First integrals of nonlinear differential equations from nonlocal constants

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Abstract. *A new method to find first integrals of nonlinear differential equations in Jacobi-type form is presented. The basic idea of our approach is to use one-parameter perturbed motions to find well-conceived nonlocal constants that are conserved along solutions. By means of such nonlocal framework we derive a set of theorems that we apply to look for the first integrals of some relevant cases, where moreover a solution is obtained. Applications also include some equations of the Painlevé–Gambier classification.*

1. Introduction

Many phenomena are mathematically described by second-order *nonlinear* Ordinary Differential Equations (ODEs). Generally, nonlinear ODEs are representative of models with a complicated behaviour and some of them emerge as *Euler–Lagrange* equations

$$\left[\partial_{\dot{x}} L\big(t, x(t), \dot{x}(t)\big)\right]^{\cdot} - \partial_x L\big(t, x(t), \dot{x}(t)\big) = 0\,, \tag{1.1}$$

for an appropriate *Lagrangian* function $L(t, x, \dot{x})$. In our notation ∂ stands for partial derivative, e.g. $\partial_{\dot{x}} = \partial/\partial\dot{x}$, whereas the upper dot means total derivative with respect to t.

Numerous methods are available in the literature to get exact solutions of equation (1.1) and, among all, finding *first integrals* allows to obtain solutions in the form of quadratures [17]. By definition, a first integral is a smooth point-function $\mathcal{I}(t, x, \dot{x})$ that is constant throughout equation (1.1) for all t, i.e. $\dot{\mathcal{I}}(t, x, \dot{x}) = 0$ along solutions. In the case of linear ODEs, a number of well defined methods to find first integrals exists. However, the same cannot be said for the nonlinear case, where their direct computation is in general an open problem. In this respect, the celebrated *Noether's Theorem* establishes a relation between invariance proprieties of a Lagrangian function and first integrals [12]. Nevertheless, finding non-trival symmetries is not always simple for many instances.

In favourable circumstances, by inspection of the Lagrangian taken into consideration, one can find particular cases for which special functions, called nonlocal constants, yield true first integrals without appealing to Noether symmetries. In particular, a *nonlocal constant* is a function that is conserved by equation (1.1)

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whose value depends not only on the state $(t, x(t), \dot{x}(t))$, but also on the whole history of the motion from its beginning at t_0. In Ref. [6] Gorni and Zampieri provide a theory to generate nonlocal constants for second order ODE Lagrangians using the concept of one-parameter perturbed motions:

Theorem 1.1. *Let* $t \rightarrow x(t)$ *be a solution to the Euler–Lagrange equation* (1.1), *and let* $x_\varepsilon(t)$, $\varepsilon \in \mathbb{R}$, *be a smooth family of perturbed motions such that* $x_0(t) = x(t)$. *Then, the following function is constant:*

$$I = \partial_{\dot{x}} L\big(t, x(t), \dot{x}(t)\big) \cdot \partial_\varepsilon x_\varepsilon(t)\big|_{\varepsilon=0} - \int_{t_0}^t \partial_\varepsilon L\big(s, x_\varepsilon(s), \dot{x}_\varepsilon(s)\big)\big|_{\varepsilon=0} \, ds. \qquad (1.2)$$

This result has been applied by Gorni and Zampieri in Refs. [6, 5, 7] to study some nonlinear mechanical systems, which include: (i) homogeneous potentials of degree $k = -2$, (ii) mechanical systems with viscous fluid resistance, (iii) mechanical system with hydraulic (quadratic) fluid resistance, (iv) conservative and dissipative Maxwell–Bloch equations of laser dynamics, and (v) geodesics for the Poincaré's half-plane. Beside that, the author extended Theorem 1.1 and related applications to the framework of higher-order theories [15, 4] and relativistic scalar field theories [16].

A viable parametrization to study second-order non-linear ODEs is the so-called *Jacobi-type* form (or, Jacobi equation) [10, 13, 8]. It is a second-degree polynomial in $\dot{x}$ like this

$$\ddot{x} + \tfrac{1}{2}\partial_x \varphi(t, x)\, \dot{x}^2 + \partial_t \varphi(t, x)\, \dot{x} + B(t, x) = 0, \qquad (1.3)$$

where the two free functions $\varphi(t, x)$ and $B(t, x)$ are both analytical in t and x. Interestingly, by taking $\varphi(t, x)$ and $B(t, x)$ appropriately, one can reproduce a great variety of nonlinear ODEs. Among all, we recall some of the second-order equations of the *Painlevé–Gambier* classification (see, e.g. Refs. [14, 3, 2]).

Painlevé–Gambier equations are attracting much attention in last years, since some problems related to their solutions, the *Painlevé transcendents*, are already under discussion [1, 11]. In particular, it is well known that the only movable singularities of these equations are poles.

In Refs. [8, 9] Guha *et al.* studied a subset of such equations, showing that *generalized Sundman transformations* can be used to obtain certain new first integrals and particular solutions. To do this, they started from (1.3) and derived some conserved functions of the form $p_1(t, \dot{x}, x)\, e^{\int_t p_2(\tau, x(\tau)) d\tau}$ for one Painlevé–Gambier equation at a time. As may be clearly seen, such functions "appear" nonlocal due to the presence of an exponential factor whose argument is an integral of $p_2(t, x)$.

Therefore, it is natural to suppose that a not yet investigated interplay between Theorem 1.1 and equation (1.3) exists. The discovery of such relation is the main result of our work. Remarkably, thanks to the Lagrangian-based approach underlying nonlocal constants like (1.2), our analysis allows to recover and generalize the results of Guha *et al.* as single formulas that have the advantage of being applicable to the whole parametrization (1.3).

As pointed before, Theorem 1.1 works with Euler–Lagrange equations only and to proceed we need an appropriate Lagrangian for equation (1.3). For our purposes comes to help a result obtained by Nucci and Tamizhmani in Ref. [13]. In that article they present a method devised by Jacobi to derive Lagrangians of many second-order ODEs. Among the examples provided, they demonstrate that equation (1.3) can be derived as the Euler–Lagrange equation for a Lagrangian

$$L(t, x, \dot{x}) \equiv \tfrac{1}{2} e^{\varphi(t,x)} \dot{x}^2 + \delta_1(t, x) \dot{x} + \delta_2(t, x), \qquad (1.4)$$

where the functions δ_1 and δ_2 satisfy the condition $\partial_t \delta_1 - \partial_x \delta_2 = e^{\varphi} B$. Expression (1.4) completes the set of information necessary to operationalize Theorem 1.1 and proceed with our analysis.

The organization of the paper is as follows. For pedagogical reasons, in Section 2 we deal with the simplest case of the autonomous Jacobi equation (1.3), or with $\partial_t \varphi = \partial_t B = 0$. We prove through Theorem 1.1 the existence of the true first integral (2.13), that we use in Subsection 2.1 to recover some Painlevé–Gambier constants and solutions obtained in Refs. [8, 9]. In Section 3.2 we extend the autonomous picture to the whole Jacobi equation (1.3) by distinguishing the two cases (i) $\partial_t \varphi = 0$, $\partial_t B \neq 0$ and (ii) $\partial_t \varphi \neq 0$, $\partial_t B \neq 0$. Here, our Theorem 3.1 (Subsection 3.1) and Theorem 3.5 (Subsection 3.2) lead to nonlocal constants whose integrand depends on t and x only (and not $\dot{x}$). In both cases, we finally investigate the effective potential of such theorems obtaining constants and solutions for some non-autonomous Painlevé–Gambier equations [8, 9] and beyond.

The main results of this work suggest that Lagrangian-based nonlocal constants like (1.2) have interesting applications to the study of nonlinear ODEs like (1.3). We believe that our approach can further be extended to, in principle, many other cases with little additional effort.

2. Autonomous Jacobi equation

In this preliminary section we are going to apply Theorem 1.1 to get nonlocal constants and related first integrals for the *autonomous* Jacobi equation

$$\ddot{x} + \tfrac{1}{2} \varphi' \dot{x}^2 + B = 0, \qquad (2.1)$$

where $\varphi = \varphi(x)$ and $B = B(x)$ do not explicitly depend on the time variable t. Here, the prime symbol stands for total derivatives with respect to x.

As anticipated in the introduction, the general Jacobi-type form (1.3) can be derived as the Euler–Lagrange equation (1.1) for a Lagrangian (1.4). Therefore, the autonomous Lagrangian for (2.1) can be expressed as

$$\mathcal{L}(x, \dot{x}) = \tfrac{1}{2} e^{\varphi} \dot{x}^2 + \delta_2, \qquad (2.2)$$

for some $\delta_2 = \delta_2(x)$ such that

$$\delta_2' = -e^{\varphi} B. \qquad (2.3)$$

Hence, writing down the nonlocal constant (1.2) in terms of (2.2), we have

$$I = e^\varphi \dot{x} \cdot \partial_\varepsilon x_\varepsilon\big|_{\varepsilon=0} - \int_{t_0}^t \left\{ \left(\tfrac{1}{2}\varphi' e^\varphi \dot{x}^2 + \delta_2'\right) \cdot \partial_\varepsilon x_\varepsilon\big|_{\varepsilon=0} + e^\varphi \dot{x} \cdot \partial_\varepsilon \dot{x}_\varepsilon\big|_{\varepsilon=0} \right\} ds, \quad (2.4)$$

where the perturbed family x_ε has not yet been included. Usefully, since x_ε is assumed to be a smooth function, we can rewrite $\partial_\varepsilon \dot{x}_\varepsilon^\pm|_{\varepsilon=0} = (\partial_\varepsilon x_\varepsilon^\pm|_{\varepsilon=0})^{\cdot}$.

In literature, some results about autonomous Painlevé–Gambier equations present constants that are nonlocal due to the presence of an integral in the argument of an exponential factor [8]. Inspired by this fact, it is quite natural for us to consider as perturbed family the x-shift parametrization

$$x_\varepsilon^\pm = x + \varepsilon a e^{\pm \int_t b\, d\tau}, \quad (2.5)$$

where $a = a(x)$ and $b = b(x)$ are free functions. The $\pm$ formalism indicates that we are simultaneously considering both families with a positive and a negative sign in the argument of the exponential. With the family (2.5) and the constraint (2.3), our nonlocal constant (2.4) can be simplified to be

$$I_\pm = a\dot{x}e^\varphi e^{\pm \int_t b\, d\tau} - \int_{t_0}^t \left\{ \left(\tfrac{1}{2}a\varphi' + a'\right) e^\varphi \dot{x}^2 \pm ab e^\varphi \dot{x} - aB e^\varphi \right\} e^{\pm \int_s b\, d\tau} ds. \quad (2.6)$$

Formally, expression (2.6) exhibits a double integration that we would like to remove. Furthermore we would also like to avoid the dependence of integrands on $\dot{x}$. A possible approach in this direction is to (i) cancel the term proportional to $\dot{x}^2$ by imposing $\tfrac{1}{2}a\varphi' + a' = 0$, which gives

$$a = e^{-\varphi/2}, \quad (2.7)$$

and (ii) reformulate the integrand in ds as a total derivative.

To establish request (ii), let us assume that $B \equiv b\bar{B}$ for some $\bar{B} = \bar{B}(x)$. Using the above definition, expression (2.6) reduces to

$$I_\pm = a\dot{x}e^\varphi e^{\pm \int_t b\, d\tau} - \int_{t_0}^t \left\{ (\pm ab\dot{x}e^\varphi) e^{\pm \int_s b\, d\tau} + (\mp a\bar{B}e^\varphi)\left(e^{\pm \int_s b\, d\tau}\right)^{\cdot} \right\} ds. \quad (2.8)$$

Thus, assuming $\pm ab\dot{x}e^\varphi = (\mp a\bar{B}e^\varphi)^{\cdot}$ as constraint, the integrand of expression (2.8) becomes a total derivative as desired

$$I_\pm = a\dot{x}e^\varphi e^{\pm \int_t b\, d\tau} \pm \int_{t_0}^t \left(a\bar{B}e^\varphi e^{\pm \int_s b\, d\tau}\right)^{\cdot} ds. \quad (2.9)$$

It is not difficult to verify that the above constraint can be rewritten as

$$b = -\tfrac{1}{2}\varphi'\bar{B} - \bar{B}'. \quad (2.10)$$

According to (2.9), we can evaluate the integral in the second term of the expression up to an additive integration constant that we assume to be zero. Then, one can use (2.7) and (2.10) to further simplify the result as follows

$$I_\pm = (\dot{x} \pm \bar{B})\, e^{\varphi/2} e^{\mp \int_t (\varphi'\bar{B}/2 + \bar{B}')\, d\tau}. \quad (2.11)$$

Note that expression (2.11) is a nonlocal constant for the autonomous Jacobi equation (2.1) that still depends on $\bar{B}$. By construction, equation (2.10) constraints $\bar{B}$ as a function of b and φ. So, multiplying equation (2.10) by $\bar{B}$ and defining $y \equiv \bar{B}^2$, we get $y' + \varphi' y + 2B = 0$. This is a first-order linear ODE, whose solution is

$$\bar{B} = \sqrt{-2e^{-\varphi} \int_x e^\varphi B \, dx} = \sqrt{2\delta_2 e^{-\varphi}} \,. \tag{2.12}$$

Theorem 2.1. *Let $e^\varphi B = -\delta_2'$ for some δ_2. Then, $\mathcal{I}$ is a first integral for the autonomous Jacobi equation $\ddot{x} + \frac{1}{2}\varphi'\dot{x}^2 + B = 0$ with*

$$\mathcal{I} = \tfrac{1}{2}\dot{x}^2 e^\varphi - \delta_2 \,. \tag{2.13}$$

Proof. Use expression (2.11) to compute $\mathcal{I} \equiv \frac{1}{2}I_+ I_- = \frac{1}{2}(\dot{x}^2 - \bar{B}^2) \, e^\varphi$. Then, substitute expression (2.12) inside $\mathcal{I}$. $\qquad\square$

2.1. Application: autonomous Painlevé–Gambier equations

There are several equations of the *Painlevé–Gambier* classification that belong to the autonomous parametrization (2.1) (see, e.g. Refs. [8, 2]).

Example 2.2. As we show in more details below, the constants we found in the previous section are useful to study some Painlevé–Gambier equations. The Painlevé–Gambier equations XVIII, XXI, and XXII can be parametrized as

$$\ddot{x} - \tfrac{1}{2}\alpha x^{-1}\dot{x}^2 + \beta x^n = 0, \tag{2.14}$$

with $\{\alpha, \beta, n\} \equiv \chi$ sets of constant parameters

$$\chi_{\text{XVIII}} = \left\{1, -4, 2\right\} \quad \chi_{\text{XXI}} = \left\{\tfrac{3}{2}, -3, 2\right\} \quad \chi_{\text{XXII}} = \left\{\tfrac{3}{2}, 1, 0\right\}. \tag{2.15}$$

First of all, a comparison between equation (2.1) and equation (2.14) yields $\varphi = -\alpha \ln x$ and $B = \beta x^n$. Then, following the definition (2.12), we deduce that $\bar{B} = \sqrt{\beta\gamma} \, x^{(n+1)/2}$ with $\gamma = 2/(\alpha - n - 1)$.

Hence, after some simple calculations, our nonlocal constants (2.11) become

$$I_\pm = \left\{\dot{x} x^{-\alpha/2} \pm \sqrt{\beta\gamma} \, x^{-1/\gamma}\right\} e^{\mp\sqrt{\beta/\gamma} \int_t x^{(n-1)/2} d\tau} \,, \tag{2.16}$$

Correspondingly, the first integral (2.13) is obtained to be

$$\mathcal{I} = \dot{x}^2 x^{-\alpha} - \beta\gamma x^{-2/\gamma} \,. \tag{2.17}$$

We mention that if we restrict to the $\mathcal{I} = 0$ hypersurface, equation (2.17) can be simply solved. In particular, after integration we get

$$x(t) = A \, (t + c_1)^\sigma \,, \tag{2.18}$$

where $A \equiv (\sqrt{\beta\gamma}/\sigma)^\sigma$ and $c_1 \equiv c/\sqrt{\beta\gamma}$, $c \in \mathbb{R}$, with $\sigma \equiv (1/\gamma - \alpha/2 + 1)^{-1}$.

When combined with coefficients (2.15), our results (2.16), (2.17) and (2.18) match exactly with some constants and solutions proposed case-by-case in Refs. [8, 9] using generalized Sundman transformations. The advantage of our approach lies in having identified a single formula for all of them.

3. Non-autonomous Jacobi equation

The procedure we discussed in the previous section has shown us the crucial in-gredients to apply Theorem 1.1 within a simplified case of Jacobi equation. Given the way to proceed, in the rest of this paper we turn to the large class of *non-autonomous* Jacobi equation

$$\ddot{x} + \tfrac{1}{2}\partial_x\varphi\,\dot{x}^2 + \partial_t\varphi\,\dot{x} + B = 0, \tag{3.1}$$

whose parameters $\varphi = \varphi(t,x)$ and $B = B(t,x)$ explicitly depend on the time variable t. As anticipated in the introduction, this class of equations can be derived as the Euler–Lagrange equation (1.1) for a Lagrangian

$$L(t,x,\dot{x}) \equiv \tfrac{1}{2}e^\varphi\dot{x}^2 + \delta_1\dot{x} + \delta_2, \tag{3.2}$$

with $\delta_1 = \delta_1(t,x)$ and $\delta_2 = \delta_2(t,x)$ satisfying

$$\partial_t\delta_1 - \partial_x\delta_2 = e^\varphi B. \tag{3.3}$$

With the above discussions in mind, let us begin evaluating expression (1.2) on Lagrangian (3.2). Expanding the derivatives of L with respect to x and $\dot{x}$ we have a nonlocal constant that looks like this

$$I = \left(e^\varphi\dot{x} + \delta_1\right)\cdot\partial_\varepsilon x_\varepsilon\big|_{\varepsilon=0} - \int_{t_0}^t \xi\,ds\,, \tag{3.4}$$

in which the integrand $\xi = \xi(s,x(s))$ is

$$\xi \equiv \left(\tfrac{1}{2}\partial_x\varphi\,e^\varphi\dot{x}^2 + \partial_x\delta_1\dot{x} + \partial_x\delta_2\right)\cdot\partial_\varepsilon x_\varepsilon\big|_{\varepsilon=0} + \left(e^\varphi\dot{x} + \delta_1\right)\cdot\partial_\varepsilon\dot{x}_\varepsilon\big|_{\varepsilon=0}. \tag{3.5}$$

Taking inspiration from the autonomous case, we choose the x-shift family

$$x_\varepsilon^\pm = x + \varepsilon a e^{\pm\int_t b_\pm\,d\tau}\,, \tag{3.6}$$

as perturbed motion, where $a = a(t,x)$ and $b_\pm = b_\pm(t,x)$ are free functions.

Using expression (3.6), as well as the property $\partial_\varepsilon\dot{x}_\varepsilon^\pm|_{\varepsilon=0} = (\partial_\varepsilon x_\varepsilon^\pm|_{\varepsilon=0})^\cdot$ with the condition (3.3), one finds that the integrand (3.5) can be expressed as

$$\xi_\pm = \left(a\delta_1 e^{\pm\int_s b_\pm\,d\tau}\right)^\cdot + (\tfrac{1}{2}a\partial_x\varphi + \partial_x a)e^\varphi\dot{x}^2 e^{\pm\int_s b_\pm\,d\tau}$$
$$+ \left\{\left[(\partial_t a \pm ab_\pm)\,e^\varphi\dot{x}\right]e^{\pm\int_s b_\pm\,d\tau} - (ae^\varphi B)\,e^{\pm\int_s b_\pm\,d\tau}\right\}. \tag{3.7}$$

Compared with the autonomous case, the above expression looks more prob-lematic to be integrated, as it now (i) explicitly depends on time, and (ii) exhibits new additional contributions. However, things become simplified by noting that various terms group themselves into total derivatives if we choose appropriately a and $b_\pm$.

To achieve this, the structure of expression (3.7) suggests first of all to neglect the term proportional to $\dot{x}^2$ by imposing $\frac{1}{2}a\partial_x\varphi + \partial_x a = 0$, which yields

$$a = e^{-\varphi/2}\,. \tag{3.8}$$

Going on, instead of working directly with $b_\pm$ it is more convenient to introduce a new variable $\bar{B}_\pm = \bar{B}_\pm(t,x)$ such that $B \equiv b_\pm \bar{B}_\pm$. Expression (3.7) thus becomes

$$\xi_\pm = \left(a\delta_1 e^{\pm\int_s b_\pm d\tau}\right)^\cdot + \left\{\left[(\partial_t a \pm ab_\pm)\,e^\varphi \dot{x}\right]e^{\pm\int_s b_\pm d\tau} + (\mp ae^\varphi \bar{B}_\pm)\left(e^{\pm\int_s b_\pm d\tau}\right)^\cdot\right\}. \tag{3.9}$$

Then, if we further require that $b_\pm$ satisfies

$$(\partial_t a \pm ab_\pm)\,e^\varphi \dot{x} = (\mp ae^\varphi \bar{B}_\pm)^\cdot\,, \tag{3.10}$$

it is immediate to check that $\xi_\pm$ becomes a total derivative as anticipated before

$$\xi_\pm = \left[a\delta_1 e^{\pm\int_s b_\pm d\tau} \mp (ae^\varphi \bar{B}_\pm)e^{\pm\int_s b_\pm d\tau}\right]^\cdot\,. \tag{3.11}$$

According to (3.11), we can evaluate the integral in the second term of expression (3.4) up to an additive integration constant that we assume to be zero. More specifically, we get

$$I_\pm = (\dot{x} \pm \bar{B}_\pm)\,ae^\varphi e^{\pm\int_t b_\pm d\tau}\,, \tag{3.12}$$

This result depends on $b_\pm$ and $\bar{B}_\pm$, whose explicit expressions are unknown so far. To reveal their form, let us consider condition (3.10) and expand it as follows

$$\mp(\partial_t a \pm ab_\pm)e^\varphi \dot{x} = \partial_t(ae^\varphi \bar{B}_\pm) + \partial_x(ae^\varphi \bar{B}_\pm)\dot{x}\,. \tag{3.13}$$

Since equation (3.13) must be true for all solutions, it can be decomposed with respect $\dot{x}$. In this way, such equation can be easily reformulated as follows

$$\begin{cases} \partial_t(ae^\varphi \bar{B}_\pm) = 0\,, \\ \partial_x(ae^\varphi \bar{B}_\pm) = \mp(\partial_t a \pm ab_\pm)e^\varphi\,. \end{cases} \tag{3.14}$$

It is immediate to check that the first equation of (3.14) is satisfied by

$$k_\pm \equiv ae^\varphi \bar{B}_\pm\,, \tag{3.15}$$

with $k_\pm = k_\pm(x)$ a time-independent function. On the other hand, using the mixed partials equality $\partial_x\partial_t k_\pm = \partial_t\partial_x k_\pm$, the second equation of (3.14) provides the relation

$$\partial_t\left[(\partial_t a \pm ab_\pm)e^\varphi\right] = 0, \tag{3.16}$$

that can be expanded to be a first-order (linear) differential equation for $b_\pm$ as follows

$$a\,\partial_t b_\pm + (\partial_t\varphi\,a + \partial_t a)\,b_\pm \pm (\partial_t\varphi\partial_t a + \partial_t\partial_t a) = 0. \tag{3.17}$$

Using standard techniques we solve equation (3.17) and impose condition (3.14) to fix the integration constant. Hence, after several calculations, we finally obtain

$$b_\pm = \mp\partial_t a - e^{-\varphi}a^{-1}\partial_x k_\pm. \tag{3.18}$$

Since $B \equiv b_{\pm}\bar{B}_{\pm}$, we multiply expression (3.18) by $\bar{B}$ and use definition (3.15) to substitute $k_{\pm}$. Consequently, our expression simplifies to

$$\bar{B}_{\pm}\partial_x \bar{B}_{\pm} + \partial_x(\ln a + \varphi)\bar{B}_{\pm}^2 \pm (\partial_t \ln a)\bar{B}_{\pm} = -B\,, \qquad (3.19)$$

that finally constraints $\bar{B}_{\pm}$ in terms of the original functions B and φ only. Once solved, the solution $\bar{B}_{\pm}$ will automatically fix $b_{\pm}(B, \varphi) = B/\bar{B}_{\pm}(B, \varphi)$ by definition.

In contrast with the autonomous case, equation (3.19) is a nonlinear differential equation for $\bar{B}_{\pm}$. Hence, without loss of generality, we choose to proceed by splitting our study depending on wether $\partial_t \varphi \sim \partial_t \ln a$ is equal to zero or not.

3.1. The $\partial_t \varphi = 0$ case

If $\partial_t \varphi = 0$, equation (3.19) becomes

$$\bar{B}_{\pm}\partial_x \bar{B}_{\pm} + \partial_x(\ln a + \varphi)\bar{B}_{\pm}^2 = -B. \qquad (3.20)$$

As in the autonomous case, substituting definition (3.8) in equation (3.20) and assuming $y_{\pm} \equiv \bar{B}_{\pm}^2$, we get

$$\partial_x y_{\pm} + (\partial_x \varphi)y_{\pm} + 2B = 0. \qquad (3.21)$$

This is a first-order linear ODE, whose solution is

$$\bar{B} = \sqrt{-2e^{-\varphi}\int_{x_0}^x (e^\varphi B)|_t\, dx} = \sqrt{2e^{-\varphi}\int_{x_0}^x (\partial_x \delta_2 - \partial_t \delta_1)|_t\, dx}. \qquad (3.22)$$

On the other hand, being $\partial_t \varphi = 0$, equation (3.18) can be replaced by

$$b_{\pm} = e^{-\varphi}a^{-1}\partial_x k_{\pm}\,. \qquad (3.23)$$

that, bearing in mind definition (3.15) and expression (3.22), finally yields

$$b \equiv b_{\pm} = -\tfrac{1}{2}\partial_x \varphi - \partial_x \bar{B}\,. \qquad (3.24)$$

Note that in this case the perturbed families $x_\varepsilon^{\pm}$ return the same value for $b_{\pm}$.

We have thus obtained all the crucial ingredients to evaluate our nonlocal constant (3.12) and, more specifically, we get

$$.I_{\pm} = (\dot{x} \pm \bar{B})\, e^{\varphi/2} e^{\mp \int_t (\partial_x \varphi/2 + \partial_x \bar{B})d\tau}. \qquad (3.25)$$

with $\bar{B}$ given by (3.22).

Theorem 3.1. *Let $\lambda = -\int e^\varphi B\, dx$ with $\partial_t \varphi = 0$. Then, I is constant along the solutions of the Jacobi equation $\ddot{x} + \frac{1}{2}\partial_x \varphi\, \dot{x}^2 + B = 0$ with*

$$I \equiv \tfrac{1}{2}\dot{x}^2 e^\varphi + \lambda - \int_{t_0}^t \partial_t \lambda\, dt. \qquad (3.26)$$

Proof. Use expression (3.25) to calculate $I \equiv \frac{1}{2}I_+ I_- = \frac{1}{2}(\dot{x}^2 - \bar{B}^2)\,e^{\varphi}$. Define $\delta_1 \equiv \partial_x \eta$ and consider expression (3.22), that can be rewritten as

$$
\begin{aligned}
\bar{B}^2 &= 2e^{-\varphi} \int_{x_0}^{x} \partial_x(\delta_2 - \partial_t \eta)\big|_t \, dx \\
&= 2e^{-\varphi}\left[\delta_2 - \partial_t \eta - \int_{t_0}^{t} \partial_t(\delta_2 - \partial_t \eta)\,dt\right] \\
&= 2e^{-\varphi}\left(\lambda - \int_{t_0}^{t} \partial_t \lambda \, dt\right),
\end{aligned}
\tag{3.27}
$$

where we defined $\lambda \equiv \delta_2 - \partial_t \eta$. Finally, substitute expression (3.27) inside I. It follows from (3.3) that $e^{\varphi}B = -\partial_x \lambda$. $\qquad\square$

Notice that, as expected, the integrand of (3.26) depends only on x and t. In addition, as we will see in the following Remark 3.2, expression (3.26) exactly recovers our previous result (2.13) when evaluated in the autonomous case.

Remark 3.2. One might wonder under which conditions expression (3.26) becomes a true first integral, in the sense of a local function of t. For this purpose, it is sufficient to require that $\partial_t \lambda = \dot{\psi}$ for some $\psi = \psi(t)$. Derive such condition with respect to x, to get $0 = \partial_x \partial_t \lambda$. Switch the partial derivatives, and use the hypothesis $\partial_x \lambda = -e^{\varphi}B$ of Theorem 3.1 to finally write $0 = \partial_t(e^{\varphi}B) = e^{\varphi}\partial_t B$, which implies $\partial_t B = 0$. Since in this subsection we also assume $\partial_t \varphi = 0$, expression (3.26) returns a true first integral only if we move to the autonomous case.

3.1.1. Application: non-autonomous Painlevé–Gambier equations

There are several equations of the *Painlevé–Gambier* classification which belong to the Jacobi parametrization (3.1) (see, e.g. Refs. [8, 2]).

Example 3.3. The Painlevé–Gambier equation IV is

$$
\ddot{x} - (6x^2 + t) = 0.
\tag{3.28}
$$

Let us first establish a comparison between equation (3.28) and Theorem 3.1 to infer that $\varphi = 0$ and $B = -(6x^2 + t)$. Since $e^{\varphi}B = -(6x^2 + t)$, we have $\lambda = -x(2x^2 + t)$. Then, expression (3.26) provides the nonlocal constant

$$
I = \frac{1}{2}\dot{x}^2 - 2x^3 - xt + \int_{t_0}^{t} x\, dt.
\tag{3.29}
$$

Example 3.4. The Painlevé–Gambier equation XX is

$$
\ddot{x} - \frac{1}{2}x^{-1}\dot{x}^2 - 4x^2 - 2tx = 0.
\tag{3.30}
$$

A comparison between equation (3.30) and Theorem 3.1 yields $\varphi = -\ln x$ and $B = -2x(2x + t)$. Since $e^{\varphi}B = -2(2x + t)$, we have $\lambda = -2x(x + t)$. Then, expression (3.26) yields

$$
I = \frac{1}{2}\dot{x}^2 x^{-1} - 2x(x + t) + 2\int_{t_0}^{t} x\, dt.
\tag{3.31}
$$

3.2. The $\partial_t\varphi \neq 0$ case

When $\partial_t\varphi \neq 0$, the mathematical structure of equation (3.19) is more complicated to be analyzed. We start by multiplying equation (3.19) by $e^{2(\ln a + \varphi)}$ and using definition (3.15) to write

$$\partial_x k_\pm^2 \pm 2\partial_t a e^\varphi k_\pm = -2Ba^2 e^{2\varphi}. \tag{3.32}$$

We take the ∂_t derivative of equation (3.32) and switch $\partial_t\partial_x k_\pm^2 = \partial_x\partial_t k_\pm^2$. Consequently, the above equation can be then written as

$$\partial_x \partial_t k_\pm^2 \pm 2\partial_t(\partial_t a e^\varphi)k_\pm \pm 2\partial_t a e^\varphi(\partial_t k_\pm) = -2\partial_t(Ba^2 e^{2\varphi}). \tag{3.33}$$

Furthermore, since $\partial_t k_\pm^2 = 2k_\pm\partial_t k_\pm = 0$ (remember that expression (3.14) fixes $\partial_t k_\pm = 0$), from equation (3.33) it follows that

$$\partial_t(\partial_t a\, e^\varphi)k_\pm = \mp\partial_t(Ba^2 e^{2\varphi}), \tag{3.34}$$

that can be solved with respect to $k_\mp$. More precisely, we obtain

$$k_\pm = \mp\frac{\partial_t(Ba^2 e^{2\varphi})}{\partial_t(\partial_t a\, e^\varphi)}. \tag{3.35}$$

which must satisfy the set of conditions (3.14).

An immediate consequence of the above result is the evaluation of $\bar{B}$. In fact, by plugging (3.35) and (3.8) in the right hand side of (3.15) we obtain

$$\bar{B}_\pm = \pm4\frac{\partial_t B + B\partial_t\varphi}{2\partial_t\partial_t\varphi + (\partial_t\varphi)^2}. \tag{3.36}$$

On the other hand, we can use expression (3.36) to calculate $b_\pm = B/\bar{B}_\pm$. Hence,

$$b_\pm = \pm\frac{1}{4}\frac{2\partial_t\partial_t\varphi + (\partial_t\varphi)^2}{\partial_t\ln B + \partial_t\varphi}. \tag{3.37}$$

Theorem 3.5. *Let $\partial_t\varphi \neq 0$ and $Be^\varphi = \rho_1 e^{\varphi/2} + \rho_2$ for some $\rho_{1,2} = \rho_{1,2}(x)$ such that*

$$\rho_1' = \partial_t(\ln \partial_t\varphi)\big(e^{\varphi/2} + \rho_2/\rho_1\big). \tag{3.38}$$

Then, I is constant along the solutions of the Jacobi equation $\ddot{x} + \frac{1}{2}\partial_x\varphi\,\dot{x}^2 + \partial_t\varphi\,\dot{x} + B = 0$ with

$$I = \left(\dot{x}e^{\varphi/2} + \frac{2\rho_1}{\partial_t(2\ln\partial_t\varphi + \varphi)}\right)\exp\left\{\frac{1}{2}\int_t \partial_t\big(2\ln\partial_t\varphi + \varphi\big)\left(1 + \frac{\rho_2}{\rho_1}e^{-\varphi/2}\right)d\tau\right\}. \tag{3.39}$$

Proof. We begin substituting (3.8) in (3.35) and imposing $\partial_t k_\pm = 0$ (see the first condition of (3.14)). As a consequence, we get $\partial_t(Be^\varphi) = \rho_1\partial_t e^{\varphi/2}$, whose solution is $B = \rho_1 e^{-\varphi/2} + \rho_2 e^{-\varphi}$ (our first hypothesis). We plug B inside (3.36) and (3.37) to evaluate $\bar{B}_\pm$ and $b_\pm$, that we use with definition (3.12) to get our final result (3.39). Note that $I_+ = I_-$, hence $I = I_\pm$. Since $k_\pm = \pm\rho_1$, our second hypothesis (3.38) is obtained from $\partial_x k_\pm = 0$ (the second condition of (3.14)). $\qquad\square$

Notice that also in this case the integrand of (3.39) depends only on x and t.

Remark 3.6. One might wonder under which conditions expression (3.39) becomes a true first integral, in the sense of a local function of t. For this purpose, since the integrand depends only on t and x, it is sufficient to require that $\partial_t(2\ln\partial_t\varphi+\varphi)\big(1+\rho_2/\rho_1 e^{-\varphi/2}\big)=\dot\psi(t)$ for some $\psi=\psi(t)$.

3.2.1. Application: non-autonomous Jacobi equation

Example 3.7. Consider the following Jacobi-type equation

$$\ddot x + \tfrac{1}{2}\dot x^2 + \dot x + \varrho\, e^{-(t+x)/2} = 0\,, \tag{3.40}$$

with ϱ a constant parameter. Since $B=\varrho\, e^{-(t+x)/2}$ and $\partial_x\varphi=\partial_t\varphi=1$, we have $\varphi=t+x$. Hence, equation (3.40) satisfies the hypotheses of Theorem 3.5 with $\rho_1=\varrho$ and $\rho_2=0$. Note that Remark 3.6 is also satisfied by $\psi=t$. Hence, expression (3.39) yields the following true first integral

$$\mathcal{I} = e^{t/2}\big\{\dot x e^{(t+x)/2} + 2\varrho\big\}. \tag{3.41}$$

Being linear in $\dot x$, $\mathcal{I}$ turns out to be a precious tool to find a solution of equation (3.40). In fact, expression (3.41) can be rewritten as $\dot{\mathcal{J}}(t,x)=0$, with

$$\mathcal{J} = 2\, e^{x/2} + \tilde{\mathcal{I}}e^{-t} - 4\varrho\, e^{-t/2}\,. \tag{3.42}$$

Hence, $\mathcal{J}$ is a first integral too. After some calculations, it can be easily checked that expression (3.42) solves directly equation (3.40) with

$$x(t) = 2\ln\big\{2\varrho\, e^{-t/2} - \tfrac{1}{2}\,\tilde{\mathcal{I}}\, e^{-t} + \tfrac{1}{2}\tilde{\mathcal{J}}\big\}. \tag{3.43}$$

Here, the $\tilde{\mathcal{I}}$ and $\tilde{\mathcal{J}}$ parameters are constants.

Example 3.8. An interesting application of Theorem 3.5 arises when $\rho_2(x)=0$. Under such condition, the hypotheses of our theorem become

$$\begin{cases} \rho_1'(x) = \partial_t\ln\partial_t\varphi(t,x)\, e^{\varphi(t,x)/2}, \\ B(t,x)e^{\varphi(t,x)} = \rho_1(x)\, e^{\varphi(t,x)/2}\,. \end{cases} \tag{3.44}$$

From the above constraints, the first differential equation can be solved easily. More precisely, we get

$$\varphi(t,x) = 2\ln\left\{\frac{2\rho_1'(x) + e^{\frac{1}{2}c_1(x)[t+c_2(x)]}}{c_1(x)}\right\}. \tag{3.45}$$

with $c_1(x)\neq 0$ and $c_2(x)$ integration functions.

On the other hand, using expression (3.45), the second equation of (3.44) gives

$$B(t,x) = \frac{c_1(x)\rho_1(x)}{2\rho_1'(x) + e^{\frac{1}{2}c_1(x)[t+c_2(x)]}}\,. \tag{3.46}$$

With the above formulas in mind, we see that once fixed $c_1(x)$, $c_2(x)$ and $\rho_1(x)$ we can deduce the class of Jacobi equations (3.1) characterized by a first integral like (3.39) with $\rho_2(x) = 0$. This is the result of the following proposition.

Proposition 3.9. *Let $c_{1,2,3} = c_{1,2,3}(x)$ be three free functions of x and define $\vartheta(t,x) \equiv c_1(c_2 + t)$. Then, $\mathcal{I}$ is a first integral for the equation*

$$\ddot{x} + \frac{4c_1 c_3'' - 4c_3' c_1' + e^{\frac{1}{2}\vartheta}\left[c_1'(\vartheta - 2) + c_1^2 c_2'\right]}{2c_1\left(2c_3' + e^{\frac{1}{2}\vartheta}\right)}\,\dot{x}^2 + \frac{c_1 e^{\frac{1}{2}\vartheta}}{2c_3' + e^{\frac{1}{2}\vartheta}}\,\dot{x} + \frac{c_1 c_3}{2c_3' + e^{\frac{1}{2}\vartheta}} = 0, \quad (3.47)$$

with

$$\mathcal{I} = e^{\frac{1}{2}c_1 t}\left[2\left(c_3' + c_3\right) + \dot{x}e^{\frac{1}{2}\vartheta}\right]c_1^{-1}. \quad (3.48)$$

Proof. Let us define $\vartheta \equiv 2\ln \partial_t \varphi + \varphi$. By choosing expression (3.45) in the definition above it turns out that $\vartheta = c_1(t + c_2)$. Hence, equation (3.47) and its first integral (3.48) are a direct consequence of (3.1) and (3.39), where φ and B have the form (3.45) and (3.46). Here, we assumed that $\rho_2 = 0$ and we defined $c_3 \equiv \rho_1$. $\qquad\square$

Let $\bar{c}_{1,2,3} \in \mathbb{R}$ and $\omega_{1,2} \in \mathbb{R}$ be some constant parameters. Interestingly, if we take $c_1 = \bar{c}_1$, $c_2 = \omega_1 \bar{c}_1^{-1} x + \bar{c}_2$ and $c_3 = \omega_2 \bar{c}_1 + \bar{c}_3 e^{-x}$, it is possible to recast expression (3.48) to look as the conservation law of one more first integral. Indeed, taking the antiderivative in time of (3.48) we obtain

$$\mathcal{K} = 2\omega_1^{-1} e^{\frac{1}{2}(\omega_1 x + \bar{c}_1)} + \tilde{\mathcal{I}}e^{-\bar{c}_1 t} - 4\omega_2 e^{-\frac{1}{2}\bar{c}_1 t}, \quad (3.49)$$

with $\dot{\mathcal{K}}(t,x) = 0$. Expression (3.49) is equivalent to a Cauchy problem for a first-order differential equation with separated variables. It is easy to see that the general solution of this equation is

$$x(t) = 2\omega_1^{-1}\ln\left\{\omega_1 e^{-\frac{1}{2}\bar{c}_1 \bar{c}_2}\left[2\omega_2 e^{-\frac{1}{2}\bar{c}_1 t} - \tfrac{1}{2}\tilde{\mathcal{I}}e^{-\bar{c}_1 t} + \tfrac{1}{2}\tilde{\mathcal{K}}\right]\right\}. \quad (3.50)$$

Here, the $\tilde{\mathcal{I}}$ and $\tilde{\mathcal{K}}$ parameters are constants.

The class of equations (3.47) includes as a particular case the equation (3.40) of Example 3.7, where $c_1 = 1$, $c_2 = x$ and $c_3 = \varrho$. Such case also corresponds to our results (3.49) and (3.50) with $\bar{c}_1 = \omega_1 = 1$, $\bar{c}_2 = \bar{c}_3 = 0$ and $\omega_2 = \varrho$.

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