\begin{aligned}
&\left[\frac{4}{4}\right]_{S_h} = \\
&\frac{0.1750000z}{1.000000000-14.51158393z+54.13271792z^2-42.36828055z^3-11.79047117z^4} \\
&- \frac{2.538907085z^2}{1.000000000-14.51158393z+54.13271792z^2-42.36828055z^3-11.79047117z^4} \\
&+ \frac{9.468632026z^3}{1.000000000-14.51158393z+54.13271792z^2-42.36828055z^3-11.79047117z^4} \\
&- \frac{7.436176858z^4}{1.000000000-14.51158393z+54.13271792z^2-42.36828055z^3-11.79047117z^4} \\
&- \frac{1.950670201z^5}{1.000000000-14.51158393z+54.13271792z^2-42.36828055z^3-11.79047117z^4},
\end{aligned}
\tag{4.31}
$$

$$
\begin{aligned}
&\left[\frac{4}{4}\right]_{V_h} \\
&= \frac{0.9499999999z}{0.9999999998-14.97506834z+59.70082036z^2-59.10246230z^3-0.5983573139z^4} \\
&- \frac{14.23573954z^2}{0.9999999998-14.97506834z+59.70082036z^2-59.10246230z^3-0.5983573139z^4} \\
&+ \frac{56.85701609z^3}{0.9999999998-14.97506834z+59.70082036z^2-59.10246230z^3-0.5983573139z^4} \\
&- \frac{56.71151428z^4}{0.9999999998-14.97506834z+59.70082036z^2-59.10246230z^3-0.5983573139z^4} \\
&- \frac{0.005501559544z^5}{0.9999999998-14.97506834z+59.70082036z^2-59.10246230z^3-0.5983573139z^4},
\end{aligned}
\tag{4.32}
$$

$$
\begin{aligned}
&\left[\frac{4}{4}\right]_{E_h} = \\
&\frac{0.1000000000z}{1.000000000-14.33716523z+52.81052051z^2-40.65776383z^3-11.51830894z^4} \\
&- \frac{1.453659627z^2}{1.000000000-14.33716523z+52.81052051z^2-40.65776383z^3-11.51830894z^4} \\
&+ \frac{5.571411910z^3}{1.000000000-14.33716523z+52.81052051z^2-40.65776383z^3-11.51830894z^4} \\
&- \frac{5.183346800z^4}{1.000000000-14.33716523z+52.81052051z^2-40.65776383z^3-11.51830894z^4} \\
&- \frac{0.09468767487z^5}{1.000000000-14.33716523z+52.81052051z^2-40.65776383z^3-11.51830894z^4},
\end{aligned}
\tag{4.33}
$$

$$
\begin{aligned}
&\left[\frac{4}{4}\right]_{I_h} \\
&= \frac{0.1200000000z}{1.000000000-2.616947449z-9.254596881z^2-0.5982149832z^3+1.582317721z^4} \\
&- \frac{0.3418421940z^2}{1.000000000-2.616947449z-9.254596881z^2-0.5982149832z^3+1.582317721z^4} \\
&- \frac{1.010538206z^3}{1.000000000-2.616947449z-9.254596881z^2-0.5982149832z^3+1.582317721z^4} \\
&+ \frac{0.1058217024z^4}{1.000000000-2.616947449z-9.254596881z^2-0.5982149832z^3+1.582317721z^4} \\
&- \frac{0.01741550710z^5}{1.000000000-2.616947449z-9.254596881z^2-0.5982149832z^3+1.582317721z^4},
\end{aligned} \tag{4.34}
$$

$$
\begin{aligned}
&\left[\frac{4}{4}\right]_{T_h} \\
&= \frac{0.8500000000z}{0.9999999999-3.294825574z-1.163710935z^2-0.5451756029z^3+0.09324692916z^4} \\
&- \frac{2.865361738z^2}{0.9999999999-3.294825574z-1.163710935z^2-0.5451756029z^3+0.09324692916z^4} \\
&- \frac{0.7805537144z^3}{0.9999999999-3.294825574z-1.163710935z^2-0.5451756029z^3+0.09324692916z^4} \\
&+ \frac{0.5693639163z^4}{0.9999999999-3.294825574z-1.163710935z^2-0.5451756029z^3+0.09324692916z^4} \\
&- \frac{0.006141723955z^5}{0.9999999999-3.294825574z-1.163710935z^2-0.5451756029z^3+0.09324692916z^4},
\end{aligned} \tag{4.35}
$$

$$
\begin{aligned}
&\left[\frac{4}{4}\right]_{R_h} = \\
&\frac{0.1500000000z}{1.000000000-10.61523753z-3.153979011z^2+1.751414950z^3+0.2961675194z^4} \\
&- \frac{1.420319130z^2}{1.000000000-10.61523753z-3.153979011z^2+1.751414950z^3+0.2961675194z^4} \\
&- \frac{2.246905406z^3}{1.000000000-10.61523753z-3.153979011z^2+1.751414950z^3+0.2961675194z^4} \\
&+ \frac{0.8104421638z^4}{1.000000000-10.61523753z-3.153979011z^2+1.751414950z^3+0.2961675194z^4} \\
&- \frac{0.004438845357z^5}{1.000000000-10.61523753z-3.153979011z^2+1.751414950z^3+0.2961675194z^4},
\end{aligned} \tag{4.36}
$$

$$
\begin{aligned}
&\left[\frac{4}{4}\right]_{E_v} \\
&= \frac{0.1000000000z}{1.000000000-3.335431054z-0.6790590667z^2+0.6136657858z^3+0.004050155030z^4} \\
&- \frac{0.3338631053z^2}{1.000000000-3.335431054z-0.6790590667z^2+0.6136657858z^3+0.004050155030z^4} \\
&- \frac{0.06708436522z^3}{1.000000000-3.335431054z-0.6790590667z^2+0.6136657858z^3+0.004050155030z^4} \\
&+ \frac{0.06265231624z^4}{1.000000000-3.335431054z-0.6790590667z^2+0.6136657858z^3+0.004050155030z^4} \\
&- \frac{0.0005332896357z^5}{1.000000000-3.335431054z-0.6790590667z^2+0.6136657858z^3+0.004050155030z^4}.
\end{aligned} \tag{4.37}
$$

Finally, applying the inverse Laplace transforms to the Pade approximant solutions in (4.31) - (4.37) yields the following approximate solutions given by the following;

$$
\begin{aligned}
S_h(t) &= 0.008021267759e^{-0.2154814030t} + 0.0014782625e^{1.428139387t} \\
&+ 0.0000554751e^{4.220069807t} + 3.593\ 10^{-7}e^{9.078856139t} + 0.1654446352, \\
V_h(t) &= 0.9408028175e^{-0.01002234779t} + 0.000002642e^{1.512421411t} \\
&+ 1.00\ 10^{-7}e^{4.306612083t} + 1.0\ 10^{-9}e^{9.166057196t} + 0.009194438534, \\
E_h(t) &= 0.09174529798e^{-0.2179141521t} + 0.00003359144e^{1.412673919t} \\
&+ 4.861\ 10^{-7}e^{4.170294470t} + 1.610\ 10^{-9}e^{8.972110993t} + 0.008220622955, \\
I_h(t) &= 0.00000102380e^{-1.886000729t} + 0.08826015413e^{-0.4930913575t} \\
&+ 0.04274513347e^{0.3676131923t} + 1.622\ 10^{-8}e^{4.628426343t} - 0.01100632753, \\
T_h(t) &= -0.1289348104e^{-0.4967191290t} + 0.9993346520e^{-0.1453263101t} \\
&+ 0.0454653058e^{0.2612685710t} + .27\ 10^{-8}e^{3.575602442t} - 0.06586516050, \\
R_h(t) &= 0.0219625305e^{-0.4938390223t} - 0.2724635881e^{-0.1494023376t} \\
&+ 0.3855134406e^{0.3686142756t} - 1.0\ 10^{-10}e^{10.88986461t} - 0.01498761703, \\
E_v(t) &= 0.00161998552e^{-0.4959164711t} + 0.2321791195e^{-0.006553937959t} \\
&+ 0.00111227874e^{0.3581018392} + 3.1\ 10^{-10}e^{3.479799624t} - 0.1316714130.
\end{aligned}
\tag{4.38}
$$

5. Discussion of Results and Graphical Illustrations

In this work, a mathematical model describing VZV dynamics has been derived. The analytical findings reveal that the model solutions are positive and bounded. The controlled R_{vzv} showed that vaccination of susceptible births and immigrants and treatment of infected humans are effective in minimizing VZV infections in human and environmental host community. Linearizing the model around the VZV - free equilibrium by obtaining the model Jacobian showed that the model system is locally and globally asymptotically stable when R_{vzv} is less than unity.

However, we use the DTM to solve the model system equations, yielding a recursive system of power series solutions. In order to enhance the convergence of the DTM solutions, a hybrid posttreatment LPDTM is employed to widen the domain of convergence of the solutions. Furthermore, we compared the results using the LPDTM technique with RK45 method via Maple computational software.

Figures 2(a) and 2(b) shows the impact of timely vaccination of susceptible newborns and influx varying ϕ and ρ. The drastic decline in both figures showed

that drastic and timely vaccination is effective in creating a level herd immunity and minimizing VZV infection in the host community. Also, Figures 3(a) - 3(c) displays the effect of the varying the direct and indirect infectious contact rate between human to human β_1 and human to environment β_2 as well as the development of infectious shingles. The gradual rise of these curves depict that further interventions strategies are needed to be forestalled in order to curtail the spread of VZV infections, while the gradual decline in Figure 3(d) describing the variation of vaccination rate τ in susceptible births and immigrants reveal that vaccination wanes overtime in vaccinated individuals.

Figures 4(a) - 4(d) and 5(a) - 5(d) reveals the favorable agreement between the two methods in obtaining the approximate solutions of the model equations while producing a very low approximation error for the approximate solutions of the model system equations. The simulations further reveal that DTM and the Laplace Pade posttreatment technique reduces the volume of computation and enhances the efficiency of the technique.

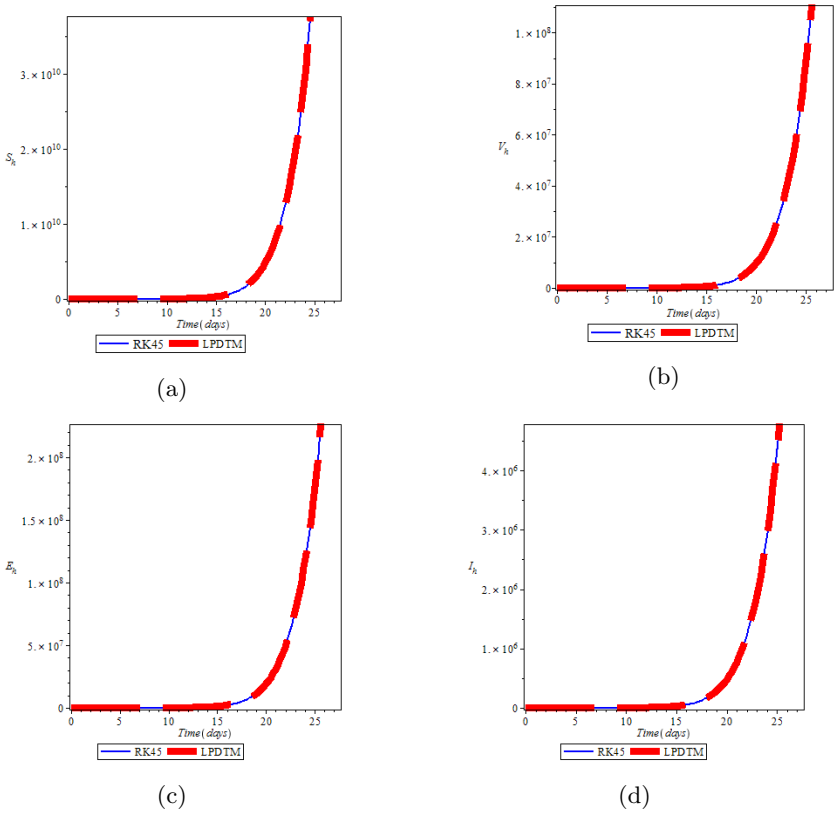


Figure 4. Model approximate solutions using LPDTM and RK45

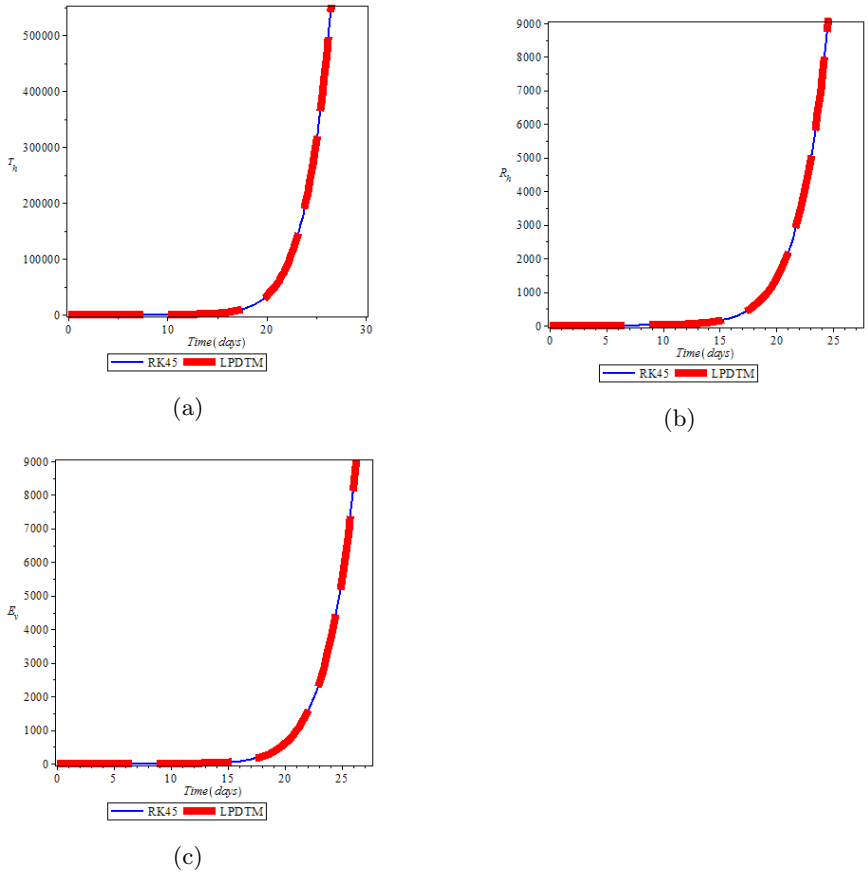


Figure 5. Model approximate solutions using LPDTM and RK45

5.1. Conclusion and Recommendations

This work proposes a mathematical model formulation of VZV dynamics incorporating essential variables and parameters. The model is shown to be positive, bounded and exist. The model is asymptotically stable locally and globally when R_{vzv} is less than unity. The DTM was employed to solve the model equations. The Laplace Pade posttreatment technique is further applied to enhance the convergence of the model solutions. This method is recommended to scientist and engineers to solve highly nonlinear models. Also, forms of interventions like timely vaccination and treatment are to be forestalled by public health practitioners to minimize VZV infections. This work also suggest application of further controls via optimal control theory to minimize reactivation of infectious shingles and concentration of the virus in environmental sources.

References

[1] Edward, S., Kuznetsov, D., and Mirau S.: Modeling and stability analysis for a varicella zoster virus model with vaccination. Applied and Computational Mathematics. 3**(4)**, 152-162, (2014); doi:10.11648/j.acm. 20140304.16.

[2] Garnet, G. P., Greenfal, B. T.: The epidemiology of varicella zoster virus infections: A mathematical model. Epidemiol. Infect 108(3), 495-511, (1992)

[3] Corbean-Vallet, A., Satonja, F. J., Jornet-Sanz, M. and Villanueva R. J.: Modeling diverse-pox dynamics with a discrete-time bayesian stochastic computational model. Complexity, pp.9, (2018); Article ID: 3060368.

[4] Korostil, I. A., Wood, J. G., and Regan D. G.: (2015). Periodicity of varicella zoster virus in the presence of immune boosting and clinical re-infection with varicella. Theoretical Biology and Medical Modeling, 12,**6**, 2015.

[5] Tang, X., Zhav, S., Chiu, A. P. Y., Ma, I., Xie, X., Mei S., Kong, D., Qin, Y., Chan, Z., Wang, X. and He, D.: Modeling the transmission and control strategies of Varicella among school children in Shenzhen, China. PLoS One, 12**(5)**, (2017); e0177514.

[6] Guzzetta, G., Poletti, P., Merler, S., and Manfiedi, P.: The epidemiology of herpes zoster after varicella immunization under different biological hypothesis; Perspectives from mathematical modeling. American Journal of Epidemiology, 8, **183**, 765 - 773, (2016)

[7] Gommel, M., Jaros, S. and Luu, D.: A mathematical model to characterize the transmission dynamics of the Varicella Zoster Virus (VZV). Proceeding of the National Conference on Undergraduate Research (NCUR), Weber State University, Ogder Utah, USA, 2012.

[8] Mahmood Hasani, Ostovar Ali Akber.: Application of the Homotopy Perturbation Method to solve influenza A(HN15) epidemic model with vaccination. Communications in Advance Computational Science with Applications, **2**, 50-53, (2015)

[9] Jahamdin. Homotopy perturbation method for a SEIR model with varying total population size. Far East Journal of Mathematical Science (FDMS), 2**84**, 187 - 198 (2014)

[10] Ogunmiloro, O. M.: Local and global asymptotic behavior of malaria filariasis co - infection in compliant and non compliant susceptible pregnant women to ante-natal medical program in the tropics e-Journal of Analysis and Applied Mathematics, 31 - 54, (2019); doi:10.2478/ejaam-2019-0003.

[11] Ogunmiloro, O. M.: Mathematical modeling of co- infection dynamics of malaria - toxoplasmosis in the tropics. Biometrical Letters, **56**, issue 2, (2019)

[12] Ogunmiloro, O. M., Abedo, F. O., and Kareem, H. A.: Numerical and stability analysis of the transmission dynamics of SVIR epidemic model with standard incidence rate. Malaysian Journal of Computing, 4**(2)**, 349 - 361, (2019)

[13] Zhou, J. K.: Differential transformation and its application for electrical circuits. Huangzhong University Press(Chinese), Wuhan, China, (1986)

[14] Batiha, A. M., Batiha, B.: Differential transformation method for a reliable treatment of the nonlinear biochemical reaction model. Advances in Difference Equations, 174, 2012.

[15] Vazquez - Leal, H.: Generalized homotopy method for solving nonlinear differential equations. Computational and Applied Mathematics, **33**, 275-288, (2014)

[16] Behammouda, B., Vazquez - Leal H. and Hernandez - Martinoz L.: Modified differential transform method for solving the model of pollution for a system of lakes. Discrete Dynamics in Nature and Society. **2014**, pp.12, (2014); Article ID 645726.

[17] Biazar, J., Eslami, M.: Differential transform method for quadratic Riccati differential equation. International Journal of Nonlinear Science, 4**(9)**, 444-447, (2010)

[18] Merdan, M.: A new application of modified differential transformation method for modeling the pollution of a system of lakes. Selcuk Journal of Applied Mathematics, 2**(111)**, 27-40, (2010)

[19] Ebaid, A. E.: A reliable aftertreatment for improving the differential transformation method and its application to nonlinear oscillators with fractional nonlinearities. Commmunications in Nonlinear Science and Numerical Simulations, **16**, 1850-1862, (2010)

[20] Ogunmiloro, O. M., Ojo, O.: Transmission dynamic response of cholera epidemic model to indirect and direct infectious contact: Stability and homotopy analysis

method. Journal of Computer Science and Computational Mathematics, **10**, 1, (2020); doi:10.20967/jcscm.2020.01.003.
[21] Zou, Li, Zhen, Wang., and Zhi Zong.: Generalized differential transform method to differential - difference equation. Physics Letters A, 45**(373)**, 4142-4151, (2009)

Received: 19 July 2020/Accepted: 3 December 2021/Published online: 7 December 2021

Oluwatayo M. Ogunmiloro
Ekiti State University, Ado - Ekiti, Ekiti State, Nigeria
oluwatayo.ogunmiloro@eksu.edu.ng

Michael O. Oke
Ekiti State University, Ado - Ekiti, Ekiti State, Nigeria
michael.oke@eksu.edu.ng

Precious O. Ojakovo
Ekiti State University, Ado - Ekiti, Ekiti State, Nigeria
preciouslouis1990@gmail.com