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Entropy solutions for some nonlinear $p(x)$-parabolic problems with degenerate coercivity

Arij Bouzelmate, Youssef Hajji* and Hassane Hjiaj

Abstract. *This paper is concerned with the study of the non-coercive $p(x)$-parabolic problems*

$$\begin{cases} u_t + Au + F(x,t,\nabla u) + \delta(x,t)|u|^{p(x)-2}u = f(x,t,u) & \text{in } Q_T,\\ u(x,t) = 0 & \text{on } S_T,\\ u(x,0) = u_0(x) & \text{in } \Omega,\end{cases}$$

where the initial condition $u_0 \in L^1(\Omega)$ and $\delta(x,t)$ is the positive function belong to $L^\infty(Q_T)$. We prove the existence of entropy solutions for this parabolic equation, and we will conclude some regularity results.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$. For $T > 0$, we denote by Q_T the cylinder $\Omega \times (0,T)$ and by S_T the later surface $\partial\Omega \times (0,T)$. In [14] Di Nardo, et al. have studied the nonlinear parabolic problem

$$\begin{cases} u_t - \Delta_p u + \operatorname{div}(c|u|^{\gamma-1}u) + b|\nabla u|^\delta = f - \operatorname{div} g & \text{in } Q_T,\\ u(x,t) = 0 & \text{on } S_T,\\ u(x,0) = u_0(x) & \text{in } \Omega,\end{cases} \tag{1.1}$$

where $f \in L^1(Q_T)$ and $g \in (L^{p'}(Q_T))^N$, with the initial data u_0 in $L^1(\Omega)$. They have proved the existence of renormalized solutions for this nonlinear parabolic problem, and in [15] they have proved the uniqueness of renormalized solution for the parabolic problem (1.1) (see [21]).

Alvino et al. have demonstrated in [3] the existence and regularity of solutions for the nonlinear elliptic problem with degenerate coercivity

$$\begin{cases} -\operatorname{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}\right) = f & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases} \tag{1.2}$$

where f is assumed to be in $L^m(\Omega)$ with $m \geq 1$, we refer the reader also to [2] and [25].

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*Corresponding author

The domain of Sobolev spaces with variable exponent has received a much attention in recent years, the impulse for which comes from their physical applications, such in electro-rheological fluids and image processing (see [12, 22]). Bendahmane et al. have studied in [9] the parabolic problem

$$\begin{cases} u_t - \text{div } (|\nabla u|^{p(x)-2}\nabla u) = f & \text{in } \quad Q_T, \\ u \quad = 0 & \text{on } \quad \Sigma_T, \\ u(x,0) = u_0(x) & \text{in } \quad \Omega, \end{cases} \tag{1.3}$$

with $f \in L^1(\Omega)$, they have proved the existence and uniqueness of the renormalized solutions to this nonlinear parabolic problem. Moreover, they have established some regularity results, (we refer the reader to [4, 5, 6, 10, 8] for more details). In [7], Azroul et al. have proved the existence of entropy solutions for the following equations, whose prototype is given by:

$$\begin{cases} u_t + Au + g(x,t,u,\nabla u) = f - \text{div}\phi(u) & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } S_T, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.4}$$

where $f \in L^1(Q_T)$, $Au = -\text{ div}(a(x,t,\nabla u) + \delta\, |u|^{p(x)-2}u$ with $\delta > 0$, and $u_0 \in L^1(\Omega)$. Moreover, Chrif et al. have proved in [13] the existence of entropy solutions for the problem (1.4) in the anisotropic Sobolev space.

In this paper, we will establish the existence of entropy solutions for the strongly nonlinear parabolic problem of the form

$$\begin{cases} u_t + Au + F(x,t,\nabla u) = f(x,t,u) & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } S_T, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.5}$$

where Au is a Leray–Lions operator with degenerate coercivity. The Carathéodory functions $F(x,t,\xi)$ and $f(x,t,s)$ satisfy only some growth conditions, and the initial data $u_0(x)$ is assumed to belongs to $L^1(\Omega)$.

This paper is organized as follows: in section 2, we recall some definitions and basic properties concerning Sobolev spaces with variable exponents. We introduce in section 3 the assumptions on the Carathéodory function $a(x,t,s,\xi)$, $F(x,t,\xi)$ and $f(x,t,s)$ for which our problem has at least one solution. Section 4 contains some important lemmas that are useful to prove our main result. The last section 5 is devoted to show the existence of entropy solutions for our quasilinear non-coercive $p(x)$-parabolic problem (1.5).

2. Preliminaries

Let Ω open bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with boundary $\partial\Omega$, we denote

$$C_+(\bar{\Omega}) = \{\text{measurable function} \quad p(\cdot)\colon \bar{\Omega} \longmapsto \mathbb{R} \quad \text{such that} \quad 1 < p_- \leq p_+ < \infty\},$$

where

$$p_- = \text{essinf}\{p(x)/\ x \in \bar{\Omega}\} \qquad \text{and} \qquad p_+ = \text{esssup}\{p(x)/\ x \in \bar{\Omega}\}.$$

We define the variable exponent Lebesgue space for $p(.) \in C_+(\bar{\Omega})$, by

$$L^{p(\cdot)}(\Omega) = \{u\colon \Omega \longmapsto \mathbb{R} \quad \text{measurable } / \ \int_\Omega |u(x)|^{p(x)}\, dx < \infty\},$$

Under the norm

$$\|u\|_{p(\cdot)} = \inf\Big\{\lambda > 0, \quad \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)}\, dx \leq \infty\Big\} \tag{2.1}$$

is a uniformly convex Banach space, and therefore reflexive. Let $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}$ with $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$, we present the generalized Hölder's inequality by

$$\Big|\int_\Omega uv\, dx\Big| \leq \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}. \tag{2.2}$$

Proposition 2.1 (see [17, 24])**.** *We define the modular $\rho(u)$ by*

$$\rho(u) = \int_\Omega |u|^{p(x)}\, dx \qquad \forall u \in L^{p(\cdot)}(\Omega),$$

then:

1. $\|u\|_{p(\cdot)} < 1 \quad (resp, = 1, > 1) \quad \Longleftrightarrow \quad \rho(u) < 1 \quad (resp, = 1, > 1)$,
2. $\|u\|_{p(\cdot)} > 1 \Longrightarrow \|u\|_{p(\cdot)}^{p_-} \leq \rho(u) \leq \|u\|_{p(\cdot)}^{p_+}$ *and* $\|u\|_{p(\cdot)} < 1 \Longrightarrow \|u\|_{p(\cdot)}^{p_+} \leq \rho(u) \leq \|u\|_{p(\cdot)}^{p_-}$,
3. $\|u_n\|_{p(\cdot)} \to 0 \Longleftrightarrow \rho(u_n) \to 0$, *and* $\|u_n\|_{p(\cdot)} \to \infty \Longleftrightarrow \rho(u_n) \to \infty$.

Now, we define the variable exponent Sobolev space by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) \quad \text{and} \quad |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

normed by

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)} \qquad \forall u \in W^{1,p(\cdot)}(\Omega). \tag{2.3}$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ for the norm topology of (2.3).

Proposition 2.2 (see [17])**.**

1. *Let $p(\cdot) \in C_+(\bar{\Omega})$, then the spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces.*

2. *We define the Sobolev exponent:* $p^*(x) = \dfrac{Np(x)}{N-p(x)}$ *for* $p(x) < N$. *If* $q(\cdot) \in C_+(\bar{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \Omega$, *then the embedding* $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$ *is continuous and compact.*

Let $T > 0$, we introduce the space V by

$$V = \Big\{u \in L^{p_-}(0,T; W_0^{1,p(\cdot)}(\Omega)) \text{ such that } u \in L^{p(\cdot)}(Q_T) \text{ and } |\nabla u| \in L^{p(\cdot)}(Q_T)\Big\}.$$

We define the modular $\rho_{1,p(\cdot)}(u)$ for any $u \in V$ by

$$\rho_{1,p(\cdot)}(u) = \int_{Q_T} |u|^{p(x)}\, dx\, dt + \int_{Q_T} |\nabla u|^{p(x)}\, dx\, dt.$$

The space V endowed by the norm

$$\|u\|_V = \|u\|_{L^{p(\cdot)}(Q_T)} + \|\nabla u\|_{L^{p(\cdot)}(Q_T)}$$

is a separable and reflexive Banach space.

Lemma 2.3 ([23]). *Let* B_0, B *and* B_1 *be some Banach spaces, with* $B_0 \subset B \subset B_1$. *Let us set*

$$Y = \big\{u \;:\; u \in L^{p_0}(0,T;B_0) \quad \textit{and} \quad u' \in L^{p_1}(0,T;B_1)\big\}$$

where $p_0 > 1$ *and* $p_1 > 1$ *are reals numbers. Assuming that the embedding* $B_0 \hookrightarrow\hookrightarrow B$ *be compact, then*

$$Y \hookrightarrow\hookrightarrow L^{p_0}(0,T;B)$$

is a compact imbedding.

Remark 2.4. Let $p_- > \dfrac{2N}{N+2}$, we set

$$B_0 = W_0^{1,p(\cdot)}(\Omega), \qquad B = L^2(\Omega) \qquad \text{and} \qquad B_1 = W^{-1,p'(\cdot)}(\Omega),$$

with $p_0 = p_-$ and $p_1 = (p_+)'$. In view of the Lemma 2.3, we obtain

$$\big\{u \;:\; u \in V \quad \text{and} \quad u' \in V^\star\big\} \subseteq Y \hookrightarrow\hookrightarrow L^1(Q_T). \tag{2.4}$$

Moreover, in view of [9], we have

$$\big\{u \;:\; u \in V \quad \text{and} \quad u' \in V^\star\big\} \subseteq C([0,T]; L^1(\Omega)). \tag{2.5}$$

3. Essential assumptions

Let $Q_T = \Omega \times (0,T)$ with $0 < T < \infty$ and taking $p(.) \in C_+(\bar{\Omega})$ such that $\frac{2N}{N+2} < p_- \leq p_+ < \infty$. We consider the Leray-Lions operator A acting from V into its dual V^* defined by

$$Au = -\text{div}\, a(x,t,u,\nabla u) + \delta(x,t)|u|^{p(x)-2}u,$$

where $\delta(x,t) \in L^\infty(Q_T)$ such that there exists a constant $\delta_0 > 0$ with $\delta(x,t) \geq \delta_0$ a.e. in Q_T, and the Carathéodory function $a(x,t,s,\xi)\colon Q_T \times \mathbb{R} \times \mathbb{R}^N \longmapsto \mathbb{R}^N$ satisfies the following conditions:

$$|a(x,t,s,\xi)| \leq \beta(K(x,t) + |s|^{p(x)-1} + |\xi|^{p(x)-1}), \tag{3.1}$$

where $K(x,t)$ is a nonnegative function lying in $L^{p'(\cdot)}(Q_T)$ and $\beta > 0$.

$$(a(x,t,s,\xi) - a(x,t,s,\xi'))(\xi - \xi') > 0 \quad \text{for any} \quad \xi \neq \xi', \tag{3.2}$$

$$a(x,t,s,\xi) \cdot \xi \geq b(x,|s|)|\xi|^{p(x)}, \tag{3.3}$$

with $b(x,\cdot)\colon \Omega \times \mathbb{R} \mapsto \mathbb{R}^+$ is a decreasing function for a.e. $x \in \Omega$ such that there exists a positive constant b_0 which verifies

$$\frac{b_0}{(1+|s|)^{\lambda(x)}} \leq b(x,|s|) \qquad \text{for any} \quad s \in \mathbb{R}, \tag{3.4}$$

where $\lambda(x)$ is a measurable function, such that $0 \leq \lambda(x) < p(x) - 1$ a.e. in Ω.

The lower order term $F(x,t,\xi)$ is a Carathéodory function which satisfies only the growth condition:

$$|F(x,t,\xi)| \leq c(x,t)|\xi|^{q(x)}, \tag{3.5}$$

where $0 < q(x) < \dfrac{p(x)(p(x)-1)}{\lambda(x)+p(x)}$ a.e. in Ω and $c(x,t) \in L^{m(x)}(Q_T)$ with

$$m(x) > \frac{p(x)(p(x)-1)}{p(x)(p(x)-q(x)-1) - \lambda(x)q(x)} \quad \text{a.e. in } \Omega.$$

The Carathéodory functions $f(x,t,s):\ Q_T \times \mathbb{R} \longmapsto \mathbb{R}$ fulfills the growth condition:

$$|f(x,t,s)| \leq f_0(x,t) + d(x,t)|s|^{\gamma(x)}, \tag{3.6}$$

where $f_0 \in L^1(Q_T)$, with $0 \leq \gamma(x) < p(x) - 1$ a.e. in Ω and $d(x,t) \in L^{r(x)}(Q_T)$ such that $r(x) \geq \dfrac{p(x)-1}{p(x)-\gamma(x)-1}$ a.e. in Ω.

We consider the quasilinear parabolic problem

$$\begin{cases} u_t + Au + F(x,t,\nabla u) = f(x,t,u) & \text{in } Q_T = \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } S_T = \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3.7}$$

with $u_0 \in L^1(\Omega)$.

4. Some technical Lemmas

Let $\mu \geq 0$, we define the time mollification u_μ of a function $u \in V$, by

$$u_\mu(x,t) = \mu \int_{-\infty}^{t} \bar{u}(x,s)\exp(\mu(s-t))\, ds \qquad \text{where} \quad \bar{u}(x,s) = u(x,s)\chi_{(0,T)}(s).$$

Proposition 4.1 (see [1]).

1. *If* $u \in L^{p(\cdot)}(Q_T)$, *then* $u_\mu \in L^{p(\cdot)}(Q_T)$, *moreover* $\dfrac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$ *and*
$$\|u_\mu\|_{L^{p(\cdot)}(Q_T)} \le \|u\|_{L^{p(\cdot)}(Q_T)}.$$
2. *If* $u \in W_0^{1,p(\cdot)}(Q_T)$, *then* $u_\mu \longrightarrow u$ *strongly in* $W_0^{1,p(\cdot)}(Q_T)$ *as* $\mu \to \infty$.
3. *If* $u_n \to u$ *strongly in* $W_0^{1,p(\cdot)}(Q_T)$, *then* $(u_n)_\mu \to u_\mu$ *strongly in* $W_0^{1,p(\cdot)}(Q_T)$ *as* $n \to \infty$.

Lemma 4.2 (see [18, Theorem 13.47]). *Let* $(u_n)_n$ *be a sequence in* $L^1(\Omega)$ *and* $u \in L^1(\Omega)$ *such that*

(i) $u_n \to u$ *a.e. in* Ω,

(ii) $u_n \ge 0$ *and* $u \ge 0$ *a.e. in* Ω,

(iii) $\displaystyle\int_\Omega u_n\,dx \to \int_\Omega u\,dx$,

then $u_n \to u$ *in* $L^1(\Omega)$.

Lemma 4.3 (see [1]). *Let* $g \in L^{p(\cdot)}(\Omega)$ *and* $g_n \in L^{p(\cdot)}(\Omega)$ *with* $\|g_n\|_{p(.)} \le C$ *for* $1 < p(x) < \infty$. *If* $g_n(x) \to g(x)$ *almost everywhere in* Ω, *then* $g_n \rightharpoonup g$ *weakly in* $L^{p(\cdot)}(\Omega)$.

Lemma 4.4 (see [7]). *Assuming that* (3.1)–(3.3) *hold, and let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence in* V *such that* $\dfrac{\partial u_n}{\partial t} \in V^\star$ *and* $u_n \rightharpoonup u$ *weakly in* V *with*

$$\int_{Q_T} \Big(a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)\Big)(\nabla u_n - \nabla u)\,dx\,dt + \int_{Q_T} \Big(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\Big)(u_n - u)\,dx\,dt \longrightarrow 0, \tag{4.1}$$

then $u_n \to u$ *in* V *for a subsequence.*

5. Main results: Existence of entropy solutions

Let $T_k(s) = \max(-k, \min(s,k))$, we set

$$S_k(r) = \int_0^r T_k(s)\,ds = \begin{cases} \frac{r^2}{2} & \text{if} \quad |r| \le k, \\ k|r| - \frac{k^2}{2} & \text{if} \quad |r| > k. \end{cases}$$

Firstly, we introduce the definition of entropy solutions for our degenerated $p(x)$-parabolic problem.

Definition 5.1. A measurable function u is called entropy solution of the parabolic problem (3.7) if $T_k(u) \in V$, $F(x,t,\nabla u) \in L^1(Q_T)$, $f(x,t,u) \in L^1(Q_T)$, and

$$\begin{aligned}&\int_\Omega S_k(u-\psi)(T)\,dx - \int_\Omega S_k(u-\psi)(0)\,dx + \int_{Q_T} \frac{\partial\psi}{\partial t} T_k(u-\psi)\,dx\,dt\\&+\int_{Q_T} a(x,t,u,\nabla u)\cdot\nabla T_k(u-\psi)\,dx\,dt + \int_{Q_T} F(x,t,\nabla u)T_k(u-\psi)\,dx\,dt\\&\leq \int_{Q_T} f(x,t,u)T_k(u-\psi)\,dx\,dt,\end{aligned} \tag{5.1}$$

for every $\psi \in V \cap L^\infty(Q_T)$ with $\dfrac{\partial\psi}{\partial t} \in V + L^1(Q_T)$.

The goal of the present paper is to prove the following existence result:

Theorem 5.2. *Assuming that the conditions (3.1)–(3.6) hold true, then the parabolic problem (3.7) has at least one entropy solution.*

6. Proof of Theorem 5.2

Step 1: Weak solutions for the approximate problems.

For any $n \in \mathbb{N}^*$, let $(u_{0,n})_n$ be a sequence in $C_0^\infty(\Omega)$ such that $u_{0,n} \to u_0$ in $L^1(\Omega)$ and $|u_{0,n}| \leq |u_0|$.

We consider the sequence of approximate problems:

$$\begin{cases}(u_n)_t + A_n u_n + F_n(x,t,\nabla u_n) = f_n(x,t,u_n) & \text{in } Q_T,\\ u_n(x,t) = 0 & \text{on } S_T,\\ u_n(x,0) = u_{0,n} & \text{in } \Omega,\end{cases} \tag{6.1}$$

where

$$A_n v = -\text{div}\, a(x,t,T_n(v),\nabla v) + \delta(x,t)|v|^{p(x)-2}v,$$

with

$$f_n(x,t,s) = T_n(f(x,t,s)) \qquad \text{and} \qquad F_n(x,t,\xi) = T_n(F(x,t,\xi)).$$

For all $u, v \in V$, we define the operator $G_n\colon V \longrightarrow V^*$ by

$$\int_0^T \langle G_n u, v\rangle\,dt = \int_{Q_T} F_n(x,t,\nabla u)v\,dx\,dt - \int_{Q_T} f_n(x,t,u)v\,dx\,dt.$$

In view of the Hölder's and the Poincaré's inequality, we have: for any $u, v \in V$

$$\begin{aligned}\Big|\int_0^T \langle G_n u, v\rangle\,dt\Big| &\leq \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\Big(\|F_n(x,t,\nabla u_n)\|_{L^{p'(\cdot)}(Q_T)}\|v\|_{L^{p(\cdot)}(Q_T)}\\&\qquad + \|f_n(x,t,u)\|_{L^{p'(\cdot)}(Q_T)}\|v\|_{L^{p(\cdot)}(Q_T)}\Big)\\&\leq 4(n^{p'_+}\text{meas}(Q_T)+1)^{\frac{1}{p'_-}}\|v\|_{L^{p(\cdot)}(Q_T)}\\&\leq C_0\|v\|_V.\end{aligned} \tag{6.2}$$

In view of the Lemma 7.2 (see appendix) and there exists at last one weak solution $u_n \in V$ of the problem (6.1) (see [19]).

Step 2: A priori estimates.

Let $\varphi(u_n) = \Big(1 - \frac{1}{(1+|u_n|)^{\theta-1}}\Big)\operatorname{sign}(u_n)$ with $\theta > 1$ small enough such that

$$0 < \frac{p(x)(p(x)-1)}{(p(x)-1)(p(x)-q(x)) - (\lambda(x)+\theta)q(x)} \le m(x).$$

By taking $\varphi(u_n)$ as a test function for the approximate problem (6.1), we have

$$\begin{aligned} &\int_0^T \Big\langle \frac{\partial u_n}{\partial t}, \varphi(u_n) \Big\rangle \, dt + (\theta-1)\int_{Q_T} \frac{a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla u_n}{(1+|u_n|)^{\theta}} \, dx\, dt \\ &\quad + \int_{Q_T} F_n(x,t,\nabla u_n)\varphi(u_n)\, dx\, dt + \int_{Q_T} \delta(x,t)|u_n|^{p(x)-2}u_n\varphi(u_n)\, dx\, dt \\ &= \int_{Q_T} f_n(x,t,u_n)\varphi(u_n)\, dx\, dt. \end{aligned} \tag{6.3}$$

We define

$$\Phi(s) = \begin{cases} |s| + \frac{1}{\theta-2}\frac{1}{(1+|s|)^{\theta-2}} + \frac{1}{2-\theta} & \text{for} \quad \theta \in]1,2[\cup]2,\infty[, \\ |s| - \log(1+|s|) & \text{for} \quad \theta = 2, \end{cases}$$

then $\varphi(s) = (\Phi(s))'$. According to the definition of $\Phi(\cdot)$, we have $\Phi(r) \ge 0$ and $|\Phi(r)| \le |r|$, then

$$\begin{aligned} \int_0^T \Big\langle \frac{\partial u_n}{\partial t}, \varphi(u_n) \Big\rangle \, dt &= \int_\Omega \int_0^T \frac{\partial u_n}{\partial t}\varphi(u_n)\, dt\, dx \\ &= \int_\Omega \Big[\Phi(u_n)\Big]_0^T \, dx \\ &= \int_\Omega \Phi(u_n(T))\, dx - \int_\Omega \Phi(u_{0,n})\, dx \\ &\ge -\|u_0\|_{L^1(\Omega)}. \end{aligned} \tag{6.4}$$

Therefore, using the growth assumptions (3.3)–(3.6), we obtain

$$\begin{aligned} &b_0(\theta-1)\int_{Q_T} \frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\lambda(x)+\theta}} \, dx\, dt + \delta_0 \int_{Q_T} |u_n|^{p(x)-1}|\varphi(u_n)|\, dx\, dt \\ &\le \|f_0\|_{L^1(Q_T)} + \|u_0\|_{L^1(\Omega)} + \int_{Q_T} c(x,t)|\nabla u_n|^{q(x)}\, dx\, dt \\ &\quad + \int_{Q_T} d(x,t)|u_n|^{\gamma(x)}\, dx\, dt. \end{aligned} \tag{6.5}$$

Using Young's inequality, we find that

$$\begin{aligned}
&\int_{Q_T} c(x,t)|\nabla u_n|^{q(x)}\,dx\,dt \\
&\quad\le C_1\int_{Q_T}|c(x,t)|^{\frac{p(x)}{p(x)-q(x)}}(1+|u_n|)^{(\lambda(x)+\theta)\frac{q(x)}{p(x)-q(x)}}\,dx\,dt \\
&\qquad+\frac{b_0(\theta-1)}{2}\int_{Q_T}\frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\lambda(x)+\theta}}\,dx\,dt \\
&\quad\le C_2\int_{Q_T}|c(x,t)|^{\frac{p(x)(p(x)-1)}{(p(x)-1)(p(x)-q(x))-(\lambda(x)+\theta)q(x)}}\,dx\,dt \\
&\qquad+\frac{\delta_0}{8}\int_{Q_T}|u_n|^{p(x)-1}\,dx\,dt+\frac{b_0(\theta-1)}{2}\int_{Q_T}\frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\lambda(x)+\theta}}\,dx\,dt,
\end{aligned}\tag{6.6}$$

and

$$\begin{aligned}
\int_{Q_T} d(x,t)|u_n|^{\gamma(x)}\,dx\,dt &\le \frac{\delta_0}{8}\int_{Q_T}|u_n|^{p(x)-1}\,dx\,dt \\
&\quad+C_3\int_{Q_T}|d(x,t)|^{\frac{p(x)-1}{p(x)-1-\gamma(x)}}\,dx\,dt.
\end{aligned}\tag{6.7}$$

Having in mind that $|\varphi(u_n)|\ge\frac12$ for $|u_n|\ge R=2^{\frac{1}{\theta-1}}-1$, then

$$\begin{aligned}
\frac{\delta_0}{2}\int_{Q_T}|u_n|^{p(x)-1}\,dx\,dt &\le \delta_0\int_{\{|u_n|\ge R\}}|u_n|^{p(x)-1}|\varphi(u_n)|\,dx\,dt \\
&\quad+\frac{\delta_0}{2}\int_{\{|u_n|<R\}}|u_n|^{p(x)-1}\,dx\,dt \\
&\le \delta_0\int_{Q_T}|u_n|^{p(x)-1}|\varphi(u_n)|\,dx\,dt+C_4.
\end{aligned}\tag{6.8}$$

By combining (6.5)–(6.8), we deduce that

$$\begin{aligned}
&\frac{b_0(\theta-1)}{2}\int_{Q_T}\frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\lambda(x)+\theta}}\,dx\,dt+\frac{\delta_0}{4}\int_{Q_T}|T_k(u_n)|^{p(x)-1}\,dx\,dt \\
&\le\|f_0\|_{L^1(Q_T)}+\|u_0\|_{L^1(\Omega)}+C_2\int_{Q_T}|c(x,t)|^{m(x)}\,dx\,dt \\
&\quad+C_3\int_{Q_T}|d(x,t)|^{r(x)}\,dx\,dt+2|Q_T|+C_4.
\end{aligned}\tag{6.9}$$

Hence,

$$\frac{b_0(\theta-1)}{2(1+k)^{\lambda_++\theta}}\int_{Q_T}|\nabla T_k(u_n)|^{p(x)}\,dx\,dt+\frac{\delta_0}{4}\int_{Q_T}|T_k(u_n)|^{p(x)-1}\,dx\,dt\le C_5.\tag{6.10}$$

It follows that: for any $k\ge1$,

$$\|T_k(u_n)\|_V^{p_-}\le\int_{Q_T}|\nabla T_k(u_n)|^{p(x)}\,dx\,dt+\int_{Q_T}|T_k(u_n)|^{p(x)}\,dx\,dt+2\le C_6k^{\theta+\lambda_+},\tag{6.11}$$

where $C_6 > 0$ is constant that doesn't depend on n and k. Then, the sequence $(T_k(u_n))_{n\in\mathbb{N}}$ is uniformly bounded in V, and there exists a subsequence still denoted $(T_k(u_n))_{n\in\mathbb{N}}$ and a measurable function $v_k \in V$ such that

$$\begin{cases} T_k(u_n) \rightharpoonup v_k & \text{weakly in} \quad V \\ T_k(u_n) \to v_k & \text{strongly in} \quad L^{p(\cdot)}(Q_T) \quad \text{and} \quad \text{a.e in} \quad Q_T. \end{cases} \tag{6.12}$$

Let $k \geq 1$, thanks to (6.10) it is obvious that

$$\begin{aligned} k^{p_- -1} \operatorname{meas}\{|u_n| > k\} &= \int_{\{|u_n|>k\}} |T_k(u_n)|^{p_- -1}\, dx\, dt \\ &\leq \int_{Q_T} |T_k(u_n)|^{p(x)-1}\, dx\, dt + |Q_T| \\ &\leq C_7. \end{aligned} \tag{6.13}$$

We deduce that

$$meas\{|u_n| > k\} \leq \frac{C_7}{k^{p_- -1}} \longrightarrow 0 \quad \text{as} \quad k \to \infty. \tag{6.14}$$

For all $\delta > 0$, we have

$$\begin{aligned} meas\,\{|u_n - u_m| > \delta\} \leq \operatorname{meas}\{|u_n| > k\} + meas\,\{|u_m| > k\} \\ + meas\,\{|T_k(u_n) - T_k(u_m)| > \delta\}\,. \end{aligned}$$

Let $\varepsilon > 0$, thanks to (6.14) we can choose $k = k(\varepsilon)$ large enough such that

$$\operatorname{meas}\{|u_n| > k\} \leq \frac{\epsilon}{3} \qquad \text{and} \qquad \operatorname{meas}\{|u_m| > k\} \leq \frac{\varepsilon}{3}. \tag{6.15}$$

Moreover, in view of (6.12) we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Q_T, then for all $k > 0$ and $\delta,\ \varepsilon > 0$ there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$\operatorname{meas}\Big\{|T_k(u_n) - T_k(u_m)| > \delta\Big\} \leq \frac{\varepsilon}{3} \qquad \forall n, m \geq n_0. \tag{6.16}$$

Thanks to (6.15) and (6.16), we conclude that : for any $\delta, \varepsilon > 0$, there exists $n_0 = n_0(\delta, \epsilon)$ such that

$$meas\,\{|u_n - u_m| > \delta\} \leq \varepsilon \qquad \text{for any } n, m \geq n_0(\delta, \varepsilon), \tag{6.17}$$

which proves that the sequence $(u_n)_n$ is a Cauchy sequence in measure, then there exists a subsequence still $(u_n)_n$ such that

$$u_n \to u \qquad \text{almost everywhere in } Q_T. \tag{6.18}$$

Consequently, thanks to (6.12) we have

$$T_k(u_n) \rightharpoonup T_k(u) \qquad \text{weakly in} \quad V, \tag{6.19}$$

and according to Lebesgue dominated convergence theorem, we get

$$T_k(u_n) \to T_k(u) \qquad \text{strongly in} \quad L^{p(\cdot)}(Q_T). \tag{6.20}$$

Step 3: Some regularity results

Let $h > k \geq 1$, we denote by $\varepsilon_j(n)$, $j = 1, 2, \ldots$ some real valued functions which converge to 0 when n goes to infinity, similarly we denote $\varepsilon_j(n,h)$ and $\varepsilon_i(n,h,\mu)$.

In this step, we will show that

$$\lim_{h\to\infty} \limsup_{n\longrightarrow\infty} \frac{1}{h} \int_{\{|u_n|\leq h\}} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla u_n \,dx\,dt = 0. \tag{6.21}$$

Let $h > 1$ and $\varphi_h(s) = \Big(2 - \dfrac{1}{(1+|s|)^{\theta-1}}\Big)T_h(s)$. By taking $\dfrac{\varphi_h(u_n)}{h}$ as a test function for the approximate problem (6.1), we have

$$\begin{aligned}
&\int_0^T \Big\langle \frac{\partial u_n}{\partial t}, \frac{\varphi_h(u_n)}{h}\Big\rangle\,dt + \frac{1}{h}\int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla\varphi_h(u_n)\,dx\,dt\\
&\quad+\frac{1}{h}\int_{Q_T} F_n(x,t,\nabla u_n)\varphi_h(u_n)\,dx\,dt + \frac{1}{h}\int_{Q_T}\delta(x,t)|u_n|^{p(x)-2}u_n\varphi_h(u_n)\,dx\,dt\\
&\quad=\frac{1}{h}\int_{Q_T} f_n(x,t,u_n)\varphi_h(u_n)\,dx\,dt.
\end{aligned} \tag{6.22}$$

In view of the growth conditions (3.3)–(3.6), and since $|T_h(u_n)| \leq |\varphi_h(u_n)| \leq 2\,|T_h(u_n)|$ we obtain

$$\begin{aligned}
&\int_0^T \Big\langle \frac{\partial u_n}{\partial t}, \frac{\varphi_h(u_n)}{h}\Big\rangle\,dt + \frac{1}{h}\int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla T_h(u_n)\,dx\,dt\\
&+\frac{b_0(\theta-1)}{h}\int_{Q_T}\frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\theta+\lambda(x)}}|T_h(u_n)|\,dx\,dt + \frac{\delta_0}{h}\int_{Q_T}|u_n|^{p(x)-1}|T_h(u_n)|\,dx\,dt\\
&\leq \frac{2}{h}\int_{Q_T} f_0(x,t)|T_h(u_n)|\,dx\,dt + \frac{2}{h}\int_{Q_T} d(x,t)|u_n|^{\gamma(x)}|T_h(u_n)|\,dx\,dt\\
&\quad+\frac{2}{h}\int_{Q_T} c(x,t)|\nabla u_n|^{q(x)}|T_h(u_n)|\,dx\,dt.
\end{aligned} \tag{6.23}$$

For the first term on the left-hand side of (6.23), we set $G_h(s) = \displaystyle\int_0^s \varphi_h(\tau)\,d\tau$ then

$$\begin{aligned}
\int_0^T \Big\langle \frac{\partial u_n}{\partial t}, \frac{\varphi_h(u_n)}{h}\Big\rangle\,dt &= \frac{1}{h}\int_\Omega\int_0^T \frac{\partial u_n}{\partial t}\varphi_h(u_n)\,dt\,dx\\
&= \frac{1}{h}\int_\Omega \Big[G_h(u_n)\Big]_0^T\,dx\\
&= \frac{1}{h}\int_\Omega G_h(u_n(T))\,dx - \frac{1}{h}\int_\Omega G_h(u_{0,n})\,dx.
\end{aligned} \tag{6.24}$$

Concerning the second and third terms on the right-hand side of (6.23), similarly to (6.7) we have

$$\frac{2}{h}\int_{Q_T} d(x,t)|u_n|^{\gamma(x)}|T_k(u_n)|\,dx\,dt \leq \frac{\delta_0}{4h}\int_{Q_T}|u_n|^{p(x)-1}|T_h(u_n)|\,dx\,dt + \frac{C_8}{h}\int_{Q_T}|d(x,t)|^{\frac{p(x)-1}{p(x)-1-\gamma(x)}}|T_h(u_n)|\,dx\,dt, \tag{6.25}$$

and

$$\begin{aligned}&\frac{2}{h}\int_{Q_T} c(x,t)|\nabla u_n|^{q(x)}|T_h(u_n)|\,dx\,dt\\ &\leq \frac{C_9}{h}\int_{Q_T}|c(x,t)|^{m(x)}|T_h(u_n)|\,dx\,dt + \varepsilon_0(h) + \frac{\delta_0}{4h}\int_{Q_T}|u_n|^{p(x)-1}|T_h(u_n)|\,dx\,dt\\ &\quad + \frac{b_0(\theta-1)}{2h}\int_{Q_T}\frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\lambda(x)+\theta}}|T_h(u_n)|\,dx\,dt.\end{aligned} \tag{6.26}$$

Combining (6.23)–(6.26), and since $\int_\Omega G_h(u_n(T))\,dx \geq 0$, we deduce that

$$\begin{aligned}&\frac{1}{h}\int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla T_h(u_n)\,dx\,dt + \frac{b_0(\theta-1)}{2h}\int_{Q_T}\frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\theta+\lambda(x)}}\\ &\quad\times|T_h(u_n)|\,dx\,dt + \frac{\delta_0}{4h}\int_{Q_T}|u_n|^{p(x)-1}|T_h(u_n)|\,dx\,dt\\ &\leq \frac{2}{h}\int_{Q_T} f_0(x,t)|T_h(u_n)|\,dx\,dt + \frac{C_8}{h}\int_{Q_T}|d(x,t)|^{\frac{p(x)-1}{p(x)-1-\gamma(x)}}|T_h(u_n)|\,dx\,dt\\ &\quad + \frac{C_9}{h}\int_{Q_T}|c(x,t)|^{m(x)}|T_h(u_n)|\,dx\,dt + \frac{1}{h}\int_\Omega G_h(u_{0,n})\,dx + \varepsilon_0(h).\end{aligned} \tag{6.27}$$

We have $f_0(x,t)$, $|d(x,t)|^{\frac{p(x)-1}{p(x)-1-\gamma(x)}}$ and $|c(x,t)|^{m(x)}$ are belongs to $L^1(Q_T)$, and since $\frac{|T_h(u_n)|}{h} \rightharpoonup 0$ weak $-*$ in $L^\infty(Q_T)$, therefore

$$\begin{aligned}\varepsilon_1(n,h) &= \frac{2}{h}\int_{Q_T} f_0(x,t)|T_h(u_n)|\,dx\,dt + \frac{C_8}{h}\int_{Q_T}|d(x,t)|^{\frac{p(x)-1}{p(x)-1-\gamma(x)}}|T_h(u_n)|\,dx\,dt\\ &\quad + \frac{C_9}{h}\int_{Q_T}|c(x,t)|^{m(x)}|T_h(u_n)|\,dx\,dt \longrightarrow 0 \quad \text{as} \quad n,h\to\infty.\end{aligned} \tag{6.28}$$

We have $\frac{G_h(u_{0,n})}{h} \leq \frac{|T_h(u_0)|}{h}|u_0| \in L^1(\Omega)$, and since $\frac{G_h(u_{0,n})}{h} \to 0$ a.e. in Ω, we conclude that

$$\varepsilon_2(n,h) = \frac{1}{h}\int_\Omega G_h(u_{0,n})\,dx \to 0 \quad \text{as} \quad n,h\to\infty. \tag{6.29}$$

Combining (6.27)–(6.29), we obtain

$$\begin{aligned}&\frac{1}{h}\int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla T_h(u_n)\,dx\,dt+\frac{b_0(\theta-1)}{2h}\int_{Q_T}\frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\theta+\lambda(x)}}\\&\quad\times|T_h(u_n)|\,dx\,dt+\frac{\delta_0}{2h}\int_{Q_T}|u_n|^{p(x)-1}|T_h(u_n)|\,dx\,dt\\&\le\varepsilon_3(n,h).\end{aligned}\tag{6.30}$$

We conclude that

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\int_{\{|u_n|\le h\}} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,dx\,dt=0.\tag{6.31}$$

Moreover, we have

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\{|u_n|\ge h\}}\frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\theta+\lambda(x)}}\,dx\,dt=0,\tag{6.32}$$

and

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\{|u_n|\ge h\}}|u_n|^{p(x)-1}\,dx\,dt=0.\tag{6.33}$$

Step 4: Equi-integrability of $(|u_n|^{p(x)-2}u)_n$ and $(f(x,t,u_n))_n$

In this part, we will prove that

$$|u_n|^{p(x)-2}u_n\to|u|^{p(x)-2}u\quad\text{strongly in}\quad L^1(Q_T),\tag{6.34}$$

and

$$f_n(x,t,u_n)\to f(x,t,u)\quad\text{strongly in}\quad L^1(Q_T).\tag{6.35}$$

Firstly, we show that $(|u_n|^{p(x)-2}u_n)_n$ is a uniformly equi-integrable in Q_T. For any measurable subset $E\subset Q_T$ and $h>0$, we have

$$\int_E|u_n|^{p(x)-1}\,dx\quad\le\int_E|T_h(u_n|^{p(x)-1}\,dx\,dt+\int_{E\cap\{|u_n|>h\}}|u_n|^{p(x)-1}\,dx\,dt.\tag{6.36}$$

In view of (6.20), it's clear that: for any $\varepsilon>0$, there exists $\beta(\varepsilon,h)$ such that

$$\int_E|T_h(u_n)|^{p(x)-1}\,dx\,dt\le\frac{\varepsilon}{2}\qquad\text{for}\quad\text{meas}(E)\le\beta(\varepsilon,h).\tag{6.37}$$

Thanks to (6.33), we obtain: for all $\varepsilon>0$, there exists $h_0(\varepsilon)$ such that

$$\int_{E\cap\{|u_n|>h\}}|u_n|^{p(x)-1}\,dx\,dt\le\frac{\varepsilon}{2}\qquad\forall h\ge h_0(\varepsilon).\tag{6.38}$$

By combining (6.36)–(6.38), we conclude that: for any $\varepsilon > 0$, there exists $\beta > 0$ such that

$$\int_E |u_n|^{p(x)-1}\, dx\, dt \le \varepsilon \qquad \text{for any} \quad E \subset \Omega \quad \text{with} \quad \text{meas}(E) \le \beta(\varepsilon). \tag{6.39}$$

Thus, the sequence $(|u_n|^{p(x)-2}u_n)_n$ is uniformly equi-integrable, and since $|u_n|^{p(x)-2}u_n \to |u|^{p(x)-2}u$ a.e. in Q_T, using Vitali theorem, we conclude (6.34). Moreover, we have

$$|f_n(x,t,u_n)| \le |f_0(x,t)| + |d(x,t)|^{\frac{p(x)-1}{p(x)-1-\gamma(x)}} + |u_n|^{p(x)-1}, \tag{6.40}$$

then, $(f_n(x,t,u_n))_n$ is uniformly equi-integrable in Q_T, and since $f_n(x,t,u_n) \to f(x,t,u)$ a.e. in Q_T, we deduce (6.35).

Step 5: Convergence of the gradient

For any $h > k \ge 1$, we set $\psi_h(s) = 1 - \dfrac{|T_{2h}(s) - T_h(s)|}{h}$ and $\omega_{n,\mu} = T_k(u_n) - (T_k(u))_\mu$.

By using $v = \omega_{n,\mu}\psi_h(u_n)$ as a test function in the approximated problem (6.1), we obtain

$$\begin{aligned}
&\int_0^T \langle \frac{\partial u_n}{\partial t}, \omega_{n,\mu}\psi_h(u_n)\rangle\, dt + \int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla u_n\, \omega_{n,\mu}\, \psi_h'(u_n)\, dx\, dt \\
&\quad + \int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot(\nabla T_k(u_n) - \nabla(T_k(u))_\mu)\psi_h(u_n)\, dx\, dt \\
&\quad + \int_{Q_T} F_n(x,t,\nabla u_n)\omega_{n,\mu}\psi_h(u_n)\, dx\, dt + \int_{Q_T} \delta(x,t)|u_n|^{p(x)-2}u_n\omega_{n,\mu}\psi_h(u_n)\, dx\, dt \\
&= \int_{Q_T} f_n(x,t,u_n)\omega_{n,\mu}\psi_h(u_n)\, dx\, dt.
\end{aligned} \tag{6.41}$$

It is clear that $\psi_h(u_n) = 0$ on the set $\{|u_n| \ge 2h\}$ and $\psi_h(u_n) = 1$ on the set $\{|u_n| \le h\}$. By using the assumptions (3.5) we have

$$\begin{aligned}
&\int_{Q_T} \frac{\partial u_n}{\partial t}\psi_h(u_n)\omega_{n,\mu}\, dx\, dt \\
&\qquad + \int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot(\nabla T_k(u_n) - \nabla(T_k(u))_\mu)\psi_h(u_n)\, dx\, dt \\
&\quad \le \int_{Q_T} c(x,t)|\nabla T_{2h}(u_n)|^{q(x)}|T_k(u_n) - (T_k(u))_\mu|\, dx\, dt \\
&\qquad + \int_{Q_T} |f_n(x,t,u_n)||T_k(u_n) - (T_k(u))_\mu|\, dx\, dt \\
&\qquad + \frac{2k}{h}\int_{\{h<|u_n|\le 2h\}} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla u_n\, dx\, dt \\
&\qquad + \|\delta(x,t)\|_\infty \int_{Q_T} |u_n|^{p(x)-1}\, |T_k(u_n) - (T_k(u))_\mu|\, dx\, dt.
\end{aligned} \tag{6.42}$$

By Lemma 7.1 (see Appendix), we have

$$\int_{Q_T} \frac{\partial u_n}{\partial t} \psi_h(u_n)\, \omega_{n,\mu}\, dx\, dt \geq \varepsilon_1(n). \tag{6.43}$$

For the first term on the right-hand side of (6.42), we have $c(x,t)|T_k(u_n) - (T_k(u))_\mu| \to 0$ a.e. in Q_T and $c(x,t)|T_k(u_n) - (T_k(u))_\mu| \leq 2kc(x,t) \in L^{\frac{p(\cdot)}{p(\cdot)-q(\cdot)}}(Q_T)$. By Lebesgue dominated convergence theorem, we have

$$c(x,t)|T_k(u_n) - (T_k(u))_\mu| \to 0 \qquad \text{strongly in} \quad L^{\frac{p(\cdot)}{p(\cdot)-q(\cdot)}}(Q_T).$$

In view of (6.19), we have $(|\nabla T_{2h}(u_n)|^{q(x)})_n$ is bounded in $L^{\frac{p(\cdot)}{q(\cdot)}}(Q_T)$, then there exists a measurable function $\zeta_{2h} \in L^{\frac{p(\cdot)}{q(\cdot)}}(Q_T)$ such that $|\nabla T_{2h}(u_n)|^{q(x)} \rightharpoonup \zeta_{2h}$ weakly in $L^{\frac{p(\cdot)}{q(\cdot)}}(Q_T)$, it follows that

$$\varepsilon_2(n,\mu) = \int_{Q_T} c(x,t)||T_k(u_n) - (T_k(u))_\mu|\nabla T_{2h}(u_n)|^{q(x)}\, dx\, dt \to 0 \quad \text{as} \quad n,\mu \to \infty. \tag{6.44}$$

On the other hand, we have $\omega_{n,\mu} \rightharpoonup 0$ weak$-*$ in $L^\infty(Q_T)$, and thanks to (6.34) and (6.35), we obtain

$$\varepsilon_3(n,\mu) = \int_{Q_T} |u_n|^{p(x)-1}\; |T_k(u_n) - (T_k(u))_\mu|\, dx\, dt \to 0 \quad \text{as} \quad n,\mu \to \infty, \tag{6.45}$$

and

$$\varepsilon_4(n,\mu) = \int_{Q_T} |f_n(x,t,u_n)||T_k(u_n) - (T_k(u))_\mu|\, dx\, dt \to 0 \quad \text{as} \quad n,\mu \to \infty. \tag{6.46}$$

From (6.31), we deduce that

$$\varepsilon_5(n,h) = \frac{1}{h}\int_{\{h<|u_n|\leq 2h\}} a(x,t,T_n(u_n),\nabla u_n)\cdot \nabla u_n\, dx\, dt \to 0 \quad \text{as} \quad n,h \to \infty. \tag{6.47}$$

By combining (6.42)–(6.47), we conclude that

$$\int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot (\nabla T_k(u_n) - \nabla (T_k(u))_\mu)\psi_h(u_n)\, dx\, dt \leq \varepsilon_6(n,\mu,h). \tag{6.48}$$

It is obvious that $a(x,t,s,0)=0$, it follows that

$$\begin{aligned}
&\int_{Q_T}(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u)))\cdot(\nabla T_k(u_n)\\
&\quad-\nabla T_k(u))\,dx\,dt+\int_{Q_T}a(x,t,T_k(u_n),\nabla T_k(u))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\,dx\,dt\\
&\quad+\int_{Q_T}a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(u)-\nabla(T_k(u))_\mu)\,dx\,dt\\
&\quad-\int_{\{k<|u_n|\le 2h\}}a(x,t,T_{2h}(u_n),\nabla T_{2h}(u_n))\cdot\nabla(T_k(u))_\mu\psi_h(u_n)\,dx\,dt\\
&\le\varepsilon_6(n,\mu,h).
\end{aligned}\tag{6.49}$$

Thanks to Lebesgue convergence theorem, we have $|a(x,t,T_k(u_n),\nabla T_k(u))|\to|a(x,t,T_k(u),\nabla T_k(u))|$ strongly in $L^{p'(\cdot)}(Q_T)$, and since $\nabla T_k(u_n)\rightharpoonup\nabla T_k(u)$ weakly in $(L^{p(\cdot)}(Q_T))^N$, we obtain

$$\begin{aligned}
\varepsilon_7(n)&=\Big|\int_{Q_T}a(x,t,T_k(u_n),\nabla T_k(u))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\,dx\,dt\Big|\\
&\le\int_{\Omega}|a(x,t,T_k(u_n),\nabla T_k(u))||\nabla T_k(u_n)-\nabla T_k(u))|\,dx\,dt\\
&\longrightarrow 0\ \text{ as }\ n\to\infty.
\end{aligned}\tag{6.50}$$

Concerning the third term on the left-hand side of (6.49), using (3.1) the sequence $(|a(x,t,T_k(u_n),\nabla T_k(u_n))|)_n$ is bounded in $L^{p'(\cdot)}(Q_T)$, then there exists $\xi_k\in L^{p'(\cdot)}(Q_T)$ such that $|a(x,t,T_k(u_n),\nabla T_k(u_n))|\rightharpoonup\xi_k$ weakly in $L^{p'(\cdot)}(Q_T)$, and since $\nabla(T_k(u))_\mu\to\nabla T_k(u)$ strongly in $(L^{p(\cdot)}(Q_T))^N$, we deduce that

$$\begin{aligned}
\varepsilon_8(n,\mu)&=\Big|\int_{Q_T}a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(u)-\nabla(T_k(u))_\mu)\,dx\,dt\Big|\\
&\longrightarrow 0\ \text{ as }\ n,\mu\to\infty.
\end{aligned}\tag{6.51}$$

For the last term on the left-hand side of (6.49), we have $|a(x,t,T_{2h}(u_n),\nabla T_{2h}(u_n))|\rightharpoonup\xi_h$ in $L^{p'(\cdot)}(Q_T)$, and since $\nabla(T_k(u))_\mu\longrightarrow\nabla T_k(u)$ strongly in $(L^{p(\cdot)}(Q_T))^N$, we can prove that

$$\begin{aligned}
\varepsilon_9(n,\mu)&=\Big|\int_{\{k<|u_n|\le 2h\}}a(x,t,T_{2h}(u_n),\nabla T_{2h}(u_n))\nabla(T_k(u))_\mu\psi_h(u_n)\,dx\,dt\Big|\\
&\le\int_{\{k<|u_n|\le 2h\}}|a(x,t,T_{2h}(u_n),\nabla T_{2h}(u_n))||\nabla(T_k(u))_\mu|\,dx\,dt\\
&\longrightarrow\int_{\{k<|u|\le 2h\}}\xi_h|\nabla T_k(u)|\,dx\,dt=0\ \text{ as }\ n,\mu\to\infty.
\end{aligned}\tag{6.52}$$

By combining (6.49)–(6.52), we get

$$\int_{Q_T} (a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))) \times(\nabla T_k(u_n) - \nabla T_k(u))\,dx\,dt \leq \varepsilon_9(n,\mu,h). \tag{6.53}$$

Thanks to (6.20), we arrive that

$$\begin{aligned}&\int_{Q_T} (a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)))\cdot(\nabla T_k(u_n) - \nabla T_k(u))\,dx\,dt\\ &\quad+\int_{Q_T} (|T_k(u_n)|^{p(x)-2}T_k(u_n) - |T_k(u)|^{p(x)-2}T_k(u))(T_k(u_n) - T_k(u))\,dx\,dt\\ &\quad\longrightarrow 0 \quad \text{as} \quad n,\mu,h\to\infty.\end{aligned} \tag{6.54}$$

In view of the Lemma 4.4, we deduce that

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{in} \quad V, \tag{6.55}$$

and

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in} \quad Q_T. \tag{6.56}$$

Therefore, we deduce that $a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla u_n \longrightarrow a(x,t,u,\nabla u)\cdot\nabla u$ a.e. in Q_T as n goes to infinity. In view of Fatou's lemma and (6.21), we conclude that

$$\begin{aligned}&\lim_{h\to\infty}\frac{1}{h}\int_{\{|u|\leq h\}} a(x,t,u,\nabla u)\cdot\nabla u\,dx\,dt\\ &\quad\leq \lim_{h\to\infty}\liminf_{n\to\infty}\frac{1}{h}\int_{\{|u_n|\leq h\}} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,dx\,dt\\ &\quad\leq \lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\int_{\{|u_n|\leq h\}} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,dx\,dt = 0.\end{aligned} \tag{6.57}$$

Step 6: Equi-integrability of the sequence $(F_n(x,t,\nabla u_n))_n$

We will show that the sequence $(F_n(x,t,\nabla u_n))_n$ converges to $F(x,t,\nabla u)$ strongly in $L^1(Q_T)$. Thanks to (6.56) we have

$$F_n(x,t,\nabla u_n) \longrightarrow F(x,t,\nabla u) \quad \text{almost everywhere in} \quad Q_T.$$

In view of Vitali's theorem, it's sufficient to prove that the sequence $(F_n(x,t,\nabla u_n))_n$ is uniformly equi-integrable.

Indeed, for any measurable subset E in Q_T and any $h > 0$ we have

$$\begin{aligned}
&\int_E |F_n(x,t,\nabla u_n)|\,dx\,dt \\
&\quad \le \int_E c(x,t)|\nabla u_n|^{q(x)}\,dx\,dt \\
&\quad \le C_{10}\int_E (|c(x,t)|+1)^{m(x)}\,dx\,dt + \int_E |u_n|^{p(x)-1}\,dx\,dt \\
&\qquad + \int_E \frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\lambda(x)+\theta}}\,dx\,dt \\
&\quad \le C_{10}\int_E (|c(x,t)|+1)^{m(x)}\,dx\,dt + \int_E |u_n|^{p(x)-1}\,dx\,dt \\
&\qquad + \int_E |\nabla T_h(u_n)|^{p(x)}\,dx + \int_{\{|u_n|>h\}} \frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\lambda(x)+\theta}}\,dx\,dt.
\end{aligned} \tag{6.58}$$

We have $|c(x,t)|^{m(x)} \in L^1(Q_T)$ and thanks to (6.39), we have : for all $\varepsilon > 0$, there exists $\beta(\varepsilon) > 0$ such that: for all $E \subset \Omega$ with $meas(E) \le \beta(\varepsilon)$, we find

$$\int_E (|c(x,t)|+1)^{m(x)}\,dx\,dt + \int_E |u_n|^{p(x)-1}\,dx\,dt \le \frac{\varepsilon}{3}. \tag{6.59}$$

For the third term on the right-hand side of (6.58), thanks to (6.55) there exists $\beta(\varepsilon) > 0$ such that:

$$\int_E |\nabla T_h(u_n)|^{p(x)}\,dx\,dt \le \frac{\varepsilon}{3} \quad \text{for all} \quad E \subset \Omega \quad \text{with} \quad \text{meas}(E) \le \beta(\varepsilon). \tag{6.60}$$

Concerning the last term of (6.58), thanks to (6.32), we have: for all $\varepsilon > 0$, there exists $h_0(\varepsilon) > 0$ such that

$$\int_{\{|u_n|>h\}} \frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\lambda(x)+\theta}}\,dx\,dt \le \frac{\varepsilon}{3} \qquad \forall h \ge h_0. \tag{6.61}$$

By combining (6.58)–(6.61), we obtain

$$\int_E |F_n(x,t,\nabla u_n)|\,dx\,dt \le \varepsilon \quad \text{for all} \quad E \subset \Omega \quad \text{with} \quad meas(E) \le \beta(\varepsilon). \tag{6.62}$$

Then, $(F_n(x,t,\nabla u_n))_n$ is uniformly equi-integrable. In view of Vitali's theorem we conclude that

$$F_n(x,t,\nabla u_n) \longrightarrow F(x,t,\nabla u) \qquad \text{strongly in} \quad L^1(Q_T). \tag{6.63}$$

Step 7: The convergence of $(u_n)_n$ in $C([0,T];L^1(\Omega))$

Let $h \geq 1$ and $0 < s \leq T$. By taking $T_1(u_n - (T_h(u))_\mu)\chi_{[0,s]}$ as a test function in for the approximate problem (6.1), we obtain

$$\begin{aligned}
&\int_\Omega \int_0^s \frac{\partial u_n}{\partial t} T_1(u_n - (T_h(u))_\mu)\,dt\,dx \\
&\quad + \int_0^s \int_\Omega a(x,t,T_n(u_n),\nabla u_n)\cdot \nabla T_1(u_n - (T_h(u))_\mu)\,dx\,dt \\
&\quad + \int_0^s \int_\Omega F_n(x,t,\nabla u_n) T_1(u_n - (T_h(u))_\mu)\,dx\,dt \\
&\quad + \int_0^s \int_\Omega \delta(x,t)|u_n|^{p(x)-2} u_n T_1(u_n - (T_h(u))_\mu)\,dx\,dt \\
&= \int_0^s \int_\Omega f_n(x,t,u_n) T_1(u_n - (T_h(u))_\mu)\,dx\,dt.
\end{aligned} \tag{6.64}$$

We have

$$\begin{aligned}
\frac{\partial u_n}{\partial t} &= \frac{\partial (u_n - (T_h(u))_\mu)}{\partial t} + \frac{\partial (T_h(u))_\mu}{\partial t} \\
&= \frac{\partial (u_n - (T_h(u))_\mu)}{\partial t} + \mu(T_h(u) - (T_h(u))_\mu),
\end{aligned}$$

it follows that

$$\begin{aligned}
&\int_\Omega \int_0^s \frac{\partial u_n}{\partial t} T_1(u_n - (T_h(u))_\mu)\,dt\,dx \\
&= \int_\Omega \int_0^s \frac{\partial (u_n - (T_h(u))_\mu)}{\partial t} T_1(u_n - (T_h(u))_\mu)\,dt\,dx \\
&\quad + \mu \int_\Omega \int_0^s (T_h(u) - (T_h(u))_\mu) T_1(u_n - (T_h(u))_\mu)\,dt\,dx.
\end{aligned} \tag{6.65}$$

Note that, for every $s \in [0,T]$ and when n tends to infinity we obtain

$$\begin{aligned}
&\int_\Omega \int_0^s (T_h(u) - (T_h(u))_\mu) T_1(u_n - (T_h(u))_\mu)\,dt\,dx \\
&\longrightarrow \int_\Omega \int_0^s (T_h(u) - (T_h(u))_\mu) T_1(u - (T_h(u))_\mu)\,dt\,dx \geq 0.
\end{aligned} \tag{6.66}$$

For the second term on the left-hand side of (6.64), we have

$$\begin{aligned}
&\int_0^s \int_\Omega a(x,t,T_n(u_n),\nabla u_n)\cdot \nabla T_1(u_n - (T_h(u))_\mu)\,dx\,dt \\
&= \int_0^s \int_\Omega \big(a(x,t,T_{h+1}(u_n),\nabla T_{h+1}(u_n)) - a(x,t,T_{h+1}(u_n),\nabla (T_h(u))_\mu)\big) \\
&\qquad\qquad \times \nabla (T_{h+1}(u_n) - (T_h(u))_\mu)\,dx\,dt \\
&\quad + \int_0^s \int_\Omega a(x,t,T_{h+1}(u_n),\nabla (T_h(u))_\mu)\cdot \nabla T_1(u_n - (T_h(u))_\mu)\,dx\,dt.
\end{aligned} \tag{6.67}$$

In view of (3.2), the first term of (6.67) is positive. Concerning the second term, we have $a(x,t,T_{h+1}(u_n),\nabla(T_h(u))_\mu) \to a(x,t,T_{h+1}(u),\nabla T_h(u))$ strongly in $(L^{p'(\cdot)}(Q_T))^N$ and since $\nabla T_1(u_n-(T_h(u))_\mu) \rightharpoonup \nabla T_1(u-T_h(u))$ weakly in $(L^{p(\cdot)}(Q_T))^N$, we have

$$\begin{aligned}\varepsilon_1(n,\mu) &= \int_0^s\int_\Omega a(x,t,T_{h+1}(u_n),\nabla(T_h(u))_\mu)\cdot\nabla T_1(u_n-(T_h(u))_\mu)\,dx\,dt\\ &\longrightarrow \int_{\{h<|u|\le h+1\}} a(x,t,T_{h+1}(u),0)\cdot\nabla u\,dx\,dt=0 \quad \text{as} \quad n,\mu\to\infty.\end{aligned} \tag{6.68}$$

Concerning the third term on the left-hand side of (6.64) , thanks to (6.63), and since $T_1(u_n-(T_h(u))_\mu) \rightharpoonup T_1(u-T_h(u))$ weak$-*$ in $L^\infty(Q_T)$, we get

$$\begin{aligned}&\int_0^s\int_\Omega |F_n(x,t,\nabla u_n)|T_1(u_n-(T_h(u))_\mu)|\,dx\,dt\\ &\quad\to \int_0^s\int_\Omega |F(x,t,\nabla u)||T_1(u-T_h(u))|\,dx\,dt \quad \text{as} \quad n,\mu\to\infty.\end{aligned} \tag{6.69}$$

On the other hand, in view of (6.34) we obtain

$$\begin{aligned}&\int_0^s\int_\Omega \delta(x,t)|u_n|^{p(x)-2}u_nT_1(u_n-(T_h(u))_\mu)\,dx\,dt\\ &\quad\to \int_0^s\int_\Omega \delta(x,t)|u|^{p(x)-2}uT_1(u-T_h(u))\,dx\,dt \quad \text{as} \quad n,\mu\to\infty,\end{aligned} \tag{6.70}$$

and thanks to (6.35) we have

$$\begin{aligned}&\int_0^s\int_\Omega |f_n(x,t,u_n)||T_1(u_n-(T_h(u))_\mu)\,dx\,dt\\ &\quad\to \int_0^s\int_\Omega |f(x,t,u)||T_1(u-T_h(u))|\,dx\,dt \quad \text{as} \quad n,\mu\to\infty.\end{aligned} \tag{6.71}$$

By combining (6.64)–(6.71), we conclude that

$$\begin{aligned}&\int_\Omega S_1(u_n(s)-(T_h(u(s)))_\mu)\,dx\\ &\le \int_0^s\int_\Omega \delta(x,t)|u|^{p(x)-2}uT_1(u-T_h(u))\,dx\,dt\\ &\quad+\int_0^s\int_\Omega |F(x,t,\nabla u)||T_1(u-T_h(u))|\,dx\,dt\\ &\quad+\int_0^s\int_\Omega |f(x,t,u)||T_1(u-T_h(u))|\,dx\,dt+\int_\Omega S_1(u_0-T_h(u_0))\,dx+\varepsilon_{10}(n,\mu).\end{aligned} \tag{6.72}$$

We have $\delta(x,t)|u|^{p(x)-2}uT_1(u-T_h(u)) \to 0$ a.e. in Q_T and

$$|\delta(x,t)|u|^{p(x)-2}uT_1(u-T_h(u))| \le \delta(x,t)|u|^{p(x)-1} \in L^1(Q_T).$$

In view of Lebesgue's dominated convergence theorem we obtain

$$\int_0^s \int_\Omega \delta(x,t)|u|^{p(x)-2}uT_1(u-(T_h(u))_\mu)\,dx\,dt \to 0 \quad \text{as} \quad h\to\infty. \tag{6.73}$$

By using the same argument, we have

$$\int_0^s \int_\Omega |F(x,t,\nabla u)||T_1(u-T_h(u))|\,dx\,dt \to 0 \quad \text{as} \quad h\to\infty, \tag{6.74}$$

and

$$\int_0^s \int_\Omega |f(x,t,u)||T_1(u-T_h(u))|\,dx\,dt \to 0 \quad \text{as} \quad h\to\infty. \tag{6.75}$$

Also, it's clear that

$$\int_\Omega S_1(u_0-T_h(u_0))\,dx \to 0 \quad \text{as} \quad h\to\infty. \tag{6.76}$$

By combining (6.72)–(6.76), we conclude that

$$\int_\Omega S_1(u_n(s)-(T_h(u(s)))_\mu)\,dx \le \varepsilon_{11}(n,\mu,h). \tag{6.77}$$

On the other hand, we have

$$\begin{aligned}\int_\Omega S_1\Big(\frac{u_n(s)-u_m(s)}{2}\Big)\,dx \le& \frac{1}{2}\Big(\int_\Omega S_1(u_n(s)-(T_h(u(s)))_\mu)\,dx \\ &+\int_\Omega S_1(u_m(s)-(T_h(u(s)))_\mu)\,dx\Big) \\ &\longrightarrow 0 \ \text{ as } \ n,m\to\infty.\end{aligned} \tag{6.78}$$

It follows that,

$$\begin{aligned}&\int_{\{|u_n(s)-u_m(s)|\le 2\}} \Big|\frac{u_n(s)-u_m(s)}{2}\Big|^2\,dx + \int_{\{|u_n(s)-u_m(s)|>2\}} \Big|\frac{u_n(s)-u_m(s)}{2}\Big|\,dx \\ &\le 2\int_\Omega S_1\Big(\frac{u_n(s)-u_m(s)}{2}\Big)\,dx \longrightarrow 0 \quad \text{as} \quad n,m\to\infty.\end{aligned} \tag{6.79}$$

We conclude that

$$\begin{aligned}&\int_\Omega |u_n(s)-u_m(s)|\,dx \\ &= \int_{\{|u_n(s)-u_m(s)|\le 2\}} |u_n(s)-u_m(s)|\,dx + \int_{\{|u_n(s)-u_m(s)|>2\}} |u_n(s)-u_m(s)|\,dx \\ &\le \Big(\int_{\{|u_n(s)-u_m(s)|\le 2\}} |u_n(s)-u_m(s)|^2\,dx\Big)^{\frac12}\cdot(\operatorname{meas}(\Omega))^{\frac12} \\ &\quad + \int_{\{|u_n(s)-u_m(s)|>2\}} |u_n(s)-u_m(s)|\,dx \longrightarrow 0 \quad \text{as} \quad n,m\to\infty.\end{aligned} \tag{6.80}$$

We deduce that

$$\int_\Omega |u_n(s) - u_m(s)|\,dx \longrightarrow 0 \quad \text{as} \quad m,n \to \infty. \tag{6.81}$$

Hence $(u_n)_n$ is a Cauchy sequence in $C([0,T];L^1(\Omega))$, thus $u \in C([0,T];L^1(\Omega))$, and we have $u_n(x,s) \to u(x,s)$ strongly in $L^1(\Omega)$ for any $0 \le s \le T$.

Step 8: Passage to the limit

Let $\psi \in V \cap L^\infty(Q_T)$ with $\dfrac{\partial \psi}{\partial t} \in V^* + L^1(Q_T)$, and $M = k + \|\psi\|_\infty$. By using $T_k(u_n - \psi)$ as a test function for the approximated problem (6.1), we get

$$\begin{aligned}&\int_0^T \langle \frac{\partial u_n}{\partial t}, T_k(u_n - \psi)\rangle\,dt + \int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot \nabla T_k(u_n-\psi)\,dx\,dt\\ &+ \int_{Q_T} \delta(x,t)|u_n|^{p(x)-2}u_n T_k(u_n-\psi)\,dx\,dt + \int_{Q_T} F_n(x,t,\nabla u_n) T_k(u_n-\psi)\,dx\,dt\\ &= \int_{Q_T} f_n(x,t,u_n)T_k(u_n-\psi)\,dx\,dt.\end{aligned} \tag{6.82}$$

On the one hand, if $|u_n| > M$ then $|u_n - \psi| \ge |u_n| - \|\psi\|_\infty > k$, therefore $\{|u_n - \psi| \le k\} \subseteq \{|u_n \le M\}$, which implies that

$$\begin{aligned}&\int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n-\psi)\,dx\,dt\\ &= \int_{\{|u_n-\psi|\le k\}} a(x,t,T_M(u_n),\nabla T_M(u_n))\cdot(\nabla T_M(u_n) - \nabla\psi)\,dx\,dt\\ &= \int_{\{|u_n-\psi|\le k\}} (a(x,t,T_M(u_n),\nabla T_M(u_n)) - a(x,t,T_M(u_n),\nabla\psi))\cdot(\nabla T_M(u_n)\\ &\quad -\nabla\psi)\,dx\,dt + \int_{\{|u_n-\psi|\le k\}} a(x,t,T_M(u_n),\nabla\psi)\cdot(\nabla T_M(u_n)-\nabla\psi)\,dx\,dt,\end{aligned}$$

since $\nabla T_M(u_n) \to \nabla T_M(u)$ strongly in $(L^{p(\cdot)}(Q_T))^N$, and in view of Fatou's Lemma we obtain

$$\begin{aligned}&\liminf_{n\to\infty} \int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n-\psi)\,dx\,dt\\ &\ge \int_{\{|u-\psi|\le k\}} (a(x,t,T_M(u),\nabla T_M(u)) - a(x,t,T_M(u),\nabla\psi)\cdot(\nabla T_M(u)-\nabla\psi)\,dx\,dt\\ &\quad + \int_{\{|u-\psi|\le k\}} a(x,t,T_M(u),\nabla\psi)\cdot(\nabla T_M(u_n)-\nabla\psi)\,dx\,dt\\ &= \int_{Q_T} a(x,t,u,\nabla u)\cdot\nabla T_k(u-\psi)\,dx\,dt.\end{aligned} \tag{6.83}$$

Now, we treat the first term on the left-hand side of (6.82), we have $\dfrac{\partial u_n}{\partial t} = \dfrac{\partial(u_n-\psi)}{\partial t} + \dfrac{\partial \psi}{\partial t}$, then

$$\begin{aligned}
&\int_0^T \langle \frac{\partial u_n}{\partial t}, T_k(u_n-\psi)\rangle\, dt \\
&= \int_0^T \langle \frac{\partial (u_n-\psi)}{\partial t}, T_k(u_n-\psi)\rangle\, dt + \int_0^T \langle \frac{\partial \psi}{\partial t}, T_k(u_n-\psi)\rangle\, dt \\
&= \int_\Omega S_k(u_n(T)-\psi(T))\, dx - \int_\Omega S_k(u_{0,n}-\psi(0))\, dx \\
&\quad + \int_{Q_T} \frac{\partial \psi}{\partial t} T_k(u_n-\psi)\, dx\, dt,
\end{aligned}$$

since $u_n \to u$ in $C([0,T];L^1(\Omega))$ then $u_n(T) \to u(T)$ in $L^1(\Omega)$, it follows that

$$\int_\Omega S_k(u_{0,n}-\psi(0))\, dx \longrightarrow \int_\Omega S_k(u_0-\psi(0))\, dx, \tag{6.84}$$

and

$$\int_\Omega S_k(u_n(T)-\psi(T))\, dx \longrightarrow \int_\Omega S_k(u(T)-\psi(T))\, dx. \tag{6.85}$$

Also, we have $\dfrac{\partial \psi}{\partial t} \in V^* + L^1(Q_T)$, and since $T_k(u_n-\psi) \rightharpoonup T_k(u-\psi)$ weakly in V and weak-$*$ in $L^\infty(Q_T)$, then

$$\int_{Q_T} \frac{\partial \psi}{\partial t} T_k(u_n-\psi)\, dx\, dt \longrightarrow \int_{Q_T} \frac{\partial \psi}{\partial t} T_k(u-\psi)\, dx\, dt. \tag{6.86}$$

On the other hand, in view of (6.34), (6.35) and (6.63), we conclude that

$$\int_{Q_T} \delta(x,t)|u_n|^{p(x)-2}u_n T_k(u_n-\psi)\, dx\, dt \longrightarrow \int_{Q_T} \delta(x,t)|u|^{p(x)-2}u T_k(u-\psi)\, dx\, dt, \tag{6.87}$$

$$\int_{Q_T} f_n(x,t,u_n) T_k(u_n-\psi)\, dx\, dt \longrightarrow \int_{Q_T} f(x,t,u) T_k(u-\psi)\, dx\, dt, \tag{6.88}$$

and

$$\int_{Q_T} F_n(x,t,\nabla u_n) T_k(u_n-\psi)\, dx\, dt \longrightarrow \int_{Q_T} F(x,t,\nabla u) T_k(u-\psi)\, dx\, dt. \tag{6.89}$$

By combining (6.82)–(6.88), we deduce that

$$\begin{aligned}
&\int_\Omega S_k(u-\psi)(T)\, dx - \int_\Omega S_k(u-\psi)(0)\, dx + \int_{Q_T} \frac{\partial \psi}{\partial t} T_k(u-\psi)\, dx\, dt \\
&+ \int_{Q_T} a(x,t,u,\nabla u)\cdot \nabla T_k(u-\psi)\, dx\, dt + \int_{Q_T} F(x,t,\nabla u) T_k(u-\psi)\, dx\, dt \\
&+ \int_{Q_T} \delta(x,t)|u|^{p(x)-2}u T_k(u-\psi)\, dx\, dt \le \int_{Q_T} f(x,t,u) T_k(u-\psi)\, dx\, dt,
\end{aligned}$$

which complete proof of the Theorem 5.2.

7. Appendix

Lemma 7.1. *Let* $\omega_{n,\mu} = T_k(u_n) - (T_k(u))_\mu$ *and* $\psi_h(s) = 1 - \dfrac{|T_{2h}(s) - T_h(s)|}{h}$, *then for any* $h \geq 1$ *we have*

$$\int_{Q_T} \frac{\partial u_n}{\partial t} \psi_h(u_n) \omega_{n,\mu}\, dx\, dt \geq \varepsilon_1(n). \tag{7.1}$$

Proof. Let $h \geq 1$, we define

$$\Psi_h(r) = \int_0^r \psi_h(s)\, ds = \begin{cases} r & \text{if} \quad |r| \leq h, \\ \dfrac{r^2 + 4hr + h^2}{2h} & \text{if} \quad -2h \leq r < -h, \\ \dfrac{-r^2 + 4hr - h^2}{2h} & \text{if} \quad h < r \leq 2h, \\ \dfrac{3h}{2} \text{sign(r)} & \text{if} \quad |r| > 2h, \end{cases}$$

We have

$$\begin{aligned} &\int_{Q_T} \frac{\partial u_n}{\partial t} \psi_h(u_n) \omega_{n,\mu}\, dx\, dt \\ &= \int_{Q_T} \frac{\partial (\Psi_h(u_n) - T_k(u_n))}{\partial t} \left(T_k(u_n) - (T_k(u))_\mu\right) dx\, dt \\ &\quad + \int_{Q_T} \frac{\partial T_k(u_n)}{\partial t} \left(T_k(u_n) - (T_k(u))_\mu\right) dx\, dt \\ &= \int_\Omega \Big[(\Psi_h(u_n) - T_k(u_n))(T_k(u_n) - (T_k(u))_\mu) \Big]_0^T dx \\ &\quad - \int_{Q_T} (\Psi_h(u_n) - T_k(u_n)) \Big(\frac{\partial T_k(u_n)}{\partial t} - \frac{\partial (T_k(u))_\mu}{\partial t} \Big) dx\, dt \\ &\quad + \int_{Q_T} \frac{\partial T_k(u_n)}{\partial t} \left(T_k(u_n) - (T_k(u))_\mu\right) dx\, dt. \end{aligned} \tag{7.2}$$

It's clear that $\Psi_h(u_n) - T_k(u_n)$ have the same sign as u_n on the set $\{|u_n| > k\}$, we have

$$\begin{aligned} &\int_\Omega \Big[(\Psi_h(u_n) - T_k(u_n))(T_k(u_n) - (T_k(u))_\mu) \Big]_0^T dx \\ &\geq - \int_{\{|u_{0,n}| > k\}} (\Psi_h(u_{0,n}) - T_k(u_{0,n}))(T_k(u_{0,n}) - (T_k(u_0))_\mu)\, dx \\ &= - \int_{\{|u_{0,n}| > k\}} (\Psi_h(u_{0,n}) - T_k(u_{0,n}))(T_k(u_{0,n}) - T_k(u_0))\, dx = \varepsilon_1(n). \end{aligned} \tag{7.3}$$

On the other hand, according to the definitions of $\Psi_h(\cdot)$ and $T_k(\cdot)$, we have $(\Psi_h(u_n) - T_k(u_n))\dfrac{\partial T_k(u_n)}{\partial t} = 0$ then

$$\begin{aligned}
&-\int_{Q_T} (\Psi_h(u_n) - T_k(u_n))\Big(\frac{\partial T_k(u_n)}{\partial t} - \frac{\partial (T_k(u))_\mu}{\partial t}\Big)\, dx\, dt \\
&= \int_0^T \int_{\{|u_n|>k\}} (\Psi_h(u_n) - T_k(u_n))\frac{\partial (T_k(u))_\mu}{\partial t}\, dx\, dt \\
&= \mu \int_0^T \int_{\{|u_n|>k\}} (\Psi_h(u_n) - T_k(u_n))(T_k(u) - (T_k(u))_\mu)\, dx\, dt \\
&= \mu \int_0^T \int_{\{|u_n|>k\}} (\Psi_h(u_n) - T_k(u_n))(T_k(u) - T_k(u_n))\, dx\, dt \\
&\quad + \mu \int_0^T \int_{\{|u_n|>k\}} (\Psi_h(u_n) - T_k(u_n))(T_k(u_n) - (T_k(u))_\mu)\, dx\, dt \\
&\geq \mu \int_0^T \int_{\{|u_n|>k\}} (\Psi_h(u_n) - T_k(u_n))(T_k(u) - T_k(u_n))\, dx\, dt = \varepsilon_2(n).
\end{aligned} \tag{7.4}$$

Concerning the last term on the right-hand side of (7.2), we obtain

$$\begin{aligned}
&\int_{Q_T} \frac{\partial T_k(u_n)}{\partial t}(T_k(u_n) - (T_k(u))_\mu)\, dx\, dt \\
&= \int_{Q_T} \frac{\partial (T_k(u_n) - (T_k(u))_\mu)}{\partial t}(T_k(u_n) - (T_k(u))_\mu)\, dx\, dt \\
&\quad + \int_{Q_T} \frac{\partial (T_k(u))_\mu}{\partial t}(T_k(u_n) - (T_k(u))_\mu)\, dx\, dt \\
&= \int_\Omega \Big[\frac{(T_k(u_n) - (T_k(u))_\mu)^2}{2}\Big]_0^T\, dx \\
&\quad + \mu \int_{Q_T} (T_k(u) - (T_k(u))_\mu)(T_k(u_n) - (T_k(u))_\mu)\, dx\, dt \\
&\geq -\int_\Omega \frac{(T_k(u_{0,n}) - T_k(u_0))^2}{2}\, dx \\
&\quad + \mu \int_{Q_T} (T_k(u) - (T_k(u))_\mu)(T_k(u_n) - (T_k(u))_\mu)\, dx\, dt \\
&\geq \varepsilon_3(n) + \mu \int_{Q_T} (T_k(u) - (T_k(u))_\mu)(T_k(u) - (T_k(u))_\mu)\, dx\, dt \geq \varepsilon_3(n).
\end{aligned} \tag{7.5}$$

By combining (7.2) and (7.3)–(7.5), we conclude that (7.1). □

Lemma 7.2. *The bounded operator $B_n = A_n + G_n$ acting from V into V^* is pseudo-monotone. Moreover, B_n is coercive in the following sense:*

$$\frac{\displaystyle\int_0^T \langle B_n u, v\rangle\, dt}{\|v\|_V} \longrightarrow \infty \qquad \textit{as} \quad \|v\|_V \to \infty \quad \textit{for} \quad v \in V.$$

Proof. In view of Hölder's inequality and the growth condition, it easy to see that the operator A_n is bounded, and by (6.2) we conclude that B_n is bounded. For the coercivity, for any $v \in V$ thanks to (6.2) we have

$$
\begin{aligned}
\int_0^T \langle B_n v, v\rangle \, dt &= \int_{Q_T} a(x,t,T_n(v),\nabla v)\cdot \nabla v \, dx\, dt + \int_{Q_T} \delta(x,t)|v|^{p(x)}\, dx\, dt \\
&\quad + \int_{Q_T} F_n(x,t,v)v \, dx\, dt - \int_{Q_T} f_n(x,t,v)v\, dx\, dt \\
&\geq \frac{b_0}{(1+n)^{\lambda_+}} \int_{Q_T} |\nabla u|^{p(x)}\, dx\, dt + \delta_0 \int_{Q_T} |v|^{p(x)}\, dx\, dt - C_0 n\|v\|_V \\
&\geq \frac{b_0}{(1+n)^{\lambda_+}} (\|\nabla u\|^{p_-}_{L^{p(\cdot)}(Q_T)} - 1) + \delta_0(\|u\|^{p_-}_{L^{p(\cdot)}(Q_T)} - 1) - C_0 n\|v\|_V \\
&\geq C_1\|v\|_V^{p_-} - C_0 n\|v\|_V - \frac{b_0}{(1+n)^{\lambda_+}} - \delta_0.
\end{aligned}
$$

We conclude that

$$
\frac{\displaystyle\int_0^T \langle B_n v, v\rangle \, dt}{\|v\|_V} \longrightarrow \infty \quad \text{as} \quad \|v\|_V \to \infty. \tag{7.6}
$$

It remains to show that B_n is pseudo-monotone. Let $(u_k)_k$ be a sequence in V such that

$$
\begin{cases}
u_k \rightharpoonup u & \text{in} \quad V, \\
B_n u_k \rightharpoonup \chi_n & \text{in} \quad V^*, \\
\limsup\limits_{k\to\infty} \langle B_n u_k, u_k\rangle \leq \langle \chi_n, u\rangle .
\end{cases} \tag{7.7}
$$

We will prove that

$$
\chi_n = B_n u \quad \text{and} \quad \langle B_n u_k, u_k\rangle \to \langle \chi_n, u\rangle \quad \text{as } k \to \infty.
$$

In view of (2.4), we have $u_k \to u$ in $L^1(Q_T)$ for a subsequence still denoted $(u_k)_k$. We have $(u_k)_k$ is a bounded sequence in V, then the sequence $(a(x,t,T_n(u_k),\nabla u_k)_k$ is bounded in $(L^{p'(\cdot)}(Q_T))^N$, then there exists a measurable function $\varphi_n \in (L^{p'(\cdot)}(Q_T))^N$ such that

$$
a(x,t,T_n(u_k),\nabla u_k) \rightharpoonup \varphi_n \quad \text{in } (L^{p'(\cdot)}(Q_T))^N \text{ as } k \to \infty. \tag{7.8}
$$

and

$$
|u_k|^{p(x)-2}u_k \rightharpoonup |u|^{p(x)-2}u \qquad \text{in} \quad L^{p'(\cdot)}(Q_T). \tag{7.9}
$$

Similarly, since $(F_n(x,t,\nabla u_k)_k$ is bounded in $L^\infty(Q_T)$, then there exists a measurable function $\psi_n \in L^\infty(Q_T)$ such that

$$
F_n(x,t,\nabla u_k) \rightharpoonup \psi_n \quad \text{weak-}* \text{ in} \quad L^\infty(Q_T) \quad \text{as} \quad k \to \infty, \tag{7.10}
$$

also, we have $|f_n(x,t,u_k)| \leq n$, and since $f_n(x,t,u_k) \to f_n(x,t,u)$ a.e. in Q_T, by Lebesgue's dominated convergence theorem, we have

$$f_n(x,t,u_k) \to f_n(x,t,u) \quad \text{in} \quad L^{p'(\cdot)}(Q_T) \quad \text{as} \quad k \to \infty. \tag{7.11}$$

For all $v \in V$, we have

$$\begin{aligned}
&\langle \chi_n, v\rangle = \lim_{k\to\infty} \langle B_n u_k, v\rangle \\
&= \lim_{k\to\infty} \int_{Q_T} a(x,t,T_n(u_k),\nabla u_k)\cdot \nabla v\,dx\,dt + \int_{Q_T} \delta(x,t)|u_k|^{p(x)-2}u_k v\,dx\,dt \\
&\quad + \lim_{k\to\infty} \int_{Q_T} F_n(x,t,\nabla u_k)v\,dx\,dt - \lim_{k\to\infty}\int_{Q_T} f_n(x,t,u_k)v\,dx\,dt \\
&= \int_{Q_T} \varphi_n\cdot\nabla v\,dx\,dt + \int_{Q_T}\delta(x,t)|u|^{p(x)-2}uv\,dx\,dt \\
&\quad + \int_{Q_T}\psi_n v\,dx\,dt - \int_{Q_T} f_n(x,t,u)v\,dx\,dt.
\end{aligned} \tag{7.12}$$

By using (7.7) and (7.12), we obtain

$$\begin{aligned}
\limsup_{k\to\infty}\langle B_n u_k, u_k\rangle &= \limsup_{k\to\infty}\Big(\int_{Q_T} a(x,t,T_n(u_k),\nabla u_k)\cdot\nabla u_k\,dx\,dt \\
&\quad + \int_{Q_T}\delta(x,t)|u_k|^{p(x)}\,dx\,dt + \int_{Q_T} F_n(x,t,\nabla u_k)u_k\,dx\,dt \\
&\quad - \int_{Q_T} f_n(x,t,u_k)u_k\,dx\,dt\Big) \\
&\leq \int_{Q_T}\varphi_n\cdot\nabla u\,dx\,dt + \int_{Q_T}\delta(x,t)|u|^{p(x)}\,dx\,dt \\
&\quad + \int_{Q_T}\psi_n u\,dx\,dt - \int_{Q_T} f_n(x,t,u)u\,dx\,dt.
\end{aligned} \tag{7.13}$$

Thanks to (7.10)–(7.11), and since $u_k \to u$ in $L^1(Q_T)$ then

$$\lim_{k\to\infty}\int_{Q_T} F_n(x,t,\nabla u_k)u_k\,dx\,dt = \int_{Q_T}\psi_n u\,dx\,dt, \tag{7.14}$$

and

$$\lim_{k\to\infty}\int_{Q_T} f_n(x,t,u_k)u_k\,dx\,dt = \int_{Q_T} f_n(x,t,u)u\,dx\,dt. \tag{7.15}$$

It follows that

$$\begin{aligned}
&\limsup_{k\to\infty}\Big(\int_{Q_T} a(x,t,T_n(u_k),\nabla u_k)\cdot\nabla u_k\,dx\,dt + \int_{Q_T}\delta(x,t)|u_k|^{p(x)}\,dx\,dt\Big) \\
&\leq \int_{Q_T}\varphi_n\cdot\nabla u\,dx\,dt + \int_{Q_T}\delta(x,t)|u|^{p(x)}\,dx\,dt.
\end{aligned} \tag{7.16}$$

On the other hand, in view of (3.2) we have

$$\begin{aligned}&\int_{Q_T}(a(x,t,T_n(u_k),\nabla u_k)-a(x,t,T_n(u_k),\nabla u))\cdot(\nabla u_k-\nabla u)\,dx\,dt\\&\quad+\int_{Q_T}\delta(x,t)(|u_k|^{p(x)-2}u_k-|u|^{p(x)-2}u)(u_k-u)\,dx\,dt\geq 0,\end{aligned}\tag{7.17}$$

hence

$$\begin{aligned}&\int_{Q_T}a(x,t,T_n(u_k),\nabla u_k)\cdot\nabla u_k\,dx\,dt+\int_{Q_T}\delta(x,t)|u_k|^{p(x)}\,dx\,dt\\&\geq\int_{Q_T}a(x,t,T_n(u_k),\nabla u_k)\cdot\nabla u\,dx\,dt\\&\quad+\int_{Q_T}\delta(x,t)|u_k|^{p(x)-2}u_k u\,dx\,dt+\int_{Q_T}\delta(x,t)|u|^{p(x)-2}u(u_k-u)\,dx\,dt\\&\quad+\int_{\Omega}a(x,t,T_n(u_k),\nabla u))\cdot(\nabla u_k-\nabla u)\,dx\,dt.\end{aligned}$$

In view of Lebesgue's dominated convergence theorem we have $T_n(u_k)\to T_n(u)$ in $L^{p(\cdot)}(\Omega)$, then $a(x,t,T_n(u_k),\nabla u)\to a(x,t,T_n(u),\nabla u)$ strongly in $(L^{p'(\cdot)}(Q_T))^N$, and using (7.8) we get

$$\begin{aligned}&\liminf_{k\to\infty}\Big(\int_{Q_T}a(x,t,T_n(u_k),\nabla u_k)\cdot\nabla u_k\,dx\,dt+\int_{Q_T}\delta(x,t)|u_k|^{p(x)}\,dx\,dt\Big)\\&\geq\int_{Q_T}\varphi_n\cdot\nabla u\,dx\,dt+\int_{Q_T}\delta(x,t)|u|^{p(x)}\,dx\,dt.\end{aligned}\tag{7.18}$$

Using (7.16), we conclude that

$$\begin{aligned}&\lim_{k\to\infty}\int_{Q_T}a(x,t,T_n(u_k),\nabla u_k)\cdot\nabla u_k\,dx\,dt\\&=\int_{Q_T}\varphi_n\cdot\nabla u\,dx\,dt+\int_{Q_T}\delta(x,t)|u|^{p(x)}\,dx\,dt.\end{aligned}\tag{7.19}$$

By combining (7.12), (7.14)–(7.15) and (7.19), we deduce that $\langle B_n u_k,u_k\rangle\to\langle\chi_n,u\rangle$ as $k\to\infty$. Now, by (7.8) and (7.19) we obtain

$$\begin{aligned}&\int_{Q_T}(a(x,t,T_n(u_k),\nabla u_k)-a(x,t,T_n(u_k),\nabla u))\cdot(\nabla u_k-\nabla u)\,dx\,dt\\&\quad+\int_{Q_T}(|u_k|^{p(x)-2}u_k-|u|^{p(x)-2}u)(u_k-u)\,dx\,dt\longrightarrow 0\qquad\text{as}\quad k\to\infty,\end{aligned}$$

Thanks to Lemma 4.4, we get $u_k\to u$ in V and $\nabla u_k\to\nabla u$ almost everywhere in Q_T, then $a(x,t,T_n(u_k),\nabla u_k)\rightharpoonup a(x,t,T_n(u),\nabla u)$ in $(L^{p'(\cdot)}(Q_T))^N$ and $F_n(x,t,\nabla u_k)\rightharpoonup F_n(x,t,\nabla u)$ in $L^{p'(x)}(Q_T)$, and thanks to (7.9), (7.11) we deduce that $\chi_n=B_n u$. □

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Arij Bouzelmate
LAR2A Laboratory, Department of Mathematics, Faculty of Sciences Tétouan, University Abdelmalek Essaadi, BP 2121, Tétouan, Morocco
abouzelmate@uae.ac.ma

Youssef Hajji
Department of Mathematics, Faculty of Sciences Tétouan, University Abdelmalek Essaadi, BP 2121, Tétouan, Morocco
youssef.hajji@etu.uae.ac.ma

Hassane Hjiaj
Department of Mathematics, Faculty of Sciences Tétouan, University Abdelmalek Essaadi, BP 2121, Tétouan, Morocco
hjiajhassane@yahoo.fr