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Other approaches for generalized Bernoulli–Euler polynomials and beyond

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Abstract. *In this paper we develop two approaches for studying a large family of generalized Bernoulli–Euler polynomials. For the determinental approach, using Little Fermat's Theorem, we establish a congruence identity and we give an explicit formulas of the generalized Bernoulli–Euler polynomials in terms of the Stirling numbers. The linear recursive approach allows us to formulate some properties of the generalized Bernoulli–Euler numbers and the generalized Bernoulli–Euler polynomials. Moreover, combinatorial formulas for these polynomials are provided.*

1. Introduction

The Bernoulli and Euler polynomials $B_n(x)$ and $E_n(x)$, respectively, as well as the Bernoulli and Euler numbers B_n and E_n, are widely used in various topics of mathematics such that number theory, complex analysis and approximation theory. Several results and relations concerning these polynomials and numbers are provided in the literature (see for instance, [1, 10]). In [20, Theorem 1.2], the Bernoulli polynomials $B_n(x)$ are expressed by a lower Hessenberg determinant. Srivastava et *al.* [21, 22] investigated a new generalization of the family of Bernoulli and Euler polynomials. They proved some interesting properties of their proposed general polynomials and derived explicit representations for them in terms of a certain generalized Hurwitz–Lerch Zeta function, and in terms of series involving the familiar Gaussian hypergeometric function. In [7] Carlitz has extended the classical Bernoulli and Euler polynomials and numbers. Especially, some properties; such as the recurrence relation between q-Bernoulli polynomials and q-Euler polynomials, have been proved in [8]. Choi et *al.* [11] defined and investigated the Apostol–Bernoulli polynomials and the Apostol–Euler polynomials $B_n^{(\alpha)}(x,\lambda,q)$, $E_n^{(\alpha)}(x,\lambda,q)$ of order α.

In [14] and [15] Luo et *al.* have introduced and studied the generalized Bernoulli and Euler polynomials $\mathfrak{B}_n(x;a,b,c)$, $\mathfrak{E}_n(x;a,b,c)$, which are defined through their associated generating functions as follows,

$$\frac{t}{b^t-a^t}c^{xt}=\sum_{n=0}^{\infty}\mathfrak{B}_n(x;a,b,c)\frac{t^n}{n!}, \qquad \text{such that} \qquad |t|<\frac{2\pi}{|\ln b-\ln a|}, \tag{1.1}$$

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$$\frac{2}{b^t + a^t} c^{xt} = \sum_{n=0}^{\infty} \mathfrak{E}_n(x; a, b, c) \frac{t^n}{n!}, \qquad \text{such that} \qquad |t| < \left| \frac{\pi}{\ln b - \ln a} \right|, \tag{1.2}$$

where a, b, c, with $a \neq b$, are given real numbers in $\mathbb{R}_+^*$. When $b = c = e$ and $a = 1$, Expressions (1.1)–(1.2) are reduced to the classical (or usual) Bernoulli and Euler polynomials $B_n(x)$, $E_n(x)$, respectively. They are denoted by $B_n(x) := \mathfrak{B}_n(x; 1, e, e)$ and $E_n(x) := \mathfrak{E}_n(x; 1, e, e)$, respectively (see [18]). In particular, the numbers $B_n = B_n(0)$ and $E_n = E_n(0)$ are nothing else but the classical Bernoulli and Euler numbers respectively.

On another side, Belbachir et *al.* [4, 5] proposed another generalization of Euler and Genocchi polynomials, which are called Euler–Genocchi polynomials, and established some of their properties, such as, those related to linear recursiveness and difference equations. Moreover, they evaluated an expression for the sum of the Stirling numbers of the second kind in terms of this family. Recently, a closed connection between Bernoulli, Euler–Genocchi numbers and some special linear difference equations of infinite order, was established in [3] and [19]. Note that these linear recursive sequences of infinite order have been introduced and studied by Rachidi *et al.* in various research papers (see, for instance, [6, 17], and references therein).

The main purpose of the present paper is to study some properties of the generalized Bernoulli and Euler polynomials $\mathfrak{B}_n(x; a, b, c)$ and $\mathfrak{E}_n(x; a, b, c)$ defined by (1.1)-(1.2), with the aid of two approaches. First, we consider the determinantal approach for exhibiting some new properties and expressions related to $\mathfrak{B}_n(x; a, b, c)$ and $\mathfrak{E}_n(x; a, b, c)$. Especially, the little Fermat Theorem, allows us to establish a congruence identity. In addition, a closed relation with the Stirling numbers is provided. Second, we study the generalized Bernoulli and Euler polynomials $\mathfrak{B}_n(x; a, b, c)$ and $\mathfrak{E}_n(x; a, b, c)$, using the second approach based on the properties of linear difference equations of infinite order. Especially, we provide combinatorial properties of the generalized Bernoulli and Euler polynomials $\mathfrak{B}_n(x; a, b, c)$ and $\mathfrak{E}_n(x; a, b, c)$, as well as their formulation in terms of Bernoulli numbers.

The paper is organized as follows. In Section 2, we present some properties of the generalized Bernoulli and Euler polynomials $\mathfrak{B}_n(x; a, b, c)$ and $\mathfrak{E}_n(x; a, b, c)$ defined by (1.1)–(1.2). We establish some explicit formulas and give the expression of the power of a variable, in addition an extension of the little Fermat's Theorem is provided. In Section 3, we evaluate the family of generalized Bernoulli and Euler polynomials $\mathfrak{B}_n(x; a, b, c)$ and $\mathfrak{E}_n(x; a, b, c)$ in terms of the Stirling numbers of the second kind. Section 4 and 5 are devoted to the generalized Bernoulli and Euler numbers $\mathfrak{B}_n(\lambda)$, $\mathfrak{B}_n(a, b)$, and the generalized Bernoulli and Euler polynomials $\mathfrak{B}_n(x; a, b, c)$, $\mathfrak{E}_n(x; a, b, c)$, using properties of the linear difference equations of infinite order. Therefore, expressions of $\mathfrak{B}_n(\lambda)$, $\mathfrak{B}_n(a, b)$ are given in terms of B_n, E_n. Moreover, the $\mathfrak{B}_n(x; a, b, c)$, $\mathfrak{E}_n(x; a, b, c)$ are formulated using $\mathfrak{B}_n(\lambda)$, $\mathfrak{B}_n(a, b)$.

2. Determinantal representation of the generalized Bernoulli and Euler polynomials

2.1. Generalized Bernoulli and Euler polynomials (1.1)–(1.2) and usual Bernoulli and Euler polynomials

Our goal, in this subsection is to give a formulation for the generalized Bernoulli and Euler polynomials in terms of the classical ones. The next result allows us to see that the generalized Bernoulli and Euler polynomials $\mathfrak{B}_n(x;a,b,c)$ and $\mathfrak{E}_n(x;a,b,c)$ can be expressed in terms of the classical Bernoulli and Euler polynomials.

Theorem 2.1. *Let a, b, c be in $\mathbb{R}_+^*$ such as $a \neq b$ and $a \neq 1$. Then, the following identities holds,*

$$\mathfrak{B}_n(x;a,b,c) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \left(\ln\left(\frac{b}{a}\right)\right)^k (\ln(a))^{n-k} B_k\left(x \ln_{b/a} c\right), \tag{2.1}$$

$$\mathfrak{E}_n(x;a,b,c) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \left(\ln\left(\frac{b}{a}\right)\right)^k (\ln(a))^{n-k} E_k\left(x \ln_{b/a} c\right), \tag{2.2}$$

for every $n \geq 0$, where $\ln_\alpha x = \frac{\ln x}{\ln \alpha}$.

Proof. We can show that Expression (1.2) can be reformulated under the form,

$$\begin{aligned}
\frac{2}{b^t + a^t} c^{xt} &= \frac{2}{e^{t\ln(b/a)} + 1} e^{t\ln(c^x/a)} \\
&= \frac{2}{e^{t\ln(b/a)} + 1} e^{xt\ln(b/a)\ln_{b/a} c} e^{-t\ln a} \\
&= \left(\sum_{n=0}^{\infty} \left(\ln\left(\frac{b}{a}\right)\right)^n E_n\left(x\ln_{b/a} c\right)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(-1)^n (\ln(a))^n \frac{t^n}{n!}\right).
\end{aligned} \tag{2.3}$$

Taking into account the right hand sides of Expression (2.3) and Expression (1.2), a direct computation permits to get the Formula (2.2). The proof of the Identity (2.1) is similar. □

Our next result concerns an explicit formula for generalized Bernoulli and Euler polynomials in terms of Bernoulli and Euler polynomials.

Theorem 2.2. *Let $a, b, c \in \mathbb{R}_+^*$ $(a \neq b)$. Then, for every $n \geq 0$, the following identities holds,*

$$\mathfrak{B}_n(x;a,b,c) = \left(\ln\left(\frac{b}{a}\right)\right)^{n-1} B_n\left(x\ln_{b/a} c - \ln_{b/a} a\right) \tag{2.4}$$

$$\mathfrak{E}_n(x;a,b,c) = \left(\ln\left(\frac{b}{a}\right)\right)^{n} E_n\left(x\ln_{b/a} c - \ln_{b/a} a\right). \tag{2.5}$$

Proof. For establishing the Formula (2.5), we can reformulate Expression (1.2) under the following form,

$$\begin{aligned}\frac{2}{b^t+a^t}c^{xt} &= \frac{2}{e^{t\ln(b/a)}+1}e^{t\ln(c^x/a)} = \frac{2}{e^{t\ln(b/a)}+1}e^{t(x\ln c-\ln a)}\\ &= \frac{2}{e^{t\ln(b/a)}+1}e^{t\ln(b/a)\left(x\ln_{\,b/a} c-\ln_{\,b/a} a\right)}\\ &= \sum_{n=0}^{\infty}\left(\ln\left(\frac{b}{a}\right)\right)^n E_n\left(x\ln_{\,b/a} c-\ln_{\,b/a} a\right)\frac{t^n}{n!}. \end{aligned} \tag{2.6}$$

Taking into account the right hand sides of Expressions (2.6) and (1.2), a direct computation permits us to obtain the Formula (2.5). The proof of the Identity (2.4) is similar. □

As a consequence of Theorem 2.2, we can give in Table 1 below the generalized Bernoulli and Euler polynomials.

In [14] and [15], it was shown the following result, concerning the generalized Bernoulli and Euler polynomials $\mathfrak{B}_n(x;a,b,c)$ and $\mathfrak{E}_n(x;a,b,c)$ defined by (1.1)-(1.2).

Theorem 2.3 (see [14, 15])**.** *Let a,b,c be in $\mathbb{R}_+^*$, with $a\neq b$. Then, for every $n\geq 0$, we have,*

$$\mathfrak{B}_n(x+y;a,b,c) = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{B}_k(x;a,b,c)y^{n-k}\left(\ln c\right)^{n-k}, \tag{2.7}$$

$$\mathfrak{E}_n(x+y;a,b,c) = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k(x;a,b,c)y^{n-k}\left(\ln c\right)^{n-k}. \tag{2.8}$$

In addition, for $x\neq 0$ and $y=0$, we derive from Expressions (2.7)–(2.8), the following formulas,

$$\mathfrak{B}_n(x;a,b,c) = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{B}_k(a,b)x^{n-k}\left(\ln c\right)^{n-k}, \tag{2.9}$$

$$\mathfrak{E}_n(x;a,b,c) = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k(a,b)x^{n-k}\left(\ln c\right)^{n-k}. \tag{2.10}$$

for every $n\geq 0$. Moreover, replacing y by $-x$ in Formulas (2.7)–(2.8), we get the following expressions,

$$\mathfrak{B}_n(a,b) = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\mathfrak{B}_k(x;a,b,c)x^{n-k}\left(\ln c\right)^{n-k},$$

$$\mathfrak{E}_n(a,b) = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\mathfrak{E}_k(x;a,b,c)x^{n-k}\left(\ln c\right)^{n-k}.$$

n	0	1	2	3
$B_n(x)$	1	$x-\frac{1}{2}$	$x^2-x+\frac{1}{6}$	$x^3-\frac{3}{2}x^2+\frac{1}{2}x$
$E_n(x)$	1	$x-\frac{1}{2}$	x^2-x	$x^3-\frac{3}{2}x^2+\frac{1}{4}$
$\mathfrak{B}_n(x;a,b,c)$	$\dfrac{1}{\ln(b/a)}$	$x\ln_{b/a}c-\ln_{b/a}a-\frac{1}{2}$	$x^2\ln\frac{b}{a}\ln^2_{b/a}c-x\left(2\ln_{b/a}a+1\right)\ln c$ $+\frac{1}{6}\ln a^5b+\ln\frac{b}{a}\ln^2_{b/a}a$	$x^3\ln^2\frac{b}{a}\ln^3_{b/a}c-3x^2\ln^2 c\left(\ln_{b/a}a+\frac{1}{2}\right)$ $+x\ln c\Big(3\ln\frac{b}{a}\ln^2_{b/a}a+\ln\sqrt{a^5b}\Big)$ $-\Big(\ln_{b/a}a\ln a+\ln a\sqrt{b}\Big)\ln a$
$\mathfrak{E}_n(x;a,b,c)$	1	$x\ln c-\frac{1}{2}\ln ab$	$x^2\ln^2 c-x\ln ab\ln c$ $+\ln a\ln b$	$x^3\ln^3 c-\frac{3}{2}x^2\ln^2 c\ln ab+3x\ln a\ln b\ln c$ $+\frac{1}{4}\ln^3\frac{b}{a}-\frac{3}{2}\ln\frac{b}{a}\ln a-\ln^3 a$

Table 1: Some special cases of $\mathfrak{B}_n(x;a,b,c)$ and $\mathfrak{E}_n(x;a,b,c)$.

Expressions (2.7) and (2.8) allow us to formulate the Bernoulli and Euler numbers $\mathfrak{B}_n(a,b)$ and $\mathfrak{E}_n(a,b)$, in terms of the generalized Bernoulli and Euler polynomials $\mathfrak{B}_k(x;a,b,c)$, $\mathfrak{E}_k(x;a,b,c)$, respectively. Using Table 1, we can obtain the following table for some values of the generalized Bernoulli and Euler numbers.

n	0	1	2	3
$\mathfrak{B}_n(a,b)$	$\dfrac{1}{\ln(b/a)}$	$-\ln_{b/a} a - \frac{1}{2}$	$\frac{1}{6}\ln a^5 b + \ln\frac{b}{a}\ln^2_{b/a} a$	$-\left(\ln_{b/a} a \ln a + \ln a\sqrt{b}\right)\ln a$
$\mathfrak{E}_n(a,b)$	1	$-\frac{1}{2}\ln ab$	$\ln a \ln b$	$\frac{1}{4}\ln^3\frac{b}{a} - \frac{3}{2}\ln\frac{b}{a}\ln a - \ln^3 a$

Table 2: Some special cases of $\mathfrak{B}_n(a,b)$ and $\mathfrak{E}_n(a,b)$.

Combining Theorem 2.2 and Theorem 2.3, namely, Expressions (2.4)–(2.5) and (2.7)–(2.8), we can state the following result.

Theorem 2.4. *Let a,b,c be in $\mathbb{R}_+^*$ with $a \neq b$. Then, for every $n \geq 0$, we have,*

$$\mathfrak{B}_n(x+y;a,b,c) = \sum_{k=0}^{n}\binom{n}{k}\left(\ln\left(\frac{b}{a}\right)\right)^{n-1} B_n\left(x\ln_{\,b/a} c - \ln_{\,b/a} a\right) y^{n-k}(\ln c)^{n-k}, \tag{2.11}$$

$$\mathfrak{E}_n(x+y;a,b,c) = \sum_{k=0}^{n}\binom{n}{k}\left(\ln\left(\frac{b}{a}\right)\right)^{n} E_n\left(x\ln_{\,b/a} c - \ln_{\,b/a} a\right) y^{n-k}(\ln c)^{n-k}. \tag{2.12}$$

Expressions (2.11)–(2.12) show that the formulas of addition (2.7)–(2.8), for the generalized Bernoulli and Euler polynomials (1.1)–(1.2), are given in terms of the classical Bernoulli and Euler polynomials.

2.2. Determinantal representation of the generalized Bernoulli and Euler polynomials (1.1)–(1.2)

Let consider the following expression

$$T(x,a,b,c,t) = \frac{2}{b^t + a^t}c^{xt} \times \frac{t}{b^t - a^t}c^{xt} = \frac{2t}{b^{2t} - a^{2t}}c^{2xt}.$$

Then, taking into account the right hand side of Expressions (1.1)–(1.2) for $a \neq 0$, a straightforward computation allows us to obtain,

$$\begin{aligned} &T(x+1,a,b,b,t) - T(x,1,b/a,b,t) \\ &\qquad = \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}\begin{vmatrix} \mathfrak{B}_{n-k}(x+1,a,b,b) & \mathfrak{E}_k(x,1,b/a,b) \\ \mathfrak{B}_{n-k}(x,1,b/a,b) & \mathfrak{E}_k(x+1,a,b,b) \end{vmatrix}\right\}\frac{t^n}{n!}, \end{aligned}$$

where $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{21} - a_{11}a_{12}$. On the other hand, we have,

$$T(x+1,a,b,b,t) - T(x,1,b/a,b,t) = 2tb^{2xt} = \sum_{n=0}^{+\infty} n2^n (x\ln b)^{n-1} \frac{t^n}{n!}.$$

Comparing the two preceding expansions of $T(x+1,a,b,b,t) - T(x,1,b/a,b,t)$, we can state the next result.

Theorem 2.5. *Let x, a, b be a real numbers such that $(a,b) \neq (0,1)$. Then, for every $n \geq 0$, we have,*

$$x^n = \frac{1}{2^{n+1}(n+1)\ln^n b} \sum_{k=0}^{n+1} \binom{n+1}{k} \begin{vmatrix} \mathfrak{B}_{n+1-k}(x+1,a,b,b) & \mathfrak{E}_k(x,1,b/a,b) \\ \mathfrak{B}_{n+1-k}(x,1,b/a,b) & \mathfrak{E}_k(x+1,a,b,b) \end{vmatrix}.$$

In particular, for $a=1$ and $b=e=\exp(1)$, we derive the following result in terms of the Bernoulli and Euler polynomials.

Corollary 2.6 (see [5])**.** *Let x be a real number and an integer $n \geq 0$. Then, we have,*

$$x^n = \frac{1}{2^{n+1}(n+1)} \sum_{k=0}^{n+1} \binom{n+1}{k} \begin{vmatrix} B_{n-(k-1)}(x+1) & E_k(x) \\ B_{n-(k-1)}(x) & E_k(x+1) \end{vmatrix}. \tag{2.13}$$

Expressions (2.9)–(2.10) allow us to obtain $\mathfrak{B}_n(x;a,b,c)$ and $\mathfrak{E}_n(x;a,b,c)$ in terms of the determinantal representation respectively. Indeed, it suffices to replace the factor x^n of Theorem 2.5 in Formulas (2.9) and (2.10), we get the following formulas:

$$\begin{aligned}\mathfrak{B}_n(x;a,b,c) &= \sum_{k=0}^{n} \binom{n}{k} \mathfrak{B}_{n-k}(a,b) \frac{1}{2^{k+1}(k+1)} (\ln_b c)^k \\ &\quad\times \sum_{j=0}^{k+1} \binom{k+1}{j} \begin{vmatrix} \mathfrak{B}_{k-(j-1)}(x+1,a,b,b) & \mathfrak{E}_j(x,1,b/a,b) \\ \mathfrak{B}_{k-(j-1)}(x,1,b/a,b) & \mathfrak{E}_j(x+1,a,b,b) \end{vmatrix},\end{aligned}$$

$$\begin{aligned}\mathfrak{E}_n(x;a,b,c) &= \sum_{k=0}^{n} \binom{n}{k} \mathfrak{E}_{n-k}(a,b) \frac{1}{2^{k+1}(k+1)} (\ln_b c)^k \\ &\quad\times \sum_{j=0}^{k+1} \binom{k+1}{j} \begin{vmatrix} \mathfrak{B}_{k-(j-1)}(x+1,a,b,b) & \mathfrak{E}_j(x,1,b/a,b) \\ \mathfrak{B}_{k-(j-1)}(x,1,b/a,b) & \mathfrak{E}_j(x+1,a,b,b) \end{vmatrix}.\end{aligned}$$

In addition, as a consequence of the Little Fermat's Theorem [2] and Corollary 2.6, we can derive a new formula of addition in a determinantal form. Before illustrating our main result, we will need to present the Fermat's little Theorem under the following lemma.

Lemma 2.7. *For p a prime number and a an integer, we have,*

$$a^p \equiv a \mod p. \tag{2.14}$$

Theorem 2.8. *Let p be a prime number and a an integer, we have*

$$\sum_{k=0}^{p+1} \binom{p+1}{k} \Delta_{p+1-k,k}(a) \equiv \frac{1}{2} \sum_{k=0}^{2} \binom{2}{k} \Delta_{2-k,k}(a) \mod p, \tag{2.15}$$

where $\Delta_{n,s}(v) := \begin{vmatrix} B_n(v+1) & E_s(v) \\ B_n(v) & E_s(v+1) \end{vmatrix}$.

Proof. Replacing $x = a$ and $n = p$ in (2.13), and combining with (2.14), permits us to get,

$$\frac{1}{2^{p+1}(p+1)} \sum_{k=0}^{p+1} \binom{p+1}{k} \Delta_{p+1-k,k}(a) \equiv \frac{1}{8} \sum_{k=0}^{2} \binom{2}{k} \Delta_{2-k,k}(a) \mod p.$$

Multiplying both sides by $2^{p+1}(p+1)$ we obtain,

$$\sum_{k=0}^{p+1} \binom{p+1}{k} \Delta_{p+1-k,k}(a) \equiv \frac{2^{p+1}(p+1)}{8} \sum_{k=0}^{2} \binom{2}{k} \Delta_{2-k,k}(a) \mod p.$$

Using the fact that $2^p \equiv 2 \mod p$ and $p+1 \equiv 1 \mod p$, Formula (2.15) holds. □

For nonnegative integer n, the Harmonic numbers $\{H_n\}_{n\geq 0}$, and the n-th generalized harmonic numbers $\{H_{n,m}\}_{n\geq 0}$, are defined by,

$$H_0 = 0, \quad H_n = \sum_{i=1}^{n} \frac{1}{i}, \quad \text{and} \quad H_{0,m} = 0, \qquad H_{n,m} = \sum_{i=1}^{n} \frac{1}{i^m},$$

for every $n \geq 1$. Different properties of the harmonic numbers have been studied recently by many mathematicians. Among them, we list some identities below,

$$\sum_{k=1}^{n-1} H_k = nH_n - n, \quad \sum_{k=m}^{n-1} \binom{k}{m} H_k = \binom{n}{m+1} \left(H_n - \frac{1}{m+1} \right) \quad \text{and}$$
$$\sum_{k=m}^{n-1} \binom{k}{m} \frac{1}{n-k} = \binom{n}{m} (H_n - H_m).$$

In the following theorem, we give a link between the Bernoulli polynomials, the Euler polynomials and the harmonic numbers.

Theorem 2.9. *Let n, m and s be three integers with $n \geq 1$ and $m \geq 1$. Then we have the following identity,*

$$1 - H_{m-1} \sum_{k=0}^{s+1} \binom{s+1}{k} \Delta_{s+1-k,k} \left(\frac{n}{2\sqrt[s]{2(s+1)}} \right)$$
$$+ \left(H_{m-1}^2 - H_{m-1,2}\right) \left[\sum_{k=0}^{s+1} \binom{s+1}{k} \Delta_{s+1-k,k} \left(\frac{n}{2\sqrt[s]{2(s+1)}} \right) \right]^2 = (-1)^{m-1} \binom{n^s - 1}{m-1}.$$

Proof. Let

$$P(n,s,m) = 1 - H_{m-1}\sum_{k=0}^{s+1}\binom{s+1}{k}\Delta_{s+1-k,k}\left(\frac{n}{2\sqrt[s]{2(s+1)}}\right) + \left(H_{m-1}^2 - H_{m-1,2}\right)\left[\sum_{k=0}^{s+1}\binom{s+1}{k}\Delta_{s+1-k,k}\left(\frac{n}{2\sqrt[s]{2(s+1)}}\right)\right]^2. \tag{2.16}$$

Substituting x by n and assuming $n = s$ in Expression (2.13), we get the following identity,

$$n^s = \sum_{k=0}^{s+1}\binom{s+1}{k}\Delta_{s+1-k,k}\left(\frac{n}{2\sqrt[s]{2(s+1)}}\right). \tag{2.17}$$

Combining (2.16) and (2.17) permit us to get,

$$\begin{aligned}
P(n,s,m) &= 1 - n^s H_{m-1} + n^{2s}\left(H_{m-1}^2 - H_{m-1,2}\right)\\
&= 1 - n^s\sum_{i=1}^{m-1}\frac{1}{i} + n^{2s}\left(\sum_{1\le i\le j\le m-1}\frac{1}{ij} - \sum_{i=1}^{m-1}\frac{1}{i^2}\right)\\
&= 1 - n^s\sum_{i=1}^{m-1}\frac{1}{i} + n^{2s}\sum_{1\le i<j\le m-1}\frac{1}{ij}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\binom{n^s-1}{m-1} &= \frac{(n^s-1)(n^s-2)\cdots(n^s-j)\cdots(n^s-(m-1))}{1.2\ldots j\ldots(m-1)}\\
&= \left(\frac{n^s}{1}-1\right)\left(\frac{n^s}{2}-1\right)\cdots\left(\frac{n^s}{j}-1\right)\cdots\left(\frac{n^s}{m-1}-1\right)\\
&= (-1)^{m-1}\left(1-\frac{n^s}{1}\right)\left(1-\frac{n^s}{2}\right)\cdots\left(1-\frac{n^s}{j}\right)\cdots\left(1-\frac{n^s}{m-1}\right)\\
&= (-1)^{m-1}\left[1 - n^s\sum_{i=1}^{m-1}\frac{1}{i} + n^{2s}\sum_{1\le i<j\le m-1}\frac{1}{ij}\right]\\
&= (-1)^{m-1}P(n,s,m).
\end{aligned}$$

Therefore, multiplying both sides by $(-1)^{m-1}$ permits us to obtain our statement. □

Theorem 2.10. *Let n, m and s be three integers with $n \geq 1$ and $m \geq 1$. Then we have the following identity,*

$$\begin{vmatrix} 0 & -\left[\sum_{k=0}^{s+1}\binom{s+1}{k}\Delta_{s+1-k,k}\left(\frac{n}{2\sqrt[s]{2(s+1)}}\right)\right]^2 & -\left[\sum_{k=0}^{s+1}\binom{s+1}{k}\Delta_{s+1-k,k}\left(\frac{n}{2\sqrt[s]{2(s+1)}}\right)\right]^4 \\ H_{m-1}^2 & H_{m-1,2} & 2 \\ H_{m-1,2} & 2H_{m-1,2}-H_{m-1}^2 & 3 \end{vmatrix}$$
$$+\begin{vmatrix} 1 & \sum_{k=0}^{s+1}\binom{s+1}{k}\Delta_{s+1-k,k}\left(\frac{n}{2\sqrt[s]{2(s+1)}}\right) & \left[\sum_{k=0}^{s+1}\binom{s+1}{k}\Delta_{s+1-k,k}\left(\frac{n}{2\sqrt[s]{2(s+1)}}\right)\right]^2 \\ 2H_{m-1} & 1 & \sum_{k=0}^{s+1}\binom{s+1}{k}\Delta_{s+1-k,k}\left(\frac{n}{2\sqrt[s]{2(s+1)}}\right) \\ H_{m-1,2} & 0 & 1 \end{vmatrix}$$
$$-2\begin{vmatrix} H_{m-1} & 1 & 0 \\ H_{m-1,2} & H_{m-1} & 1 \\ H_{m-1}H_{m-1,2} & H_{m-1,2} & H_{m-1} \end{vmatrix}\left[\sum_{k=0}^{s+1}\binom{s+1}{k}\Delta_{s+1-k,k}\left(\frac{n}{2\sqrt[s]{2(s+1)}}\right)\right]^2 = \binom{n^s-1}{m-1}^2.$$

Proof. Let

$Q(n,s,m) =$

$$\begin{vmatrix} 0 & -\left[\sum_{k=0}^{s+1}\binom{s+1}{k}\Delta_{s+1-k,k}\left(\frac{n}{2\sqrt[s]{2(s+1)}}\right)\right]^2 & -\left[\sum_{k=0}^{s+1}\binom{s+1}{k}\Delta_{s+1-k,k}\left(\frac{n}{2\sqrt[s]{2(s+1)}}\right)\right]^4 \\ H_{m-1}^2 & H_{m-1,2} & 2 \\ H_{m-1,2} & 2H_{m-1,2}-H_{m-1}^2 & 3 \end{vmatrix}$$
$$+\begin{vmatrix} 1 & \sum_{k=0}^{s+1}\binom{s+1}{k}\Delta_{s+1-k,k}\left(\frac{n}{2\sqrt[s]{2(s+1)}}\right) & \left[\sum_{k=0}^{s+1}\binom{s+1}{k}\Delta_{s+1-k,k}\left(\frac{n}{2\sqrt[s]{2(s+1)}}\right)\right]^2 \\ 2H_{m-1} & 1 & \sum_{k=0}^{s+1}\binom{s+1}{k}\Delta_{s+1-k,k}\left(\frac{n}{2\sqrt[s]{2(s+1)}}\right) \\ H_{m-1,2} & 0 & 1 \end{vmatrix}$$
$$-2\begin{vmatrix} H_{m-1} & 1 & 0 \\ H_{m-1,2} & H_{m-1} & 1 \\ H_{m-1}H_{m-1,2} & H_{m-1,2} & H_{m-1} \end{vmatrix}\left[\sum_{k=0}^{s+1}\binom{s+1}{k}\Delta_{s+1-k,k}\left(\frac{n}{2\sqrt[s]{2(s+1)}}\right)\right]^2. \quad (2.18)$$

Combining (2.17) and (2.18) permit us to get

$$\begin{aligned} Q(n,s,m) &= \begin{vmatrix} 0 & -n^{2s} & -n^{4s} \\ H_{m-1}^2 & H_{m-1,2} & 2 \\ H_{m-1,2} & 2H_{m-1,2}-H_{m-1}^2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & n^s & n^{2s} \\ 2H_{m-1} & 1 & n^s \\ H_{m-1,2} & 0 & 1 \end{vmatrix} \\ &\quad - 2\begin{vmatrix} H_{m-1} & 1 & 0 \\ H_{m-1,2} & H_{m-1} & 1 \\ H_{m-1}H_{m-1,2} & H_{m-1,2} & H_{m-1} \end{vmatrix} n^{2s} \\ &= \begin{vmatrix} H_{m-1}^2 & 2 \\ H_{m-1,2} & 3 \end{vmatrix} n^{2s} - \begin{vmatrix} H_{m-1}^2 & H_{m-1,2} \\ H_{m-1,2} & 2H_{m-1,2}-H_{m-1}^2 \end{vmatrix} n^{4s} + 1 - 2H_{m-1}n^s \\ &\quad - 2\left[\begin{vmatrix} H_{m-1} & 1 \\ H_{m-1,2} & H_{m-1} \end{vmatrix} H_{m-1} - \begin{vmatrix} 1 & 0 \\ H_{m-1,2} & H_{m-1} \end{vmatrix} H_{m-1,2} \right. \\ &\quad \left. + \begin{vmatrix} 1 & 0 \\ H_{m-1} & 1 \end{vmatrix} H_{m-1}H_{m-1,2}\right] n^{3s} \\ &= 1 - 2H_{m-1}n^s + \left(3H_{m-1}^2 - 2H_{m-1,2}\right)n^{2s} - 2\left(H_{m-1}^3 - H_{m-1}H_{m-1,2}\right)n^{3s} \\ &\quad + \left(H_{m-1}^4 - 2H_{m-1}^2H_{m-1,2} + H_{m-1,2}^2\right)n^{4s}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\binom{n^s-1}{m-1}^2 &= \frac{(n^s-1)^2(n^s-2)^2\cdots(n^s-j)^2\cdots(n^s-(m-1))^2}{1^2.2^2\ldots j^2\ldots(m-1)^2}\\
&= \left(\frac{n^s}{1}-1\right)^2\left(\frac{n^s}{2}-1\right)^2\cdots\left(\frac{n^s}{j}-1\right)^2\cdots\left(\frac{n^s}{m-1}-1\right)^2\\
&= \left(1-\frac{n^s}{1}\right)^2\left(1-\frac{n^s}{2}\right)^2\cdots\left(1-\frac{n^s}{j}\right)^2\cdots\left(1-\frac{n^s}{m-1}\right)^2\\
&= \left[1-n^s\sum_{i=1}^{m-1}\frac{1}{i}+n^{2s}\sum_{1\le i<j\le m-1}\frac{1}{ij}\right]^2\\
&= \left[1-H_{m-1}n^s+\left(H_{m-1}^2-H_{m-1,2}\right)n^{2s}\right]^2\\
&= 1-H_{m-1}n^s+\left(H_{m-1}^2-H_{m-1,2}\right)n^{2s}-H_{m-1}n^s+H_{m-1}^2n^{2s}\\
&\quad-\left(H_{m-1}^3-H_{m-1}H_{m-1,2}\right)n^{3s}+\left(H_{m-1}^2-H_{m-1,2}\right)n^{2s}\\
&\quad-\left(H_{m-1}^3-H_{m-1}H_{m-1,2}\right)n^{3s}+\left(H_{m-1}^4+H_{m-1,2}^2-2H_{m-1}^2H_{m-1,2}\right)n^{4s}\\
&= 1-2H_{m-1}n^s+\left(3H_{m-1,2}-2H_{m-1,2}\right)n^{2s}-2\left(H_{m-1}^3-H_{m-1}H_{m-1,2}\right)n^{3s}\\
&\quad+\left(H_{m-1}^4+H_{m-1,2}^2-2H_{m-1}^2H_{m-1,2}\right)n^{4s}\\
&= Q(n,s,m).
\end{aligned}$$

□

3. Explicit formulas in terms of the Stirling numbers

In this section, we prove some explicit formulas for the generalized Bernoulli and Euler polynomials in terms of the Stirling numbers of the second kind. Firstly, we give some definitions of remarkable numbers and polynomials well-known in the literature. The partial Bell polynomials $B_{n,k}(x_1,x_2,\ldots,x_n)$ are defined by,

$$\sum_{n=k}^{\infty}B_{n,k}(x_1,x_2,\ldots,x_{n-k+1})\frac{t^n}{n!}=\frac{1}{k!}\left(\sum_{m=1}^{\infty}x_m\frac{t^m}{m!}\right)^k,$$

(see [12, 16]), also by considering the explicit formula,

$$B_{n,k}(x_1,x_2,\ldots,x_{n-k+1})=\sum_{\sigma(n,k)}\frac{n!}{\prod_{i=1}^{n-k+1}m_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{m_i},$$

given in [12, p. 134], where $\sigma(n,k)$ denotes the set of all integer solutions of the system,

$$\begin{cases} m_1+2m_2+\cdots+nm_n=n;\\ m_1+m_2+\cdots+m_n=k.\end{cases}$$

For n and k nonnegative integers with $(k \leq n)$, the Stirling numbers of the second kind $S(n,k)$ are defined by the following formula $S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$ (see [12, p. 206]). Moreover, their associated generating functions is given by,

$$\sum_{n\geq k} S(n,k)\frac{t^n}{n!} = \frac{(e^t-1)^k}{k!}.$$

The following result represents one of our main results of the paper.

Theorem 3.1. *Let a,b,c be in $\mathbb{R}_+^*$, with $a \neq b$, $x \in \mathbb{R}$ and $\mathfrak{B}_n(x;a,b,c)$ be the generalized Bernoulli polynomials. Then, we have,*

$$\begin{aligned}\mathfrak{B}_n(x;a,b,c) = \sum_{k=1}^{n}\binom{n+k}{n}^{-1} \sum_{\substack{s+t=k\\ i+j=n}}\ \sum_{r\leq s, m\leq t}(-1)^{t-m-r}\binom{n+k}{s-r, t-m, i+r, j+m}\\ \times C_{i,j,k}(x;a,b,c)S(i+r,r)S(j+m,m),\end{aligned}$$

for $n \in \mathbb{N}$, where $\quad C_{i,j,k}(x;a,b,c) := \dfrac{\ln^j(a/c^x)\ln^i(b/c^x)}{\ln^{k+1}(b/a)}.$

Especially, for $b=c=e$ and $a=1$, we can derive the following corollary.

Corollary 3.2. *Let $B_n(x)$ be the Bernoulli polynomials. Then, we have,*

$$\begin{aligned}B_n(x) = \sum_{k=1}^{n}\binom{n+k}{n}^{-1} \sum_{\substack{s+t=k\\ i+j=n}}\ \sum_{r\leq s, m\leq t}(-1)^{t+j-(m+r)}\binom{n+k}{s-r, t-m, i+r, j+m}\\ \times x^j(1-x)^i S(i+r,r)S(j+m,m),\end{aligned}$$

for $n \in \mathbb{N}$.

For establishing the preceding main result, namely, Theorem 3.1, we need some preliminary lemmas.

Lemma 3.3 (see [12]). *For every $n \geq k \geq 1$, we have,*

$$B_{n,k}(abx_1, ab^2x_2, \ldots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}).$$

for $a,b \in \mathbb{C}$, with $a \neq 0$, $b \neq 0$.

Lemma 3.4 (see [12]). *For $n \geq k \geq 1$, we have,*

$$\begin{aligned}B_{n,k}(x_1+y_1, \ldots, x_{n-k+1}+y_{n-k+1}) = \\ \sum_{\substack{s+t=k\\ i+j=n}}\binom{n}{i,j}B_{i,s}(x_1,\ldots,x_{i-s+1})B_{j,t}(y_1,\ldots,y_{j-t+1}).\end{aligned}$$

Lemma 3.5 (see [23]). *For every $n \geq k \geq 1$, we have,*

$$B_{n,k}\left(\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-k+2}\right) = \frac{n!}{(n+k)!}\sum_{i=0}^{k}(-1)^{k-i}\binom{n+k}{k-i}S(n+i,i).$$

On the other side, the Faà di Bruno Formula for computing higher order derivatives of composite functions, can be stated in terms of the partial Bell polynomials $B_{n,k}$,as follows,

Theorem 3.6 ([12]). *For every $n \geq k \geq 1$, we have,*

$$\frac{d^n}{dt^n}(g \circ f)(t) = \sum_{k=1}^{n} g^{(k)}(f(t))B_{n,k}(f^{'}(t), f^{''}(t), \ldots, f^{(n-k+1)}(t)). \tag{3.1}$$

Proof of Theorem 3.1. We can reformulate Expression (1.1) under the form,

$$\frac{t}{b^t - a^t}c^{xt} = \frac{t}{e^{t\ln(b/c^x)} - e^{t\ln(a/c^x)}} = \frac{1}{\int_{\ln(a/c^x)}^{\ln(b/c^x)} e^{tu}du}. \tag{3.2}$$

Putting $g(y) = \dfrac{1}{y}$ and $f(t) = \displaystyle\int_{\ln(a/c^x)}^{\ln(b/c^x)} e^{tu}du$. By using the Faà di Bruno Formula (3.1), on the right-hand side of (3.2), we get

$$\frac{d^n}{dt^n}\left(\frac{t}{b^t - a^t}c^{xt}\right) = \sum_{k=1}^{n}\frac{(-1)^k k!}{\left(\int_{\ln(a/c^x)}^{\ln(b/c^x)} e^{tu}du\right)^{k+1}} \times B_{n,k}\left(\int_{\ln(a/c^x)}^{\ln(b/c^x)} ue^{tu}du, \ldots, \int_{\ln(a/c^x)}^{\ln(b/c^x)} u^{n-k+1}e^{tu}du\right).$$

We show that when $t \mapsto 0$ in the above formula and by Taylor-Maclaurin series expansion in (1.1), we obtain $\mathfrak{B}_n(x;a,b,c) = \left[\dfrac{d^n}{dt^n}\left(\dfrac{t}{b^t - a^t}c^{xt}\right)\right]_{t=0}$, which implies that we have,

$$\mathfrak{B}_n(x;a,b,c) = \sum_{k=1}^{n}\frac{(-1)^k k!}{(\ln\frac{b}{c^x} - \ln\frac{a}{c^x})^{k+1}}B_{n,k}\left(\int_{\ln(a/c^x)}^{\ln(b/c^x)} udu, \ldots, \int_{\ln(a/c^x)}^{\ln(b/c^x)} u^{n-k+1}du\right).$$

Therefore, we arrive to have the following formula,

$$\mathfrak{B}_n(x;a,b,c) =$$
$$\sum_{k=1}^{n}\frac{(-1)^k k!}{\ln^{k+1}(b/a)}B_{n,k}\left(\frac{1}{2}\left[\ln^2\frac{b}{c^x} - \ln^2\frac{a}{c^x}\right], \ldots, \frac{1}{n-k+2}\left[\ln^{n-k+2}\frac{b}{c^x} - \ln^{n-k+2}\frac{a}{c^x}\right]\right).$$

By applying Lemma 3.4, we obtain,

$$\begin{aligned}
&\mathfrak{B}_n(x;a,b,c)\\
&=\sum_{k=1}^{n}\frac{(-1)^k k!}{\ln^{k+1}(b/a)}\sum_{\substack{s+t=k\\ i+j=n}}\binom{n}{i,j}B_{i,s}\left(\frac{1}{2}\ln^2\left(\frac{b}{c^x}\right),\ldots,\frac{1}{i-s+2}\ln^{i-s+2}\left(\frac{b}{c^x}\right)\right)\\
&\quad\times B_{j,t}\left(-\frac{1}{2}\ln^2\left(\frac{a}{c^x}\right),\ldots,-\frac{1}{j-t+2}\ln^{j-t+2}\left(\frac{a}{c^x}\right)\right).
\end{aligned}$$

And with the aid of Lemma 3.3 and Lemma 3.5, we have

$$\begin{aligned}
\mathfrak{B}_n(x;a,b,c)&=\sum_{k=1}^{n}\frac{(-1)^k k!}{\ln^{k+1}(b/a)}\sum_{\substack{s+t=k\\ i+j=n}}\binom{n}{i,j}\ln^i\left(\frac{b}{c^x}\right)\\
&\quad\times B_{i,s}\left(\frac{1}{2},\ldots,\frac{1}{i-s+2}\right)(-1)^t\ln^j\left(\frac{a}{c^x}\right)B_{j,t}\left(\frac{1}{2},\ldots,\frac{1}{j-t+2}\right)\\
&=\sum_{k=1}^{n}\sum_{\substack{s+t=k\\ i+j=n}}\sum_{r=0}^{s}\sum_{m=0}^{t}(-1)^{t-(m+r)}\binom{n}{i,j}\binom{i+s}{s-r}\binom{j+t}{t-m}\\
&\quad\times\frac{k!i!j!}{(i+s)!(j+t)!}\frac{1}{\ln^{k+1}(b/a)}\ln^j\left(\frac{a}{c^x}\right)\ln^i\left(\frac{b}{c^x}\right)S(i+r,r)S(j+m,m).
\end{aligned}$$

Finally, we obtain,

$$\begin{aligned}
\mathfrak{B}_n(x;a,b,c)&=\sum_{k=1}^{n}\sum_{\substack{s+t=k\\ i+j=n}}\sum_{r=0}^{s}\sum_{m=0}^{t}(-1)^{t-(m+r)}\frac{\binom{n+k}{s-r,t-m,i+r,j+m}}{\binom{n+k}{n}}\\
&\quad\times\frac{\ln^j(a/c^x)\ln^i(b/c^x)}{\ln^{k+1}(b/a)}S(i+r,r)S(j+m,m).
\end{aligned}$$

Therefore, we can express the generalized Euler polynomials in terms of the Stirling numbers of second kind, using closed relation between the generalized Bernoulli and Euler polynomials. Indeed, for every even integer h, it is well-known that,

$$\mathfrak{E}_n(hx;a,b,c)=\frac{(-2)h^n}{n+1}\sum_{j=0}^{h-1}\mathfrak{B}_{n+1}\left(x+\frac{j(\ln b-\ln a)+(h-1)\ln a}{h\ln c};a,b,c\right),$$

(see [13]). Particularly, for $h=2$, we derive,

$$\mathfrak{E}_n(2x;a,b,c)=\frac{-2^{n+1}}{n+1}\left[\mathfrak{B}_{n+1}\left(x+\frac{1}{2}\ln_c a;a,b,c\right)-\mathfrak{B}_{n+1}\left(x+\frac{1}{2}\ln_c b;a,b,c\right)\right].$$

Hence, we can reformulate Theorem 3.1 as follows.

Theorem 3.7. *Let a, b, c in $\mathbb{R}_+^*$, with $a \neq b$, and $\mathfrak{E}_n(x; a, b, c)$ be the generalized Euler polynomials. Then, for every nonnegative integer n, we have,*

$$\begin{aligned}&\mathfrak{E}_n(x; a, b, c)\\ &= \frac{-2^{n+1}}{n+1} \sum_{k=1}^{n+1} \binom{n+k+1}{n+1}^{-1} \sum_{\substack{s+t=k \\ i+j=n+1}} \sum_{r\leq s, m\leq t} (-1)^{t-(m+r)} \binom{n+k+1}{s-r, t-m, i+r, j+m}\\ &\quad \times S(i+r, r) S(j+m, m) \left[C_{i,j,k}\left(\frac{2x + \ln_c a}{2}; a, b, c\right) - C_{i,j,k}\left(\frac{2x + \ln_c b}{2}; a, b, c\right)\right],\end{aligned}$$

where $\quad C_{i,j,k}(x; a, b, c) := \frac{\ln^j(a/c^x)\ln^i(b/c^x)}{\ln^{k+1}(b/a)}.$

In particular, for $b = c = e$ and $a = 1$, we reach the following corollary.

Corollary 3.8. *Let $E_n(x)$ be the Euler polynomials. Then, for every nonnegative integer n, we have,*

$$\begin{aligned}E_n(x) = \frac{2^{n+1}}{n+1} \sum_{k=1}^{n+1} \binom{n+k+1}{n+1}^{-1} \sum_{\substack{s+t=k \\ i+j=n+1}} \sum_{r\leq s, m\leq t} \epsilon_{m,r}(t, j) \binom{n+k+1}{s-r, t-m, i+r, j+m}\\ \times S(i+r, r) S(j+m, m) \begin{vmatrix} x^j & \left(\frac{1}{2} - x\right)^i \\ \left(\frac{1}{2} + x\right)^j & (1-x)^i \end{vmatrix},\end{aligned}$$

where $\epsilon_{m,r}(t, j) = (-1)^{t+j+1-(m+r)}$.

4. Linear recursive approach for generalized Bernoulli polynomials

4.1. Preliminary on the linear recursiveness of infinite order

Let $\{a_i\}_{i\geq 0}$ and $\{\alpha_i\}_{i\geq 0}$ be two sequences of real or complex numbers, such that for every $N \in \mathbb{N}$ there exists $i > N$ such that $a_i \neq 0$. The former sequence is called the *coefficient sequence* and the latter the *initial sequence*. Consider the sequence $\{w_n\}_{n\in\mathbb{Z}}$ defined by setting $w_{-n} = \alpha_n$ for $n \geq 0$, and

$$w_n = \sum_{i=0}^{\infty} a_i w_{n-i-1} \qquad \text{for} \qquad n \geq 1. \tag{4.1}$$

Expression (4.1) represents a series, thus the general term involves infinitely many terms. Therefore, we have to worry about the convergence of this series (for more details see [6, 17]). In [6] a necessary and sufficient condition, labeled $[(C_\infty)]$, on the existence of w_n $(n \geq 1)$, is formulated as follows: *The series $\sum_{i=0}^{\infty} a_{i+n-1}\alpha_{-i}$ converges for all $n \geq 1$* (see [6, Proposition 2.1]). In particular, if $\alpha_j = 0$, for all $j \geq k+1$, then the condition (C_∞) is trivially verified, and we have $v_{n+1} = \sum_{j=0}^{n+k} a_j w_{n-j}$, for all $n \geq 0$. Under some hypothesis on the two sequences $\{a_j\}_{j\geq 0}$

and $\{\alpha_j\}_{j\geq 0}$ (respectively), it was proved in [9] that $\{w_n\}_{n\in\mathbb{Z}}$ takes the following combinatoric form $w_n = \sum_{s=1}^{n} A_s \rho(n-s,0)$ with $A_s = \sum_{m=0}^{+\infty} a_{s+m-1}\alpha_m$, where

$$\rho(n,0) = \sum_{k_0+2k_1+\cdots+nk_{n-1}=n} \frac{(k_0+\cdots+k_{n-1})!}{k_0!\cdots k_{n-1}!} a_0^{k_0}\cdots a_{n-1}^{k_{n-1}}, \tag{4.2}$$

with $\rho(0,0) = 1$ and $\rho(-k,0) = 0$ for every $k \geq 1$. Especially, the sequence $\{v_n\}_{n\in\mathbb{Z}}$ defined by (4.2), namely, $v_n = \rho(n,0)$, for every $n \geq 1$, with $\rho(0,0) = 1$ and $\rho(-k,0) = 0$ for every $k \geq 1$, satisfies the recursive relation (4.1) of infinite order. A straightforward computation shows that the generating function of $\{v_n\}_{n\in\mathbb{Z}}$ is,

$$f(t) = \sum_{n=0}^{\infty} v_n t^n = \frac{1}{Q(t)}, \tag{4.3}$$

where $Q(t) = 1 - \sum_{j=0}^{\infty} a_j t^{j+1}$ is the so-called *the characteristic function of the sequence* (4.1) (for more details see [17]). Conversely, let $Q(t)$ be a complex function which is analytic in open disk $D(0;R)$. Suppose that Q takes the following power series form $Q(t) = 1 - \sum_{j=0}^{\infty} a_j t^{j+1}$, in $D(0;R)$. Since $Q(0) = 1 \neq 0$, then $f(t) = 1/Q(t)$ has a Taylor expansion in a certain disk $D(0;R)$ centered at 0, which is of the form

$$f(t) = \frac{1}{1 - \sum_{j=0}^{\infty} a_j t^{j+1}} = \sum_{n=0}^{\infty} w_n t^n. \tag{4.4}$$

And the identity $Q(t)f(t) = 1$ implies that we have $w_{n+1} = \sum_{j=0}^{n} a_j w_{n-j}$, for all $n \geq 0$, where $w_0 = 1$ and $w_{-j} = 0$ for all $j \geq 1$. Hence, $\{w_n\}_{n\in\mathbb{Z}}$ is nothing else but the sequence $\{v_n\}_{n\in\mathbb{Z}}$ defined by (4.3).

4.2. Generalized Bernoulli numbers $B_n(\lambda)$ by recursiveness of infinite order

Let $\{B_n(\lambda)\}_{n\geq 0}$ be the sequence of Bernoulli numbers defined by their associated generating function,

$$\frac{t}{e^{\lambda t} - 1} = \sum_{n=0}^{+\infty} B_n(\lambda) \frac{t^n}{n!}. \tag{4.5}$$

For $\lambda = 1$, Expression (4.5) allows us to get the usual Bernoulli numbers, namely, $B_n(1) = B_n$. Also, we have $e^{\lambda t} - 1 = \lambda t \left[1 + \sum_{n=0}^{+\infty} \frac{\lambda^{n+1}}{(n+2)!} t^{n+1}\right]$. The left-side of (4.5) can be written as follows $\frac{t}{e^{\lambda t} - 1} = \frac{1}{Q_\lambda(t)}$, where $Q_\lambda(t) = 1 - \sum_{n=0}^{+\infty} b_n(\lambda) t^{n+1}$, with $b_n(\lambda) = -\frac{\lambda^{n+1}}{(n+2)!}$.

Moreover, we have,

$$\frac{t}{e^{\lambda t}-1}=\frac{1}{\lambda}\frac{1}{Q_\lambda(t)}=\frac{1}{\lambda}\sum_{n=0}^{+\infty}v_n(\lambda)t^n=\sum_{n=0}^{+\infty}n!\frac{v_n(\lambda)}{\lambda}\frac{t^n}{n!}, \tag{4.6}$$

where $\{v_n(\lambda)\}_{n\in\mathbb{Z}}$ is a sequence (4.1) of coefficients $b_n(\lambda)=-\frac{\lambda^{n+1}}{(n+2)!}$ and initial conditions $v_0(\lambda)=1$ and $v_{-k}(\lambda)=0$, for every $k\geq 1$. Comparing with the right sides of (4.5)–(4.6) we derive the following result.

Theorem 4.1. *The Bernoulli numbers are expressed in terms of the linear recursive sequence of infinite order* (4.1) *as follows,*

$$B_n(\lambda)=n!\times\frac{v_n(\lambda)}{\lambda}, \tag{4.7}$$

where $\{v_n(\lambda)\}_{n\in\mathbb{Z}}$ *is a sequence* (4.1) *of coefficients* $b_n(\lambda)=-\frac{\lambda^{n+1}}{(n+2)!}$ *and initial conditions* $v_0(\lambda)=1$ *and* $v_{-k}(\lambda)=0$, *for every* $k\geq 1$. *In addition, the combinatorial formula of the Bernoulli numbers is given by*

$$B_n(\lambda)=\frac{n!}{\lambda}\sum_{k_0+2k_1+\cdots+nk_{n-1}=n}(-1)^{k_0+\cdots+k_{n-1}}\frac{(k_0+\cdots+k_{n-1})!}{k_0!\cdots k_{n-1}!}\prod_{j=0}^{n-1}\left[\frac{\lambda^{j+1}}{(j+2)!}\right]^{k_j}. \tag{4.8}$$

Moreover, the sequence $\left\{\frac{B_n(\lambda)}{n!}\right\}_{n\geq 0}$ *satisfies the recursive relation* (4.1),

$$\frac{B_{n+1}(\lambda)}{(n+1)!}=a_0\frac{B_n(\lambda)}{n!}+a_1\frac{B_{n-1}(\lambda)}{(n-1)!}+\cdots+a_n\frac{B_0(\lambda)}{0!}. \tag{4.9}$$

We can show easily that $\displaystyle\prod_{j=0}^{n-1}\left[\frac{\lambda^{j+1}}{(j+2)!}\right]^{k_j}=\lambda^{\sum_{j=0}^{n-1}(j+1)k_j}\prod_{j=0}^{n-1}\left[\frac{1}{(j+2)!}\right]^{k_j}$. Since $\displaystyle\sum_{j=0}^{n-1}(j+1)k_j=n$ and

$$B_n=B_n(1)=n!\sum_{k_0+2k_1+\cdots+nk_{n-1}=n}\frac{(k_0+\cdots+k_{n-1})!}{k_0!\cdots k_{n-1}!}\prod_{j=0}^{n-1}a_j^{k_j},$$

where $a_j=-\frac{1}{(j+2)!}$, we derive that Expression (4.9) takes the following form,

$$B_n(\lambda)=\lambda^{n-1}B_n(1)=n!\lambda^{n-1}\sum_{k_0+2k_1+\cdots+nk_{n-1}=n}\frac{(k_0+\cdots+k_{n-1})!}{k_0!\cdots k_{n-1}!}\prod_{j=0}^{n-1}a_j^{k_j}. \tag{4.10}$$

Formulas (4.8) and (4.10) represent the combinatorial expression for generalized Bernoulli numbers $B_n(\lambda)$. Meanwhile, formula (4.9) gives a recursive process for generating the generalized Bernoulli numbers $B_n(\lambda)$. It seems for us that Expressions (4.8), (4.9) and (4.10), for the generalized Bernoulli numbers $B_n(\lambda)$, are not known in the literature under this form.

Remark 4.2. The function $f_\lambda(t) = \frac{t}{e^{\lambda t}-1}$ satisfies $f_\lambda(-t) = t + f_\lambda(t)$, therefore the equality $f_\lambda(-t) = \sum_{n=0}^{\infty}(-1)^n v_n(\lambda)t^n = t + \sum_{n=0}^{\infty} v_n(\lambda)t^n$ implies that $v_{2n+1} = 0$ for every $n \geq 1$. Thus, Expression (4.7) shows that $B_{2n+1}(\lambda) = 0$ for every $n \geq 1$. Since all the odd Bernoulli numbers vanish except $B_1(\lambda) = -\frac{1}{2}$.

Remark 4.3. For $\lambda = 1$ in the preceding data, we recover results on the usual Bernoulli numbers established in [3] and [19].

4.3. Recursive approach for generalized Bernoulli numbers $B_n(a, b)$

Let first consider the generalized Bernoulli numbers $B_n(a, b)$ defined by the following generating function,

$$\frac{t}{b^t - a^t} = \frac{t}{e^{\beta t} - e^{\alpha t}} = \sum_{n=0}^{+\infty} B_n(a, b)\frac{t^n}{n!}\ , \tag{4.11}$$

where$a$, b are positive numbers, $\alpha = \ln a$ and $\beta = \ln b$. Set $\lambda = \beta - \alpha = \ln(b) - \ln(a)$. Let $F_{a,b}(t)$ be the function $F_{a,b}(t) = \frac{t}{b^t - a^t} = \frac{t}{e^{\beta t}-e^{\alpha t}}$. We show easily that $F_{a,b}(t) = \frac{te^{-\alpha t}}{e^{\lambda t}-1}$. The basic Taylor series $e^{-\alpha t} = \sum_{p=0}^{+\infty}(-1)^p\alpha^p\frac{t^p}{p!}$ and $\frac{t}{e^{\lambda t}-1} = \frac{1}{\lambda}\frac{1}{1-\sum_{j=0}^{+\infty} b_j t^j}$, where $b_j = -\frac{\lambda^{j+1}}{(j+2)!}$, implies that we have,

$$F_{a,b}(t) = \sum_{n=0}^{+\infty}\left[\sum_{s=0}^{n}\frac{(-\alpha)^s}{s!}\frac{v_{n-s}(\lambda)}{\lambda}\right]t^n = \sum_{n=0}^{+\infty} n!\left[\sum_{s=0}^{n}\frac{(-\alpha)^s}{s!}\frac{v_{n-s}(\lambda)}{\lambda}\right]\frac{t^n}{n!}, \tag{4.12}$$

where $\{v_n(\lambda)\}_{n\in\mathbb{Z}}$ is a sequence (4.1) of coefficients $b_n(\lambda) = -\frac{\lambda^{n+1}}{(n+2)!}$ and initial conditions $v_0(\lambda) = 1$ and $v_{-k}(\lambda) = 0$, for every $k \geq 1$. Using Expressions (4.7), (4.10), and comparing with (4.11), we derive the following result.

Theorem 4.4. *Let $a, b > 0$ and set $\alpha = \ln a$, $\lambda = \ln b - \ln a$. The generalized Bernoulli numbers $B_n(a, b)$ given by (4.11), satisfy the following properties,*
1) For every $n \geq 1$, the linear recursive relation is verified,

$$\frac{B_n(a, b)}{n!} = \sum_{s=0}^{n}\frac{(-\alpha)^s}{s!}\frac{v_{n-s}(\lambda)}{\lambda},$$

where $\{v_n(\lambda)\}_{n\in\mathbb{Z}}$ is a sequence (4.1) of coefficients $b_n(\lambda) = -\frac{\lambda^{n+1}}{(n+2)!}$ and initial conditions $v_0(\lambda) = 1$ and $v_{-k}(\lambda) = 0$, for every $k \geq 1$.

2) For every $n \geq 1$, the combinatorial expression of the Bernoulli numbers $B_n(a,b)$ is,

$$B_n(a,b) = n!\lambda^{n-1}\sum_{s=0}^{n}(-1)^{n-s}\frac{\alpha^{n-s}}{(n-s)!}\sum_{k_0+2k_1+\cdots+nk_{n-1}=n}\frac{(k_0+\cdots+k_{n-1})!}{k_0!\cdots k_{n-1}!}\prod_{j=0}^{n-1}a_j^{k_j},$$

where $a_j = -\frac{1}{(j+2)!}$.

Taking into account, results of Theorems 4.1, 4.4 and Expression (4.10), we give below the expression of the $B_n(a,b)$ in terms of the usual Bernoulli numbers B_n.

Corollary 4.5. *Under the data of Theorem 4.4, the expression of the $B_n(a,b)$ in terms of the usual Bernoulli numbers B_n is,*

$$B_n(a,b) = (\ln b/a)^{n-1}\sum_{s=0}^{n}\binom{n}{s}(-\ln a)^{n-s}B_s, \tag{4.13}$$

for every $n \geq 1$.

Moreover, the generalized Bernoulli numbers $B_n(a,b)$ satisfy the following recursive relation,

$$B_{n+1}(a,b) = (-1)^{n+1}\frac{(\ln a)^{n+1}}{\ln b/a}\sum_{j=0}^{n}\frac{(\ln b/a)^{j+1}}{(j+1)!}\binom{n+1}{j+1}B_{n-j}(a,b),$$

for every $n \geq 0, b \neq a$ and $(a,b) \neq (1,1)$.

4.4. Recursive approach for generalized Bernoulli polynomials $\mathfrak{B}_n(x;a,b,c)$

Let a, b, $c > 0$, with $a \neq b$, and set $\alpha = \ln a$, $\beta = \ln b$, $\gamma = \ln c$ and $\lambda = \ln b - \ln a$. The associated generalized Bernoulli polynomials $\mathfrak{B}_n(x;a,b,c)$ are defined by (1.1), namely, $\frac{t}{b^t - a^t}c^{xt} = \frac{t}{e^{\beta t} - e^{\alpha t}}e^{\gamma xt} = \sum_{n=0}^{+\infty}\mathfrak{B}_n(x;a,b,c)\frac{t^n}{n!}$. Following Expression (4.12) we have $\frac{t}{b^t - a^t} = \sum_{n=0}^{+\infty} n!\left[\sum_{s=0}^{n}\frac{(-\alpha)^s}{s!}\frac{v_{n-s}(\lambda)}{\lambda}\right]\frac{t^n}{n!}$, where $\{v_n(\lambda)\}_{n\in\mathbb{Z}}$ is a sequence (4.1) of coefficients $b_n(\lambda) = -\frac{\lambda^{n+1}}{(n+2)!}$ and initial conditions $v_0(\lambda) = 1$ and $v_{-k}(\lambda) = 0$, for every $k \geq 1$. Since $e^{\gamma xt} = \sum_{n=0}^{+\infty}\gamma^n x^n\frac{t^n}{n!}$, we derive,

$$\frac{t}{b^t - a^t}c^{xt} = \left[\sum_{n=0}^{+\infty}\gamma^n x^n\frac{t^n}{n!}\right]\left[\sum_{n=0}^{+\infty}\Omega_n(\lambda)\frac{t^n}{n!}\right] = \sum_{n=0}^{+\infty} n!\left[\sum_{k+m=n}\frac{\gamma^k}{k!}x^k\frac{\Omega_m(\lambda)}{m!}\right]\frac{t^n}{n!},$$

where $\Omega_n(\lambda) = n!\left[\sum_{s=0}^{n} \frac{(-\alpha)^s}{s!}\frac{v_{n-s}(\lambda)}{\lambda}\right]$. Comparing the former formula with Expression (1.1), we derive that the generalized Bernoulli polynomials $\mathfrak{B}_n(x; a, b, c)$ are given by,

$$\mathfrak{B}_n(x; a, b, c) = n! \sum_{k+m=n} \frac{\gamma^k}{k!}\frac{\Omega_m(\lambda)}{m!} x^k = n! \sum_{k=0}^{n} \frac{\gamma^k}{k!}\frac{\Omega_{n-k}(\lambda)}{(n-k)!} x^k.$$

Theorem 4.6. *Under the preceding data, the generalized Bernoulli polynomials $\mathfrak{B}_n(x; a, b, c)$ are given by one of the following three equivalent formulas,*

$$\mathfrak{B}_n(x; a, b, c) = \sum_{k=0}^{n} (n-k)! \binom{n}{k} \gamma^k \left[\sum_{s=0}^{n-k} \frac{(-\alpha)^s}{s!}\frac{v_{n-k-s}(\lambda)}{\lambda}\right] x^k,$$

$$\mathfrak{B}_n(x; a, b, c) = \sum_{k=0}^{n} (n-k)! \binom{n}{k} \gamma^k \left[\sum_{s=0}^{n-k} \frac{(-\alpha)^s}{s!}\frac{B_{n-k-s}(\lambda)}{(n-k-s)!}\right] x^k,$$

$$\mathfrak{B}_n(x; a, b, c) = \sum_{k=0}^{n} \binom{n}{k} \gamma^k \left[\sum_{s=0}^{n-k} \binom{n-k}{s} (-\alpha)^s B_{n-k-s}(\lambda)\right] x^k,$$

where

$$B_n(\lambda) = \frac{n!}{\lambda} \sum_{k_0+2k_1+\cdots+nk_{n-1}=n} (-1)^{k_0+\ldots+k_{n-1}} \frac{(k_0+\cdots+k_{n-1})!}{k_0!\cdots k_{n-1}!} \prod_{j=0}^{n-1} \left[\frac{\lambda^{j+1}}{(j+2)!}\right]^{k_j}.$$

Taking into account, results of Theorems 4.1, 4.4, Corollary 4.5 and Expressions (4.10), (4.13), we can show that the generalized Bernoulli polynomials $\mathfrak{B}_n(x; a, b, c)$ can be formulated in terms of the generalized Bernoulli numbers $B_n(a, b)$ and the usual Bernoulli numbers B_n.

Corollary 4.7. *In terms of the generalized Bernoulli numbers $B_n(a, b)$, the generalized Bernoulli polynomials $\mathfrak{B}_n(x; a, b, c)$ are given by,*

$$\mathfrak{B}_n(x; a, b, c) = \sum_{k=0}^{n} \binom{n}{k} \gamma^k B_{n-k}(a, b) x^k,$$

for every $n \geq 1$, where $\gamma = \ln c$. In terms of the usual Bernoulli numbers B_n, we have the following formula,

$$\mathfrak{B}_n(x; a, b, c) = \sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k-1} \gamma^k \left[\sum_{s=0}^{n-k} \binom{n-k}{s} (-\alpha)^{n-k-s} B_s\right] x^k,$$

for every $n \geq 1$.

Through Expression (4.10), with $\lambda = 1$, and the Corollary 4.7, we can arrive at the combinatorial expression of the generalized Bernoulli polynomials.

Corollary 4.8. *The combinatorial formula of the generalized Bernoulli polynomials $\mathfrak{B}_n(x;a,b,c)$ is given by,*

$$\mathfrak{B}_n(x;a,b,c) = \sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k-1}\gamma^k \left[\sum_{s=0}^{n-k} s!\binom{n-k}{s}(-\alpha)^{n-k-s}B_s\right] x^k,$$

for every $n \geq 1$, where $\gamma = \ln c$, $B_0 = 1$ and

$$B_s = s! \sum_{k_0+2k_1+\cdots+sk_{s-1}=s} \frac{(k_0+\cdots+k_{n-1})!}{k_0!\cdots k_{s-1}!} \prod_{j=0}^{s-1} a_j^{k_j}.$$

5. Linear recursive approach for generalized Euler polynomials

5.1. Generalized Euler numbers $E_n(\lambda)$ and $E_n(a,b)$ by recursiveness of order ∞

The generalized Euler numbers $E_n(\lambda)$ are defined by their generating function as follows,

$$\frac{2}{e^{\lambda t}+1} = \sum_{n=0}^{+\infty} E_n(\lambda)\frac{t^n}{n!}. \tag{5.1}$$

We can show easily that for $\lambda = 1$, we get the usual Euler numbers, namely, $E_n(1) = E_n$. The process used for expressing the Bernoulli numbers in terms of linear recursive relations (4.1), can also be applied for Euler numbers E_n. Indeed, we have

$$\frac{2}{e^{\lambda t}+1} = \frac{2}{2+\sum_{n=0}^{+\infty}\lambda^n\frac{t^n}{n!}} = \frac{1}{1-\sum_{n=0}^{+\infty} b_n(\lambda)t^{n+1}}, \text{ where } b_n(\lambda) = -\frac{\lambda^{n+1}}{2[(n+1)!]}.$$

Therefore, with the aid of Expression (4.4), we derive

$$\frac{2}{e^{\lambda t}+1} = \sum_{n=0}^{+\infty} w_n(\lambda)t^n = \sum_{n=0}^{+\infty} n!w_n(\lambda)\frac{t^n}{n!},$$

where $\{w_n(\lambda)\}_{n\in\mathbb{Z}}$ is a sequence (4.1) of coefficients $b_n(\lambda) = -\frac{\lambda^{n+1}}{2[(n+1)!]}$ and initial conditions $w_0 = 1$, $w_{-k} = 0$ for $k \geq 1$. Using (4.4) and comparing with (5.1), we obtain,

$$E_n(\lambda) = n!w_n(\lambda) = n! \sum_{k_0+2k_1+\cdots+nk_{n-1}=n} \frac{(k_0+\cdots+k_{n-1})!}{k_0!\cdots k_{n-1}!} \prod_{j=0}^{n-1}\left[\frac{-\lambda^{j+1}}{2[(j+1)!]}\right]^{k_j}. \tag{5.2}$$

For $\lambda = 1$, we get the following expression for Euler numbers,

$$E_n = E_n(1) = n!w_n(1) = n! \sum_{k_0+2k_1+\cdots+nk_{n-1}=n} \frac{(k_0+\cdots+k_{n-1})!}{k_0!\cdots k_{n-1}!} \prod_{j=0}^{n-1} b_j^{k_j}, \tag{5.3}$$

where $b_j = \frac{-1}{2[(j+1)!]}$. Expression (5.3) has been established in [3]. Since $b_j(\lambda) = \lambda^{j+1}[\frac{-1}{2[(n+1)!]}] = \lambda^{j+1} b_j$, we derive that, $\prod_{j=0}^{n-1} b_j^{k_j}(\lambda) = \lambda^{\sum_{j=0}^{n-1}(j+1)k_j} \prod_{j=0}^{n-1} b_j^{k_j} = \lambda^n \prod_{j=0}^{n-1} b_j^{k_j}$, for $\sum_{j=0}^{n-1}(j+1)k_j = n$. Therefore, Expressions (5.2)-(5.3), imply that,

$$E_n(\lambda) = n!\lambda^n \sum_{k_0+2k_1+\cdots+nk_{n-1}=n} \frac{(k_0+\cdots+k_{n-1})!}{k_0!\cdots k_{n-1}!} \prod_{j=0}^{n-1} b_j^{k_j}, \text{ i.e., } E_n(\lambda) = \lambda^n E_n.$$

Let $a > 0$, $b > 0$, with $a \neq b$, and consider the generalized Euler numbers $E_n(a, b)$ defined as follows,

$$\frac{2}{b^t + a^t} = \sum_{n=0}^{+\infty} E_n(a, b) \frac{t^n}{n!}. \tag{5.4}$$

On the other hand, let $\lambda = \ln b - \ln a$, $\alpha = \ln a$ we have $\dfrac{2}{b^t + a^t} = \dfrac{2}{e^{\lambda t} + 1} e^{-\alpha t}$, therefore,

$$\frac{2}{e^{\lambda t}+1} e^{-\alpha t} = \left[\sum_{n=0}^{+\infty} n! w_n(\lambda) \frac{t^n}{n!}\right] \left[\sum_{n=0}^{+\infty} (-\alpha)^n \frac{t^n}{n!}\right] = \left[\sum_{n=0}^{+\infty} E_n(\lambda) \frac{t^n}{n!}\right] \left[\sum_{n=0}^{+\infty} (-\alpha)^n \frac{t^n}{n!}\right],$$

where $E_n(\lambda) = n! w_n(\lambda)$ and $\{w_n(\lambda)\}_{n\in\mathbb{Z}}$ is a sequence (4.1) of coefficients $b_n(\lambda) = -\frac{\lambda^{n+1}}{2[(n+1)!]}$ and initial conditions $w_0 = 1$, $w_{-k} = 0$ for $k \geq 1$. Therefore, we have,

$$\frac{2}{b^t + a^t} = \sum_{n=0}^{+\infty} n! \left[\sum_{p=0}^{n} \frac{E_p(\lambda)}{p!} \frac{(-\alpha)^{n-p}}{(n-p)!}\right] \frac{t^n}{n!} = \sum_{n=0}^{+\infty} \left[\sum_{p=0}^{n} \binom{n}{p} E_p(\lambda)(-\alpha)^{n-p}\right] \frac{t^n}{n!}.$$

Comparing with Expression (5.4), we derive,

$$E_n(a, b) = \sum_{p=0}^{n} \binom{n}{p} E_p(\lambda)(-\alpha)^{n-p} = \sum_{p=0}^{n} \binom{n}{p} \lambda^p (-\alpha)^{n-p} E_p, \tag{5.5}$$

since $E_n(\lambda) = \lambda^n E_n$, where E_n are the usual Euler numbers, namely.

$$E_n(a, b) = \sum_{p=0}^{n} \binom{n}{p} (\ln b - \ln a)^p (-\ln a)^{n-p} E_p. \tag{5.6}$$

5.2. Recursive approach for the generalized Euler polynomials $E_n(x; a, b, c)$

Let a, b, $c > 0$ be real numbers, with $a \neq b$, $c \neq 1$, and set $\alpha = \ln a$, $\beta = \ln b$, $\gamma = \ln c$ and $\lambda = \ln b - \ln a$. The associated generalized Euler polynomials

$\mathfrak{E}_n(x;a,b,c)$ are defined by (1.2), namely, $\dfrac{2}{b^t+a^t}c^{xt} = \sum_{n=0}^{+\infty} \mathfrak{E}_n(x;a,b,c)\dfrac{t^n}{n!}$. Following Expression (5.4) we have $\dfrac{2}{b^t+a^t} = \sum_{n=0}^{+\infty} E_n(a,b)\dfrac{t^n}{n!}$, we derive,

$$\frac{2}{b^t+a^t}c^{xt} = \left[\sum_{n=0}^{+\infty} E_n(a,b)\frac{t^n}{n!}\right]\left[\sum_{n=0}^{+\infty} \gamma^n x^n \frac{t^n}{n!}\right] = \sum_{n=0}^{+\infty}\left[\sum_{p=0}^{n}\binom{n}{p}E_{n-p}(a,b)\gamma^p x^p\right]\frac{t^n}{n!},$$

where $E_n(a,b)$ is given by Expressions (5.5)–(5.6).

Theorem 5.1. *Let a, b, $c > 0$ be real numbers, with $a \neq b$ and $c \neq 1$. Then, the generalized Euler polynomials $\mathfrak{E}_n(x;a,b,c)$ are given by,*

$$\mathfrak{E}_n(x;a,b,c) = \sum_{p=0}^{n}\binom{n}{p}E_{n-p}(a,b)(\ln c)^p x^p, \text{ for every } n \geq 0, \tag{5.7}$$

where $E_n(a,b)$ is given by Expressions (5.5)–(5.6), or equivalently,

$$\mathfrak{E}_n(x;a,b,c) = \sum_{p=0}^{n}\binom{n}{p}\left[\sum_{k=0}^{n-p}\binom{n-p}{k}(\ln b - \ln a)^k(-\ln a)^{n-p-k}E_k\right](\ln c)^p x^p, \tag{5.8}$$

where E_k are the usual Euler numbers.

Expression (5.7) shows that the generalized Euler polynomials $\mathfrak{E}_n(x;a,b,c)$ are expressed in terms of the generalized Euler numbers $E_n(a,b)$ and $\ln c$. Meanwhile, Expression (5.8) shows that the generalized Euler polynomials $\mathfrak{E}_n(x;a,b,c)$ are expressed in terms of the usual Euler numbers E_n and, the real numbers $\ln a$, $\ln b$ and $\ln c$. In the best of our knowledge these two formulas are not current in the literature.

Moreover, utilizing Expression (5.3), namely, $E_k = k!\sum_{\mathcal{S}_k}\binom{s_0+\cdots+s_{k-1}}{s_0,\ldots,s_{k-1}}\prod_{j=0}^{k-1} b_j^{s_j}$, where $\mathcal{S}_k = \{(s_0,s_1,\ldots,s_{k-1}); s_0+2s_1+\cdots+ks_{k-1}=k\}$ and $\binom{s_0+\cdots+s_{k-1}}{s_0,\ldots,s_{k-1}} = \frac{(s_0+\cdots+s_{k-1})!}{s_0!\cdots s_{n-1}!}$, we get the following corollary.

Corollary 5.2. *Let $a,b,c > 0$ be real numbers, with $a \neq b$ and $c \neq 1$. Then, the combinatorial expression of the generalized Euler polynomials $\mathfrak{E}_n(x;a,b,c)$ is given by,*

$$\mathfrak{E}_n(x;a,b,c) = \sum_{p=0}^{n}\binom{n}{p}\left[\sum_{k=0}^{n-p} k!\binom{n-p}{k}\lambda^k(-\alpha)^{n-p-k}\sum_{\mathcal{S}_k}\binom{s_0+\cdots+s_{k-1}}{s_0,\ldots,s_{k-1}}\prod_{j=0}^{k-1} b_j^{s_j}\right]\gamma^p x^p,$$

where $\mathcal{S}_k = \{(s_0,s_1,\ldots,s_{k-1}); s_0+2s_1+\cdots+ks_{k-1}=k\}$ and
$\displaystyle\binom{s_0+\cdots+s_{k-1}}{s_0,\ldots,s_{k-1}} = \frac{(s_0+\cdots+s_{k-1})!}{s_0!\cdots s_{n-1}!}$.

6. Concluding remarks and perspectives

In the preceding sections, we had used two approaches for studying the generalized Bernoulli and Euler polynomials, namely, the determinantal approach and the linear recursive sequences of order infinity. These approaches have allowed us to establish some new explicit compact formulas, for the generalized Bernoulli and Euler polynomials (1.1)–(1.2), in terms of generalized Bernoulli numbers and generalized Euler numbers, or the usual Bernoulli and Euler numbers. In addition, new properties were established and other known identities are recovered.

It seems for us that our approaches, for the generalized Bernoulli and Euler polynomial (1.1)–(1.2), are not current in the literature. On the other hand, it appears to us that these approaches can be applied to the generalized Genocchi polynomials.

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