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# A regularity criterion of 3D incompressible MHD system with mixed pressure-velocity-magnetic field

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**Abstract.** *This work focuses on the 3D incompressible magnetohydrodynamic (MHD) equations with mixed pressure-velocity-magnetic field in view of Lorentz spaces. Our main result shows the weak solution is regular, provided that*

$$\frac{\pi}{\left(e^{-|x|^2}+|u|+|b|\right)^{\theta}} \in L^{p}(0,T;L^{q,\infty}), \quad \textit{where } \frac{2}{p}+\frac{3}{q}=2-\theta \textit{ and } 0\leq\theta\leq 1.$$

## 1. Introduction

We are interested in the regularity of weak solutions to the viscous incompressible magnetohydrodynamics (MHD) equations;

$$\begin{cases} \partial_t u+(u\cdot\nabla)u-(b\cdot\nabla)b-\Delta u+\nabla\pi=0, \quad x\in\mathbb{R}^3,\ t\in(0,T), \\ \partial_t b+(u\cdot\nabla)b-(b\cdot\nabla)u-\Delta b=0, \quad x\in\mathbb{R}^3,\ t\in(0,T), \\ \nabla\cdot u=\nabla\cdot b=0, \quad x\in\mathbb{R}^3,\ t\in(0,T), \\ u(x,0)=u_0(x) \text{ and } b(x,0)=b_0(x), \quad x\in\mathbb{R}^3, \end{cases} \tag{1.1}$$

where $u, b$ and $\pi$ stand for the velocity field, the magnetic field, and the scalar pressure, respectively, and $T\in(0,\infty)$ is an arbitrary existence time. Equations (1.1) describe the motion of electrically conducting fluid in the presence of a magnetic field, where the fluid and magnetic field interact strongly with each other, and then their dynamics are coupled. They present applications in several physical situations; for instance, liquid metals, cosmic plasmas, ionized fluids in astrophysics, geophysics, and high-speed aerodynamics (see, e.g., [2, 5, 18]). Moreover, taking $b=0$ and $b_0=0$, we obtain the 3D Navier–Stokes equations which describe the motion of viscous incompressible fluids. Thus, due to their mathematical and physical interest, they have been studied by a number of mathematicians, physicists, engineers etc.

An important open problem for system (1.1) and 3D Navier–Stokes equations is the global regularity of weak solutions. In this address, there is a rich literature

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with several types of regularity criterion in different functional spaces (Lebesgue, weak-$L^p$; Besov, among others). For example, see [3, 4, 6, 7, 9, 10, 11, 16, 17, 21, 22] and references therein. Note that the literatures listed here are far from being complete, we refer the readers to see for example [8, 12, 13, 14, 15] for expositions and more references. As a matter of fact, Beirão da Veiga and Yang [1] provided a regularity criterion for weak-solutions $(u, \pi)$ of the 3D Navier–Stokes equations with divergence-free initial data $u_0 \in L^2 \cap L^4$. They proved that weak solutions $(u, \pi)$ are regular by assuming the pressure-velocity condition

$$\frac{\pi}{\left(e^{-|x|^2} + |u|\right)^{\theta}} \in L^p(0, T; L^{q,\infty}), \tag{1.2}$$

where $0 \leq \theta \leq 1$, $1 \leq p, q \leq \infty$, $\frac{2}{p} + \frac{3}{q} = 2 - \theta$ and $L^{q,\infty}$ stands for the weak-$L^q$ (c.f. [20]).

Motivated by [1], the present work extends the above criterion to the system (1.1). More precisely,

**Theorem 1.1.** *Suppose that* $(u_0, b_0) \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ *in the sense of distribution. Let* $(u, b)$ *be a weak solution to the MHD equations on some interval* $[0, T]$ *with* $0 < T < \infty$. *Assume that* $0 \leq \theta \leq 1$ *and that*

$$\frac{\pi}{\left(e^{-|x|^2} + |u| + |b|\right)^{\theta}} \in L^p(0, T; L^{q,\infty}(\mathbb{R}^3)), \quad \text{where } \frac{2}{p} + \frac{3}{q} = 2 - \theta \tag{1.3}$$

*then the weak* $(u, b)$ *is regular on* $(0, T]$.

**Remark 1.2.** A special consequence of Theorem 1.1 and its proof is the regularity criterion of the 3D Navier–Stokes equations with the mixed pressure-velocity in Lorentz spaces. This generalizes those of [1].

In order to derive the regularity criterion of weak solutions to the MHD equations (1.1), we introduce the definition of weak solution.

Next, let us write

$$w^{\pm} = u \pm b, \quad w_0^{\pm} = u_0 \pm b_0.$$

We reformulate equation (1.1) as follows. Formally, if the first equation of MHD equations (1.1) plus and minus the second one, respectively, then MHD equations (1.1) can be re-written as:

$$\begin{cases} \partial_t w^+ - \Delta w^+ + (w^- \cdot \nabla) w^+ + \nabla \pi = 0, \\ \partial_t w^- - \Delta w^- + (w^+ \cdot \nabla) w^- + \nabla \pi = 0, \\ \operatorname{div} w^+ = 0, \quad \operatorname{div} w^- = 0, \\ w^+(x, 0) = w_0^+(x), \quad w^-(x, 0) = w_0^-(x). \end{cases} \tag{1.4}$$

The advantage is that the equations become symmetric.

## 2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. In order to do it, we first recall the following estimates for the pressure in terms of $u$ and $b$ (see e.g., [8]):

$$\|\pi\|_{L^q} \le C\left(\|u\|_{L^{2q}}^2 + \|b\|_{L^{2q}}^2\right), \quad \text{with} \quad 1 < q < \infty. \tag{2.1}$$

We are now in position to prove our main result.

*Proof.* Multiplying the first and the second equations of (1.4) by $|w^+|^2 w^+$ and $|w^-|^2 w^-$ , respectively, integrating by parts and summing up, we have

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}(\|w^+\|_{L^4}^4 + \|w^-\|_{L^4}^4) + \int_{\mathbb{R}^3}(\left|\nabla w^+\right|^2\left|w^+\right|^2 + \left|\nabla w^-\right|^2\left|w^-\right|^2)dx \\
&+\frac{1}{2}\int_{\mathbb{R}^3}(\left|\nabla\left|w^+\right|^2\right|^2 + \left|\nabla\left|w^-\right|^2\right|^2)dx \\
=& -\int_{\mathbb{R}^3}\nabla\pi\cdot(w^+\left|w^+\right|^2 + w^-\left|w^-\right|^2)dx \\
=& \int_{\mathbb{R}^3}\pi\cdot\operatorname{div}(w^+\left|w^+\right|^2 + w^-\left|w^-\right|^2)dx \\
\le& \int_{\mathbb{R}^3}|\pi|\left(\left|w^+\right| + \left|w^-\right|\right)(\nabla\left|w^+\right|^2 + \nabla\left|w^-\right|^2)dx \\
\le& C\int_{\mathbb{R}^3}|\pi|^2\left(\left|w^+\right| + \left|w^-\right|\right)^2dx + \frac{1}{4}\int_{\mathbb{R}^3}(\left|\nabla\left|w^+\right|^2\right|^2 + \left|\nabla\left|w^-\right|^2\right|^2)dx.
\end{aligned}$$

Notice that $u = \frac{1}{2}(w^+ + w^-)$ and $b = \frac{1}{2}(w^+ - w^-)$, then the above inequality means that

$$\begin{aligned}
&\frac{d}{dt}(\|u\|_{L^4}^4 + \|b\|_{L^4}^4) + 2\left\|\nabla|u|^2\right\|_{L^2}^2 + 2\left\|\nabla|b|^2\right\|_{L^2}^2 \\
&+2\left\||u|\,|\nabla u|\right\|_{L^2}^2 + 2\left\||b|\,|\nabla b|\right\|_{L^2}^2 + 2\left\||u|\,|\nabla b|\right\|_{L^2}^2 + 2\left\||b|\,|\nabla u|\right\|_{L^2}^2 \\
\le& C\int_{\mathbb{R}^3}|\pi|^2\left(|u| + |b|\right)^2dx = K,
\end{aligned} \tag{2.2}$$

where we have used

$$\left|w^+\right| + \left|w^-\right| \le \left|w^+ + w^-\right| + \left|w^+ - w^-\right|.$$

For $K$, borrowing the arguments in [1], we set

$$V = e^{-|x|^2} + |u| + |b| \quad \text{and} \quad \widetilde{\pi} = \frac{\pi}{\left(e^{-|x|^2} + |u| + |b|\right)^\theta}.$$

By the Hölder inequality and the following interpolation in Lorentz space (see [20])

$$\|f^\alpha\|_{L^{p,q}(\mathbb{R}^3)} \le C\,\|f\|_{L^{\alpha p,\alpha q}(\mathbb{R}^3)}^\alpha \quad \text{for} \;\; \alpha > 0,\; p > 0,\; q > 0,$$

we have

$$\begin{aligned}
K &= \int_{\mathbb{R}^3} |\pi|^\lambda V^{-\lambda\theta} |\pi|^{2-\lambda} V^{\lambda\theta}(|u|+|b|)^2 dx \\
&\le \int_{\mathbb{R}^3} |\widetilde{\pi}|^\lambda |\pi|^{2-\lambda} V^{2+\lambda\theta} dx \\
&\le \left\| |\widetilde{\pi}|^\lambda \right\|_{L^{\frac{q}{\lambda},\infty}} \left\| |\pi|^{2-\lambda} \right\|_{L^{s,\frac{2}{2-\lambda}}} \left\| V^{2\lambda} \right\|_{L^{r,\frac{2}{\lambda}}} \\
&= \|\widetilde{\pi}\|_{L^{q,\infty}}^\lambda \|\pi\|_{L^{s(2-\lambda),2}}^{2-\lambda} \left\| V^2 \right\|_{L^{\lambda r,2}}^\lambda ,
\end{aligned}$$

where

$$\frac{\lambda}{q} + \frac{1}{s} + \frac{1}{r} = 1 \quad \text{and} \quad \lambda = \frac{2}{2-\theta}.$$

By (2.1), we have

$$\begin{aligned}
K &\le \|\widetilde{\pi}\|_{L^{q,\infty}}^\lambda \left( \left\| |u|^2 \right\|_{L^{s(2-\lambda),2}} + \left\| |b|^2 \right\|_{L^{s(2-\lambda),2}} \right)^{2-\lambda} \left\| V^2 \right\|_{L^{\lambda r,2}}^\lambda \\
&\le C \|\widetilde{\pi}\|_{L^{q,\infty}}^\lambda \left\| V^2 \right\|_{L^{s(2-\lambda),2}}^{2-\lambda} \left\| V^2 \right\|_{L^{\lambda r,2}}^\lambda .
\end{aligned}$$

By the interpolation and Sobolev inequalities in Lorentz spaces, it follows that

$$\begin{cases}
\left\| V^2 \right\|_{L^{s(2-\lambda),2}} \le C \left\| V^2 \right\|_{L^{2,2}}^{1-\delta_1} \left\| V^2 \right\|_{L^{6,2}}^{\delta_1} \le C \left\| V^2 \right\|_{L^2}^{1-\delta_1} \left\| \nabla V^2 \right\|_{L^2}^{\delta_1} , \\
\left\| V^2 \right\|_{L^{\lambda r,2}} \le C \left\| V^2 \right\|_{L^{2,2}}^{1-\delta_2} \left\| V^2 \right\|_{L^{6,2}}^{\delta_2} \le C \left\| V^2 \right\|_{L^2}^{1-\delta_2} \left\| \nabla V^2 \right\|_{L^2}^{\delta_2} ,
\end{cases} \tag{2.3}$$

where $0 < \delta_1, \delta_2 < 1$ and

$$\frac{1}{s(2-\lambda)} = \frac{1-\delta_1}{2} + \frac{\delta_1}{6}, \quad \frac{1}{\lambda r} = \frac{1-\delta_2}{2} + \frac{\delta_2}{6}.$$

Hence from (2.3) and Young inequality, it follows that

$$\begin{aligned}
K &\le C \|\widetilde{\pi}\|_{L^{q,\infty}}^\lambda \left\| V^2 \right\|_{L^2}^{(2-\lambda)(1-\delta_1)+\lambda(1-\delta_2)} \left\| \nabla V^2 \right\|_{L^2}^{(2-\lambda)\delta_1+\lambda\delta_2} \\
&\le C \|\widetilde{\pi}\|_{L^{q,\infty}}^{\frac{2\lambda}{2-(2-\lambda)\delta_1-\lambda\delta_2}} \left\| V^2 \right\|_{L^2}^2 + \frac{1}{2} \left\| \nabla V^2 \right\|_{L^2}^2 .
\end{aligned}$$

Due to the definition of $V$, we see that

$$\left\| V^2 \right\|_{L^2}^2 \le C(1 + \| |u| + |b| \|_{L^2}^2 + \left\| |u|^2 + |b|^2 \right\|_{L^2}^2),$$

and

$$\left\| \nabla V^2 \right\|_{L^2}^2 \le C \left( 1 + \| |u| + |b| \|_{L^2}^2 + \| \nabla (|u| + |b|) \|_{L^2}^2 + \left\| \nabla (|u|^2 + |b|^2) \right\|_{L^2}^2 \right).$$

Consequently, we get

$$
\begin{aligned}
K \leq\ & C \|\widetilde{\pi}\|_{L^{q,\infty}}^{\frac{2\lambda}{2-(2-\lambda)\delta_1-\lambda\delta_2}} (1 + \||u| + |b|\|_{L^2}^2 + \left\||u|^2 + |b|^2\right\|_{L^2}^2) \\
& + C(1 + \||u| + |b|\|_{L^2}^2 + \|\nabla(|u| + |b|)\|_{L^2}^2) + \frac{1}{2}\left\|\nabla(|u|^2 + |b|^2)\right\|_{L^2}^2 \\
\leq\ & C \|\widetilde{\pi}\|_{L^{q,\infty}}^{\frac{2\lambda}{2-(2-\lambda)\delta_1-\lambda\delta_2}} (1 + \|u\|_{L^2}^2 + \|b\|_{L^2}^2 + \|u\|_{L^4}^4 + \|b\|_{L^4}^4) \\
& + C(1 + \|u\|_{L^2}^2 + \|b\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \frac{1}{2}\left\|\nabla |u|^2\right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla |b|^2\right\|_{L^2}^2.
\end{aligned}
$$

Since $(u, b)$ is a weak solution to (1.1), then $(u, b)$ satisfies

$$(u, b) \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)).$$

Inserting the above estimates into (2.2), we obtain

$$
\begin{aligned}
& \frac{d}{dt}(\|u\|_{L^4}^4 + \|b\|_{L^4}^4) + \left\|\nabla |u|^2\right\|_{L^2}^2 + \left\|\nabla |b|^2\right\|_{L^2}^2 \\
& + 2\,\||u|\,|\nabla u|\|_{L^2}^2 + 2\,\||b|\,|\nabla b|\|_{L^2}^2 + 2\,\||u|\,|\nabla b|\|_{L^2}^2 + 2\,\||b|\,|\nabla u|\|_{L^2}^2 \\
\leq\ & C \|\widetilde{\pi}\|_{L^{q,\infty}}^{\frac{2\lambda}{2-(2-\lambda)\delta_1-\lambda\delta_2}} (1 + \|u\|_{L^2}^2 + \|b\|_{L^2}^2 + \|u\|_{L^4}^4 + \|b\|_{L^4}^4) \\
& + C(1 + \|u\|_{L^2}^2 + \|b\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \\
\leq\ & C \|\widetilde{\pi}\|_{L^{q,\infty}}^{\frac{2\lambda}{2-(2-\lambda)\delta_1-\lambda\delta_2}} (1 + \|u\|_{L^4}^4 + \|b\|_{L^4}^4) + C(1 + \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2),
\end{aligned}
$$

Using Gronwall's inequality with the assumption (1.3), we deduce that

$$(u, b) \in L^\infty(0, T; L^4(\mathbb{R}^3)) \subset L^8(0, T; L^4(\mathbb{R}^3)).$$

We complete the proof of Theorem 1.1. □

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