Rend. Mat. Appl. (7),
Volume 45 (2024), 181–199

Rendiconti di Matematica
e delle sue Applicazioni

Optimality and duality for a weakly efficient solution of bilevel multiobjective fractional programming problems with extremal-value function

Ahmed Rikouane

Abstract. The purpose of this paper is to establish necessary and sufficient optimality conditions and some duality results for weakly efficient solutions of a constrained bilevel multiobjective fractional programming problem (P) with an extremal-value function. Using parametric approach, the problem (P) is first equivalently transformed into a parametric problem (P^{μ}) with $\mu \in \mathbb{R}^p$, for which we construct then a dual problem. This is achieved in terms of conjugate duality theory. Under appropriate assumptions, the weak and strong duality results for (P^{μ}) are presented. These results permit us to give dual characterizations for the weakly efficient solutions of the problem (P).

1. Introduction

Bilevel programming problems are hierarchical optimization problems in which their constraints and/or the objective function of the so-called upper level problem is determined implicitly by the solution set of another parametric optimization problem called the lower level problem. This class of optimisation problems plays an important role in a variety of fields, such as electricity markets [25], transportation planning and management problems [26], medical engineering [10] and optimal allocation of water resources [2]. When the set of solutions of the lower level problem is a singleton, the bilevel problem is called bilevel programming problem with extremal-value function. Among the large number papers treat bilevel problems [1, 9, 24]. Dempe [9] studied a bilevel programming problem with an extremal-value function and developed necessary and sufficient optimality conditions. Aboussoror and Adly [1] considered a bilevel nonlinear optimization problem with an extremal value function and obtained necessary and sufficient optimality conditions under constraint qualifications and via the Fenchel-Lagrange duality approach. Recently, Wang et al. [24] considered a bilevel multiobjective program with extremal-value function, and obtained optimality conditions and duality results under a generalized Slater-type constraint qualification.

Fractional multiobjective programming problem has many applications such management science, operational research, economics and information theory (see Stancu-Minasian [19, 20]). Many researchers have made contributions in the field

2020 Mathematics Subject Classification: 90C25, 90C32, 90C46.
Keywords: Weakly efficient solution, optimality conditions, conjugate duality, bilevel programming, multiobjective fractional programming.

of fractional multiobjective programming, one can see for example [4, 5, 6, 12, 16, 18, 21] and the references therein. Bector et al. [4] presented a duals and duality results for multiobjective fractional problems under some differentiability assumptions. Singh and Hanson [18] established various duality results associating properly efficient solutions for a multiobjective fractional programming problem. Kim et al. [12] gave ϵ-optimality conditions for multiobjective fractional optimization problems by using the concept of epigraphs of conjugate functions in terms of ϵ-subdifferentials computed at a optimal solution. Ram U. Verma [6] reformulated the multiobjective fractional programming problem as a scalar optimization problem and obtained a necessary and sufficient ϵ-optimality conditions for it. Recently, Moustaid et al. [16] established sequential approximate weak optimality conditions for multiobjective fractional programming problems via sequential subdifferential calculus.

The aim of this paper is to extend the approach in [1, 24] to a bilevel multiobjective fractional programming problem (see [14])

$$(P) \quad \mathrm{v}-\min_{x\in C}\left\{\frac{f_1(x,v(x))}{g_1(x,v(x))},\dots,\frac{f_p(x,v(x))}{g_p(x,v(x))}\right\},$$

where $C := \{x \in X,\ G(x,v(x)) \leq_{\mathbb{R}^q_+} 0\}$ and $v(x)$ is the optimal value of the following problem parametrized by x

$$(P_x) \qquad \min_{y\in A} f(x,y).$$

The notation "v−min" refers to a vector minimum problem. By using the parametric approach of Dinkelbach [8], we transform the problem (P) into the nonfractional multiobjective bilevel convex optimization problem (P^μ) with parameter $\mu \in \mathbb{R}^p$ (see, for instance [3, 5, 6, 11, 12]), and by applying a linear scalarizing functional to the objective function of the problem (P^μ). A necessary and sufficient optimality conditions are established. The structure of the scalar problem gives an idea about how to construct a multiobjective dual problem of the problem (P^μ). Under a standard constraint qualification and some convexity and monotonicity assumptions, the weak and strong duality results for (P^μ) are proved. These results permit us to give dual characterizations for the weakly efficient solutions of the problem (P).

This paper is organized as follows. In Section 2, we recall some notions and definitions and we give some preliminary results which are used later. In Section 3, by using the parametric approach, we attach to the bilevel multiobjective fractional problem (P) a parametric problem (P^μ) with $\mu \in \mathbb{R}^p$, and by using the scalar method, we associate the problem (P^μ) to a scalar problem (P^μ_λ), $\lambda \in \mathbb{R}^p$. A strong duality theorem and optimality conditions are established. In Section 4, we construct a dual problem to (P^μ) by using the dual of the scalarized problem. Under appropriate assumptions, the weak and strong duality results are presented. In Section 5, we present an example illustrating the main result of this study.

2. Preliminaries and basic definitions

In this section, we give some definitions and preliminary results which will be used throughout this paper. Let X be nonempty subset of $\mathbb{R}^n$. We denote by $\mathrm{ri}(X)$ the relative interior of the set X, by $x^T y$ the inner product of the vectors $x = (x_1, \dots, x_p)$, $y = (y_1, \dots, y_p) \in \mathbb{R}^p$ and by $\mathbb{R}^p_+$ the non-negative orthant of $\mathbb{R}^p$, defined by

$$\mathbb{R}^p_+ := \{u = (u_1, \dots, u_p) \in \mathbb{R}^p,\ u_i \geq 0,\ i = 1, \dots, p\}.$$

For $x, y \in \mathbb{R}^p$ we denote $x \leq_{\mathbb{R}^p_+} y$ (or $y \geq_{\mathbb{R}^p_+} x$) if $y - x \in \mathbb{R}^p_+$.

For a function $f :\ \mathbb{R}^n \longrightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, the set defined by

$$\mathrm{dom}(f) := \{x \in \mathbb{R}^n,\ f(x) < +\infty\}$$

denotes the effective domain of f. We say that f is proper if $\mathrm{dom}(f) \neq \emptyset$ and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$. We shall denote by δ_X and σ_X the indicator and the support functions of a nonempty subset $X \subset \mathbb{R}^n$, respectively, defined on $\mathbb{R}^n$ by

$$\delta_X(x) := \begin{cases} 0, & \text{if } x \in X \\ +\infty, & \text{otherwise} \end{cases}, \quad \sigma_X(p^*) := \sup_{x \in X} p^{*T} x,\ \forall p^* \in \mathbb{R}^n.$$

The normal cone to X at $\bar{x} \in X$ is defined by

$$N(\bar{x}, X) := \{x^* \in \mathbb{R}^n,\ x^{*T}(x - \bar{x}) \leq 0,\ \forall x \in X\}.$$

A function $f \colon \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ is said to be $\mathbb{R}^n_+$-increasing if for each $x, y \in \mathbb{R}^n$, we have

$$x \leq_{\mathbb{R}^n_+} y \Longrightarrow f(x) \leq f(y).$$

The function defined by

$$f^* \colon \mathbb{R}^n \longrightarrow \overline{\mathbb{R}},\ \ f^*(x^*) := \sup_{x \in \mathbb{R}^n} \{x^{*T} x - f(x)\}.$$

is called the conjugate function of f. We have the so-called Young-Fenchel inequality

$$f^*(x^*) + f(x) \geq x^{*T} x,\ \forall x, x^* \in \mathbb{R}^n. \tag{2.1}$$

It is well known that for a non-negative real number λ,

$$(\lambda f)^*(x^*) := \begin{cases} \lambda f^*(\frac{x^*}{\lambda}), & \text{if } \lambda > 0, \\ \\ \delta_{\{0\}}(x^*), & \text{if } \lambda = 0. \end{cases}$$

Let $K \subseteq \mathbb{R}^p$ be a convex cone. The dual cone K^* of K is defined by

$$K^* := \{x^* \in \mathbb{R}^p,\ x^{*T} x \geq 0,\ \forall x \in K\}.$$

Lemma 2.1 (cf. [7, Lemma 2.1])**.** *Let $K \subseteq \mathbb{R}^p$ be a convex cone and $h : \mathbb{R}^p \longrightarrow \overline{\mathbb{R}}$ a proper and K-increasing function. Then $h^*(x^*) = +\infty$ for all $x^* \notin K^*$.*

Let $g : \mathbb{R}^q \longrightarrow \mathbb{R}^p \cup \{+\infty_{\mathbb{R}^p}\}$ be a given vector valued function. The function g is called $\mathbb{R}^p_+$-convex if for all $x, y \in \mathbb{R}^q$ and all $t \in [0, 1]$ we have

$$g(tx + (1 - t)y) \leq_{\mathbb{R}^p_+} tg(x) + (1 - t)g(y).$$

Furthermore, g is called $(\mathbb{R}^q_+, \mathbb{R}^p_+)$-increasing if for each $x, y \in \mathbb{R}^q$ we have

$$x \leq_{\mathbb{R}^q_+} y \Longrightarrow g(x) \leq_{\mathbb{R}^p_+} g(y).$$

Let $h : \mathbb{R}^n \longrightarrow \mathbb{R}^q \cup \{+\infty_{\mathbb{R}^q}\}$ be a mapping, then the composed mapping $g \circ h : \mathbb{R}^n \longrightarrow \mathbb{R}^p \cup \{+\infty_{\mathbb{R}^p}\}$ is defined by

$$(g \circ h)(x) := \begin{cases} g(h(x)), & \text{if } x \in \operatorname{dom}(h), \\ \\ +\infty_{\mathbb{R}^p}, & \text{otherwise.} \end{cases}$$

It is easy to see that if $g \colon \mathbb{R}^q \longrightarrow \mathbb{R}^p \cup \{+\infty_{\mathbb{R}^p}\}$ is $\mathbb{R}^p_+$-convex, $(\mathbb{R}^q_+, \mathbb{R}^p_+)$-increasing on $\operatorname{dom}(g)$ and $h \colon \mathbb{R}^n \longrightarrow \mathbb{R}^q \cup \{+\infty_{\mathbb{R}^q}\}$ is $\mathbb{R}^q_+$-convex with $h(\operatorname{dom}(h)) \subseteq \operatorname{dom}(g)$, then the composed mapping $g \circ h$ is $\mathbb{R}^p_+$-convex.

In this paper we adopt the conventions that $0 \times (\pm\infty) = 0$ and $a \times (\pm\infty) = \pm\infty$, for all $a > 0$.

Let us recall the following lemmas can be found in [13, 17].

Lemma 2.2 (cf. [17, Theorem 16.4])**.** *Let $g_i \colon \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ $(i = 1, \ldots, m)$ be proper convex functions. If $\cap_{i=1}^m ri(dom(g_i)) \neq \emptyset$, then*

(i) $\left(\sum_{i=1}^m g_i\right)^* (x^*) = \inf \{\sum_{i=1}^m g_i^*(x_i^*) : \ x^* = \sum_{i=1}^m x_i^*\}$;

(ii) for all $x^* \in \mathbb{R}^n$, the infimum in (i) is attained.

Lemma 2.3 ([13])**.** *Let $h = (h_1, \ldots, h_n)$ with $h_i \colon \mathbb{R}^m \longrightarrow \mathbb{R}$ $(i = 1, \ldots, n)$ be convex functions, and $g : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ be proper convex and $\mathbb{R}^n_+$-increasing function. If $h(\cap_{i=1}^n dom(h_i)) \cap int(dom(g)) \neq \emptyset$, then*

$$(g \circ h)^*(x^*) = \inf_{r \in \mathbb{R}^n_+} \left\{ g^*(r) + \left(\sum_{i=1}^n r_i h_i \right)^* (x^*) \right\},$$

where for any $x^* \in \mathbb{R}^m$ the infimum is attained.

Let $\overline{x}$ be a feasible point of (P) i.e., $\overline{x} \in C$ and $v(\overline{x})$ is the optimal value of the lower level problem $(P_{\overline{x}})$. We will denote the set of the feasible solution of (P) as Ω, that is

$$\Omega := \{x \in X, \ G(x, v(x)) \leq_{\mathbb{R}^q_+} 0 \text{ and } v(x) \text{ is the optimal value of } (P_x)\}.$$

Definition 2.4. An element $\overline{x} \in \Omega$ is said to be
- efficient solution for (P) if there is no $x \in \Omega$ such that

$$\frac{f_i(x, \upsilon(x))}{g_i(x, \upsilon(x))} \leq \frac{f_i(\overline{x}, \upsilon(\overline{x}))}{g_i(\overline{x}, \upsilon(\overline{x}))}, \text{ for each } i \in \{1, \dots, p\},$$
$$\frac{f_j(x, \upsilon(x))}{g_j(x, \upsilon(x))} < \frac{f_j(\overline{x}, \upsilon(\overline{x}))}{g_j(\overline{x}, \upsilon(\overline{x}))}, \text{ for some one } j \in \{1, \dots, p\};$$

- weakly efficient solution for (P) if there is no $x \in \Omega$ such that

$$\frac{f_i(x, \upsilon(x))}{g_i(x, \upsilon(x))} < \frac{f_i(\overline{x}, \upsilon(\overline{x}))}{g_i(\overline{x}, \upsilon(\overline{x}))}, \text{ for each } i \in \{1, \dots, p\}.$$

3. Problem formulation

In this section, we consider the following bilevel multiobjective fractional programming problem with an extremal-value function

$$(P) \qquad \mathrm{v} - \min_{x \in C} \left\{ \frac{f_1(x, \upsilon(x))}{g_1(x, \upsilon(x))}, \dots, \frac{f_p(x, \upsilon(x))}{g_p(x, \upsilon(x))} \right\},$$

where $C := \{x \in X,\ G(x, \upsilon(x)) \leq_{\mathbb{R}^q_+} 0\} \neq \emptyset$ and $\upsilon(x)$ is the optimal value of the lower level problem

$$(P_x) \qquad \min_{y \in A} f(x, y),$$

Herein, X is a nonempty convex subset of $\mathbb{R}^n$, A is a nonempty subset of $\mathbb{R}^m$ compact and convex, $f_i, -g_i \colon \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$, $i = 1, \dots, p$, $G_j \colon \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$, $j = 1, \dots, q$ are convex functions and $\mathbb{R}^{n+1}_+$-increasing, $f \colon \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$ is a convex function. Moreover, we assume that for any $x \in C$, $f_i(x, \upsilon(x)) \geq 0$ and $g_i(x, \upsilon(x)) > 0,\ i = 1, \dots, p$.

We mention that the functions $f_i,\ g_i,\ i = 1, \dots, p$ and $G_j,\ j = 1, \dots, q$ are all continuous since $\mathrm{int}(\mathrm{dom}(f_i)) = \mathrm{int}(\mathrm{dom}(g_i)) = \mathbb{R}^{n+1},\ i = 1, \dots, p$ and $\mathrm{int}(\mathrm{dom}(G_j)) = \mathbb{R}^{n+1},\ j = 1, \dots, q$. Moreover, one can see that the function $\upsilon : \mathbb{R}^n \longrightarrow \mathbb{R}$ is finite, convex, continuous and for each $x \in \mathbb{R}^n$ there exists $y \in A$ such that $\upsilon(x) = f(x, y)$.

The following notation will be considered in what follows

$$\mu_i := \frac{f_i(\overline{x}, \upsilon(\overline{x}))}{g_i(\overline{x}, \upsilon(\overline{x}))},\ \overline{x} \in \Omega \text{ and } \mu = (\mu_1, \dots, \mu_p) \in \mathbb{R}^p_+.$$

Using the parametric approach of Dinkelbach [8], we transform the bilevel multiobjective fractional programming problem (P) into an equivalently bilevel vector convex nonfractional programming problem defined as follows

$$(P^\mu) \qquad \mathrm{v} - \min_{x \in C} \{f_1(x, \upsilon(x)) - \mu_1 g_1(x, \upsilon(x)), \dots, f_p(x, \upsilon(x)) - \mu_p g_p(x, \upsilon(x))\},$$

where

$$\overline{x} \in \Omega = \{x \in X,\ G(x, \upsilon(x)) \leq_{\mathbb{R}^q_+} 0 \text{ and } \upsilon(x) \text{ is the optimal value of } (P_x)\}.$$

Let us consider the following auxiliary functions defined by (see [16])

$$\begin{array}{rccl} F_i: & \mathbb{R}^n \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ & (x, y) & \longrightarrow & F_i(x, y) := f_i(x, y) - \mu_i g_i(x, y) \end{array} \qquad (i = 1, \ldots, p)$$

Then, the problem (P^μ) may be written equivalently as

$$(P^\mu) \qquad \mathrm{v} - \min_{x \in C} F(x, \upsilon(x)),$$

where

$$\begin{array}{rccl} F: & \mathbb{R}^n \times \mathbb{R} & \longrightarrow & \mathbb{R}^p \\ & (x, y) & \longrightarrow & F(x, y) := (F_1(x, y), \ldots, F_p(x, y)). \end{array}$$

Let $h : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ be a function defined by (see [1])

$$h(x) := (h_1(x), \ldots, h_{n+1}(x)) = (x, \upsilon(x)),$$

It is clear that the function h is $\mathbb{R}^{n+1}$-convex, continuous and $h(\mathrm{dom}(h)) \subseteq \mathbb{R}^{n+1}$ since the function $\upsilon \colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is finite, convex and continuous. Then, the problem (P^μ_λ) will be like the following

$$(P^\mu) \qquad \mathrm{v} - \min_{x \in C} F(h(x)).$$

Definition 3.1. An element $\overline{x} \in \Omega$ is said to be a weakly efficient solution for (P^μ) if there is no $x \in \Omega$ such that

$$F_i(h(x)) < F_i(h(\overline{x})), \text{ for each } i \in \{1, \ldots, p\}.$$

In order to characterize the weakly efficient solutions of (P^μ), we consider the scalar problem corresponding to (P^μ) as follows

$$(P^\mu_\lambda) \quad \begin{cases} \min \lambda^T F(h(x)), \\ x \in C, \end{cases}$$

where $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p_+ \backslash \{0\}$.

A point $\overline{x} \in \Omega$ is called an optimal solution of the scalar problem (P^μ_λ) if

$$\lambda^T F(h(\overline{x})) \leq \lambda^T F(h(x)),\ \forall x \in \Omega.$$

We will need the following lemma.

Lemma 3.2 (cf. [18, Theorem 2])**.** *The point $\overline{x} \in \Omega$ is an efficient solution of (P) if and only if $\overline{x}$ is an efficient solution of (P^μ) where $\mu = (\mu_1, \ldots, \mu_p)$ and $\mu_i = \frac{f_i(\overline{x}, \upsilon(\overline{x}))}{g_i(\overline{x}, \upsilon(\overline{x}))}$, $i = 1, \ldots, p$.*

The relationship linking (P), (P^{μ}) and (P^{μ}_{λ}) which will be useful for our purposes are stated in the following result.

Lemma 3.3. *Suppose that $\overline{x} \in \Omega$. Then the following statements are equivalent*

(i) $\overline{x}$ is a weakly efficient solution for problem (P) ;

(ii) $\overline{x}$ is a weakly efficient solution for problem (P^{μ}) where $\mu_i = \frac{f_i(\overline{x},\upsilon(\overline{x}))}{g_i(\overline{x},\upsilon(\overline{x}))}$, $i = 1, \ldots, p$;

(iii) $\overline{x}$ is an optimal solution for problem (P^{μ}_{λ}) where $\mu_i = \frac{f_i(\overline{x},\upsilon(\overline{x}))}{g_i(\overline{x},\upsilon(\overline{x}))}$, $i = 1, \ldots, p$ and for some $\lambda \in \mathbb{R}^p_+ \backslash \{0\}$.

Proof. $(i) \Longrightarrow (ii)$ Assume that $\overline{x}$ is not a weakly efficient solution of (P^{μ}), then there exists $x \in \Omega$ such that

$$(f_i(x,\upsilon(x)) - \mu_i g_i(x,\upsilon(x))) < 0 = (f_i(\overline{x},\upsilon(\overline{x})) - \mu_i g_i(\overline{x},\upsilon(\overline{x}))),\ \forall i \in \{1,\ldots,p\}. \tag{3.1}$$

Since $g_i(x,\upsilon(x)) > 0$, it follows from (3.1) that

$$\frac{f_i(x,\upsilon(x))}{g_i(x,\upsilon(x))} < \mu_i = \frac{f_i(\overline{x},\upsilon(\overline{x}))}{g_i(\overline{x},\upsilon(\overline{x}))},\ \forall i \in \{1,\ldots,p\},$$

this leads to a contradiction.
$(ii) \Longrightarrow (i)$ The proof of is similar to $(i) \Longrightarrow (ii)$, so that (i) is equivalent to (ii).
$(ii) \Longrightarrow (iii)$ Let $\overline{x} \in \Omega$ be a weakly efficient solution of (P^{μ}) where $\mu_i = \frac{f_i(\overline{x},\upsilon(\overline{x}))}{g_i(\overline{x},\upsilon(\overline{x}))}$, $i = 1, \ldots, p$. Then, there exists no $x \in \Omega$ such that

$$F_i(h(x)) < F_i(h(\overline{x})) = 0,\ \forall i \in \{1,\ldots,p\},$$

i.e,

$$F(h(\Omega)) \cap (-\mathrm{int}\mathbb{R}^p_+) = \emptyset.$$

Since $F(h(\Omega))$ is $\mathbb{R}^p_+$-convex ($F(h(\Omega)) + \mathbb{R}^p_+$ is convex), therefore, it follows from separation theorem that there exists $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p_+ \backslash \{0\}$ such that

$$\lambda^T F(h(x)) = \sum_{i=1}^{p} \lambda_i F_i(h(x)) \geq 0 = \sum_{i=1}^{p} \lambda_i F_i(h(\overline{x})) = \lambda^T F(h(\overline{x})).$$

Thus, $\overline{x}$ is an optimal solution of (P^{μ}_{λ}).
$(iii) \Longrightarrow (ii)$ Assume that $\overline{x} \in \Omega$ is not a weakly efficient solution of (P^{μ}), then there exists $x \in \Omega$ such that

$$F_i(h(x)) < 0 = F_i(h(\overline{x})),\ \forall i \in \{1,\ldots,p\}.$$

For $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p_+ \backslash \{0\}$, it follows

$$\sum_{i=1}^{p} \lambda_i F_i(h(x)) < 0 = \sum_{i=1}^{p} \lambda_i F_i(h(\overline{x})).$$

This contradicts the fact that $\overline{x}$ is an optimal solution of (P^{μ}_{λ}). □

Obviously, the problem (P_λ^μ) is a composed convex optimization problem. Then, we can construct its Lagrangian dual problem as follows (see [1, 17, 22, 23]).

$$(D_\lambda^\mu) \ \sup_{r\in\mathbb{R}_+^q} \inf_{x\in\mathbb{R}^n} \left\{(\lambda^T F + r^T G)(h(x)) + \delta_X(x)\right\}, \text{ where } \lambda \in \mathbb{R}_+^p\backslash\{0\}.$$

It is easy to check that h_i is a finite convex function for every $i \in \{1,\dots,n+1\}$, and since $\lambda \in \mathbb{R}_+^p\backslash\{0\}$ and $r \in \mathbb{R}_+^q$, therefore it follows that the function $\lambda^T F + r^T G$ is finite valued convex and $\mathbb{R}_+^{n+1}$-increasing. It follows from Lemmas 2.2 and 2.3 that the dual (D_λ^μ) will be, then

$$(D_\lambda^\mu) \quad \sup_{(r,s,w,t)\in Y} \left\{-\sum_{i=1}^p \lambda_i F_i^*(w_i) - (r^T G)^*\Big(s - \sum_{i=1}^p \lambda_i w_i\Big) - (s^T h)^*(t) - \sigma_X(-t)\right\}$$

where

$$Y := \big\{(r,s,w,t),\ r\in\mathbb{R}_+^q, s\in\mathbb{R}_+^{n+1}, t\in\mathbb{R}^n, w=(w_1,\dots,w_p), \\ w_i \in \mathbb{R}^{n+1}, i=1,\dots,p\big\}.$$

Since the functions f_i, $(-\mu_i g_i)$, $i=1,\dots,p$ satisfy all the assumptions of Lemma 2.2, then for each $i\in\{1,\dots,p\}$, we have

$$\begin{aligned} F_i^*(w_i) &= (f_i + (-\mu_i g_i))^*(w_i) = \inf\{f_i^*(u_i) + (-\mu_i g_i)^*(v_i),\ u_i + v_i = w_i\} \\ &= -\sup\{-f_i^*(u_i) - (-\mu_i g_i)^*(v_i),\ u_i + v_i = w_i\}. \end{aligned}$$

Hence, the problem (D_λ^μ) rewrites as

$$\sup_{(r,s,u,v,t)\in Y^\lambda} \Big\{ -\sum_{i=1}^p \lambda_i[f_i^*(u_i) + (-\mu_i g_i)^*(v_i)] - (r^T G)^*\Big(s - \sum_{i=1}^p \lambda_i(u_i+v_i)\Big) \\ - (s^T h)^*(t) - \sigma_X(-t)\Big\}$$

where

$$Y^\lambda := \big\{(r,s,u,v,t),\ r\in\mathbb{R}_+^q, s\in\mathbb{R}_+^{n+1}, t\in\mathbb{R}^n, u=(u_1,\dots,u_p), \\ v=(v_1,\dots,v_p), u_i, v_i\in\mathbb{R}^{n+1}, i=1,\dots,p\big\}.$$

We denote by $\mathrm{val}(P_\lambda^\mu)$ and $\mathrm{val}(D_\lambda^\mu)$ the optimal values of the problem (P_λ^μ) and (D_λ^μ) , respectively. The weak duality always holds, i.e.

$$\mathrm{val}(P_\lambda^\mu) \geq \mathrm{val}(D_\lambda^\mu). \tag{3.2}$$

In the following, to derive the strong duality theorem and the optimality conditions, we need a so-called constraint qualification

$$(CQ)\ \exists \overline{x} \in \mathrm{ri}(X) \text{ such that } \begin{cases} G_i(h(\overline{x})) \leq 0, \text{ if } i \in L \\ \\ G_i(h(\overline{x})) < 0, \text{ if } i \in N, \end{cases}$$

where $L = \{i \in \{1, \ldots, q\} : \ G_i \circ h$ is an affine function$\}$ and $N = \{1, \ldots, q\} \backslash L$.

In what follows, we will need the following strong duality theorem (see [23, Theorem 3.2]).

Theorem 3.4 (Strong duality for (P^{μ}_{λ})). *If (CQ) is fulfilled and $val(P^{\mu}_{\lambda})$ is finite, then the problem (D^{μ}_{λ}) has an optimal solution and it holds*

$$val(P^{\mu}_{\lambda}) = val(D^{\mu}_{\lambda}).$$

For $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p_+ \backslash \{0\}$, let $I_\lambda = \{i \in \{1, \ldots, p\}, \ \lambda_i > 0\}$. Now, we derive a necessary and sufficient optimality conditions for the problem (P^{μ}_{λ}) and its dual (D^{μ}_{λ}).

Theorem 3.5. *(1) Let $\overline{x} \in C$ and $\mu = (\mu_1, \ldots, \mu_p) \in \mathbb{R}^p_+$ with $\mu_i = \frac{f_i(\overline{x}, v(\overline{x}))}{g_i(\overline{x}, v(\overline{x}))}$, $i = 1, \ldots p$. Suppose that the constraint qualification (CQ) is satisfied at $\overline{x}$. If $\overline{x}$ is a weakly efficient solution of (P^{μ}), then there exist $\lambda \in \mathbb{R}^p_+ \backslash \{0\}$, $\overline{r} \in \mathbb{R}^q_+$, $\overline{s} \in \mathbb{R}^{n+1}_+$, $\overline{t} \in \mathbb{R}^n$ and $\overline{u}_i, \overline{v}_i \in \mathbb{R}^{n+1}_+$, $i \in I_\lambda$ such that the following optimality conditions hold*

(i) $f_i(h(\overline{x})) + f_i^*(\overline{u}_i) = \overline{u}_i^T h(\overline{x}), \ \forall i \in I_\lambda,$

(ii) $(-\mu_i g_i)(h(\overline{x})) + (-\mu_i g_i)^*(\overline{v}_i) = \overline{v}_i^T h(\overline{x}), \ \forall i \in I_\lambda,$

(iii) $\overline{r}^T G(h(\overline{x})) + (\overline{r}^T G)^*\big(\overline{s} - \sum_{i \in I_\lambda} \lambda_i(\overline{u}_i + \overline{v}_i)\big) = \big(\overline{s} - \sum_{i \in I_\lambda} \lambda_i(\overline{u}_i + \overline{v}_i)\big)^T h(\overline{x}),$

(iv) $\overline{s}^T h(\overline{x}) + (\overline{s}^T h)^*(\overline{t}) = \overline{t}^T \overline{x},$

(v) $\sigma_X(-\overline{t}) = -\overline{t}^T \overline{x},$

(vi) $\overline{r}^T G(h(\overline{x})) = 0.$

(2) Let $\overline{x} \in C$, $\mu = (\mu_1, \ldots, \mu_p) \in \mathbb{R}^p_+$ with $\mu_i = \frac{f_i(\overline{x}, v(\overline{x}))}{g_i(\overline{x}, v(\overline{x}))}$, $i = 1, \ldots p$. If $\overline{x} \in \Omega$ such that for some $\lambda \in \mathbb{R}^p_+ \backslash \{0\}$, $\overline{r} \in \mathbb{R}^q_+$, $\overline{s} \in \mathbb{R}^{n+1}_+$, $\overline{t} \in \mathbb{R}^n$ and $\overline{u}_i, \overline{v}_i \in \mathbb{R}^{n+1}_+$, $i \in I_\lambda$, the conditions $(i) - (vi)$ are satisfied, then $\overline{x}$ is a weakly efficient solution of (P^{μ}).

Proof. (1) Let $\overline{x}$ be a weakly solution of (P^{μ}). According to Lemma 3.3, there exists $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p_+ \backslash \{0\}$ such that $\overline{x}$ is an optimal solution of the scalar problem (P^{μ}_{λ}). Because the constraint qualification (CQ) is fulfilled, by the Theorem 3.4, there exist $\overline{r} \in \mathbb{R}^q_+$, $\overline{s} \in \mathbb{R}^{n+1}_+$, $\overline{t} \in \mathbb{R}^n$ and $\overline{u}_i, \overline{v}_i \in \mathbb{R}^{n+1}_+$, $i \in I_\lambda$, such that

$$\sum_{i \in I_\lambda} \lambda_i F_i(h(\overline{x})) = -\sum_{i \in I_\lambda} \lambda_i [f_i^*(\overline{u}_i) + (-\mu_i g_i)^*(\overline{v}_i)] - (\overline{r}^T G)^* \Big(\overline{s} - \sum_{i \in I_\lambda} \lambda_i(\overline{u}_i + \overline{v}_i)\Big) \\ - (\overline{s}^T h)^*(\overline{t}) - \sigma_X(-\overline{t}). \tag{3.3}$$

The last equality is equivalent to

$$\begin{aligned}0 = &\left\{\sum_{i\in I_\lambda} \lambda_i[f_i(h(\overline{x})) + f_i^*(\overline{u}_i) - \overline{u}_i^T h(\overline{x})]\right\}\\ &+ \left\{\sum_{i\in I_\lambda} \lambda_i[(-\mu_i g_i)(h(\overline{x})) + (-\mu_i g_i)^*(\overline{v}_i) - \overline{v}_i^T h(\overline{x})]\right\}\\ &+ \left\{\overline{r}^T G(h(\overline{x})) + (\overline{r}^T G)^*\Big(\overline{s} - \sum_{i\in I_\lambda} \lambda_i(\overline{u}_i + \overline{v}_i)\Big) - \Big(\overline{s} - \sum_{i\in I_\lambda} \lambda_i(\overline{u}_i + \overline{v}_i)\Big)^T h(\overline{x})\right\}\\ &+ \left\{\overline{s}^T h(\overline{x}) + (\overline{s}^T h)^*(\overline{t}) - \overline{t}^T\overline{x}\right\} + \left\{\sigma_X(-\overline{t}) + \overline{t}^T\overline{x}\right\} - \overline{r}^T G(h(\overline{x})).\end{aligned} \tag{3.4}$$

It follows that from the Young-Fenchel inequality (2.1), the following relations hold:

$$\begin{cases} f_i(h(\overline{x})) + f_i^*(\overline{u}_i) - \overline{u}_i^T h(\overline{x}) \geq 0, \ \forall i \in I_\lambda;\\ (-\mu_i g_i)(h(\overline{x})) + (-\mu_i g_i)^*(\overline{v}_i) - \overline{v}_i^T h(\overline{x}) \geq 0, \ \forall i \in I_\lambda;\\ \overline{r}^T G(h(\overline{x})) + (\overline{r}^T G)^*\Big(\overline{s} - \sum_{i\in I_\lambda} \lambda_i(\overline{u}_i + \overline{v}_i)\Big) - \Big(\overline{s} - \sum_{i\in I_\lambda} \lambda_i(\overline{u}_i + \overline{v}_i)\Big)^T h(\overline{x}) \geq 0;\\ \overline{s}^T h(\overline{x}) + (\overline{s}^T h)^*(\overline{t}) - \overline{t}^T\overline{x} \geq 0;\\ \sigma_X(-\overline{t}) + \overline{t}^T\overline{x} \geq 0. \end{cases} \tag{3.5}$$

Since $\overline{r} \in \mathbb{R}_+^q$ and $\overline{x} \in C$, there is $-\overline{r}^T G(h(\overline{x})) \geq 0$. By the inequalities (3.5), it follows that all the terms of the sum in (3.4) must be equal to zero. Then, the equalities $(i) - (vi)$ hold.

(2) Assume that $\overline{x} \in \Omega$ such that for some $\lambda \in \mathbb{R}_+^p\backslash\{0\}$, $\overline{r} \in \mathbb{R}_+^q$, $\overline{s} \in \mathbb{R}_+^{n+1}$, $\overline{t} \in \mathbb{R}^n$ and $\overline{u}_i, \overline{v}_i \in \mathbb{R}_+^{n+1}$, $i \in I_\lambda$, the conditions $(i) - (vi)$ are satisfied. Then, we have

$$\begin{aligned}-\sum_{i\in I_\lambda} \lambda_i f_i^*(\overline{u}_i) - \sum_{i\in I_\lambda} \lambda_i(-\mu_i g_i)^*(\overline{v}_i) - (\overline{r}^T G)^*\big(\overline{s} - \sum_{i\in I_\lambda} \lambda_i(\overline{u}_i + \overline{v}_i)\big)\\ - (\overline{s}^T h)^*(\overline{t}) - \sigma_X(-\overline{t}) = \sum_{i\in I_\lambda} \lambda_i F_i(h(\overline{x})),\end{aligned}$$

and, since $\mathrm{val}(P_\lambda^\mu) = \inf(P_\lambda^\mu)$ and $\mathrm{val}(D_\lambda^\mu) = \max(D_\lambda^\mu)$, we get

$$\begin{aligned}\mathrm{val}(D_\lambda^\mu) \geq &- \sum_{i\in I_\lambda} \lambda_i f_i^*(\overline{u}_i) - \sum_{i\in I_\lambda} \lambda_i(-\mu_i g_i)^*(\overline{v}_i) - (\overline{r}^T G)^*\Big(\overline{s} - \sum_{i\in I_\lambda} \lambda_i(\overline{u}_i + \overline{v}_i)\Big)\\ &- (\overline{s}^T h)^*(\overline{t}) - \sigma_X(-\overline{t})\\ =& \sum_{i\in I_\lambda} \lambda_i F_i(h(\overline{x})) \geq \mathrm{val}(P_\lambda^\mu).\end{aligned}$$

This proves the equality (3.3) results and shows that $\overline{x}$ is an optimal solution to (P_λ^μ). Apply Lemma 3.3, $\overline{x}$ is a weakly efficient solution of (P^μ). □

4. The multiobjective dual problem

For a given $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{R}^p_+ \backslash \{0\}$ let be $|\lambda| := \sum_{i=1}^p \lambda_i$. In this section, we establish under constraint qualifications the weak and strong duality theorems of the problem (P^μ) and its dual multiobjective optimization problem (D^μ) defined by

$$(D^\mu) \quad \begin{cases} \text{v} - \max H(r,s,u,v,t,\lambda,\alpha), \\ \text{s.t} \quad (r,s,u,v,t,\lambda,\alpha) \in B \end{cases}$$

where

$$H(r,s,u,v,t,\lambda,\alpha) := \begin{pmatrix} H_1(r,s,u,v,t,\lambda,\alpha) \\ . \\ . \\ . \\ H_p(r,s,u,v,t,\lambda,\alpha) \end{pmatrix}$$

with

$$H_i(r,s,u,v,t,\lambda,\alpha) := -\, f_i^*(u_i) - (-\mu_i g_i)^*(v_i) - \frac{1}{|\lambda|}\Big[(r^T G)^*\Big(s - \sum_{i \in I_\lambda} \lambda_i (u_i + v_i)\Big) + (s^T h)^*(t) + \sigma_X(-t)\Big] + \alpha_i,\ \forall i \in \{1, \dots, p\},$$

and the set of constraints

$$B := \Big\{ (r,s,u,v,t,\lambda,\alpha) :\ r \in \mathbb{R}^q_+, s \in \mathbb{R}^{n+1}_+, t \in \mathbb{R}^n, u = (u_1, \dots, u_p), v = (v_1, \dots, v_p), u_i, v_i \in \mathbb{R}^{n+1}_+, i = 1, \dots, p, \lambda \in \mathbb{R}^p_+ \backslash \{0\}, \alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{R}^p, \sum_{i=1}^p \lambda_i \alpha_i = 0 \Big\}.$$

Definition 4.1. An element $(\overline{r}, \overline{s}, \overline{u}, \overline{v}, \overline{t}, \overline{\lambda}, \overline{\alpha}) \in B$ is said to be a weakly efficient solution of the problem (D^μ), if there exists no $(r,s,u,v,t,\lambda,\alpha) \in B$ such that

$$H_i(r,s,u,v,t,\lambda,\alpha) > H_i(\overline{r}, \overline{s}, \overline{u}, \overline{v}, \overline{t}, \overline{\lambda}, \overline{\alpha}) \text{ for all } i = 1, \dots, p.$$

The following theorem states the weak duality assertion between the bilevel multiobjective problem (P^μ) and its dual (D^μ).

Theorem 4.2 (Weak duality). *There is no $x \in \Omega$ and no $(r,s,u,v,t,\lambda,\alpha) \in B$ such that $F_i(h(x)) < H_i(r,s,u,v,t,\lambda,\alpha)$ for all $i = 1, \dots, p$.*

Proof. In order to prove the theorem, assume that there exist $(r,s,u,v,t,\lambda,\alpha) \in B$ and $x \in \Omega$ such that $F_i(h(x)) < H_i(r,s,u,v,t,\lambda,\alpha)$ for all $i = 1, \dots, p$. This implies that

$$\sum_{i=1}^p \lambda_i F_i(h(x)) < \sum_{i=1}^p \lambda_i H_i(r,s,u,v,t,\lambda,\alpha),\ \forall \lambda \in \mathbb{R}^p_+ \backslash \{0\}. \tag{4.1}$$

On the other hand,

$$\begin{aligned}\sum_{i=1}^{p}\lambda_i H_i(r,s,u,v,t,\lambda,\alpha) &= \sum_{i\in I_\lambda}\lambda_i H_i(r,s,u,v,t,\lambda,\alpha)\\ &= \sum_{i\in I_\lambda}\lambda_i\Big[-f_i^*(u_i)-(-\mu_i g_i)^*(v_i)\\ &\quad -\frac{1}{|\lambda|}\Big((r^TG)^*\Big(s-\sum_{i=1}^{p}\lambda_i(u_i+v_i)\Big)\\ &\quad +(s^Th)^*(t)+\sigma_X(-t)\Big)+\alpha_i\Big],\end{aligned}$$

and as $|\lambda|=\sum_{i=1}^{p}\lambda_i=\sum_{i\in I_\lambda}\lambda_i$ and $\sum_{i\in I_\lambda}\lambda_i\alpha_i=\sum_{i=1}^{p}\lambda_i\alpha_i=0$, we obtain

$$\begin{aligned}\sum_{i\in I_\lambda}\lambda_i H_i(r,s,u,v,t,\lambda,\alpha) = &-\sum_{i\in I_\lambda}\lambda_i f_i^*(u_i)-\sum_{i\in I_\lambda}\lambda_i(-\mu_i g_i)^*(v_i)\\ &-(r^TG)^*\Big(s-\sum_{i\in I_\lambda}\lambda_i(u_i+v_i)\Big)\\ &-(s^Th)^*(t)-\sigma_X(-t).\end{aligned}$$

From the Young-Fenchel inequality (2.1), the following relations hold

$$\begin{cases} -\sum_{i\in I_\lambda}\lambda_i f_i^*(u_i)\le \sum_{i\in I_\lambda}\lambda_i f_i(h(x))-\sum_{i\in I_\lambda}\lambda_i u_i^T h(x);\\ -\sum_{i\in I_\lambda}\lambda_i(-\mu_i g_i)^*(v_i)\le \sum_{i\in I_\lambda}\lambda_i(-\mu_i g_i)(h(x))-\sum_{i\in I_\lambda}\lambda_i v_i^T h(x);\\ -(r^TG)^*\Big(s-\sum_{i\in I_\lambda}\lambda_i(u_i+v_i)\Big)\le r^TG(h(x))-\Big(s-\sum_{i\in I_\lambda}\lambda_i(u_i+v_i)\Big)^T h(x);\\ -(s^Th)^*(t)\le s^Th(x)-t^Tx;\\ -\sigma_X(-t)\le t^Tx. \end{cases} \tag{4.2}$$

and adding them up, we have

$$\begin{aligned}\sum_{i\in I_\lambda}\lambda_i H_i(r,s,u,v,t,\lambda,\alpha) = &-\sum_{i\in I_\lambda}\lambda_i f_i^*(u_i)-\sum_{i\in I_\lambda}\lambda_i(-\mu_i g_i)^*(v_i)\\ &-(r^TG)^*\Big(s-\sum_{i\in I_\lambda}\lambda_i(u_i+v_i)\Big)\\ &-(s^Th)^*(t)-\sigma_X(-t).\\ \le &\sum_{i\in I_\lambda}\lambda_i F_i(h(x))+r^TG(h(x)).\end{aligned}$$

Since $r^T G(h(x)) \leq 0$, there follows the inequality

$$\sum_{i=1}^{p} \lambda_i H_i(r, s, u, v, t, \lambda, \alpha) \leq \sum_{i \in I_\lambda} \lambda_i F_i(h(x)) = \sum_{i=1}^{p} \lambda_i F_i(h(x)). \tag{4.3}$$

Then, the inequality (4.3) contradicts the relation (4.1). Thus, the weak duality between (P^μ) and (D^μ) holds. $\square$

The following theorem provides the strong duality between the problem (P^μ) and its dual (D^μ).

Theorem 4.3. *(Strong duality) Let (CQ) be fulfilled and $\overline{x} \in \Omega$ be a weakly efficient solution to (P^μ). Then a weakly efficient solution $(\overline{r}, \overline{s}, \overline{u}, \overline{v}, \overline{t}, \overline{\lambda}, \overline{\alpha}) \in B$ to the dual problem (D^μ) exists and for all $i = 1, \ldots, p$ applies $F_i(h(\overline{x})) = H_i(\overline{r}, \overline{s}, \overline{u}, \overline{v}, \overline{t}, \overline{\lambda}, \overline{\alpha})$.*

Proof. Since $\overline{x} \in \Omega$ is a weakly efficient solution for (P^μ) and the constraint qualification (CQ) is fulfilled at $\overline{x}$, from Theorem 3.5, there exist $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p_+ \backslash \{0\}$, $r \in \mathbb{R}^q_+$, $s \in \mathbb{R}^{n+1}_+$, $t \in \mathbb{R}^n$ and $u_i, v_i \in \mathbb{R}^{n+1}_+$, $i \in I_\lambda$ such that the optimality conditions $(i) - (vi)$ are fulfilled. For $i \notin I_\lambda$. Since f_i, $(-\mu_i g_i)$ are proper convex functions (because f_i, $(-\mu_i g_i)$ are finite convex functions), the functions f_i^*, $(-\mu_i g_i)^*$ are proper and convex (see [17]). According to Lemma 2.1, there exist $\widetilde{u}_i, \widetilde{v}_i \in \mathbb{R}^{n+1}_+$ such that $f_i^*(\widetilde{u}_i) \in \mathbb{R}$ and $(-\mu_i g_i)^*(\widetilde{v}_i) \in \mathbb{R}$. We consider the following notations

$$\overline{\lambda} := \lambda,\ \overline{r} := r,\ \overline{s} := s,\ \overline{t} := t,\ \overline{u}_i := \begin{cases} u_i,\ i \in I_\lambda, \\ \widetilde{u}_i,\ i \notin I_\lambda, \end{cases} \quad \overline{v}_i := \begin{cases} v_i,\ i \in I_\lambda, \\ \widetilde{v}_i,\ i \notin I_\lambda, \end{cases}$$

and

$$\overline{\alpha}_i := \begin{cases} \frac{1}{|\lambda|}\left[(r^T G)^*\left(s - \sum_{i \in I_\lambda} \lambda_i (u_i + v_i)\right) + (s^T h)^*(t) + \sigma_X(-t)\right] \\ \qquad +(u_i + v_i)^T h(\overline{x}),\ i \in I_\lambda, \\ \\ \frac{1}{|\lambda|}\left[(r^T G)^*\left(s - \sum_{i \in I_\lambda} \lambda_i (u_i + v_i)\right) + (s^T h)^*(t) + \sigma_X(-t)\right] + f_i(h(\overline{x})) \\ \qquad +f_i^*(\widetilde{u}_i) + (-\mu_i g_i)(h(\overline{x})) + (-\mu_i g_i)^*(\widetilde{v}_i),\ i \notin I_\lambda. \end{cases}$$

It is clear that $\overline{\alpha}_i \in \mathbb{R}$ for all $i = 1, \ldots, p$. By using the conditions $(i) - (vi)$

established in Theorem 3.5, we obtain

$$
\begin{aligned}
\sum_{i=1}^{p} \overline{\lambda}_i \overline{\alpha}_i =& \sum_{i \in I_{\overline{\lambda}}} \overline{\lambda}_i \overline{\alpha}_i \\
=&(\overline{r}^T G)^* \Big(\overline{s} - \sum_{i \in I_{\overline{\lambda}}} \overline{\lambda}_i (\overline{u}_i + \overline{v}_i) \Big) + (\overline{s}^T h)^* (\overline{t}) + \sigma_X(-\overline{t}) \\
&+ \sum_{i \in I_{\overline{\lambda}}} \overline{\lambda}_i (\overline{u}_i + \overline{v}_i)^T h(\overline{x}) \\
=&(\overline{r}^T G)^* \Big(\overline{s} - \sum_{i \in I_{\overline{\lambda}}} \overline{\lambda}_i (\overline{u}_i + \overline{v}_i) \Big) + \overline{x}^T \overline{t} - \overline{s}^T h(\overline{x}) - \overline{x}^T \overline{t} \\
&+ \sum_{i \in I_{\overline{\lambda}}} \overline{\lambda}_i (\overline{u}_i + \overline{v}_i)^T h(\overline{x}) \\
=&\overline{r}^T G(h(\overline{x})) + (\overline{r}^T G)^* \Big(\overline{s} - \sum_{i \in I_{\overline{\lambda}}} \overline{\lambda}_i (\overline{u}_i + \overline{v}_i) \Big) \\
&- \Big(\overline{s} - \sum_{i \in I_{\overline{\lambda}}} \overline{\lambda}_i (\overline{u}_i + \overline{v}_i) \Big)^T h(\overline{x}) \\
=&0.
\end{aligned}
$$

We proved that $(\overline{r}, \overline{s}, \overline{u}, \overline{v}, \overline{t}, \overline{\lambda}, \overline{\alpha}) \in B$.
Now, we show that $F_i(h(\overline{x})) = H_i(\overline{r}, \overline{s}, \overline{u}, \overline{v}, \overline{t}, \overline{\lambda}, \overline{\alpha})$ for all $i = 1, \ldots, p$. For $i \in I_{\overline{\lambda}}$, from Theorem 3.5, we have,

$$
\begin{aligned}
H_i(\overline{r}, \overline{s}, \overline{u}, \overline{v}, \overline{t}, \overline{\lambda}, \overline{\alpha}) =& - f_i^*(\overline{u}_i) - (-\mu_i g_i)^*(\overline{v}_i) - \frac{1}{|\overline{\lambda}|} \Big[(\overline{r}^T G)^* \Big(\overline{s} - \sum_{i \in I_{\overline{\lambda}}} \overline{\lambda}_i (\overline{u}_i + \overline{v}_i) \Big) \\
&+ (\overline{s}^T h)^*(\overline{t}) + \sigma_X(-\overline{t}) \Big] + \overline{\alpha}_i \\
=& - f_i^*(u_i) - (-\mu_i g_i)^*(v_i) - \frac{1}{|\lambda|} \Big[(r^T G)^* \Big(s - \sum_{i \in I_\lambda} \lambda_i (u_i + v_i) \Big) \\
&+ (s^T h)^*(t) + \sigma_X(-t) \Big] + \frac{1}{|\lambda|} \Big[(r^T G)^* \Big(s - \sum_{i \in I_\lambda} \lambda_i (u_i + v_i) \Big) \\
&+ (s^T h)^*(t) + \sigma_X(-t) \Big] + (u_i + v_i)^T h(\overline{x}) \\
=& - f_i^*(u_i) - (-\mu_i g_i)^*(v_i) + (u_i + v_i)^T h(\overline{x}) \\
=&F_i(h(\overline{x})).
\end{aligned}
$$

For $i \notin I_{\overline{\lambda}}$, by definition of $\overline{\alpha}$, we have

$$\begin{aligned}
H_i(\overline{r},\overline{s},\overline{u},\overline{v},\overline{t},\overline{\lambda},\overline{\alpha}) =& - f_i^*(\overline{u}_i) - (-\mu_i g_i)^*(\overline{v}_i) - \frac{1}{|\overline{\lambda}|}\Big[(\overline{r}^T G)^*\Big(\overline{s} - \sum_{i\in I_{\overline{\lambda}}} \overline{\lambda}_i(\overline{u}_i+\overline{v}_i)\Big) \\
&+ (\overline{s}^T h)^*(\overline{t}) + \sigma_X(-\overline{t})\Big] + \overline{\alpha}_i \\
=& - f_i^*(\widetilde{u}_i) - (-\mu_i g_i)^*(\widetilde{v}_i) - \frac{1}{|\lambda|}\Big[(r^T G)^*\Big(s - \sum_{i\in I_{\lambda}} \lambda_i(u_i+v_i)\Big) \\
&+ (s^T h)^*(t) + \sigma_X(-t)\Big] + f_i^*(\widetilde{u}_i) + f_i(h(\overline{x})) + (-\mu_i g_i)^*(\widetilde{v}_i) \\
&+ (-\mu_i g_i)(h(\overline{x})) + \frac{1}{|\lambda|}\Big[(r^T G)^*\Big(s - \sum_{i\in I_{\lambda}} \lambda_i(u_i+v_i)\Big) \\
&+ (s^T h)^*(t) + \sigma_X(-t)\Big] \\
=& f_i(h(\overline{x})) + (-\mu_i g_i)(h(\overline{x})) = F_i(h(\overline{x})).
\end{aligned}$$

According to Theorem 4.2, it follows that $(\overline{r},\overline{s},\overline{u},\overline{v},\overline{t},\overline{\lambda},\overline{\alpha})$ is a weakly efficient solution of (D^μ) and for all $i = 1,\dots,p$, we have

$$H_i(\overline{r},\overline{s},\overline{u},\overline{v},\overline{t},\overline{\lambda},\overline{\alpha}) = F_i(h(\overline{x})).$$

□

Now, we give example illustrating Theorem 4.3.

5. An example

Consider the following bilevel multiobjective fractional problem

$$(P) \quad \mathrm{v}-\min_{x\in C}\left\{\frac{f_1(x,\upsilon(x))}{g_1(x,\upsilon(x))}, \frac{f_2(x,\upsilon(x))}{g_2(x,\upsilon(x))}\right\}, \text{ with } C := \{x \in X,\ G(x,\upsilon(x)) \le 0\},$$

where $f_i, g_i\colon \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$, $(i = 1,2)$, $G\colon \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ and $f\colon \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ are defined as follows

$$\begin{aligned}
f_1(x,t) &:= 2x + t + 3, & g_1(x,t) &:= -2x - t + 1, \\
f_2(x,t) &:= 2x + t + 4, & g_2(x,t) &:= -2x - t + 2, \\
f(x,y) &:= -x + y^2 - 1, & G(x,t) &:= 2x + t.
\end{aligned}$$

Let $X := \mathbb{R}_+$, $A := [0,1]$. For any $x \in \mathbb{R}$, we have

$$\upsilon(x) = \inf_{y\in A} f(x,y) = -x - 1 \text{ and } h(x) = (x,\upsilon(x)) = (x,-x-1).$$

Hence, the functions $f_i,\ g_i,\ i=1,2$ and G become

$$\begin{aligned} f_1(x,\upsilon(x)) &= x+2, & g_1(x,\upsilon(x)) &= -x+2,\\ f_2(x,\upsilon(x)) &= x+3, & g_2(x,\upsilon(x)) &= -x+3,\\ G(x,\upsilon(x)) &= x-1. \end{aligned}$$

Then, the problem (P) rewrite as

$$(P)\quad \mathrm{v}-\min_{x\in C}\left(\frac{x+2}{-x+2},\frac{x+3}{-x+3}\right),$$

where $C=\{x\in X,\ x-1\le 0\}=[0,1]$.

The corresponding parametric problem (P^μ) will be

$$(P^\mu)\quad \mathrm{v}-\min_{x\in C} F(h(x)) = \left((1+\mu_1)x+2-2\mu_1,(1+\mu_2)x+3-3\mu_2\right),$$

where $\mu=(\mu_1,\mu_2)\in\mathbb{R}^2_+$.
Clearly, f is convex, $f_i,-g_i,\ \ i=1,2$ and G are convex functions and $\mathbb{R}^2_+$-increasing. Moreover, one can see that $f_i(x,\upsilon(x))\ge 0$ and $g_i(x,\upsilon(x))>0$ for all $x\in C$, $i=1,2$.

To formulate the dual problem (D^μ), we need to determine the conjugate functions as presented below

$$f_1^*(x^*,t^*)=\begin{cases}-3, & x^*=2,\ t^*=1,\\ +\infty, & \text{otherwise},\end{cases}$$

$$f_2^*(x^*,t^*)=\begin{cases}-4, & x^*=2,\ t^*=1,\\ +\infty, & \text{otherwise},\end{cases}$$

$$(-\mu_1 g_1)^*(x^*,t^*)=\begin{cases}\mu_1, & x^*=2\mu_1,\ t^*=\mu_1,\\ +\infty, & \text{otherwise},\end{cases}$$

$$(-\mu_2 g_2)^*(x^*,t^*)=\begin{cases}2\mu_2, & x^*=2\mu_2,\ t^*=\mu_2,\\ +\infty, & \text{otherwise},\end{cases}$$

$$(rG)^*(x^*,t^*)=\begin{cases}0, & x^*=2r,\ t^*=r,\\ +\infty, & \text{otherwise},\end{cases}$$

$$(s^T h)^*(x^*)=\begin{cases}s_2, & x^*=s_1-s_2,\\ +\infty, & \text{otherwise},\end{cases}$$

$$\sigma_X(x^*)=\begin{cases}0, & x^*\le 0,\\ +\infty, & \text{otherwise}.\end{cases}$$

Consequently, the dual problem (D^μ) takes on the following form

$$(D^\mu)\quad \mathrm{v}-\max_{(r,s,u,v,t,\lambda,\alpha)\in B} H(r,s,u,v,t,\lambda,\alpha)=\begin{pmatrix}H_1(r,s,u,v,t,\lambda,\alpha)\\ H_2(r,s,u,v,t,\lambda,\alpha)\end{pmatrix},$$

where

$$H_i(r,s,u,v,t,\lambda,\alpha) = -\,f_i^*(u_i) - (-\mu_i g_i)^*(v_i) - \frac{1}{|\lambda|}\Big[(rG)^*\Big(s - \sum_{i=1}^{2}\lambda_i(u_i+v_i)\Big) + (s^T h)^*(t) + \sigma_X(-t)\Big] + \alpha_i,\ (i=1,2).$$

Thus the two objective functions of the dual problem are greater than $-\infty$ if and only if $u_1=(2,1),\ u_2=(2,1),\ v_1=(2\mu_1,\mu_1),\ v_2=(2\mu_2,\mu_2),\ s-\sum_{i=1}^{2}\lambda_i(u_i+v_i)=(2r,r),\ t=s_1-s_2\geq 0$. Then, the dual problem (D^μ) is written as follows

$$(D)\quad \text{v}-\max_{(r,s,u,v,t,\lambda,\alpha)\in B}\begin{pmatrix} H_1(r,s,u,v,t,\lambda,\alpha)\\ H_2(r,s,u,v,t,\lambda,\alpha)\end{pmatrix},$$

where

$$H_1(r,s,u,v,t,\lambda,\alpha) = 3-\mu_1-\frac{s_2}{\lambda_1+\lambda_2}+\alpha_1,$$
$$H_2(r,s,u,v,t,\lambda,\alpha) = 4-2\mu_2-\frac{s_2}{\lambda_1+\lambda_2}+\alpha_2,$$

and

$$B := \Big\{(r,s,u,v,t,\lambda,\alpha):\ r\geq 0,\ s=(s_1,s_2)\in\mathbb{R}^2_+,\ t=s_1-s_2\geq 0,\ u=(u_1,u_2), v=(v_1,v_2), u_1=(2,1),\ u_2=(2,1),\ v_1=(2\mu_1,\mu_1),\ v_2=(2\mu_2,\mu_2), s-\sum_{i=1}^{2}\lambda_i(u_i+v_i)=(2r,r), \alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2, \lambda=(\lambda_1,\lambda_2)\in\mathbb{R}^2_+\backslash\{0\},\ \sum_{i=1}^{2}\lambda_i\alpha_i=0\Big\}.$$

It is easy to check that the feasible point $\overline{x}=0$ is a weakly efficient solution to the problem (P^μ) and that (CQ) is fulfilled (since $G(h(\overline{x}))=G(0,-1)=-1\leq 0$). By the Theorem 4.3, there exists an efficient solution $(\overline{r},\overline{u},\overline{v},\overline{s},\overline{t},\overline{\lambda},\overline{\alpha})\in B$ to the dual problem (D^μ) such that

$$F_i(h(\overline{x})) = H_i(\overline{r},\overline{s},\overline{u},\overline{v},\overline{t},\overline{\lambda},\overline{\alpha}),\ \forall i\in\{1,2\},$$

where

$$\mu=(\mu_1,\mu_2)=\left(\frac{f_1(\overline{x},v(\overline{x}))}{g_1(\overline{x},v(\overline{x}))},\frac{f_2(\overline{x},v(\overline{x}))}{g_2(\overline{x},v(\overline{x}))}\right)=(1,1).$$

An efficient solution for (D^μ) can be found through simple calculations as follows $\overline{r}=0,\ \overline{s}=(\overline{s}_1,\overline{s}_2)=(4,2),\ \overline{t}=\overline{s}_1-\overline{s}_2=2,\ \overline{u}=(\overline{u}_1,\overline{u}_2)=((2,1),(2,1)),\ \overline{v}=(\overline{v}_1,\overline{v}_2)=((2,1),(2,1)),\ \overline{\lambda}=(\overline{\lambda}_1,\overline{\lambda}_2)=(1,1)$ and $\overline{\alpha}=(0,0)$.

Acknowledgements. The author would like to express their sincere thanks to an anonymous reviewer for his valuable comments and suggestions.

References

[1] Aboussoror, A. and Adly, S.: A Fenchel-Lagrange Duality Approach for a Bilevel Programming problem with Extremal-Value Function, J. Optim Theory Appl., (2011), **149**, 254–268; doi:10.1007/s10957-011-9831-5

[2] Ahmad, I., Zhang, F., Liu, J., Anjum, M.N., Zaman, M., Tayyab, M., Waseem, M., Farid, H.U.: A linear bi-level multi-objective program for optimal allocation of water resources, Plos one, (2018), **13**, 1–25; doi:10.1371/journal.pone.0192294

[3] Bank, B., Guddat, J., Klatte, D., Kummer, B., Tammer, K.: Non-Linear Parametric Optimization, Akademie-Verlag Berlin, (1982).

[4] Bector, C.R., Chandra, S. and Singh, C.: Duality on multiobjective fractional programming. In Lecture Notes in Economics and Mathematical Systems, Springer Verlag, Berlin, (1980), **345**, 232–241.

[5] Bhatnagar, P.: Optimality and Duality for an Efficient Solution of Multiobjective Nonlinear Fractional Programming Problem Involving Semilocally Convex Functions, Int. J. Appl. Oper. Res., (2012), **2**, 21–29.

[6] Bot, R. I., Chares, R. and Wanka G.: Duality for multiobjective fractional programming problems, Nonlinear Anal. Forum., (2006), **11**, 185–201.

[7] Bot, R. I., Hodrea, I. B. and Wanka, G.: Optimality conditions for weak efficiency to vector optimization problems with composed convex functions, Cent. Eur. J. Math., (2008), **6**, 453–468; doi:10.2478/s11533-008-0036-6

[8] Dinklebach, W.: On nonlinear fractional programming, Manage. Sci., (1967), **137**, 492–498.

[9] Dempe, S.: A necessary and sufficient optimality condition for bilevel programming problem, Optimization, (1992), **25**, 341–354; doi:10.1080/02331939208843831

[10] Eichfelder, G.: Multiobjective bilevel optimization, Math. Program., (2010), **123**, 419–449; doi:10.1007/s10107-008-0259-0

[11] Ibaraki, T.: Parametric approaches to fractional programs, Mathematical Programming, (1983), **26**, 345–362.

[12] Kim, M. H., Kim, G. S. and Lee, G. M.: On ϵ-optimality conditions for multiobjective fractional optimization problems, Fixed Point Theory Appl.**2011**, (2011), **6**, 1–13; doi:10.1186/1687-1812-2011-6

[13] Kutateladze, S.S.: Changes of variables in the Young transformation, Soviet Math. Dokl., (1977), **18**, 545–548.

[14] Kohli, B.: Optimality Conditions Using Convexifactors for a Multiobjective Fractional Bilevel Programming Problem. Tamkang J. Math. 2023, **54**, 21–41; doi:10.5556/j.tkjm.54.2023.3830

[15] Liu, J.C. and Yokoyama, K.: ϵ-Optimality and duality for multiobjective fractional programming, Comput. Math. Appl., (1999), **12**, 119–128.

[16] Moustaid, M.B., Rikouane, A., Laghdir, M. and Dali, I.: Sequential approximate weak optimality conditions for multiobjective fractional programming problems via sequential calculus rules for the Brøndsted-Rockafellar approximate subdifferential, Rend. Circ. Mat. Palermo (2), (2022), **71**, 737–754.

[17] Rockaffellar, R. T.: Convex Analysis, Princeton University Press, Princeton, (1970).

[18] Singh, C. and Hanson, M.A.: Multiobjective Fractional Programming Duality Theory, Nav. Res. Logist., (1991), **38**, 925–933; doi:10.1002/nav.3800380610

[19] Stancu-Minasian, I.M.: Fractional programming: Theory, Methods and Applications, **409**, Springer Science and Business Media, (2012).

[20] Stancu-Minasian, I. M.: A ninth bibliography of fractional programming, Optimization, (2019), **68**, 2125–2169; doi:10.1080/02331934.2019.1632250

[21] Verma, R. U.: Weak ϵ-efficiency conditions for multiobjective fractional programming, Appl. Math. Comput., (2013), **219**, 6819–6827; doi:10.1016/j.amc.2012.12.087

[22] Wanka, G. and Bot, R. I.: A new duality aproach for multiobjective convex optimization problems, J. Nonlinear Convex Anal., (2002), **3**, 41–57.

[23] Wanka, G., Bot, R. I. and Vargyas, E.: Conjugate duality for multiobjective composed optimization problems, Acta Math. Hungar., (2007), **116**, 177–196; doi:10.1007/s10474-007-4273-0

[24] Wang, H., Zhang, R. and Vargyas, E.: Duality for Multiobjective Bilevel Programming Problems with Extremal-Value Function, J. Math. Res. Appl., (2015), **35**, 311–320.
[25] Zhang, G., Zhang, G., Gao, Y., Lu, J.: Competitive strategic bidding optimization in electricity markets using bilevel programming and swarm technique. IEEE Transactions on Industrial Electronics, (2011), **58**, 2138–2146.
[26] Yin, Y.: Multiobjective bilevel optimization for transportation planning and management problems, J. Adv. Transp., (2002), **36**, 93–105; doi:10.1002/atr.5670360106

Received: 29 October 2022/Accepted: 11 October 2023/Published online: 3 November 2023

Department of Mathematics, IMI Laboratory, Faculty of Sciences, Ibnou Zohr University, BP 8106 - City Dakhla, Agadir, Morocco.

rikouaneah@gmail.com