Rend. Mat. Appl. (7),
Volume 44 (2023), 281–307

Rendiconti di Matematica
e delle sue Applicazioni

Solvability of some Stefan type problems with L^1-data

Youssef Akdim, Mohammed El Ansari* and Soumia Lalaoui Rhali

Abstract. *In this paper, we focus on some class of Stefan type problems. We prove the existence and uniqueness of renormalized solution in anisotropic Sobolev spaces with data belonging to L^1-data, based on the properties of the renormalized solutions and the generalized monotonicity method in the functional spaces.*

1. Introduction

As of late, anisotropic elliptic equations have received much attention in different fields due to their time dependent versions [5, 8, 14, 17, 22]. These latter have been used as mathematical models to describe the spread of an epidemic disease [9]. Also, these evolution models arise in fluid dynamics when the media has different conductivities in different directions [5, 6], and electrorheological fluids as an important class of non-Newtonian fluids [31].

Our main task is to study the behavior of solutions for a class of Stefan-type problems that have the form:

$$(E,f)\begin{cases}\beta(u)-div(a(x,Du)+F(u))\ni f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N(N\geq 1)$ and $\partial\Omega$ its Lipschitz boundary if $N\geq 2$, a right-hand side f which is assumed to belong to $L^\infty(\Omega)$ or $L^1(\Omega)$ for (E,f). Furthermore, $F:\mathbb{R}\to\mathbb{R}^{\mathbb{N}}$ is locally Lipschitz continuous and $\beta\colon\mathbb{R}\to 2^{\mathbb{R}}$ is a set valued, maximal monotone mapping such that $0\in\beta(0)$ and
$a\colon\Omega\times\mathbb{R}^{\mathbb{N}}\to\mathbb{R}^{\mathbb{N}}$ is a Carathéodory function satisfying the following assumptions:

$(\mathbf{H}_1)-$ Coerciveness: there exists a positive constant λ such that

$$\sum_{i=1}^{N} a_i(x,\xi).\xi_i \geq \lambda\sum_{i=1}^{N}|\xi_i|^{p_i}$$

holds for all $\xi\in\mathbb{R}^N$ and almost every $x\in\Omega$, and for the assumptions on p_i see Section 2.1.

$(\mathbf{H}_2)-$ Growth restriction:

2020 Mathematics Subject Classification: 35J60, 35A01.
Keywords: Anisotrpic Sobolev spaces, Trunctions, maximal monotone graphe, Stefan type problems.
*Corresponding author

$$|a_i(x,\xi)| \leq \gamma(d_i(x) + |\xi_i^{p_i-1}|)$$

for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^N$, γ is a positive constant for $i = 1,\dots,N$, d_i is a positive function in $L^{p_i'}(\Omega)$.

($\mathbf{H}_3$)− Monotonicity in $\xi \in \mathbb{R}^N$:

$$(a(x,\xi) - a(x,\eta)).(\xi - \eta) \geq 0,$$

for almost every $x \in \Omega$ and for $\xi, \eta \in \mathbb{R}^N$.

Due to the possible jumps of β, problem (E,f) belongs to the class of Stefan problems for wich there exists a large number of references, among them [19], [24]. Here we use the notion of renormalized solution developed by DiPerna and Lions [21], for first order equations for L^1-data in [29], and for Radon measure data in [20]. It was then extended to the study of various problems of partial differential equations of parabolic, elliptic-parabolic and hyperbolic type, we refer for instance to [15, 13].

Our problem has been studied in variable exponents spaces and Orlicz spaces by Wittbold et al. [33, 25] and in weighted Sobolev spaces by Akdim and Allalou [3]. Other works in this direction are in [4, 11, 2].

In this work, we prove an existence result of (E,f) in anisotropic Sobolev spaces, this notion was introduced by Nikolskii [30] and Troisi [32]. The main tools in our proofs are Poincaré inequality and embedding theorems in anisotropic Sobolev spaces [27, 18].

The paper is organized as follows: In Section 2, we recall the standard framework of anisotropic Sobolev spaces and some notations which will be used frequently. In Section 3, we introduce the notion of weak and also renormalized solutions for the problem (E,f) for any L^1-data. In Section 4, we give our main results on the existence and uniqueness of renormalized solutions and we discuss the existence of weak solutions. We devote Section 5 to the existence of renormalized solutions in the case where $f \in L^\infty(\Omega)$. Based on this result, the existence and uniqueness of a renormalized solution in the case where $f \in L^1(\Omega)$ is shown in Section 6. In Section 7, we will prove the existence of a weak solution. Finally, we give an example to illustrate our results.

2. Function spaces

2.1. Anisotropic Sobolev spaces

Let Ω be a bounded open subset of $\mathbb{R}^N$, $(N \geq 2)$ and let $1 \leq p_1,\dots,p_N < \infty$ be N a real numbers, $p^+ = \max(p_1,\dots,p_N)$, $p^- = \min(p_1,\dots,p_N)$ and $\overrightarrow{p} = (p_1,\dots,p_N)$.

The anisotropic spaces (see [32])

$$W^{1,\overrightarrow{p}}(\Omega) = \{u \in W^{1,1}(\Omega) : \partial_{x_i} u \in L^{p_i}(\Omega), i = 1,\dots,N\}.$$

is a Banach space with respect to norm

$$\|u\|_{W^{1,\overrightarrow{p}}(\Omega)}= \|u\|_{L^1(\Omega)}+\sum_{i=1}^{N}\|\partial_{x_i}u\|_{L^{p_i}(\Omega)}. \tag{2.1}$$

The space $W_0^{1,\overrightarrow{p}}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to this norm. The dual space of anisotropic Sobolev space $W_0^{1,\overrightarrow{p}}(\Omega)$ is equivalent to $W^{-1,\overrightarrow{p'}}(\Omega)$, where $\overrightarrow{p'}=(p'_1,\dots,p'_N)$ and $p'_i=\dfrac{p_i}{p_i-1}$ for all $i=1,\dots,N$.

The expression

$$|||u|||=\|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)}=\sum_{i=1}^{N}\|\partial_{x_i}u\|_{L^{p_i}(\Omega)}$$

is a norm defined on $W_0^{1,\overrightarrow{p}}(\Omega)$ and equivalent to the norm (2.1).

We recall now a Poincaré-type inequality:
Let $u\in W_0^{1,\overrightarrow{p}}(\Omega)$, then for every $q\geq 1$ there exists a constant C_p (depending on q and i (see [23]), such that

$$\|u\|_{L^q(\Omega)}\leq C_p\|\partial_{x_i}u\|_{L^{p_i}(\Omega)} \quad \text{for } \ i=1,\dots,N. \tag{2.2}$$

Moreover a Sobolev-type inequality holds.
Let us denote by $\overline{p}$ the harmonic mean of these numbers, i.e. $\dfrac{1}{\overline{p}}=\dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\dfrac{1}{p_i}$. Let $u\in W_0^{1,\overrightarrow{p}}(\Omega)$, it follows from [32] that there exists a constant C_s such that

$$\|u\|_{L^q(\Omega)}\leq C_s\prod_{i=1}^{N}\|\partial_{x_i}u\|^{\frac{1}{N}}_{L^{p_i}(\Omega)}, \tag{2.3}$$

where $q=\overline{p}^*=\frac{N\overline{p}}{N-\overline{p}}$ if $\overline{p}<N$ or $q\in[1,+\infty[$ if $\overline{p}\geq N$. On the right-hand side of (2.3) it is possible to replace the geometric mean by the arithmetic mean:
let $a_1,\dots,a_N$ be positive numbers, it holds

$$\prod_{i=1}^{N}a_i^{\frac{1}{N}}\leq\frac{1}{N}\sum_{i=1}^{N}a_i,$$

which implies by (2.3) that

$$\|u\|_{L^q(\Omega)}\leq\frac{C_s}{N}\sum_{i=1}^{N}\|\partial_{x_i}u\|_{L^{p_i}(\Omega)}. \tag{2.4}$$

Note that when the following inequality holds

$$\overline{p}<N, \tag{2.5}$$

inequality (2.4) implies the continuous embedding of the space $W_0^{1,\overrightarrow{p}}(\Omega)$ into $L^q(\Omega)$ for every $q \in [1, \overline{p}^*]$. On the other hand, the continuity of the embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^{p^+}(\Omega)$ with $p^+ := \max\{p_1, \ldots, p_N\}$ relies on inequality (2.2).

It may happen that $\overline{p}^* < p^+$ if the exponents p_i are close enough, then $p_\infty := \max\{\overline{p}^*, p^+\}$ turns out to be the critical exponent in the anisotropic Sobolev embedding (see [32]).

Proposition 2.1. *If the condition (2.5) holds, then for $q \in [1, p_\infty]$ there is a continuous embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^q(\Omega)$. For $q < p_\infty$ the embedding is compact.*

$$W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega). \tag{2.6}$$

2.2. Notations and functions

Before we discuss the concept of solution we introduce some notations and functions that will be frequently used.

We begin by introducing the truncature operator. For given constant $k > 0$ we define the cut function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(r) = \begin{cases} -k, & \text{if } r \leq -k, \\ r, & \text{if } |r| < k, \\ k, & \text{if } r \geq k, \end{cases}$$

and for $r \in \mathbb{R}$, let us define the functions: $r \to r^+ := \max(r, 0)$ and $r \to sign_0(r)$

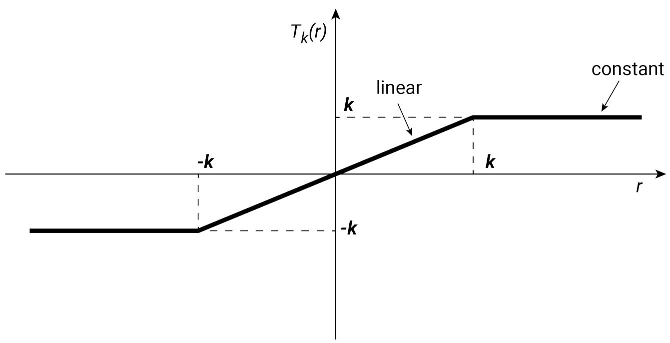

Figure 1: Trunction function

the usual sign function which is defined by

$$r \to sign_0(r) := \begin{cases} -1, & \text{on }]-\infty, 0[, \\ 1, & \text{on }]0, \infty[, \\ 0, & \text{if } r = 0. \end{cases}$$

and

$$r \to sign_0^+(r) := \begin{cases} 1, & \text{if } r > 0, \\ 0, & \text{if } r \leq 0. \end{cases}$$

Let $h_l : \mathbb{R} \to \mathbb{R}$ be defined by $h_l(r) := \min((l+1-|r|)^+, 1)$ for each $r \in \mathbb{R}$.

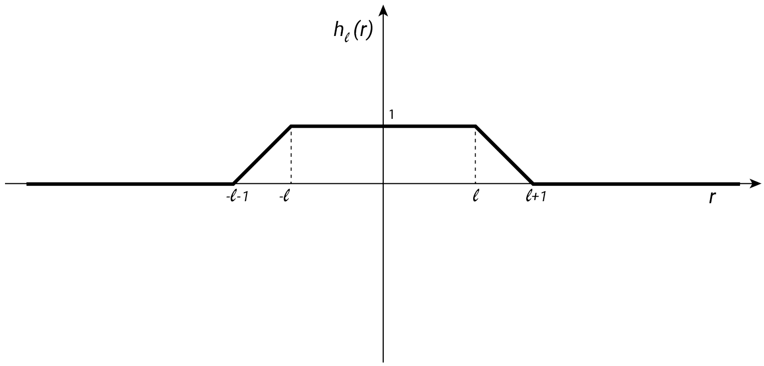

Figure 2: Function $h_l(r)$

For $\delta > 0$, we define $H_\delta^+ : \mathbb{R} \to \mathbb{R}$ by

$$H_\delta^+(r) := \begin{cases} 0, & \text{if } r < 0, \\ \frac{1}{\delta} r, & \text{if } 0 \leq r \leq \delta, \\ 1, & \text{if } r > \delta. \end{cases}$$

and $H_\delta \colon \mathbb{R} \to \mathbb{R}$ by

$$H_\delta(r) := \begin{cases} -1, & \text{if } r < -\delta, \\ \frac{1}{\delta} r, & \text{if } -\delta \leq r \leq \delta, \\ 1, & \text{if } r > \delta. \end{cases}$$

3. Notion of solutions

3.1. Weak solutions

Definition 3.1. A weak solution of (E, f) is a pair of functions $(u, b) \in W_0^{1,\vec{p}}(\Omega) \times L^1(\Omega)$ satisfaying $F(u) \in (L^1_{loc}(\Omega))^N, b \in \beta(u)$ almost everywhere in Ω and

$$b - div(a(x, Du) + F(u)) = f \ \ in \ \ D'(\Omega). \tag{3.1}$$

3.2. Renormalized solutions

Definition 3.2. A renormalized solution of (E, f) is a pair of functions (u, b) satisfying the following conditions:

($\mathbf{R}_1$) $u \colon \Omega \to \mathbb{R}$ is measurable, $b \in L^1(\Omega), u(x) \in D(\beta(x))$ and $b(x) \in \beta(u(x))$ for a.e. $x \in \Omega$.

($\mathbf{R}_2$) For each $k > 0, T_k(u) \in W_0^{1,\overrightarrow{p}}(\Omega)$ and

$$\int_\Omega b.h(u)\phi + \int_\Omega (a(x, D(u)) + F(u)).D(h(u)\phi) = \int_\Omega fh(u)\phi, \tag{3.2}$$

holds for all $h \in C_c^1(\mathbb{R})$ and all $\phi \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$.

($\mathbf{R}_3$) $\displaystyle\int_{\{k<|u|<k+1\}} a(x, Du).Du \longrightarrow 0$ as $k \longrightarrow \infty$.

4. Main results

In this section, we will first state the existence and uniqueness of renormalized solutions for (E, f). Then, we will prove that the renormalized solution of (E, f) is a weak solution.

Theorem 4.1. *For $f \in L^1(\Omega)$, there exists at least one renormalized solution (u, b) of (E, f).*

Theorem 4.2. *Let $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ be strictly monotone for almost every $x \in \Omega$. For $f \in L^1(\Omega)$, let $(u, b), (\widetilde{u}, \widetilde{b})$ be renormalized solutions of (E, f). Then $u = \widetilde{u}$ and $b = \widetilde{b}$.*

Proposition 4.3. *Let (u, b) be a renormalized solution of (E, f) for $f \in L^\infty(\Omega)$. Then $u \in W^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$ and thus, in particular, u is a weak solution of (E, f).*

To prove Theorem 4.1, we will introduce and solve some approximating problems. To this end, for $f \in L^1(\Omega)$ and $m, n \in \mathbb{N}$ we define $f_{m,n} : \Omega \to \mathbb{R}$ by

$$f_{m,n}(x) = \max(\min(f(x), m), -n)$$

for almost every $x \in \Omega$. Clearly, $f_{m,n} \in L^\infty(\Omega)$ for each $m, n \in \mathbb{N}$, $|f_{m,n}(x)| \leq |f(x)|$ a.e. in Ω, hence $\lim\limits_{n\to\infty} \lim\limits_{m\to\infty} f_{m,n} = f$ in $L^1(\Omega)$ and almost everywhere in Ω. The next theorem will give us existence of renormalized solutions $(u_{m,n}, b_{m,n})$ of $(E, f_{m,n})$ for each $m, n \in \Omega$.

5. Case where $f \in L^\infty(\Omega)$

Theorem 5.1. *For $f \in L^\infty(\Omega)$, there exists at least one renormalized solution (u, b) of (E, f).*

The following section will be devoted to prove Theorem 5.1, and we will divide the proof into several steps.

5.1. Approximate solution for L^∞- data

First we will introduce the approximate problem to (E, f) for $f \in L^\infty(\Omega)$ and for which the existence can be proved by standard variational arguments. For $0 < \varepsilon \leq 1$, let $\beta_\varepsilon : \mathbb{R} \longmapsto \mathbb{R}$ be the Yosida approximation of β (see [16]). We introduce the operators

$$A_{1,\varepsilon} : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega),$$
$$u \to \beta_\varepsilon(T_{1/\varepsilon}(u)) + \varepsilon \arctan(u) - diva(x, Du)$$

and

$$A_{2,\varepsilon} : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega),$$
$$u \to -divF(T_{1/\varepsilon}(u)).$$

Because of $(\mathbf{H}_2) - (\mathbf{H}_3)$, $A_{1/\varepsilon}$ is well-defined and monotone (see [28] for instance). Since $\beta_\varepsilon \circ T_{1/\varepsilon}$ is bounded and continuous and thanks to the growth condition $(\mathbf{H}_2)$ on a, it follows that $A_{1,\varepsilon}$ is hemicontinuous (see [28]). From the continuity and boundedness of $F \circ T_{1/\varepsilon}$, it follows that $A_{2,\varepsilon}$ is strongly continuous. Therefore the operator $A_\varepsilon := A_{1,\varepsilon} + A_{2,\varepsilon}$ is pseudomonotone. Using the monotonicity of β_ε, the Gauss-Green Theorem for Sobolev functions and the boundary condition on the convection term $\int_\Omega F(T_{1/\varepsilon}(u)).Du$, we show by using similar arguments as in [12] that A_ε is coercive and bounded. Then it follows from [28] Theorem 2.7 that A_ε is surjective, i.e., for each $0 < \varepsilon \leq 1$ and $f \in W^{-1,\overrightarrow{p}'}(\Omega)$ there exists a solution $u_\varepsilon \in W_0^{1,\overrightarrow{p}}(\Omega)$ of the problem

$$(E_\varepsilon, f) \begin{cases} \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) + \varepsilon \arctan(u_\varepsilon) - div(a(x, Du_\varepsilon) + F((T_{1/\varepsilon}(u_\varepsilon)))) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

such that the following identity holds for all $\phi \in W_0^{1,\overrightarrow{p}}(\Omega)$

$$\int_\Omega (\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) + \varepsilon \arctan(u))\phi + \int_\Omega (a(x, D(u_\varepsilon)) + F(T_{1/\varepsilon}(u_\varepsilon))).D\phi =< f, \phi > \tag{5.1}$$

where $< .,. >$ denotes the duality pairing between $W_0^{1,\overrightarrow{p}}(\Omega)$ and $W^{-1,\overrightarrow{p}'}(\Omega)$.

Proposition 5.2. *For $0 < \varepsilon \leq 1$ fixed and $f, \widetilde{f} \in L^\infty(\Omega)$, let $u_\varepsilon, \widetilde{u}_\varepsilon \in W_0^{1,\overrightarrow{p}}(\Omega)$ be solutions of (E_ε, f) and $(E_\varepsilon, \widetilde{f})$, respectively, then, the follwing comparison principle holds:*

$$\varepsilon \int_\Omega (\arctan(u_\varepsilon) - \arctan(\widetilde{u}_\varepsilon))^+ \leq \int_\Omega (f - \widetilde{f}) sign_0^+(u_\varepsilon - \widetilde{u}_\varepsilon). \tag{5.2}$$

Proof. We use the test function $\varphi = H_\delta^+(u_\varepsilon - \widetilde{u}_\varepsilon)$ in the weak formulation (5.1) for u_ε and $\widetilde{u}_\varepsilon$. Substracting the resulting inequalties, we obtain

$$I_{l,\delta}^1 + I_{l,\delta}^2 + I_{l,\delta}^3 + I_{l,\delta}^4 = I_{l,\delta}^5,$$

where

$$
\begin{aligned}
I_{l,\delta}^1 &= \int_\Omega (\beta_\varepsilon T_{\frac{1}{\varepsilon}}(u_\varepsilon) - \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(\widetilde{u}_\varepsilon)))H_\delta^+(u_\varepsilon - \widetilde{u}_\varepsilon) \geq 0,\\
I_{l,\delta}^2 &= \int_\Omega (\varepsilon \arctan(u_\varepsilon) - \varepsilon \arctan(\widetilde{u}_\varepsilon))H_\delta^+(u_\varepsilon - \widetilde{u}_\varepsilon),\\
I_{l,\delta}^3 &= \int_\Omega a(x, Du_\varepsilon) - a(x, D\widetilde{u}_\varepsilon).DH_\delta^+(u_\varepsilon - \widetilde{u}_\varepsilon) \geq 0,\\
I_{l,\delta}^4 &= \int_\Omega (F(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - F(T_{\frac{1}{\varepsilon}}(\widetilde{u}_\varepsilon))).DH_\delta^+(u_\varepsilon - \widetilde{u}_\varepsilon) \geq 0,\\
I_{l,\delta}^5 &= \int_\Omega (f - \widetilde{f})H_\delta^+(u_\varepsilon - \widetilde{u}_\varepsilon).
\end{aligned}
$$

Passing to the limit with $\delta \to 0$, (5.2) follows since $H_\delta^+ \to sign_0^+$. □

Remark 5.3. Let $f, \widetilde{f} \in L^\infty(\Omega)$ be such that $f \leq \widetilde{f}$ almost everywhere in Ω, $\varepsilon > 0$ and u_ε, $\widetilde{u}_\varepsilon \in W_0^{1,\overrightarrow{p}}(\Omega)$ be solutions of (E_ε, f) and $(E_\varepsilon, \widetilde{f})$, respectively, then an immediate consequence of Propsition 5.2 is that $u_\varepsilon \leq \widetilde{u}_\varepsilon$ almost everywhere in Ω. Furthermore, from the monotonocity of $\beta_\varepsilon \circ T_{1/\varepsilon}$ it follows that also

$$\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \leq \beta_\varepsilon(T_{1/\varepsilon}(\widetilde{u}_\varepsilon))$$

a.e. in Ω.

5.2. A priori estimates

Lemma 5.4. *For $0 < \varepsilon \leq 1$ and $f \in L^\infty(\Omega)$ let $u_\varepsilon \in W_0^{1,\overrightarrow{p}}(\Omega))$ be a solution of (E_ε, f). Then:*

i) There exists a constant $C_1 = C_1(||f||_\infty, \lambda, p_i, N) > 0$, not depending on ε, such that

$$|||u_\varepsilon||| \leq C_1. \tag{5.3}$$

ii) for all $0 < \varepsilon \leq 1$, we have

$$||\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))||_\infty \leq ||f||_\infty \tag{5.4}$$

iii) for all $0 < \varepsilon \leq 1$ and all $l, k > 0$, we have

$$\int_{\{l\leq|u|\leq k+l\}} a(x, Du_\varepsilon).Du_\varepsilon \leq k \int_{\{|u_\varepsilon|>l\}} |f|. \tag{5.5}$$

Proof. *i)* Taking u_ε as a test function in (5.1) we obtain

$$
\begin{aligned}
&\int_\Omega (\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) + \varepsilon \arctan(u_\varepsilon))u_\varepsilon dx + \int_\Omega a(x, Du_\varepsilon).Du_\varepsilon dx\\
&+ \int_\Omega F(T_{1/\varepsilon}(u_\varepsilon)).Du_\varepsilon dx = \int_\Omega f u_\varepsilon dx.
\end{aligned}
$$

As the first term on the left-hand side is nonnegative and the integral over the convection term vanishes by (H_1), we have

$$\lambda\sum_{i=1}^{N}\int_\Omega|\frac{\partial u_\varepsilon}{\partial x_i}|^{p_i}dx \le \sum_{i=1}^{N}\int_\Omega a_i(x,Du_\varepsilon).\frac{\partial u_\varepsilon}{\partial x_i}dx$$
$$\le \int_\Omega f u_\varepsilon dx \le C||f||_\infty(\sum_{i=1}^{N}\int_\Omega|\frac{\partial u_\varepsilon}{\partial x_i}|^{p_i}dx)^{1/p_i}.$$

due to Hölder inequality. Thus $|||u_\varepsilon|||^{p_i} \le C_2\,|||u_\varepsilon|||$, where C_2 is a positive constant. Then we can deduce that u_ε remains bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$ i.e.,

$$|||u_\varepsilon||| \le C_1.$$

ii) Taking $\frac{1}{\delta}[T_{k+\delta}(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)))-T_k(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)))]$ as a test function in (5.1), passing to the limit as $\delta\to 0$ and choosing $k>||f||_\infty$, we obtain *ii*).

iii) For $k,l>0$ fixed we take $T_k(u_\varepsilon-T_l(u_\varepsilon))$ as a test function in (5.1). Using $\int_\Omega a(x,Du_\varepsilon).DT_k(u_\varepsilon-T(u_\varepsilon))dx = \int_{\{l<|u_\varepsilon|<l+k\}} a(x,Du_\varepsilon).Du_\varepsilon dx$, and as the first term on the left-hand side is nonnegative and the convection term vanishes, we get

$$\int_{\{l<|u_\varepsilon|<k+l\}} a(x,Du_\varepsilon).Du_\varepsilon \le \int_\Omega fT_k(u_\varepsilon-T_l(u_\varepsilon))dx \le \int_{\{|u_\varepsilon|>l\}}|f|dx. \tag{5.6}$$

□

Remark 5.5. For $k>0$, from *iii*) in Lemma 5.4, we deduce that

$$|\{|u_\varepsilon|\ge l\}| \le \frac{C_2}{l^{1-\frac{1}{\overline{p}}}} \tag{5.7}$$

$$\int_{\{l\le|u_\varepsilon|\le k+l\}} a(x,Du_\varepsilon).Du_\varepsilon \le k||f||_\infty|\{|u_\varepsilon|>l\}| \le \frac{C_2(k)}{l^{\frac{1}{\overline{p}}-1}} \tag{5.8}$$

for any $0<\varepsilon\le 1$ and a constant $C_2(k)>0$ not depending on ε.

Indeed, let $l>0$ large enough we have:

$$l|\{|u_\varepsilon|\ge l\}| = \int_{\{|u_\varepsilon|\ge l\}}|T_l(u_\varepsilon)|dx \le C\left(\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial T_l(u_\varepsilon)}{\partial x_i}\right|^{p_i}\right)^{1/p_i} \le C_2 l^{1/\overline{p}}$$

which implies that $|\{|u_\varepsilon|\ge l\}| \le C_2 l^{1/\overline{p}-1}$. Then

$$\lim_{l\to+\infty}|\{|u_\varepsilon|\ge l\}| = 0.$$

Therefore, (5.7) follows from (5.8).

5.3. Basic convergence results

Lemma 5.6. *For $0 < \varepsilon \leq 1$ and $f \in L^\infty(\Omega)$, let $u_\varepsilon \in W_0^{1,\overrightarrow{p}}(\Omega)$ be a solution of (E_ε, f). There exist $u \in W_0^{1,\overrightarrow{p}}(\Omega), b \in L^\infty(\Omega)$ such that for a not relabeled subsequence of $(u_\varepsilon)_{0<\varepsilon\leq 1}$ as $\varepsilon \to 0$:*

$$u_\varepsilon \rightharpoonup u \qquad \text{in } W_0^{1,\overrightarrow{p}}(\Omega) \text{ and a.e. in } \Omega, \tag{5.9}$$

$$T_k(u_\varepsilon) \rightharpoonup T_k(u) \quad \text{in } W_0^{1,\overrightarrow{p}}(\Omega) \text{ and strongly in } L^q(\Omega), \tag{5.10}$$

$$\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \rightharpoonup b \quad \text{in } L^\infty(\Omega). \tag{5.11}$$

Moreover, for any $k > 0$,

$$DT_k(u_\varepsilon) \rightharpoonup DT_k(u) \quad \text{in } \prod_{i=1}^{N} L^{p_i}(\Omega), \tag{5.12}$$

$$a(x, DT_k(u_\varepsilon)) \rightharpoonup a(x, DT_k(u)) \quad \text{in } \prod_{i=1}^{N} L^{p_i'}(\Omega). \tag{5.13}$$

Proof. By combining Lemma 5.4 and Rem. 5.5, we obtain (5.11). From (5.7), (5.3) and (2.6), we deduce with a classical argument (see [1]) that for a subsequence still indexed by ε, (5.9) – (5.10) and (5.12) hold as ε tends to 0, where u is a mesurable function defined on Ω.

It is left to prove (5.13). For this, by ($\mathbf{H}_2$) and (5.3) it follows that given any subsequence of $(a(x, DT_k(u_\varepsilon))_\varepsilon)$, there exists a subsequence, still denoted by $(a(x, DT_k(u_\varepsilon))_\varepsilon)$, such that $a(x, DT_k(u_\varepsilon)) \rightharpoonup \Phi_k$ in $\prod_{i=1}^{N} L^{p_i'}(\Omega)$. We will prove that $\Phi_k = a(x, DT_k(u))$ a.e. on Ω. The proof consists of three steps.

Step 1: For every $h \in W^{1,\infty}(\mathbb{R}), h \leq 0$ and $supp(h)$ compact, we will prove that

$$\limsup_{\varepsilon \to 0} \int_\Omega a(x, DT_k(u_\varepsilon)).D[h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))]dx \leq 0. \tag{5.14}$$

Taking $h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))$ as a test function in (5.1), we have

$$\begin{aligned}
&\int_\Omega (\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon) + \varepsilon \arctan(u_\epsilon))h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)) \\
&+ \int_\Omega a(x, D(u_\varepsilon)).D[h_\varepsilon(T_k(u_\varepsilon) - T_k(u))] + \int_\Omega F(T_{1/\varepsilon}(u_\varepsilon)).D[h_\varepsilon(T_k(u_\varepsilon) - T_k(u))] \\
&= \int_\Omega fh(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)).
\end{aligned} \tag{5.15}$$

Using $|h_\varepsilon(T_k(u_\varepsilon)-T_k(u))|\leq 2k||h||_\infty$, by Lebesgue's dominated convergence theorem we find that $\lim\limits_{\varepsilon\to 0}\int_\Omega fh(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))=0$ and then

$$\lim_{\varepsilon\to 0}\int_\Omega F(T_{1/\varepsilon}(u_\varepsilon)).D[h_\varepsilon(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))]=0.$$

By using the same arguments as in [4], we can prove that

$$\limsup_{\varepsilon\to 0}\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)).[h(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))]dx\geq 0.$$

Passing to the limit in (5.15) and using the above results, we obtain (5.14).

Step 2: We now prove that for every $k>0$,

$$\limsup_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon)).[D(T_k(u_\varepsilon)-DT_k(u))]dx\leq 0. \tag{5.16}$$

Indeed, for $k>l$, take $h_l(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))$ as a test function in (5.1). Letting $\varepsilon\to 0$ and then $l\to\infty$, we obtain

$$\limsup_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon)).D[h_l(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))]dx=E_1+E_2+E_3.$$

where

$$E_1=\int_{\{|u_\varepsilon|\leq k\}}h_l(u_\varepsilon)a(x,DT_k(u_\varepsilon)).[DT_k(u_\varepsilon)-DT_k(u)]dx,$$

$$E_2=\int_{\{|u_\varepsilon|>k\}}h_l(u_\varepsilon)a(x,DT_k(u_\varepsilon)).(-DT_k(u))]dx,$$

$$E_3=\int_\Omega h_l'(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))a(x,DT_k(u_\varepsilon)).Du_\varepsilon dx.$$

Since $l>k$, on the set $\{|u_\varepsilon|\leq k\}$ we have $h_l(u_\varepsilon)=1$ so that we can write

$$\limsup_{\varepsilon\to 0}E_1=\limsup_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon)).(DT_k(u_\varepsilon)-DT_k(u))dx.$$

For E_2, using Lebesgue's dominated convergence theorem, we get

$$\lim_{\varepsilon\to 0}E_2=\int_{\{|u_\varepsilon|>k\}}h_l(u)\Phi_{l+1}.DT_k(u)dx=0.$$

For E_3, we have

$$-\int_\Omega h_l'(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))a(x,DT_k(u_\varepsilon))Du_\varepsilon dx$$

$$\leq 2k\int_{\{l<|u_\varepsilon|\leq l+1\}}a(x,Du_\varepsilon)Du_\varepsilon dx.$$

Using (5.8), we deduce that

$$\limsup_{l\to\infty}\limsup_{\varepsilon\to 0}\left(-\int_\Omega h_l'(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))a(x,DT_k(u_\varepsilon)).Du_\varepsilon dx\right)\le 0.$$

Applying (5.14) with h replaced by $h_l, l>k$, we get

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon)).[DT_k(u_\varepsilon)-DT_k(u)]dx\\&\le\limsup_{\varepsilon\to 0}(-\int_\Omega h_l'(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))a(x,DT_k(u_\varepsilon)).Du_\varepsilon dx).\end{aligned}$$

Now letting $l\to\infty$, (5.16) yields.

Step 3: In this step, we prove by monotonicity arguments that for $k>0$, $\Phi_k=a(x,DT_k(u))$ for almost every $x\in\Omega$. Let $\phi\in D(\Omega)$ and $\alpha\in\mathbb{R}$. Using (5.16), we have

$$\alpha\lim_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon)).D\phi dx\ge\alpha\int_\Omega a(x,D(T_k(u)-\alpha\phi)).D\phi dx.$$

Dividing by $\alpha>0$ and $\alpha<0$ and letting $\alpha\to 0$, we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon)).D\phi dx=\int_\Omega a(x,DT_k(u)).D\phi dx. \tag{5.17}$$

This means that for all $k>0$, $\int_\Omega\Phi_k.D\phi dx=\int_\Omega a(x,DT_k(u))D\phi$, and then $\Phi_k=a(x,DT_k(u))$ in $D'(\Omega)$ for all $k>0$. Hence $\Phi_k=a(x,DT_k(u))$ a.e. in Ω and then $a(x,DT_k(u_\varepsilon))\rightharpoonup a(x,DT_k(u))$ weakly in $\prod_{i=1}^{N}L^{p_i'}(\Omega)$. □

Remark 5.7. As an immediate consequence of (5.16) and $(\mathbf{H}_3)$ we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon)-a(x,DT_k(u)).(DT_k(u_\varepsilon)-T_k(u))=0. \tag{5.18}$$

Let us see finally that

$$\lim_{l\to\infty}\int_{l<|u|<l+1}a(x,Du).Dudx=0. \tag{5.19}$$

Indeed, for any $l\ge 0$ fixed we have

$$\begin{aligned}\int_{l<|u|<l+1}a(x,D(u_\varepsilon).D(u_\varepsilon)dx&=\int_\Omega a(x,DT_{l+1}(u_\varepsilon).(DT_{l+1}(u_\varepsilon)-DT_l(u_\varepsilon))dx\\&=\int_\Omega a(x,DT_{l+1}(u_\varepsilon)).DT_{l+1}(u_\varepsilon)dx-\int_\Omega a(x,DT_l(u_\varepsilon)).DT_l(u_\varepsilon)dx.\end{aligned}$$

By (5.18) and passing to the limit as $\varepsilon \to 0$ for fixed $l \geq 0$ we obtain

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \int_{\{l<|u_\varepsilon|<l+1\}} a(x, D(u_\varepsilon)).D(u_\varepsilon)dx \\
&= \int_\Omega a(x, DT_{l+1}(u)).DT_{l+1}(u)dx - \int_\Omega a(x, DT_l(u)).DT_l(u)dx \\
&= \int_{\{l<|u|<l+1\}} a(x, Du).D(u)dx.
\end{aligned} \tag{5.20}$$

Therefore, taking $l \to +\infty$ in (5.20) and using the estimate (5.8) shows that satisfies (R_3).

5.4. Proof of the existence result

We are now in position to conclude the proof of our main result presented in Theorem 5.1:

Proof. Let $h \in C_c^1(\mathbb{R})$ and $\varphi \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$. Taking $h_l(u_\varepsilon)h(u)\varphi$ as a test function in (5.1), we obtain

$$I^1_{\varepsilon,l} + I^2_{\varepsilon,l} + I^3_{\varepsilon,l} + I^4_{\varepsilon,l} = I^5_{\varepsilon,l} \tag{5.21}$$

where

$$\begin{aligned}
I^1_{\varepsilon,l} &= \int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))h_l(u_\varepsilon)h(u)\varphi, \\
I^2_{\varepsilon,l} &= \varepsilon \int_\Omega \arctan(u_\varepsilon)h_l(u_\varepsilon)h(u)\varphi, \\
I^3_{\varepsilon,l} &= \int_\Omega a(x, Du_\varepsilon).D(h_l(u_\varepsilon)h(u)\varphi), \\
I^4_{\varepsilon,l} &= \int_\Omega F(T_{1/\varepsilon}(u_\varepsilon)).D(h_l(u_\varepsilon)h(u)\varphi), \\
I^5_{\varepsilon,l} &= \int_\Omega f h_l(u_\varepsilon)h(u)\varphi.
\end{aligned}$$

Step 1: Letting $\varepsilon \to 0$ obviously, we have

$$\lim_{\varepsilon \to 0} I^2_{\varepsilon,l} = 0. \tag{5.22}$$

Using the convergence results (5.9), (5.11) from Lemma 5.6 we can immediately calculate the following limits:

$$\lim_{\varepsilon \to 0} I^1_{\varepsilon,l} = \int_\Omega b h_l(u)h(u)\varphi, \tag{5.23}$$

$$\lim_{\varepsilon \to 0} I^5_{\varepsilon,l} = \int_\Omega f h_l(u)h(u)\varphi. \tag{5.24}$$

We write $I^3_{\varepsilon,l} = I^{3,1}_{\varepsilon,l} + I^{3,2}_{\varepsilon,l}$ where

$$I^{3,1}_{\varepsilon,l} = \int_\Omega h'_l(u_\varepsilon)a(x, Du_\varepsilon).Du_\varepsilon h(u)\varphi,\ I^{3,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon)a(x, Du_\varepsilon).D(h(u)\varphi).$$

Using (5.8), we get the estimate

$$|\lim_{\varepsilon\to 0} I^{3,1}_{\varepsilon,l}| \le ||h||_\infty||\varphi||_\infty.C_2 l^{-(1-1/\bar{p})}. \tag{5.25}$$

By Lebesgue's dominated convergence theorem it follows that for any $i \in \{1, \dots, N\}$, we have

$$h_l(u_\varepsilon)\frac{\partial}{\partial x_i}(h(u)\varphi) \to h_l(u)\frac{\partial}{\partial x_i}(h(u)\varphi) \quad in\ L^{p_i}\ as\ \varepsilon \to 0.$$

Keeping in mind that $I^{3,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon)a(x, DT_{l+1}(u_\varepsilon)).D(h(u)\varphi)$ and by using (5.13), we get

$$\lim_{\varepsilon\to 0} I^{3,2}_{\varepsilon,l} = \int_\Omega h_l(u)a(x, DT_{l+1}(u)).D(h(u)\varphi). \tag{5.26}$$

Let us write $I^4_{\varepsilon,l} = I^{4,1}_{\varepsilon,l} + I^{4,2}_{\varepsilon,l}$, where

$$I^{4,1}_{\varepsilon,l} = \int_\Omega h'_l(u_\varepsilon)F(T_{1/\varepsilon}(u_\varepsilon)).Du_\varepsilon h(u)\varphi,$$

$$I^{4,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon)F(T_{1/\varepsilon}(u_\varepsilon)).D(h(u)\varphi).$$

For any $l \in \mathbb{N}$, there exists $\varepsilon_0(l)$ such that for all $\varepsilon < \varepsilon_0(l)$,

$$I^{4,1}_{\varepsilon,l} = \int_\Omega h'_l(T_{l+1}(u_\varepsilon))F(T_{l+1}(u_\varepsilon)).h(u)\varphi. \tag{5.27}$$

Using the Gauss–Green theorem for Sobolev functions in (5.27), we get for all $\varepsilon < \varepsilon_0(l)$,

$$I^{4,1}_{\varepsilon,l} = -\int_\Omega \int_0^{T_{l+1}(u_\varepsilon)} h'_l(r)F(r)dr.D(h(u)\varphi). \tag{5.28}$$

Now, using (5.9) and the Gauss–Green Theorem, after letting $\varepsilon \to 0$, we get

$$\lim_{\varepsilon\to} I^{4,1}_{\varepsilon,l} = \int_\Omega h'_l(u)F(u).Duh(u)\varphi. \tag{5.29}$$

Choosing ε small enough, we can write

$$I^{4,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon)F(T_{l+1}(u_\varepsilon)).D(h(u)\varphi), \tag{5.30}$$

and conclude that

$$\lim_{\varepsilon\to 0} I^{4,2}_{\varepsilon,l} = \int_\Omega h_l(u)F(u).D(h(u)\varphi). \tag{5.31}$$

Step 2: Passage to the limit with $l \to \infty$. Combining (5.21) and (5.22) – (5.31) we deduce that

$$I_l^1 + I_l^2 + I_l^3 + I_l^4 + I_l^5 = I_l^6 \tag{5.32}$$

where

$$I_l^1 = \int_\Omega b h_l(u) h(u)\varphi, \qquad I_l^2 = \int_\Omega h_l(u) a(x, DT_{l+1}(u)).D(h(u)\varphi),$$

$$|I_l^3| \le C_2 |l^{-(1-1/\bar{p})}||h||_\infty ||\varphi||_\infty, \qquad I_l^4 = \int_\Omega h_l(u) F(u).D(h(u)\varphi),$$

$$I_l^5 = \int_\Omega h_l'(u) F(u).Du h(u)\varphi, \qquad I_l^6 = \int_\Omega f h_l(u) h(u)\varphi.$$

Obviously, we have

$$\lim_{\varepsilon\to\infty} I_l^3 = 0. \tag{5.33}$$

Choosing $m > 0$ such that $supp\ h \subset [-m, m]$, we can replace u by $T_m(u)$ in $I_l^1, I_l^2, \ldots, I_l^6$, and

$$h_l'(u) = h_l'(T_m(u)) = 0 \ \ if \ \ l+1 > m, \ \ h_l(u) = h_l(T_m(u)) = 0 \ \ if \ \ l > m.$$

Therefore, letting $l \to \infty$ and combining (5.32) with (5.33) we obtain

$$\int_\Omega b h(u)\varphi + \int_\Omega (a(x, Du) + F(u)).D(h(u)\varphi) = \int_\Omega f h(u)\varphi \tag{5.34}$$

for all $h \in C_c^1(\mathbb{R})$ and all $\varphi \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$.

Step 3: Subdifferential argument. It is left to prove that $u(x) \in D(\beta(x))$ and $b(x) \in \beta(u(x))$ for almost all $x \in \Omega$. Since β is a maximal monotone graph, there exist a convex, l.s.c and proper function $j : \mathbb{R} \to [0, \infty]$, such that

$$\beta(r) = \partial j(r) \quad \text{for all} \ \ r \in \mathbb{R}.$$

According to [16], for $0 < \varepsilon \le 1$, $j_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by $j_\varepsilon(r) = \int_0^r \beta_\varepsilon(s)ds$ has the following properties as in [33]

i) For any $0 < \varepsilon \le 1, j_\varepsilon$ is convex and differentiable for all $r \in \mathbb{R}$, such that

$$j_\varepsilon'(r) = \beta_\varepsilon(r) \text{ for all } r \in \mathbb{R} \text{ and any } 0 < \varepsilon \le 1.$$

ii) $j_\varepsilon(r) \to j(r)$ for all $r \in \mathbb{R}$ as $\varepsilon \to 0$.

From *i)*, it follows that for any $0 < \varepsilon \le 1$

$$j_\varepsilon(r) \ge j_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) + (r - T_{1/\varepsilon}(u_\varepsilon))\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \tag{5.35}$$

holds for all $r \in \mathbb{R}$ and almost everywhere in Ω.

Let $E \subseteq \Omega$ be an arbitrary measurable set and χ_E its characteristic function. We fix $\varepsilon_0 > 0$. Multiplying (5.35) by $h_l(u_\varepsilon)\chi_E$, integrating over Ω and using *ii)*, we obtain

$$j(r)\int_E h_l(u_\varepsilon) \ge \int_E j_{\varepsilon_0}(T_{l+1}(u_\varepsilon))h_l(u_\varepsilon) + (r - T_{l+1}h_l(u_\varepsilon)\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \tag{5.36}$$

for all $r \in \mathbb{R}$ and all $0 < \varepsilon < \min(\varepsilon_0, 1/l)$.

As $\varepsilon \to 0$, taking into account that E is arbitarily chosen, we obtain from (5.36)

$$j(r)h_l(u) \geq j_{\varepsilon_0}(T_{l+1}(u))h_l(u) + bh_l(u)(r - T_{l+1}(u)) \tag{5.37}$$

for all $r \in \mathbb{R}$ and almost everywhere in Ω.

Passing to the limit with $l \to \infty$ and then with $\varepsilon_0 \to 0$ in (5.37) finally yields

$$j(r) \geq j(u(x)) + b(x)(r - u(x)) \tag{5.38}$$

for all $r \in \mathbb{R}$ and almost everywhere in Ω, hence $u \in D(\beta)$ and $b \in \beta(u)$ for almost everywhere in Ω. With this last step the proof of Theorem 5.1 is concluded. □

6. Case where $f \in L^1(\Omega)$

6.1. Approximate solution for L^1-data

The comparison principle from proposition 5.2 will be the tool in second approximation procedure. For $f \in L^1(\Omega)$ and $m, n \in \mathbb{N}$ let $f_{m,n} \in L^\infty(\Omega)$ be defined as in Section 3. Using Theorem 5.1, we deduce that for any $m, n \in \mathbb{N}$, there exists $u_{m,n} \in W_0^{1,\overrightarrow{p}}(\Omega), b_{m,n} \in L^\infty(\Omega)$, such that $(u_{m,n}, b_{m,n})$ is a renormalized solution of $(E, f_{m,n})$. Therefore

$$\int_\Omega b_{m,n}h(u_{m,n})\phi + \int_\Omega (a(x, Du_{m,n}) + F(u_{m,n})).D(h(u_{m,n})\phi) = f_{m,n}h(u_{m,n})\phi \tag{6.1}$$

holds for all $m,\, n \in \mathbb{N}$, $h \in C_c^1(\mathbb{R}), \phi \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$. In the next lemma, we give a priori estimates that will be important in the the following:

Lemma 6.1. *For $m, n \in \mathbb{N}$, let $(u_{m,n}, b_{m,n})$ be a renormalized solution of $(E, f_{m,n})$. Then:*

i) For any $k > 0$ we have,

$$\sum_{i=1}^N \int_\Omega |DT_k(u_{m,n})|^{p_i} \leq \frac{k}{\gamma}\|f\|_1 \tag{6.2}$$

ii) for any $k > 0$, there exists a constant $C_3(k) > 0$, not depending on $m, n \in \mathbb{N}$, such that

$$\sum_{i=1}^N \int_\Omega |DT_k(u_{m,n})|^{p_i} \leq C_3(k). \tag{6.3}$$

iii) For $m, n \in \mathbb{N}$, we have:

$$\|b_{m,n}\|_1 \leq \|f\|_1. \tag{6.4}$$

Proof. For $l, k > 0$, we choose $h_l(u_{m,n})T_k(u_{m,n})$ as a test function in (6.1). Then i) and ii) follows with similar arguments as used in the proof of Lemma 5.4. To prove iii), we neglet the positve term

$$\int_\Omega a(x, DT_k(u_{m,n}))DT_k(u_{m,n})$$

and keep

$$\int_\Omega b_{m,n}T_k(u_{m,n}) \leq \int_\Omega f_{m,n}T(u_{m,n}). \tag{6.5}$$

Since $b_{m,n} \in \beta(u_{m,n})$ a.e. in Ω, it follows from (6.5) that

$$\int_{|u_{m,n}|>k} |b_{m,n}| \leq \int_\Omega |f|. \tag{6.6}$$

and we deduce iii) by passing to the limit with $k \to 0$. □

By definition we have

$$f_{m,n} \leq f_{m+1,n} \quad \textit{and} \quad f_{m,n+1} \leq f_{m,n}. \tag{6.7}$$

From Propostion 5.2 it follows that

$$u^\varepsilon_{m,n} \leq u^\varepsilon_{m+1,n} \quad \textit{and} \quad u^\varepsilon_{m,n+1} \leq u^\varepsilon_{m,n}, \tag{6.8}$$

almost everywhere in Ω for any $m, n \in \mathbb{N}$ and all $\varepsilon > 0$. Hence passing to the limit with $\varepsilon \to 0$ in (6.8) yields

$$u_{m,n} \leq u_{m+1,n} \quad \textit{and} \quad u_{m,n+1} \leq u_{m,n}, \tag{6.9}$$

almost everywhere in Ω for any $m, n \in \mathbb{N}$.

Setting $b_\varepsilon := \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))$, using (6.8), Remark 5.3 and the fact that $b^\varepsilon_{m,n} \rightharpoonup b_{m,n}$ in $L^\infty(\Omega)$ and since this convergence preserves order we get

$$b_{m,n} \leq b_{m+1,n} \quad \textit{and} \quad b_{m,n+1} \leq b_{m,n} \tag{6.10}$$

almost everywhere in Ω for any $m, n \in \mathbb{N}$. By (6.10) and (6.4), for any $n \in \mathbb{N}$ there exist $b^n \in L^1(\Omega)$ such that $b_{m,n} \to b^n$ as $m \to \infty$ in $L^1(\Omega)$ and almost everywhere and $b \in L^1(\Omega)$, such that $b^n \to b$ as $n \to \infty$ in $L^1(\Omega)$ and almost every where in Ω. By (6.9), the sequence $(u_{m,n})_m$ is monotone increasing, hence, for any $n \in \mathbb{N}, u_{m,n} \to u^n$ almost everywhere in Ω, where $u^n\colon \Omega \to \overline{\mathbb{R}}$ is a mesurable function. In order to show that u is finite almost everywhere we will give an estimate on the level sets of $u_{m,n}$ in the next lemma:

Lemma 6.2. *For $m, n \in \mathbb{N}$, let $(u_{m,n}, b_{m,n})$ be a renormalized solution of $(E, f_{m,n})$. Then, there exists a constant $C_4 > 0$, not depending on $m, n \in \mathbb{N}$, such that*

$$|\{|u_{m,n}| \geq l\}| \leq C_4 l^{\frac{1}{p}-1} \tag{6.11}$$

for all $l \geq 0$.

Proof. With the same arguments as in remark 5.5 we obtain

$$|\{|u_{m,n}| \geq l\}| \leq C(\overline{p}, N) l^{1/\overline{p}-1} (\sum_{i=1}^{N} \int_{\Omega} |DT_k(u_{m,n})|^{p_i} + |\Omega|) \tag{6.12}$$

for all $m, n \in \mathbb{N}$ where $C(\overline{p}, N)$ is the constant from Sobolev embedding in (2.6). Now we plug (6.2) into (6.12) to obtain (6.11). Note that, as $(u_{m,n})_m$ is pointwise increasing with respect to m,

$$\lim_{m \to \infty} |\{u_{m,n} \geq l\}| = |\{u^n \geq l\}| \tag{6.13}$$

and

$$\lim_{m \to \infty} |\{u_{m,n} \leq -l\}| = |\{u^n \leq -l\}|. \tag{6.14}$$

Combining (6.11) with (6.13) and (6.14) we get

$$|\{u^n \leq -l\}| + |\{u^n > l\}| \leq C_4 l^{\frac{1}{\overline{p}}-1} \tag{6.15}$$

for any $l \geq 1$, hence u^n is finite almost everywhere for $n \in \mathbb{N}$. By the same arguments we get

$$|\{u < -l\}| + |\{u > l\}| \leq C_4 l^{\frac{1}{\overline{p}}-1} \tag{6.16}$$

from (6.15), hence u is finite almost eveyrywhere. Now, since $b_{m,n} \in \beta(u_{m,n})$ almost everywhere in Ω it follows by a subdifferential argument that $b^n \in \beta(u^n)$ and $b \in \beta(u)$ a.e. in Ω.

□

Remark 6.3. If $(u_{m,n}, b_{m,n})$ is renormalized solution of $(E, f_{m,n})$, using $h_\nu(u_{m,n}) T_k(u_{m,n} - T_l(u_{m,n}))$ as a test function in (6.1), neglecting positive terms and passing to the limit with $\nu \to \infty$ we obtain

$$\int_{\{l<|u_{m,n}|<l+k\}} a(x, Du_{m,n}).Du_{m,n} \leq k \left(\int_{\{|u_{m,n}|>l\} \cap \{|f|<\sigma\}} |f| + \int_{\{|f|>\sigma\}} |f| \right) \tag{6.17}$$

for any $k, \sigma > 0, l$. Now applying (6.11) to (6.17), we find that

$$\int_{\{l<|u_{m,n}|<l+k\}} a(x, Du_{m,n}).Du_{m,n} \leq \sigma k C_4 l^{\frac{1}{\overline{p}}-1} + k \int_{\{|f|>\sigma\}} |f| \tag{6.18}$$

holds for any $k, \sigma > 0, l \geq 0$ uniformly in $m, n \in \mathbb{N}$.

6.2. Basic convergence results

Lemma 6.4. *For $m, n \in \mathbb{N}$ let $(u_{m,n}, b_{m,n})$ be a renormalized soltuion of $(E, F_{m,n})$. There exists a subsequence $(m(n))_n$ such that setting $f_n := f_{m(n),n}, b_n := b_{m(n),n}$, $u_n := u_{m(n),n}$ we have*

$$u_n \to u \quad \textit{almost everywhere in } \Omega. \tag{6.19}$$

Moreover, for any $k > 0$,

$$T_k(u_n) \to T_k(u) \ \ in \ \ L^{\overrightarrow{p}}(\Omega), \tag{6.20}$$

$$DT_k(u_n) \rightharpoonup DT_k(u) \ \ in \ \ \prod_{i=1}^{N} L^{p_i}(\Omega), \tag{6.21}$$

$$a(x, DT_k(u_n)) \rightharpoonup a(x, DT_k(u)) \ \ in \ \ \prod_{i=1}^{N} L^{p_i'}(\Omega), \tag{6.22}$$

as $n \to \infty$.

Proof. We construct a subsequence $(m(n))_n$, such that

$$\arctan(u_{m(n),n}) \to \arctan(u),$$

$$b_n := b_{m(n),n} \to b,$$

$$f_n := f_{m(n),n} \to f$$

as $n \to \infty$ in $L^1(\Omega)$ and almost everywhere in Ω. It follows that (6.19) and (6.20) hold. Combining (6.20) with (6.3) we get $T_k(u) \in W_0^{1,\overrightarrow{p}}(\Omega), T_k(u_n) \to T_k(u) \in W_0^{1,\overrightarrow{p}}(\Omega)$ and (6.21) holds for any $k > 0$. From (6.2) and (H_2), it follows that for fixed $k > 0$, given any subsequence of $(a(x, DT_k(u_n)))_n$ there exists a subsequence, still denoted by such that $a(x, DT_k(u_n))_n$, such that

$$a(x, DT_k(u_n))_n \rightharpoonup \Phi_k \quad in \ \ \prod_{i=1}^{N} L^{p_i'}(\Omega)$$

as $n \to \infty$. Since $h_l(u_n)(T_k(u_n) - T_k(u))$ is an admissible test function in (6.1),

$$\lim_{n\to\infty} \sup \int_\Omega a(x, DT_k(u_n)) D(T_k(u_n) - T_k(u)) \leq 0. \tag{6.23}$$

Then, (6.22) follows with the same arguments as int the proof of Lemma 5.6. □

Remark 6.5. With the same arguments as in Remark 5.7, we have

$$\lim_{n\to\infty} \int_\Omega a(x, DT_k(u_n) - a(x, DT_k(u))).D(T_k(u_n) - T_k(u)) = 0, \tag{6.24}$$

$$\lim_{l\to\infty} \int_{\{l<|u|<l+1\}} a(x, Du).Du = 0. \tag{6.25}$$

6.3. Conclusion of the proof of Theorem 4.1

It is left to prove that (u, b) satisfies

$$\int_\Omega bh(u)\phi + \int_\Omega (a(x, Du) + F(u)).D(h(u)\phi) = \int_\Omega fh(u)\phi. \tag{6.26}$$

for all $h \in C_c^1(\mathbb{R})$ and $\phi \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$. To this end, we take $h \in C_c^1(\mathbb{R})$ and $\phi \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$ arbitrary and plug $h_l(u_n)h(u)\phi$ into (6.1) to obtain

$$I_{n,l}^1 + I_{n,l}^2 + I_{n,l}^3 = I_{n,l}^4, \tag{6.27}$$

where

$$\begin{aligned}
I_{n,l}^1 &= \int_\Omega b_n h_l(u_n)h(u)\phi,\\
I_{n,l}^2 &= \int_\Omega a(x, Du_n).D(h_l(u_n)h(u)\phi),\\
I_{n,l}^3 &= \int_\Omega F(u_n).D(h_l(u_n)h(u)\phi),\\
I_{n,l}^4 &= \int_\Omega f_n h_l(u_n)h(u)\phi.
\end{aligned}$$

Step 1. Passing to the limit as $n \to \infty$, applying the convergence results from Lemma 6.4 we get

$$\lim_{n\to\infty} I_{n,l}^1 = \int_\Omega bh_l(u)h(u)\phi, \quad \lim_{n\to\infty} I_{n,l}^4 = \int_\Omega fh_l(u)h(u)\phi. \tag{6.28}$$

Let us write

$$I_{n,l}^2 = I_{n,l}^{2,1} + I_{n,l}^{2,2}, \tag{6.29}$$

where

$$I_{n,l}^{2,1} = \int_\Omega h_l(u_n)a(x, Du_n).D(h(u)\phi), \quad I_{n,l}^{2,2} = \int_\Omega h_l'(u_n)a(x, Du_n).Du_n h(u)\phi. \tag{6.30}$$

With similar arguments as in the proof of (5.26) it follows that

$$\lim_{n\to\infty} I_{n,l}^{2,1} = \int_\Omega h_l(u)a(x, Du).D(h(u)\phi). \tag{6.31}$$

By (6.18), we get the estimate

$$|\lim_{n\to\infty} I_{n,l}^{2,2}| \le \|h\|_\infty \|\phi\|_\infty \left(\delta C_4 l^{\frac{1}{\bar{p}}-1} + \int_{\{|f|>\delta\}} |f|\right), \tag{6.32}$$

for all $n \in \mathbb{N}$ and all $l \ge 1$, $\delta > 0$. Next, we write

$$I_{n,l}^3 = I_{n,l}^{3,1} + I_{n,l}^{3,2},$$

where

$$\lim_{n\to\infty} I_{n,l}^{3,1} = \int_\Omega h_l(u)F(u).D(h(u)\phi), \ \lim_{n\to\infty} I_{n,l}^{3,2} = \int_\Omega h_l'(u)F(u).Duh(u)\phi, \quad (6.33)$$

follows with the same arguments as in (5.27) – (5.31).
Step 2. Passing to the limit as $l \to \infty$. Combining (6.27) with (6.28)–(6.33) we get for all $\delta > 0$ and all $l \geq 1$

$$I_l^1 + I_l^2 + I_l^3 + I_l^4 + I_l^5 = I_l^6, \quad (6.34)$$

where

$$I_l^1 = \int_\Omega bh_l(u)h(u)\phi, \quad I_l^2 = \int_\Omega h_l(u)a(x, DT_{l+1}(u)).D(h(u)\phi)$$

$$|I_l^3| \leq \|h\|_\infty \|\phi\|_\infty \big(\delta C_4 l^{\frac{1}{p}-1} + \int_{\{|f|>\delta\}} |f|\big),$$

for any $\delta > 0$ and

$$I_l^4 = \int_\Omega h_l'(u)F(u)h(u)\phi Du, \ I_l^5 = \int_\Omega h_l(u)F(u).D(h(u)\phi), \ |I_l^6| = \int_\Omega fh_l(u)h(u)\phi.$$

Choosing $m > 0$ such that $supp h \subset [-m, m]$, we can replace u by $T_m(u)$ in $I_l^1, I_l^2, \ldots, I_l^6$ hence

$$\lim_{l\to\infty} I_l^1 = \int_\Omega bh(u)\phi, \qquad \lim_{l\to\infty} I_l^2 = \int_\Omega a(x, Du).D(h(u)\phi), \quad (6.35)$$

$$\lim_{l\to\infty} |I_l^3| \leq \|h\|_\infty \|\phi\|_\infty \int_{\{|f|>\sigma\}} |f|, \qquad \lim_{l\to\infty} I_l^4 = 0, \quad (6.36)$$

$$\lim_{l\to\infty} |I_l^5| = \int_\Omega F(u).D(h(u)\phi), \qquad \lim_{l\to\infty} |I_l^6| = \int_\Omega fh(u)\phi, \quad (6.37)$$

for all $\delta > 0$. Combining (6.34) with (6.35)-(6.37) we finally deduce that (6.1) holds for all $h \in C_C^1(\mathbb{R})$ and all $\phi \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$.
Hence (u, b) satisfies (**R1**), (**R2**) and (**R3**) and the proof of the theorem is completed.

6.4. Proof of Theorem 4.2 (Uniqueness)

Lemma 6.6. *For $f, \widetilde{f} \in L^1(\Omega)$ let (u, b), $(\widetilde{u}, \widetilde{b})$ be the renormalized solutions of (E, f) and $(E, \widetilde{f})$ respectively, then*

$$\int_\Omega (b - \widetilde{b})Sign_0^+(u - \widetilde{u})dx \leq \int_\Omega (f - \widetilde{f})Sign_0^+(u - \widetilde{u})dx, \quad (6.38)$$

Proof. For $\delta > 0$ let H_δ^+ be a Lipschitz approximation of the $sign_0^+$ function. Since (u, b), $(\widetilde{u}, \widetilde{b})$ are renormalized solutions, it follows that

$$T_{l+1}(u), T_{l+1}(\widetilde{u}) \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega) \text{ for all } l > 0.$$

Hence $H_\delta^+(T_{l+1}(u) - T_{l+1}(\widetilde{u})) \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$ for $l, \delta > 0$.
Now, we choose $H_\delta^+(T_{l+1}(u) - T_{l+1}(\widetilde{u}))$ as a test function in the renormalized formulation with $h = h_l$ for (u, b) and for $(\widetilde{u}, \widetilde{b})$ respectively. Subtracting the resulting equalities, we obtain

$$I_{l,\delta}^1 + I_{l,\delta}^2 + I_{l,\delta}^3 + I_{l,\delta}^4 + I_{l,\delta}^5 = I_{l,\delta}^6, \tag{6.39}$$

where $K = \{0 < T_{l+1}(u) - T_{l+1} < \delta\}$ and

$$I_{l,\delta}^1 = \int_\Omega (bh_l(u) - \widetilde{b}h_l(\widetilde{u}))H_\delta^+(T_{l+1}(u) - T_{l+1}(\widetilde{u}))dx,$$

$$I_{l,\delta}^2 = \int_\Omega (h_l'(u)a(x, Du).Du - h_l'(\widetilde{u})a(x, D\widetilde{u}).D\widetilde{u}).H_\delta^+(T_{l+1}(u) - T_{l+1}(\widetilde{u}))dx,$$

$$I_{l,\delta}^3 = \frac{1}{\delta}\int_K (h_l(u)a(x, Du) - h_l(\widetilde{u})a(x, D\widetilde{u})).D(T_{l+1}(u) - T_{l+1}(\widetilde{u}))dx,$$

$$I_{l,\delta}^4 = \int_\Omega (h_l'(u)F(u).Du - h_l'(\widetilde{u})F(\widetilde{u}).D\widetilde{u})H_\delta^+(T_{l+1}(u) - T_{l+1}(\widetilde{u}))dx,$$

$$I_{l,\delta}^5 = \frac{1}{\delta}\int_K (h_l(u)F(u) - h_l(\widetilde{u})F(\widetilde{u})).D(T_{l+1}(u) - T_{l+1}(\widetilde{u}))dx,$$

$$I_{l,\delta}^6 = \int_\Omega (fh_l(u) - \widetilde{f}h_l(\widetilde{u}))H_\delta^+(T_{l+1}(u) - T_{l+1}(\widetilde{u}))dx.$$

Using the same arguments as in [33] i.e., neglecting the nonnegative part of $I_{l,\delta}^3$ and using the fact that F is locally Lipschitz continuous, we can pass to the limit as $\delta \to 0$.
Using the energy dissipation condition $(\mathbf{R_3})$ we can pass the limit as $l \to \infty$ and obtain (6.38).

Now we are in position to give the proof of Theorem 4.2:
Assuming $f = \widetilde{f}$, from Lemma 6.6 we get

$$\int_\Omega (b - \widetilde{b})sign_0^+(u - \widetilde{u})dx \leq 0, \tag{6.40}$$

hence $(b - \widetilde{b})sign_0^+(u - \widetilde{u}) = 0$ almost everywhere in Ω. Now, let us write $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 := \{x \in \Omega : sign_0^+(u(x) - \widetilde{u}(x)) = 0\}$, $\Omega_2 := \{x \in \Omega : (b(x) - \widetilde{b}(x)) = 0\}$. Since $r \mapsto \beta(r)$ is strictly increasing for $x \in \Omega$, we can define the function $\beta^{-1} \colon \mathbb{R} \to \mathbb{R}$ such that $\beta^{-1}(r) = s$ for all $(s, r) \in \mathbb{R}^2$ such that $r \in \beta(s)$. For a.e. $x \in \Omega_2$ we have $b(x) = \widetilde{b}(x)$, hence $u(x) = \beta^{-1}(b(x)) = \beta^{-1}(\widetilde{b}(x)) = \widetilde{u}(x)$. Therefore, $u(x) = \widetilde{u}(x)$ a.e. in Ω_2 and $sign_0^+(u - \widetilde{u}) = 0$. Interchanging the roles of u and $\widetilde{u}$ and respeating the arguments, we get $sign_0^+(\widetilde{u} - u) = 0$ a.e. in Ω and we finally arrive at $u = \widetilde{u}$ a.e. in Ω. Now, we write the renormalized formulation for (u, b) and $(\widetilde{u}, \widetilde{b})$ respectively. Substracting the resulting equalities, we obtain

$$\int_\Omega (b - \widetilde{b})h(u)\varphi dx = 0$$

for all $h \in C_c^1(\mathbb{R})$ and all $\varphi \in C_c^\infty(\Omega)$. Choosing $h(u) = h_l(u)$ and passing to the limit with $l \to \infty$ we find $b = \widetilde{b}$ a.e. in Ω.

7. Proof of Proposition 4.3

Note that for $\varepsilon, k > 0$, $h_l(u)\frac{1}{\varepsilon}T_\varepsilon(u - T_k(u))$ as a test function in (3.2). Neglecting positive terms and passing to the limit with $l \to \infty$, we obtain

$$\frac{1}{\varepsilon}\sum_{i=1}^{N}\int_{k<|u|<k+\varepsilon} |Du|^{p_i} \leq \|f\|_N(\phi(k))^{(N-1)/N}, \tag{7.1}$$

where $\phi(k) := |\{|u| > k\}|$ for $k > 0$. Now we use similar arguments as in [33]. We apply the continuous embedding of $W_0^{1,1}(\Omega)$ into $L^{N/N-1}(\Omega)$ and the Hölder inequality to get

$$\frac{1}{\varepsilon C_N}\|T_\varepsilon(u - T_k(u))\|_{\frac{N}{N-1}} \leq \left(\frac{\phi(k) - \phi(k+\varepsilon)}{\varepsilon}\right)^{1/(p^-)'}\left(\frac{1}{\varepsilon}\int_{k<|u|<k+\varepsilon} |Du|^{p^-}\right)^{1/p^-}, \tag{7.2}$$

where $C_N > 0$ is the constant coming from the Sobolev embedding.
Notice that

$$\frac{1}{\varepsilon}\sum_{i=1}^{N}\int_{k<|u|<k+\varepsilon} |Du|^{p^-} \leq \frac{\phi(k) - \phi(k+\varepsilon)}{\varepsilon} + \frac{1}{\varepsilon}\sum_{i=1}^{N}\int_{k<|u|<k+\varepsilon} |Du|^{p_i}, \tag{7.3}$$

hence, from (7.1), (7.2) and (7.3) we deduce that

$$\begin{aligned}&\frac{1}{\varepsilon C_N}\|T_\varepsilon(u - T_k(u))\|_{\frac{N}{N-1}}\\ &\leq \left(\frac{\phi(k) - \phi(k+\varepsilon)}{\varepsilon}\right)^{1/(p^-)'}\left(\frac{\phi(k) - \phi(k+\varepsilon)}{\varepsilon} + \|f\|_N(\phi(k))^{(N-1)/N}\right)^{1/(p^-)}.\end{aligned} \tag{7.4}$$

From (7.4) and Young's inequality with $\alpha > 0$ it follows that

$$\frac{1}{C_N C}(\phi(k+\varepsilon))^{(N-1)/N} - \frac{\alpha^{p^-}}{p^- C}\|f\|_N(\phi(k))^{(N-1)/N} - \frac{\phi(k) - \phi(k+\varepsilon)}{\varepsilon} \leq 0, \tag{7.5}$$

where

$$C := \left(\frac{1}{\alpha^{(p^-)'}(p^-)'} + \frac{\alpha^{p^-}}{p^-}\right) > 0.$$

The mapping $(0,\infty) \ni k \to \phi(k)$ is non-increasing and therefore of bounded variation, hence it is differentiable almost everywhere on $(0,\infty)$ with $\phi' \in L^1_{loc}(0,\infty)$.

Since it is also continuous from the right, we can pass to the limit with $\varepsilon \downarrow 0$ in (7.5) to find

$$C''(\phi(k))^{(N-1)/N} + \phi'(k) \leq 0 \tag{7.6}$$

for almost every $k > 0$ and $\alpha > 0$ choosen small enough such that

$$C'' := \left(\frac{C_N}{C} - \frac{\alpha^{p^-}}{p^- C} \|f\|_N \right) > 0.$$

Now, the conclusion of the proof follows by contradiction. We assume that $\phi(k) > 0$ for each $k > 0$. For $k > 0$ fixed, we choose $k_0 < k$. From (7.6) it follows that

$$\frac{1}{N} C'' + \frac{d}{ds}((\phi(s))^{(1/N)}) \leq 0 \tag{7.7}$$

for almost all $s \in (k_0, k)$. The left hand side of (7.7) is in $L^1(k_0, k)$, hence we integrate (7.7) over $[k_0, k]$. Moreover, since ϕ is non-increasing, integrating (7.7) over (k_0, k) we get

$$(\phi(k))^{1/N} \leq \phi(k_0)^{1/N} + \frac{1}{N} C''(k_0 - k) \tag{7.8}$$

and from (7.8) the contradiction follows.

8. Example

This section is devoted to an example for illustrating our results. Let us consider the special case:

$$\beta(r) = (r-1)^+ - (r-1)^-, \quad F\colon \mathbb{R} \to (F_i)_{i=1,\dots,N} \in \mathbb{R}^N,$$

where F is locally lipshitz continuous function, and

$$a_i(x, \xi) = \sum_{i=1}^{N} |\xi_i|^{p_i - 1} sgn(\xi_i), \quad i = 1, \dots, N,$$

the $a_i(x, \xi)$ are Carathéodory function satisfying the growth condition $(\mathbf{H}_2)$, and the coercivity $(\mathbf{H}_1)$. On the other the monotonicity condition is verified. In fact

$$\begin{aligned}
&\sum_{i=1}^{N} \left(a_i(x, \xi) - a_i(x, \tilde{\xi}) \right) (\xi_i - \tilde{\xi}_i) \\
&= \sum_{i=1}^{N} \left(|\xi_i|^{p_i - 1} sgn(\xi_i) - |\tilde{\xi}_i|^{p_i - 1} sgn(\tilde{\xi}_i) \right) (\xi_i - \tilde{\xi}_i) \geq 0,
\end{aligned}$$

for almost all $x \in \Omega$ and for all $\xi, \tilde{\xi} \in \mathbb{R}^N$. This last inequality can not be strict, since for $\xi \neq \tilde{\xi}$ with $\xi_N \neq \tilde{\xi}_N$ and $\xi = \tilde{\xi}$, $i = 1, \dots, N-1$. The corresponding expression is zero.

Therefore, for all $f \in L^1(\Omega)$, the following problem:

$$\begin{cases} T_k(u) \in W_0^c(\Omega) \quad \text{for} \quad (k>0); b \in L^1(\Omega) \quad \text{and} \quad b(x) \in \beta(u(x)), \\ \lim_{l\to\infty} \int_{\{l<|u|<l+1\}} a(x,Du).Dudx = 0, \\ \int_\Omega bh(u)\varphi dx + \int_\Omega h(u) \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i-1} sgn\Big(\frac{\partial u}{\partial x_i}\Big).\frac{\partial \varphi}{\partial x_i} dx \\ + \int_\Omega h'(u) \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i-1} sgn\Big(\frac{\partial u}{\partial x_i}\Big).\frac{\partial \varphi}{\partial x_i} dx + \int_\Omega F(u).D(h(u)\varphi)dx \\ = \int_\Omega f.D(h(u)\phi), \quad \forall \varphi \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega) \quad and \quad h \in C_c^1(\mathbb{R}), \end{cases}$$

has at least one renormalized solution.

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Received: 22 September 2022/Accepted: 13 July 2023/Published online: 18 July 2023

Youssef Akdim
University of Sidi Mohamed Ben Abdellah, LAMA Laboratory, Faculty of Sciences D'har El Mahraz, P.O. Box 1796 Atlas Fez, Morocco.
youssef.akdim@usmba.ac.ma

Mohammed El Ansari
University of Sidi Mohamed Ben Abdellah, Laboratory of Engineering Sciences, Multidisciplinary Faculty of Taza, P.O. Box 1223 Taza, Morocco.
mohammed.elansari1@usmba.ac.ma

Soumia Lalaoui Rhali
University of Sidi Mohamed Ben Abdellah, Laboratory of Engineering Sciences, Multidisciplinary Faculty of Taza, P.O. Box 1223 Taza, Morocco.
soumia.lalaoui@usmba.ac.ma