Rend. Mat. Appl. (7),
Volume 44 (2023), 77–158

Rendiconti di Matematica
e delle sue Applicazioni

A nonautonomous dynamical system applied to dengue seasonality

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Abstract. *Seasonality due to environmental influences often affects contact between species for food or shelter as well as the spread and persistence of diseases from those vector species. Epidemic models may capture seasonality patterns in a phenomenological way by making the epidemiological parameters and the population demographics are time-periodic. A mathematical model with these features for the dengue fever is analyzed, to such an extent that the threshold between uniform persistence and extinction of the disease is established, that is: there exists a unique positive disease-free periodic solution being globally asymptotically stable when the basic reproductive number is greater that one, but it is unstable when the basic reproductive number is less than one, in whose situation there exists at least one non-trivial positive periodic solution and dengue fever is endemic in the community. At last, numerical simulations are carried out to illustrate the theoretical results.*

1. Introduction

Dengue fever (DENV) is the arthropod-transmitted disease with the highest morbi-mortality in the world, also one of the most frequent causes of hospitalization and significant interruption of income potential in endemic areas (an estimated 390 million people become infected every year, 500 000 people suffering from severe dengue require hospitalization and 2.5% die), it affects the tropical and subtropical countries of Asia, the Pacific Islands, the Caribbean islands, Africa and Central and South America [1]. There are macrofactors to explain the increase of DENV on a global scale: climatics (global warming) and social, such as the increase in world population, the tendency to disorderly urbanization, international travel and poverty expressed in problems of housing, education, water supply, solid waste collection and others, as well as the lack of effective national and international programs against this disease and its vector; currently, vector control is the predominant strategy to prevent the spread of DENV because there are no effective, economical or tetravalent vaccine and treatment for disease [2].

DENV belongs to the family *Flaviviridae* and there are four serotypes formally recognized: DEN-1, DEN-2, DEN-3 and DEN-4 [3], although in October 2013 a fifth sylvatic serotype (DENV-5) has been detected during screening of viral samples taken from a 37 year old farmer admitted in hospital in Sarawak state of

2020 Mathematics Subject Classification: 37C60, 34C25, 37C25.
Keywords: *Aedes aegypti* and dengue, seasonality, periodic system, stability, extinction, persistence.

Malaysia in the year 2007 [4]; the infection by a serotype 1 to 4 confers permanent immunity against this serotype and only for a few months against the rest of the serotypes; if a person is infected by one of the four serotypes, they will never be infected by the same serotype (homologous immunity), but lose immunity to the other three serotypes (heterologous immunity) in approximately 12 weeks and then becomes more susceptible to developing dengue hemorrhagic fever [5].

The primary vector of DENV is *Aedes aegypti* and the secondary vector is *Aedes albopictus*, both can feed at any time during the day and acquires the virus through the bite to a sick person during his period of viremia, which goes from a day before the onset of fever to an average of 5 or 6 days after the start of the same, being able to reach up to 9–10 days exceptionally [6]. Seasonal variations in climatic factors, such as temperature, humidity and rainfall significantly influence the mosquito development and several studies suggest that entomological parameters are temperature sensitive as the dengue fever normally occurs in tropical and subtropical regions [7]; the high temperature increases the lifespan of mosquitoes and shortens the extrinsic incubation period of the dengue virus, increasing the number of infected mosquitoes, the rainfall provides places for eggs and for larva development, thereby affecting the distribution and abundance of vectors seasonally [8].

The mathematical modeling approach is an important tool to explore the complex dynamics of any real-world problem, including infectious diseases, whose mathematical models can be used to plan control and mitigation measures in a community in the face of any future epidemic [9, 10]. It was shown that seasonality plays a major role in the size of the mosquito population, which influences the decision of effective strategies to control the disease [11, 12], therefore, it is critical to incorporate seasonal effects into dengue transmission modeling.

A natural and important problem associated with epidemic models is to estimate whether an infection can invade and persist in a population, and then determine a measure of the effort required to control it, a threshold value used for this is the basic reproduction number (BRN). Diekmann et al., van den Driessche and Watmough [13]–[15] presented a general approach for the calculus of the BRN for autonomous ordinary differential equations models with compartmental structure. In the past twenty years, many authors have extended the definition of the BRN to periodic environments, we highlight authors like Bacaër and Guernaoui (2006), Wang and Zhao (2008), Thieme (2009), Bacaër (2011), Inaba (2012), Bacaër and Ait Dads (2012), Wang and Zhao (2017) [16]–[22].

Motivated by the above discussion, initially, a host-vector compartmental model is formulated to represent disease transmission, including seasonality of mosquito recruitment, mosquito mortality, and human-mosquito contacts in general way —many regions have shown seasonal patterns, leading researchers to develop mathematical models with periodic transmission rates [23, 24] and periodic demographic rates due to mosquito life-cycle [25, 26]—. This research is focused on global extinction and uniform persistence of the dengue in uncontrolled dynamics, where the BRN addressed in [16, 17] serves as threshold value —conditions that ensure the uniform persistence of a given disease in a periodic environment has been also studied in [27, 28]—.

This paper is structured as follows: the next two sections are dedicated to the methodological approach of the proposed model. In subsection 4.1, the existence of a dengue-free periodic solution is discovered. In subsection 4.2, the qualitative properties of the model are completely determined by the BRN. In subsections 4.3, 5.1 and 5.2, it is proved that the BRN serves as a threshold parameter that determines the global stability of the disease-free periodic solution (DFS), the instability of the DFS gives rise to the existence of at least a endemic positive ω-periodic solution and persistence is guaranteed through a analysis of the flow in the boundary. In section 6, the BRN is derived numerically by solving a matrix eigenvalue problem and the analytical findings are illustrated by numerical simulations. In Section 7, some final comments about the results are done.

2. Notation and basic concepts

Throughout the manuscript the reader should know the following: (t) is allowed to be omitted in time-dependent expressions, bold icons in a mathematical environment exclusively denote vectors or matrices, capital letters are used to represent matrices or scalars, the superscript $\top$ indicates transposition of a matrix, $\Re(\lambda)$ is the real part of λ, $\rho(A)$ denotes the spectral radius of a matrix A, I_n is the $n\times n$ identity matrix, O_n is the $n\times n$ null matrix, O_n^* is the matrix of order n with each element being 1, and $\mathfrak{diag}\,\mathbf{v}$ denotes a diagonal matrix with $\mathrm{v}_i = \mathrm{v}_{ii}$ located on the main diagonal; *e.g.*, the expression $\mathbf{a} = \begin{bmatrix} a_1 & a_2 \dots a_n \end{bmatrix}^\top$ is the same as

$$\mathbf{a}^\top = \begin{bmatrix} a_1 & a_2 \dots a_n \end{bmatrix} \text{ or } \mathbf{a}(t) = \begin{bmatrix} a_1(t) & a_2(t) \dots a_n(t) \end{bmatrix}^\top$$

if it denotes a vector function in t from an sub-interval of $\mathbb{R} = \mathbb{R}_+ \cup \mathbb{R}_-$ to $\mathbb{R}^n$, where $\mathbb{R}_+ := [0\,,\,\infty)$ and $\mathbb{R}_- := (-\infty\,,\,0)$.

Let $\left(\mathbb{R}^n, \mathbb{R}^n_+\right)$ be the standard ordered n-dimensional Euclidean space with a norm $\|\cdot\|$. If $\mathbf{a}, \mathbf{b}, \mathbf{0} \in \mathbb{R}^n$, $\mathbf{0}$ is the null vector and $\mathsf{Int}\,(\cdot)$ is the interior of a set, it is written: $\mathbf{a} \geq \mathbf{b}$ provided $\mathbf{a} - \mathbf{b} \in \mathbb{R}^n_+$, $\mathbf{a} > \mathbf{b}$ provided $\mathbf{a} - \mathbf{b} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$, $\mathbf{a} \gg \mathbf{b}$ provided $\mathbf{a} - \mathbf{b} \in \mathsf{Int}\left(\mathbb{R}^n_+\right) \setminus \{\mathbf{0}\}$. Each element of the canonical basis of $\mathbb{R}^n$ is symbolized by $\mathbf{e}_i$ when 1 occupies its component i, and to refer to component i of any other vector, say $\mathbf{v}$, we write it as a dot product: $\mathbf{v} \cdot \mathbf{e}_i = v_i$.

Inequalities between vectors are considered in their usual coordinate-wise sense, *i.e.*, for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$:

$$\mathbf{a} \geq \mathbf{b} \Longleftrightarrow a_i \geq b_i, \quad i = 1, 2, \dots, n.$$

Particularly, $\mathbf{a}$ is positive if $\mathbf{a} > \mathbf{0}$, $\mathbf{b}$ is nonnegative if $\mathbf{b} \geq \mathbf{0}$, and $\mathbf{1} > 0$ is the vector of which all components are identically 1.

We say that $A = \begin{bmatrix} a_{ij} \end{bmatrix} \in \mathbb{M}_{\overline{m}}(\mathbb{R})$ (where $\mathbb{M}$ is the vector space of all real matrices of order $\overline{m}$) is cooperative if all its off-diagonal elements are non-negative and we say that A is irreducible if it cannot be placed into block upper-triangular form by simultaneous row/column permutations, or if its index set $\{1, 2, \dots, n\}$ cannot be split into two complementary sets (without common indices) $\{c_1, c_2, \ldots, c_{n_1}\}$ and $\{\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_{n_2}\}$ $(n = n_1 + n_2)$ such that $a_{c_\beta \bar{c}_\nu} = 0$, for all $1 \leq \beta \leq n_1$, $1 \leq \nu \leq n_2$.

Suppose $X(t)$ is a real and differentiable matrix function of t. We define

$$e^{X(t)} = \sum_{n=0}^{\infty} \frac{(X(t))^n}{n!}.$$

When $(dX/dt)\, X(t) = X(t)\, (dX/dt)$, where the derivative is taken entry-wise, it holds

$$\frac{d}{dt} e^{X(t)} = e^{X(t)} \left(\frac{d}{dt} X(t) \right) = \left(\frac{d}{dt} X(t) \right) e^{X(t)} \tag{2.1}$$

For a non-negative, continuous ω-periodic function $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}$, let

$$\sigma^u = \sup_{t\in[0,\,\omega]} \sigma(t) = \sigma_{\sup}, \quad \sigma^l = \inf_{t\in[0\,,\,\omega]} \sigma(t) = \sigma_{\inf},\ \sigma^\infty = \limsup_{t\longrightarrow\infty} \sigma(t), \quad \sigma_\infty = \liminf_{t\longrightarrow\infty} \sigma(t).$$

The floor $(\lfloor\cdot\rfloor)$, ceiling $(\lceil\cdot\rceil)$, sign and indicatrix $(\chi_{\mathcal{D}}(\cdot),\, \mathcal{D} \subseteq \mathbb{R})$ of s are defined as:

$$\lfloor s\rfloor = \max\{n \in \mathbb{Z}_+ \,|\, n \leq s\},\ \lceil s\rceil = \min\{n \in \mathbb{Z}_+ \,|\, s \leq n\},\ \mathrm{sgn}(s) = \begin{cases} -1\,, & s < 0 \\ 0\,, & s = 0 \\ 1\,, & s > 0 \end{cases},\ \chi_{\mathcal{D}}(s) = \begin{cases} 1\,, & s \in \mathcal{D} \\ 0\,, & s \notin \mathcal{D} \end{cases};$$

with $\mathbb{Z}_+ := \{1, 2, 3, \ldots, n, \ldots\}$. The long-term average of σ on the interval $[\tau\,,\, t+\tau]$ is defined as:

$$\langle\sigma\rangle = \lim_{\tau\to\infty} \frac{1}{\tau} \int_t^{t+\tau} \sigma(\tau) d\tau.$$

3. Mathematical model formulation

It is considered a mathematical model of a dengue serotype that is spread in a community due to the ecological interaction of humans and mosquitoes of the *Ae. aegypti* species. The human population is divided into classes or states that contain susceptible, latent, symptomatic, and immune individuals; for its part, the vector population is described using an analogous SEI model (non-carrier, non-infectious carrier and infectious carrier); all are born susceptible and non-carriers in both populations (there is no vertical transmission) and confined to a particular geographic area.

Dengue is mainly transmitted by the daily bite of mosquitoes, whose life cycle is influenced by the seasonality of the climatic variables: the density of adult vectors is usually higher during the wet season [29, 30] and it is known that ambient temperature regulates dengue transmission through its effects on adult longevity, blood feeding activity and virus incubation within the mosquito [31]. The rainy season is key for the mosquitoes to lay their batches of eggs and to emerge that population of culicids that hibernated, even the number of mosquitoes falls to low levels and outside this season the natural mortality of the vectors decreases [32, 33]; the temporality of these processes determines a greater or lesser number of dengue cases in positive correlation with the abundance of the vectors [34, 35], for this reason, the vector rates of emergence, mortality and bite are considered periodic functions in time.

On the other hand, it is plausible to assume that people experience the same rate of change in births and deaths not induced by dengue, given that the death rate from dengue is less than 1% under adequate medical care and the human population practically does not change on the time scale of several generations of mosquitoes [36, 37].

Being more specific, the proposed model is valid under the following assumptions:

1. Preserving some resemblance regarding the symptomatology of the disease in the hosts (humans), we use the following nomenclature:
    - susceptible population/non-carrier population, subscript S, comprising those individuals capable of catching the disease;
    - non-infectious infected population/non-infectious carrier population, subscript E, comprising those mosquitoes temporarily unable of transmitting the disease;
    - symptomatic population/infectious carrier population, subscript I, comprising those individuals capable of transmitting the disease; and
    - recovered or inmune population, subscript R, including those individuals who acquire permanent immunity against infection.
2. All vector population measures refer to densities of female mosquitoes.
3. Alternative dengue virus hosts are not considered as blood sources.
4. Dengue-induced mortality in humans or vectors is not considered.
5. Carrier vectors probably transmit the virus throughout the life-span.
6. The total population of hosts is constant (births balance deaths).
7. Mosquito demographics and human-vector contact are modeled including time-periodic parameters.

Most of the terms of the model can be understood with the following list:

- $m(t)$: natural mortality rate of adult mosquitoes at time t.
- h: natural mortality rate of humans.
- $\hat{l}$: rate of humans who develop dengue symptoms.

- r: human recovery rate.
- $b(t)$: average number of bites per mosquito per day at time t.
- p: probability of transmission from a symptomatic human to a non-carrier mosquito.
- q: probability of transmission of an infectious carrier mosquito by bite on a susceptible human.
- c: transfer rate of mosquitoes from non-infectious carrier to infectious carrier.
- $\Delta(t)$: mosquito recruitment rate (by birth and immigration) at time t.
- $\mathrm{H}(t)$: average number of people in the community at time t.

The effective contact rates between the two populations, defined as the average number of contacts (blood feeding) per day that will cause inoculation of a serotype from one party if the other party is infectious, depends on several factors: the bite rate of mosquitoes, the probabilities of transmission between species and the number of individuals in both populations. Mosquitoes and people who have recently acquired the virus move to the states of non-infectious carriers and exposed at rates

$$\Lambda_{\mathrm{M}}(\mathrm{H}_{\mathrm{I}}, \mathrm{M}_{\mathrm{S}}) = \left(\frac{qb(t)\mathrm{H}_{\mathrm{I}}(t)}{\mathrm{H}(t)}\right)\mathrm{M}_{\mathrm{S}}(t) \text{ and } \Lambda_{\mathrm{S}}(\mathrm{H}_{\mathrm{H}}, \mathrm{M}_{\mathrm{I}}) = \left(\frac{pb(t)\mathrm{H}_{\mathrm{H}}(t)}{\mathrm{H}(t)}\right)\mathrm{M}_{\mathrm{I}}(t) \quad (3.1)$$

respectively. The incidence terms (3.1) called *standard* or *frequency-dependent* are interpreted as follows: if $b(t)$ is the bite rate for mosquitoes, a proportion of the number of bites that are not potentially contagious to humans is $\mathrm{M}_{\mathrm{S}}(t)/\mathrm{H}(t)$ and a proportion of the number of bites that are potentially contagious to humans is $\mathrm{M}_{\mathrm{I}}(t)/\mathrm{H}(t)$, so there are $b(t)\mathrm{M}_{\mathrm{S}}(t)/\mathrm{H}(t)$ bites per human per unit of time that are not potentially contagious to humans and $b(t)\mathrm{M}_{\mathrm{I}}(t)/\mathrm{H}(t)$ bites per human per unit time that are potentially contagious to humans; now, since there are $\mathrm{H}_{\mathrm{I}}(t)$ symptomatic people and $\mathrm{M}_{\mathrm{I}}(t)$ infectious carrier mosquitoes, the number of blood intakes taken from viremic people is $b(t)\mathrm{H}_{\mathrm{I}}(t)\mathrm{M}_{\mathrm{S}}(t)/\mathrm{H}(t)$ and the number of blood intakes taken by infectious carrier mosquitoes is $b(t)\mathrm{H}_{\mathrm{S}}(t)\mathrm{M}_{\mathrm{I}}(t)/\mathrm{H}(t)$, however, only corresponding fractions p and q from these bites successfully extract and inoculate the virus.

The standard incidence applies because a non-carrier mosquito can bite a finite number of people in a unit of time in a large human population until it obtains enough blood to provide protein for egg production [38]. The non-infectious carrier mosquito becomes an infectious carrier at a rate c, where $1/c$ is the extrinsic incubation period; analogously, the exposed human becomes symptomatic when the intrinsic incubation period is completed, which occurs at a rate $\hat{l}$, where $1/\hat{l}$ is the intrinsic incubation period; finally, infectious humans recover at a rate r, where $1/r$ is the length of the recovery period.

Transmission dynamics is interpreted according to the compartment diagram in Figure 1; this diagram is made up of seven state compartments and the flows between classes, four compartments for the human population and three for the vector population.

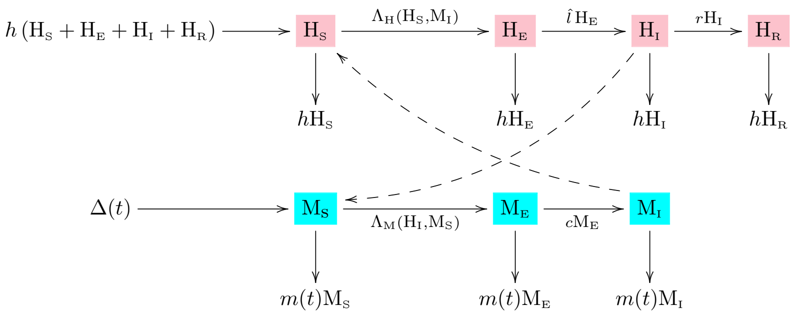


Figure 1: Flowchart of the model (3.2), the broken lines represent interactions involved in new infections.

The above explanations lead to the following nonlinear nonautonomous system of ordinary differential equations:

$$\begin{cases} \dot{\mathrm{H}}_{\mathrm{S}} = h\mathrm{H} - \dfrac{qb(t)\mathrm{M_I}\mathrm{H_S}}{\mathrm{H}} - h\mathrm{H_S} \\ \dot{\mathrm{H}}_{\mathrm{E}} = \dfrac{qb(t)\mathrm{M_I}\mathrm{H_S}}{\mathrm{H}} - (\hat{l}+h)\mathrm{H_E} \\ \dot{\mathrm{H}}_{\mathrm{I}} = \hat{l}\mathrm{H_E} - (h+r)\mathrm{H_I} \\ \dot{\mathrm{H}}_{\mathrm{R}} = r\mathrm{H_I} - h\mathrm{H_R} \\ \dot{\mathrm{M}}_{\mathrm{S}} = \Delta(t) - \dfrac{pb(t)}{\mathrm{H}}\mathrm{H_I}\mathrm{M_S} - m(t)\mathrm{M_S} \\ \dot{\mathrm{M}}_{\mathrm{E}} = \dfrac{pb(t)}{\mathrm{H}}\mathrm{H_I}\mathrm{M_S} - (c+m(t))\mathrm{M_E} \\ \dot{\mathrm{M}}_{\mathrm{I}} = c\mathrm{M_E} - m(t)\mathrm{M_I} \end{cases} \tag{3.2}$$

subject to the initial conditions at $t = t_0 \geq 0$: $\mathrm{H_S}(t_0) > 0$, $\mathrm{H_E}(t_0) > 0$, $\mathrm{H_I}(t_0) > 0$, $\mathrm{H_R}(t_0) > 0$, $\mathrm{M_S}(t_0) > 0$, $\mathrm{M_E}(t_0) > 0$, $\mathrm{M_I}(t_0) > 0$; the constant parameters verify that $h > 0$, $\hat{l} > 0$, $r > 0$, $c > 0$, and $(p,q) \in [0\,,1]^2$; the rates $\Delta(t)$, $b(t)$ and $m(t)$ are continuously differentiable, positive, real-valued, ω-periodic functions. The state space of epidemiological interest is Π defined by:

$$\Pi = \left\{ \mathbf{x} = \begin{bmatrix} \mathbf{x}_{\mathrm{H}} \\ \mathbf{x}_{\mathrm{M}} \end{bmatrix} \in \mathbb{R}^7_+ : \mathrm{H_S}+\mathrm{H_E}+\mathrm{H_I}+\mathrm{H_R} = \mathrm{H} = \mathrm{const} \wedge 0 \leq \mathrm{M_S}+\mathrm{M_E}+\mathrm{M_I} \leq \frac{\Delta^u}{m^l} \right\}, \tag{3.3}$$

where $\mathbf{x}_{\mathrm{H}} = \left[\mathrm{H_S} \quad \mathrm{H_E} \quad \mathrm{H_I} \quad \mathrm{H_R}\right]^\top$ and $\mathbf{x}_{\mathrm{M}} = \left[\mathrm{M_S} \quad \mathrm{M_E} \quad \mathrm{M_I}\right]^\top$.

The system can be written as:

$$\dot{\mathbf{x}} = \mathbf{F}(t, \mathbf{x}(t)), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \in \Pi, \quad (t_0, t) \in \mathbb{R}_+^2 \tag{3.4}$$

Since the dynamic system is useful for predicting the behavior of a physical system in its current state (the densities of humans and mosquitoes during a dengue epidemic), its solutions must exist, be non-negative, remain uniformly bounded, and be unique.

Proposition 3.1.

(i) *Each solution $\boldsymbol{\chi}(t, \mathbf{x}_0(t_0))$ of the system (3.2) with initial condition*

$$\mathbf{x}_0 = \left[\mathrm{H_S}(t_0) \;\; \mathrm{H_E}(t_0) \;\; \mathrm{H_I}(t_0) \;\; \mathrm{H_R}(t_0) \; \mathrm{M_S}(t_0) \;\; \mathrm{M_E}(t_0) \;\; \mathrm{M_I}(t_0)\right]^\top > \mathbf{0},$$

remains positive for all $t \geq t_0 \geq 0$ and ultimately bounded.

(ii) *For any initial data $\mathbf{x}_0 \in \Pi$, the system (3.2) has a unique globally defined solution $\boldsymbol{\chi}(t, \mathbf{x}_0)$.*

Proof. Visit Appendix B. □

4. Threshold dynamics

4.1. The dengue free solution (DFS)

In the case that Δ, b and m are non-constant, bounded, continuous, periodic functions, there is no equilibrium point $\tilde{\mathbf{c}}$ for the system (3.2). This means that the equation $\mathbf{F}(t, \tilde{\mathbf{c}}) = \mathbf{0}$ cannot be satisfied since, by definition, all components of $\tilde{\mathbf{c}}$ are constants. So there is no dengue-free equilibrium point, although there could be a disease-free solution (DFS): if $\mathrm{H_E}(t) = \mathrm{H_I}(t) = \mathrm{M_E}(t) = \mathrm{M_I}(t) = 0$ for all $t \in \mathbb{R}_+$, then the differential equation of non-carrier mosquitoes becomes

$$\frac{d}{dt}\mathrm{M_S}(t) = \Delta(t) - m(t)\mathrm{M_S}(t); \quad \mathrm{M_S}(t_0) > 0, \, \forall t_0 \geq 0. \tag{4.1}$$

Below are several lemmas that will be helpful in proving the main results.

Lemma 4.1. *If $w \colon [0, \infty) \longmapsto \mathbb{R}$ is a periodic function of period ω, n is any positive integer and t is any real number, then*

(i) $\displaystyle\int_t^{t+n\omega} w(t)dt = n\int_0^\omega w(t)dt;$ **(ii)** $\displaystyle\langle w \rangle = \frac{1}{\omega}\int_0^\omega w(t)dt.$

Proof. Visit Appendix A.1. □

Lemma 4.2.

$$\max\left\{\left(\mathrm{M_S}-\overline{\mathrm{M}}_\mathrm{S}\right)^{\infty},\ \left(\mathrm{H_S}-\overline{\mathrm{H}}(0)\right)^{\infty}\right\}\leq 0.$$

Proof. Visit Appendix A.2. □

Lemma 4.3 implies that the solution of (4.1) is bounded by positive constants, in addition the coefficient $m(t)$ and the inhomogeneous term $\Delta(t)$ are periodic (of period ω) and continuous, for these reasons it follows from the theory of linear differential equations [42, Theorem 1.1, p. 408] that a ω-periodic solution exists if and only if (4.1) has at least one bounded solution. In the next Lemma, we show that (4.1) has a unique globally attractive periodic solution.

Lemma 4.3. *There exist constants* $\mathrm{N}^l>0$ *and* $\mathrm{N}^u>0$ *such that* $\mathrm{N}^l\leq \mathrm{M_S}(t)\leq \mathrm{N}^u$, *for all* $t\in\mathbb{R}_+$.

Proof. Visit Appendix A.3. □

In the next lemma, it is shown that (4.1) has a unique globally attractive periodic solution.

Lemma 4.4. *The solution* $\mathrm{M_S}(t)$ *of the initial scalar value problem* (4.1) *converges uniformly to a unique* ω*-periodic solution* $\overline{\mathrm{M}}_\mathrm{S}(t)$.

Proof. The initial value problem (4.1) has the solution

$$\begin{aligned}\mathrm{M_S}(t)=\mathrm{M_S}(t_0)\exp\left(-\int_{t_0}^{t}m(\tau)d\tau\right)\\ +\exp\left(-\int_{t_0}^{t}m(\tau)d\tau\right)\int_{t_0}^{t}\exp\left(\int_{t_0}^{\zeta}m(\tau)d\tau\right)\Delta(\zeta)d\zeta.\end{aligned}\tag{4.2}$$

A recursive relationship between the average number of non-carriers mosquitoes at $t_k=t_0+k\omega$ ($k\in\mathbb{Z}_+$) is given by:

$$\begin{aligned}\mathrm{M}_{k+1}&=\mathrm{M_S}(t_{k+1})\\ &=\mathrm{M}_k\exp\left(-\int_{t_k}^{t_{k+1}}m(\tau)d\tau\right)\\ &\quad+\exp\left(-\int_{t_k}^{t_{k+1}}m(\tau)d\tau\right)\int_{t_k}^{t_{k+1}}\exp\left(\int_{t_k}^{\zeta}m(\tau)d\tau\right)\Delta(\zeta)d\zeta.\end{aligned}\tag{4.3}$$

Due to $m(\tau)$ is a periodic function and that the integral is invariant under translation, then

$$\int_{t_k}^{\zeta} m(\tau)d\tau = \int_{t_0}^{\zeta-k\omega} m(\tau+k\omega)d\tau = \int_{t_0}^{\zeta-k\omega} m(\tau)d\tau\,,$$
$$\int_{t_k}^{t_{k+1}} m(\tau)d\tau = \int_{t_0}^{t_1} m(\tau)d\tau\,,$$

and

$$\begin{aligned}\mathrm{M}_{k+1} &= \mathrm{M}_k \exp\left(-\int_{t_0}^{t_1} m(\tau)d\tau\right) \\ &\quad + \exp\left(-\int_{t_0}^{t_1} m(\tau)d\tau\right)\int_{t_k}^{t_{k+1}} \exp\left(\int_{t_0}^{\zeta-k\omega} m(\tau)d\tau\right)\Delta(\zeta)d\zeta.\end{aligned}$$

Taking the change of variable $\eta = \zeta - k\omega$ and because $\Delta(\zeta)$ is a periodic function, then

$$\mathrm{M}_{k+1} = \mathrm{M}_k \exp\big(-\omega\langle m\rangle\big) + \exp\big(-\omega\langle m\rangle\big)\int_{t_0}^{t_0+\omega} \exp\left(\int_{t_0}^{\eta} m(\tau)d\tau\right)\Delta(\eta)d\eta \quad (k\in\mathbb{Z}_+).$$

This defines a mapping $\mathcal{S}$ such that $\mathcal{S}(\mathrm{M}_k) = \mathrm{M}_{k+1}$; if M_{k_1} and M_{k_2} are different values of M_k, then

$$\big|\mathcal{S}(\mathrm{M}_{k_1}) - \mathcal{S}(\mathrm{M}_{k_2})\big| \le \exp\big(-\omega\langle m\rangle\big)\big|\mathrm{M}_{k_1} - \mathrm{M}_{k_2}\big|.$$

So $\mathcal{S}$ is a contraction mapping and in virtue of the Banach fixed point theorem [57] has a unique fixed point $\mathrm{M}^*_{\mathrm{S}}(t_{k^*})$ such that $\mathcal{S}\big(\mathrm{M}_{\mathrm{S}}(t_{k^*+1})\big) = \mathcal{S}\big(\mathrm{M}_{\mathrm{S}}(t_{k^*})\big) = \mathrm{M}_{\mathrm{S}}(t_{k^*})$, equivalently $\mathcal{S}\big(\mathrm{M}_{\mathrm{S}}(t_0+k^*\omega)\big) = \mathrm{M}_{\mathrm{S}}(t_0+(k^*+1)\omega)$. This fixed point can be found for any solution M_{S} of the differential equation with arbitrary initial time t_0^*. The fixed point has the form:

$$\mathrm{M}^*_{\mathrm{S}}(t_0) = \big(\exp\big(\omega\,\langle m\rangle\big)-1\big)^{-1}\int_{t_0^*}^{t_0^*+\omega} \exp\left(\int_{t_0^*}^{\eta} m(\tau)d\tau\right)\Delta(\eta)d\eta. \tag{4.4}$$

This fixed point is a continuously differentiable function with respect to t_0^* and leads to define the function

$$\overline{\mathrm{M}}_{\mathrm{S}}(t) = \big(\exp\big(\omega\,\langle m\rangle\big)-1\big)^{-1}\int_{t}^{t+\omega} \exp\left(\int_{t}^{\eta} m(\tau)d\tau\right)\Delta(\eta)d\eta,$$

which satisfies the property:

$$\begin{aligned}\overline{\mathrm{M}}_{\mathrm{S}}(t+\omega) &= \big(\exp\big(\omega\,\langle m\rangle\big)-1\big)^{-1}\int_{t+\omega}^{t+2\omega} \exp\left(\int_{t+\omega}^{\eta} m(\tau)d\tau\right)\Delta(\eta)d\eta \\ &= \big(\exp\big(\omega\,\langle m\rangle\big)-1\big)^{-1}\int_{t+\omega}^{t+\omega+\omega} \exp\left(\int_{t}^{\eta-\omega} m(\tau+\omega)d\tau\right)\Delta(\eta-\omega)d\eta \\ &= \overline{\mathrm{M}}_{\mathrm{S}}(t).\end{aligned}$$

Hence, $\overline{\mathrm{M}}_{\mathrm{S}}$ is periodic with period ω, or what is the same $\overline{\mathrm{M}}_{\mathrm{S}}(t) = \overline{\mathrm{M}}_{\mathrm{S}}(t + k\omega)$, $\forall k \in \mathbb{Z}_{+}$. Applying the substitution $\zeta = \eta + k\omega$ one arrives to:

$$\begin{aligned}
\overline{\mathrm{M}}_{\mathrm{S}}(t) &= \big(\exp\big(\omega\left\langle m\right\rangle\big) - 1\big)^{-1} \int_{t+k\omega}^{t+(k+1)\omega} \exp\left(\int_{t}^{\zeta - k\omega} m(\tau)d\tau\right) \Delta(\zeta - k\omega)d\zeta \\
&= \frac{\displaystyle\int_{t+k\omega-k\omega}^{t+(k+1)\omega-k\omega} \exp\left(\int_{t}^{(\zeta+k\omega)-k\omega} m(\tau)d\tau\right) \Delta((\zeta + k\omega) - k\omega)d\zeta}{2e^{\omega\langle m\rangle/2}(e^{\omega\langle m\rangle/2} - e^{-\omega\langle m\rangle/2})/2} \\
&= \frac{\displaystyle\int_{t}^{t+\omega} \exp\left(\int_{t}^{\zeta} m(\tau)d\tau\right) \Delta(\zeta)d\zeta}{2e^{\omega\langle m\rangle/2}(e^{\omega\langle m\rangle/2} - e^{-\omega\langle m\rangle/2})/2} \\
&= \frac{\displaystyle\int_{t}^{t+\omega} \exp\left(\int_{t}^{\zeta} m(\tau)d\tau\right) \Delta(\zeta)d\zeta}{2e^{\omega\langle m\rangle/2}\sinh\left(\omega\left\langle m\right\rangle/2\right)} \\
&= \frac{1}{2}\mathrm{csch}\left(\frac{\omega}{2}\left\langle m\right\rangle\right) \exp\left(-\frac{\omega}{2}\left\langle m\right\rangle\right) \int_{t}^{t+\omega} \exp\left(\int_{t}^{\zeta} m(\tau)d\tau\right) \Delta(\zeta)d\zeta
\end{aligned} \tag{4.5}$$

What follows is to prove that all the solutions of (4.1) converges uniformly to the periodic solution (4.5) and $\overline{\mathrm{M}}_{\mathrm{S}}(t)$ is unique. The derivative of $\mathrm{N}(t) = \big|\mathrm{M}_{\mathrm{S}}(t) - \overline{\mathrm{M}}_{\mathrm{S}}(t)\big|$ is

$$\frac{d}{dt}\mathrm{N}(t) = \mathrm{sgn}\Big(\mathrm{M}_{\mathrm{S}}(t) - \overline{\mathrm{M}}_{\mathrm{S}}(t)\Big)\Big(\big(\Delta(t) - m(t)\mathrm{M}_{\mathrm{S}}(t)\big) - \big(\Delta(t) - m(t)\overline{\mathrm{M}}_{\mathrm{S}}(t)\big)\Big) = -m(t)\mathrm{N}(t).$$

The differential equation $\dfrac{d}{dt}\mathrm{N}(t) = -m(t)\mathrm{N}(t)$ with $\mathrm{N}(t_0) = \big|\mathrm{M}_{\mathrm{S}}(t_0) - \overline{\mathrm{M}}_{\mathrm{S}}(t_0)\big|$ has solution

$$\begin{aligned}
\mathrm{N}(t) &= \mathrm{N}(t_0)\exp\left(-\int_{t_0}^{t} m(\tau)d\tau\right) \\
\Rightarrow \mathrm{N}(t_0 + k\omega) &= \mathrm{N}(t_0)\exp\left(-\int_{t_0}^{t_0+k\omega} m(\tau)d\tau\right) \le \mathrm{N}(t_0)\exp\Big(-k\omega\big\langle m\big\rangle\Big).
\end{aligned}$$

Since $\exp\Big(-\omega\big\langle m\big\rangle\Big) < 1$, $\exp\Big(-k\omega\big\langle m\big\rangle\Big) \to 0$ as $k \to \infty$. Hence $\big|\mathrm{N}(t_0 + k\omega)\big| = \mathrm{N}(t_0 + k\omega) \to 0$ as $k \to \infty$. So given $\epsilon_0 > 0$ there exists n_0 such that $\mathrm{N}(t_0 + k\omega) \le \epsilon_0$ for all $k \ge k_0$. We shall now prove that $\mathrm{N}(t) \to 0$ as $t \to \infty$. So we must show that given $\epsilon > 0$ there exists τ_0 such that $\mathrm{N}(t) \le \epsilon$ for all $t \ge \tau_0$. Choosing k such that k is the largest integer satisfying $t_0 + k\omega \le t$, it is had to

$$\begin{aligned}
\mathrm{N}(t_0 + t) &= \mathrm{N}(t_0 + k\omega)\exp\left(-\int_{t_0+k\omega}^{t_0+t} m(\tau)d\tau\right) \\
&\le \mathrm{N}(t_0 + k\omega)\exp\left(-\int_{t_0+k\omega}^{t} m(\tau)d\tau\right).
\end{aligned}$$

Given $\epsilon > 0$, as $m(t)$ is a bounded and continuous function on $\underline{I} = [t_0+k\omega\, ,\, t_0+(k+1)\omega]$, choose $\epsilon_0 = \epsilon \exp\left(\inf_{t\in\underline{I}} \int_{t_0+k\omega}^{t} m(\tau)d\tau\right)$. Now choose k_0 large enough such that $\mathrm{N}(t+k\omega) \leq \epsilon_0$ for all $k \geq k_0$. So

$$\mathrm{N}(t_0+t) \leq \epsilon_0 \exp\left(-\inf_{t\in\underline{I}} \int_{t_0+k\omega}^{t} m(\tau)d\tau\right) = \epsilon.$$

Hence, given $\epsilon > 0$ there exists $\tau_0 = t_0 + k_0\omega$ such that $\mathrm{N}(t) \leq \epsilon$ for all $t > \tau_0$. So, as $t \to \infty$,

$$\mathrm{N}(t) \to 0 \Leftrightarrow \mathrm{M_S}(t) \to \overline{\mathrm{M}}_\mathrm{S}(t).$$

Finally, it is found that there cannot be two periodic solutions: for positive periodic solutions $\overline{\mathrm{M}}_\mathrm{S}(t)$ and $\overline{\overline{\mathrm{M}}}_\mathrm{S}(t)$ there exist $\{\,\bar{t}_k\,\}$ and $\{\,\bar{\bar{t}}_k\,\}$, when $\bar{t}_k \to +\infty$ and $\bar{\bar{t}}_k \to +\infty$ $(k \to +\infty)$, $\lim_{k\to+\infty} \overline{\mathrm{M}}_\mathrm{S}(\,\bar{t}_k) = \overline{\mathrm{M}}_\mathrm{S}(t)$ and $\lim_{k\to+\infty} \overline{\overline{\mathrm{M}}}_\mathrm{S}(\,\bar{\bar{t}}_k) = \overline{\overline{\mathrm{M}}}_\mathrm{S}(t)$, which is contradictory with $\lim_{t\to+\infty} \left|\overline{\overline{\mathrm{M}}}_\mathrm{S}(t) - \overline{\mathrm{M}}_\mathrm{S}(t)\right| = 0$. □

Proposition 4.5. *System (3.2) has a unique continuously differentiable dengue-free periodic solution (DFS) of period ω given by*

$$\mathbf{x^0} = \left[\overline{\mathrm{H}}_\mathrm{S}(t) \quad 0 \quad 0 \quad 0 \quad \overline{\mathrm{M}}_\mathrm{S}(t) \quad 0 \quad 0\right]^\top, \tag{4.6}$$

where $\overline{\mathrm{H}}_\mathrm{S}(t) = \mathrm{H}(0)$ *and* $\overline{\mathrm{M}}_\mathrm{S}(t) = \dfrac{\displaystyle\int_t^{t+\omega} \exp\left(\int_t^{\zeta} m(\tau)d\tau\right)\Delta(\zeta)d\zeta}{2\sinh\left(\frac{\omega}{2}\langle m\rangle\right)\exp\left(\frac{\omega}{2}\langle m\rangle\right)}$.

Any DFS to (3.2) approaches this one as time becomes large.

Proof. By Lemmas 4.3 and 4.4, the Cauchy problem (4.1) admits a unique globally attractive positive periodic solution (4.5) in the infection free environment ($\mathrm{H_I}(t) = \mathrm{H_E}(t) = \mathrm{M_E}(t) = \mathrm{M_I}(t) = 0\ \forall t \in \mathbb{R}_+$). Since $d\mathrm{H_R}(t)/dt = -h\mathrm{H_R}(t)$, then $\mathrm{H_R}(t) = 0$ is an asymptotically stable equilibrium solution for the recovered population, moreover, $\mathrm{H_S}(t) = \mathrm{H} - \mathrm{H_E}(t) - \mathrm{H_I}(t) - \mathrm{H_R}(t) \to \mathrm{H}$ as $t \to \infty$. Notice that (4.6) ultimately lies in (3.3) because the inequality

$$\begin{aligned}&\int_{t_0}^{t} \exp\left(\int_{t_0}^{\zeta} m(\tau)d\tau\right)\Delta(\zeta)d\zeta \\ &\quad\leq \int_{t_0}^{t} \exp\left(\int_{t_0}^{\zeta} m(\tau)d\tau\right)\frac{m(\zeta)}{m^l}\Delta^u d\zeta = \frac{\Delta^u}{m^l}\exp\left(\int_{t_0}^{t} m(\tau)d\tau\right) - \frac{\Delta^u}{m^l}\end{aligned}$$

applied to (4.2) evidences that

$$\mathrm{M_S}(t) < \frac{\Delta^u}{m^l}\exp\left(-\int_{t_0}^{t} m(\tau)d\tau\right) + \exp\left(-\int_{t_0}^{t} m(\tau)d\tau\right)\left(\frac{\Delta^u}{m^l}\exp\left(\int_{t_0}^{t} m(\tau)d\tau\right) - \frac{\Delta^u}{m^l}\right) = \frac{\Delta^u}{m^l}.$$

Thus, the system (3.2) admits a unique DF periodic solution given by (4.6). □

4.2. Basic reproductive number (R_0)

4.2.1. R_0 calculation

Utilizing a notation similar to that in [17], we sort the compartments so that the first three compartments correspond to infected and carrier individuals. Let

$$\overline{\mathbf{x}} = \begin{bmatrix} \mathrm{H_E} & \mathrm{M_E} & \mathrm{H_I} & \mathrm{M_I} & \mathrm{H_S} & \mathrm{M_S} & \mathrm{H_R} \end{bmatrix} = \begin{bmatrix} \overline{x}_1 & \overline{x}_2 & \overline{x}_3 & \overline{x}_4 & \overline{x}_5 & \overline{x}_6 & \overline{x}_7 \end{bmatrix}^\top$$

and define

- F_i: rate of secondary infections in compartment i.
- V_i^+: rate of transfer individuals into compartment i by others means.
- V_i^-: rate of transfer individuals out of compartment i.

System (3.2) can be written in the form:

$$\frac{d\overline{\mathbf{x}}}{dt} = F(t,\overline{\mathbf{x}}) - V(t,\overline{\mathbf{x}}) = \mathbf{f}(t,\overline{\mathbf{x}}), \tag{4.7}$$

where

$$\begin{cases} F(t,\overline{\mathbf{x}}) = \begin{bmatrix} F_1(t,\overline{\mathbf{x}}) & F_2(t,\overline{\mathbf{x}}) & \dots & F_7(t,\overline{\mathbf{x}}) \end{bmatrix}^\top = \begin{bmatrix} \frac{qb(t)}{\mathrm{H}}\overline{x}_4\overline{x}_5 & \frac{pb(t)}{\mathrm{H}}\overline{x}_3\overline{x}_6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\top, \\ V(t,\overline{\mathbf{x}}) = \begin{bmatrix} V_1(t,\overline{\mathbf{x}}) & V_2(t,\overline{\mathbf{x}}) & \dots & V_7(t,\overline{\mathbf{x}}) \end{bmatrix} = V^-(t,\overline{\mathbf{x}}) - V^+(t,\overline{\mathbf{x}}), \\ V^-(t,\overline{\mathbf{x}}) = \begin{bmatrix} (\hat{l}+h)\overline{x}_1 \\ (c+m(t))\overline{x}_2 \\ (h+r)\overline{x}_3 \\ m(t)\overline{x}_4 \\ \frac{qb(t)}{\mathrm{H}}\overline{x}_4\overline{x}_5 + h\overline{x}_5 \\ \frac{pb(t)}{\mathrm{H}}\overline{x}_3\overline{x}_6 + m(t)\overline{x}_6 \\ h\overline{x}_7 \end{bmatrix}, V^+(t,\overline{\mathbf{x}}) = \begin{bmatrix} 0 \\ 0 \\ \hat{l}\overline{x}_1 \\ c\overline{x}_2 \\ h\mathrm{H} \\ \Delta(t) \\ r\overline{x}_3 \end{bmatrix}, \mathbf{f}(t,\overline{\mathbf{x}}) = \begin{bmatrix} f_1(t,\overline{\mathbf{x}}) \\ f_2(t,\overline{\mathbf{x}}) \\ f_3(t,\overline{\mathbf{x}}) \\ f_4(t,\overline{\mathbf{x}}) \\ f_5(t,\overline{\mathbf{x}}) \\ f_6(t,\overline{\mathbf{x}}) \\ f_7(t,\overline{\mathbf{x}}) \end{bmatrix}. \end{cases} \tag{4.8}$$

The following partial derivative matrices will be required soon:

$$\begin{aligned} \overline{F}(t) &= \left[\frac{\partial F_i(t,\mathbf{x}^0)}{\partial \overline{x}_j}\right]_{1\le i,j\le 4} = \begin{bmatrix} 0 & 0 & 0 & qb(t) \\ 0 & 0 & \frac{pb(t)\overline{\mathrm{M}}_\mathrm{S}(t)}{\mathrm{H}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \text{and} \quad \overline{V}(t) &= \left[\frac{\partial V_i(t,\mathbf{x}^0)}{\partial \overline{x}_j}\right]_{1\le i,j\le 4} = \begin{bmatrix} h+\hat{l} & 0 & 0 & 0 \\ 0 & c+m(t) & 0 & 0 \\ -\hat{l} & 0 & h+r & 0 \\ 0 & -c & 0 & m(t) \end{bmatrix} \end{aligned} \tag{4.9}$$

For a compartmental epidemiological model based on an autonomous system, the BRN is defined as the expected number of secondary cases produced by a typical infected individual during its entire infectious period in a population consisting only of susceptibles [15], and mathematically it is the spectral radius of a so-called *next generation matrix* (which is independent of time). Multiple researchers have investigated the rich nonlinear effects caused by periodically varying rates in epidemic models to the point of generalizing the definition of the BRN for periodic dynamic systems as mentioned in the introductory section. Especially, Wang and Zhao in [17] extended the Bacaër's innovative method [16] for a large class of epidemic models in periodic environments. They established the *next-infection integral operator*:

Let $\mathcal{P}_\omega = \mathcal{P}_\omega\left(\mathbb{R}, \mathbb{R}^4\right)$ be the ordered Banach space of all ω-periodic functions $\boldsymbol{\phi} : \mathbb{R} \mapsto \mathbb{R}^4$, which is equipped with the maximum norm

$$\|\boldsymbol{\phi}\| = \max_{1\leq i\leq 4} \sup\{|\phi_i(t)| \mid t \in [0, \omega]\}$$

and the positive generator cone $\mathcal{P}_\omega^+ = \{\boldsymbol{\phi} \in \mathcal{P}_\omega : \boldsymbol{\phi}(t) \geq 0, \forall t \in \mathbb{R}\}$. Now define a linear operator $\mathcal{L} : \mathcal{P}_\omega \mapsto \mathcal{P}_\omega$ by

$$(\mathcal{L}\boldsymbol{\phi})(t) = \int_0^\infty Y(t, t-s)\overline{F}(t-s)\boldsymbol{\phi}(t-s)ds. \tag{4.10}$$

They call $\mathcal{L}$ the next-infection integral operator following the motivation of van den Driessche and Watmough, and then the spectral radius of $\mathcal{L}$ is given by

$$\mathrm{R}_0 := \rho(\mathcal{L}) \tag{4.11}$$

for the periodic system (3.2). In the equation (4.10), $\boldsymbol{\phi}(s) \in \mathcal{P}_\omega$ represents the initial distribution of infectious individuals in this periodic environment and $Y(t, s)$ is the evolution operator of the linear ω-periodic system

$$\frac{d\mathbf{z}}{dt} = -\overline{V}(t)\mathbf{z}, \tag{4.12}$$

which means the 3×3 matrix Y satisfies

$$\frac{dY(t,s)}{dt} = -\overline{V}(t)Y(t,s),\; Y(s,s) = I_3$$

for each $t \geq s$, $s \in \mathbb{R}$. Then $\mathcal{L}\boldsymbol{\phi}$ is the distribution of accumulative new infections at time t produced by all those infected individuals $\boldsymbol{\phi}(s)$ introduced before t, with kernel $\overline{U}(t, s) = Y(t, t-s)\overline{F}(t-s)$, whose element $\overline{U}_{ij}(t, s)$ in row i and column j represents the expected number of individuals in compartment I_i that one individual in compartment I_j generates at the beginning of an epidemic per unit time at time t if it has been in compartment I_j for s units of time, with $I_1 = \mathrm{H_E}$, $I_2 = \mathrm{M_E}$, $I_2 = \mathrm{H_I}$ and $I_4 = \mathrm{M_I}$ [21].

Remark 4.6. Since $Y(t, s)$ is the evolution operator of (4.12), according to standard theory of linear periodic systems [43, Sec. III.7], there exists constants $\Theta_0 > 0$ and $\hat{\omega} < 0$ such that

$$\|Y(t,s)\| \leq \Theta_0 e^{\hat{\omega}(t-s)},\; \forall t \geq s \text{ with } s \in \mathbb{R}.$$

Following the setting of [17], we verify the following assumptions that show that the proposed non-autonomous compartmental epidemic model is well-posed and makes biological sense:

(A1) For $1 \leq i \leq n$, the functions $F_i(t, \overline{\mathbf{x}})$, $V_i^+(t, \overline{\mathbf{x}})$ and $V_i^-(t, \overline{\mathbf{x}})$ are nonnegative and continuous on $\mathbb{R} \times \mathbb{R}^n$ and continuously differential with respect to $\overline{\mathbf{x}}$.

(A2) There is a real number $\omega > 0$ such that for each $1 \leq i \leq n$, the functions are ω-periodic in t. (This is new for periodic models.)

(A3) If $\overline{x}_i = 0$ then $V_i^- = 0$. In particular, we define that X_s is a disease-free subspace, so that if $\overline{\mathbf{x}} \in \mathrm{X}_s$, then $V_i^- = 0$ for $i = 1, \ldots, \tilde{m}$ ($\tilde{m}$ is the number of compartments of infected and carrier individuals.)

(A4) $F_i = 0$ for $i > \tilde{m}$.

(A5) If $\overline{\mathbf{x}} \in \mathrm{X}_s$, then $F_i = 0$ and $V_i^+ = 0$ for $i = 1, \ldots, \tilde{m}$.

(A6) Define an $(n - \tilde{m}) \times (n - \tilde{m})$ matrix $\tilde{M}(t) = \left[\dfrac{\partial f_i(t, \mathbf{x}^0)}{\partial \overline{x}_j}\right]_{\tilde{m}+1 \leq i,j \leq n}$ and let $\Phi_{\tilde{M}}(t)$ be the monodromy matrix of the linear ω-periodic system $d\mathbf{y}/dt = \tilde{M}(t)\mathbf{y}$, we then have that $\rho\big(\Phi_{\tilde{M}}(\omega)\big) < 1$.

(A7) $\rho\big(\Phi_{-\overline{V}}(\omega)\big) < 1$, where $\Phi_{-\overline{V}}(t)$ is the monodromy matrix of (4.12).

With $\tilde{m} = 4$ and $n = 7$, it is simple to check the assumptions (A1)-(A5) from observation of the vectors F and V in (4.8). Now it remains to verify conditions (A6) and (A7). We know that (3.2) has the disease-free periodic solution (4.6), so to verify assumption (A6) we define

$$\tilde{M}(t) = \left[\frac{\partial f_i(t, \mathbf{x}^0)}{\partial \overline{x}_j}\right]_{5 \leq i,j \leq 7} = \begin{bmatrix} -\frac{qb(t)}{\mathrm{H}}\overline{x}_2 - h & 0 & 0 \\ 0 & -\frac{pb(t)}{\mathrm{H}}\overline{x}_1 - m(t) & 0 \\ 0 & 0 & -h \end{bmatrix}\Bigg|_{\mathbf{x}=\mathbf{x}^0} \tag{4.13}$$

and solving the system $d\mathbf{y}/dt = \tilde{M}(t)\mathbf{y}$ yields the principal fundamental matrix:

$$\Phi_{\tilde{M}}(t) = \begin{bmatrix} e^{-ht} & 0 & 0 \\ 0 & \exp\left(-\int_0^t m(\tau)d\tau\right) & 0 \\ 0 & 0 & e^{-ht} \end{bmatrix}.$$

Clearly $\Phi_{\tilde{M}}^{-1}(0) = I_3$, and the monodromy matrix is the principal fundamental matrix evaluated at the period, $\Phi_{\tilde{M}}(\omega)$, thus (A6) is satisfied.

From matrices (4.9) and the evolution operator of the linear system (4.12), that is,

$$\frac{dY(t,s)}{dt} = \begin{bmatrix} -(\hat{l} + h) & 0 & 0 & 0 \\ 0 & -(c + m(t)) & 0 & 0 \\ \hat{l} & 0 & -(h + r) & 0 \\ 0 & c & 0 & -m(t) \end{bmatrix} Y(t,s),\ Y(s,s) = I_3 \tag{4.14}$$

for each $t \geq s$ $(s \in \mathbb{R})$, (A7) above must be verified. The eigenvalues of $-\overline{V}(t)$ (some time dependent and different from each other) are

$$s_0 = -(\hat{l} + h), \quad s_1 = -(c + m(t)), \quad s_2 = -(h + r) \quad \text{and} \quad s_3 = -m(t) \tag{4.15}$$

A quadruple of corresponding eigenvectors is

$$\begin{bmatrix} (s_0 - s_2)\hat{l}^{-1} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ (s_1 - s_3)c^{-1} \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

We create the matrices

$$P = \begin{bmatrix} (s_0 - s_2)\hat{l}^{-1} & 0 & 0 & 0 \\ 0 & (s_1 - s_3)c^{-1} & 0 & 0 \\ l & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} \hat{l}(s_0 - s_2)^{-1} & 0 & 0 & 0 \\ 0 & c(s_1 - s_3)^{-1} & 0 & 0 \\ -\hat{l}(s_0 - s_2)^{-1} & 0 & 1 & 0 \\ 0 & -c(s_1 - s_3)^{-1} & 0 & 1 \end{bmatrix}.$$

Remark 4.7. Obviously P and P^{-1} are constant matrices because $\hat{l}$, c, r, $s_0 - s_2 = r - \hat{l}$ and $s_1 - s_3 = -c$ are constants quantity.

Under the coordinate transformation $\boldsymbol{\eta} = P^{-1}\mathbf{z}$, taking into account Remark 4.7, we obtain the uncoupled linear system

$$\dot{\boldsymbol{\eta}} = -P^{-1}\overline{V}(t)P\boldsymbol{\eta} = \begin{bmatrix} s_0 & 0 & 0 & 0 \\ 0 & s_1 & 0 & 0 \\ 0 & 0 & s_2 & 0 \\ 0 & 0 & 0 & s_2 \end{bmatrix} \boldsymbol{\eta}$$

whose general solution is given by $\boldsymbol{\eta}(t) = E(t)\mathbf{d}$, where $\mathbf{d}$ is a constant vector, and

$$E(t) = \operatorname{diag}\left[e^{s_0 t}, \ \exp\left(\int_0^t s_1(\tau)d\tau\right), \ e^{s_2 t}, \ \exp\left(\int_0^t s_3(\tau)d\tau\right)\right].$$

Since $\mathbf{z} = P\boldsymbol{\eta}$ and $\mathbf{d} = P^{-1}\mathbf{c}$, it follows that (4.12) has a fundamental matrix:

$$\Psi(t) = PE(t)P^{-1}$$
$$= \begin{bmatrix} e^{s_0 t} & 0 & 0 & 0 \\ 0 & \exp\left(\int_0^t s_1(\tau)d\tau\right) & 0 & 0 \\ \dfrac{\hat{l}(e^{s_0 t} - e^{s_2 t})}{s_0 - s_2} & 0 & e^{s_2 t} & 0 \\ 0 & \left(\dfrac{c}{s_1 - s_3}\right)\left(\exp\left(\int_0^t s_3(\tau)d\tau\right) - \exp\left(\int_0^t s_1(\tau)d\tau\right)\right) & 0 & \exp\left(\int_0^t s_3(\tau)d\tau\right) \end{bmatrix}$$

with inverse

$$\Psi^{-1}(t) = \begin{bmatrix} e^{-s_0 t} & 0 & 0 & 0 \\ 0 & \exp\left(-\int_0^t s_1(\tau)d\tau\right) & 0 & 0 \\ \dfrac{\hat{l}\left(e^{-s_2 t} - e^{-s_0 t}\right)}{s_2 - s_0} & 0 & e^{-s_2 t} & 0 \\ 0 & \left(\dfrac{c}{s_3 - s_1}\right)\left(\exp\left(-\int_0^t s_3(\tau)d\tau\right) - \exp\left(-\int_0^t s_1(\tau)d\tau\right)\right) & 0 & \exp\left(-\int_0^t s_3(\tau)d\tau\right) \end{bmatrix},$$

but $\Psi(t)$ is not the principal fundamental matrix at $t = s$. Note that

$$\Psi(t)\Psi^{-1}(s) = \begin{bmatrix} e^{s_0(t-s)} & 0 & 0 & 0 \\ 0 & \exp\left(\int_s^t s_1(\tau)d\tau\right) & 0 & 0 \\ \dfrac{\hat{l}\left(e^{s_0(t-s)} - e^{s_2(t-s)}\right)}{s_0 - s_2} & 0 & e^{s_2(t-s)} & 0 \\ 0 & \left(\dfrac{c}{s_1 - s_3}\right)\left(\exp\left(\int_s^t s_1(\tau)d\tau\right) - \exp\left(\int_s^t s_3(\tau)d\tau\right)\right) & 0 & \exp\left(\int_s^t s_3(\tau)d\tau\right) \end{bmatrix},$$

is also a fundamental matrix which satisfies $\Psi(s)\Psi(s)^{-1} = I_4$, consequently we define the evolution operator by

$$Y(t,s) = \begin{bmatrix} e^{s_0(t-s)} & 0 & 0 & 0 \\ 0 & \exp\left(\int_s^t s_1(\tau)d\tau\right) & 0 & 0 \\ \dfrac{\hat{l}\left(e^{s_0(t-s)} - e^{s_2(t-s)}\right)}{s_0 - s_2} & 0 & e^{s_2(t-s)} & 0 \\ 0 & \dfrac{\exp\left(\int_s^t s_1(\tau)d\tau\right) - \exp\left(\int_s^t s_3(\tau)d\tau\right)}{\left(\dfrac{c}{s_1 - s_3}\right)^{-1}} & 0 & \exp\left(\int_s^t s_3(\tau)d\tau\right) \end{bmatrix} \quad (4.16)$$

The monodromy matrix $\Phi_{-\overline{V}}(t)$ of the system (4.12) equals $Y(t,0)$ $\forall t \geq 0$. So one need only consider the monodromy matrix evaluated at the period; the roots of the equation $\det\left[\Phi_{-\overline{V}}(\omega) - \lambda I\right] = 0$ are $\lambda_j = e^{s_j \omega}$, $\big(j = 1, 2, 3\big)$. Since $\Re(s_j) < 0$ for all j, the spectral radius becomes $\max_j \{|\exp(s_j\omega)|\} < 1$ and clearly (A7) is true.

Remark 4.8. Based on the fulfillment of assumptions (A1) – (A7), it is possible to calculate, at least numerically, the BRN for the epidemic model. (See Appendix F).

Remark 4.9. $\overline{F} : \mathbb{R}_+ \mapsto \mathbb{M}_4(\mathbb{R})$ and $Y(t,\, s)\colon [s,\, \infty) \times \mathbb{R}_+ \mapsto \mathbb{M}_4(\mathbb{R})$ are nonnegative matrix functions: that $\overline{F}(t)$ is positive is evident from the preliminary assumptions, meanwhile, the elements of $Y(t,s) = \left[\overline{y}_{ij}(t,s)\right]_{4\times 4}$ satisfy $\overline{y}_{ij}(t,s) \geq 0$

for $(i\,,\,j) \notin \{(3\,,\,1),(4\,,\,2)\}$, but $\overline{y}_{ij}(t,s) \geq 0$ for $(i\,,\,j) \in \{(3\,,\,1),(4\,,\,2)\}$ requires more clarification. Indeed,

$$\overline{u}_{42}(t,\,s) = \frac{\exp(s_0(t-s)) - \exp(s_2(t-s))}{s_0 - s_2} = \exp(s_0\,s)\left(\frac{\exp((s_2-s_0)(t-s))-1}{s_2-s_0}\right) \geq 0$$

and

$$\begin{aligned}
\overline{u}_{31}(t,\,s) &= \frac{\exp\left(\int_s^t s_1(\tau)d\tau\right) - \exp\left(\int_s^t s_3(\tau)d\tau\right)}{(s_1(t)-s_3(t))/c} \\
&= \frac{\exp\left(\int_{t-s}^t s_1(\tau)d\tau\right)\left(1-\exp\left(\int_s^t (s_3(\tau)-s_1(\tau))d\tau\right)\right)}{-c/c} \\
&= \frac{\exp\left(\int_{t-s}^t s_1(\tau)d\tau\right)}{(\exp(c\,s)-1)^{-1}} \geq 0
\end{aligned}$$

for all $t \geq s$, $s \in \mathbb{R}_+$. The periodicity of $Y(t,\,s)$ induced by each component of the matrix with common period ω is argued with the Lemma 4.1(i) and the definition of the parameters.

In order to characterize R_0 for periodic systems, we consider the following linear ω-periodic system

$$\dot{\mathbf{y}} = \left(\frac{1}{\overline{\lambda}}\overline{F}(t) - \overline{V}(t)\right)\mathbf{y}(t), \quad \forall t \in \mathbb{R}_+,\ \overline{\lambda} \in (0,\infty). \tag{4.17}$$

Let $W(t,s,\overline{\lambda})$, $t \geq s$, $s \in \mathbb{R}$, be the evolution operator of the system (4.17) on $\mathbb{R}^3$. It is clear that

$$W(t,0,1) = \Phi_{\overline{F}-\overline{V}}(t), \quad \forall t \in \mathbb{R}_+.$$

Lemma 4.10 and Theorem 4.11 clarify the valuable connection between the linear operator (4.10) and the linearization of system (3.2) near (4.6). These results were proved in [17, Lemma 2.1 and Theorem 2.2] and we omit their proofs here.

Lemma 4.10. *Assume that (A1)–(A7) hold.*

(i) If $\rho\left(W(\omega,0,\overline{\lambda})\right) = 1$ has a positive solution $\overline{\lambda}_0$, then $\overline{\lambda}_0$ is an eigenvalue of $\mathcal{L}$, and hence $\mathsf{R}_0 > 0$.

(ii) If $\mathsf{R}_0 > 0$, then $\overline{\lambda} = \mathsf{R}_0$ is the unique solution of $\rho\left(W(\omega,0,\overline{\lambda})\right) = 1$.

(iii) $\mathsf{R}_0 = 0$ if and only if $\rho\left(W(\omega,0,\overline{\lambda})\right) < 1$, for all $\overline{\lambda} > 0$.

Proposition 4.11 ([17]). *Assume that (A1)-(A7) hold.*

(i) $\mathsf{R}_0 = 1 \Longleftrightarrow \rho\left(\Phi_{\overline{F}-\overline{V}}\right) = 1$.

(ii) $\mathsf{R}_0 < 1 \Longleftrightarrow \rho\left(\Phi_{\overline{F}-\overline{V}}\right) < 1$.

(iii) $\mathsf{R}_0 > 1 \Longleftrightarrow \rho\left(\Phi_{\overline{F}-\overline{V}}\right) > 1$.

4.2.2. Boundedness of the BRN

This subsection provides an upper bound for R_0. For any given $\lambda \in (\hat{\omega}, \infty) \subset \mathbb{R}$ ($\hat{\omega}$ is the same as mentioned in Remark 4.6) and $t \geq 0$, let E_λ be an operator on $\mathcal{P}_\omega$ defined by

$$(E_\lambda \boldsymbol{\varphi})(\vartheta) = \exp\left(\tilde{d}_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)\left(\tilde{S}\boldsymbol{\varphi}(\vartheta) + \tilde{d}_2(\lambda)\left(\tilde{\Lambda}_t\,\boldsymbol{\varphi}(t) - \tilde{S}\boldsymbol{\varphi}(\vartheta)\right)\right), \forall\, \vartheta \in [0\,,\, t],\ \boldsymbol{\varphi} \in \mathcal{P}_\omega, \tag{4.18}$$

which includes:

- $\tilde{d}_0(\lambda) = \min\{0,\, 1-\lambda\}$, $\tilde{d}_1(\lambda) = |\hat{\omega}| \tanh\left(\tilde{d}_0(\lambda)\right)$ (functions of λ, non-increasing and non-positive);
- $\tilde{d}_2(\lambda) = \max\{0,\, 1-\lambda\}$ (function of λ, non-increasing and non-negative);
- $\tilde{\Lambda}_t \equiv \tilde{\Lambda}(t) = \operatorname{diag}\left[\tilde{\delta}_1(t)\,,\quad \tilde{\delta}_2(t)\,,\quad \tilde{\delta}_3(t)\,,\quad \tilde{\delta}_4(t)\right]$, where

$$\tilde{\delta}_j(t) = \begin{cases} 1 & \text{si} \quad \boldsymbol{\varphi}(t) = \mathbf{0} \\ -1 & \text{si} \quad \begin{cases} (\boldsymbol{e}_j \cdot \boldsymbol{\varphi})_{\sup} \leq 0 \\ \boldsymbol{\varphi}(t) \neq \mathbf{0} \end{cases} \\ \dfrac{\max\limits_{i\in\mathcal{A}_1} (\boldsymbol{e}_i \cdot \boldsymbol{\varphi})_{\sup}}{\min\limits_{i\in\mathcal{A}_1} (\boldsymbol{e}_i \cdot \boldsymbol{\varphi})_{\inf}};\ i \in \mathcal{A}_1 & \text{si} \quad (\boldsymbol{e}_i \cdot \boldsymbol{\varphi})_{\inf} > 0 \\ \dfrac{\operatorname{sgn}(\boldsymbol{e}_i \cdot \boldsymbol{\varphi}(t))\tilde{\rho}}{\min\limits_{1\leq i\leq 4} |\boldsymbol{e}_i \cdot \boldsymbol{\varphi}(t)|};\ i \in \mathcal{A}_2 & \text{si} \quad \begin{cases} \boldsymbol{\varphi}(t) \neq \mathbf{0} \\ (\boldsymbol{e}_i \cdot \boldsymbol{\varphi})_{\sup} > 0 \\ (\boldsymbol{e}_i \cdot \boldsymbol{\varphi})_{\inf} < 0 \end{cases} \end{cases} \tag{4.19}$$

  such that $\mathcal{A}_1, \mathcal{A}_2 \neq \emptyset$ belong to the power set of $\mathcal{A} = \{1,\, 2,\, 3,\, 4\}$, and $j \in \mathcal{A}$;
- $\tilde{\rho} = \max\left\{\max\limits_{1\leq i\leq 4} (\boldsymbol{e}_i \cdot \boldsymbol{\varphi})_{\sup}\ ,\ \min\limits_{1\leq i\leq 4} \left|(\boldsymbol{e}_i \cdot \boldsymbol{\varphi})_{\inf}\right|\right\}$;
- $\tilde{S} = \operatorname{diag}\left[\chi_{\mathbb{R}\setminus\{0\}}\,(\boldsymbol{e}_1 \cdot \boldsymbol{\varphi})\,(t)\,,\quad \chi_{\mathbb{R}\setminus\{0\}}\,(\boldsymbol{e}_2 \cdot \boldsymbol{\varphi})\,(t)\,,\quad \chi_{\mathbb{R}\setminus\{0\}}\,(\boldsymbol{e}_3 \cdot \boldsymbol{\varphi})\,(t)\,,\quad \chi_{\mathbb{R}\setminus\{0\}}\,(\boldsymbol{e}_4 \cdot \boldsymbol{\varphi})\,(t)\right]$.

Now for all $t \in \mathbb{R}_+$, we introduce a family of linear operators:

$$(\mathcal{L}_\lambda \boldsymbol{\phi})(t) = \frac{2 + |\hat{\omega}|}{1 + \lambda + |\hat{\omega}|} \int_0^\infty \exp\left(d_1(\lambda)s\right) Y(t, t-s) \overline{F}(t-s) (E_\lambda \boldsymbol{\phi})(t-s)\, ds, \quad \boldsymbol{\phi} \in \mathcal{P}_\omega. \tag{4.20}$$

Remark 4.12. Regarding the functions of $t \in \mathbb{R}$ included in formula (4.18), we emphasize that $|\sin(\pi(\cdot-\theta)/\omega)| : t \mapsto \mathbb{R}$ is continuous and ω-periodic, also $\chi_{\mathbb{R}\setminus\{0\}}(\boldsymbol{e}_j \cdot \boldsymbol{\varphi}(\cdot)) : t \mapsto \{0, 1\}$ and $\min_{1\le i\le 4}|\boldsymbol{e}_i \cdot \boldsymbol{\varphi}(\cdot)| : t \mapsto \mathbb{R}$ are ω-periodic if $\boldsymbol{\varphi}(t)$ is ω-periodic, $\chi_{\mathbb{R}\setminus\{0\}}(\boldsymbol{e}_j \cdot \boldsymbol{\varphi}(\cdot))$ is discontinuous at each $t = t^* + n\omega$ ($n \in \mathbb{Z}_+$) if $\boldsymbol{e}_j \cdot \boldsymbol{\varphi}(\cdot) : t \mapsto \mathbb{R}$ crosses the t-axis (that is, $\boldsymbol{e}_j \cdot \boldsymbol{\varphi}(t^*) = 0$), and $\min_{1\le i\le 4}|\boldsymbol{e}_i \cdot \boldsymbol{\varphi}(\cdot)| : t \mapsto \mathbb{R}$ is continuous if $\boldsymbol{\varphi} : t \mapsto \mathbb{R}$ is continuous. Hence, $E_\lambda \boldsymbol{\phi} \in \mathcal{P}_\omega$ for all $\boldsymbol{\phi} \in \mathcal{P}_\omega$.

Lemma 4.13. *For each $\lambda > \hat{\omega}$, the operator $\mathcal{L}_\lambda$ is positive, continuous and compact on $\mathcal{P}_\omega$.*

Proof. Visit Appendix A.4. □

Lemma 4.14. *Consider $\tilde{d}_0(\lambda) = \min\{0, 1-\lambda\}$, $\tilde{d}_1(\lambda) = |\hat{\omega}| \tanh\left(\tilde{d}_0(\lambda)\right)$, $\tilde{d}_2(\lambda) = \max\{0\,,\, 1-\lambda\}$, $\tilde{d}_3 = \tilde{d}_5(\lambda) - \tilde{d}_5(\lambda_0)$, $\tilde{d}_4(\lambda) = -\min\{\tilde{d}_5(\lambda), \tilde{d}_5(\lambda_0)\}$ y $\tilde{d}_5(\lambda)$ functions of λ. The following propositions are valid:*

(i) $(\exists \lambda_0 \in \mathbb{R})\, (\forall \lambda \in \mathbb{R})\,,\ \max\left\{\left|\tilde{d}_0(\lambda) - \tilde{d}_0(\lambda_0)\right|,\ |2\hat{\omega}|^{-1}\left|\tilde{d}_1(\lambda) - \tilde{d}_1(\lambda_0)\right|,\ \left|\tilde{d}_2(\lambda) - \tilde{d}_2(\lambda_0)\right|\right\} \le |\lambda - \lambda_0|\,.$

(ii) $\left(\forall \tilde{d}_3 \in \mathbb{R}\right)(\forall \tau \in \mathbb{R}_+)\,,\ \left|\exp\left(-\tilde{d}_5(\lambda)\tau\right) - \exp\left(-\tilde{d}_5(\lambda_0)\tau\right)\right| \le |\tilde{d}_3| \exp\left(\tilde{d}_4(\lambda)\right)\tau.$

Proof. Visit Appendix A.5. □

Let $\mu(\lambda)$ be the spectral radius of $\mathcal{L}_\lambda$, that is, $\mu(\lambda) := \rho(\mathcal{L}_\lambda)$. Then we have the following results on properties of the function $\mu(\lambda)$.

Proposition 4.15.

(i) *The mapping $\lambda \mapsto \mu(\lambda)$ is continuous and nonincreasing on $(\hat{\omega}\,, +\infty)$, and $\mu(\infty) = 0$.*

(ii) *$\mu(\lambda) = 1$ has at most one solution in $(\hat{\omega}\,, +\infty)$ if $\mathrm{R}_0 \ge 1$, and $\mathrm{R}_0 \le \mu(0)$.*

Proof.
(i) Let $\overline{\Theta}_1, \overline{\Theta}_2, \Theta_1, \Theta_2, \Theta_3, \Theta_4$ be positive constants, $\tilde{d}_6(\lambda, j) = 1 - j + j\tilde{d}_2(\lambda)$ and $\lambda_0 \in (\hat{\omega}\,, +\infty)$ all given, and choose $0 < \delta \ll 1$ such that $[\lambda_0 - \delta, \lambda_0 + \delta] \subset (\hat{\omega}\,, +\infty)$. Appendix E.1 presents the derivation of

$$\|E_\lambda - E_{\lambda_0}\| \le \overline{\Theta}_2\, |\lambda - \lambda_0|\,, \quad \forall\, \lambda \in [\lambda_0 - \delta, \lambda_0 + \delta] \tag{4.21}$$

As a second finding (see Appendix E.2), for any $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$ we have to

$$\|\mathcal{L}_\lambda - \mathcal{L}_{\lambda_0}\| \le \frac{\Theta_4\, |\lambda - \lambda_0|}{1-\delta}\left(\left(\tilde{d}_0(\lambda_0 - \delta) + \hat{\omega}\right)^{-2} + \left(|\tilde{d}_4(\lambda_0)| + |\hat{\omega}|\right)^{-1}\right) \tag{4.22}$$

This implies that $\lim_{\lambda\to\lambda_0} \|\mathcal{L}_\lambda - \mathcal{L}_{\lambda_0}\| = 0$. By the continuity of spectral radius for compact linear operators [47, Theorem 2.1(a)], one obtains that $\lim_{\lambda\to\lambda_0} \mu(\lambda) = \mu(\lambda_0)$. Thus, $\mu(\lambda)$ is continuous on $(\hat{\omega}\,, +\infty)$. It is easy to verify that

$$(\mathcal{L}_{\lambda_1}\boldsymbol{\phi})\,(t) \geq (\mathcal{L}_{\lambda_2}\boldsymbol{\phi})\,(t), \quad \forall\,\hat{\omega} < \lambda_1 \leq \lambda_2,\ t\in\mathbb{R},\ \boldsymbol{\phi}\in\mathcal{P}_\omega^+$$

utilizing the implications below:

$$\begin{aligned}
&1 < 1+\lambda_1+\hat{\omega} \leq 1+\lambda_1+\hat{\omega} \implies \frac{1}{1+\lambda_1+\hat{\omega}} \geq \frac{1}{1+\lambda_2+\hat{\omega}}\\
&\wedge\ \tilde{d}_2(\lambda_1)\,(\Lambda_t-1) \geq \tilde{d}_2(\lambda_2)\,(\Lambda_t-1) \implies (E_{\lambda_1}\boldsymbol{\phi})\,(t) \geq (E_{\lambda_2}\boldsymbol{\phi})\,(t)\\
&\wedge\ e^{d_1(\lambda_1)s} \geq e^{d_1(\lambda_2)s} \implies e^{d_1(\lambda_1)s}Y(t,t-s)\overline{F}(t-s) \geq e^{d_1(\lambda_2)s}Y(t,t-s)\overline{F}(t-s)\\
&\because\ \hat{\omega} < \lambda_1 \leq \lambda_2 \wedge \lambda \mapsto \tilde{d}_i(\lambda) \text{ is non-increasing } (i=1,\,2) \wedge \boldsymbol{\phi}(t) \geq \mathbf{0}\ \forall\,t\in\mathbb{R}.
\end{aligned}$$

Since each $\mathcal{L}_\lambda$ is a positive and bounded linear operator on $\mathcal{P}_\omega$ according to Lemma 4.13, Theorem 1.1 in [48] implies that $\mu(\lambda) = \rho\,(\mathcal{L}_\lambda)$ is a nonincreasing function of λ on $(\hat{\omega}\,,+\infty)$. Notice that

$$\begin{aligned}
\|\mathcal{L}_\lambda\| &\leq \frac{2+|\hat{\omega}|}{1+\lambda+|\hat{\omega}|}\int_0^\infty \exp\left(\tilde{d}_1(\lambda)s\right)\Theta_0\|\overline{F}\|\left(1+\tilde{d}_2(\hat{\omega})\left(\overline{\Theta}_1\|\tilde{\Lambda}\|+1\right)\right)\exp\left(\hat{\omega}s\right)ds\\
&\leq \overline{\Theta}_3\frac{2+|\hat{\omega}|}{1+\lambda+|\hat{\omega}|}\int_0^\infty \exp\left(\left(\tilde{d}_1(\lambda)+\hat{\omega}\right)s\right)ds\\
&= \overline{\Theta}_3\,(1+\lambda+|\hat{\omega}|)^{-1}\int_0^\infty \exp\left(\left(\tilde{d}_1(\lambda)+\hat{\omega}\right)s\right)ds, \quad \forall t\geq s\geq 0,\,\forall\,\lambda\geq\hat{\omega}
\end{aligned}$$

$$\begin{aligned}
\therefore\ \|\mathcal{L}_\lambda\| &\leq \frac{\overline{\Theta}_3(2+|\hat{\omega}|)}{|\tilde{d}_1(\lambda)+\hat{\omega}|\,(1+\lambda+|\hat{\omega}|)}\lim_{\zeta\to+\infty}\left(-\exp\left(\left(\tilde{d}_1(\lambda)+\lambda\right)s\right)\Big|_{s=0}^{s=\zeta}\right)\\
&= \frac{\overline{\Theta}_3(2+|\hat{\omega}|)}{|\tilde{d}_1(\lambda)+\hat{\omega}|\,(1+\lambda+|\hat{\omega}|)}\lim_{\zeta\to+\infty}\left(1-\exp\left(\left(\tilde{d}_1(\lambda)+\lambda\right)s\right)\right)\\
&= \frac{\overline{\Theta}_3(2+|\hat{\omega}|)}{|\tilde{d}_1(\lambda)+\hat{\omega}|\,(1+\lambda+|\hat{\omega}|)}, \quad \forall\lambda\geq\hat{\omega}.
\end{aligned}$$

In view of $0\leq\mu(\lambda)=\rho(\mathcal{L}_\lambda)\leq\|\mathcal{L}_\lambda\|$ and the squeeze theorem [56, Thm. 1.10.1]: $\mu(+\infty) = \lim_{\lambda\to+\infty}\mu(\lambda) = 0$.

(ii) Notice that the operator $\mathcal{L}_\lambda$ may not be strongly positive (that is, it does not map positive functions into strictly positive functions). To show $\mu(\lambda)=1$ exists, we use argument of perturbation. For $\epsilon^*>0$, consider the modified operator

$$(\mathcal{L}_{\lambda,\,\epsilon^*}\boldsymbol{\phi})(t) = \frac{2+|\hat{\omega}|}{1+\lambda+|\hat{\omega}|}\int_0^\infty \exp\left(\tilde{d}_1(\lambda)s\right)Y(t,t-s)F_{\epsilon^*}(t-s)(E_\lambda\boldsymbol{\phi})(t-s)\,ds, \tag{4.23}$$

with $\overline{F}_{\epsilon^*}(t) = \overline{F}(t)+\epsilon^*\mathrm{O}_4^*$ and its spectral radius $\mu_{\epsilon^*}(\lambda) = \rho\,(\mathcal{L}_{\lambda,\,\epsilon^*})$. Since $\overline{F}_{\epsilon^*}(t-s)$ and the other matrices of the kernel are greater than O_4, then $\mathcal{L}_{\lambda,\,\epsilon^*}$ is continuous, compact and strongly positive. By the upper semicontinuity of the spectrum [49, Section IV.3.1] and the continuity of a finite system of eigenvalues ([49, Section IV.3.5]:

$$\lim_{\epsilon^*\to 0^+} \mu_{\epsilon^*}(\lambda) = \mu(\lambda) \tag{4.24}$$

In the case where $\mu_{\epsilon^*}(\lambda) > 0$, the Krein–Rutman theorem [50, Thm. 5.4.33] for strongly positive compact linear operator implies that $\mathcal{L}_{\lambda,\,\epsilon^*}\mathbf{v} = \mu_{\epsilon^*}(\lambda)\mathbf{v}$ for some $\mathbf{v} > 0$ in $\mathcal{P}_\omega^+ \setminus \{\mathbf{0}\}$. By virtue of (A.5) with $\overline{F}(t)$ replaced by $\overline{F}_{\epsilon^*}(t) = \overline{F}(t) + \epsilon^* \mathrm{O}_4^*$, $\alpha^*(\lambda) = \dfrac{2+|\hat{\omega}|}{1+\lambda+|\hat{\omega}|}$ and $\overline{c}(\lambda) = 1 + \tilde{d}_2(\lambda)\left(\dfrac{\max\limits_{1\le i\le 4} (\boldsymbol{e}_i\cdot\mathbf{v})_{\sup}}{\min\limits_{1\le i\le 4} (\boldsymbol{e}_i\cdot\mathbf{v})_{\inf}} - 1\right)$:

$$\begin{aligned}
\mu_{\epsilon^*}(\lambda)\mathbf{v}'(t) &= \alpha^*(\lambda)\overline{F}_{\epsilon^*}(t)\left(e^{\tilde{d}_1(\lambda)|\sin(0)|}I_4\right)\left(\mathbf{v}(t)I_4 + \tilde{d}_2(\lambda)\left(\left(\frac{\max\limits_{1\le i\le 4}(\boldsymbol{e}_i\cdot\mathbf{v})_{\sup}}{\min\limits_{1\le i\le 4}(\boldsymbol{e}_i\cdot\mathbf{v})_{\inf}}\right)\mathbf{v}(t) - \mathbf{v}(t)I_4\right)\right)\\
&\quad -\alpha^*(\lambda)\mu_{\epsilon^*}(\lambda)\left(\tilde{d}_1(\lambda)I_4 + \overline{V}(t)\right)\mathbf{v}(t)\\
&= \alpha^*(\lambda)\overline{F}_{\epsilon^*}(t)\left(\mathbf{v}(t)I_4 + \tilde{d}_2(\lambda)\left(\left(\frac{\max\limits_{1\le i\le 4}(\boldsymbol{e}_i\cdot\mathbf{v})_{\sup}}{\min\limits_{1\le i\le 4}(\boldsymbol{e}_i\cdot\mathbf{v})_{\inf}}\right)\mathbf{v}(t) - \mathbf{v}(t)I_4\right)\right)\\
&\quad -\alpha^*(\lambda)\mu_{\epsilon^*}(\lambda)\left(\tilde{d}_1(\lambda)I_4 + \overline{V}(t)\right)\mathbf{v}(t)\\
&= \alpha^*(\lambda)\overline{c}(\lambda)\overline{F}_{\epsilon^*}(t)\mathbf{v}(t) - \mu_{\epsilon^*}(\lambda)\left(\tilde{d}_1(\lambda)I_4 + \overline{V}(t)\right)\mathbf{v}(t)
\end{aligned}$$

$$\Leftrightarrow \frac{d}{dt}\mathbf{v}(t) = \left(\frac{\alpha^*(\lambda)\overline{c}(\lambda)}{\mu_{\epsilon^*}(\lambda)}\right)\overline{F}_{\epsilon^*}(t)\mathbf{v}(t) - \alpha^*(\lambda)\left(\overline{V}(t) + |\tilde{d}_1(\lambda)|I_4\right)\mathbf{v}(t).$$

Let $\lambda^* = \left(\dfrac{\alpha^\star(\lambda)\overline{c}(\lambda)}{\mu_{\epsilon^*}(\lambda)}\right)^{-1}$ and $\overline{\mathbf{y}}(t) = \exp\left(X_\lambda(t)\right)\mathbf{v}(t)$, where

$$X_\lambda(t) = \int_0^t \left((1-\alpha^\star(\lambda))\overline{V}(\tau) - \alpha^\star(\lambda)|\tilde{d}_1(\lambda)|I_4\right) \tag{4.25}$$

It is easy to check (2.1) with (4.25), then:

$$\begin{aligned}
\frac{d}{dt}\mathbf{v}(t) &= \exp\left(-X_\lambda(t)\right)\left(\frac{d}{dt}\overline{\mathbf{y}}(t) - \alpha^\star(\lambda)\left(\overline{V}(t) + |\tilde{d}_1(\lambda)|I_4\right)\mathbf{y}(t) + \overline{V}(t)\mathbf{y}(t)\right)\\
&= \exp\left(-X_\lambda(t)\right)\left(\left(\left(\frac{\alpha^\star(\lambda)\overline{c}(\lambda)}{\mu_{\epsilon^*}(\lambda)}\right)\overline{F}_{\epsilon^*}(t)\right)\overline{\mathbf{y}}(t) - \alpha^\star(\lambda)\left(\overline{V}(t) + |\tilde{d}_1(\lambda)|I_4\right)\overline{\mathbf{y}}(t)\right)
\end{aligned} \tag{4.26}$$

$$\Leftrightarrow \quad \frac{d}{dt}\overline{\mathbf{y}}(t) = \left(\frac{1}{\lambda^*}\overline{F}_{\epsilon^*}(t) - \overline{V}(t)\right)\overline{\mathbf{y}}(t), \quad \forall t\in\mathbb{R},\ \lambda^*\in\mathbb{R}_+.$$

Setting $\overline{\mathbf{y}}(0) := \overline{\mathbf{y}}_0$, then $W_{\epsilon^*}(t,\,0,\,1/\lambda^*)\,\mathbf{y}_0 = \overline{\mathbf{y}}(t)$, $\forall t \ge 0$, and hence $\overline{\mathbf{y}}_0 \in \mathcal{C}_\omega \setminus \{\mathbf{0}\}$ since $\overline{\mathbf{y}}(\cdot) \not\equiv \mathbf{0}$ on $\mathbb{R}$. Clearly, the ω-periodicity of $\mathbf{v}(t)$ yields

$$W_{\epsilon^*}(\omega,\,0,\,\lambda^*)\,\mathbf{y}_0 = \overline{\mathbf{y}}(\omega) = \exp\left(X_\lambda(\omega)\right)\mathbf{v}(\omega) = \exp\left(X_\lambda(\omega)\right)\mathbf{v}(0) = \exp\left(X(\omega)\right)\mathbf{y}_0.$$

It follows that $X_\lambda(\omega)$ and $W_{\epsilon^*}(\omega,\, 0,\, c^*(\lambda))$ have the same eigenvalues; in particular, $\rho\,(X_1(\omega)) = 1$ is an eigenvalue of $W_{\epsilon^*}(\omega,\, 0,\, \mu_{\epsilon^*}(1))$ with eigenvector $\mathbf{y}_0 > 0$. Then, the Krein–Rutman theorem [50, Theorem 5.4.33] implies that $\rho\,(W_{\epsilon^*}(\omega,\, 0,\, \mu_{\epsilon^*}(1))) = 1$.

According to the continuity of the spectrum for matrices [49, Section II.5.8], equation (4.24) and Lemma 4.10(ii), we get $\rho\,(W(\omega,\, 0,\, \mu(1))) = 1$ where $\mu(1) = \mathrm{R}_0$ is the unique solution. Thus, part (i) above and assumption $\mathrm{R}_0 \geq 1$ ensure the existence of $\lambda \geq \lambda_*$ with $\mu(\lambda) = 1$. Finally, the part (i) and $\mathrm{R}_0 = \mu(1)$ imply that $0 < \mathrm{R}_0 \leq \mu(0)$.

□

There is a relationship of Proposition 4.15 with previous results:

Set $s_i^*(t) = s_i(t) + \tilde{d}_1(\lambda) \quad (i = 1,\, 2,\, 3)$. Because of the ω-periodicities of $\boldsymbol{\phi}$ and $\overline{F}$ a more practical form of (4.20) is

$$
\begin{aligned}
&(\mathcal{L}_\lambda \boldsymbol{\phi})(t)\\
&= \int_0^\infty \exp\left(\tilde{d}_1(\lambda) s\right) Y(t, t-s)\overline{F}(t-s)(E_\lambda \boldsymbol{\phi})(t-s)\, ds\\
&= \lim_{n\to\infty} \left(\sum_{j=0}^{n} \int_{j\omega}^{(j+1)\omega} \exp\left(\tilde{d}_1(\lambda) s\right) Y(t, t-s)\overline{F}(t-s)(E_\lambda \boldsymbol{\phi})(t-s)\, ds \right)\\
&= \lim_{n\to+\infty} \left(\sum_{j=0}^{n} \int_0^{\omega} \exp\left(\tilde{d}_1(\lambda)(s+j\omega)\right) Y(t, t-s-j\omega)\overline{F}(t-s-j\omega)(E_\lambda \boldsymbol{\phi})(t-s-j\omega) ds \right) \qquad (4.27)\\
&= \lim_{n\to\infty} \int_0^{\omega} \left(\sum_{j=0}^{n} \exp\left(\tilde{d}_1(\lambda)(s+j\omega)\right) Y(t, t-s-j\omega)\overline{F}(t,\, s) \right) (E_\lambda \boldsymbol{\phi})(t-s) ds\\
&= \int_0^{\omega} U_\lambda(t,\, s)(E_\lambda \boldsymbol{\phi})(t-s) ds
\end{aligned}
$$

in which, recalling (4.9), (4.15), (4.16) and the formula for infinite geometric series [51], the matrix function $U_\lambda(t,\, s)$ converges to

$$
U_\lambda(t, s) = \left(\sum_{j=0}^{\infty} Y(t, t-s-j\omega) \right) \overline{F}(t-s) \qquad (4.28)
$$

$$
= \sum_{j=0}^{\infty} \begin{bmatrix} 0 & 0 & 0 & q\, b(t-s) \exp\left(s_0^*(s+j\omega)\right) \\ 0 & 0 & p\, b(t-s) \exp\left(\int_{t-s}^{t+j\omega} s_1^*(\tau) d\tau \right) \dfrac{\overline{\mathrm{M}}_{\mathrm{S}}(t-s)}{\mathrm{H}} & 0 \\ 0 & 0 & 0 & \dfrac{\exp\left(s_0^*(s+j\omega)\right) - \exp\left(s_2^*(s+j\omega)\right)}{(s_0 - s_2) \left(\hat{l}\, q\, b(t-s) \right)^{-1}} \\ 0 & 0 & \dfrac{\left(\exp\left(\int_{t-s}^{s+j\omega} s_1^*(\tau) d\tau \right) - \exp\left(\int_{t-s}^{t+j\omega} s_3^*(\tau) d\tau \right) \right)}{(s_1 - s_3) \left((c\, p\, b(t-s)) \dfrac{\overline{\mathrm{M}}_{\mathrm{S}}(t-s)}{\mathrm{H}} \right)^{-1}} & 0 \end{bmatrix}
$$

$$
\begin{aligned}
&= \begin{bmatrix}
0 & 0 & 0 & q\,b(t-s)\displaystyle\sum_{j=0}^{\infty}\exp\big(s_0^*(s+j\omega)\big) \\
0 & 0 & p\,b(t-s)\displaystyle\sum_{j=0}^{\infty}\exp\left(\int_{t-s}^{t+j\omega} s_1^*(\tau)d\tau\right)\frac{\overline{\mathrm{M}}_{\mathrm{S}}(t-s)}{\mathrm{H}} & 0 \\
0 & 0 & 0 & \dfrac{\displaystyle\sum_{j=0}^{\infty}\exp\left(s_0^*(s+j\omega)\right)-\sum_{j=0}^{\infty}\exp\left(s_2^*(s+j\omega)\right)}{(s_0-s_2)\left(\hat{l}\,q\,b(t-s)\right)^{-1}} \\
0 & 0 & \dfrac{\displaystyle\sum_{j=0}^{\infty}\exp\left(\int_{t-s}^{s+j\omega} s_1^*(\tau)d\tau\right)-\sum_{j=0}^{\infty}\exp\left(\int_{t-s}^{t+j\omega} s_3^*(\tau)d\tau\right)}{(s_1-s_3)\left(c\,p\,b(t-s)\dfrac{\overline{\mathrm{M}}_{\mathrm{S}}(t-s)}{\mathrm{H}}\right)^{-1}} & 0
\end{bmatrix} \\
&= \begin{bmatrix}
0 & 0 & 0 & \dfrac{q\,b(t-s)\exp\left(s_0^*s\right)}{1-\exp\big(\omega s_0^*\big)} \\
0 & 0 & p\,b(t-s)\displaystyle\sum_{j=0}^{\infty}\exp\left(\int_{t-s}^{t} s_1^*(\tau)d\tau + j\int_0^{\omega} s_1^*(\tau)d\tau\right)\frac{\overline{\mathrm{M}}_{\mathrm{S}}(t-s)}{\mathrm{H}} & 0 \\
0 & 0 & 0 & \dfrac{\dfrac{\exp\left(s_0^*s\right)}{1-\exp\left(\omega s_0^*\right)}-\dfrac{\exp\left(s_2^*s\right)}{1-\exp\left(\omega s_2^*\right)}}{(s_0-s_2)\left(\hat{l}\,q\,b(t-s)\right)^{-1}} \\
0 & 0 & \dfrac{\dfrac{\displaystyle\sum_{j=0}^{\infty}\exp\left(j\int_0^{\omega} s_1^*(\tau)d\tau\right)}{\exp\left(-\displaystyle\int_{t-s}^{t} s_1^*(\tau)d\tau\right)}-\dfrac{\displaystyle\sum_{j=0}^{\infty}\exp\left(j\int_0^{\omega} s_3^*(\tau)d\tau\right)}{\exp\left(-\displaystyle\int_{t-s}^{t} s_3^*(\tau)d\tau\right)}}{(s_1-s_3)\left(c\,p\,b(t-s)\dfrac{\overline{\mathrm{M}}_{\mathrm{S}}(t-s)}{\mathrm{H}}\right)^{-1}} & 0
\end{bmatrix} \\
&= \begin{bmatrix}
0 & 0 & 0 & \dfrac{q\,b(t-s)\exp\left(s_0^*s\right)}{1-\exp\big(\omega s_0^*\big)} \\
0 & 0 & p\,b(t-s)\left(\dfrac{\exp\left(\displaystyle\int_{t-s}^{t} s_1^*(\tau)d\tau\right)}{1-\exp\left(\displaystyle\int_0^{\omega} s_1^*(\tau)d\tau\right)}\right)\dfrac{\overline{\mathrm{M}}_{\mathrm{S}}(t-s)}{\mathrm{H}} & 0 \\
0 & 0 & 0 & \dfrac{\dfrac{\exp\left(s_0^*s\right)}{1-\exp\left(\omega s_0^*\right)}-\dfrac{\exp\left(s_2^*s\right)}{1-\exp\left(\omega s_2^*\right)}}{(s_0-s_2)\left(\hat{l}\,q\,b(t-s)\right)^{-1}} \\
0 & 0 & \dfrac{\dfrac{\exp\left(\displaystyle\int_{t-s}^{t} s_1^*(\tau)d\tau\right)}{1-\exp\left(\displaystyle\int_0^{\omega} s_1^*(\tau)d\tau\right)}-\dfrac{\exp\left(\displaystyle\int_{t-s}^{t} s_3^*(\tau)d\tau\right)}{1-\exp\left(\displaystyle\int_0^{\omega} s_3^*(\tau)d\tau\right)}}{(s_1-s_3)\left(c\,p\,b(t-s)\dfrac{\overline{\mathrm{M}}_{\mathrm{S}}(t-s)}{\mathrm{H}}\right)^{-1}} & 0
\end{bmatrix} \\
&= \big[U_{\lambda,\,ij}(t,\,s)\big]_{1\le i\,,\,j<4}\,.
\end{aligned}
\tag{4.29}
$$

In Proposition 4.15, $\mu(0)$ is the spectral radius of the equivalent operator:

$$
\begin{aligned}
&(\mathcal{L}_0\boldsymbol{\phi})(t)\\
&= \int_0^\omega U_0(t,\,s)(E_0\boldsymbol{\phi})(t-s)ds\\
&= \int_0^\infty U_0(t,\,s)\left(\exp\left(\tilde{d}_1(0)\left(s+\left|\sin\left(\frac{\pi s}{\omega}\right)\right|\right)\right)\left(\tilde{S}\boldsymbol{\phi}(t-s)+\tilde{d}_2(0)\left(\tilde{\Lambda}_t\,\boldsymbol{\phi}(t)-\tilde{S}\boldsymbol{\phi}(t-s)\right)\right)\right)ds\\
&= \int_0^\omega U_0(t,\,s)\left(\exp\left(\tanh\left(\min\{0\,,\,1\}\right)\left(s+\left|\sin\left(\frac{\pi s}{\omega}\right)\right|\right)\right)\times\right.\\
&\qquad\qquad\left.\times\left(\tilde{S}\boldsymbol{\phi}(t-s)+\max\{0\,,\,1\}\left(\tilde{\Lambda}_t\,\boldsymbol{\phi}(t)-\tilde{S}\boldsymbol{\phi}(t-s)\right)\right)\right)ds\\
&= \int_0^\omega U_0(t,\,s)\left(\tilde{S}\boldsymbol{\phi}(t-s)+(1)\left(\tilde{\Lambda}_t\,\boldsymbol{\phi}(t)-\tilde{S}\boldsymbol{\phi}(t-s)\right)\right)ds\\
&= \int_0^\omega U_0(t,\,s)\left(\tilde{\Lambda}_t\,\boldsymbol{\phi}(t)\right)ds\\
&= \left(\int_0^\omega U_0(t,\,s)\,ds\right)\tilde{\Lambda}_t\,\boldsymbol{\phi}(t)\\
&= \begin{bmatrix}
0 & 0 & 0 & \tilde{\delta}_4(t)\displaystyle\int_0^\omega U_{0,\,14}(t,\,s)ds\\
0 & 0 & \tilde{\delta}_3(t)\displaystyle\int_0^\omega U_{0,\,23}(t,\,s)ds & 0\\
0 & 0 & 0 & \tilde{\delta}_4(t)\displaystyle\int_0^\omega U_{0,\,34}(t,\,s)ds\\
0 & 0 & \tilde{\delta}_3(t)\displaystyle\int_0^\omega U_{0,\,43}(t,\,s)ds & 0
\end{bmatrix}\boldsymbol{\phi}(t).
\end{aligned}
$$

Repeatedly applying the operator $\mathcal{L}_0$:

$$
\begin{aligned}
(\mathcal{L}_0\boldsymbol{\phi})(t) &= \left[u^*_{ij}(t)\right]_{1\le i\,,\,j<4}\boldsymbol{\phi}(t) = U^*(t)\boldsymbol{\phi}(t),\\
(\mathcal{L}_0^2\boldsymbol{\phi})(t) &= \left[u^*_{ij}(t)\right]^2_{1\le i\,,\,j<4}\boldsymbol{\phi}(t)\\
&= \begin{bmatrix}
0 & 0 & u^*_{14}(t)u^*_{43}(t) & 0\\
0 & 0 & 0 & u^*_{23}(t)u^*_{34}(t)\\
0 & 0 & u^*_{34}(t)u^*_{43}(t) & 0\\
0 & 0 & 0 & u^*_{43}(t)u^*_{34}(t)
\end{bmatrix}\boldsymbol{\phi}(t)\\
&= U^{**}(t)\boldsymbol{\phi}(t) = \left[u^{**}_{ij}(t)\right]_{1\le i\,,\,j<4}\boldsymbol{\phi}(t),\\
(\mathcal{L}_0^3\boldsymbol{\phi})(t) &= \left[u^*_{ij}(t)\right]^3_{1\le i\,,\,j<4}\boldsymbol{\phi}(t) = \left(u^*_{34}(t)u^*_{43}(t)\right)^1 U^*(t)\boldsymbol{\phi}(t),\\
(\mathcal{L}_0^4\boldsymbol{\phi})(t) &= \left[u^*_{ij}(t)\right]^4_{1\le i\,,\,j<4}\boldsymbol{\phi}(t) = \left(u^*_{34}(t)u^*_{43}(t)\right)^1 U^{**}(t)\boldsymbol{\phi}(t),\\
(\mathcal{L}_0^5\boldsymbol{\phi})(t) &= \left[u^*_{ij}(t)\right]^5_{1\le i\,,\,j<4}\boldsymbol{\phi}(t) = \left(u^*_{34}(t)u^*_{43}(t)\right)^2 U^*(t)\boldsymbol{\phi}(t),\\
(\mathcal{L}_0^6\boldsymbol{\phi})(t) &= \left[u^*_{ij}(t)\right]^6_{1\le i\,,\,j<4}\boldsymbol{\phi}(t) = \left(u^*_{34}(t)u^*_{43}(t)\right)^2 U^{**}(t)\boldsymbol{\phi}(t),\\
&\;\;\vdots
\end{aligned}
$$

results in the following recurrence formula:

$$
(\mathcal{L}_0^k\boldsymbol{\phi})(t) = \begin{cases}
\left(u^*_{34}(t)u^*_{43}(t)\right)^{(k-1)/2}(\mathcal{L}_0\boldsymbol{\phi})(t) & \text{for} \quad k = 2n-1 \text{ and } n = 1,2,3,\ldots\\[2ex]
\left(u^*_{34}(t)u^*_{43}(t)\right)^{(k-2)/2}(\mathcal{L}_0^2\boldsymbol{\phi})(t) & \text{for} \quad k = 2n \text{ and } n = 1,2,3,\ldots.
\end{cases}
$$

Applying the supremum norm we arrive at

$$\|\mathcal{L}_0^k\boldsymbol{\phi}\| = \begin{cases} \left\|(u_{34}^*(t)u_{43}^*(t))^{(k-1)/2}\,\mathcal{L}_0\,\boldsymbol{\phi}\right\| & \text{for} \quad k=2n-1 \text{ and } n=1,2,3,\dots \\ \left\|(u_{34}^*(t)u_{43}^*(t))^{(k-2)/2}\,\mathcal{L}_0^2\,\boldsymbol{\phi}\right\| & \text{for} \quad k=2n \text{ and } n=1,2,3,\dots \end{cases}$$
$$= \begin{cases} \sup\limits_{t\in[0\,,\,\omega]} \left|u_{34}^*(t)u_{43}^*(t)\right|^{(k-1)/2}\|\mathcal{L}_0\,\boldsymbol{\phi}\| & \text{for} \quad k=2n-1 \text{ and } n=1,2,3,\dots \\ \sup\limits_{t\in[0\,,\,\omega]} \left|u_{34}^*(t)u_{43}^*(t)\right|^{(k-2)/2}\|\mathcal{L}_0^2\,\boldsymbol{\phi}\| & \text{for} \quad k=2n \text{ and } n=1,2,3,\dots \end{cases} \tag{4.30}$$

As a consequence of the definition of matrix norm for (4.30):

$$\|\mathcal{L}_0\boldsymbol{\phi}\| \le \|U^*\|\|\boldsymbol{\phi}\| \ \wedge\ \|\mathcal{L}_0^2\boldsymbol{\phi}\| \le \|U^{**}\|\|\boldsymbol{\phi}\|$$

so that by taking the supremum over all $\boldsymbol{\phi}$ of norm one:

$$\|\mathcal{L}_0^k\| \le \begin{cases} \sup\limits_{t\in[0\,,\,\omega]} \left|u_{34}^*(t)u_{43}^*(t)\right|^{(k-1)/2}\|U^*\| & \text{for} \quad k=2n-1 \text{ and } n=1,2,3,\dots \\ \sup\limits_{t\in[0\,,\,\omega]} \left|u_{34}^*(t)u_{43}^*(t)\right|^{(k-2)/2}\|U^{**}\| & \text{for} \quad k=2n \text{ and } n=1,2,3,\dots \end{cases}$$

On the other hand, when choosing the particular $\boldsymbol{\phi}(t) \equiv \mathbf{1} \in \mathcal{P}_\omega$, we obtain

$$\begin{aligned} \|\mathcal{L}_0\mathbf{1}\| &= \sup_{t\in[0\,,\,\omega]} \max\left\{\left|u_{14}^*(t)\right|, \left|u_{23}^*(t)\right|, \left|u_{34}^*(t)\right|, \left|u_{43}^*(t)\right|\right\} = \|U^*\|, \\ \|\mathcal{L}_0^2\mathbf{1}\| &= \sup_{t\in[0\,,\,\omega]} \max\left\{\left|u_{14}^{**}(t)\right|, \left|u_{23}^{**}(t)\right|, \left|u_{34}^{**}(t)\right|, \left|u_{43}^{**}(t)\right|\right\} = \|U^{**}\|. \end{aligned}$$

Hence, the norm of $\mathcal{L}_0^k$ is

$$\|\mathcal{L}_0^k\| = \begin{cases} \sup\limits_{t\in[0\,,\,\omega]} \left|u_{34}^*(t)u_{43}^*(t)\right|^{(k-1)/2}\|U^*\| & \text{for} \quad k=2n-1 \text{ and } n=1,2,3,\dots \\ \sup\limits_{t\in[0\,,\,\omega]} \left|u_{34}^*(t)u_{43}^*(t)\right|^{(k-2)/2}\|U^{**}\| & \text{for} \quad k=2n \text{ and } n=1,2,3,\dots \end{cases} \tag{4.31}$$

Extracting the k-th root in (4.31) we arrive at a sequence $\left\{a_k = \|\mathcal{L}^k\|^{1/k}\right\}$ which is the union of the sub-succession of odd-position terms $\left\{a_{2k-1}\right\}$ and the sub-succession of even-position terms $\left\{a_{2k}\right\}$; such sub-sequences converge to the same value $l^* = \left(\sup\limits_{t\in[0\,,\,\omega]} \left|u_{34}^*(t)u_{43}^*(t)\right|\right)^{1/2}$:

$$\lim_{n\to\infty}\|\mathcal{L}_0^k\|^{1/k} = \begin{cases} \lim\limits_{k\to\infty}\left(\sup\limits_{t\in[0\,,\,\omega]} \left|u_{34}^*(t)u_{43}^*(t)\right|\right)^{(k-1)/(2k)} \|U^*\|^{1/k} = l^* & \text{for} \quad k \text{ odd} \\ \lim\limits_{k\to\infty}\left(\sup\limits_{t\in[0\,,\,\omega]} \left|u_{34}^*(t)u_{43}^*(t)\right|\right)^{(k-2)/(2k)} \|U^{**}\|^{1/k} = l^* & \text{for} \quad k \text{ even.} \end{cases}$$

It is enough to prove that $a_k \to l^*$ as $k \to \infty$. Let $\epsilon^* > 0$. By the definition of limit there are $n_1, n_2 \in \mathbb{Z}_+$ such that

$$|a_{2k+1} - l^*| < \epsilon^* \ \ \forall k \geq n_1; \quad |a_{2k} - l^*| < \epsilon^* \ \ \forall k \geq n_2.$$

Now whether k is even or odd, if $n_0 = \max\left\{2n_1 + 1\,,\, 2n_2\right\}$, we have

$$\begin{aligned}|a_k - l^*| < \epsilon^*,\ \forall k \geq n_2 \quad &\Leftrightarrow \lim_{k\to\infty} \|\mathcal{L}_0^k\|^{1/k} = l^* \Leftrightarrow \rho\left(\mathcal{L}_0\right) = \left(\sup_{t\in[0\,,\,\omega]} \left|u_{34}^*(t)u_{43}^*(t)\right|\right)^{1/2} \\ &\Leftrightarrow \mu^2(0) = \sup_{t\in[0\,,\,\omega]} \left|\tilde{\delta}_3(t)\tilde{\delta}_4(t)\right| \sup_{t\in[0\,,\,\omega]} \mathrm{R}^*(t),\end{aligned}$$

where

$$\mathrm{R}^*(t) = \int_0^\omega U_{0,\,34}(t,\,s)ds \int_0^\omega U_{0,\,44}(t,\,s)ds \tag{4.32}$$

or

$$\mathrm{R}^*(t) = \frac{\displaystyle\int_0^\omega \left(\left(\frac{\exp\left(\int_{t-s}^t s_1(\tau)d\tau\right)}{1-\exp\left(\int_0^\omega s_1(\tau)d\tau\right)} - \frac{\exp\left(\int_{t-s}^t s_3^*(\tau)d\tau\right)}{1-\exp\left(\int_0^\omega s_3(\tau)d\tau\right)}\right)\left(\frac{c\,p\,b(t-s)\dfrac{\overline{\mathrm{M}}_{\mathrm{S}}(t-s)}{\mathrm{H}}}{(s_1-s_3)}\right)\right)ds}{\left(\displaystyle\int_0^\omega \left(\frac{\dfrac{\exp\left(s_0 s\right)}{1-\exp\left(\omega s_0\right)} - \dfrac{\exp\left(s_2 s\right)}{1-\exp\left(\omega s_2\right)}}{(s_0-s_2)\left(\hat{l}\,q\,b(t-s)\right)^{-1}}\right)ds\right)^{-1}}.$$

Notice that some of

$$\boldsymbol{\nu}_*(t) = \begin{bmatrix}\nu_{1*}(t) & \nu_{2*}(t) & 0 & \nu_{4*}(t)\end{bmatrix}^\top \text{ and } \boldsymbol{\nu}(t) = \begin{bmatrix}\nu_1(t) & \nu_2(t) & \nu_3(t) & 0\end{bmatrix}^\top$$

($\boldsymbol{\nu}_*, \boldsymbol{\nu} \in \mathcal{C}_\omega$) cannot be an eigenvector of $\mathcal{L}_0$ if $\left(\boldsymbol{e}_3 \cdot \boldsymbol{\nu}_*\right)_{\sup} > 0 \geq \left(\boldsymbol{e}_3 \cdot \boldsymbol{\nu}_*\right)_{\inf}$ and $\left(\boldsymbol{e}_4 \cdot \boldsymbol{\nu}\right)_{\sup} > 0 \geq \left(\boldsymbol{e}_4 \cdot \boldsymbol{\nu}\right)_{\inf}$. In effect:

$$\begin{aligned}
&\quad (\mathcal{L}_0\boldsymbol{\nu}_*)(t) = \mu(0)\boldsymbol{\nu}_*(t) \\
&\Leftrightarrow \begin{bmatrix}\phi_{4*}(t)u_{14}^*(t) & 0 & \nu_{4*}(t)u_{34}^*(t) & 0\end{bmatrix} = \mu(0)\begin{bmatrix}\nu_{1*}(t) & \nu_{2*}(t) & 0 & \nu_{4*}(t)\end{bmatrix} \\
&\Leftrightarrow \mu(0)\nu_{1*}(t) = \nu_{4*}(t)u_{14}^*(t),\, \mu(0)\nu_{2*}(t) = 0,\, 0 = \nu_{4*}(t)u_{34}^*(t),\, \mu(0)\nu_{4*}(t) = 0 \ \ \forall t \\
&\Longrightarrow\ \nu_{2*}(t) = 0,\, \nu_{4*}(t) = 0,\, \nu_{1*}(t) = 0,\, \nu_{2*}(t) = 0 \ \ \forall t \quad \because \mu(0) > 0 \\
&\Leftrightarrow \boldsymbol{\nu}_*(t) = \mathbf{0} \ \ \forall t; \\
&\quad (\mathcal{L}_0\boldsymbol{\nu})(t) = \mu(0)\boldsymbol{\nu}(t) \\
&\Leftrightarrow \begin{bmatrix}0 & \nu_3(t)u_{23}^*(t) & 0 & \nu_3(t)u_{43}^*(t)\end{bmatrix} = \mu(0)\begin{bmatrix}\nu_1(t) & \nu_2(t) & \nu_3(t) & 0\end{bmatrix} \\
&\Leftrightarrow \mu(0)\nu_1(t) = 0,\, \mu(0)\nu_2(t) = \nu_3(t)u_{23}^*(t),\, \mu(0)\nu_3(t) = 0,\, \mu(0)0 = \nu_3(t)u_{43}^*(t) \ \ \forall t \\
&\Longrightarrow\ \nu_1(t) = 0,\, \nu_3(t) = 0,\, \nu_2(t) = 0,\, \nu_4(t) = 0 \ \ \forall t \quad \because \mu(0) > 0 \\
&\Leftrightarrow \boldsymbol{\nu}(t) = \mathbf{0} \ \ \forall t.
\end{aligned}$$

This reveals that the eigenvector associated with the eigenvalue $\mu(0)$, say $\boldsymbol{\phi}_0(t) = \sum_{i=1}^{4} \phi_{0i}(t)\boldsymbol{e}_i$, has $\phi_{03}(t), \phi_{04}(t) \neq 0$, and from (4.19) we see that

$$\tilde{\delta}_j(t) \equiv \text{constant} = \begin{cases} -1 & \text{if} \quad (\boldsymbol{e}_j \cdot \boldsymbol{\phi}_0)_{\sup} < 0 \\ \dfrac{\max\limits_{1\le i\le 4} (\boldsymbol{e}_i \cdot \boldsymbol{\phi}_0)_{\sup}}{\min\limits_{1\le i\le 4} (\boldsymbol{e}_i \cdot \boldsymbol{\phi}_0)_{\inf}} & \text{if} \quad (\boldsymbol{e}_i \cdot \boldsymbol{\phi}_0)_{\inf} > 0 \end{cases} \quad (j = 3,\, 4) \tag{4.33}$$

Remark 4.16. By Proposition 4.15, $\mathrm{R}_0 \le \mu(0)$, in turn it is possible deduce that the spectral radius of $\mathcal{L}$ has an upper bound with radicand (4.34), that is, $\overline{\mathrm{R}}_0 \le \sqrt{\overline{\mathrm{R}}_0^{\inf}}$, which depends on the components of the eigenvector corresponding to $\mu(0)$ in the form (4.33).

$$\overline{\mathrm{R}}_0^{\inf} = \frac{\left(\dfrac{\mathrm{R}^*_{\sup}}{\mathrm{R}^*_{\inf}}\right)\left(\dfrac{p\,q\,\hat{l}\,\hat{\delta}_3\tilde{\delta}_4 b_{\inf}\exp(\omega\, c)}{s_0 s_2\,(1-\exp(\omega\langle s_3\rangle))\,\mathrm{H}}\right)\inf\limits_{t\in[0\,,\,\omega]}\displaystyle\int_0^{\omega} b(t-\eta)\exp\left(\int_t^{t-\eta}(c+m(\tau))d\tau\right)(\overline{\mathrm{M}}_{\mathrm{S}}(t-\eta))d\eta}{(1-\exp(-\omega\langle c+m\rangle))} \tag{4.34}$$

Appendix C contains the proof of this assert.

Remark 4.17. Biologically, the term $\overline{\mathrm{M}}_{\mathrm{S}}(t-s)/\mathrm{H}$, $t \ge s$, in the matrix (4.28), expresses seasonal variations of the so-called "vector density", defined as the average number of vectors (female mosquitoes) per human host [7]. Due to the cyclical pattern of mosquito population density, in winter the density of vectors drops to very low levels and inertially the incidence of dengue, below the $\mathsf{R}_0 = 1$ threshold for transmission; followed by winter and before summer, the vector density begins to increase until it reaches a critical level at which the threshold crosses $\mathsf{R}_0 = 1$ and a wave of transmission begins. Control campaigns have been mainly interested in reducing this important ratio, through larval control measures and elimination of breeding sites, in order to set vector densities below the threshold of epidemic transmission [64].

Remark 4.18. It is possible to show that R_0 for the periodic environment converges to the standard basic reproduction number for the time-averaged non-autonomous epidemic system, that is, the one in which the parameters in system (3.2) are replaced by their long-time averages.

Lemma 4.3 implies that

$$\mathrm{N}^l \le \liminf_{t\to\infty} \mathrm{M}_{\mathrm{S}}(t) \le \limsup_{t\to\infty} \mathrm{M}_{\mathrm{S}}(t) \le \mathrm{N}^u,$$

where we can choose

$$\mathrm{N}^l = \left(\frac{\Delta}{m}\right)_{\infty} \quad \text{and} \quad \mathrm{N}^u = \left(\frac{\Delta}{m}\right)^{\infty}.$$

Since the time-averaged non-autonomous epidemic system has an free-dengue equilibrium point, $\tilde{\mathbf{x}}^{\mathbf{0}} = \begin{bmatrix} \mathrm{H} & 0 & 0 & 0 & \overline{\mathrm{M}}_{\mathrm{S}} & 0 & 0 \end{bmatrix}^{\top}$, one has in that model that $\mathrm{N}^u = \mathrm{N}^l$, and then $\overline{\mathrm{M}}_{\mathrm{S}} = \langle \Delta \rangle / \langle m \rangle$.

On the other hand, since R_0 is an eigenvalue of $\mathcal{L}$, there is a non-negative non-zero function $\boldsymbol{v}(t) \in \mathcal{P}_\omega$ such that

$$\int_0^\omega U_1(t,s)\,\boldsymbol{v}(t-s)ds = \mathsf{R}_0\,\boldsymbol{v}(t).$$

Notice that if $\overline{\mathrm{M}}_{\mathrm{S}}(t) = \langle \Delta \rangle / \langle m \rangle$, then $U_1(t,\,s)$ does not depend on t, i.e. $U(t-s) = U(s)$. In this case, considering a constant function $\boldsymbol{v}(t)$ equal to a nonnegative eigenvector of the nonnegative matrix $\int_0^\omega U_1(s)\,ds$, we see that R_0 is the spectral radius of this matrix, which is generally called the next-generation matrix [65, p.74]. More precisely:

$$\begin{aligned}
\overline{\mathsf{R}}_0 &= \left(\int_0^\omega \left(\frac{\dfrac{\exp(s_0 s)}{1-\exp(\omega s_0)} - \dfrac{\exp(s_2 s)}{1-\exp(\omega s_2)}}{(s_0 - s_2)\left(\hat{l}\,q\,\langle b \rangle\right)^{-1}} \right) ds \int_0^\omega \left(\frac{\dfrac{\exp(s_1 s)}{1-\exp(\omega s_1)} - \dfrac{\exp(s_3 s)}{1-\exp(\omega s_3)}}{(s_1 - s_3)\left(c\,p\,\langle b \rangle\,\overline{\mathrm{M}}_{\mathrm{S}}/\mathrm{H}\right)^{-1}} \right) ds \right)^{1/2} \\
&= \left(\left(\frac{\dfrac{\exp(s_0 s)\big|_{s=0}^{s=\omega}}{s_0(1-\exp(\omega s_0))} - \dfrac{\exp(s_2 s)\big|_{s=0}^{s=\omega}}{s_2(1-\exp(\omega s_2))}}{(s_0 - s_2)\left(\hat{l}\,q\,\langle b \rangle\right)^{-1}} \right) \int_0^\omega \left(\frac{\dfrac{\exp(s_1 s)\big|_{s=0}^{s=\omega}}{s_1(1-\exp(\omega s_1))} - \dfrac{\exp(s_3 s)\big|_{s=0}^{s=\omega}}{s_3(1-\exp(\omega s_3))}}{(s_1 - s_3)\left(p\,\langle b \rangle\,\overline{\mathrm{M}}_{\mathrm{S}}/\mathrm{H}\right)^{-1}} \right) ds \right)^{1/2} \\
&= \left(\left(\frac{\dfrac{1}{-s_0} - \dfrac{1}{-s_2}}{(s_0 - s_2)\left(\hat{l}\,q\,\langle b \rangle\right)^{-1}} \right) \left(\frac{\dfrac{1}{-s_1} - \dfrac{1}{-s_3}}{(s_1 - s_3)\left(p\,\langle b \rangle\,\overline{\mathrm{M}}_{\mathrm{S}}/\mathrm{H}\right)^{-1}} \right) \right)^{1/2} \\
&= \left(\left(\frac{\hat{l}\,q\,\langle b \rangle}{s_0 s_2} \right) \left(\frac{p\,\langle b \rangle\,\overline{\mathrm{M}}_{\mathrm{S}}/\mathrm{H}}{s_1 s_3} \right) \right)^{1/2} = \sqrt{\overline{\mathrm{R}}_{\mathrm{H}} \overline{\mathrm{R}}_{\mathrm{M}}}\ ,
\end{aligned}$$

where

$$\overline{\mathrm{R}}_{\mathrm{H}} = \frac{\langle b \rangle\,p\,\hat{l}}{(h+r)(h+\hat{l})\mathrm{H}} \tag{4.35}$$

and

$$\overline{\mathrm{R}}_{\mathrm{M}} = \frac{\langle b \rangle\,q\,c}{\langle m \rangle (c + \langle m \rangle)} \frac{\langle \Delta(t) \rangle}{\langle m \rangle}. \tag{4.36}$$

To avoid misunderstanding, it is recalled that some authors call R_0 what appears here as R_0^2, this point is discussed briefly in [66, Section 2.1]. The definition of R_0 is consistent with that given for autonomous compartmental epidemic models [67].

Remark 4.19. The factor (4.35) is the number of humans infected by a carrier vector during its period of portability of the dengue virus strain in a population of susceptibles only, and the factor (4.36) is the number of vectors that become carriers by biting an infected human, all of them being non-carriers. Suppose that an infectious mosquito is introduced into populations of humans and mosquitoes, susceptible and non-carriers, exclusively. This infectious carrier mosquito, mean virus incubation period of $1/c$, bites an average number of $\left(q\langle b\rangle/\langle m\rangle\right)\left(\langle\Delta\rangle/\langle m\rangle\right)$ susceptible humans; $q\langle b\rangle/\langle m\rangle$ is the mean number of bites per mosquito. Afterwards, these humans that exceeded the extrinsic incubation period with probability $\hat{l}/(h+\hat{l})$, are bitten on average by $p\,\langle b\rangle/\,((h+r)\mathrm{H})$ non-carrier mosquitoes during the infectious period $1/(h+r)$ of humans. Finally, the probability that these non-infectious carrier mosquitoes survive the extrinsic incubation period and become infectious carrier mosquitoes is given by $c/(c+\langle m\rangle)$. The introduction of an infectious human follows a similar interpretation. Therefore, $\mathrm{R}_0^2=\overline{\mathrm{R}}_{\mathrm{H}}\overline{\mathrm{R}}_{\mathrm{M}}$ is the average number of secondary infected individuals produced by an index human or index mosquito in full infecting capacity, introduced into a dengue-free ecosystem of humans and mosquitoes [13].

4.3. Extinction of the infection

The question of whether the seasonal basic reproductive number has a disease threshold behavior, more precisely, the outbreak of disease will only involve a very small number of individuals and the infection cannot establish itself, is resolved in propositions (4.20) and (4.22).

Proposition 4.20. *Let the BRN and the DFS be defined as* (4.11) *and* (4.6) *for system* (3.2). *If* $\mathrm{R}_0<1$ *then the DFS is locally asymptotically stable, whereas if* $\mathrm{R}_0>1$ *then the DFS is unstable.*

Proof. The local stability of the DFS of (3.2) is determined by linearizing the system (4.7) near $\overline{\mathbf{x}}^{\mathbf{0}}=\begin{bmatrix}0 & 0 & 0 & 0 & \mathrm{H} & \overline{\mathrm{M}}_{\mathrm{s}} & 0\end{bmatrix}^{\top}$. Set the disturbance $\overline{\mathbf{x}}=\mathbf{y}+\overline{\mathbf{x}}^{\mathbf{0}}$ or $\overline{\mathrm{x}}_1=\mathrm{y}_1+0$, $\overline{\mathrm{x}}_2=\mathrm{y}_2+0$, $\overline{\mathrm{x}}_3=\mathrm{y}_3+0$, $\overline{\mathrm{x}}_4=\mathrm{y}_4+0$, $\overline{\mathrm{x}}_5=\mathrm{y}_5+\mathrm{H}$, $\overline{\mathrm{x}}_6=\mathrm{y}_6+\overline{\mathrm{M}}_{\mathrm{s}}$, $\overline{\mathrm{x}}_7=\mathrm{y}_7+0$. Since $\dot{\mathbf{y}}+\dot{\overline{\mathbf{x}}}^{\mathbf{0}}=\mathbf{f}(t,\mathbf{y}+\overline{\mathbf{x}}^{\mathbf{0}})$ y $\dot{\overline{\mathbf{x}}}^{\mathbf{0}}=\mathbf{f}(t,\overline{\mathbf{x}}^{\mathbf{0}})$, the system becomes $\dot{\mathbf{y}}=\mathbf{f}(t,\mathbf{y}+\overline{\mathbf{x}}^{\mathbf{0}})-\mathbf{f}(t,\overline{\mathbf{x}}^{\mathbf{0}})=\mathbf{g}(t,\mathbf{y})$. Expanding the components of $\mathbf{g}$ into a Maclaurin series gives

$$\dot{\mathrm{y}}_i=\frac{\partial\mathrm{g}_i(t,\mathbf{0})}{\partial\mathrm{y}_{j_1}}\mathrm{y}_{j_1}+\frac{1}{2}\frac{\partial^2\mathrm{g}_i(t,\mathbf{0})}{\partial\mathrm{y}_{j_1}\partial\mathrm{y}_{j_2}}\mathrm{y}_{j_1}\mathrm{y}_{j_2}+\frac{1}{3!}\frac{\partial^3\mathrm{g}_i(t,\mathbf{0})}{\partial\mathrm{y}_{j_1}\partial\mathrm{y}_{j_2}\partial\mathrm{y}_{j_3}}\mathrm{y}_{j_1}\mathrm{y}_{j_2}\mathrm{y}_{j_3}+\cdots$$

The above equation can be written as

$$\dot{\mathbf{y}} = \boldsymbol{D}_{\overline{\mathbf{x}}}\mathbf{f}(t,\mathbf{y})\mathbf{y} + o(|\mathbf{y}|)$$

such that $\dfrac{o(|\mathbf{y}|)}{|\mathbf{y}|} \longrightarrow 0$ as $\mathbf{y} \longrightarrow \mathbf{0}$ uniformly in t. The variational system with respect to the solution $\overline{\mathbf{x}}^{\mathbf{0}}$ is

$$\dot{\mathbf{y}} = \mathcal{J}(t)\mathbf{y} \quad \text{where} \quad \mathcal{J}(t) = \begin{bmatrix} \overline{F}(t) - \overline{V}(t) & O_4 \\ \mathcal{J}_3(t) & \mathcal{J}_4(t); \end{bmatrix} \tag{4.37}$$

In turn (4.37) includes the matrices $\overline{F}(t)$ and $\overline{V}(t)$ defined in (4.9), the matrix $\mathcal{J}_4(t) = \tilde{M}(t)$ defined in (4.13), and

$$\mathcal{J}_3(t) = \begin{bmatrix} 0 & 0 & 0 & -\dfrac{qb(t)}{\mathrm{H}}\mathrm{H} \\ 0 & 0 & -\dfrac{pb(t)}{\mathrm{H}}\overline{\mathrm{M}}_{\mathrm{S}}(t) & 0 \\ 0 & 0 & r & 0 \end{bmatrix}.$$

In view of assumption (A6), $\rho\big(\Phi_{\mathcal{J}_4}(\omega)\big) < 1$, so that the stability of system (4.7) depends on the eigenvalues of $\Phi_{\overline{F}-\overline{V}}(\omega)$; if $\rho\big(\Phi_{\overline{F}-\overline{V}}(\omega)\big) < 1$ then $\overline{\mathbf{x}}^{\mathbf{0}}$ is uniformly asymptotically stable, but if $\rho\big(\Phi_{\overline{F}-\overline{V}}(\omega)\big) > 1$ then $\overline{\mathbf{x}}^{\mathbf{0}}$ is unstable [52]. Thus, thanks to Proposition 4.11, the DFS (4.6) is locally asymptotically stable if $\mathsf{R}_0 < 1$ and unstable if $\mathsf{R}_0 > 1$. □

The following result is useful for our subsequent comparison arguments.

Lemma 4.21. *Let $A(t)$ be a continuous, cooperative, irreducible and ω-periodic matrix function, let $\Phi_{A(\cdot)}(t)$ be the principal fundamental matrix solution of $\dot{\overline{\mathbf{x}}} = A(t)\mathbf{x}$ and let $\ln\left(\rho\left(\Phi_{A(\cdot)}(\omega)\right)\right)^{\frac{1}{\omega}} = \overline{p}$, then there exists a positive ω-periodic function $\mathbf{v}(t)$ such that $e^{\overline{p}t}\mathbf{v}(t)$ is a solution of $\dot{\overline{\mathbf{x}}} = A(t)\overline{\mathbf{x}}$.*

Proof. Visit Appendix A.6 □

We are now in conditions to state a result about global stability of the DFS.

Proposition 4.22. *The DFS (4.5) of the system (3.2) is globally asymptotically stable if $\mathrm{R}_0 < 1$, and unstable if $\mathrm{R}_0 > 1$.*

Proof. By Proposition 4.11, if $\mathrm{R}_0 < 1$ then $\mathbf{x^0} = \begin{bmatrix}\mathrm{H} & 0 & 0 & 0 & \overline{\mathrm{M}}_{\mathrm{S}}(t) & 0 & 0\end{bmatrix}^\top$ is locally asymptotically stable, so it is sufficient prove that $\mathbf{x^0}$ attracts all non-negative solutions $\mathbf{x}(t)$ of (3.2). Given $\epsilon > 0$, by Lemma 4.2 we have

$$\left(\mathrm{M}_{\mathrm{S}} - \overline{\mathrm{M}}_{\mathrm{S}}\right)^\infty = \lim_{t\to\infty} \sup_{\tau\geq t} \left(\mathrm{M}_{\mathrm{S}}(\tau) - \overline{\mathrm{M}}_{\mathrm{S}}(\tau)\right) = L \leq 0,$$

then there exists a $\overline{N} > 0$ such that for all $\tau_3 > \overline{N}$,

$$-\epsilon < \sup_{t\geq\tau_3} \left(\mathrm{M}_{\mathrm{S}}(t) - \overline{\mathrm{M}}_{\mathrm{S}}(t)\right) - L < \epsilon,$$

this implies that $\sup_{t\geq\tau_3} \left(\mathrm{M}_{\mathrm{S}}(t) - \overline{\mathrm{M}}_{\mathrm{S}}(t)\right) < L + \epsilon \leq \epsilon$. Then, from definition of supremum [55], we have $\mathrm{M}_{\mathrm{S}}(t) \leq \overline{\mathrm{M}}_{\mathrm{S}}(t) + \epsilon$ for all $t \geq \tau_3$. With this last result and $\overline{x}_5/\mathrm{H} < 1$ we can associate to (4.7) the following system of differential inequalities:

$$\begin{cases} \dot{\overline{x}}_1 \leq qb(t)\overline{x}_4 - (\hat{l} + h)\overline{x}_1 \\ \dot{\overline{x}}_2 \leq \dfrac{pb(t)\left(\overline{\mathrm{M}}_{\mathrm{S}}(t) + \epsilon\right)}{\mathrm{H}}\overline{x}_3 - (c + m(t))\overline{x}_2 \\ \dot{\overline{x}}_3 \leq \hat{l}\,\overline{x}_1 - (h + r)\overline{x}_3 \\ \dot{\overline{x}}_4 \leq c\overline{x}_2 - m(t)\overline{x}_4. \end{cases} \tag{4.38}$$

Let

$$M_1(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{pb(t)}{\mathrm{H}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Consider the perturbated subsystem:

$$\begin{cases} \dot{\overline{w}}_1 = qb(t)\overline{w}_4 - (\hat{l} + h)\overline{w}_1 \\ \dot{\overline{w}}_2 = \dfrac{pb(t)\overline{\mathrm{M}}_{\mathrm{S}}(t)}{\mathrm{H}}\overline{w}_3 - (c + m(t))\overline{w}_2 + \epsilon\left(\dfrac{pb(t)}{\mathrm{H}}\right)\overline{w}_3 \\ \dot{\overline{w}}_3 = \hat{l}\overline{w}_1 - (h + r)\overline{w}_3 \\ \dot{\overline{w}}_4 = c\overline{w}_2 - m(t)\overline{w}_4 \end{cases} \tag{4.39}$$

which can be rewritten as

$$\begin{bmatrix}\dot{\overline{w}}_1 & \dot{\overline{w}}_2 & \dot{\overline{w}}_3 & \dot{\overline{w}}_4\end{bmatrix}^\top = \left(\overline{F}(t) - \overline{V}(t) + \epsilon M_1(t)\right)\begin{bmatrix}w_1 & w_2 & w_3 & w_4\end{bmatrix}^\top$$

with $\overline{F}$ and $\overline{V}$ defined in (4.9).

Notice that $\overline{F}-\overline{V}+\epsilon M_1(t)$ is ω-periodic, cooperative, irreducible and continuous, then, by Lemma 4.21, the function $se^{\overline{p}(t-\tau_0)}\mathbf{v}(t-\tau_0)$ with $\overline{p}=\ln\left(\rho\left(\Phi_{A(\cdot)}(\omega)\right)\right)^{\frac{1}{\omega}}$ is also a solution of system (4.39) with initial condition $s\mathbf{v}(0)$ at $t=\tau_0$ for all $s>0$. Choose a $\overline{t}>t_1$ and $\overline{s}>0$ such that $\begin{bmatrix}\overline{x}_1(\overline{t}) & \overline{x}_2(\overline{t}) & \overline{x}_3(\overline{t}) & \overline{x}_4(\overline{t})\end{bmatrix}^\top \leq \overline{s}\mathbf{v}(0)$, then from (4.38),

$$\frac{d}{dt}\begin{bmatrix}\overline{x}_1 & \overline{x}_2 & \overline{x}_3 & \overline{x}_4\end{bmatrix}^\top \leq \left(\overline{F}-\overline{V}\right)\begin{bmatrix}\overline{x}_1 & \overline{x}_2 & \overline{x}_3 & \overline{x}_4\end{bmatrix}^\top + \epsilon M_1\begin{bmatrix}\overline{x}_1 & \overline{x}_2 & \overline{x}_3 & \overline{x}_4\end{bmatrix}^\top;$$

and applying comparison principle [54, Theorem B.1]:

$$\begin{bmatrix}\overline{x}_1 & \overline{x}_2 & \overline{x}_3 & \overline{x}_4\end{bmatrix}^\top \leq \overline{s}e^{\overline{p}(t-\overline{t})}\mathbf{v}(t-\overline{t})$$

for all $t\geq\overline{t}$.

From Proposition 4.11 we conclude that $\rho\left(\Phi_{\overline{F}-\overline{V}}\right)<1$ if and only if $\mathrm{R}_0<1$, and by the continuity of the spectrum for matrices [49, Section II.5.8], there exists a $\epsilon>0$ small enough such that $\rho\left(\Phi_{\overline{F}-\overline{V}+\epsilon M_1(t)}\right)<1$, consequently $\overline{p}<0$. Then, utilizing positivity of solutions and squeeze theorem [56, Theorem 1.10.1]:

$$0\leq\lim_{t\to\infty}\overline{\mathrm{x}}_3(t)=\lim_{t\to\infty}\mathrm{H_I}(t)\leq\lim_{t\to\infty}\overline{s}e^{\overline{p}(t-\overline{t})}v_1(t-\overline{t})=0. \tag{4.40}$$

Similarly for $\mathrm{M_I}$, $\mathrm{H_E}$ y $\mathrm{M_E}$:

$$\begin{aligned}&0\leq\lim_{t\to\infty}\overline{\mathrm{x}}_3(t)=\lim_{t\to\infty}\mathrm{H_I}(t)\leq\lim_{t\to\infty}\overline{s}e^{\overline{p}(t-\overline{t})}v_1(t-\overline{t})=0\\ &\therefore\lim_{t\to\infty}\overline{\mathrm{x}}_3(t)=\lim_{t\to\infty}\mathrm{H_I}(t)=0\end{aligned} \tag{4.41}$$

We also need prove that $\mathrm{H_S}(t)$ approaches to H as $t\to\infty$. At infection free solution, $\overline{\mathrm{H}}_\mathrm{R}(t)=0$, where $\overline{\mathrm{H}}_\mathrm{R}$ satisfies the equation

$$\frac{d}{dt}\left(\mathrm{H_R}-\overline{\mathrm{H}}_\mathrm{R}\right)=r\mathrm{H_I}-h\left(\mathrm{H_R}-\overline{\mathrm{H}}_\mathrm{R}\right).$$

Due to (4.40) and given $\epsilon_1>0$, we can find a $\tau_4>0$ such that $\mathrm{H_I}<\epsilon_1$ for $t>\tau_4$, then

$$\frac{d}{dt}\mathrm{H_R}\leq r\epsilon_1-h\mathrm{H_R}.$$

Multiplying in both sides by e^{ht} and integrating this inequality over $[\tau_4\,,\,t]$ we get

$$\mathrm{H_R}(t)\leq\mathrm{H_R}(\tau_4)e^{-h(t-\tau_4)}+\frac{r\epsilon_1}{h}\left(1-e^{-h(t-\tau_4)}\right)\text{ and }\mathrm{H_R^\infty}\leq\frac{r\epsilon_1}{h}.$$

Since ϵ_1 is arbitrarily small then $\mathrm{H_R^\infty}\leq 0$. For $\epsilon_2>0$, we can find $\tau_5>0$ such that $\mathrm{H_R}(t)\leq\epsilon_2/2$ for $t\geq\tau_5$. In addition, from (4.40) and (4.41) we can find $\overline{\tau}_4>0$ with $\mathrm{H_E}+\mathrm{H_I}<\epsilon_2/2$ for $t>\overline{\tau}_4$. Let $t>\tau_6=\max\{\overline{\tau}_4,\tau_5\}$, then

$$\mathrm{H_S}(t)=\mathrm{H}-\mathrm{H_E}(t)-\mathrm{H_I}(t)-\mathrm{H_R}(t)\geq\mathrm{H}-\epsilon_2,$$

or $\mathrm{H_S}(t) - \mathrm{H} \geq -\epsilon_2$, with ϵ_2 arbitrarily small, this implies that $(\mathrm{H_S} - \mathrm{H}(0))_\infty \geq 0$. When comparing and utilizing Lemma 4.2:

$$0 \geq (\mathrm{H_S} - \mathrm{H}(0))^\infty \geq (\mathrm{H_S} - \mathrm{H}(0))_\infty \geq 0 \therefore \lim_{t\to\infty} \mathrm{H_S}(t) = \mathrm{H_S}(0).$$

Finally, since $\mathrm{M}(t)$ (total size of mosquito population) is a solution of equation (4.1), we conclude that $\lim_{t\to+\infty} \left(\mathrm{M}(t) - \overline{\mathrm{M}}_\mathrm{S}(t)\right) = 0$ and

$$\mathrm{M_S}(t) - \overline{\mathrm{M}}_\mathrm{S}(t) = \mathrm{M}(t) - \overline{\mathrm{M}}_\mathrm{S}(t) - \mathrm{M_E}(t) - \mathrm{M_I}(t) \to 0 \ \text{ as } t \to \infty$$

if and only if $\lim_{t\to\infty} \mathrm{M_S}(t) = \overline{\mathrm{M}}_\mathrm{S}(t)$. Therefore, the DFS is globally attractive. □

5. Disease persistence analysis

5.1. Existence of an endemic periodic solution

In order to prepare the arguments that ensure the existence of an endemic periodic solution of the system (3.2), we establish new lemmas and show that if $\mathrm{R}_0^{\inf} = \overline{\mathrm{R}}_0^{\inf}(0,1)$ (see Definition 5.1) is greater than 1, then there are minimum threshold values for $\mathrm{H_I}(t)$ and $\mathrm{M_I}(t)$ such that if $\mathrm{H_E}(0) > 0$ or $\mathrm{H_I}(0) > 0$ or $\mathrm{M_I}(0) > 0$ or $\mathrm{M_E}(0) > 0$ then $\mathrm{H_I}(t)$ and $\mathrm{M_I}(t)$ will rise above those threshold values and, thereafter, the times spent continuously beneath them can be bounded above by bounds that depend only on the parameters of the model and not on the initial conditions; moreover, the times taken to rise initially above the thresholds can be bounded above by bounds that depend only on the initial data $\mathrm{H_I}(0)$ and $\mathrm{M_I}(0)$ and the model parameters.

We need the following definitions:

Definition 5.1. Driven by Remark 4.16, two elements are introduced:

$$\overline{\mathrm{R}}_0^{\sup}(\lambda, j) := \sup_{t\in[n\omega\,,\,n^*\omega]} \mathrm{R}(t,\lambda,j) \ \text{ and } \ \overline{\mathrm{R}}_0^{\inf}(\lambda, j) := \inf_{t\in[n\omega\,,\,n^*\omega]} \mathrm{R}(t,\lambda,j), \tag{5.1}$$

where for $j \in \{0,\, 1\}$ and $(n\,,n^*) \in \mathbb{Z}_+^2$,

$$\mathrm{R}(t,\lambda,j) = \frac{\mathrm{K}^j \displaystyle\int_0^\omega b(t-\eta)\exp\left(\int_t^{t-\eta} (c+\lambda+m(\tau))d\tau\right)\left(\overline{\mathrm{M}}_\mathrm{S}(t-\eta)\right)^j d\eta}{1-\exp\left(-\omega\left\langle c+\lambda+m\right\rangle\right)} \tag{5.2}$$

and

$$\mathrm{K} = \left(\frac{\mathrm{R}^*_{\sup}}{\mathrm{R}^*_{\inf}}\right)\left(\frac{p\,q\,\hat{l}\,\tilde{\delta}_3\tilde{\delta}_4 b_{\inf}\exp\left(\omega\, c\right)}{s_0 s_2\left(1-\exp\left(\omega\langle s_3\rangle\right)\right)\mathrm{H}}\right).$$

Remark 5.2. Notice that $\mathrm{R}(t,\lambda,j)$ is periodic in t for a fixed λ and hence

$$\begin{aligned} &\sup\{\mathrm{R}(t,\lambda,j) : n\omega \le t \le n^*\omega\} = \sup\{\mathrm{R}(t,\lambda,j) : 0 \le t \le \omega\} \\ \textit{and} \quad &\inf\{\mathrm{R}(t,\lambda,j) : n\omega \le t \le n^*\omega\} = \inf\{\mathrm{R}(t,\lambda,j) : 0 \le t \le \omega\} \end{aligned}$$

for all $(n\,,n^*) \in \mathbb{Z}_+^2$.

Remark 5.3. Consider the equation $\overline{Q}(\lambda) = \lambda^2 - (p^* + q^*)\lambda + c_0(\lambda) = 0$, where

$$p^* = (\mathrm{N}^u + 1)(\mathrm{K}_2 + 1)\mathrm{H}/(pq), \;\; q^* = \hat{l} + h + 1, \;\; \text{and } c_0(\lambda) = \tfrac{1}{2}p^*q^*\big(1 - \overline{\mathrm{R}}_0^{\inf}(\lambda, 1)\big).$$

If $\mathrm{R}_0^{\inf} = \overline{\mathrm{R}}_0^{\inf}(0,1) > 1$, then $c_0(0) < 0$ and $\overline{Q}(\lambda) = 0$ has a unique positive root, say $\lambda = \lambda_1$, such that

$$\frac{1}{2}\left(\frac{p^*}{\lambda_1 - p^*}\right)\left(\frac{q^*}{\lambda_1 - q^*}\right)\big(1 + \overline{\mathrm{R}}_0^{\inf}(\lambda_1, 1)\big) = 1$$

and

$$\lambda_1 = \frac{p^* + q^* + \sqrt{(p^*+q^*)^2 + 2c_0(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1) - 1)}}{2} \ge \frac{p^* + q^* + |p^* + q^*|}{2} = p^* + q^*.$$

Definition 5.4.

$$\overline{\mathrm{R}}(\overline{\theta}) = \frac{\Big(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1) - 1\Big)(1-\overline{\theta})/\overline{\mathrm{R}}_0^{\sup}(\lambda_1,0)}{4(\mathrm{N}^u+1)\mathrm{K}\Big(1 + pb^u\Delta^u/\big(\mathrm{H}(c+m^l)m^l\big) + qb^u\mathrm{H}/(\hat{l}+h) + r/h\Big)}$$

and

$$0 < \Theta(\overline{\theta}) < \min\left\{\frac{\overline{\mathrm{R}}(\overline{\theta})}{2}\,,\ \frac{\Delta^u}{m^l}\left(1 - \frac{pb^u\beta_{\mathrm{H}}(\overline{\theta})}{\mathrm{H}(c+m^l)}\right),\ \mathrm{H}\left(1 - \frac{\beta_{\mathrm{H}}(\overline{\theta})}{h/r}\right),\ \mathrm{H}\left(1 - \frac{qb^u\beta_{\mathrm{M}}(\overline{\theta})}{(\hat{l}+h)}\right)\right\}.$$

Here $0 < \overline{\theta} < 1$ is a arbitrarily small number and $\big[\beta_{\mathrm{H}}(\overline{\theta}) \;\; \beta_{\mathrm{M}}(\overline{\theta})\big]^\top \in \mathbb{R}_+^2$ is a vector such that

$$0 < \beta_{\mathrm{H}}(\overline{\theta}) < \min\left\{\overline{\mathrm{R}}(\overline{\theta})\,,\ \frac{\mathrm{H}(c+m^l)}{pb^u}\,,\ \frac{h}{r}\right\} \;\wedge\; 0 < \beta_{\mathrm{M}}(\overline{\theta}) < \min\left\{\overline{\mathrm{R}}(\overline{\theta})\,,\ \frac{\hat{l}+h}{qb^u}\right\}.$$

Remark 5.5. For the moment we shall suppose that $\overline{\theta} > 0$ is fixed and write $\overline{\mathrm{R}}$, Θ, β_{H} and β_{M} instead of $\overline{\mathrm{R}}(\overline{\theta})$, $\Theta(\overline{\theta})$, $\beta_{\mathrm{H}}(\overline{\theta})$ and $\beta_{\mathrm{M}}(\overline{\theta})$, respectively. In Definition 5.4, $\overline{\mathrm{R}}(\overline{\theta})$ is a well defined positive number and $\Theta(\overline{\theta})$, β_{H} and β_{M} are in well defined ranges. Given $\overline{\theta} > 0$ we suppose that $\min\{\mathrm{H}_{\mathrm{I}}(0)\,,\mathrm{M}_{\mathrm{I}}(0)\} > 0$ and find upper bounds for the times t and τ for which $\big[\mathrm{H}_{\mathrm{I}}(t) \;\; \mathrm{M}_{\mathrm{I}}(\tau)\big] < \big[\beta_{\mathrm{H}}(\overline{\theta}) \;\; \beta_{\mathrm{M}}(\overline{\theta})\big]$. We do this by supposing that $\big[\mathrm{H}_{\mathrm{I}}(t) \;\; \mathrm{M}_{\mathrm{I}}(t)\big] < \big[\beta_{\mathrm{H}}(\overline{\theta}) \;\; \beta_{\mathrm{M}}(\overline{\theta})\big]$ indefinitely continuously and derive a contradiction. The important result holds that $\big[\mathrm{H}_{\mathrm{I}}(t) \;\; \mathrm{M}_{\mathrm{I}}(\tau)\big]$ must eventually rise again above $\big[\beta_{\mathrm{H}}(\overline{\theta}) \;\; \beta_{\mathrm{M}}(\overline{\theta})\big]$ and the times t and τ taken to do this are bounded above by times depending only on Θ, β_{H}, β_{M} and the model parameters. Without loss of generality, if the solution $\boldsymbol{\chi}(t,\mathbf{x}_0)$ to the problem (3.4) utilizes the initial values $\mathrm{H_I}(0) > \beta_\mathrm{H}$ and $\mathrm{M_I}(0) > \beta_\mathrm{M}$ for a scenario in which $\mathrm{H_I}(t)$ drops beneath $\beta_\mathrm{H}(\overline{\theta})$ for the first time at time $t=\eta_0$ and $\mathrm{M_I}(t)$ drops beneath $\beta_\mathrm{M}(\overline{\theta})$ for the first time at time $\tau=\eta_1$, we can assume that $\eta_0=\eta_1=0$.

We also need four preliminary lemmas:

Lemma 5.6. *Suppose that* $\mathrm{H_I}(t)\le\beta_\mathrm{H}$ *for all* $t\ge 0$, *then there exists times* $T_0>0$ *and* $T_1>0$ *such that*

$$\begin{cases} \mathrm{H_R}(t) < \dfrac{r\beta_\mathrm{H}}{h} + \Theta \ \ \textit{for all}\ \ t > T_0 \\ \mathrm{M_E}(t) < \dfrac{pb^u\beta_\mathrm{H}}{\mathrm{H}(c+m^l)}\dfrac{\Delta^u}{m^l} + \Theta \ \ \textit{for all}\ \ t > T_1, \end{cases}$$

where T_0 *and* T_1 *depends only on* Θ, β_H *and the model parameters.*

Proof. For $t\ge t_0$, the boundedness of the rates, the phase region (3.3) and the assumption $\mathrm{H_I}(t)\le\beta_\mathrm{H}$ convert the differential equation of non-infectious carrier mosquitoes into

$$\begin{aligned} &\frac{d}{dt}\mathrm{M_E}(t) \le \frac{qb^u\beta_\mathrm{H}}{\mathrm{H}}\frac{\Delta^u}{m^l} - \left(c+m^l\right)\mathrm{M_E}(t) \\ \Leftrightarrow\ &\frac{d}{dt}\left(\mathrm{M_E}(t)\exp\left((c+m^l)t\right)\right) \le \frac{qb^u\beta_\mathrm{H}}{\mathrm{H}}\frac{\Delta^u}{m^l}\exp\left((c+m^l)t\right). \end{aligned}$$

Integrating this inequality and accommodating terms:

$$\begin{aligned} \mathrm{M_E}(t) &\le \frac{\mathrm{M_E}(t_0)}{\exp\left((c+m^l)(t-t_0)\right)} + \frac{pb^u\beta_\mathrm{H}}{\mathrm{H}(c+m^l)}\frac{\Delta^u}{m^l}\left(1-\exp\left(-(c+m^l)(t-t_0)\right)\right) \\ &\le \frac{pb^u\beta_\mathrm{H}}{\mathrm{H}(c+m^l)}\frac{\Delta^u}{m^l} + \exp\left(-(c+m^l)(t-t_0)\right)\frac{\Delta^u}{m^l}\left(1-\frac{pb^u\beta_\mathrm{H}}{\mathrm{H}(c+m^l)}\right) \\ &< \frac{pb^u\beta_\mathrm{H}}{\mathrm{H}(c+m^l)}\frac{\Delta^u}{m^l} + \Theta \end{aligned}$$

provided that $\Theta > \exp\left(-(c+m^l)(t-t_0)\right)\dfrac{\Delta^u}{m^l}\left(1-\dfrac{pb^u\beta_\mathrm{H}}{\mathrm{H}(c+m^l)}\right)$, equivalently

$$t > T_0 = t_0 + \frac{1}{c+m^l}\ln\left(\frac{1}{\Theta}\frac{\Delta^u}{m^l}\left(1-\frac{pb^u\beta_\mathrm{H}}{\mathrm{H}(c+m^l)}\right)\right) > 0 \text{ whenever } \beta_\mathrm{H} < \frac{\mathrm{H}(c+m^l)}{pb^u}.$$

Again the set (3.3) and assumption $\mathrm{H_I}(t)\le\beta_\mathrm{H}$ convert the differential equation of the recovered humans into

$$\frac{d}{dt}\mathrm{H_R}(t) \le r\beta_\mathrm{H}\mathrm{H} - h\mathrm{H_R}(t) \Leftrightarrow \frac{d}{dt}\left(\mathrm{H_R}(t)\exp\left(h\,t\right)\right) \le r\beta_\mathrm{H}\mathrm{H}\exp\left(h\,t\right)$$

for $t \geq t_0$. Integrating this inequality we get

$$\begin{aligned}\mathrm{M_E}(t) &\leq \frac{\mathrm{H_R}(t_0)}{\exp\big(h(t-t_0)\big)} + \frac{r\beta_{\mathrm{H}}\mathrm{H}}{h}\Big(1-\exp\big(-h(t-t_0)\big)\Big)\\ &\leq \frac{r\beta_{\mathrm{H}}\mathrm{H}}{h} + \exp\big(-h(t-t_0)\big)\mathrm{H}\left(1-\frac{r\beta_{\mathrm{H}}}{h}\right)\\ &< \frac{r\beta_{\mathrm{H}}\mathrm{H}}{h} + \Theta\end{aligned}$$

whenever $\Theta > \exp\big(-h(t-t_0)\big)\mathrm{H}\left(1-\frac{r\beta_{\mathrm{H}}}{h}\right)$, equivalently

$$t > T_1 = t_0 + \frac{1}{h}\ln\left(\frac{\mathrm{H}}{\Theta}\left(1-\frac{r\beta_{\mathrm{H}}}{h}\right)\right) > 0 \text{ cuando } \beta_{\mathrm{H}} < \min\left\{\frac{\mathrm{H}(c+m^l)}{pb^u}, \frac{h}{r}\right\}.$$

Since $\Theta < \min\left\{\frac{\Delta^u}{m^l}\left(1-\frac{pb^u\beta_{\mathrm{H}}}{\mathrm{H}(c+m^l)}\right), \mathrm{H}\left(1-\frac{r\beta_{\mathrm{H}}}{h}\right)\right\}$, then there exists a $T_j > 0$ $(j = 0, 1)$ which depends only on Θ, β_{H} and the model parameters. □

Lemma 5.7. *Suppose that* $\mathrm{M_I}(t) \leq \beta_{\mathrm{M}}$ *for all* $t \geq 0$, *then there exists a time* $T_2 > 0$ *such that*

$$\mathrm{H_E}(t) < \frac{qb^u\beta_{\mathrm{M}}\mathrm{H}}{h+\hat{l}} + \Theta \textit{ for all } t > T_2,$$

where T_2 *depends only on* Θ, β_{M} *and the model parameters.*

Proof. Visit Appendix A.7. □

Lemma 5.8. *Suppose that* $\mathrm{H_I}(0) = \beta^*_{\mathrm{H}}$ *and define a time* $T_3 = (n_0+1)\omega$, *where* $n_0 = \lceil T_4/\omega\rceil$ *and* $T_4 = (\ln 4)\mathrm{S}_1/\Delta^l$. *Let*

$$\begin{cases}\mathrm{E}^*_1 = \mathrm{E}^*_2\exp\big(-(c+m^u)T_3\big) > 0\\ \mathrm{E}^*_2 = \big(p\beta^*_{\mathrm{H}}/\mathrm{H}\big)\mathrm{S}_1\exp\big(-(h+r)n_0\omega\big)\displaystyle\int_0^\omega b(t)\exp\big(-(h+r)t\big)dt > 0\\ \mathrm{S}_1 = \big(\Delta^l/2\big)/\big(pb^u+m^u\big),\end{cases}$$

then $\mathrm{M_E}(T_3) \geq \mathrm{E}^*_1 > 0$. *Both* E^*_1 *and* T_3 *depend only on* β^*_{H} *and the model parameters.*

Proof. From the differential equation of the non-carrier mosquitoes:

$$\frac{d}{dt}\mathrm{M_S}(t) \geq \Delta^l - \big((\Delta^l/2)/\mathrm{S}_1\big)\mathrm{M_S}(t) \Leftrightarrow \frac{d}{dt}\big(\mathrm{M_S}(t)\exp\big((\Delta^l/2)t/\mathrm{S}_1\big)\big) \geq \Delta^l\exp\big((\Delta^l/2)t/\mathrm{S}_1\big)$$

Integrating this inequality over $[0\,,\,t]$,

$$\begin{aligned}\mathrm{M_S}(t) &\geq \mathrm{M_S}(0)\exp\big(-(\Delta^l/2)t/\mathrm{S}_1\big) + \frac{\Delta^l}{(\Delta^l/2)/\mathrm{S}_1}\Big(1-\exp\big(-(\Delta^l/2)t/\mathrm{S}_1\big)\Big)\\ &\geq 2\mathrm{S}_1\Big(1-\exp\big(-(\Delta^l/2)t/\mathrm{S}_1\big)\Big) \end{aligned} \tag{5.3}$$

As $\mathrm{H_I}(0) = \beta^*_{\mathrm{H}}$ then $\mathrm{H_I}(t) \geq \beta^*_{\mathrm{H}} e^{-(h+r)t}$ from the third equation of system (3.2); hence, utilizing the sixth equation of (3.2) and the minorant function (5.3) (monotonically increasing),

$$\begin{aligned} f_0(t) = \frac{d}{dt}\Big(\mathrm{M_E}(t)\exp\big((c+m^u)t\big)\Big) &\geq \big(pb(t)\beta^*_{\mathrm{H}}/\mathrm{H}\big)\mathrm{S}_1\exp\big((c+m^u-(h+r))t\big)\\ &\geq \big(pb(t)\beta^*_{\mathrm{H}}/\mathrm{H}\big)\mathrm{S}_1\exp\big(-(h+r)t\big) = f_1(t). \end{aligned}$$

for $t \geq T_4$. If $f_0(t) \geq f_1(t)$ is integrated between $n_0\omega$ and $(n_0+1)\omega$, then

$$\begin{aligned} \mathrm{M_E}(T_3) &\geq \exp\big(-(c+m^u)T_3\big)\big(p\beta^*_{\mathrm{H}}/\mathrm{H}\big)\mathrm{S}_1\int_{n_0\omega}^{(n_0+1)\omega} b(t)\exp\big(-(h+r)t\big)dt\\ &\geq \exp\big(-(c+m^u)T_3\big)\big(p\beta^*_{\mathrm{H}}/\mathrm{H}\big)\mathrm{S}_1\exp\big(-(h+r)n_0\omega\big)\int_0^{\omega} b(t)\exp\big(-(h+r)t\big)dt = \mathrm{E}^*_1 > 0 \end{aligned}$$

as requested. □

Lemma 5.9. *Suppose that* $\mathrm{M_I}(0) = \beta^*_{\mathrm{M}}$ *and define a time* $\overline{T}_3 = (n_1+1)\omega$*, where* $n_1 = \lceil \overline{T}_4/\omega \rceil$ *and* $\overline{T}_4 = (\ln 4)\overline{\mathrm{S}}_1/\big(h\mathrm{H}\big)$*. Let*

$$\begin{cases} \overline{\mathrm{E}}^*_1 = \overline{\mathrm{E}}^*_2\exp\big(-(\hat{l}+h)\overline{T}_3\big) > 0\\ \overline{\mathrm{E}}^*_2 = \big(q\beta^*_{\mathrm{M}}/\mathrm{H}\big)\overline{\mathrm{S}}_1\exp\big(-m^u n_1\omega\big)\int_0^{\omega} b(t)\exp\big(-m^u t\big)dt > 0\\ \overline{\mathrm{S}}_1 = (h\mathrm{H}/2)/\big(qb^u\Delta^u/(m^l\mathrm{H})+h\big), \end{cases}$$

then $\mathrm{H_E}(\overline{T}_3) \geq \overline{\mathrm{E}}^*_1 > 0$*. Both* $\overline{\mathrm{E}}^*_1$ *and* $\overline{T}_3$ *depend only on* β^*_{M} *and the model parameters.*

Proof. Visit Appendix A.8. □

We proceed to the first proposition in this section, which gives a lower bound $\big[\overline{\theta}_0 \;\; \overline{\theta}_1\big]$ for $\big[\mathrm{H}_{\mathrm{I},\infty} \;\; \mathrm{M}_{\mathrm{I},\infty}\big]$ together with upper bounds on the initial times for which $\big[\mathrm{H_I}(t) \;\; \mathrm{M_I}(\tau)\big]$ remains beneath $\big[\overline{\theta}_0 \;\; \overline{\theta}_1\big]$.

Proposition 5.10. *If* $\mathrm{R}_0^{\inf} = \overline{\mathrm{R}}_0^{\inf}(0,1) > 1$ *then there exists a vector* $\big[\overline{\theta}_0 \;\; \overline{\theta}_1\big]^\top \in \mathbb{R}^2_+ \setminus \{\mathbf{0}\}$ *such that for all* $\big[\theta_0 \;\; \theta_1\big]^\top \in \mathbb{R}^2_+ \setminus \{\mathbf{0}\}$ *if* $\mathrm{H_I}(0) \geq \theta_0$ *and* $\mathrm{M_I}(0) \geq \theta_1$ *then* $\mathrm{H_I}(t) \geq \overline{\theta}_0$ *for all* $t \geq T^\star_0(\theta_0)$ *and* $\mathrm{M_I}(t) \geq \overline{\theta}_1$ *for all* $t \geq T^\star_1(\theta_1)$*, where* $T^\star_i(\theta_i)$ *(i =0, 1) depends only on* θ_i *and the model parameters.*

Proof. It will be proven that $\mathrm{R}_0^{\inf} > 1$ forces the function $\left[\mathrm{H_I}(t)\ \ \mathrm{M_I}(\tau)\right]$ to rise to at least the level $\left[\overline{\theta}_0\ \ \overline{\theta}_1\right]$ at certain minimum times for t and τ if it remains continuously below $\left[\overline{\theta}_0\ \ \overline{\theta}_1\right]$. Through Lemma 4.2, given $\epsilon > 0$, there exists T_5 such that

$$\left(\mathrm{M_S} - \overline{\mathrm{M}}_\mathrm{S}\right)_\infty - \epsilon < \mathrm{M_S}(t) - \overline{\mathrm{M}}_\mathrm{S}(t)$$

for all $t \geq T_5$. Thus Lemma 5.6 implies that for $t > \overline{T}_5 = \max\{T_0, T_1, T_5\}$,

$$\begin{aligned}\overline{\mathrm{M}}_\mathrm{S}(t) - \mathrm{M_S}(t) &\leq \mathrm{M_E}(t) + \mathrm{M_I}(t) - \left(\mathrm{M_S} - \overline{\mathrm{M}}_\mathrm{S}\right)_\infty + \epsilon \\ &\leq \Theta + \beta_\mathrm{H}\left(1 + \frac{pb^u\Delta^u}{\mathrm{H}(c+m^l)m^l}\right) - \left(\mathrm{M_S} - \overline{\mathrm{M}}_\mathrm{S}\right)_\infty + \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary,

$$\mathrm{M_S}(t) \geq \overline{\mathrm{M}}_\mathrm{S}(t) - 2\Theta - \beta_\mathrm{H}\left(1 + \frac{pb^u\Delta^u}{\mathrm{H}(c+m^l)m^l}\right) = \overline{\mathrm{M}}_\mathrm{S}(t) - s_0 \tag{5.4a}$$

Also, utilizing lemmas 5.6 and 5.7, it is shown that

$$\begin{aligned}\overline{\mathrm{H}}_\mathrm{S}(t) &= \mathrm{H} - \overline{\mathrm{H}}_\mathrm{E}(t) - \overline{\mathrm{H}}_\mathrm{I}(t) - \overline{\mathrm{H}}_\mathrm{R}(t) \\ &\geq \mathrm{H} - 2\Theta - \beta_\mathrm{H}\left(1 + \frac{r}{h}\right) - \frac{\beta_\mathrm{M} qb^u\mathrm{H}}{h+\hat{l}} = \mathrm{H} - s_1\end{aligned} \tag{5.4b}$$

for $t > \overline{T}_5$. By definition 5.4:

$$\begin{aligned}\max\{s_0, s_1\} &< \overline{\mathrm{R}}\left(1 + \frac{pb^u\Delta^u}{\mathrm{H}(c+m^l)m^l} + \frac{qb^u\mathrm{H}}{h+\hat{l}} + \frac{r}{h}\right) \quad \because \Theta < \frac{\overline{\mathrm{R}}}{2} \wedge \max\{\beta_\mathrm{H}, \beta_\mathrm{M}\} < \overline{\mathrm{R}} \\ &< 2\overline{\mathrm{R}}\left(1 + \frac{pb^u\Delta^u}{\mathrm{H}(c+m^l)m^l} + \frac{qb^u\mathrm{H}}{h+\hat{l}} + \frac{r}{h}\right) \\ &< \frac{(\overline{\mathrm{R}}_0^{\inf}(\lambda_1, 1) - 1)(1-\overline{\theta})}{2(\mathrm{N}^u + 1)\mathrm{K}\overline{\mathrm{R}}_0^{\sup}(\lambda_1, 0)}\end{aligned} \tag{5.4c}$$

Choose $t_0 > T_6 = \max\{T_4, \overline{T}_4, \overline{T}_5\}$ and four consequences arise:

$$\begin{aligned}\frac{d}{dt}\mathrm{M_E}(t) \geq -(c+m^u)\mathrm{M_E}(t) &\implies \mathrm{M_E}(t) \geq \mathrm{M_E}(T_4)\exp\left(-(c+m^u)(t-T_4)\right) \\ &\implies \mathrm{M_E}(t_0) \geq \mathrm{E}_1^*\exp\left(-(c+m^u)(t_0-T_4)\right)\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}\frac{d}{dt}\mathrm{H_E}(t) \geq -(h+r)\mathrm{H_E}(t) &\implies \mathrm{H_E}(t) \geq \mathrm{H_E}(\overline{T}_4)\exp\left(-(\hat{l}+h)(t-\overline{T}_4)\right) \\ &\implies \mathrm{H_E}(t_0) \geq \overline{\mathrm{E}}_1^*\exp\left(-(\hat{l}+h)(t_0-\overline{T}_4)\right)\end{aligned} \tag{5.6}$$

utilizing lemmas 5.8 and 5.9 together with the sixth and second equations of the system (3.2);

$$\begin{aligned}\frac{d}{dt}\mathrm{H_I}(t) \geq -(h+r)\mathrm{H_I}(t) &\implies \mathrm{H_I}(t) \geq \mathrm{H_I}(0)\exp\bigl(-(h+r)t\bigr)\\ &\implies \mathrm{H_I}(t_0) \geq \beta_{\mathrm{H}}^{*}\exp\bigl(-(h+r)t_0\bigr)\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}\frac{d}{dt}\mathrm{M_I}(t) \geq -m^u\mathrm{M_I}(t) &\implies \mathrm{M_I}(t) \geq \mathrm{M_I}(0)\exp\bigl(-m^u t\bigr)\\ &\implies \mathrm{M_I}(t_0) \geq \beta_{\mathrm{M}}^{*}\exp\bigl(-m^u t_0\bigr)\end{aligned} \tag{5.8}$$

utilizing the third and the seventh equations of the system (3.2). Given a small enough $\overline{\theta}$, choose β_j $(j=0,1)$ and an arbitrary $k \in \mathbb{Z}_+$, together satisfying that

$$\frac{\displaystyle\inf_{t\in[k\omega\,,\,n\omega]}\int_0^{\omega} b(t-\eta)\exp\left(\int_t^{t-\eta}(c+\lambda_1+m(\tau))d\tau\right)\overline{\mathrm{M}}_{\mathrm{S}}(t-\eta)d\eta}{\displaystyle\frac{1}{\mathrm{K}_2}\left(\sum_{j=0}^{k-1}\exp\Bigl(-j\omega\langle c+\lambda_1+m\rangle\Bigr)\right)^{-1}} > (1-\overline{\theta})\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1) \tag{5.9}$$

and

$$0<\beta_j<\min\left\{\frac{\exp\bigl(-(\hat{l}+h)(s_2^*-\overline{T}_4)\bigr)}{\alpha^*\bigl(\mathrm{E}_1^*\bigr)^{-1}}\,,\,\frac{\exp\bigl(-(c+m^u)(s_2^*-T_4)\bigr)}{\alpha^*\bigl(\mathrm{E}_1^*\bigr)^{-1}}\,,\,\frac{\exp\bigl(-(h+r)s_2^*\bigr)}{\alpha^*\bigl(\beta_{\mathrm{H}}\bigr)^{j-1}(\theta_0)^{-j}}\,,\,\frac{\exp\bigl(-m^u s_2^*\bigr)}{\alpha^*\bigl(\beta_{\mathrm{M}}\bigr)^{j-1}(\theta_1)^{-j}}\,,\,\alpha^*\,,\,\frac{1}{\overline{s}_2}\right\}, \tag{5.10}$$

where

$$\begin{cases}\mathrm{K}_2 = \left(\dfrac{\mathrm{R}_{\sup}^*}{\mathrm{R}_{\inf}^*}\right)\left(\dfrac{p\,q\,\hat{l}\,\tilde{\delta}_3\tilde{\delta}_4 b_{\inf}\exp\left(\omega\, c\right)}{s_0 s_2\left(1-\exp\left(\omega\langle s_3\rangle\right)\right)\mathrm{H}}\right),\\ s_2^* = t_0+k\omega \ (t_0>\max\{T_4,\,\overline{T}_4\,,\,\overline{T}_5\}),\\ \overline{s}_2 = p^*q^*\exp\left((p^*+q^*)k\omega\right),\\ \alpha^* = \min\left\{\beta_{\mathrm{H}}\,,\,\beta_{\mathrm{M}}\right\},\\ \lambda_1^* = \lambda_1-(p^*+q^*),\\ p^*,\ q^* \text{ were previously defined in (5.3).}\end{cases} \tag{5.11}$$

It is asserted that provided that $\mathrm{H_I}(t)$ remains continuously below the level β_{H} and provided that $\mathrm{M_I}(t)$ remains continuously below the level β_{M}, then

$$\begin{aligned}&\overline{s}_0(\tau) = \min\left\{\mathrm{H_I}(t_0+\tau)\,,\,\mathrm{M_I}(t_0+\tau)\right\} \geq \beta_0\exp\bigl(\lambda_1(\tau-k\omega)/2\bigr)\\ \wedge\ \ &\overline{s}_1(\tau) = \min\left\{\mathrm{H_E}(t_0+\tau)\,,\,\mathrm{M_E}(t_0+\tau)\right\} \geq \beta_0^2\exp\bigl(\lambda_1^*(\tau-k\omega)/2\bigr)\end{aligned}$$

for define τ_0 such that

$$\tau_0 = \inf\Bigl\{\eta^*\geq 0 \,\Big|\, \overline{s}_0(\tau)\geq\beta_0\exp\bigl(\lambda_1(\tau-k\omega)/2\bigr)\ \wedge\ \overline{s}_1(\tau)\geq\beta_0^2\exp\bigl(\lambda_1^*(\tau-k\omega)/2\bigr) \text{ for } \tau\in[0\,,\,\eta^*]\Bigr\}.$$

By continuity, $\tau_0 > 0$, and if $\tau_0 < \infty$ then

$$\overline{s}_0(\tau_0) = \beta_0 \exp\big(\lambda_1(\tau_0 - k\omega)/2\big) \;\vee\; \overline{s}_1(\tau_0) = \beta_0^2 \exp\big(\lambda_1^*(\tau - k\omega)/2\big) \tag{5.12}$$

It will be shown that (5.12) leads to a contradiction by treating with the two assumptions $\big(\mathrm{H_I}(0) = \beta^*_{\mathrm{H}} = \beta_{\mathrm{H}},\ \mathrm{M_I}(0) = \beta^*_{\mathrm{M}} = \beta_{\mathrm{M}};\ \mathrm{H_I}(0) = \beta^*_{\mathrm{H}} \in (\theta_0\,,\,\beta_{\mathrm{H}}),$ $\mathrm{M_I}(0) = \beta^*_{\mathrm{M}} \in (\theta_1\,,\,\beta_{\mathrm{M}})\big)$ and their cases ($\tau_0 \le k\omega$, $\tau_0 > k\omega$) separately. Before, to facilitate the justification of the inequalities (5.13) to (5.20), a general argument is explained in the Appendix D.1, in addition, the extensive calculations to derive the inequalities (5.19) and (5.20) led to its realization in the appendices D.2 and D.3.

Suppose that $\mathrm{H_I}(0) = \beta^*_{\mathrm{H}} = \beta_{\mathrm{H}}$, $\mathrm{M_I}(0) = \beta^*_{\mathrm{M}} = \beta_{\mathrm{M}}$, $\mathrm{H_I}(t) \le \beta_{\mathrm{H}}$ and $\mathrm{M_I}(t) \le \beta_{\mathrm{M}}$ for $t \ge 0$.

- For $\tau_0 \le k\omega$:

$$\mathrm{H_I}(t_0+\tau_0) \ge \beta_{\mathrm{H}} \exp\big(-(h+r)(t_0+\tau_0)\big) > \beta_0 \exp\big(\lambda_1(\tau_0 - k\omega)/2\big) \tag{5.13}$$

$$\mathrm{M_I}(t_0+\tau_0) \ge \beta_{\mathrm{M}} \exp\big(-m^u(t_0+\tau_0)\big) > \beta_0 \exp\big(\lambda_1(\tau_0 - k\omega)/2\big) \tag{5.14}$$

$$\begin{aligned}
&\mathrm{M_E}(t_0+\tau_0) \ge \mathrm{E}_1^*\exp\big(-(c+m^u)(t_0 - T_4 + \tau_0)\big)\\
&\quad > \beta_0 \exp\left(\lambda_1(\tau_0 - k\omega)/2\right)\\
&\quad > \beta_0 \exp\left(\lambda_1(\tau_0 - k\omega)/2 - (p^*+q^*)(\tau_0 - k\omega)/2 - (p^*+q^*)k\omega/2\right)\\
&\quad > \beta_0\ \exp\left(\lambda_1^\star(\tau_0 - k\omega)/2\right)\left(p^*q^*\exp\left((p^*+q^*)(\tau_0 - k\omega)/2\right)\right)^{-1}\\
&\quad = \beta_0\ \exp\left(\lambda_1^\star(\tau_0 - k\omega)/2\right)/\overline{s}_2\\
&\quad > \beta_0^2\ \exp\left(\lambda_1^\star(\tau_0 - k\omega)/2\right) \quad \because\ 1/(p^*q^*) > 1 \wedge 1/\overline{s}_2 \ge \min\{-,-,-,-,-,\ 1/\overline{s}_2\} > \beta_0
\end{aligned} \tag{5.15}$$

$$\begin{aligned}
\mathrm{H_E}(t_0+\tau_0) \ge \overline{\mathrm{E}}_1^*\exp\big(-(\hat{l}+h)(t_0 - \overline{T}_4 + \tau_0)\big) &> \beta_0 \exp\left(\lambda_1(\tau_0 - k\omega)/2\right)\\
&> \beta_0^2\exp\left(\lambda_1^\star(\tau_0 - k\omega)/2\right)
\end{aligned} \tag{5.16}$$

- Let $n_2 = \lceil (t_0+\tau_0)/\omega \rceil$. For $\tau_0 > k\omega$:

$$\begin{aligned}
\mathrm{H_I}(t_0+\tau_0) &= \exp\left(-\tau_0(h+r)\right)\left(\mathrm{H_I}(t_0) + \hat{l}\int_{t_0}^{t_0+\tau_0} \exp\big((\zeta - t_0)(h+r)\big)\mathrm{H_E}(\zeta)d\zeta\right)\\
&> \mathrm{H_I}(t_0)\exp\big(-(t_0+\tau_0)(h+r)\big)\\
&> \beta_0 \exp\big(\lambda_1(\tau_0 - k\omega)\big)
\end{aligned} \tag{5.17}$$

$$\begin{aligned}
\mathrm{M_I}(t_0+\tau_0) &= \exp\left(-\textstyle\int_{t_0}^{t_0+\tau_0} m(\tau)d\tau\right)\left(\mathrm{M_I}(t_0) + c\textstyle\int_{t_0}^{t_0+\tau_0}\exp\left(\textstyle\int_{t_0}^{\zeta} m(\tau)d\tau\right)\mathrm{M_E}(\zeta)d\zeta\right)\\
&> \mathrm{M_I}(t_0)\exp\big(-(t_0+\tau_0)m^u\big)\\
&> \beta_0 \exp\big(\lambda_1(\tau_0 - k\omega)\big)
\end{aligned} \tag{5.18}$$

$$\mathrm{M_E}(t_0+\tau_0) > \beta_0^2 \exp\left(\lambda_1^\star(\tau_0 - k\omega)\right)(\lambda_1 - p^*)(\lambda_1 - q^*)(1 - \overline{\theta}) \tag{5.19}$$

$$\mathrm{H_E}(t_0+\tau_0) > \beta_0^2 \exp\left(\lambda_1^\star(\tau_0 - k\omega)\right)(\lambda_1 - p^*)(\lambda_1 - q^*)(1-\overline{\theta}) \tag{5.20}$$

Since $(\lambda_1 - p^*)(\lambda_1 - q^*) > 1$ and inequalities (5.17) to (5.20) hold, in accordance with Remark 5.3 and $\overline{\theta}$ is arbitrarily small, choose $\overline{\theta}$ so that

$$(\lambda_1 - p^*)(\lambda_1 - q^*)(1-\overline{\theta}) > 1$$

and

$$\min\left\{\mathrm{H_E}(t_0+\tau_0)\,,\,\mathrm{M_E}(t_0+\tau_0)\right\} > \beta_0^2 \exp\left(\lambda_1^\star\left(\tau_0 - k\omega\right)\right) \tag{5.21}$$

Hence, inequalities (5.21) contradict proposition (5.12), so we deduce that $\tau_0 = \infty$, and assuming that $\mathrm{H_I}(t)$ always lies below β_H and $\mathrm{M_I}(t)$ always lies below β_M:

$$\begin{gathered}\min\left\{\mathrm{H_I}(t_0+\tau)\,,\,\mathrm{M_I}(t_0+\tau)\right\} \geq \beta_0 \exp\left(\lambda_1(\tau - k\omega)\right)\\ \wedge\ \min\left\{\mathrm{H_E}(t_0+\tau)\,,\,\mathrm{M_E}(t_0+\tau)\right\} \geq \beta_0^2 \exp\left(\lambda_1^\star(\tau - k\omega)\right)\end{gathered}$$

for all $\tau \geq 0$. Particulary, since

$$\beta_0 \exp\left(\lambda_1(\tau - k\omega)\right) \geq \max\left\{\beta_\mathrm{H}\,,\beta_\mathrm{M}\right\} \equiv \tau \geq k\omega + \frac{1}{\lambda_1}\ln\left(\max\left\{\frac{\beta_\mathrm{H}}{\beta_0}\,,\frac{\beta_\mathrm{M}}{\beta_0}\right\}\right),$$

$\mathrm{H_I}(t)$ and $\mathrm{M_I}(t)$ must rise again above their initial conditions β_H and β_M by a time at most $\overline{T}_0 = T_6 + \overline{\tau}_0$ for $\mathrm{H_I}(t)$ and at most $\overline{T}_1 = T_6 + \overline{\tau}_1$ for $\mathrm{M_I}(t)$, where $\overline{\tau}_i = \ln\left(\beta_\mathrm{H}^{1-i}\beta_\mathrm{M}^{i} e^{k\omega\lambda_1}\beta_0^{-1}\right)^{1/\lambda_1}$ for $i \in \{0,1\}$ and each $\overline{T}_i$ depends only on β_H, β_M, Θ and the model parameters. Moreover,

$$\mathrm{H_I}(t) \geq \overline{\theta}_0 = \beta_\mathrm{H}\exp\left(-(h+r)\overline{T}_0\right) \quad \wedge \quad \mathrm{M_I}(t) \geq \overline{\theta}_1 = \beta_\mathrm{M}\exp\left(-m^u\overline{T}_1\right) \quad \forall\, t \geq 0 \tag{5.22}$$

Next suppose that $\beta_\mathrm{H} > \mathrm{H_I}(0) = \beta_\mathrm{H}^* > \theta_0 > 0$ and $\beta_\mathrm{M} > \mathrm{M_I}(0) = \beta_\mathrm{M}^* > \theta_1 > 0$. It is no longer necessarily true that $\mathrm{H_I}(t) \geq \overline{\theta}_0$ and $\mathrm{M_I}(t) \geq \overline{\theta}_1$ for all $t \geq 0$. Let $j = 1$ in (5.10) and set $\overline{\tau}_{i+2} = \ln\left(\beta_\mathrm{H}^{1-i}\beta_\mathrm{M}^{i} e^{k\omega\lambda_1}\beta_1^{-1}\right)^{1/\lambda_1}$ for $i \in \{0,1\}$.

A simple modification of the previous argument to arrive at (5.22) reveals that

$$\begin{gathered}\min\left\{\mathrm{H_I}(t_0+\tau)\,,\,\mathrm{M_I}(t_0+\tau)\right\} \geq \beta_1 \exp\left(\lambda_1(\tau - k\omega)\right)\\ \wedge\ \min\left\{\mathrm{H_E}(t_0+\tau)\,,\,\mathrm{M_E}(t_0+\tau)\right\} \geq \beta_1 \exp\left(\lambda_1^\star(\tau - k\omega)\right)\end{gathered}$$

for all $\tau \geq 0$. So, whenever $\min\{\theta_0\,,\,\theta_1\} > 0$, $\mathrm{H_I}(t)$ rises above β_H by a time at most $T_0^\star(\theta_0) = T_6 + \overline{\tau}_2$ and $\mathrm{M_I}(t)$ rises above β_H by a time at most $T_1^\star(\theta_1) = T_6 + \overline{\tau}_3$, whose $T_i^\star(\theta_i)$ depends only on θ_i and the model parameters, not on the initial conditions. Hence, if $\mathrm{H_I}(0) > \theta_0$ and $\mathrm{M_I}(0) > \theta_1$ then $\left[\mathrm{H_I}(t)\ \ \mathrm{M_I}(\tau)\right]$ is bounded below by a strictly positive bound $\left(\left[\overline{\theta}_0\ \ \overline{\theta}_1\right]\right)$ for any $\left(t\,,\,\tau\right) \in \left[T_0^\star(\theta_0)\,,\,\infty\right) \times \left[T_1^\star(\theta_1)\,,\,\infty\right)$, and for times exceeding $T_0^\star(\theta_0)$ and $T_1^\star(\theta_1)$, $\left[\mathrm{H_I}(t)\ \ \mathrm{M_I}(\tau)\right]$ never spend periods longer than $\overline{T}_0$ and $\overline{T}_1$ continuously below the level $\left[\beta_\mathrm{H}\ \ \beta_\mathrm{M}\right]$. □

This section ends with the proof of the existence of at least one positive periodic solution of the system (3.2), with the help of the fixed point theory.

Proposition 5.11. *The system* (3.2) *has a positive* ω*-periodic solution inside* Π *if* $\mathrm{R}_0^{\inf} = \overline{\mathrm{R}}_0^{\inf}(0,1) > 1$.

Proof. The set $\mathbb{R}^7$ with the norm $\|\mathbf{x}\| = \sqrt{\mathrm{H}_{\mathrm{S}}^2 + \mathrm{H}_{\mathrm{E}}^2 + \mathrm{H}_{\mathrm{I}}^2 + \mathrm{H}_{\mathrm{R}}^2 + \mathrm{M}_{\mathrm{S}}^2 + \mathrm{M}_{\mathrm{E}}^2 + \mathrm{M}_{\mathrm{I}}^2}$ is a Banach space [57]. Let $\Pi_1 \subset \Pi_2 \subset \Pi_3$ be subsets of the Banach space $\left(\mathbb{R}^7, \|\cdot\|\right)$. Define the sets:

$$\Pi_0 = \left\{\mathbf{x} \in \Pi\colon\ \mathrm{H}_{\mathrm{I}} \geq \overline{\theta}_0,\, \mathrm{M}_{\mathrm{I}} \geq \overline{\theta}_1\right\},\ \Pi_1 = \left\{\mathbf{x} \in \Pi\colon\ \mathrm{H}_{\mathrm{I}} > \frac{\overline{\theta}_0}{2},\, \mathrm{M}_{\mathrm{I}} > \frac{\overline{\theta}_1}{2}\right\},\ \Pi_2 = \Pi.$$

Notice that Π_0 and Π_2 are compact, $\Pi_2 \setminus \Pi_1$ is closed, and each Π_i is convex. Define the mapping:

$$\begin{aligned} \Phi_\star\colon\ \Pi_2 &\mapsto \Pi_2 \\ \mathbf{x}_0 &\mapsto \Phi_\star(\mathbf{x}_0) = \mathbf{x}(\omega,\, 0,\, \mathbf{x}_0) \end{aligned}$$

which represents the solution of the Cauchy problem (3.4) at time ω with

$$\mathbf{x}_0 = \begin{bmatrix}\mathrm{H}_{\mathrm{S}}(0) & \mathrm{H}_{\mathrm{E}}(0) & \mathrm{H}_{\mathrm{I}}(0) & \mathrm{H}_{\mathrm{R}}(0) & \mathrm{M}_{\mathrm{S}}(0) & \mathrm{M}_{\mathrm{E}}(0) & \mathrm{M}_{\mathrm{I}}(0)\end{bmatrix}^\top$$

at time $t = 0$. The mapping $\Phi_\star$ is continuous since the right-hand of the system (3.2) is differentiable, moreover Π is positively invariant (Proposition 3.1) and thus $\Phi_\star$ maps Π_2 into itself. Then for any $\overline{n} \in \mathbb{Z}_+$, $\Phi_\star^{\overline{n}}(\Pi_1) \subset \Pi_2$.

Now suppose that $\mathbf{x}_0 \in \Pi_1$. Then for $t \geq T\left(\overline{\theta}_0/2\right)$, $\mathrm{H}_{\mathrm{I}}(t) \geq \overline{\theta}_0$, and for $t \geq T\left(\overline{\theta}_1/2\right)$, $\mathrm{M}_{\mathrm{I}}(t) \geq \overline{\theta}_1$ (Proposition 5.10), hence if $n_3\omega > \max\left\{T\left(\overline{\theta}_0/2\right), T\left(\overline{\theta}_1/2\right)\right\}$ then $\Phi_\star^n(\Pi_1) \subset \Pi_0$ for all $n > n_3$. We can apply Horn's fixed point theorem [58] to conclude that $\Phi_\star$ has a fixed point in Π_0, that is, system (3.2) has a positive ω-periodic solution. □

5.2. Persistence results

Persistence is an important property of dynamic systems that model phenomena in ecology, epidemiology, among others. Persistence addresses the long-term survival of some or all components of a system (for example, some or all of the species that interact in an ecosystem), even when the population size of the specie(s) is quite low at times. In the epidemiology of infectious diseases persistence has two faces: persistence (or endemicity) of the disease and survival of the host population. For general information and classic references on the topic discussed here, we refer to [59, 60, 61].

Let $\Pi^{(0)}$ be a closed subset of Π with boundary of extinction $\partial\Pi^{(0)}$. The system (3.2) is said to be uniformly (strongly) persistent if for all $\boldsymbol{\chi}(t, \mathbf{x}_0) \in \operatorname{Int}\Pi^{(0)}$, there exists a constant $\mu_\star > 0$ such that

$$\liminf_{t\to\infty} d\left(\boldsymbol{\chi}(t,\mathbf{x}_0)\,,\, \partial\Pi^{(0)}\right) > \mu_\star \tag{5.23}$$

The function $d\left(\mathbf{x}(t)\,,\, S\right)$ denotes the distance from a point $\mathbf{x}(t)$ to a subset S. Conditions are presented for uniform strong persistence from now on. As a first step, we prove that $\Pi^{(1)}$ is a uniform strong repeller for $\Pi^{(2)} = \Pi^{(0)} \setminus \Pi^{(1)}$.

Definition 5.12 (Uniform strong repeller)**.** The set

$$\Pi^{(1)} = \left\{ \begin{bmatrix} \mathbf{y}_{\mathrm{H}} \\ \mathbf{y}_{\mathrm{M}} \end{bmatrix} : \mathbf{y}_{\mathrm{H}} \in \Pi_{\mathrm{H}} \, , \, \mathbf{y}_{\mathrm{M}} \in \Pi_{\mathrm{M}} \right\} \tag{5.24}$$

with

$$\Pi_{\mathrm{H}} = \left\{ \begin{bmatrix} \mathrm{H_S} & \mathrm{H_E} & \mathrm{H_I} \end{bmatrix}^\top : 0 \leq \mathrm{H_S} \leq \mathrm{H} \, , \, 0 \leq \mathrm{H_E} \leq \mathrm{H} \, , \, \mathrm{H_I} = 0 \, , \, 0 \leq \mathrm{H_S} + \mathrm{H_E} \leq \mathrm{H} \right\}$$

and

$$\Pi_{\mathrm{M}} = \left\{ \begin{bmatrix} \mathrm{M_S} & \mathrm{M_E} & \mathrm{M_I} \end{bmatrix}^\top : 0 \leq \mathrm{M_S} \leq \frac{\Delta^u}{m^l} \, , \, 0 \leq \mathrm{M_E} \leq \frac{\Delta^u}{m^l} \, , \, \mathrm{M_I} = 0 \, , \, 0 \leq \mathrm{M_S} + \mathrm{M_E} \leq \frac{\Delta^u}{m^l} \right\},$$

is said to be a uniform strong repeller for the set

$$\Pi^{(2)} = \left\{ \begin{bmatrix} \mathbf{y}_{\mathrm{H}} \\ \mathbf{y}_{\mathrm{M}} \end{bmatrix} : \mathbf{y}_{\mathrm{H}} \in \Pi_{\mathrm{H}}^* \, , \mathbf{y}_{\mathrm{M}} \in \Pi_{\mathrm{M}}^* \right\} \tag{5.25}$$

with

$$\Pi_{\mathrm{H}}^* = \left\{ \begin{bmatrix} \mathrm{H_S} & \mathrm{H_E} & \mathrm{H_I} \end{bmatrix}^\top : 0 \leq \mathrm{H_S} \leq \mathrm{H} \, , \, 0 \leq \mathrm{H_E} \leq \mathrm{H} \, , \, 0 < \mathrm{H_I} \leq \mathrm{H} \, , \, 0 \leq \mathrm{H_S} + \mathrm{H_E} + \mathrm{H_I} \leq \mathrm{H} \right\}$$

and

$$\Pi_{\mathrm{M}}^* = \left\{ \begin{bmatrix} \mathrm{M_S} & \mathrm{M_E} & \mathrm{M_I} \end{bmatrix}^\top : 0 \leq \mathrm{M_S} \leq \frac{\Delta^u}{m^l} \, , \, 0 \leq \mathrm{M_E} \leq \frac{\Delta^u}{m^l} \, , \, 0 < \mathrm{M_I} \leq \frac{\Delta^u}{m^l} \, , \, 0 \leq \mathrm{M_S} + \mathrm{M_E} + \mathrm{M_I} \leq \frac{\Delta^u}{m^l} \right\},$$

if $\min \left\{ \mathrm{H}_{\mathrm{I},\infty} \, , \, \mathrm{M}_{\mathrm{I},\infty} \right\} > 0$.

Corollary 5.13. *If* $\mathrm{R}_0^{\mathrm{inf}} = \overline{\mathrm{R}}_0^{\mathrm{inf}}(0,1) > 1$ *then* $\Pi^{(1)}$ *is a uniform strong repeller for* $\Pi^{(2)}$*.*

Proof. Occurring that $\mathrm{R}_0^{\mathrm{inf}} > 1$, Proposition 5.10 implies that $\mathrm{H_I}(t) \geq \overline{\theta}_0$ y $\mathrm{M_I}(t) \geq \overline{\theta}_1$. Since $\mathrm{H}_{\mathrm{I},\infty} \geq \overline{\theta}_0 > 0$ and $\mathrm{M}_{\mathrm{I},\infty} \geq \overline{\theta}_1 > 0$, the result is followed. □

Definition 5.14 (Uniform persistence)**.** Dengue modeling dynamics is said to be uniformly persistent if each state solution is strictly bounded away from zero and, moreover, each bound depends only on the model parameters after suficiently long time.

As a second step, we prove that the disease is uniformly persistent if $\mathrm{R}_0^{\mathrm{inf}} = \overline{\mathrm{R}}_0^{\mathrm{inf}}(0,1) > 1$. This requires demonstrating that $\mathrm{H_S}(t)$, $\mathrm{H_E}(t)$, $\mathrm{H_R}(t)$, $\mathrm{M_S}(t)$, and $\mathrm{M_E}(t)$ are similarly bounded away from zero for large times.

Lemma 5.15. *The recovered population, the non-infectious carrier population, the non-carrier population, the exposed population and the susceptible population are bounded away from zero, that is:*

(a) $\mathrm{H}_{\mathrm{R},\infty} \geq \overline{\theta}_4 > 0$

(b) $\mathrm{H}_{\mathrm{S},\infty} \geq \overline{\theta}_3 > 0$

(c) $\mathrm{H}_{\mathrm{E},\infty} \geq \overline{\theta}_2 > 0$

(d) $\mathrm{M}_{\mathrm{S},\infty} \geq \theta_3 > 0$

(e) $\mathrm{M}_{\mathrm{E},\infty} \geq \theta_2 > 0$

where $\overline{\theta}_4$, θ_2, θ_3, $\overline{\theta}_2$ and $\overline{\theta}_3$ depend only on the model parameters, not on the initial conditions.

Proof. Visit Appendix A.9. □

We are in position to introduce the result of the uniform persistence of the disease.

Proposition 5.16. *Let $\mathrm{R}_0^{\inf} = \overline{\mathrm{R}}_0^{\inf}(0,1) > 1$, then there exists a $\mu_\star > 0$ (independent of the initial conditions) such that any solution $\boldsymbol{\chi}(t, \mathbf{x}_0) = \mathbf{x}(t)$ of the system (3.2) with $\mathbf{x}_0 \in \Pi$ satisfies $\liminf_{t\to\infty} \mathbf{x}(t) \geq \mu_\star \mathbf{1}$.*

Proof. Lemma 5.15 and Theorem 5.10 point out that

$$\min\left\{\mathrm{H}_{\mathrm{S},\infty}\,,\, \mathrm{H}_{\mathrm{E},\infty}\,,\, \mathrm{H}_{\mathrm{I},\infty}\,,\, \mathrm{H}_{\mathrm{R},\infty}\,,\, \mathrm{M}_{\mathrm{S},\infty}\,,\, \mathrm{M}_{\mathrm{E},\infty}\,,\, \mathrm{M}_{\mathrm{I},\infty}\right\} \geq \mu_\star > 0$$

whenever $\mathrm{R}_0^{\inf} = \overline{\mathrm{R}}_0^{\inf}(0,1) > 1$, moreover, $\mu_\star$ depends only on the model parameters. This means that the disease is uniformly persistent if $\mathrm{R}_0^{\inf} > 1$. □

As a consequence of Remark 4.16 and propositions 4.22, 5.10 and 5.16 we achieve the following corollary. Corollary 5.17 expresses that (4.11) serves as a threshold value for the global extinction and uniform persistence of the disease.

Corollary 5.17. *The following statements are valid:*

1. *If $\mathrm{R}_0 < 1$ then the disease-free periodic solution, $\mathbf{x}^{\mathbf{0}}(t) \in \Pi$, is globally asymptotically stable for the system (3.2).*

2. *If $\mathrm{R}_0 > 1$ then $\mathbf{x}^{\mathbf{0}}(t)$ is unstable, the system (3.2) admits a positive ω-periodic solution, and there exists a real number $\mu_\star > 0$ such that the solution $\boldsymbol{\chi}(t, \mathbf{x}_0)$ satisfies*

$$\min\left\{\mathrm{H}_{\mathrm{S},\infty}\,,\, \mathrm{H}_{\mathrm{E},\infty}\,,\, \mathrm{H}_{\mathrm{I},\infty}\,,\, \mathrm{H}_{\mathrm{R},\infty}\,,\, \mathrm{M}_{\mathrm{S},\infty}\,,\, \mathrm{M}_{\mathrm{E},\infty}\,,\, \mathrm{M}_{\mathrm{I},\infty}\right\} \geq \mu_\star$$

for any $\mathbf{x}_0 \in \Pi$.

6. Numerical simulations

Some simulations are provided to illustrate the analytical results. The ordinary differential equations (3.2) are integrated with the built-in MATLAB routine ode45, entering the values of the parameters and the initial conditions reported by the Table 1 into the script.

Numerous theoretical studies have modeled seasonality utilizing periodic forcings to describe vital processes and the transmission of parasites or viruses [23, 68]. The population of adult mosquitoes, fluctuates on a temporary scale at the rate of an average large number of eggs hatched per unit of time, survivors of development through the intermediate aquatic stages (larvae and pupae), so we suppose a birth function in the form [69]:

$$\Delta(t) = \delta\left(1 - \varepsilon_2 \sin\left(\frac{2\pi t}{365} + \psi_2\right)\right).$$

It is assumed that the contact rate undergoes a simple harmonic oscillation [70]:

$$b(t) = \bar{b}\left(1 + \varepsilon_1 \cos\left(\frac{2\pi t}{365} + \psi_1\right)\right).$$

Here, Δ and b rates are periodic functions of time with a common period $\omega =$ 365 days, or 1 year. The phase shifts ψ_1, $\psi_2 \in [-2\pi\,,\, 2\pi]$ play no dynamical role, they are included to align Δ and b when comparing model time series with data [84]. The coefficients $\bar{b}$ and δ represent the base transmission rate and the average vector recruitment rate, respectively. The parameters ε_1, $\varepsilon_2 \in (0\,,\,1)$ measure the degree of seasonality of the rates [85]. The variation of the mosquito mortality rate is assumed constant, $m(t) \equiv \overline{m}$ over $\mathbb{R}_+$ (baseline mortality mosquito rate), in order to reduce the computational effort, also $\psi_1 < 0$ and $\psi_2 = 0$ are assumed so that at the beginning of the year the contact rate is always at a local minimum and recruitment rate is always at a local maximum.

A MATLAB code of the algorithm presented in Appendix F was implemented; the numerical evaluation with the data from Table 1, $d = 100$ and $n = 5000$ generated a $4n \times 4n$ matrix $\hat{A}$ in the form of (F.3) with which a reasonable approximation of R_0 by (F.5) was determined. In Figure 2, we plot R_0 when the parameter $\overline{m}$ is variable and the other parameters remain fixed. Consistent with the biological interpretation of R_0, R_0 is inversely proportional to $\overline{m}$, the graph is seen as the branch of an equilateral hyperbola in the first quadrant passing through the points $\mathsf{R}_0 \approx 0.9088$ if $\overline{m} = 1/10$, $\mathsf{R}_0 \approx 1$ if $\overline{m} = 1/10.75$ and $\mathsf{R}_0 \approx 1.5409$ if $\overline{m} = 1/15$. Thus, whenever the vector mortality rate is the most large that $1/10.75$ mosquitoes per day, dengue persists in the community.

Table 1: Parameters and initial data described in the model and their ranges of possible values.

Par.	Value(s)	Range	Source	Dim.	Par.	Value(s)	Range	Source	Dless.
$\overline{m}$	See text	1/20 – 1/4	[71, 72]	day^{-1}	h	$(75.58 \times 365)^{-1}$		[81]	day^{-1}
r	1/7	1/7 – 1/2	[73, 74]	day^{-1}	l	1/5.5	1/11 – 1	[82, 80]	day^{-1}
$\bar{b}$	1/3	0.3 – 1	[75, 76]	day^{-1}	δ	35000		Assumed	Dless.
p	0.51	0.5 – 1	[77, 78]	Dless.	$(\varepsilon_1 , \varepsilon_2)$	(0.6 , 0.2)	0 – 1	Assumed	Dless.
q	0.42	0.1 – 1	[77, 79]	Dless.	(ψ_1 , ψ_2)	$(-3 , 0)$		Assumed	Dless.
c	0.10	0.08 – 0.13	[73, 80]	day^{-1}	H	304218		[83]	Dless.

Initial conditions	H_{S0}	H_{E0}	H_{I0}	H_{R0}	M_{S0}	M_{E0}	M_{I0}
IC1	92261	3	0	211954	419967	0	33
IC2	92258	4	2	211954	419968	32	0
IC3	92263	5	3	211947	419969	0	31
IC4	92266	6	4	211942	419970	30	0

Abbreviation: Par./Parameter, Dless./Dimensionless, Dim./Dimension.

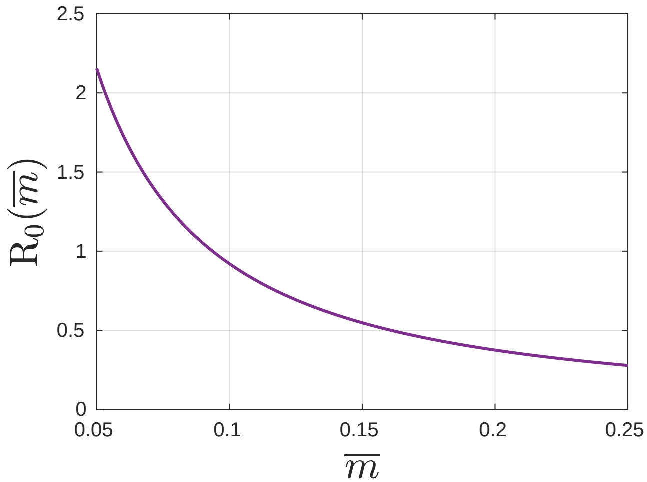
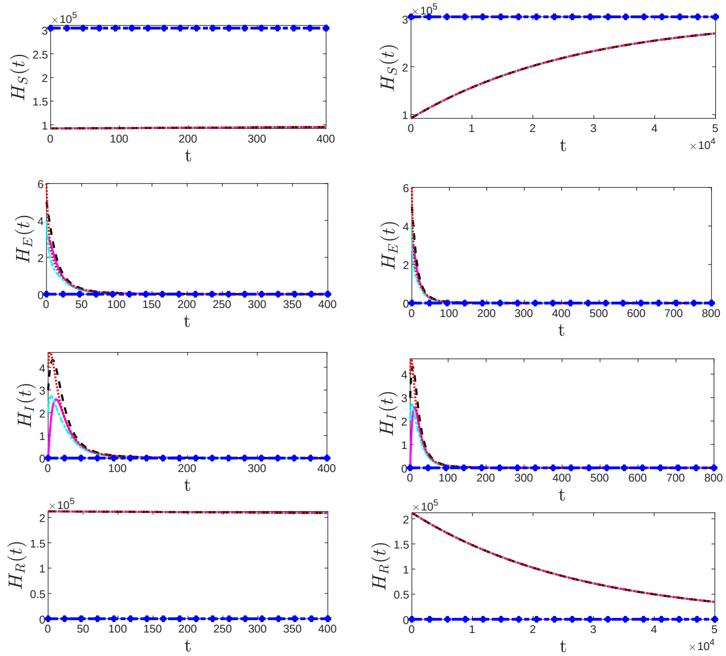


Figure 2: Graph of the BRN when $\overline{m}$ varies.

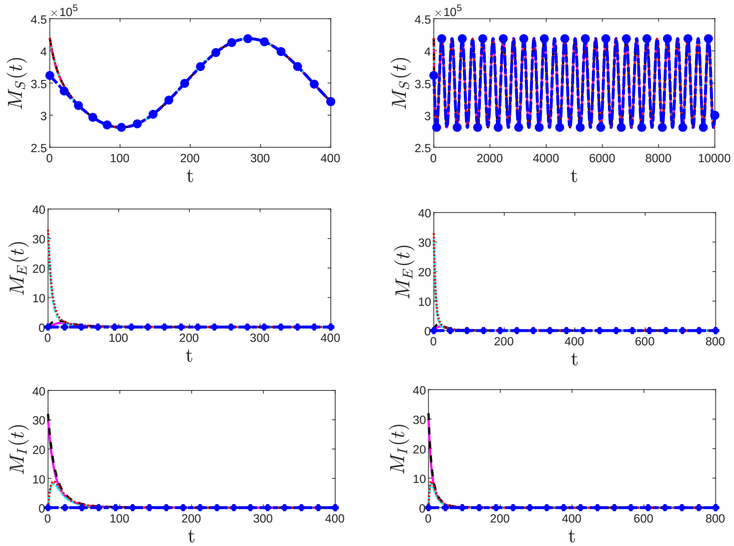
The numerical results in figures 3 and 4 show us four solutions of the system (3.2) when $R_0 < 1$ and $R_0 > 1$, respectively. The planes on the left-hand side in each figure illustrate the evolution of disease states in humans and mosquitoes during a calendar year, while in the planes on the right-hand side we have extended the time to more than a year in the simulations to numerically demonstrate propositions 4.5 and 4.22. When $R_0 < 1$, the effect of seasonal variation can be described as follows: in the human population, dengue outbreaks occur in less than three weeks shortly after the disease is introduced or reactivated in the community, beyond this period the number of infected people vary in a similar way to an exponential decrease and the disease disappears, qualitatively the same occurs in the mosquito population; meanwhile, the rate of variation in both the susceptible population (which grows) and the recovered population (which decreases) is slow, being necessary to extend the time scale to appreciate an asymptotic behavior of the trajectories of these population; relative to mosquito population densities that make transmission possible, they are higher in the first season of the year that covers the first three months with the highest incidence of dengue in humans, but after this season their sizes become too small to cause an epidemic. See Figure 3a-b.

When $R_0 > 1$, the effect of seasonal variation can be described as follows: the highest annual daily number of latent cases occurs around 230 days, and a week later the peak of the symptomatic subpopulation occurs; the average number of susceptible people grows monotonously from the beginning of the outbreak, until several years later this growth stops and dampened oscillations between 100000 and 150000 appear. On the contrary, the recovered population seems to decrease progressively from the beginning of the outbreak throughout the calendar year, although simulating several years, the decrease comes to a halt and oscillations between 140,000 and 200,000 appear. The simulations also show that in the calendar year, dengue outbreaks are in phase with the abundance of carrier mosquitoes, exhibit a lag with the bottleneck of non-carrier mosquitoes, and transmission is unfavorable when the non-carrier vector population begins to decline. See Figure 4a-b.

The global dynamics therefore consists of an initial phase in which the dynamics of the outbreaks produce a large initial epidemic, which exhausts most of the susceptible population and the prevalence reaches lower levels, as shown by the overlapping infectivity curves of different amplitudes in Figure 4: the underdamping of the size of the susceptible population predicts an endemo-epidemic pattern with outbreaks every 3 to 5 years, that is, a new but less strong epidemic, and so on until it converges to the state endemic.

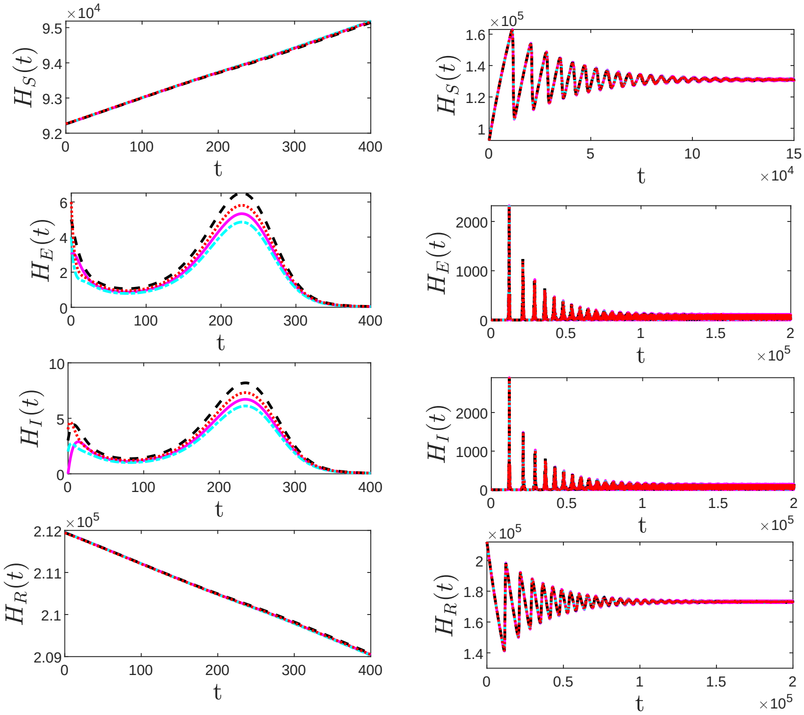


(a) Distribution of the human population.

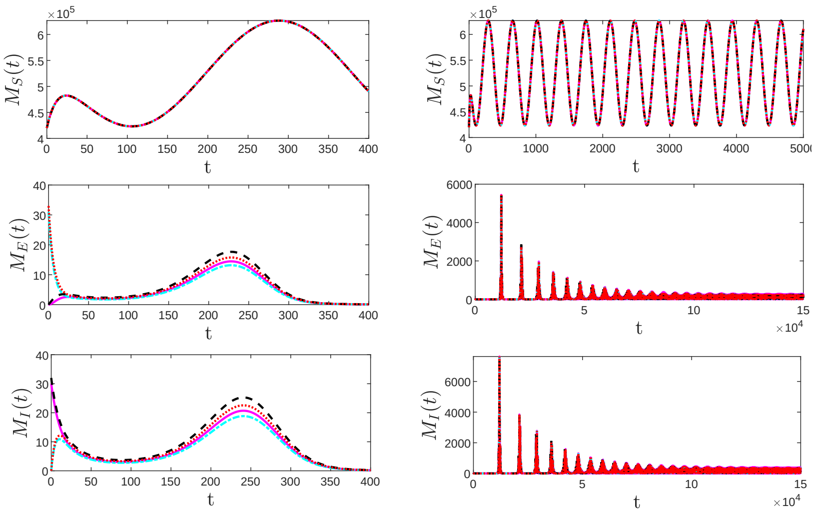


(b) Distribution of the mosquito population.

Figure 3: Trajectories with initial conditions IC1, IC2, IC3 and IC4 (see Table 1) when $\mathsf{R}_0 < 1$. The other solutions converge to the dengue-free solution.



(a) Distribution of the human population.



(b) Distribution of the mosquito population.

Figure 4: Trajectories with initial conditions IC1, IC2, IC3 and IC4 (see Table 1) when $\mathsf{R}_0 > 1$. The long-term simulation illustrate that the disease is endemic.

7. Conclusion

It was formulated and analyzed a non-autonomous deterministic mathematical model describing dengue transmission in a periodic environment, considering transmission rates and vector demographics to be continuous, positive, differentiable and periodic general functions of time. A key parameter for dynamics is the basic reproductive number (BRN), whose analytical formula, unlike the non-seasonal model, is rarely available, therefore adequate approximations were derived in the seasonal case, which allowed the results of this research to be achieved (Corollary (5.17)).

Standard methodology based in a general method developed by Wang and Zhao [17] employing the BRN mathematical definition in a periodic environment proposed by Bacaër and Guernaoui [16], shown that the disease-free solution is globally asymptotically stable if the BRN is less than unity and in this case the disease will ultimately die out. The epidemiological implication of this result is that the disease can be effectively controlled if the control strategies implemented in the community can bring (and maintain) the BRN to a value less than unity. In other words, this result shows that bringing (and maintaining) the BRN to a value less than unity is necessary for the effective control of the disease in the community.

It was also set a threshold condition for uniform persistence: when $\mathsf{R}_0 > 1$ there are a certain minimum long-term value for the number of symptomatic people ($\overline{\theta}_0$) and a certain minimum long-term value for the number of infectious carrier mosquitoes ($\overline{\theta}_1$), both depending only on the parameters of the model, not on the initial conditions, such that if the initial number of people with infection or carrier vectors is(are) strictly positive then eventually the number of symptomatic people rise to the level of at least a certain minimum value β_{H} and the number of infectious carriers mosquitoes must rise to the level of at least a certain minimum value β_{M}, and the maximum times taken before they do it again depends only on $\mathrm{H_I}(0)$, $\mathrm{M_I}(0)$ and the parameters of the model; once $\mathrm{H_I}$ and $\mathrm{M_I}$ have risen above β_{H} and β_{M} they will never subsequently fall below $\overline{\theta}_0$ and $\overline{\theta}_1$. This result enables to show the existence of a positive periodic solution for the system (3.2).

Consequently, when $\mathsf{R}_0 > 1$, dengue is uniformly persistent if initially present. A biological explanation for the result is although the horizontal transmission of dengue between humans and mosquitoes is a determining factor in the epidemiology of this disease, it has also been shown that *Aedes aegypti* is capable of transmitting the dengue virus to the progeny after it has been been invaded by the virus [86, 87], suggesting an important mechanism of sustained virus circulation in vector populations during adverse periods for horizontal transmission. Simulation analyzes about epidemic and endemic dynamics confirm that the disease is completely dies out if $\mathsf{R}_0 < 1$ and persists if $\mathsf{R}_0 > 1$.

The formulated model plays a fundamental role, since it reproduces well its main qualitative and quantitative characteristics such as endemic, oscillation and the observed inter-epidemic period. However, the model captures a series of additional characteristics, ranging from sustained five-year oscillations prior to control, motivated by the fact that the life cycle of the mosquito, depending on the annual climatic seasons, has a periodicity of one year.

Acknowledgements. I take this opportunity to pay homage to my director and friend Anibal Muñoz Loaiza, a prominent researcher, postdoctoralist in Mathematics and founder of the Grupo de Modelación Matemática en Epidemiología (GMME). Professor Muñoz Loaiza's contribution to the world of science transcends local borders and placed him as a benchmark for the country and the world, therefore, it has not been easy for me to assimilate his unexpected departure. His extensive intellectual and scientific production continues to be an icon and a benchmark for the training of new generations where rigor and academic discipline will be his best legacy. *RIP.*

I also wish to thank the anonymous reviewers for their careful reading of this manuscript and for their comments and suggestions that improved the presentation of the results.

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Appendices

A. Proofs of lemmas

A.1. Proof of Lemma 4.1

(i) The equality is valid under the property of additivity in the interval $t \leq k\omega \leq t + n\omega$ and the concept of periodic function:

$$\begin{aligned}
\int_t^{t+n\omega} w(\tau)d\tau &= \int_t^{k\omega} w(\tau)d\tau + \int_{k\omega}^{t+\omega} w(\tau)d\tau \\
&= \int_{t+n\omega}^{(k+n)\omega} w(\tau - n\omega)d\tau + \int_{k\omega}^{t+n\omega} w(\tau)d\tau \\
&= \int_{k\omega-k\omega}^{(k+n)\omega-k\omega} w(\tau + k\omega)d\tau \\
&= \int_0^{n\omega} w(\tau)d\tau \\
&= \sum_{j=1}^{n} \int_{(j-1)\omega}^{j\omega} w(\tau)d\tau \\
&= \sum_{j=1}^{n} \int_0^{\omega} w(\tau)d\tau = n \int_0^{\omega} w(\tau)d\tau.
\end{aligned}$$

(ii) Let $\tau \in [k\omega\,,\,(k+1)\omega)$, then $\tau = k\omega + \rho$, where $\rho \in [0\,,\,\omega)$ and $k \in \mathbb{Z}$. Further,

$$\begin{aligned}
\int_t^{t+\tau} w(\tau)d\tau &= \int_t^{t+k\omega} w(\tau)d\tau + \int_{t+k\omega}^{t+k\omega+\rho} w(\tau)d\tau \\
&= k\int_0^{\omega} w(\tau)d\tau + \int_0^{\rho} w(\tau + t)d\tau.
\end{aligned}$$

Dividing by $\tau = k\omega + \rho$ and taking the limit as $k \to \infty$:

$$\langle w \rangle = \lim_{k\to\infty} \left(\frac{k}{k\omega + \rho} \int_0^{\omega} w(\tau)d\tau + \frac{1}{k\omega + \rho} \int_0^{\rho} w(\tau + t)d\tau \right) = \frac{1}{\omega} \int_0^{\omega} w(\tau)d\tau.$$

A.2. Proof of Lemma 4.2

The differential equation of non-carrier mosquitoes implies that

$$\begin{aligned}\frac{d}{dt}\mathrm{M_S}(t) &= \Delta(t) - \frac{pb(t)}{\mathrm{H}}\mathrm{H_I}(t)\mathrm{M_S}(t) - m(t)\mathrm{M_S}(t)\\ &\leq \Delta(t) - m(t)\mathrm{M_S}(t).\end{aligned}$$

Since $\mathbf{x^0} = \begin{bmatrix}\mathrm{H} & 0 & 0 & 0 & \overline{\mathrm{M}}_\mathrm{S}(t) & 0 & 0\end{bmatrix}$ is a solution of the system (3.2), then

$$\frac{d}{dt}\overline{\mathrm{M}}_\mathrm{S}(t) = \Delta(t) - m(t)\overline{\mathrm{M}}_\mathrm{S}.$$

Therefore,

$$\frac{d}{dt}\big(\mathrm{M_S}(t) - \overline{\mathrm{M}}_\mathrm{S}(t)\big) \leq -m(t)\big(\mathrm{M_S}(t) - \overline{\mathrm{M}}_\mathrm{S}(t)\big).$$

Integrating this inequality over $[t_0\,,\, t]$, one find

$$\mathrm{M_S}(t_n) - \overline{\mathrm{M}}_\mathrm{S}(t_n) \leq \big(\mathrm{M_S}(0) - \overline{\mathrm{M}}_\mathrm{S}(0)\big)\exp\Big(-\int_0^{t_n} m(\tau)d\tau\Big).$$

According to fluctuation Lemma [41], there is a sequence $\{t_n\}$ such that $t_n \to \infty$ and $\big(\mathrm{M_S}(t_n) - \overline{\mathrm{M}}_\mathrm{S}(t_n)\big) \to \big(\mathrm{M_S}(t_n) - \overline{\mathrm{M}}_\mathrm{S}(t_n)\big)^\infty$ as $n \to \infty$. Letting $n \to \infty$ and knowing that $\big(\mathrm{M_S} - \overline{\mathrm{M}}_\mathrm{S}\big)(t)$ is bounded for all $t \geq 0$ (Lemma 4.3), it follows that

$$\big(\mathrm{M_S} - \overline{\mathrm{M}}_\mathrm{S}\big)^\infty \leq 0.$$

Arguing as previosly, it follows that

$$\frac{d}{dt}\big(\mathrm{H_S}(t) - \mathrm{H}(0)\big) \leq -h\big(\mathrm{H_S}(t) - \mathrm{H}(0)\big) \implies (\mathrm{H_S} - \mathrm{H}(0))^\infty \leq 0.$$

Lemma 4.2 now follows straightforwardly.

A.3. Proof of Lemma 4.3

The change of variable $\mathrm{M_S}(t) = 1/\mathrm{N_S}(t)$ transforms equation (4.1) into

$$\frac{d}{dt}\mathrm{N_S}(t) = \Big(m(t) - \Delta(t)\mathrm{N_S}(t)\Big)\mathrm{N_S}(t), \quad \mathrm{N_S}(t_0) = \frac{1}{\mathrm{M_S}(t_0)} \tag{A.1}$$

Let $\Delta^l \leq \Delta(t) \leq \Delta^u$ and $m^l \leq m(t) \leq m^u$, where Δ^l, m^l, Δ^u, $m^u > 0$. Notice that $\frac{d}{dt}\mathrm{N_S}(t) \leq \big(m^u - \Delta^l\mathrm{N_S}(t)\big)\mathrm{N_S}(t)$, then $\frac{d}{dt}\mathrm{N_S}(t) < 0$ if $\mathrm{N_S}(t) > m^u/\Delta^l$; hence $\mathrm{N_S}(t) \leq \max\{\mathrm{N_S}(0)\,,\, m^u/\Delta^l\} = 1/\mathrm{N}^l > 0$ for all $t \in \mathbb{R}_+$.

To prove that $\mathrm{N_S}(t)$ is bounded below by a positive constant, choose $\sigma > 0$ such that $\mathrm{N_S}(0) \geq \sigma$ and

$$\langle m \rangle - \sigma \Delta^u - \sigma = \rho_0 > 0. \tag{A.2}$$

By assuming that $\mathrm{N_S}(t)$ is not bounded below then, for each $0 < \theta < \sigma$, there exists an interval $[\tau_1, \tau_2]$ such that $\mathrm{N_S}(\tau_1) = \sigma$, $\mathrm{N_S}(\tau_2) = \theta$, and $\mathrm{N_S}(t) < \sigma$ for $t \in (\tau_1, \tau_2)$. Now notice that $\frac{d}{dt}\mathrm{N_S}(t) \geq \big(m^l - \sigma\Delta^u\big)\mathrm{N_S}(t)$ for $t \in [\tau_1, \tau_2]$. If $m^l - \sigma\Delta^u \geq 0$ it follows that $\theta = \mathrm{N_S}(\tau_2) \geq \mathrm{N_S}(t) \geq \mathrm{N_S}(\tau_1) = \sigma$ for $t \in [\tau_1, \tau_2]$, therefore, the concordant inequality must be $m^l - \sigma\Delta^u = -\rho_1 < 0$. In this case we have $\mathrm{N_S}(t) \geq \sigma e^{-\rho_1(t-\tau_1)}$ for $t \in [\tau_1, \tau_2]$. At $t = \tau_2$, $\theta \geq \sigma e^{-\rho_1(\tau_2-\tau_1)}$ or equivalently

$$\ln{(\sigma/\theta)}^{1/\rho_1} \leq \tau_2 - \tau_1. \tag{A.3}$$

Since θ can be chosen arbitrarily close to zero, Lemma 4.1 and inequality (A.3) imply that $T = \tau_2 - \tau_1$ can be sufficiently large so that

$$\frac{1}{T}\int_t^{t+T} m(\tau) d\tau > \langle m \rangle - \sigma. \tag{A.4}$$

For θ chosen sufficiently small such that inequality (A.4) holds on the interval $\tau_1 \leq t \leq \tau_2$, $\frac{d}{dt}\mathrm{N_S}(t) \geq \big(m(t) - \sigma\Delta^u\big)\mathrm{N_S}(t)$ for $t \in [\tau_1, \tau_2]$. Thus,

$$\mathrm{N_S}(\tau_2) \geq \mathrm{N_S}(\tau_1)\exp\left(\int_{\tau_1}^{\tau_2} \left(m(t) - \sigma\Delta^u\right) dt\right) > \mathrm{N_S}(\tau_1)e^{\rho_0 T}.$$

But because of the choice of σ in equation (A.2), the preceding inequality leads again to the contradiction $\theta > \sigma$. It only remains to conclude that $\mathrm{N_S}(t) \geq \min\{\mathrm{N_S}(0)\,,\, m^l/\Delta^u\} = 1/\mathrm{N}^u > 0$ for all $t \in \mathbb{R}_+$. Finally, boundedness of $\mathrm{N_S}(t)$ implies that $\mathrm{M_S}(t)$ is bounded by positive constants: $\mathrm{N}^l \leq \mathrm{M_S}(t) \leq \mathrm{N}^u$.

A.4. Proof of Lemma 4.13

Let $\lambda > \hat{\omega}$ be given. E_λ complies with the facts that is

Positive Automatically by definition.

Linear For any scalars α_1, α_2:

$$
\begin{aligned}
&(E_\lambda\,(\alpha_1\boldsymbol{\varphi}_1+\alpha_2\boldsymbol{\varphi}_2))\,(\vartheta)\\
&=\exp\left(\tilde d_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)\left(\tilde S\,(\alpha_1\boldsymbol{\varphi}_1+\alpha_2\boldsymbol{\varphi}_2)\,(\vartheta)+\tilde d_2(\lambda)\left(\tilde\Lambda_t\,(\alpha_1\boldsymbol{\varphi}_1+\alpha_2\boldsymbol{\varphi}_2)\,(t)-\tilde S\,(\alpha_1\boldsymbol{\varphi}_1+\alpha_2\boldsymbol{\varphi}_2)\,(\vartheta)\right)\right)\\
&=\exp\left(\tilde d_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)\left(\alpha_1\tilde S\boldsymbol{\varphi}_1(\vartheta)+\alpha_2\tilde S\boldsymbol{\varphi}_2(\vartheta)+\tilde d_2(\lambda)\left(\tilde\Lambda_t\,(\alpha_1\boldsymbol{\varphi}_1(t)+\alpha_2\boldsymbol{\varphi}_2(t))-\alpha_1\tilde S\boldsymbol{\varphi}_1(\vartheta)-\alpha_2\tilde S\boldsymbol{\varphi}_2(\vartheta)\right)\right)\\
&=\exp\left(\tilde d_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)\left(\alpha_1\tilde S\boldsymbol{\varphi}_1\tilde S(\vartheta)+\alpha_2\tilde S\boldsymbol{\varphi}_2(\vartheta)+\tilde d_2(\lambda)\left(\tilde\Lambda_t\,(\alpha_1\boldsymbol{\varphi}_1(t))+\tilde\Lambda_t\,(\alpha_2\boldsymbol{\varphi}_2(t))\right.\right.\\
&\qquad\left.\left.-\alpha_1\boldsymbol{\varphi}_1\tilde S(\vartheta)-\alpha_2\tilde S\boldsymbol{\varphi}_2(\vartheta)\right)\right)\\
&=\exp\left(\tilde d_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)\left(\alpha_1\tilde S\boldsymbol{\varphi}_1(\vartheta)+\tilde d_2(\lambda)\left(\tilde\Lambda_t\,(\alpha_1\boldsymbol{\varphi}_1(t))-\alpha_1\tilde S\boldsymbol{\varphi}_1(\vartheta)\right)+\alpha_2\tilde S\boldsymbol{\varphi}_2(\vartheta)\right.\\
&\qquad\left.+\tilde d_2(\lambda)\left(\tilde\Lambda_t\,(\alpha_2\boldsymbol{\varphi}_2(t))-\alpha_2\tilde S\boldsymbol{\varphi}_2(\vartheta)\right)\right)\\
&=\exp\left(\tilde d_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)\left(\alpha_1\tilde S\boldsymbol{\varphi}_1(\vartheta)+\alpha_1\tilde d_2(\lambda)\left(\tilde\Lambda_t\,(\boldsymbol{\varphi}_1(t))-\tilde S\boldsymbol{\varphi}_1(\vartheta)\right)\right.\\
&\qquad\left.+\alpha_2\tilde S\boldsymbol{\varphi}_2(\vartheta)+\alpha_2\tilde d_2(\lambda)\left(\tilde\Lambda_t\,(\boldsymbol{\varphi}_2(t))-\tilde S\boldsymbol{\varphi}_2(\vartheta)\right)\right)\\
&=\alpha_1\,(E_\lambda\boldsymbol{\varphi}_1)\,(\vartheta)+\alpha_2\,(E_\lambda\boldsymbol{\varphi}_2)\,(\vartheta),\quad\forall\,\vartheta\in[0\,,\,t],\quad\{\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2\}\subset\mathcal P_\omega.
\end{aligned}
$$

Bounded For certain positive constant $\overline{\Theta}_1$:

$$
\begin{aligned}
\|E_\lambda\| &= \sup_{\|\boldsymbol{\varphi}\|=1}\|(E_\lambda\boldsymbol{\varphi})\|\\
&= \sup_{\|\boldsymbol{\varphi}\|=1}\sup_{0\le\vartheta\le t}\|(E_\lambda\boldsymbol{\varphi})\,(\vartheta)\|\\
&= \sup_{\|\boldsymbol{\varphi}\|=1}\sup_{0\le\vartheta\le t}\left\|\exp\left(\tilde d_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)\left(\tilde S\boldsymbol{\varphi}(\vartheta)+\tilde d_2(\lambda)\left(\tilde\Lambda_t\,\boldsymbol{\varphi}(t)-\tilde S\boldsymbol{\varphi}(\vartheta)\right)\right)\right\|\\
&\le \frac{\displaystyle\sup_{0\le\vartheta\le t}\left(\exp\left(\tilde d_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)\right)}{\left(\displaystyle\sup_{\|\boldsymbol{\varphi}\|=1}\sup_{0\le\vartheta\le t}\left(\|\tilde S\|\|\boldsymbol{\varphi}(\vartheta)\|+\tilde d_2(\lambda)\left(\|\tilde\Lambda_t\|\|\boldsymbol{\varphi}(t)\|+\|\tilde S\|\|\boldsymbol{\varphi}(\vartheta)\|\right)\right)\right)^{-1}}\\
&\le \sup_{0\le\vartheta\le t}\left(\exp\left(\tilde d_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)\right)\sup_{\|\boldsymbol{\varphi}\|=1}\left(\|\boldsymbol{\varphi}\|+\tilde d_2(\lambda)\left(\|\tilde\Lambda_t\|\overline{\Theta}_1+\|\boldsymbol{\varphi}\|\right)\right)\\
&\qquad\qquad\because\|\tilde S\|=\max_{1\le i\le 4}|\mathrm{sgn}\,\varphi_i(t)|\le 1.\\
\|E_\lambda\| &\le \exp(0)\left(1+\tilde d_2(\hat\omega)\left(\overline{\Theta}_1\|\tilde\Lambda_t\|+1\right)\right)\qquad\because t\ge\theta,\ \tilde d_2(\hat\omega)\ge\tilde d_2(\lambda)\ge 0,\ \tilde d_1(\lambda)\le 0.\\
&= 1+\tilde d_2(\hat\omega)\left(\overline{\Theta}_1\|\tilde\Lambda_t\|+1\right),\quad\forall\,\lambda\in(\hat\omega,\,\infty).
\end{aligned}
$$

These facts together with Remarks 4.9 and 4.12 imply that $\mathcal L_\lambda$ is positive in the sense that $\mathcal L_\lambda(\mathcal P_\omega^+)\subset\mathbb R_+^n$. Checking that

$$
\begin{aligned}
\|Y(t,t-s)\overline F(t-s)E_\lambda\| &\le \Theta_0\|E_\lambda\|\left(\sup_{t\in[0\,,\,\omega]}\|\overline F(t)\|\right)\exp(\hat\omega s)\\
&\le \Theta_0\|\overline F\|\left(1+\tilde d_2(\hat\omega)\left(\overline{\Theta}_1\sup_{t\in[0\,,\,\omega]}\|\tilde\Lambda(t)\|+1\right)\right)\exp(\hat\omega s)\\
&\le \Theta_0\|\overline F\|\left(1+\tilde d_2(\hat\omega)\left(\overline{\Theta}_1\|\tilde\Lambda\|+1\right)\right)\exp(\lambda s),\quad\forall\lambda\ge\hat\omega,\ \forall t\ge s,\ s\in\mathbb R_+,
\end{aligned}
$$

for some $\Theta_0>0$, it turns out that $\mathcal L_\lambda$ is bounded and therefore continuous on $\mathcal P_\omega$ [45, Theorem 2.7-9].

In view of

$$(\mathcal{L}_\lambda \boldsymbol{\phi})(t) = \frac{2+|\hat{\omega}|}{1+\lambda+|\hat{\omega}|}\left(\int_{-\infty}^{t} \exp\left(\tilde{d_1}(\lambda)(t-s)\right) Y(t,s)\overline{F}(s)(E_\lambda \boldsymbol{\phi})(s)\, ds\right), \quad \forall\, t \in \mathbb{R},\ \boldsymbol{\phi} \in \mathcal{P}_\omega,$$

one obtains

$$\frac{d}{dt}(\mathcal{L}_\lambda \boldsymbol{\phi})(t) \quad = \frac{2+|\hat{\omega}|}{1+\lambda+|\hat{\omega}|}\left(\overline{F}(t)(E_\lambda \boldsymbol{\phi})(t) - \left(|\tilde{d_1}(\lambda)|I_4 + \overline{V}(t)\right)(\mathcal{L}_\lambda \boldsymbol{\phi})(t)\right) \tag{A.5}$$

It then follows that for any $a > 0$, there exists $K_4 = K_4(a) > 0$ such that $\left|\frac{d}{dt}(\mathcal{L}_\lambda \boldsymbol{\phi})(t)\right| \le K_4$ for all $t \in [0\,,\omega]$ and $\boldsymbol{\phi} \in \mathcal{P}_\omega$ with $\|\boldsymbol{\phi}\| \le a$. Thus, the Ascoli–Arzelà theorem [46, Theorem 1.22] implies that $\mathcal{L}_\lambda$ is compact on $\mathcal{P}_\omega$.

A.5. Proof of Lemma 4.14

The definition of hyperbolic tangent (increasing monotonic function on $\mathbb{R}$ with codomain $(-1,\,1)$), the formulas

$$\min\{a_2,\,a_1\} = \frac{a_2+a_1-|a_2-a_1|}{2} \text{ and } \max\{a_2,\,a_1\} = \frac{a_2+a_1+|a_2-a_1|}{2},$$

and the property $\big||a_2|-|a_1|\big| \le |a_2-a_1|$ will be helpful here.

(i) It is straightforward to deduce this statement no matter what real values are assigned to λ given $\lambda_0 \in \mathbb{R}$ with the help of the triangular inequality:

$$\begin{aligned}\left|\frac{(1-\lambda)\pm|\lambda-1|}{2} - \frac{(1-\lambda_0)\pm|\lambda_0-1|}{2}\right| &= \left|\frac{-(\lambda-\lambda_0)\pm(|\lambda-1|-|\lambda_0-1|)}{2}\right| \\ &\le \left(\big||\lambda-1|-|\lambda_0-1|\big| + |\lambda-\lambda_0|\right)/2 \\ &\le \left(|(\lambda-1)-(\lambda_0-1)| + |\lambda-\lambda_0|\right)/2 \\ &= \left(|\lambda-\lambda_0| + |\lambda-\lambda_0|\right)/2 \\ &= |\lambda-\lambda_0|.\end{aligned} \tag{A.6}$$

On the other hand,

$$
\begin{aligned}
&\tanh\left(\tilde{d}_0(\lambda)\right) - \tanh\left(\tilde{d}_0(\lambda_0)\right) \\
&\quad = \frac{\sinh\left(\tilde{d}_0(\lambda)\right)\cosh\left(\tilde{d}_0(\lambda_0)\right) - \cosh\left(\tilde{d}_0(\lambda)\right)\sinh\left(\tilde{d}_0(\lambda_0)\right)}{\cosh\left(\tilde{d}_0(\lambda)\right)\cosh\left(\tilde{d}_0(\lambda_0)\right)} \\
&\quad = \frac{\left(\exp\left(\tilde{d}_0(\lambda)\right) - \exp\left(-\tilde{d}_0(\lambda)\right)\right)\left(\exp\left(\tilde{d}_0(\lambda_0)\right) + \exp\left(-\tilde{d}_0(\lambda_0)\right)\right)}{\cosh\left(\tilde{d}_0(\lambda)\right)\cosh\left(\tilde{d}_0(\lambda_0)\right)} \\
&\qquad - \frac{\left(\exp\left(\tilde{d}_0(\lambda)\right) + \exp\left(-\tilde{d}_0(\lambda)\right)\right)\left(\exp\left(\tilde{d}_0(\lambda_0)\right) - \exp\left(-\tilde{d}_0(\lambda_0)\right)\right)}{\cosh\left(\tilde{d}_0(\lambda)\right)\cosh\left(\tilde{d}_0(\lambda_0)\right)} \\
&\quad = \frac{\exp\left(\tilde{d}_0(\lambda) - \tilde{d}_0(\lambda_0)\right) - \exp\left(-\left(\tilde{d}_0(\lambda) - \tilde{d}(\lambda_0)\right)\right)}{\cosh\left(\tilde{d}_0(\lambda)\right)\cosh\left(\tilde{d}_0(\lambda_0)\right)} \\
&\qquad - \frac{-\exp\left(\tilde{d}_0(\lambda) - \tilde{d}_0(\lambda_0)\right) + \exp\left(-\left(\tilde{d}_0(\lambda) - \tilde{d}_0(\lambda_0)\right)\right)}{\cosh\left(\tilde{d}_0(\lambda)\right)\cosh\left(\tilde{d}_0(\lambda_0)\right)} \\
&\quad = \frac{4\sinh\left(\tilde{d}_0(\lambda) - \tilde{d}_0(\lambda_0)\right)}{\left(\exp\left(\tilde{d}_0(\lambda)\right) + \exp\left(-\tilde{d}_0(\lambda)\right)\right)\left(\exp\left(\tilde{d}_0(\lambda_0)\right) + \exp\left(-\tilde{d}_0(\lambda_0)\right)\right)} \\
&\quad = \frac{2\sinh\left(\tilde{d}_0(\lambda) - \tilde{d}_0(\lambda_0)\right)}{\cosh\left(\tilde{d}_0(\lambda) - \tilde{d}_0(\lambda_0)\right) + \cosh\left(\tilde{d}_0(\lambda) + \tilde{d}_0(\lambda_0)\right)},
\end{aligned}
$$

and thereby:

$$
\begin{aligned}
\left|\tilde{d}_1(\lambda) - \tilde{d}_1(\lambda_0)\right| &\leq 2|\hat{\omega}| \left|\frac{\sinh\left(\tilde{d}_0(\lambda) - \tilde{d}_0(\lambda_0)\right)}{\cosh\left(\tilde{d}_0(\lambda) - \tilde{d}_0(\lambda_0)\right)}\right| \\
&= 2|\hat{\omega}| \left|\tanh\left(\tilde{d}_0(\lambda) - \tilde{d}_0(\lambda_0)\right)\right| \\
&= 2|\hat{\omega}| \begin{cases} \tanh\left(\tilde{d}_0(\lambda) - \tilde{d}_0(\lambda_0)\right) & \text{if} \quad \tanh\left(\tilde{d}_0(\lambda) - \tilde{d}_0(\lambda_0)\right) \geq 0; \\ -\tanh\left(\tilde{d}_0(\lambda) - \tilde{d}_0(\lambda_0)\right) & \text{if} \quad \tanh\left(\tilde{d}_0(\lambda) - \tilde{d}_0(\lambda_0)\right) < 0. \end{cases} \\
&= 2|\hat{\omega}| \begin{cases} \tanh\left(\tilde{d}_0(\lambda) - \tilde{d}_0(\lambda_0)\right) & \text{if} \quad \tilde{d}_0(\lambda) - \tilde{d}_0(\lambda_0) \geq 0; \\ \tanh\left(-\left(\tilde{d}_0(\lambda) - \tilde{d}_0(\lambda_0)\right)\right) & \text{if} \quad \tilde{d}_0(\lambda) - \tilde{d}_0(\lambda_0) < 0. \end{cases} \\
&= 2|\hat{\omega}| \tanh\left|\tilde{d}_0(\lambda) - \tilde{d}_0(\lambda_0)\right| \\
&\leq 2|\hat{\omega}| \tanh|\lambda - \lambda_0| \qquad \because \text{ tanh is increasing and inequality (A.6).}
\end{aligned}
$$

The derivative of $\tilde{d}_5(\lambda) = \tanh|\lambda - \lambda_0| - |\lambda - \lambda_0|$:

$$
\frac{d}{d\lambda}\tilde{d}_6(\lambda) = \operatorname{sgn}(\lambda - \lambda_0)\left(\operatorname{sech}^2(\lambda - \lambda_0) - 1\right) = -\operatorname{sgn}(\lambda - \lambda_0)\tanh^2(\lambda - \lambda_0)
$$

reveals that

$$\left(\forall\lambda<\lambda_0\,,\frac{d}{d\lambda}\tilde{d}_6(\lambda)>0\right)\wedge\left(\forall\lambda>\lambda_0\,,\frac{d}{d\lambda}\tilde{d}_6(\lambda)<0\right)\implies \max_{\lambda\in\mathbb{R}}\tilde{d}_6(\lambda)=\tilde{d}_6(\lambda_0)=0.$$

Equivalently,

$$\tanh|\lambda-\lambda_0|\leq|\lambda-\lambda_0|\;\wedge\;\frac{1}{2|\hat{\omega}|}\left|\tilde{d}_1(\lambda)-\tilde{d}_1(\lambda_0)\right|\leq|\lambda-\lambda_0| \tag{A.7}$$

Clearly the left sides in (A.6) and (A.7) are less than the maximum of the three quantities and said maximum is less than the bound $|\lambda-\lambda_0|$.

(ii) The derivative of $\tilde{d}_7(\tau)=\left|1-\exp\left(-|\tilde{d}_3|\tau\right)\right|-|\tilde{d}_3|\tau$:

$$\frac{d}{d\tau}\tilde{d}_7(\tau)=|\tilde{d}_3|\operatorname{sgn}\left(1-\exp\left(-|\tilde{d}_3|\tau\right)\right)\exp\left(-|\tilde{d}_3|\tau\right)-|\tilde{d}_3| \qquad (\tau>0).$$

reveals that

$$\left(\forall\tilde{d}_3\in\mathbb{R}\right)(\forall\tau\in\mathbb{R}_+)\,,\;\frac{d}{d\tau}\tilde{d}_7(\tau)=\tilde{d}_3\left(\exp\left(-|\tilde{d}_3|\tau\right)-1\right)\leq 0.$$

Therefore,

$$\tilde{d}_7(\tau)\leq\tilde{d}_7(0)=0\implies\left(\forall\tilde{d}_3\in\mathbb{R}\right)(\forall\tau\in\mathbb{R}_+)\,,\left|1-\exp\left(-|\tilde{d}_3|\tau\right)\right|\leq|\tilde{d}_3|\tau \tag{A.8}$$

Consider

$$\tilde{d}_8(\tau)=\left|\exp\left(-\tilde{d}_5(\lambda)\tau\right)-\exp\left(-\tilde{d}_5(\lambda_0)\tau\right)\right| \tag{A.9}$$

For $\tilde{d}_5(\lambda)-\tilde{d}_5(\lambda_0)=\tilde{d}_3\leq 0$, $\exp\left(-\tilde{d}_5(\lambda)\tau\right)$ is factorized in (A.9):

$$\tilde{d}_8(\tau)=\left|1-\exp\left(\tilde{d}_3\tau\right)\right|\exp\left(-\tilde{d}_5(\lambda)\tau\right) \tag{A.10}$$

For $\tilde{d}_5(\lambda)-\tilde{d}_5(\lambda_0)=\tilde{d}_3>0$, $\exp\left(-\tilde{d}_5(\lambda)\tau\right)$ is factorized in (A.9):

$$\tilde{d}_8(\tau)=\left|1-\exp\left(-\tilde{d}_3\tau\right)\right|\exp\left(-\tilde{d}_5(\lambda_0)\tau\right) \tag{A.11}$$

Expressions (A.10) and (A.11) are compacted with

$$\tilde{d}_8(\tau)=\left|1-\exp\left(-|\tilde{d}_3|\tau\right)\right|\exp\left(\tilde{d}_4(\lambda)\right),$$

Therefore, utilizing (A.8):

$$\tilde{d}_8(\tau)\leq|\tilde{d}_3|\exp\left(\tilde{d}_4(\lambda)\right)\tau.$$

A.6. Proof of Lemma 4.21

The solutions of systems $\dot{\overline{\mathbf{x}}} = A\overline{\mathbf{x}}$ and $\dot{\overline{\mathbf{z}}} = -A\dot{\overline{\mathbf{z}}}$ are linked by the property that their inner product $\langle \overline{\mathbf{z}}(t), \overline{\mathbf{x}}(t)\rangle$ remains constant, as shown by the following simple calculation:

$$\frac{d}{dt}\langle \overline{\mathbf{z}}(t), \overline{\mathbf{x}}(t)\rangle = \langle \dot{\overline{\mathbf{x}}}, \overline{\mathbf{x}}\rangle + \langle \overline{\mathbf{x}}, \dot{\overline{\mathbf{x}}}\rangle = \left(-A^\top \overline{\mathbf{z}}\right)^\top \overline{\mathbf{x}} + \overline{\mathbf{z}}^\top A\overline{\mathbf{x}} = 0.$$

Then, for any solution of $\dot{\overline{\mathbf{x}}} = A(t)\overline{\mathbf{x}}$, $\Phi^\top_{A(\cdot)}(t)\overline{\mathbf{z}} = \overline{\mathbf{z}}$ for all $t \geq 0$. In particular, $\Phi^\top_{A(\cdot)}(\omega)\overline{\mathbf{z}} = \overline{\mathbf{z}}(\omega)$. Since $t \mapsto A(t)$ is continuous, cooperative, irreducible and ω-periodic, then it follows from ([53], Lemma 2) that $\Phi_{A(\cdot)}(\omega) > 0$, and by virtue of ([53], Lemma 1), 1 is the principal eigenvalue of $\Phi^\top_{A(\cdot)}(\omega)$. Consequently, $\Phi^\top_{A(\cdot)}(t)$ also has the principal eigenvalue 1 and a corresponding eigenvector $\overline{\mathbf{x}}^* > 0$. Thus, $\Phi^\top_{A(\cdot)}(\omega)\overline{\mathbf{x}}^* = \overline{\mathbf{x}}^*$.

By the change of variable $\overline{\mathbf{x}}(t) = e^{\overline{p}\,t}\mathbf{v}(t)$, the linear system $\dot{\mathbf{x}} = A(t)\mathbf{x}$ is reduced to

$$\begin{aligned}&\dot{\mathbf{v}}(t) = e^{-\overline{p}\,t}\left(\dot{\overline{\mathbf{x}}} - \overline{p}\,\mathbf{x}\right) = e^{-\overline{p}\,t}\left(A(t)\mathbf{x} - \overline{p}\,\mathbf{x}(t)\right) = \left(A(t)e^{-\overline{p}\,t}\mathbf{x} - \overline{p}\,e^{-\overline{p}\,t}\mathbf{x}\right)\\ \Leftrightarrow\quad &\dot{\mathbf{v}} = \left(A(t) - \overline{p}\,I\right)\mathbf{v}.\end{aligned} \tag{A.12}$$

Thus, $\overline{\mathbf{v}}(t) := \Phi_{(A(\cdot)-\overline{p}\,I)}\overline{\mathbf{x}}^*$ is a positive solution of (A.12), where $\Phi_{(A(\cdot)-\overline{p}\,I)}(t)$ is the principal fundamental matrix of (A.12). Notice that $e^{\overline{p}\,t}\Phi_{(A(\cdot)-\overline{p}\,I(t))}(t)$ is also a fundamental matrix of $\dot{\overline{\mathbf{x}}} = A(t)\overline{\mathbf{x}}$:

$$\begin{aligned}\frac{d}{dt}\left(e^{\overline{p}\,t}\Phi_{(A(\cdot)-\overline{p}\,I)}(t)\right) &= e^{\overline{p}\,t}\left(\dot{\Phi}_{(A(\cdot)-\overline{p}\,I)}(t) + \overline{p}\,\Phi_{(A(\cdot)-\overline{p}\,I)}(t)\right)\\ &= e^{\overline{p}\,t}\left(\left(A(\cdot) - \overline{p}\,I\right)\Phi_{(A(t)-\overline{p}\,I)}(t) + \overline{p}\,\Phi_{(A(\cdot)-\overline{p}\,I)}(t)\right)\\ &= A(t)\left(e^{\overline{p}\,t}\Phi_{(A(\cdot)-\overline{p}\,I)}(t)\right).\end{aligned}$$

The uniqueness of the principal fundamental matrix implies that
$e^{\overline{p}\,t}\Phi_{(A(\cdot)-\overline{p}\,I(t))}(t) = \Phi_{A(\cdot)}(t)$. Moreover,

$$\overline{\mathbf{v}}(\omega) = \Phi_{(A(\cdot)-\overline{p}\,I)}(\omega)\overline{\mathbf{x}}^* = e^{-\overline{p}\,\omega}\Phi_{(A(\cdot))}(\omega)\overline{\mathbf{x}}^* = e^{-\overline{p}\,\omega}\rho\left(\Phi_{A(\cdot)}(\omega)\right)\overline{\mathbf{x}}^* = \overline{\mathbf{x}}^* = \overline{\mathbf{v}}(0).$$

Thus, $\overline{\mathbf{v}}(t)$ is a positive ω-periodic solution of (2.1), and hence, $\overline{\mathbf{x}}(t) = e^{\overline{p}\,t}\mathbf{v}(t)$ is a solution of $\dot{\overline{\mathbf{x}}} = A(t)\overline{\mathbf{x}}$.

A.7. Proof of Lemma 5.7

The set (3.3) and assumptions $\mathrm{M_I}(t) \leq \beta_\mathrm{M}$, $\mathrm{H} \geq 1$ convert the differential equation of the exposed people into

$$\frac{d}{dt}\mathrm{H_E}(t) \leq qb^u\beta_\mathrm{M}\mathrm{H} - (\hat{l} + h)\mathrm{H_E}(t) \Leftrightarrow \frac{d}{dt}\left(\mathrm{H_E}(t)\exp\left((\hat{l} + h)t\right)\right) \leq r\beta_\mathrm{M}\exp\left((\hat{l} + h)t\right)$$

for $t \geq t_0$. Integrating this inequality and accommodating terms, it is deduced that

$$\begin{aligned} \mathrm{H_E}(t) &\leq \frac{\mathrm{H_E}(t_0)}{\exp\big((\hat{l}+h)(t-t_0)\big)} + \frac{qb^u\beta_{\mathrm{M}}\mathrm{H}}{h+\hat{l}}\Big(1-\exp\big(-(\hat{l}+h)(t-t_0)\big)\Big) \\ &\leq \frac{qb^u\beta_{\mathrm{M}}\mathrm{H}}{h+\hat{l}} + \exp\big(-(\hat{l}+h)(t-t_0)\big)\mathrm{H}\left(1-\frac{qb^u\beta_{\mathrm{M}}}{h+\hat{l}}\right) \\ &< \frac{qb^u\beta_{\mathrm{M}}\mathrm{H}}{h+\hat{l}} + \Theta \end{aligned}$$

provided that $\Theta > \exp\big(-(\hat{l}+h)(t-t_0)\big)\mathrm{H}\left(1-\frac{qb^u\beta_{\mathrm{M}}}{h+\hat{l}}\right)$, equivalently

$$t > T_1 = t_0 + \frac{1}{h}\ln\left(\frac{\mathrm{H}}{\Theta}\left(1-\frac{qb^u\beta_{\mathrm{M}}}{h+\hat{l}}\right)\right) > 0 \text{ whenever } \beta_{\mathrm{H}} < \frac{h+\hat{l}}{qb^u}.$$

Since $\Theta < \mathrm{H}\left(1-\frac{qb^u\beta_{\mathrm{M}}}{h+\hat{l}}\right)$, then there exists a $T_2 > 0$ which depends only on Θ, β_{M} and the model parameters.

A.8. Proof of Lemma 5.9

From the differential equation of the exposed humans:

$$\frac{d}{dt}\mathrm{H_S}(t) \geq h\mathrm{H} - \big((h\mathrm{H}/2)/\overline{\mathrm{S}}_1\big)\mathrm{H_S}(t) \Leftrightarrow \frac{d}{dt}\big(\mathrm{H_S}(t)\exp\big((h\mathrm{H}/2)t/\overline{\mathrm{S}}_1\big)\big) \geq h\mathrm{H}\exp\big((h\mathrm{H}/2)t/\overline{\mathrm{S}}_1\big).$$

Integrating this inequality over $[0\,,\,t]$,

$$\begin{aligned} \mathrm{H_S}(t) &\geq \mathrm{H_S}(0)\exp\big(-(h\mathrm{H}/2)t/\overline{\mathrm{S}}_1\big) + \frac{h\mathrm{H}}{(h\mathrm{H}/2)/\overline{\mathrm{S}}_1}\Big(1-\exp\big(-(h\mathrm{H}/2)t/\overline{\mathrm{S}}_1\big)\Big) \\ &\geq 2\overline{\mathrm{S}}_1\Big(1-\exp\big(-(h\mathrm{H}/2)t/\overline{\mathrm{S}}_1\big)\Big) \end{aligned} \tag{A.13}$$

As $\mathrm{M_I}(0) = \beta^*_{\mathrm{M}}$ then $\mathrm{M_I}(t) \geq \beta^*_{\mathrm{M}}\exp\big(-m^u t\big)$ from the seventh equation of system (3.2); hence, utilizing the second equation of (3.2) and the minorant function (A.13) (monotonically increasing),

$$\begin{aligned} \overline{f}_0(t) = \frac{d}{dt}\Big(\mathrm{H_E}(t)\exp\big((\hat{l}+h)t\big)\Big) &\geq \big(qb(t)\beta^*_{\mathrm{M}}/\mathrm{H}\big)\overline{\mathrm{S}}_1\exp\big((h+\hat{l}-m^u)t\big) \\ &\geq \big(qb(t)\beta^*_{\mathrm{M}}/\mathrm{H}\big)\overline{\mathrm{S}}_1\exp\big(-m^u t\big) = \overline{f}_1(t). \end{aligned}$$

for $t \geq \overline{T}_4$. If $\overline{f}_0(t) \geq \overline{f}_1(t)$ is integrated between $n_1\omega$ and $(n_1+1)\omega$, then

$$\begin{aligned} \mathrm{H_E}(\overline{T}_3) &\geq \exp\big(-(\hat{l}+h)\overline{T}_3\big)\big(q\beta^*_{\mathrm{M}}/\mathrm{H}\big)\overline{\mathrm{S}}_1\int_{n_1\omega}^{(n_1+1)\omega} b(t)\exp\big(-m^u t\big)dt \\ &\geq \exp\big(-(\hat{l}+h)\overline{T}_3\big)\big(q\beta^*_{\mathrm{M}}/\mathrm{H}\big)\overline{\mathrm{S}}_1\exp\big(-m^u n_1\omega\big)\int_0^{\omega} b(t)\exp\big(-m^u t\big)dt = \overline{\mathrm{E}}^*_1 > 0 \end{aligned}$$

as required.

A.9. Proof of Lemma 5.15

(a) Remembering Proposition 5.10 and the fourth equation of the system (3.2), let

$$\mathrm{H}_{\mathrm{I},\infty} \geq \overline{\theta}_0 \quad \text{and} \quad \frac{d}{dt}\mathrm{H}_{\mathrm{R}}(t) \geq r\mathrm{H}_{\mathrm{I},\infty} - h\mathrm{H}_{\mathrm{R}}(t).$$

Now, given $\overline{\epsilon} > 0$, there exists $\overline{t}_5$ such that $\mathrm{H}_{\mathrm{I}}(t) \geq \overline{\theta}_0 - \overline{\epsilon}$ for all $t \geq \overline{t}_5$. Then for all $t \geq \overline{t}_5$,

$$\frac{d}{dt}\mathrm{H}_{\mathrm{R}}(t) \geq r(\overline{\theta}_0 - \overline{\epsilon}) - h\mathrm{H}_{\mathrm{R}}(t).$$

This inequality is integrated over $[t_5\,,\,t] \subset \left[\overline{t}_5\,,\,t\right]$ to find

$$\begin{aligned}
\mathrm{H}_{\mathrm{R}}(t) &\geq \mathrm{H}_{\mathrm{R}}(t_5)\exp\bigl(-h(t-t_5)\bigr) + \frac{r}{h}(\overline{\theta}_0 - \overline{\epsilon})\left(1 - \exp\bigl(-h(t-t_5)\bigr)\right) \\
&= \left(\mathrm{H}_{\mathrm{R}}(t_5) - \frac{r}{h}(\overline{\theta}_0 - \overline{\epsilon})\right)\exp\bigl(-h(t-t_5)\bigr) + \frac{r}{h}(\overline{\theta}_0 - \overline{\epsilon}) \\
&\geq \overline{\theta}_4 - \left|\mathrm{H}_{\mathrm{R}}(t_5) - \overline{\theta}_4\right|\exp\bigl(-h(t-t_5)\bigr),
\end{aligned}$$

where $\overline{\theta}_4 = r(\overline{\theta}_0 - \overline{\epsilon})/h$. Choose t_5 large enough so that $\left|\mathrm{H}_{\mathrm{R}}(t_5) - \overline{\theta}_4\right|\exp\bigl(-h(t-t_5)\bigr) \leq \overline{\epsilon}$ for $t \geq t_6$, thus $\mathrm{H}_{\mathrm{R}}(t) \geq \overline{\theta}_6 - \overline{\epsilon}$ for $t \geq \overline{t}_6$ and $\inf_{t \geq t_6} \mathrm{H}_{\mathrm{R}}(t) \geq \overline{\theta}_4 - \overline{\epsilon}$ for $t_6 \geq \overline{t}_6$. Letting $t_6 \to \infty$ it follows that $\mathrm{H}_{\mathrm{R},\infty} \geq \overline{\theta}_4 - \overline{\epsilon}$ and, since $\overline{\epsilon}$ can be chosen arbitrarily small, $\mathrm{H}_{\mathrm{R},\infty} \geq \overline{\theta}_4$.

(b) By the fluctuation lemma [41], there exists a sequence $\{\tau_n\}$ such that $\tau_n \to \infty$, $\mathrm{H}_{\mathrm{S}}(\tau_n) \to \mathrm{H}_{\mathrm{S},\infty}$ and $\frac{d}{dt}\mathrm{H}_{\mathrm{S}}(\tau_n) \to 0$ as $n \to \infty$. It follows from the first equation of (3.2) that

$$\frac{d}{dt}\mathrm{H}_{\mathrm{S}}(\tau_n) + h\mathrm{H}_{\mathrm{S}}(\tau_n) + \frac{qb(\tau_n)\mathrm{M}_{\mathrm{I}}(\tau_n)\mathrm{H}_{\mathrm{S}}(\tau_n)}{\mathrm{H}} = h\mathrm{H}.$$

It is easy to see that $0 \leq \mathrm{M}_{\mathrm{I}}(\tau_n) \leq \Delta_{\sup}/m_{\inf}$ y $b_{\inf} \leq \mathrm{b}_{\mathrm{I}}(\tau_n) \leq b_{\sup}$. Taking this into account and letting $n \to \infty$, we obtain

$$\mathrm{H}_{\mathrm{S},\infty} \geq \frac{hm_{\inf}\mathrm{H}^2}{hm_{\inf}\mathrm{H} + qb_{\sup}\Delta_{\sup}} = \overline{\theta}_3.$$

(c) There exists t_3 such that for $t \geq t_3$, $\mathrm{H}_{\mathrm{S}}(t) \geq \mathrm{H}_{\mathrm{S},\infty}/\sqrt{2}$ and $\mathrm{M}_{\mathrm{I}}(t) \geq \mathrm{M}_{\mathrm{I},\infty}/\sqrt{2}$. Employing the system (3.2) and the region (3.3),

$$\frac{d}{dt}\mathrm{H}_{\mathrm{E}}(t) + (\hat{l} + h)\mathrm{H}_{\mathrm{E}}(t) \geq \mathrm{E}_3(t) = \frac{qb(t)\mathrm{M}_{\mathrm{I},\infty}\mathrm{H}_{\mathrm{S},\infty}}{2\mathrm{H}} \tag{A.14}$$

for $t \geq t_3$. Pick k such that $k\omega \geq t_3$, multiply (A.14) by $\exp\big((\hat{l}+h)t\big)$ and integrate over $[k\omega\,,\,(k+1)\omega]$:

$$\frac{\mathrm{H_E}\big((k+1)\,\omega\big)}{\exp\big(-(\hat{l}+h)\,(k+1)\,\omega\big)} \geq \frac{\mathrm{H_E}\big(k\omega\big)}{\exp\big(-(\hat{l}+h)k\omega\big)} + \int_{k\omega}^{(k+1)\omega} \mathrm{E}_3(t)\exp\big((\hat{l}+h)t\big)dt.$$

So,

$$\mathrm{H_E}\big((k+1)\,\omega\big) \geq \int_0^{\omega} \mathrm{E}_3(t)\exp\big((\hat{l}+h)(t-\omega)\big)dt = \overline{\mathrm{E}}_3 > 0,$$

utilizing the fact that the integrand is a positive continuous function so the integral is strictly positive. Moreover $\overline{\mathrm{E}}_3$ depends only on the model parameters. But

$$\frac{d}{dt}\mathrm{H_E}(t) \geq -(\hat{l}+h)\mathrm{H_E}(t).$$

So for $(k+1)\omega \leq t \leq (k+2)\omega$ and $-t \geq -(k+2)\omega$,

$$\mathrm{H_E}(t) \geq \mathrm{H_E}\big((k+1)\,\omega\big)\exp\big(-(\hat{l}+h)\big(t-(k+1)\omega\big) \geq \overline{\theta}_2 = \overline{\mathrm{E}}_3\exp\big(-(\hat{l}+h)\omega\big).$$

Hence, $\mathrm{H_E}(t) \geq \overline{\theta}_2$ for all $t \geq (k+1)\omega$.

(d) By the fluctuation lemma [41], there exists a sequence $\{\tau_n\}$ such that $\tau_n \to \infty$, $\mathrm{M_S}(\tau_n) \to \mathrm{M_{S,\infty}}$ and $\frac{d}{dt}\mathrm{M_S}(\tau_n) \to 0$ as $n \to \infty$. It follows from the fifth equation of (3.2) that

$$\frac{d}{dt}\mathrm{M_S}(\tau_n) + m(\tau_n)\mathrm{M_S}(\tau_n) + \frac{pb(\tau_n)\mathrm{H_I}(\tau_n)\mathrm{M_S}(\tau_n)}{\mathrm{H}} = \Delta(\tau_n).$$

It is easy to see that $0 \leq \mathrm{H_I}(\tau_n) \leq \mathrm{H}$, $\Delta_{\inf} \leq \Delta(\tau_n) \leq \Delta_{\sup}$, $m_{\inf} \leq m_{\mathrm{I}}(\tau_n) \leq m_{\sup}$ and $b_{\inf} \leq \mathrm{b_I}(\tau_n) \leq b_{\sup}$. Taking this into account and letting $n \to \infty$, one obtains

$$\mathrm{M_{S,\infty}} \geq \frac{\Delta_{\inf}}{m_{\sup} + pb_{\sup}} = \theta_3.$$

(e) There exists $\bar{t}_3$ such that for $t \geq \bar{t}_3$, $\mathrm{M_S}(t) \geq \mathrm{M_{S,\infty}}/\sqrt{2}$ and $\mathrm{H_I}(t) \geq \mathrm{H_{I,\infty}}/\sqrt{2}$. Employing the system (3.2) and the region (3.3),

$$\frac{d}{dt}\mathrm{M_E}(t) + (c+m^u)\mathrm{M_E}(t) \geq \mathrm{E}_3^*(t) = \frac{pb(t)\mathrm{H_{I,\infty}}\mathrm{M_{S,\infty}}}{2\mathrm{H}}. \tag{A.15}$$

para $t \geq \bar{t}_3$. Picking k such that $k\omega \geq \bar{t}_3$ then by multiplying (A.15) by $\exp\big((c+m^u)\,t\big)$ and integrating over $[k\omega\,,\,(k+1)\omega]$:

$$\frac{\mathrm{M_E}\big((k+1)\,\omega\big)}{\exp\big(-(c+m^u)\,(k+1)\,\omega\big)} \geq \frac{\mathrm{M_E}\big(k\omega\big)}{\exp\big(-(c+m^u)k\omega\big)} + \int_{k\omega}^{(k+1)\omega} \mathrm{E}_3^*(t)\exp\big((c+m^u)t\big)dt.$$

So,

$$\mathrm{M_E}\big((k+1)\,\omega\big) \geq \int_0^{\omega} \mathrm{E}_3^*(t)\exp\big((c+m^u)(t-\omega)\big)dt = \overline{\mathrm{E}}_3^* > 0,$$

utilizing the fact that the integrand is a positive continuous function so the integral is strictly positive. Moreover $\overline{\mathrm{E}}_3^*$ depends only on the model parameters. But

$$\frac{d}{dt}\mathrm{M_E}(t) \geq -(c+m^u)\mathrm{M_E}(t).$$

So for $(k+1)\omega \leq t \leq (k+2)\omega$ and $-t \geq -(k+2)\omega$,

$$\mathrm{M_E}(t) \geq \mathrm{M_E}\big((k+1)\,\omega\big)\exp\big(-(c+m^u)\big(t-(k+1)\omega\big) \geq \theta_2 = \overline{\mathrm{E}}_3\exp\big(-(c+m^u)\omega\big).$$

Hence, $\mathrm{M_E}(t) \geq \theta_2$ for all $t \geq (k+1)\omega$.

B. Proof of Proposition 3.1

(i) The continuity of the right side of (3.4) in its arguments ensures the existence of at least one solution ([39], Theorem 2.3). Now, to confirm the positivity of the solution for any admissible pair of controls $\mathbf{u}(t) = \begin{bmatrix} u_{\mathrm{H}}(t) & u_{\mathrm{M}}(t)\end{bmatrix} \in \Gamma$ and for all $t \in [0,\, t_{\mathrm{f}}]$, let

$$\xi(t) = \min\big\{\mathrm{H_S}(t),\,\mathrm{H_E}(t),\,\mathrm{H_I}(t),\,\mathrm{H_R}(t),\,\mathrm{M_S}(t),\,\mathrm{M_E}(t),\,\mathrm{M_I}(t)\big\}$$

for all $t \geq t_0$. It is assumed that $\tau > 0$ exists such that $\xi(\tau) \notin \mathbb{R}_+ \setminus \{0\}$ and $\xi(t) > 0$ for each $t \in [t_0, \tau)$. If $\xi(t) = \mathrm{H_S}(t)$, then $\mathrm{H_S}(t) > 0$. Therefore, from the first equation of the system (3.2) one has

$$\begin{aligned}&\frac{d}{dt}\mathrm{H_S}(t) > -\left((1-\varpi_1 u_{\mathrm{H}}(t))\frac{qb(t)\mathrm{M_I}}{\mathrm{H}} + h\right)\mathrm{H_S}\\ \Leftrightarrow &\frac{d}{dt}\left(\mathrm{H_S}(t)\exp\left(-\int_{t_0}^{t}\left((1-\varpi_1 u_{\mathrm{H}}(t))\frac{qb(\tau)\mathrm{M_I}}{\mathrm{H}} + h\right)dt\right)\right).\end{aligned}$$

It follows that

$$\begin{aligned}&\mathrm{H_S}(t)\exp\left(-\int_{t_0}^{\tau}\left((1-\varpi_1 u_{\mathrm{H}}(t))\frac{qb(\tau)\mathrm{M_I}}{\mathrm{H}}+h\right)dt\right)\\&\quad>\mathrm{H_S}(t_0)\exp\left(-\int_{t_0}^{t_0}\left((1-\varpi_1 u_{\mathrm{H}}(t))\frac{qb(\tau)\mathrm{M_I}}{\mathrm{H}}+h\right)dt\right)\\&\Leftrightarrow\mathrm{H_S}(\tau)>\mathrm{H_S}(t_0)\exp\left(-\int_{t_0}^{\tau}\left((1-\varpi_1 u_{\mathrm{H}}(t))\frac{qb(\tau)\mathrm{M_I}}{\mathrm{H}}+h\right)dt\right)>0,\end{aligned}$$

which leads to a contradiction. If $\xi(t)=\mathrm{H_E}(t)$, then $\mathrm{H_E}(t)>0$. Therefore, from the second equation of the system (3.2) one has

$$\frac{d}{dt}\mathrm{H_E}(t)>-(h+l)\mathrm{H_E}(t).$$

It follows that

$$\mathrm{H_E}(\tau)>\mathrm{H_E}(t_0)\exp\left(-(\hat{l}+h)(\tau-t_0)\right)>0,$$

which leads to a contradiction. If $\xi(t)=\mathrm{H_I}(t)$, then $\mathrm{H_I}(t)>0$. Therefore, from the third equation of the system (3.2) one has

$$\frac{d}{dt}\mathrm{H_I}(t)>-(h+r)\mathrm{H_I}(t).$$

It follows that

$$\mathrm{H_E}(\tau)>\mathrm{H_E}(t_0)\exp\left(-(h+l)(\tau-t_0)\right)>0,$$

which leads to a contradiction. If $\xi(t)=\mathrm{H_R}(t)$, then $\mathrm{H_R}(t)>0$. Therefore, from the fourth equation of the system (3.2) one has

$$\frac{d}{dt}\mathrm{H_R}(t)>-h\mathrm{H_R}(t).$$

It follows that

$$\mathrm{H_R}(\tau)>\mathrm{H_R}(t_0)\exp\left(-h(\tau-t_0)\right)>0,$$

which leads to a contradiction. Similar contradictions are obtained if $\xi(t)=\mathrm{M_S}(t)$, $\xi(t)=\mathrm{M_E}(t)$ and $\xi(t)=\mathrm{M_I}(t)$. Therefore $\tau=+\infty$, and positivity of $\boldsymbol{\chi}(t,\mathbf{x}_0)$ is guaranteed for all $t\geq t_0$.

The total size of the human population is

$$\mathrm{H}\equiv\mathrm{H}(t)=\mathrm{H_S}(t)+\mathrm{H_E}(t)+\mathrm{H_I}(t)+\mathrm{H_R}(t).$$

Adding the first four equations of the system (3.2) then $\dot{\mathrm{H}} = 0$, therefore the value of $\mathrm{H}(t)$ is constant with $\mathrm{H}(t) = \mathrm{H}(t_0) > 0$ for all $t \geq t_0$.

Likewise, the total size of the adult mosquito population

$$\mathrm{M} \equiv \mathrm{M}(t) = \mathrm{M}_{\mathrm{S}}(t) + \mathrm{M}_{\mathrm{E}}(t) + \mathrm{M}_{\mathrm{I}}(t),$$

satisfies

$$\frac{d}{dt}\mathrm{M}(t) = (1 - u_{\mathrm{M}}(t))\Delta(t) - (m(t) + \varpi_2 u_{\mathrm{M}}(t))\mathrm{M}(t) \tag{B.1}$$

and

$$\dot{\mathrm{M}} \leq 1\Delta^u - (m^l + 0\varpi_2)\mathrm{M} = \Delta^u - m^l\mathrm{M}.$$

Let $\mathrm{W}(t)$ be the solution of the differential equation $\dot{\mathrm{W}} = \Delta^u - m^l\mathrm{W}$ with $\mathrm{W}(t_0) = \mathrm{M}(t_0)$, then

$$\mathrm{W}(t) = \left(\mathrm{W}(t_0) - \frac{\Delta^u}{m^l}\right)\exp\left(-m^l(t - t_0)\right) + \frac{\Delta^u}{m^l}.$$

W grows continuously strictly over $[t_0, \infty)$ if $\mathrm{W}(t_0) < \Delta^u/m^l$. Consequently, according to the standard comparison Lemma (see, for example, Lemma 3.4 in [40]), the solution of (B.1) is defined for all $t \geq t_0$ and satisfies

$$\mathrm{M}(t) \leq \left(\mathrm{M}(t_0) - \frac{\Delta^u}{m^l}\right)\exp\left(-m^l(t - t_0)\right) + \frac{\Delta^u}{m^l} \tag{B.2}$$

if $0 \leq \mathrm{M}(t_0) \leq \Delta^u/m^l$. Therefore, $\boldsymbol{\chi}(t, \mathbf{x}_0)$ is bounded; in particular the image of $[0, t_{\mathrm{f}}]$ under $\boldsymbol{\chi}$ is bounded. Else, for any initial condition outside Π, every forward solution of the equation (B.2) converges to Δ^u/m^l $(t \to +\infty)$, and therefore $\mathrm{H}(t)$ and $\mathrm{M}(t)$ are ultimately bounded, that is: there exists $t_1 > 0$ and a positive constant $L_1 = \max\{\mathrm{H}, \Delta^u/m^l\}$ such that $\forall t > t_1$, $\mathrm{H}(t) \leq L_1$ and $\mathrm{M}(t) \leq L_1$.

(ii) The uniqueness of the solution can be ensured by showing that $\mathbf{F}$ in (3.4) satisfies a Lipschitz condition. The system (3.2) is rewritten as follows:

$$\mathbf{F}(t, \mathbf{x}, \mathbf{u}) = \mathbf{F}_1(t, \mathbf{u})\mathbf{x} + \mathbf{F}_2(t, \mathbf{x}, \mathbf{u}), \tag{B.3}$$

where

$$\mathbf{F}_1(t, \mathbf{u}) = \begin{bmatrix} -h & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -h-l & 0 & 0 & 0 & 0 & 0 \\ 0 & l & -h-r & 0 & 0 & 0 & 0 \\ 0 & 0 & r & -h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -m(t) - \varpi_2 u_{\mathrm{M}}(t) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c - m(t) - \varpi_2 u_{\mathrm{M}}(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & c & -m(t) - \varpi_2 u_{\mathrm{M}}(t) \end{bmatrix}$$

and

$$\mathbf{F}_2(t,\mathbf{x},\mathbf{u}) = \begin{bmatrix} h\mathrm{H} - (1-u_{\mathrm{H}}(t))\dfrac{qb(t)x_1x_7}{\mathrm{H}} & (1-\varpi_1 u_{\mathrm{H}}(t))\dfrac{qb(t)x_1x_7}{\mathrm{H}} & 0 & 0 & 0 & 0 & 0\end{bmatrix}^\top + \begin{bmatrix} 0 & 0 & 0 & (1-u_{\mathrm{M}}(t))\Delta(t) - (1-\varpi_1 u_{\mathrm{H}}(t))\dfrac{pb(t)x_5x_3}{\mathrm{H}} & (1-\varpi_1 u_{\mathrm{H}}(t))\dfrac{pb(t)x_5x_3}{\mathrm{H}} & 0\end{bmatrix}^\top$$

are continuous over any time sub-interval of $\mathbb{R}_+$, whose components are bounded by positive constants. A constant $\overline{L} > 0$ must be found such that

$$\|\mathbf{F}(t,\mathbf{x}_2,\mathbf{u}) - \mathbf{F}(t,\mathbf{x}_1)\|_1, \mathbf{u} \leq \overline{L}\|\mathbf{x}_2 - \mathbf{x}_1\|_1.$$

Here $\|\cdot\|_1$ denotes the 1-norm of vectors and matrices. Starting from the definition of $\mathbf{F}$:

$$\begin{aligned}
&\|\mathbf{F}(t,\mathbf{x}_2,\mathbf{u}) - \mathbf{F}(t,\mathbf{x}_1,\mathbf{u})\|_1 \\
&= (1-u_{\mathrm{H}}(t))\frac{2b(t)}{\mathrm{H}}\left(q\left|x_{12}x_{72} - x_{11}x_{71}\right| + p\left|x_{52}x_{32} - x_{51}x_{31}\right|\right) + \|\mathbf{F}_1(t,\mathbf{u})\|_1\|\mathbf{x}_2 - \mathbf{x}_1\|_1 \\
&= (1-u_{\mathrm{H}}(t))\frac{2b(t)}{\mathrm{H}}\left|x_{12}(x_{72}-x_{71}) + x_{11}(x_{72}-x_{71}) + x_{72}(x_{12}-x_{11}) + x_{71}(x_{12}-x_{11})\right| \\
&\qquad + (1-u_{\mathrm{H}}(t))\frac{2b(t)}{\mathrm{H}}\left|x_{52}(x_{32}-x_{31}) + x_{51}(x_{32}-x_{31}) + x_{32}(x_{52}-x_{51}) + x_{31}(x_{52}-x_{51})\right| \\
&\quad + \|\mathbf{F}_1(t,\mathbf{u})\|_1\|\mathbf{x}_2 - \mathbf{x}_1\|_1 \\
&= (1-u_{\mathrm{H}}(t))\frac{(x_{12}+x_{11})\left|x_{72}-x_{71}\right| + (x_{72}+x_{71})\left|x_{12}-x_{11}\right|}{\mathrm{H}/(qb^u)} + \|\mathbf{F}_1(t,\mathbf{u})\|_1\|\mathbf{x}_2 - \mathbf{x}_1\|_1 \\
&\quad + (1-u_{\mathrm{H}}(t))\frac{(x_{52}+x_{51})\left|x_{32}-x_{31}\right| + (x_{32}+x_{31})\left|x_{52}-x_{51}\right|}{\mathrm{H}/(pb^u)} \\
&\leq \frac{2\mathrm{H}\left|x_{72}-x_{71}\right| + 2(\Delta^u/m^l)\left|x_{12}-x_{11}\right|}{\mathrm{H}/(qb^u)} + \left|x_{42}-x_{41}\right| + \left|x_{62}-x_{61}\right| + \left|x_{22}-x_{21}\right| \\
&\quad + \frac{2(\Delta^u/m^l)\left|x_{32}-x_{31}\right| + 2\mathrm{H}\left|x_{52}-x_{51}\right|}{\mathrm{H}/(pb^u)} + \sup_{0\leq t\leq\omega}\|\mathbf{F}_1(t,\mathbf{u}(t))\|_1\|\mathbf{x}_2 - \mathbf{x}_1\|_1,
\end{aligned}$$

a Lipchitz constant can be taken as

$$L = \max\left\{2\mathrm{H}qb^u\,,\ \|\mathbf{F}_1\|\,,\ 2qb^u[\Delta^u/m^l + \alpha(\Delta^u/m^l)^2]\,,\ 3\,,\ 2pb^u(\Delta^u/m^l)/\mathrm{H}\,,\ 2pb^u\right\} > 0,$$

Thus, $\mathbf{F}(t,\mathbf{x},\mathbf{u})$ is uniformly Lipchitz continuous in $\mathbf{x} \in \Pi$, where $\mathbf{u}(\cdot)$ is piecewise continuos function of t. The conditios of theorems 3.2 and 3.3 in ([40], p.93) are satisfied, therefore the system (3.2) has a unique solution $\boldsymbol{\chi}(t,\mathbf{x}_0)$ in its maximum interval of existence $[t_0,+\infty)$.

C. Proof of Remark 4.16

It will be proved that $\overline{\mathrm{R}}_0 \leq \sqrt{\overline{\mathrm{R}}_0^{\inf}}$. The simplifying expressions

$$\begin{cases} \mathrm{K}^{**} = \dfrac{p\,q\,\hat{l}\,\left(s_0 s_2 \mathrm{H}\right)^{-1}}{1-\exp\left(\omega\langle s_3\rangle\right)}, \\ \mathrm{K}^{*} = \dfrac{\left|\tilde{\delta}_3\tilde{\delta}_4\right| b_{\inf}\exp\left(\omega\langle s_3\rangle\right)}{\left(1-\exp\left(-c\right)\right)^{-1}}, \\ \mathrm{K} = \left(\dfrac{\mathrm{R}^*_{\sup}}{\mathrm{R}^*_{\inf}}\right)\left(\dfrac{p\,q\,\hat{l}\,\left|\tilde{\delta}_3\tilde{\delta}_4\right| b_{\inf}\exp\left(\omega\, c\right)}{s_0 s_2\left(1-\exp\left(\omega\langle s_3\rangle\right)\right)\mathrm{H}}\right), \\ \mathrm{Q}^*(t) = \dfrac{\displaystyle\int_0^{\omega} b(t-\eta)\exp\left(\int_t^{t-\eta}\left(c+m(\tau)\right)d\tau\right)\left(\overline{\mathrm{M}}_{\mathrm{S}}(t-\eta)\right)d\eta}{1-\exp\left(-\omega\left\langle c+m\right\rangle\right)}, \\ \mu^2(0) = \left|\tilde{\delta}_3\tilde{\delta}_4\right| \displaystyle\sup_{t\in[0\,,\,\omega]}\mathrm{R}^*(t), \quad \overline{\mathrm{R}}_0^{\sup} = \mathrm{K}\sup_{t\in[0\,,\,\omega]}\mathrm{Q}^*(t), \quad \overline{\mathrm{R}}_0^{\inf} = \mathrm{K}\inf_{t\in[0\,,\,\omega]}\mathrm{Q}^*(t) \end{cases}$$

will be utilized in the estimation, first of an interval that contains $\mathrm{R}_0^{\sup}$ and $\mathrm{R}_0^{\inf}$, second of that new bound for R_0. Unequalizing (4.32) and substituting (4.15) and (5.1), one finds:

$$\begin{aligned}
&\inf_{t\in[0,\,\omega]} \mathrm{R}_0^*(t) \\
&= \inf_{0\le t\le\omega}\int_0^{\omega}\left(\frac{\dfrac{\exp\left(s_0 s\right)}{1-\exp\left(\omega s_0\right)}-\dfrac{\exp\left(s_2 s\right)}{1-\exp\left(\omega s_2\right)}}{\left(s_0-s_2\right)\left(\hat{l}\,q\,b(t-s)\right)^{-1}}\right)ds\ \inf_{0\le t\le\omega}\int_0^{\omega}\left(\frac{\dfrac{\exp\left(\int_{t-s}^{t}s_1(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_1\rangle\right)}-\dfrac{\exp\left(\int_{t-s}^{t}s_3(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_3\rangle\right)}}{\left(s_1-s_3\right)\left(c\,p\,b(t-s)\overline{\mathrm{M}}_{\mathrm{S}}(t-s)/\mathrm{H}\right)^{-1}}\right)ds \\
&= \inf_{0\le t\le\omega}\int_{t-\omega}^{t}\left(\frac{\dfrac{\exp\left(s_0(t-s)\right)}{1-\exp\left(\omega s_0\right)}-\dfrac{\exp\left(s_2(t-s)\right)}{1-\exp\left(\omega s_2\right)}}{\left(s_0-s_2\right)\left(\hat{l}\,q\,b(s)\right)^{-1}}\right)ds\ \inf_{0\le t\le\omega}\int_0^{\omega}\left(\frac{\dfrac{\exp\left(\int_{t-s}^{t}s_1(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_1\rangle\right)}-\dfrac{\exp\left(\int_{t-s}^{t}s_3(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_3\rangle\right)}}{-c\left(c\,p\,b(t-s)\overline{\mathrm{M}}_{\mathrm{S}}(t-s)/\mathrm{H}\right)^{-1}}\right)ds \\
&\le \inf_{0\le t\le\omega}\int_{t-\omega}^{t}\left(\frac{\dfrac{\exp\left(s_0(t-s)\right)}{1-\exp\left(\omega s_0\right)}-\dfrac{\exp\left(s_2(t-s)\right)}{1-\exp\left(\omega s_2\right)}}{\left(s_0-s_2\right)\left(\hat{l}\,q\,b_{\sup}\right)^{-1}}\right)ds\ \inf_{0\le t\le\omega}\int_0^{\omega}\left(\frac{\dfrac{\exp\left(\int_{t-s}^{t}s_3(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_3\rangle\right)}-\dfrac{\exp\left(\int_{t-s}^{t}s_1(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_1\rangle\right)}}{\left(p\,b(t-s)\overline{\mathrm{M}}_{\mathrm{S}}(t-s)/\mathrm{H}\right)^{-1}}\right)ds \\
&\le \inf_{0\le t\le\omega}\left(\frac{\dfrac{\exp\left(s_0(t-s)\right)\Big|_{s=t-\omega}^{s=t}}{s_0\left(1-\exp(\omega s_0)\right)}-\dfrac{\exp\left(s_2(t-s)\right)\Big|_{s=t-\omega}^{s=t}}{s_2\left(1-\exp(\omega s_2)\right)}}{\left(s_0-s_2\right)\left(l\,q\,b_{\sup}\right)^{-1}}\right)\inf_{0\le t\le\omega}\int_0^{\omega}\left(\frac{\dfrac{\exp\left(\int_{t-s}^{t}s_3(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_3\rangle\right)}-\dfrac{\exp\left(\int_{t-s}^{t}s_1(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_1\rangle\right)}}{\left(p\,b(t-s)\overline{\mathrm{M}}_{\mathrm{S}}(t-s)/\mathrm{H}\right)^{-1}}\right)ds \\
&= \left(\frac{\dfrac{1}{-s_0}-\dfrac{1}{-s_2}}{\left(s_0-s_2\right)\left(\hat{l}\,q\,b_{\sup}\right)^{-1}}\right)\inf_{0\le t\le\omega}\int_0^{\omega}\left(\frac{\dfrac{\exp\left(\int_{t-s}^{t}s_3(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_3\rangle\right)}-\dfrac{\exp\left(\int_{t-s}^{t}s_1(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_1\rangle\right)}}{\left(p\,b(t-s)\overline{\mathrm{M}}_{\mathrm{S}}(t-s)/\mathrm{H}\right)^{-1}}\right)ds
\end{aligned}$$

$$= \inf_{0\le t\le\omega}\int_0^{\omega}\left(\frac{\frac{\hat{l}\,q\,b_{\sup}}{s_0s_2}\exp\left(\int_{t-s}^{t}s_1(\tau)d\tau\right)}{\left(1-\exp\left(\omega\langle s_1\rangle\right)\right)\left(\left(p\,b(t-s)\overline{\mathrm{M}}_{\mathrm{S}}(t-s)/\mathrm{H}\right)^{-1}\right)}\left(\frac{\exp\left(\int_{t-s}^{t}\left(s_3(\tau)-s_1(\tau)\right)d\tau\right)\left(1-\exp\left(\omega\langle s_1\rangle\right)\right)}{1-\exp\left(\omega\langle s_3\rangle\right)}-1\right)\right)ds$$

$$\le \frac{\hat{l}\,q\,b_{\sup}}{s_0s_2}\inf_{0\le t\le\omega}\int_0^{\omega}\left(\frac{\frac{\exp\left(\int_{t-s}^{t}s_1(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_1\rangle\right)}\left(\frac{\exp\left(c\,\omega\right)(1)}{1-\exp\left(\omega\langle s_3\rangle\right)}\right)}{\left(p\,b(t-s)\overline{\mathrm{M}}_{\mathrm{S}}(t-s)/\mathrm{H}\right)^{-1}}\right)ds$$

$$= b_{\sup}\exp\left(\omega\,c\right)\mathrm{K}^{**}\inf_{0\le t\le\omega}\mathrm{Q}^*(t).$$

$$\sup_{t\in[0,\,\omega]}\mathrm{R}_0^*(t) =$$

$$\sup_{0\le t\le\omega}\int_0^{\omega}\left(\frac{\frac{\exp\left(s_0s\right)}{1-\exp\left(\omega s_0\right)}-\frac{\exp\left(s_2s\right)}{1-\exp\left(\omega s_2\right)}}{\left(s_0-s_2\right)\left(\hat{l}\,q\,b(t-s)\right)^{-1}}\right)ds\ \sup_{0\le t\le\omega}\int_0^{\omega}\left(\frac{\frac{\exp\left(\int_{t-s}^{t}s_1(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_1\rangle\right)}-\frac{\exp\left(\int_{t-s}^{t}s_3(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_3\rangle\right)}}{\left(s_1-s_3\right)\left(c\,p\,b(t-s)\overline{\mathrm{M}}_{\mathrm{S}}(t-s)/\mathrm{H}\right)^{-1}}\right)ds$$

$$= \sup_{0\le t\le\omega}\int_{t-\omega}^{t}\left(\frac{\frac{\exp\left(s_0(t-s)\right)}{1-\exp\left(\omega s_0\right)}-\frac{\exp\left(s_2(t-s)\right)}{1-\exp\left(\omega s_2\right)}}{\left(s_0-s_2\right)\left(\hat{l}\,q\,b(s)\right)^{-1}}\right)ds\ \sup_{0\le t\le\omega}\int_0^{\omega}\left(\frac{\frac{\exp\left(\int_{t-s}^{t}s_1(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_1\rangle\right)}-\frac{\exp\left(\int_{t-s}^{t}s_3(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_3\rangle\right)}}{-c\left(c\,p\,b(t-s)\overline{\mathrm{M}}_{\mathrm{S}}(t-s)/\mathrm{H}\right)^{-1}}\right)ds$$

$$\ge \sup_{0\le t\le\omega}\int_{t-\omega}^{t}\left(\frac{\frac{\exp\left(s_0(t-s)\right)}{1-\exp\left(\omega s_0\right)}-\frac{\exp\left(s_2(t-s)\right)}{1-\exp\left(\omega s_2\right)}}{\left(s_0-s_2\right)\left(\hat{l}\,q\,b_{\inf}\right)^{-1}}\right)ds\ \sup_{0\le t\le\omega}\int_0^{\omega}\left(\frac{\frac{\exp\left(\int_{t-s}^{t}s_3(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_3\rangle\right)}-\frac{\exp\left(\int_{t-s}^{t}s_1(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_1\rangle\right)}}{\left(p\,b(t-s)\overline{\mathrm{M}}_{\mathrm{S}}(t-s)/\mathrm{H}\right)^{-1}}\right)ds$$

$$\ge \sup_{0\le t\le\omega}\left(\frac{\frac{\exp\left(s_0(t-s)\right)\Big|_{s=t-\omega}^{s=t}}{s_0\left(1-\exp(\omega s_0)\right)}-\frac{\exp\left(s_2(t-s)\right)\Big|_{s=t-\omega}^{s=t}}{s_2\left(1-\exp(\omega s_2)\right)}}{\left(s_0-s_2\right)\left(\hat{l}\,q\,b_{\inf}\right)^{-1}}\right)\sup_{0\le t\le\omega}\int_0^{\omega}\left(\frac{\frac{\exp\left(\int_{t-s}^{t}s_3(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_3\rangle\right)}-\frac{\exp\left(\int_{t-s}^{t}s_1(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_1\rangle\right)}}{\left(p\,b(t-s)\overline{\mathrm{M}}_{\mathrm{S}}(t-s)/\mathrm{H}\right)^{-1}}\right)ds$$

$$= \left(\frac{\frac{1}{-s_0}-\frac{1}{-s_2}}{\left(s_0-s_2\right)\left(\hat{l}\,q\,b_{\inf}\right)^{-1}}\right)\sup_{0\le t\le\omega}\int_0^{\omega}\left(\frac{\frac{\exp\left(\int_{t-s}^{t}s_3(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_3\rangle\right)}-\frac{\exp\left(\int_{t-s}^{t}s_1(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_1\rangle\right)}}{\left(p\,b(t-s)\overline{\mathrm{M}}_{\mathrm{S}}(t-s)/\mathrm{H}\right)^{-1}}\right)ds$$

$$= \frac{\hat{l}\,q\,b_{\inf}}{s_0s_2}\sup_{0\le t\le\omega}\int_0^{\omega}\frac{\frac{\exp\left(\int_{t-s}^{t}s_1(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_1\rangle\right)}\left(\frac{\exp\left(\int_{t-s}^{t}\left(s_3(\tau)-s_1(\tau)\right)d\tau\right)\left(1-\exp\left(\omega\langle s_1\rangle\right)\right)}{1-\exp\left(\omega\langle s_3\rangle\right)}-1\right)}{\left(p\,b(t-s)\overline{\mathrm{M}}_{\mathrm{S}}(t-s)/\mathrm{H}\right)^{-1}}ds$$

$$\geq \frac{\hat{l}\, q\, b_{\inf}}{s_0 s_2} \sup_{0\leq t\leq \omega} \int_0^\omega \frac{\frac{\exp\left(\int_{t-s}^{t} s_1(\tau)d\tau\right)}{1-\exp\left(\omega\langle s_1\rangle\right)} \left(\frac{(1)\left(1-\exp\left(-c+\omega\langle s_3\rangle\right)\right)-\left(1-\exp\left(\omega\langle s_3\rangle\right)\right)}{1-\exp\left(\omega\langle s_3\rangle\right)}\right)}{\left(p\, b(t-s)\overline{\mathrm{M}}_{\mathrm{S}}(t-s)/\mathrm{H}\right)^{-1}} ds$$

$$= b_{\inf} \exp\left(\omega\langle s_3\rangle\right)\left(1-\exp\left(-c\right)\right) \mathrm{K}^{**} \sup_{0\leq t\leq \omega} \mathrm{Q}^*(t).$$

Consequently,

$$\inf_{t\in[0,\,\omega]} \mathrm{R}^*(t) \leq \frac{b_{\sup}\mathrm{K}^{**} \inf_{t\in[0,\,\omega]} \mathrm{Q}^*(t)}{\exp\left(-\omega\, c\right)} \wedge \frac{b_{\inf}\left(1-\exp\left(-c\right)\right)\mathrm{K}^{**} \sup_{0\leq t\leq\omega} \mathrm{Q}^*(t)}{\exp\left(-\omega\langle s_3\rangle\right)} \leq \sup_{t\in[0,\,\omega]} \mathrm{R}^*(t)$$

$$\Leftrightarrow \frac{\mathrm{R}^*_{\inf}}{b_{\sup}\mathrm{K}^{**}\exp\left(\omega\, c\right)} \leq \inf_{t\in[0,\,\omega]} \mathrm{Q}^*(t) \leq \sup_{t\in[0,\,\omega]} \mathrm{Q}^*(t) \leq \frac{\mathrm{R}^*_{\sup}\exp\left(-\omega\langle s_3\rangle\right)}{b_{\inf}\mathrm{K}^{**}\left(1-\exp\left(-c\right)\right)}$$

$$\Leftrightarrow \frac{\mathrm{R}^*_{\inf}}{b_{\sup}\mathrm{K}^{**}\exp\left(\omega\, c\right)} \leq \inf_{t\in[0,\,\omega]} \mathrm{Q}^*(t) \leq \sup_{t\in[0,\,\omega]} \mathrm{Q}^*(t) \leq \frac{\left|\tilde{\delta}_3\tilde{\delta}_4\right|\mathrm{R}^*_{\sup}\exp\left(-\omega\langle s_3\rangle\right)}{\left|\tilde{\delta}_3\tilde{\delta}_4\right| b_{\inf}\mathrm{K}^{**}\left(1-\exp\left(-c\right)\right)}$$

$$\Leftrightarrow \frac{\mathrm{R}^*_{\inf}}{b_{\sup}\mathrm{K}^{**}\exp\left(\omega\, c\right)} \leq \inf_{t\in[0,\,\omega]} \mathrm{Q}^*(t) \leq \sup_{t\in[0,\,\omega]} \mathrm{Q}^*(t) \leq \frac{\mu^2(0)\exp\left(-\omega\langle s_3\rangle\right)}{\left|\tilde{\delta}_3\tilde{\delta}_4\right| b_{\inf}\mathrm{K}^{**}\left(1-\exp\left(-c\right)\right)}$$

$$\Leftrightarrow \frac{\mathrm{K}^*\mathrm{R}^*_{\inf}}{b_{\sup}\exp\left(\omega\, c\right)} \leq \mathrm{K}^* \inf_{t\in[0,\,\omega]} \mathrm{Q}^*(t) \leq \mathrm{K}^* \sup_{t\in[0,\,\omega]} \mathrm{Q}^*(t) \leq \mu^2(0)$$

$$\Leftrightarrow \mu^2(0) \leq \left(\frac{\mu^2(0) b_{\sup}}{\mathrm{K}^*\mathrm{R}^*_{\inf}\exp\left(-\omega\, c\right)}\right)\mathrm{K}^*\mathrm{Q}^*_{\inf} \leq \left(\frac{\mu^2(0) b_{\sup}}{\mathrm{K}^*\mathrm{R}^*_{\inf}\exp\left(-\omega\, c\right)}\right)\mathrm{K}^*\mathrm{Q}^*_{\sup} \leq \frac{\mu^4(0) b_{\sup}}{\mathrm{K}^*\mathrm{R}^*_{\inf}\exp\left(-\omega\, c\right)}$$

$$\Leftrightarrow \mu^2(0) \leq \mathrm{K} \inf_{t\in[0,\,\omega]} \mathrm{Q}^*(t) \leq \mathrm{K} \sup_{t\in[0,\,\omega]} \mathrm{Q}^*(t) \leq \mu^4(0)/\left(\mathrm{K}^*\mathrm{R}^*_{\inf}\exp\left(-\omega\, c\right)\right)$$

$$\Leftrightarrow \mu(0) \leq \sqrt{\mathrm{R}_0^{\inf}} \leq \sqrt{\mathrm{R}_0^{\sup}} \leq \mu^2(0)/\sqrt{\mathrm{K}^*\mathrm{R}^*_{\inf}\exp\left(-\omega\, c\right)}.$$

D. Supplement of Proposition 5.10

D.1. Outline of the proof.

- Label and group the constants at which inequalities (5.13)-(5.20) are evaluated, symbolically:

$$\varsigma_j = \left(x^\star(t_0+\tau_0)\,,\, \alpha_j\,,\, s^\star\,,\, t_0^\star\right) \quad (j=0,\,1),$$

where

$$\varsigma_j \in \Big\{\left(\mathrm{H}_{\mathrm{E}}(t_0+\tau_0)\,,\, \overline{\mathrm{E}}_1^*\,,\, -(\hat{l}+h)\,,\, s_2^*-\overline{T}_4\right),\, \left(\mathrm{M}_{\mathrm{E}}(t_0+\tau_0)\,,\, \mathrm{E}_1^*\,,\, -(c+m^u)\,,\, s_2^*-T_4\right),$$
$$\left(\mathrm{H}_{\mathrm{I}}(t_0+\tau_0)\,,\, (\beta_{\mathrm{H}})^{1-j}(\theta_0)^j\,,\, -(h+r)\,,\, s_2^*\right),\, \left(\mathrm{M}_{\mathrm{I}}(t_0+\tau_0)\,,\, (\beta_{\mathrm{M}})^{1-j}(\theta_1)^j\,,\, -m^u\,,\, s_2^*\right),$$
$$\left(\mathrm{H}_{\mathrm{I}}(t_0+\tau_0)\,,\, \mathrm{H}_{\mathrm{I}}(t_0)\,,\, -(h+r)\,,\, s_2^*\right),\, \left(\mathrm{M}_{\mathrm{I}}(t_0+\tau_0)\,,\, \mathrm{M}_{\mathrm{I}}(t_0)\,,\, -m^u\,,\, s_2^*\right)\Big\}.$$

- Originate relations (D.1) utilizing interval (5.10), definition of ς_j and inequalities (5.5)–(5.6). Two things about the interval (5.10) are highlighted:

the candidates for minimum (except $1/\overline{s}_2$ and α^*) in the upper extreme contain an exponential factor whose argument is negative; $\beta_j < \alpha^* < \beta_{\mathrm{H}}$ and $\beta_j < \alpha^* < \beta_{\mathrm{M}}$. Therefore, $\alpha^*\beta_j$ will be mayorized after removing $k\omega$ or $\exp(s^\star t_0^\star)$ from $\exp(s^\star t_0^\star)$.

$$1 < \min\left\{\frac{\beta_{\mathrm{H}}}{\beta_j}\,,\,\frac{\beta_{\mathrm{M}}}{\beta_j}\right\} = \frac{\alpha^*}{\beta_j}\,,\quad \beta_j^{*2} := \alpha^*\beta_j < \alpha_j \in \left\{\beta_{\mathrm{H}}\left(\frac{\theta_0}{\beta_{\mathrm{H}}}\right)^j,\, \beta_{\mathrm{M}}\left(\frac{\theta_1}{\beta_{\mathrm{M}}}\right)^j,\, \overline{\mathrm{E}}_1^*\,,\, \mathrm{E}_1^*\,,\, \mathrm{H}_{\mathrm{E}}(t_0)\,,\, \mathrm{M}_{\mathrm{E}}(t_0)\right\} \tag{D.1}$$

- Evaluated at tuple ς_j:

$$\begin{aligned}
&\frac{\alpha^*}{\beta_j}\exp\big(\lambda_1(t_0^\star + k\omega)\big) > 1\\
&\quad\equiv\ \beta_j^{\star 2}\exp\big(\lambda_1(t_0^\star + k\omega)\big) > \beta_j^2\\
&\quad\equiv\ \big(\beta_j^\star\big)^{s^\star/\lambda_1}\exp\big(s^\star(t_0^\star + k\omega)/2\big) > (\beta_j)^{s^\star/\lambda_1}\\
&\quad\equiv\ \big(\beta_j^\star\big)^{1-(\lambda_1 - s^\star)/\lambda_1}\exp\big(s^\star(t_0^\star + k\omega)/2\big) > (\beta_j)^{1-(\lambda_1 - s^\star)/\lambda_1}\\
&\quad\equiv\ \beta_j^\star > (\beta_j)^{1-(\lambda_1 - s^\star)/\lambda_1}\big(\beta_j^\star\big)^{(\lambda_1 - s^\star)/\lambda_1}\exp\big(-s^\star(t_0^\star + k\omega)/2\big)\\
&\quad\equiv\ \beta_j^\star > \beta_j\exp\big(-s^\star(t_0^\star + k\omega)/2\big)\left(\frac{\beta_j^\star}{\beta_j}\right)^{(\lambda_1 - s^\star)/\lambda_1}
\end{aligned}\tag{D.2}$$

- Since $\big[\mathrm{H_I}(t)\ \ \mathrm{M_I}(t)\big] \le \big[\beta_{\mathrm{H}}\ \ \beta_{\mathrm{M}}\big]$ indefinitely continuously was assumed, τ_0 must satisfy the condition:

$$0 < \tau_0 \le k\omega + \frac{1}{\lambda_1}\min\left\{\ln\left(\frac{\beta_{\mathrm{H}}}{\beta_j}\right)\,,\,\ln\left(\frac{\beta_{\mathrm{M}}}{\beta_j}\right)\right\} = k\omega + \frac{1}{\lambda_1}\ln\left(\frac{\alpha^*}{\beta_j}\right).$$

Notice that τ_0, if it exists, must be in a neighborhood of $k\omega$. This same condition has other representations:

$$\begin{aligned}
&\tau_0 \le k\omega + \frac{1}{\lambda_1}\ln\left(\frac{\alpha^*}{\beta_j}\right)\\
&\quad\equiv \min\{\beta_{\mathrm{H}}\,,\,\beta_{\mathrm{M}}\} \ge \beta_j\exp\big(\lambda_1(\tau_0 - k\omega)\big)\\
&\quad\equiv (\lambda_1 - s^\star)\,\tau_0 \le (\lambda_1 - s^\star)\left(k\omega + \ln\left(\frac{\alpha^*}{\beta_j}\right)^{1/\lambda_1}\right)\\
&\quad\equiv \lambda_1\tau_0 - \lambda_1 k\omega \le s^\star\tau_0 - s^\star k\omega + \ln\left(\left(\frac{\alpha^*}{\beta_j}\right)^{(\lambda_1 - s^\star)/\lambda_1}\right)\\
&\quad\equiv \exp\left(\lambda_1\left(\tau_0 - k\omega\right)\right) \le \exp\left(s^\star\left(\tau_0 - k\omega\right)\right)\left(\frac{\alpha^*}{\beta_j}\right)^{(\lambda_1 - s^\star)/\lambda_1}
\end{aligned}$$

$$\therefore\quad \tau_0 \le k\omega + \frac{1}{\lambda_1}\ln\left(\frac{\alpha^*}{\beta_j}\right) \equiv \exp\left(\lambda_1\left(\tau_0 - k\omega\right)/2\right) \le \exp\left(s^\star\left(\tau_0 - k\omega\right)/2\right)\left(\sqrt{\frac{\alpha^*}{\beta_j}}\right)^{(\lambda_1 - s^\star)/\lambda_1} \tag{D.3}$$

- The inequalities (5.5)-(5.8) in coordination with the homologues (D.1)-(D.3) result in:

$$
\begin{aligned}
& x^\star(t_0+\tau_0) \geq \alpha_j \exp\big(s^\star(t_0^\star+\tau_0)\big) \\
& \quad \Longrightarrow\ x^\star(t_0+\tau_0) > \beta_j^\star \exp\big(s^\star(t_0^\star+\tau_0)\big) \\
& \quad \Longrightarrow\ x^\star(t_0+\tau_0) > \beta_j^\star \sqrt{\exp\big(s^\star(t_0^\star+\tau_0)\big)} \\
& \quad \equiv\ x^\star(t_0+\tau_0) > \frac{\beta_j \exp\big(s^\star(t_0^\star+\tau_0)/2\big)}{\exp\big(s^\star(t_0^\star+k\omega)/2\big)} \left(\frac{\beta_j^\star}{\beta_j}\right)^{(\lambda_1-s^\star)/\lambda_1} \\
& \quad \equiv\ x^\star(t_0+\tau_0) > \beta_j\ \exp\big(s^\star(\tau_0-k\omega)/2\big) \left(\frac{\beta_j^\star}{\beta_j}\right)^{(\lambda_1-s^\star)/\lambda_1} \\
& \quad \equiv\ x^\star(t_0+\tau_0) > \beta_j\ \exp\big(s^\star(\tau_0-k\omega)/2\big) \left(\frac{\sqrt{\alpha^*\beta_j}}{\beta_j}\right)^{(\lambda_1-s^\star)/\lambda_1} \\
& \quad \Longrightarrow\ x^\star(t_0+\tau_0) > \beta_j\ \exp\big(s^\star(\tau_0-k\omega)/2\big) \left(\sqrt{\frac{\alpha^*}{\beta_j}}\right)^{(\lambda_1-s^\star)/\lambda_1} \\
& \quad \Longrightarrow\ x^\star(t_0+\tau_0) > \beta_j \exp\left(\lambda_1(\tau_0-k\omega)/2\right).
\end{aligned}
$$

D.2. Deduction of the inequality (5.19).

$$
\begin{aligned}
& \mathrm{M_E}(t_0+\tau_0) \\
& = \exp\left(-\int_{t_0}^{t_0+\tau_0}(c+m(\tau))d\tau\right)\left(\mathrm{M_E}(t_0)+\frac{p}{\mathrm{H}}\int_{t_0}^{t_0+\tau_0} b(\zeta)\exp\left(\int_{t_0}^{\zeta}(c+m(\tau))d\tau\right)\mathrm{H_I}(\zeta)\mathrm{M_S}(\zeta)d\zeta\right) \\
& \geq \mathrm{M_E}(t_0)\exp\left(-\tau_0(c+m^u)\right)+\frac{p}{\mathrm{H}}\int_{t_0}^{t_0+\tau_0} b(\zeta)\exp\left(\int_{t_0+\tau_0}^{\zeta}(c+m(\tau))d\tau\right)\mathrm{H_I}(\zeta)\mathrm{M_S}(\zeta)d\zeta \\
& > \beta_0\left(\frac{p}{\mathrm{H}}\right)\int_{t_0}^{t_0+\tau_0} b(\zeta)\left(\overline{\mathrm{M}}_\mathrm{S}(\zeta)-\frac{(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)-1)(1-\bar\theta)}{2(\mathrm{N}^u+1)\big(\mathrm{K}_2\overline{\mathrm{R}}_0^{\sup}(\lambda_1,0)\big)}\right)\exp\left(\int_{t_0+\tau_0}^{\zeta}(c+m(\tau))d\tau+\lambda_1(\zeta-t_0-k\omega)\right)d\zeta \\
& \qquad \because \text{Inequalities (5.4a), (5.4c) and (5.17); } t_0\leq\zeta\leq t_0+\tau_0 \equiv 0\leq \zeta-t_0\leq\tau_0 \equiv \exp(\zeta-t_0)\leq\exp(\tau_0) \\
& > \beta_0\left(\frac{p}{\mathrm{H}}\right)\exp\left(\lambda_1(\tau_0-k\omega)\right)\int_{t_0}^{t_0+\tau_0} b(\zeta)\left(\overline{\mathrm{M}}_\mathrm{S}(\zeta)-\frac{(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)-1)(1-\bar\theta)}{2(\mathrm{N}^u+1)\big(\mathrm{K}_2\overline{\mathrm{R}}_0^{\sup}(\lambda_1,0)\big)}\right)\exp\left(\int_{t_0+\tau_0}^{\zeta}(c+\lambda_1+m(\tau))d\tau\right)d\zeta \\
& > \int_0^{\tau_0}\frac{p\beta_0 b(t_0+\tau_0-\eta)}{\mathrm{H}\exp(-\lambda_1(\tau_0-k\omega))}\left(\overline{\mathrm{M}}_\mathrm{S}(t_0+\tau_0-\eta)-\frac{(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)-1)(1-\bar\theta)}{2(\mathrm{N}^u+1)\big(\mathrm{K}_2\overline{\mathrm{R}}_0^{\sup}(\lambda_1,0)\big)}\right)\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta}(c+\lambda_1+m(\tau))d\tau\right)d\eta \\
& \qquad \because \text{Change of variable: } \zeta=t_0+\tau_0-\eta \\
& > \beta_0\left(\frac{p}{\mathrm{H}}\right)\exp\left(\lambda_1(\tau_0-k\omega)\right)\int_0^{k\omega} b(t_0+\tau_0-\eta)\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta}(c+\lambda_1+m(\tau))d\tau\right)\overline{\mathrm{M}}_\mathrm{S}(t_0+\tau_0-\eta)d\eta \\
& \quad -\left(\frac{\beta_0 p\exp\left(\lambda_1(\tau_0-k\omega)\right)(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)-1)(1-\bar\theta)}{2(\mathrm{N}^u+1)\mathrm{H}\mathrm{K}_2\overline{\mathrm{R}}_0^{\sup}(\lambda_1,0)}\right)\int_0^{n_2\omega}\frac{\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta}(c+\lambda_1+m(\tau))d\tau\right)}{(b(t_0+\tau_0-\eta))^{-1}}d\eta \\
& \qquad \because \tau_0\geq k\omega \wedge n_2\omega\geq t_0+\tau_0\geq\tau_0 \\
& = \beta_0\,(p/\mathrm{H})\exp\left(\lambda_1(\tau_0-k\omega)\right)\sum_{j=0}^{k-1}\int_0^{\omega}\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-(\eta+j\omega)}(c+\lambda_1+m(\tau))d\tau\right)b(t_0+\tau_0-\eta)\overline{\mathrm{M}}_\mathrm{S}(t_0+\tau_0-\eta)d\eta \\
& \quad -\left(\frac{\beta_0 p\exp\left(\lambda_1(\tau_0-k\omega)\right)(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)-1)(1-\bar\theta)}{2(\mathrm{N}^u+1)\mathrm{H}\mathrm{K}_2\overline{\mathrm{R}}_0^{\sup}(\lambda_1,0)}\right)\sum_{j=0}^{n_2-1}\int_0^{\omega}\frac{\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-(\eta+j\omega)}(c+\lambda_1+m(\tau))d\tau\right)}{(b(t_0+\tau_0-\eta))^{-1}}d\eta
\end{aligned}
$$

$$= \int_0^\omega \frac{\left(\sum_{j=0}^{k-1} \exp\left(\int_{t_0+\tau_0-\eta}^{t_0+\tau_0-\eta-j\omega} (c+\lambda_1+m(\tau))d\tau\right)\right) \exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta} (c+\lambda_1+m(\tau))d\tau\right)}{\left(\beta_0\,(p/\mathrm{H})\exp\left(\lambda_1(\tau_0-k\omega)\right)\right)^{-1}\left(b(t_0+\tau_0-\eta)\overline{\mathrm{M}}_{\mathrm{S}}(t_0+\tau_0-\eta)\right)^{-1}} d\eta$$

$$- \int_0^\omega \frac{\sum_{j=0}^{n_2-1} \exp\left(\int_{t_0+\tau_0-\eta}^{t_0+\tau_0-\eta-j\omega} (c+\lambda_1+m(\tau))d\tau\right) b(t_0+\tau_0-\eta)\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta} (c+\lambda_1+m(\tau))d\tau\right)}{2(\mathrm{N}^u+1)\mathrm{HK}_2\overline{\mathrm{R}}_0^{\sup}(\lambda_1,0)\left(\beta_0 p\exp\left(\lambda_1(\tau_0-k\omega)\right)(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)-1)(1-\overline{\theta})\right)^{-1}} d\eta$$

$$= \int_0^\omega \frac{\left(\sum_{j=0}^{k-1} \exp\left(-j\omega\left\langle c+\lambda_1+m\right\rangle\right)\right) \exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta} (c+\lambda_1+m(\tau))d\tau\right)}{\left(\beta_0\,(p/\mathrm{H})\exp\left(\lambda_1(\tau_0-k\omega)\right)\right)^{-1}\left(b(t_0+\tau_0-\eta)\overline{\mathrm{M}}_{\mathrm{S}}(t_0+\tau_0-\eta)\right)^{-1}} d\eta$$

$$- \int_0^\omega \frac{\sum_{j=0}^{n_2-1} \exp\left(-j\omega\left\langle c+\lambda_1+m\right\rangle\right) b(t_0+\tau_0-\eta)\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta} (c+\lambda_1+m(\tau))d\tau\right)}{2(\mathrm{N}^u+1)\mathrm{HK}_2\overline{\mathrm{R}}_0^{\sup}(\lambda_1,0)\left(\beta_0 p\exp\left(\lambda_1(\tau_0-k\omega)\right)(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)-1)(1-\overline{\theta})\right)^{-1}} d\eta \quad \because \text{Lemma 4.1}$$

$$= \frac{\sum_{j=0}^{k-1}\exp\left(-j\omega\left\langle c+\lambda_1+m\right\rangle\right)}{\left(b(t_0+\tau_0-\eta)\overline{\mathrm{M}}_{\mathrm{S}}(t_0+\tau_0-\eta)\right)^{-1}} \inf_{k\omega\le t_0+\tau_0\le n_2\omega} \int_0^\omega \frac{\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta} (c+\lambda_1+m(\tau))d\tau\right)}{\left(b(t_0+\tau_0-\eta)\overline{\mathrm{M}}_{\mathrm{S}}(t_0+\tau_0-\eta)\right)^{-1}} d\eta$$

$$- \frac{\frac{1-\exp\left(-n_2\omega\left\langle c+\lambda_1+m\right\rangle\right)}{1-\exp\left(-\omega\left\langle c+\lambda_1+m\right\rangle\right)} \sup_{k\omega\le t_0+\tau_0\le n_2\omega}\int_0^\omega b(t_0+\tau_0-\eta)\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta} (c+\lambda_1+m(\tau))d\tau\right)}{2(\mathrm{N}^u+1)\mathrm{HK}_2\overline{\mathrm{R}}_0^{\sup}(\lambda_1,0)\left(\beta_0 p\exp\left(\lambda_1(\tau_0-k\omega)\right)(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)-1)(1-\overline{\theta})\right)^{-1}} d\eta$$

$\because$ Observation 5.2 and sum of the finite geometric series

$$> \frac{\sum_{j=0}^{k-1}\exp\left(-j\omega\left\langle c+\lambda_1+m\right\rangle\right)}{\left(b(t_0+\tau_0-\eta)\overline{\mathrm{M}}_{\mathrm{S}}(t_0+\tau_0-\eta)\right)^{-1}} \inf_{k\omega\le t_0+\tau_0\le n_2\omega} \int_0^\omega \frac{\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta} (c+\lambda_1+m(\tau))d\tau\right)}{\left(b(t_0+\tau_0-\eta)\overline{\mathrm{M}}_{\mathrm{S}}(t_0+\tau_0-\eta)\right)^{-1}} d\eta$$

$$- \frac{\left(1-\exp\left(-\omega\left\langle c+\lambda_1+m\right\rangle\right)\right)^{-1} \sup_{k\omega\le t_0+\tau_0\le n_2\omega}\int_0^\omega b(t_0+\tau_0-\eta)\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta} (c+\lambda_1+m(\tau))d\tau\right)}{2(\mathrm{N}^u+1)\mathrm{HK}_2\overline{\mathrm{R}}_0^{\sup}(\lambda_1,0)\left(\beta_0 p\exp\left(\lambda_1(\tau_0-k\omega)\right)(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)-1)(1-\overline{\theta})\right)^{-1}} d\eta$$

$$> \frac{\beta_0 p\exp\left(\lambda_1(\tau_0-k\omega)\right)(1-\overline{\theta})\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)}{(\mathrm{N}^u+1)\mathrm{HK}_2} - \frac{\beta_0 p\exp\left(\lambda_1(\tau_0-k\omega)\right)(1-\overline{\theta})(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)-1)}{2(\mathrm{N}^u+1)\mathrm{HK}_2}$$

$\because$ Definition 5.1 and inequality (5.9)

$$= \frac{\beta_0 p\exp\left(\lambda_1(\tau_0-k\omega)\right)(1-\overline{\theta})\left(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)+1\right)}{2(\mathrm{N}^u+1)\mathrm{HK}_2}$$

$$> \frac{\beta_0 pq\exp\left(\lambda_1(\tau_0-k\omega)\right)(1-\overline{\theta})(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)+1)}{2(\mathrm{N}^u+1)(\mathrm{K}_2+1)\mathrm{H}}\left(\frac{\exp\left(\left((\mathrm{N}^u+1)(\mathrm{K}_2+1)\mathrm{H}/(pq)+\hat{l}+h+1\right)(k\omega-\tau_0)\right)}{\exp\left(\left((\mathrm{N}^u+1)(\mathrm{K}_2+1)\mathrm{H}/(pq)+\hat{l}+h+1\right)k\omega\right)}\right)$$

$$= \frac{\beta_0}{\overline{s}_2}\exp\left(\lambda_1^\star(\tau_0-k\omega)\right)(\lambda_1-p^*)(\lambda_1-q^*)(1-\overline{\theta})\times\frac{1}{2}\left(\frac{p^*}{\lambda_1-p^*}\right)\left(\frac{q^*}{\lambda_1-q^*}\right)\left(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)+1\right)$$

$\because$ Equalities (5.11) and Observation 5.3

$$> \beta_0^2\exp\left(\lambda_1^\star(\tau_0-k\omega)\right)(\lambda_1-p^*)(\lambda_1-q^*)(1-\overline{\theta}) \qquad \because 1 > 1/\overline{s}_2 \ge \min\{-,-,-,-,-,1/\overline{s}_2\} > \beta_0$$

D.3. Deduction of the inequality (5.20).

$$\mathrm{H_E}(t_0+\tau_0)=\exp\left(-\int_{t_0}^{t_0+\tau_0}(\hat{l}+h)d\tau\right)\left(\mathrm{H_E}(t_0)+q\int_{t_0}^{t_0+\tau_0}b(\zeta)\exp\left(\int_{t_0}^{\zeta}(\hat{l}+h)d\tau\right)\frac{\mathrm{M_I}(\zeta)\mathrm{H_S}(\zeta)}{\mathrm{H}}d\zeta\right)$$
$$\geq \exp\left(-\tau_0(\hat{l}+h)\right)\left(\mathrm{H_E}(t_0)+\frac{q}{\mathrm{H}}\int_{t_0}^{t_0+\tau_0}b(\zeta)\exp\left(-\int_{\zeta}^{t_0+\tau_0}(c+m(\tau))d\tau\right)\mathrm{M_I}(\zeta)\mathrm{H_S}(\zeta)d\zeta\right)$$

$$\because \exp\left(\int_{t_0}^{\zeta}(\hat{l}+h)d\tau\right)\geq 1\geq \exp\left(-\int_{\zeta}^{t_0+\tau_0}(c+m(\tau))d\tau\right)$$

$$>\frac{\exp\left(-\tau_0(\hat{l}+h)\right)}{\mathrm{H}/(\beta_0 q)}\int_{t_0}^{t_0+\tau_0}b(\zeta)\left(\mathrm{H}-\frac{(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)-1)(1-\bar{\theta})}{2(\mathrm{N}^u+1)\mathrm{K}_2\overline{\mathrm{R}}_0^{\sup}(\lambda_1,0)}\right)\exp\left(\int_{t_0+\tau_0}^{\zeta}(c+m(\tau))d\tau+\lambda_1(\zeta-t_0-k\omega)\right)d\zeta$$

$$+\frac{\beta_0}{\mathrm{K}_3}\exp\left(-\tau_0(\hat{l}+h)\right)$$

$\because$ Inequalities (5.4b), (5.4c) and (5.17)

$$\because t_0\leq\zeta\leq t_0+\tau_0\equiv 0\leq\zeta-t_0\leq\tau_0\equiv\exp(\zeta-t_0)\leq\exp(\tau_0)$$

$$>\frac{\beta_0 q\exp\left(-\tau_0(\hat{l}+h)\right)}{\mathrm{H}}\int_{t_0}^{t_0+\tau_0}b(\zeta)\left(\frac{\overline{\mathrm{M}}_\mathrm{S}(\zeta)}{\overline{\mathrm{M}}_\mathrm{S}(\zeta)}-\frac{(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)-1)(1-\bar{\theta})}{2(\mathrm{N}^u+1)\mathrm{K}_2\overline{\mathrm{R}}_0^{\sup}(\lambda_1,0)}\right)\exp\left(\int_{t_0+\tau_0}^{\zeta}(c+\lambda_1+m(\tau))d\tau\right)d\zeta$$

$\because\ \mathrm{H}\geq 1$

$$>\frac{\beta_0 q\exp\left(-\tau_0(\hat{l}+h)\right)}{\mathrm{H}\exp\left(-\lambda_1(\tau_0-k\omega)\right)}\int_{t_0}^{t_0+\tau_0}b(\zeta)\left(\frac{\overline{\mathrm{M}}_\mathrm{S}(\zeta)}{\mathrm{N}^u}-\frac{(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)-1)(1-\bar{\theta})}{2(\mathrm{N}^u+1)\mathrm{K}_2\overline{\mathrm{R}}_0^{\sup}(\lambda_1,0)}\right)\exp\left(\int_{t_0+\tau_0}^{\zeta}(c+\lambda_1+m(\tau))d\tau\right)d\zeta$$

$\because$ Lemma 4.3

$$>\frac{\beta_0 q\exp\left(-\tau_0(\hat{l}+h)\right)}{\mathrm{H}\exp\left(-\lambda_1(\tau_0-k\omega)\right)}\int_0^{\tau_0}\left(\frac{\overline{\mathrm{M}}_\mathrm{S}(t_0+\tau_0-\eta)}{\mathrm{N}^u}-\frac{(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)-1)(1-\bar{\theta})}{2(\mathrm{N}^u+1)\mathrm{K}_2\overline{\mathrm{R}}_0^{\sup}(\lambda_1,0)}\right)\frac{\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta}(c+\lambda_1+m(\tau))d\tau\right)}{(b(t_0+\tau_0-\eta))^{-1}}d\eta$$

$\because$ Change of variable: $\zeta=t_0+\tau_0-\eta$.

$$>\frac{\beta_0 q\exp\left(-\tau_0(\hat{l}+h)\right)}{\mathrm{N}^u\mathrm{H}\exp\left(-\lambda_1(\tau_0-k\omega)\right)}\int_0^{k\omega}b(t_0+\tau_0-\eta)\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta}(c+\lambda_1+m(\tau))d\tau\right)\overline{\mathrm{M}}_\mathrm{S}(t_0+\tau_0-\eta)d\eta$$
$$-\left(\frac{(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)-1)(1-\bar{\theta})}{2(\mathrm{N}^u+1)\mathrm{H}\mathrm{K}_2\overline{\mathrm{R}}_0^{\sup}(\lambda_1,0)}\right)\int_0^{n_2\omega}\frac{\exp\left(-\tau_0(\hat{l}+h)\right)\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta}(c+\lambda_1+m(\tau))d\tau\right)d\eta}{\exp\left(-\lambda_1(\tau_0-k\omega)\right)(\beta_0 q b(t_0+\tau_0-\eta))^{-1}}$$

$$\because \tau_0>k\omega\ \wedge\ n_2\omega\geq t_0+\tau_0>\tau_0$$

$$=\frac{\beta_0 q\exp\left(-\tau_0(\hat{l}+h)\right)}{\mathrm{N}^u\mathrm{H}\exp\left(-\lambda_1(\tau_0-k\omega)\right)}\sum_{j=0}^{k-1}\int_0^{\omega}b(t_0+\tau_0-\eta)\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-(\eta+j\omega)}(c+\lambda_1+m(\tau))d\tau\right)\overline{\mathrm{M}}_\mathrm{S}(t_0+\tau_0-\eta)d\eta$$
$$-\left(\frac{(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)-1)(1-\bar{\theta})}{2(\mathrm{N}^u+1)\mathrm{H}\mathrm{K}_2\overline{\mathrm{R}}_0^{\sup}(\lambda_1,0)}\right)\sum_{j=0}^{n_2-1}\int_0^{n_2\omega}\frac{\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-(\eta+j\omega)}(c+\lambda_1+m(\tau))d\tau\right)d\eta}{\exp\left(-\lambda_1(\tau_0-k\omega)\right)\exp\left(\tau_0\left(\hat{l}+h\right)\right)(\beta_0 q b(t_0+\tau_0-\eta))^{-1}}$$
$$=\int_0^{\omega}\frac{\left(\sum_{j=0}^{k-1}\exp\left(\int_{t_0+\tau_0-\eta}^{t_0+\tau_0-\eta-j\omega}(c+\lambda_1+m(\tau))d\tau\right)\right)\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta}(c+\lambda_1+m(\tau))d\tau\right)}{\left(\beta_0 q\exp\left(-\tau_0(\hat{l}+h)\right)\right)^{-1}\left(\mathrm{N}^u\mathrm{H}\exp\left(-\lambda_1(\tau_0-k\omega)\right)\right)\left(b(t_0+\tau_0-\eta)\overline{\mathrm{M}}_\mathrm{S}(t_0+\tau_0-\eta)\right)^{-1}}d\eta$$
$$-\int_0^{\omega}\frac{\sum_{j=0}^{n_2-1}\exp\left(\int_{t_0+\tau_0-\eta}^{t_0+\tau_0-\eta-j\omega}(c+\lambda_1+m(\tau))d\tau\right)b(t_0+\tau_0-\eta)\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta}(c+\lambda_1+m(\tau))d\tau\right)}{2(\mathrm{N}^u+1)\mathrm{H}\mathrm{K}_2\overline{\mathrm{R}}_0^{\sup}(\lambda_1,0)\exp\left(\tau_0\left(\hat{l}+h\right)\right)\left(\beta_0 q\exp\left(\lambda_1(\tau_0-k\omega)\right)(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)-1)(1-\bar{\theta})\right)^{-1}}d\eta$$

$$
\begin{aligned}
&= \int_0^{\omega} \frac{\left(\sum_{j=0}^{k-1} \exp\left(-j\omega\left\langle c+\lambda_1+m\right\rangle\right)\right) \exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta}(c+\lambda_1+m(\tau))d\tau\right)}{\left(\beta_0 q \exp\left(-\tau_0(\hat{l}+h)\right)\right)^{-1}(\mathrm{N}^u\mathrm{H}\exp\left(-\lambda_1(\tau_0-k\omega)\right))\left(b(t_0+\tau_0-\eta)\overline{\mathrm{M}}_{\mathrm{S}}(t_0+\tau_0-\eta)\right)^{-1}} d\eta \\
&- \int_0^{\omega} \frac{\sum_{j=0}^{n_2-1} \exp\left(-j\omega\left\langle c+\lambda_1+m\right\rangle\right) b(t_0+\tau_0-\eta)\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta}(c+\lambda_1+m(\tau))d\tau\right)}{2(\mathrm{N}^u+1)\mathrm{HK}_2\overline{\mathrm{R}}_0^{\sup}(\lambda_1,0)\exp\left(\tau_0\left(\hat{l}+h\right)\right)\left(\beta_0 q\exp\left(\lambda_1(\tau_0-k\omega)\right)(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)-1)(1-\overline{\theta})\right)^{-1}} d\eta \\
&\qquad \because \text{Lemma 4.1} \\
&= \frac{\exp\left(-\tau_0(\hat{l}+h)\right)\inf_{k\omega\le t_0+\tau_0\le n_2\omega}\int_0^{\omega}\frac{\overline{\mathrm{M}}_{\mathrm{S}}(t_0+\tau_0-\eta)}{(b(t_0+\tau_0-\eta))^{-1}}\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta}(c+\lambda_1+m(\tau))d\tau\right)d\eta}{\mathrm{N}^u\mathrm{H}\exp\left(-\lambda_1(\tau_0-k\omega)\right)\left(\beta_0 q\sum_{j=0}^{k-1}\exp\left(-j\omega\left\langle c+\lambda_1+m\right\rangle\right)\right)^{-1}} \\
&- \int_0^{\omega} \frac{\left(\frac{1-\exp\left(-n_2\omega\left\langle c+\lambda_1+m\right\rangle\right)}{1-\exp\left(-\omega\left\langle c+\lambda_1+m\right\rangle\right)}\right) b(t_0+\tau_0-\eta)\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta}(c+\lambda_1+m(\tau))d\tau\right)}{2(\mathrm{N}^u+1)\mathrm{HK}_2\overline{\mathrm{R}}_0^{\sup}(\lambda_1,0)\exp\left(\tau_0\left(\hat{l}+h\right)\right)\left(\beta_0 q\exp\left(\lambda_1(\tau_0-k\omega)\right)(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)-1)(1-\overline{\theta})\right)^{-1}} d\eta \\
&\qquad \because \text{ Remark 5.2 and sum of the finite geometric series} \\
&> \frac{\exp\left(-\tau_0(\hat{l}+h)\right)\inf_{k\omega\le t_0+\tau_0\le n_2\omega}\int_0^{\omega}\frac{\overline{\mathrm{M}}_{\mathrm{S}}(t_0+\tau_0-\eta)}{(b(t_0+\tau_0-\eta))^{-1}}\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta}(c+\lambda_1+m(\tau))d\tau\right)d\eta}{\mathrm{N}^u\mathrm{H}\exp\left(-\lambda_1(\tau_0-k\omega)\right)\left(\beta_0 q\sum_{j=0}^{k-1}\exp\left(-j\omega\left\langle c+\lambda_1+m\right\rangle\right)\right)^{-1}} \\
&- \int_0^{\omega} \frac{\left(1-\exp\left(-\omega\left\langle c+\lambda_1+m\right\rangle\right)\right)^{-1} b(t_0+\tau_0-\eta)\exp\left(\int_{t_0+\tau_0}^{t_0+\tau_0-\eta}(c+\lambda_1+m(\tau))d\tau\right)}{2(\mathrm{N}^u+1)\mathrm{HK}_2\overline{\mathrm{R}}_0^{\sup}(\lambda_1,0)\exp\left(\tau_0\left(\hat{l}+h\right)\right)\left(\beta_0 q\exp\left(\lambda_1(\tau_0-k\omega)\right)(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)-1)(1-\overline{\theta})\right)^{-1}} d\eta \\
&> \frac{\beta_0 q\exp\left(\lambda_1(\tau_0-k\omega)\right)(1-\overline{\theta})\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)}{(\mathrm{N}^u+1)\mathrm{HK}_2\exp\left(\tau_0(\hat{l}+h)\right)} - \frac{\beta_0 q\exp\left(\lambda_1(\tau_0-k\omega)\right)(1-\overline{\theta})\left(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)-1\right)}{2(\mathrm{N}^u+1)\mathrm{HK}_2\exp\left(\tau_0(\hat{l}+h)\right)} \\
&\qquad \because \text{ Definition 5.1 and inequality (5.9)} \\
&= \frac{\beta_0 q\exp\left(\lambda_1(\tau_0-k\omega)\right)(1-\overline{\theta})\left(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)+1\right)}{2(\mathrm{N}^u+1)\mathrm{HK}_2\exp\left(\tau_0(\hat{l}+h)\right)} \\
&> \frac{\beta_0 pq\exp\left(\lambda_1(\tau_0-k\omega)\right)(1-\overline{\theta})(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)+1)}{2(\mathrm{N}^u+1)(\mathrm{K}_2+1)\mathrm{H}}\left(\frac{\exp\left(\left((\mathrm{N}^u+1)(\mathrm{K}_2+1)\mathrm{H}/(pq)+\hat{l}+h+1\right)(k\omega-\tau_0)\right)}{\exp\left(\left((\mathrm{N}^u+1)(\mathrm{K}_2+1)\mathrm{H}/(pq)+\hat{l}+h+1\right)k\omega\right)}\right) \\
&= \frac{\beta_0}{\overline{s}_2}\exp\left(\lambda_1^{\star}(\tau_0-k\omega)\right)(\lambda_1-p^*)(\lambda_1-q^*)(1-\overline{\theta})\times\frac{1}{2}\left(\frac{p^*}{\lambda_1-p^*}\right)\left(\frac{q^*}{\lambda_1-q^*}\right)\left(\overline{\mathrm{R}}_0^{\inf}(\lambda_1,1)+1\right) \\
&\qquad \because \text{ Equalities (5.11) and Remark 5.3} \\
&> \beta_0^2\exp\left(\lambda_1^{\star}(\tau_0-k\omega)\right)(\lambda_1-p^*)(\lambda_1-q^*)(1-\overline{\theta}) \qquad \because 1 > 1/\overline{s}_2 \ge \min\{-,-,-,-,-,1/\overline{s}_2\} > \beta_0.
\end{aligned}
$$

E. Supplement of Proposition 4.15

E.1. Deduction of the inequality (4.21).

$$
\begin{aligned}
&\|E_\lambda - E_{\lambda_0}\| = \sup_{\|\boldsymbol{\varphi}\|=1}\sup_{0\le\vartheta\le t}\|E_\lambda\boldsymbol{\varphi}(\theta) - E_{\lambda_0}\boldsymbol{\varphi}(\theta)\| \\
&= \sup_{\|\boldsymbol{\varphi}\|=1}\sup_{0\le\vartheta\le t}\Bigg\|\sum_{j=0}^{1}\Bigg(\Big((1-j)\tilde{S}\boldsymbol{\varphi}(\vartheta) + j\left(\tilde{\Lambda}_t\,\boldsymbol{\varphi}(t) - \tilde{S}\boldsymbol{\varphi}(\vartheta)\right)\Big)\times \\
&\qquad\qquad \times\frac{\tilde{d}_6(\lambda,\,j)}{\exp\left(-\tilde{d}_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)} - \frac{\tilde{d}_6(\lambda_0,\,j)}{\exp\left(-\tilde{d}_1(\lambda_0)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)}\Bigg)\Bigg\| \\
&\le \sup_{\|\boldsymbol{\varphi}\|=1}\sup_{0\le\vartheta\le t}\sum_{j=0}^{1}\Bigg(\left|\frac{\tilde{d}_6(\lambda,\,j)}{\exp\left(-\tilde{d}_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)} - \frac{\tilde{d}_6(\lambda_0,\,j)}{\exp\left(-\tilde{d}_1(\lambda_0)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)}\right|\times \\
&\qquad\qquad \times\left\|(1-2j)\tilde{S}\boldsymbol{\varphi}(\vartheta) + j\tilde{\Lambda}_t\,\boldsymbol{\varphi}(t)\right\|\Bigg) \\
&\le \sup_{\|\boldsymbol{\varphi}\|=1}\sup_{0\le\vartheta\le t}\sum_{j=0}^{1}\Bigg(\exp\left(\tilde{d}_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)\left|\tilde{d}_6(\lambda,\,j) - \frac{\tilde{d}_6(\lambda_0,\,j)}{\exp\left(\left(\tilde{d}_1(\lambda)-\tilde{d}_1(\lambda_0)\right)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)}\right|\times \\
&\qquad\qquad \times\left(|2j-1|\|\tilde{S}\|\|\boldsymbol{\varphi}(\vartheta)\| + j\|\tilde{\Lambda}_t\|\|\boldsymbol{\varphi}(t)\|\right)\Bigg) \\
&\le \sup_{\|\boldsymbol{\varphi}\|=1}\sup_{0\le\vartheta\le t}\sum_{j=0}^{1}\Bigg(\exp\left(\tilde{d}_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)\left|\tilde{d}_6(\lambda,\,j) - \tilde{d}_6(\lambda_0,\,j)\right|\left(|2j-1|\|\boldsymbol{\varphi}(\vartheta)\| + j\overline{\Theta}_1\|\tilde{\Lambda}_t\|\right) \\
&\qquad + \exp\left(\tilde{d}_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)\left|\tilde{d}_6(\lambda_0,\,j) - \frac{\tilde{d}_6(\lambda_0,\,j)}{\exp\left(\left(\tilde{d}_1(\lambda)-\tilde{d}_1(\lambda_0)\right)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)}\right|\times \\
&\qquad\qquad \times\left(|2j-1|\|\boldsymbol{\varphi}(\vartheta)\| + j\overline{\Theta}_1\tilde{\Lambda}_t\right)\Bigg) \\
&\qquad\qquad \because \|\tilde{S}\| = \max_{1\le i\le 4}|\operatorname{sgn}\varphi_i(t)| \le 1. \\
&= \sup_{\|\boldsymbol{\varphi}\|=1}\sup_{0\le\vartheta\le t}\sum_{j=0}^{1}\Bigg(j\exp\left(\tilde{d}_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)\left|\tilde{d}_2(\lambda) - \tilde{d}_2(\lambda_0)\right|\left(|2j-1|\|\boldsymbol{\varphi}(\vartheta)\| + j\overline{\Theta}_1\|\tilde{\Lambda}_t\|\right) \\
&\qquad + \tilde{d}_6(\lambda_0,\,j)\exp\left(\tilde{d}_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)\left|1 - \exp\left(-\left(\tilde{d}_1(\lambda)-\tilde{d}_1(\lambda_0)\right)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)\right|\times \\
&\qquad\qquad \times\left(|2j-1|\|\boldsymbol{\varphi}(\vartheta)\| + j\overline{\Theta}_1\|\tilde{\Lambda}_t\|\right)\Bigg) \\
&\le \sup_{\|\boldsymbol{\varphi}\|=1}\sup_{0\le\vartheta\le t}\sum_{j=0}^{1}\Bigg(j\exp\left(\tilde{d}_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)|\lambda-\lambda_0|\left(|2j-1|\|\boldsymbol{\varphi}(\vartheta)\| + j\overline{\Theta}_1\|\tilde{\Lambda}_t\|\right) \\
&\qquad + \tilde{d}_6(\lambda_0,\,j)\exp\left(\tilde{d}_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\left|\tilde{d}_1(\lambda)-\tilde{d}_1(\lambda_0)\right|\left(|2j-1|\|\boldsymbol{\varphi}(\vartheta)\| + j\overline{\Theta}_1\|\tilde{\Lambda}_t\|\right)\Bigg) \\
&\qquad\qquad \because \text{ Lemma 4.14.} \\
&\le \sup_{\|\boldsymbol{\varphi}\|=1}\sup_{0\le\vartheta\le t}\sum_{j=0}^{1}\Bigg(j\exp\left(\tilde{d}_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)|\lambda-\lambda_0|\left(|2j-1|\|\boldsymbol{\varphi}(\vartheta)\| + j\overline{\Theta}_1\tilde{\Lambda}_t\right) \\
&\qquad + 2\tilde{d}_6(\lambda_0,\,j)\exp\left(\tilde{d}_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right||\lambda-\lambda_0|\left(|2j-1|\|\boldsymbol{\varphi}(\vartheta)\| + j\overline{\Theta}_1\|\tilde{\Lambda}_t\|\right)\Bigg) \\
&\qquad\qquad \because \text{ Lemma 4.14.} \\
&= |\lambda-\lambda_0|\sup_{\|\boldsymbol{\varphi}\|=1}\sum_{j=0}^{1}\Bigg(j\sup_{0\le\vartheta\le t}\exp\left(\tilde{d}_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)\left(|2j-1|\sup_{0\le\vartheta\le t}\|\boldsymbol{\varphi}(\vartheta)\| + j\overline{\Theta}_1\|\tilde{\Lambda}_t\|\right) \\
&\qquad + 2\tilde{d}_6(\lambda_0,\,j)\sup_{0\le\vartheta\le t}\exp\left(\tilde{d}_1(\lambda)\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\right)\sup_{0\le\vartheta\le t}\left|\sin\left(\frac{\pi(t-\theta)}{\omega}\right)\right|\left(|2j-1|\sup_{0\le\vartheta\le t}\|\boldsymbol{\varphi}(\vartheta)\| + j\overline{\Theta}_1\|\tilde{\Lambda}_t\|\right)\Bigg) \\
&\le |\lambda-\lambda_0|\sup_{\|\boldsymbol{\varphi}\|=1}\sum_{j=0}^{1}\Bigg(j\exp(0)\left(|2j-1|\|\boldsymbol{\varphi}\| + j\overline{\Theta}_1\|\tilde{\Lambda}_t\|\right) + 2\tilde{d}_6(\hat{\omega},\,j)\exp(0)\,(1)\left(|2j-1|\|\boldsymbol{\varphi}\| + j\overline{\Theta}_1\sup_{0\le t\le\omega}\|\tilde{\Lambda}(t)\|\right)\Bigg) \\
&\qquad\qquad \because \tilde{d}_1(\lambda)\le 0,\ 0\le\tilde{d}_2(\lambda)\le\tilde{d}_2(\hat{\omega}),\ 0\le|\sin(\pi(t-\theta)/\omega)|\le 1,\ \tilde{d}_6(\lambda,\,j) = 1-j+j\tilde{d}_2(\lambda). \\
&\le |\lambda-\lambda_0|\sum_{j=0}^{1}\left(j + 2\tilde{d}_6(\hat{\omega},\,j)\right)\left(|2j-1| + j\overline{\Theta}_1\|\tilde{\Lambda}\|\right) \\
&= \overline{\Theta}_2\,|\lambda-\lambda_0|.
\end{aligned}
$$

E.2. Deduction of the inequality 4.22.

$$
\begin{aligned}
&\|\mathcal{L}_\lambda - \mathcal{L}_{\lambda_0}\| \\
&\le \Theta_1 \int_0^\infty \left\|(1+\lambda+|\hat\omega|)^{-1} \exp\left(\tilde d_1(\lambda)s\right) E_\lambda - (1+\lambda_0+|\hat\omega|)^{-1} \exp\left(\tilde d_1(\lambda_0)s\right) E_{\lambda_0}\right\| \exp(\hat\omega s)\, ds \\
&= \frac{\Theta_1 \int_0^\infty \left\|(1+\lambda_0+|\hat\omega|)\left(\exp\left(\tilde d_1(\lambda)s\right) - \exp\left(\tilde d_1(\lambda_0)s\right)\right) E_\lambda - (\lambda-\lambda_0)\exp\left(\tilde d_1(\lambda_0)s\right)(E_\lambda - E_{\lambda_0})\right\| \exp(\hat\omega s)\, ds}{(1+\lambda_0+|\hat\omega|)(1+\lambda+|\hat\omega|)} \\
&\le \frac{\Theta_1 \|E_\lambda\| \int_0^\infty \left|\exp\left(-|\tilde d_1(\lambda)|s\right) - \exp\left(-|\tilde d_1(\lambda_0)|s\right)\right| \exp(\hat\omega s)\, ds}{(1+\lambda_0+|\hat\omega|)(1-\delta)} \\
&\quad + \frac{\Theta_1 |\lambda-\lambda_0|\, \|E_\lambda - E_{\lambda_0}\| \int_0^\infty \exp\left(\tilde d_1(\lambda_0)s\right)\exp(\hat\omega s) ds}{(1+\lambda_0+|\hat\omega|)(1-\delta)} \\
&\le \Theta_2 \int_0^\infty \left|\tilde d_1(\lambda) - \tilde d_1(\lambda_0)\right| s \exp\left(\left(\tilde d_4(\lambda)+\hat\omega\right)s\right) ds + \delta(1-\delta)^{-1}\Theta_3|\lambda-\lambda_0| \int_0^\infty \exp\left(\left(\tilde d_1(\lambda_0)+\hat\omega\right)s\right) ds \\
&\qquad\qquad \because \text{ Lemma 4.14(ii) with } \tilde d_4(\lambda) = -\min\{|\tilde d_1(\lambda)|, |\tilde d_1(\lambda_0)|\} \text{ and inequality (4.21).} \\
&\le \Theta_2 \int_0^\infty \left|\tilde d_1(\lambda) - \tilde d_1(\lambda_0)\right| s \exp\left(\left(\tilde d_4(\lambda)+\hat\omega\right)s\right) ds + \delta(1-\delta)^{-1}\Theta_3|\lambda-\lambda_0| \int_0^\infty \exp\left(\left(\tilde d_1(\lambda_0)+\hat\omega\right)s\right) ds \\
&\le 2\Theta_2|\omega|\,|\lambda-\lambda_0| \int_0^\infty s\exp\left(\left(\tilde d_4(\lambda_0-\delta)+\hat\omega\right)s\right) ds + \delta(1-\delta)^{-1}\Theta_3|\lambda-\lambda_0| \int_0^\infty \exp\left(\left(\tilde d_4(\lambda_0)+\hat\omega\right)s\right) ds. \\
&\qquad\qquad \because \min\{|\tilde d_1(\lambda)|, |\tilde d_1(\lambda_0)|\} \le |\tilde d_1(\lambda_0)| = -\tilde d_1(\lambda_0) \text{ and Lemma 4.14(i).} \\
&\qquad\qquad \because \lambda_0-\delta \le \lambda \le \lambda_0+\delta \text{ and } \lambda \mapsto \tilde d_1(\lambda) \text{ is non-increasing.} \\
&\le (1+\delta(1-\delta)^{-1})\Theta_4|\lambda-\lambda_0| \lim_{\zeta\to+\infty}\int_0^\zeta \left(s\exp\left(\left(\tilde d_1(\lambda_0-\delta)+\hat\omega\right)s\right) + \exp\left(\left(\tilde d_1(\lambda_0)+\hat\omega\right)s\right)\right) ds \\
&= \frac{\Theta_4|\lambda-\lambda_0|}{1-\delta}\lim_{\zeta\to+\infty}\left(\frac{\exp\left(\left(\tilde d_4(\lambda_0-\delta)+\hat\omega\right)s\right)\left(\left(\tilde d_4(\lambda_0-\delta)+\hat\omega\right)s-1\right)}{\left(\tilde d_4(\lambda_0-\delta)+\hat\omega\right)^2} + \frac{\exp\left(\left(\tilde d_4(\lambda_0)+\hat\omega\right)s\right)}{\left(\tilde d_4(\lambda_0)+\hat\omega\right)}\Bigg|_{s=0}^{s=\zeta}\right) \\
&= \frac{\Theta_4|\lambda-\lambda_0|}{1-\delta}\lim_{\zeta\to+\infty}\left(\frac{\exp\left(\left(\tilde d_1(\lambda_0-\delta)+\hat\omega\right)\zeta\right)\left(\left(\tilde d_1(\lambda_0-\delta)+\hat\omega\right)\zeta-1\right)+1}{\left(\tilde d_1(\lambda_0-\delta)+\hat\omega\right)^2} + \frac{\exp\left(\left(\tilde d_4(\lambda_0)+\hat\omega\right)\zeta\right)-1}{\tilde d_4(\lambda_0)+\hat\omega}\right) \\
&= \frac{\Theta_4|\lambda-\lambda_0|}{1-\delta}\left(\left(\tilde d_0(\lambda_0-\delta)+\hat\omega\right)^{-2} + \left(|\tilde d_4(\lambda_0)|+|\hat\omega|\right)^{-1}\right).
\end{aligned}
$$

F. Numerical computation of R_0

To numerically compute the BRN, the linear operator in (4.10) is rewritten into the form of equation (3) in [62], where an algorithm is proposed for the R_0 computation of periodic systems of ordinary differential equations. This numerical algorithm is described below for the operator in (4.27) when $\lambda = 1$, since $\mathcal{L} = \mathcal{L}_1$.

The interval $[0, \omega]$ is partitioned uniformly into n nodes labeled as $t_i = i \cdot \omega$ for $i = 0, \dots, n-1$. Using the trapezoidal rule [63], one of the most common numerical integration techniques, the integral in (4.27) can be approximated with second-order accuracy:

$$
(\mathcal{L}\boldsymbol{\phi})(t) \approx \frac{\omega}{n}\left(\sum_{i=1}^{n-1} U_1(t, t_i)\boldsymbol{\phi}(t-t_i) + \frac{1}{2}U_1(t, t_0)\boldsymbol{\phi}(t-t_0) + \frac{1}{2}U_1(t, t_n)\boldsymbol{\phi}(t-t_n)\right),
$$

where

$$U_1(t,s) = \begin{bmatrix} 0 & 0 & 0 & \dfrac{q\,b(t-s)\exp\left(s_0 s\right)}{1-\exp\left(\omega s_0^*\right)} \\ 0 & 0 & p\,b(t-s)\left(\dfrac{\exp\left(\int_{t-s}^{t} s_1(\tau)d\tau\right)}{1-\exp\left(\int_0^{\omega} s_1(\tau)d\tau\right)}\right)\dfrac{\overline{\mathrm{M}}_{\mathrm{S}}(t-s)}{\mathrm{H}} & 0 \\ 0 & 0 & 0 & \dfrac{\dfrac{\exp\left(s_0\right)}{1-\exp\left(\omega s_0\right)}-\dfrac{\exp\left(s_2 s\right)}{1-\exp\left(\omega s_2\right)}}{\left(s_0-s_2\right)\left(\hat{l}\,q\,b(t-s)\right)^{-1}} \\ 0 & 0 & \dfrac{\dfrac{\exp\left(\int_{t-s}^{t} s_1(\tau)d\tau\right)}{1-\exp\left(\int_0^{\omega} s_1(\tau)d\tau\right)}-\dfrac{\exp\left(\int_{t-s}^{t} s_3(\tau)d\tau\right)}{1-\exp\left(\int_0^{\omega} s_3(\tau)d\tau\right)}}{\left(s_1-s_3\right)\left(c\,p\,b(t-s)\dfrac{\overline{\mathrm{M}}_{\mathrm{S}}(t-s)}{\mathrm{H}}\right)^{-1}} & 0 \end{bmatrix}$$

Since $\boldsymbol{\phi}(t)$ is ω-periodic, it is clear that $\boldsymbol{\phi}(t,\, t_0) = \boldsymbol{\phi}(t - t_n)$. For convenience, write

$$\tilde{U}_1(t,\, t_0) = \frac{1}{2}\left(U_1(t,\, t_0) + U_1(t,\, t_n)\right).$$

Then,

$$(\mathcal{L}\boldsymbol{\phi})(t) \approx \frac{\omega}{n}\left(\tilde{U}_1(t,\, t_0)\boldsymbol{\phi}(t-t_0) + \sum_{i=1}^{n-1} U_1(t,\, t_i)\boldsymbol{\phi}(t-t_i)\right).$$

Now $(\mathcal{L}\boldsymbol{\phi})(t) = \lambda^\star\boldsymbol{\phi}$ can be written as a matrix equation:

$$\frac{\omega}{n}\begin{bmatrix}\tilde{U}_1(t,\, t_0) & U_1(t,\, t_1) & U_1(t,\, t_2) & \dots & U_1(t,\, t_{n-1})\end{bmatrix}\begin{bmatrix}\boldsymbol{\phi}(t-t_0)\\ \boldsymbol{\phi}(t-t_1)\\ \boldsymbol{\phi}(t-t_2)\\ \vdots \\ \boldsymbol{\phi}(t-t_{n-1})\end{bmatrix} = \lambda^\star\boldsymbol{\phi}(t_j).$$

Setting $t = t_j$ $(0 \le j \le n-1)$ in the above equation yields

$$\frac{\omega}{n}\begin{bmatrix}\tilde{U}_1(t_j,\, t_0) & U_1(t_j,\, t_1) & U_1(t_j,\, t_2) & \dots & U_1(t_j,\, t_{n-1})\end{bmatrix}\begin{bmatrix}\boldsymbol{\phi}(t_j-t_0)\\ \boldsymbol{\phi}(t_j-t_1)\\ \boldsymbol{\phi}(t_j-t_2)\\ \vdots \\ \boldsymbol{\phi}(t_j-t_{n-1})\end{bmatrix} = \lambda^\star\boldsymbol{\phi}(t_j) \qquad \text{(F.1)}$$

Again, by the periodicity of $\boldsymbol{\phi}(t)$, it follows that

$$\begin{array}{ccc} \boldsymbol{\phi}(t_j-t_0) = \boldsymbol{\phi}(t_j), & \boldsymbol{\phi}(t_j-t_1) = \boldsymbol{\phi}(t_{j-1}), & \dots, \\ \boldsymbol{\phi}(t_j-t_{j-1}) = \boldsymbol{\phi}(t_1), & \boldsymbol{\phi}(t_j-t_j) = \boldsymbol{\phi}(t_0), & \boldsymbol{\phi}(t_j-t_{j+1}) = \boldsymbol{\phi}(t_{n-1}), \\ \dots, & \boldsymbol{\phi}(t_j-t_{n-2}) = \boldsymbol{\phi}(t_{j+2}), & \boldsymbol{\phi}(t_j-t_{n-1}) = \boldsymbol{\phi}(t_{j+1}), \end{array}$$

and the terms in (F.1) can be rearranged to obtain

$$\frac{\omega}{n}\begin{bmatrix}U_1(t_j,t_j) & \dots & \tilde{U}_1(t_j,t_0) & U_1(t_j,t_{n-1}) & \dots & U_1(t_j,t_{j+1})\end{bmatrix}\begin{bmatrix}\boldsymbol{\phi}(t_0)\\ \boldsymbol{\phi}(t_1)\\ \vdots\\ \boldsymbol{\phi}(t_j)\\ \vdots\\ \boldsymbol{\phi}(t_{n-2})\\ \boldsymbol{\phi}(t_{n-1})\end{bmatrix} = \lambda^\star \boldsymbol{\phi}(t_j) \tag{F.2}$$

Note that this equation holds for all $j = 0, \dots, n-1$ and, hence, it generates a matrix system. The coefficient matrix is denoted $\hat{A}$ and is given by

$$\hat{A} = \left[\hat{A}_{ij} = \frac{1}{2}(1+\operatorname{sgn}(i-j))U_1(t_{i-1}, t_{i-j}) + \frac{1}{2}(1-\operatorname{sgn}(i-j))U_1(t_{i-1}, t_{n+i-j})\right]_{1\le i,j\le n} \tag{F.3}$$

The numerical method transforms the integral operator eigenvalue problem into a matrix eigenvalue problem of the form:

$$\frac{\omega}{n}\hat{A}\tilde{\boldsymbol{\phi}} = \lambda^\star \tilde{\boldsymbol{\phi}}, \tag{F.4}$$

where $\hat{A}$ defined in (F.3) is a $(4n)\times(4n)$ matrix and

$$\tilde{\boldsymbol{\phi}} = \begin{bmatrix}\boldsymbol{\phi}(t_0) & \boldsymbol{\phi}(t_1) & \dots & \boldsymbol{\phi}(t_{n-1})\end{bmatrix}^\top$$

is a $(4n)\times 1$ vector. Consequently, to compute the BRN it suffices to find the maximum $\lambda^\star$ such that (F.4) is valid, that is,

$$\mathsf{R}_0 \approx \frac{\omega}{n}\rho(\hat{A}) \tag{F.5}$$

Received: 1 February 2022/Accepted: 5 December 2022/Published online: 19 December 2022

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