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Approximate solutions to hyperbolic partial differential equation with fractional differential and fractional integral forcing functions

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Abstract. *This manuscript deals with a hyperbolic partial differential equation with fractional differential and fractional integral forcing functions. Semidiscretization method is used to establish a unique strong solution and also approximate solutions. Error estimates and continuous dependence of the strong solution on the given conditions have also been discussed. At the end, we illustrated the results with an example.*

1. Introduction

The fractional calculus is a versatile method to demonstrate different types of physical developments in science and engineering. Many real-world problems such as earthquake design, allometry in biology, control processing, diffusion processes, etc., can be modelled as fractional differential equations.

In this manuscript, we study the existence of a unique strong solution and also approximate solutions to the following hyperbolic partial differential equation with fractional differential and fractional integral forcing functions

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f\Big(t, u, \frac{\partial u}{\partial t}, D_t^{\alpha} u, I_t^{\beta} u\Big), \quad t \in (0, T],\ x \in (0, 1), \tag{1.1}$$

for a given initial and integral conditions as

$$u(0, x) = U_0(x) \ \text{ and } \ \frac{\partial u}{\partial t}(0, x) = U_1(x) \quad \text{on} \quad (0, 1), \tag{1.2}$$

$$\int_0^1 u(t, x)dx = \int_0^1 xu(t, x)dx = 0, \quad t \in [0, T], \tag{1.3}$$

by usings Rothe's method, where $T > 0$, $0 < \alpha < \frac{1}{4}$, $\beta > 1$, and $D_t^{\alpha} u$ is the Caputo fractional α^{th} order derivative of u with respect to t, $I_t^{\beta} u$ is the Riemann–Liouville fractional integral of u with respect to t of order β, and f, U_0 and U_1 are some appropriate given functions.

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Equations of the kind (1.1) have been used in various physical situation, such as, in the theory of transmission lines and wave-guides, fractional-order hyperbolic telegraph equations, and time-fractional diffusion-wave equations, for instance, we refer [15, 17, 24, 35, 40, 41, 45] and references listed therein.

Khan *et al.* [24] established analytical solutions to fractional-order hyperbolic telegraph equations by using natural transformation based decomposition technique. Mollahasani *et al.* [35] provided an approach based on hybrid functions of Legendre polynomials and Block-Pulse-Functions to solve fractional order telegraph equation. Shah *et al.* [40] established analytical solutions of diffusion equations of fractional order using natural transform technique.

Many authors opted several numerical methodology for solving fractional order differential equations. Zhou and Xu [45] studied numerical solutions of diffusion-wave equations of fractional order using Chebyshev wavelets collocation method. Zhao *et al.* [43] considered integro-differential equations of fractional order with weakly singular kernels and established numerical results by using piecewise polynomial collocation methods. Hu *et al.* [22] constructed an algorithm for solving the parabolic equation of Caputo-type with fractional Laplacian in one dimensional space by applying finite difference method. Mockary *et al.* [34] solved fractional partial differential equations numerically by using operational matrices of Chebyshev polynomials. Li and Chen [27] discussed many types of numerical methods like finite difference methods, Galerkin finite element methods and spectral methods for solving partial differential equations of fractional order.

The Rothe's semidiscretization technique is a very powerful tool which allows to establish the existence of a unique solution and also helps in constructing algorithms for approximate solutions. Rothe's method has always been an interesting topic among researchers to solve functional and fractional differential equations, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16, 17, 18, 19, 20, 21, 23, 25, 26, 28, 29, 30, 31, 32, 33, 36, 37, 38, 42]. Bahuguna and Jaiswal [1] considered an integro-differential equation of fractional order in a Banach space and established the existence and uniqueness of a strong solution by applying Rothe's time discretization method. Chaoui and Hallaci [13] considered a diffusion equation of fractional order with a second-order differential Volterra operator and fractional integral condition, and obtained existence of a unique weak solution and some regularity results by using Rothe's method. Bahuguna and Jaiswal [2] considered a fractional differential equation of Riemann–Liouville type in a Banach space whose dual space is uniformly convex, and proved existence of a unique strong solution by considering the method of semi-discretization in time.

In this manuscript, we use Rothe's time-discretization technique to study approximate solutions and the existence of a unique strong solution of (1.1)–(1.3). We shall also discuss some error estimates and continuous dependence of the strong solution on the given conditions. This article has been organized in five sections (2–6). In Section 2, we provide few notations and presumptions. Section 3 presents discretization technique and some estimates. Section 4 presents convergent results and error estimates. Section 5 has been given full consideration in establishing the main results. An example to support the results is discussed in Section 6.

2. Preliminaries and assumptions

Let $T > 0$, and let I_T be the interval $[0, T]$. Let $L^2(0,1)$ be the real Hilbert of all real square integrable functions defined on $(0,1)$ and let $C_0(0,1)$ be the space all continuous functions defined on $(0,1)$ with compact support in $(0,1)$. Let S_B be the completion of $C_0(0,1)$ with the inner product defined by

$$\langle u_1, u_2 \rangle_B = \int_0^1 \xi_x u_1(x) \xi_x u_2(x) dx, \tag{2.1}$$

where $\xi_x \upsilon(x) = \int_0^x \upsilon(z)dz$ for $x \in (0,1)$. If $w \in L^2(0,1)$, then

$$\|w\|_B^2 \leq \frac{1}{2}\|w\|^2. \tag{2.2}$$

Also, consider the Hilbert space W introduced in [32] and defined by

$$W = \left\{ \vartheta \in L^2(0,1) : \int_0^1 \vartheta(\omega) d\omega = \int_0^1 \omega \vartheta(\omega) d\omega = 0 \right\}.$$

Define $u \colon I_T \to S_B$ and $f \colon I_T \times S_B \times S_B \times S_B \times S_B \to S_B$ by

$$u(t)(x) = u(t, x),$$
$$f(t, \phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t))(x) = f(t, \phi_1(t,x), \phi_2(t,x), \phi_3(t,x), \phi_4(t,x)),$$

respectively. Consider the presumptions given below:

(H1) $f \colon I_T \times S_B \times S_B \times S_B \times S_B \to S_B$ holds the condition that $\exists\, l_f > 0$ such that

$$\|f(t_1, \upsilon_1, \nu_1, \phi_1, \psi_1) - f(t_2, \upsilon_2, \nu_2, \phi_2, \psi_2)\|_B$$
$$\leq l_f(|t_1 - t_2| + \|\upsilon_1 - \upsilon_2\|_B + \|\nu_1 - \nu_2\|_B + \|\phi_1 - \phi_2\|_B + \|\psi_1 - \psi_2\|_B)$$

for all $t_1, t_2 \in I_T$, for all $\upsilon_1, \nu_1, \phi_1, \psi_1, \upsilon_2, \nu_2, \phi_2, \psi_2 \in S_B$.

(H2) U_0, $U_1 \in W$, and U_0 is a twice differentiable function.

(H3) $0 < \alpha < \frac{1}{4}$ and $1 < \beta$.

Definition 2.1 ([44]). Let $\alpha > 0$. Then, the Caputo fractional derivative of function $g(t)$ of order α is defined by

$$(D^\alpha g)(t) = \begin{cases} \dfrac{1}{\Gamma(\vartheta - \alpha)} \displaystyle\int_0^t \frac{g^{(\vartheta)}(\omega)}{(t-\omega)^{\alpha+1-\vartheta}} d\omega, & \vartheta - 1 < \alpha < \vartheta \in \mathbb{N}, \\ \dfrac{d^\vartheta g}{dt^\vartheta}(t), & \alpha = \vartheta \in \mathbb{N}. \end{cases} \tag{2.3}$$

Definition 2.2 ([44]). Let $\beta > 0$. Then, the Riemann–Liouville fractional integral of function $h(t)$ of order β is defined by

$$(I^{\beta}h)(t) = \frac{1}{\Gamma(\beta)} \int_0^t h(\omega)(t-\omega)^{\beta-1} d\omega. \tag{2.4}$$

Definition 2.3. A function $u\colon I_T \to S_B$ defined by $u(t)(x) = u(t,x)$ is called a strong solution of (1.1)–(1.3) on I_T if u is continuous on I_T, u and u' are differentiable a.e. on $(0,T]$, $u(0) = U_0$, $\dfrac{du}{dt} = U_1$ at $t = 0$, $u(t) \in W$ for all $t \in I_T$, and

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f\left(t, u(t), \frac{du}{dt}, D_t^{\alpha}u, I_t^{\beta}u\right) \quad \text{a.e. } t \in I_T. \tag{2.5}$$

3. Discretization method and priori estimates

Assume that (H1)–(H3) hold in the rest of this manuscript. Divide I_T into n subintervals $[t_{j-1}^n, t_j^n]$ with $h_n = \frac{T}{n}$ for every $n \in \mathbb{N}$, where $t_0^n = 0$, $t_j^n = jh_n$, for $j = 1, 2, \ldots, n$. Let $u_0^n = U_0(x)$ and $u_{-1}^n = U_0(x) - h_n U_1(x)$ for all $n \in \mathbb{N}$. Defining $\{u_j^n\}$ successively as the unique solution of the equation

$$\delta^2 u_j^n - \frac{\partial^2 u_j^n}{\partial x^2} = f\left(t_j^n, u_{j-1}^n, \delta u_{j-1}^n, \delta^{\alpha} u_j^n, \zeta^{\beta} u_j^n\right), \tag{3.1}$$

$$\int_0^1 u_j^n dx = \int_0^1 x u_j^n dx = 0, \tag{3.2}$$

where

$$u_j^n = u(t_j^n, x), \quad \delta u_j^n = \frac{u_j^n - u_{j-1}^n}{h_n}, \quad \delta^2 u_j^n = \frac{u_j^n - 2u_{j-1}^n + u_{j-2}^n}{h_n^2},$$

$$\delta^{\alpha} u_j^n = \frac{(t_j^n)^{1-\alpha}}{\Gamma(2-\alpha)} \frac{u_{j-1}^n - u_{j-2}^n}{h_n} = \frac{(t_j^n)^{1-\alpha}}{\Gamma(2-\alpha)} \delta u_{j-1}^n, \quad \zeta^{\beta} u_j^n = \frac{(t_j^n)^{\beta}}{\Gamma(1+\beta)} u_{j-1}^n.$$

The existence of a unique solution u_j^n of (3.1)–(3.2) is a consequence of [7, Lemma 3.1].

Lemma 3.1. *For $n \in \mathbb{N}$, $n \geq 2$, $j = 2, \ldots, n$:*

(i) $|(t_j^n)^{1-\alpha} - t^{1-\alpha}| \leq (1-\alpha) h_n^{1-\alpha}$ *for all* $t \in [t_{j-1}^n, t_j^n]$,

(ii) $|(t_j^n)^{\beta} - t^{\beta}| \leq \beta h_n T^{\beta-1}$ *for all* $t \in [t_{j-1}^n, t_j^n]$,

(iii) $\|\delta^{\alpha} u_j^n - \delta^{\alpha} u_{j-1}^n\|_B \leq \frac{1}{\Gamma(2-\alpha)} \Big(T^{1-\alpha} h_n \|\delta^2 u_{j-1}^n\|_B + (1-\alpha) h_n^{1-\alpha} \|\delta u_{j-1}^n\|_B \Big)$,

(iv) $\|\zeta^{\beta} u_j^n - \zeta^{\beta} u_{j-1}^n\|_B \leq \frac{1}{\Gamma(1+\beta)} \Big(T^{\beta} h_n \|\delta u_{j-1}^n\|_B + \beta h_n T^{\beta-1} \|u_{j-1}^n\|_B \Big)$.

Moreover, if $n \in \mathbb{N}$, *then* $|(t_1^n)^{1-\alpha} - t^{1-\alpha}| \leq h_n^{1-\alpha}$ *and* $|(t_1^n)^{\beta} - t^{\beta}| \leq \beta h_n T^{\beta-1}$ *for all* $t \in [t_0^n, t_1^n]$.

Proof. By the mean value theorem, there exists $c_j^n \in (t, t_j^n)$ such that

$$(t_j^n)^{1-\alpha} - t^{1-\alpha} = (1-\alpha)(t_j^n - t)(c_j^n)^{-\alpha}.$$

Therefore,

$$\begin{aligned} |(t_j^n)^{1-\alpha} - t^{1-\alpha}| &\leq (1-\alpha)h_n(c_j^n)^{-\alpha} \\ &\leq (1-\alpha)h_n(t_{j-1}^n)^{-\alpha} \\ &= (1-\alpha)h_n(j-1)^{-\alpha}h_n^{-\alpha} \\ &\leq (1-\alpha)h_n^{1-\alpha}. \end{aligned}$$

Similarly, for $t \in [t_{j-1}^n, t_j^n]$, we have

$$|(t_j^n)^{\beta} - t^{\beta}| \leq \beta h_n (t_j^n)^{\beta-1} \leq \beta h_n T^{\beta-1}.$$

Consider,

$$\begin{aligned} &\|\delta^{\alpha} u_j^n - \delta^{\alpha} u_{j-1}^n\|_B \\ &\quad = \frac{1}{\Gamma(2-\alpha)} \|(t_j^n)^{1-\alpha} \delta u_{j-1}^n - (t_{j-1}^n)^{1-\alpha} \delta u_{j-2}^n\|_B \\ &\quad \leq \frac{1}{\Gamma(2-\alpha)} \Big((t_{j-1}^n)^{1-\alpha} \|\delta u_{j-1}^n - \delta u_{j-2}^n\|_B \\ &\qquad\qquad + |(t_j^n)^{1-\alpha} - (t_{j-1}^n)^{1-\alpha}| \|\delta u_{j-1}^n\|_B \Big) \\ &\quad \leq \frac{1}{\Gamma(2-\alpha)} \Big(T^{1-\alpha} h_n \|\delta^2 u_{j-1}^n\|_B + (1-\alpha) h_n^{1-\alpha} \|\delta u_{j-1}^n\|_B \Big). \end{aligned}$$

Now, consider

$$\begin{aligned} &\|\zeta^{\beta} u_j^n - \zeta^{\beta} u_{j-1}^n\|_B \\ &= \frac{1}{\Gamma(1+\beta)} \|(t_j^n)^{\beta} u_{j-1}^n - (t_{j-1}^n)^{\beta} u_{j-2}^n\|_B \\ &\leq \frac{1}{\Gamma(1+\beta)} \Big((t_{j-1}^n)^{\beta} \|u_{j-1}^n - u_{j-2}^n\|_B + |(t_j^n)^{\beta} - (t_{j-1}^n)^{\beta}| \|u_{j-1}^n\|_B \Big) \\ &\leq \frac{1}{\Gamma(1+\beta)} \Big(T^{\beta} h_n \|\delta u_{j-1}^n\|_B + \beta h_n T^{\beta-1} \|u_{j-1}^n\|_B \Big). \end{aligned}$$

If $n \in \mathbb{N}$ and $t \in [t_0^n, t_1^n]$, then

$$|(t_1^n)^{1-\alpha} - t^{1-\alpha}| = (t_1^n)^{1-\alpha} - t^{1-\alpha} \leq (t_1^n)^{1-\alpha} = h_n^{1-\alpha}$$

and by the mean value theorem, there exists $c_1^n \in (t, t_1^n)$ such that

$$|(t_1^n)^{\beta} - t^{\beta}| = (t_1^n - t)\beta(c_1^n)^{\beta-1} \leq \beta h_n T^{\beta-1}.$$

□

Lemma 3.2. *Let* $n, m \in \mathbb{N}$, *let* $1 \le j \le n$ *and* $1 \le l \le m$.
If $[t_{j-1}^n, t_j^n] \cap [t_{l-1}^m, t_l^m] \neq \emptyset$, *then*

(i) $|(t_j^n)^{1-\alpha} - (t_l^m)^{1-\alpha}| \le h_n^{1-\alpha} + h_m^{1-\alpha}$,

(ii) $|(t_j^n)^{\beta} - (t_l^m)^{\beta}| \le \beta(h_n + h_m)T^{\beta-1}$.

Proof. Let $t \in [t_{j-1}^n, t_j^n] \cap [t_{l-1}^m, t_l^m]$. Suppose $j \ge 2$ and $l \ge 2$, then, by Lemma 3.1,

$$\begin{aligned} |(t_j^n)^{1-\alpha} - (t_l^m)^{1-\alpha}| &\le |(t_j^n)^{1-\alpha} - t^{1-\alpha}| + |t^{1-\alpha} - (t_l^m)^{1-\alpha}| \\ &\le (1-\alpha)(h_n^{1-\alpha} + h_m^{1-\alpha}) \\ &\le h_n^{1-\alpha} + h_m^{1-\alpha} \end{aligned} \tag{3.3}$$

and

$$|(t_j^n)^{\beta} - (t_l^m)^{\beta}| \le |(t_j^n)^{\beta} - t^{\beta}| + |t^{\beta} - (t_l^m)^{\beta}| \le \beta(h_n + h_m)T^{\beta-1}. \tag{3.4}$$

If $j = 1$ and $l \ge 2$, then, by Lemma 3.1,

$$\begin{aligned} |(t_j^n)^{1-\alpha} - (t_l^m)^{1-\alpha}| &= |(t_1^n)^{1-\alpha} - (t_l^m)^{1-\alpha}| \\ &\le |(t_1^n)^{1-\alpha} - t^{1-\alpha}| + |t^{1-\alpha} - (t_l^m)^{1-\alpha}| \\ &\le h_n^{1-\alpha} + (1-\alpha)h_m^{1-\alpha} \\ &\le h_n^{1-\alpha} + h_m^{1-\alpha} \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} |(t_j^n)^{\beta} - (t_l^m)^{\beta}| &= |(t_1^n)^{\beta} - (t_l^m)^{\beta}| \\ &\le |(t_1^n)^{\beta} - t^{\beta}| + |t^{\beta} - (t_l^m)^{\beta}| \\ &\le \beta(h_n + h_m)T^{\beta-1}. \end{aligned} \tag{3.6}$$

Similarly, if $j \ge 2$ and $l = 1$, then, by Lemma 3.1,

$$|(t_j^n)^{1-\alpha} - (t_l^m)^{1-\alpha}| = |(t_j^n)^{1-\alpha} - (t_1^m)^{1-\alpha}| \le h_n^{1-\alpha} + h_m^{1-\alpha} \tag{3.7}$$

and

$$|(t_j^n)^{\beta} - (t_l^m)^{\beta}| = |(t_j^n)^{\beta} - (t_1^m)^{\beta}| \le \beta(h_n + h_m)T^{\beta-1}. \tag{3.8}$$

If $j = 1$ and $l = 1$, then,

$$|(t_j^n)^{1-\alpha} - (t_l^m)^{1-\alpha}| = |(t_1^n)^{1-\alpha} - (t_1^m)^{1-\alpha}| \le (t_1^n)^{1-\alpha} + (t_1^m)^{1-\alpha} = h_n^{1-\alpha} + h_m^{1-\alpha} \tag{3.9}$$

and

$$\begin{aligned} |(t_j^n)^{\beta} - (t_l^m)^{\beta}| &= |(t_1^n)^{\beta} - (t_1^m)^{\beta}| \\ &\le |(t_1^n)^{\beta} - t^{\beta}| + |t^{\beta} - (t_1^m)^{\beta}| \\ &\le \beta(h_n + h_m)T^{\beta-1}. \end{aligned} \tag{3.10}$$

Therefore, by (3.3), (3.5), (3.7) and (3.9),

$$|(t_j^n)^{1-\alpha} - (t_l^m)^{1-\alpha}| \le h_n^{1-\alpha} + h_m^{1-\alpha}$$

and by (3.4), (3.6), (3.8) and (3.10),

$$|(t_j^n)^{\beta} - (t_l^m)^{\beta}| \le \beta(h_n + h_m)T^{\beta-1}$$

for all $j = 1, 2, \dots, n$ and $l = 1, 2, \dots, m$. □

Lemma 3.3. *For $n \in \mathbb{N}$, $j = 1, 2, \dots, n$,*

$$\|\delta u_j^n\|_B + \|u_j^n\| \le C, \tag{3.11}$$

where C is a positive constant, independent of both n and j.

Proof. Notice that $\|u_0^n\| = \|U_0\|$ and $\|\delta u_0^n\| = \|U_1\|$ for all $n \in \mathbb{N}$. If $v \in W$, then

$$\left\langle \frac{\partial^2 u_j^n}{\partial x^2}, v \right\rangle_B = -\int_0^1 u_j^n v dx = -\langle u_j^n, v\rangle, \tag{3.12}$$

therefore, by (3.1),

$$\langle \delta^2 u_j^n, v\rangle_B + \langle u_j^n, v\rangle = \left\langle f\left(t_j^n, u_{j-1}^n, \delta u_{j-1}^n, \delta^\alpha u_j^n, \zeta^\beta u_j^n\right), v\right\rangle_B. \tag{3.13}$$

Put $v = \delta u_j^n$ in (3.13), then

$$\langle \delta^2 u_j^n, \delta u_j^n\rangle_B + \langle u_j^n, \delta u_j^n\rangle = \left\langle f\left(t_j^n, u_{j-1}^n, \delta u_{j-1}^n, \delta^\alpha u_j^n, \zeta^\beta u_j^n\right), \delta u_j^n\right\rangle_B.$$

Therefore,

$$\langle \delta u_j^n - \delta u_{j-1}^n, \delta u_j^n\rangle_B + \langle u_j^n, u_j^n - u_{j-1}^n\rangle = h_n\left\langle f\left(t_j^n, u_{j-1}^n, \delta u_{j-1}^n, \delta^\alpha u_j^n, \zeta^\beta u_j^n\right), \delta u_j^n\right\rangle_B.$$

Since

$$2\langle \varpi_1, \varpi_1 - \varpi_2\rangle = \|\varpi_1\|^2 - \|\varpi_2\|^2 + \|\varpi_1 - \varpi_2\|^2, \tag{3.14}$$

therefore,

$$\begin{aligned} &\|\delta u_j^n\|_B^2 + \|u_j^n\|^2 \\ &\le \|\delta u_{j-1}^n\|_B^2 + \|u_{j-1}^n\|^2 + 2h_n\left\|f\left(t_j^n, u_{j-1}^n, \delta u_{j-1}^n, \delta^\alpha u_j^n, \zeta^\beta u_j^n\right)\right\|_B \|\delta u_j^n\|_B. \end{aligned}$$

Thus,

$$\begin{aligned} &\|\delta u_j^n\|_B^2 + \|u_j^n\|^2 \\ &\le 2h_n \sum_{i=1}^{j} \left(\left\|f\left(t_i^n, u_{i-1}^n, \delta u_{i-1}^n, \delta^\alpha u_i^n, \zeta^\beta u_i^n\right)\right\|_B \|\delta u_i^n\|_B\right) + \|U_0\|^2 + \|U_1\|_B^2 \\ &\le h_n \sum_{i=1}^{j} \left\|f\left(t_i^n, u_{i-1}^n, \delta u_{i-1}^n, \delta^\alpha u_i^n, \zeta^\beta u_i^n\right)\right\|_B^2 + h_n \sum_{i=1}^{j} \|\delta u_i^n\|_B^2 + \|U_0\|^2 + \|U_1\|_B^2. \end{aligned} \tag{3.15}$$

Since

$$
\begin{aligned}
&\left\|f\left(t_j^n, u_{j-1}^n, \delta u_{j-1}^n, \delta^\alpha u_j^n, \zeta^\beta u_j^n\right)\right\|_B^2 \\
&\leq 2\left\|f\left(t_j^n, u_{j-1}^n, \delta u_{j-1}^n, \delta^\alpha u_j^n, \zeta^\beta u_j^n\right) - f(0,0,0,0,0)\right\|_B^2 + 2\|f(0,0,0,0,0)\|_B^2 \\
&\leq 2l_f^2\Big[|t_j^n| + \|u_{j-1}^n\|_B + \|\delta u_{j-1}^n\|_B + \frac{(t_j^n)^{1-\alpha}}{\Gamma(2-\alpha)}\|\delta u_{j-1}^n\|_B + \frac{(t_j^n)^\beta}{\Gamma(1+\beta)}\|u_{j-1}^n\|_B\Big]^2 \\
&\quad + 2\|f(0,0,0,0,0)\|_B^2 \\
&\leq 16l_f^2\left[T^2 + \|u_{j-1}^n\|_B^2 + \|\delta u_{j-1}^n\|_B^2 + \frac{T^{2-2\alpha}\|\delta u_{j-1}^n\|_B^2}{(\Gamma(2-\alpha))^2} + \frac{T^{2\beta}\|u_{j-1}^n\|_B^2}{(\Gamma(1+\beta))^2}\right] \\
&\quad + 2\|f(0,0,0,0,0)\|_B^2,
\end{aligned}
$$

therefore, by (3.15),

$$
\begin{aligned}
&\|\delta u_j^n\|_B^2 + \|u_j^n\|^2 \\
&\leq 16l_f^2T^3 + \Big(16l_f^2T\Big(1 + \frac{T^{2\beta}}{(\Gamma(1+\beta))^2}\Big) + 1\Big)\|U_0\|^2 \\
&\quad + \Big(16l_f^2T\Big(1 + \frac{T^{2-2\alpha}}{(\Gamma(2-\alpha))^2}\Big) + 1\Big)\|U_1\|_B^2 + 2T\|f(0,0,0,0,0)\|_B^2 \\
&\quad + h_n\Big(16l_f^2\Big(2 + \frac{T^{2-2\alpha}}{(\Gamma(2-\alpha))^2} + \frac{T^{2\beta}}{(\Gamma(1+\beta))^2}\Big) + 1\Big)\sum_{i=1}^{j}(\|\delta u_i^n\|_B^2 + \|u_i^n\|_B^2).
\end{aligned}
$$

Thus,

$$
\|u_j^n\|^2 + \|\delta u_j^n\|_B^2 \leq a_1 + b_1 h_n \sum_{i=1}^{j}(\|u_i^n\|^2 + \|\delta u_i^n\|_B^2), \tag{3.16}
$$

where

$$
\begin{aligned}
a_1 &= 16l_f^2T^3 + \Big(16l_f^2T\Big(1 + \frac{T^{2\beta}}{(\Gamma(1+\beta))^2}\Big) + 1\Big)\|U_0\|^2 \\
&\qquad + \Big(16l_f^2T\Big(1 + \frac{T^{2-2\alpha}}{(\Gamma(2-\alpha))^2}\Big) + 1\Big)\|U_1\|_B^2 + 2T\|f(0,0,0,0,0)\|_B^2, \\
b_1 &= 16l_f^2\Big(2 + \frac{T^{2-2\alpha}}{(\Gamma(2-\alpha))^2} + \frac{T^{2\beta}}{(\Gamma(1+\beta))^2}\Big) + 1.
\end{aligned}
$$

By Gronwall's inequality,

$$
\begin{aligned}
\|u_j^n\|^2 + \|\delta u_j^n\|_B^2 &\leq \frac{a_1}{1 - b_1h_n}\exp\left(\frac{b_1(j-1)h_n}{1 - b_1h_n}\right) \\
&\leq \frac{a_1}{1 - b_1h_n}\exp\left(\frac{b_1T}{1 - b_1h_n}\right).
\end{aligned} \tag{3.17}
$$

Therefore, $\|u_j^n\| + \|\delta u_j^n\|_B$ is uniformly bounded for all $n \in \mathbb{N}$ and $1 \leq j \leq n$. □

Lemma 3.4. *For $n \in \mathbb{N}$, $j = 1, 2, \ldots n$,*

$$\|\delta^2 u_j^n\|_B + \|\delta u_j^n\| \leq C, \tag{3.18}$$

where C is a positive constant, independent of both n and j.

Proof. Put $v = \delta^2 u_j^n$ in (3.13), then

$$\begin{aligned} &\langle \delta^2 u_j^n - \delta^2 u_{j-1}^n, \delta^2 u_j^n \rangle_B + \langle u_j^n - u_{j-1}^n, \delta^2 u_j^n \rangle \\ &\quad = \Big\langle f\Big(t_j^n, u_{j-1}^n, \delta u_{j-1}^n, \delta^\alpha u_j^n, \zeta^\beta u_j^n\Big) \\ &\qquad - f\Big(t_{j-1}^n, u_{j-2}^n, \delta u_{j-2}^n, \delta^\alpha u_{j-1}^n, \zeta^\beta u_{j-1}^n\Big), \delta^2 u_j^n \Big\rangle_B. \end{aligned} \tag{3.19}$$

By (3.14),

$$\begin{aligned} \|\delta^2 u_j^n\|_B^2 + \|\delta u_j^n\|^2 \leq \|\delta^2 u_{j-1}^n\|_B^2 + \|\delta u_{j-1}^n\|^2 + 2\Big[\Big\| f\Big(t_j^n, u_{j-1}^n, \delta u_{j-1}^n, \delta^\alpha u_j^n, \zeta^\beta u_j^n\Big) \\ - f\Big(t_{j-1}^n, u_{j-2}^n, \delta u_{j-2}^n, \delta^\alpha u_{j-1}^n, \zeta^\beta u_{j-1}^n\Big)\Big\|_B \|\delta^2 u_j^n\|_B\Big]. \end{aligned}$$

Therefore, for $j = 2, 3, \ldots, n$,

$$\begin{aligned} &\|\delta^2 u_j^n\|_B^2 + \|\delta u_j^n\|^2 \\ &\leq \|\delta^2 u_1^n\|_B^2 + \|\delta u_1^n\|^2 + 2\sum_{i=2}^{j} \Big[\Big\| f\Big(t_i^n, u_{i-1}^n, \delta u_{i-1}^n, \delta^\alpha u_i^n, \zeta^\beta u_i^n\Big) \\ &\qquad - f\Big(t_{i-1}^n, u_{i-2}^n, \delta u_{i-2}^n, \delta^\alpha u_{i-1}^n, \zeta^\beta u_{i-1}^n\Big)\Big\|_B \|\delta^2 u_i^n\|_B\Big]. \end{aligned}$$

By using Cauchy inequality for $\epsilon = h_n$, we get

$$\begin{aligned} &\|\delta^2 u_j^n\|_B^2 + \|\delta u_j^n\|^2 \\ &\leq \|\delta^2 u_1^n\|_B^2 + \|\delta u_1^n\|^2 + h_n \sum_{i=2}^{j} \Big\|\delta^2 u_i^n\Big\|_B^2 \\ &\quad + \frac{1}{h_n} \sum_{i=2}^{j} \Big[\Big\| f\Big(t_i^n, u_{i-1}^n, \delta u_{i-1}^n, \delta^\alpha u_i^n, \zeta^\beta u_i^n\Big) \\ &\qquad - f\Big(t_{i-1}^n, u_{i-2}^n, \delta u_{i-2}^n, \delta^\alpha u_{i-1}^n, \zeta^\beta u_{i-1}^n\Big)\Big\|_B^2\Big]. \end{aligned} \tag{3.20}$$

By (H1) and Lemma 3.1, we obtain

$$\begin{aligned}
&\Big\| f\Big(t_j^n, u_{j-1}^n, \delta u_{j-1}^n, \delta^\alpha u_j^n, \zeta^\beta u_j^n\Big) - f\Big(t_{j-1}^n, u_{j-2}^n, \delta u_{j-2}^n, \delta^\alpha u_{j-1}^n, \zeta^\beta u_{j-1}^n\Big)\Big\|_B^2 \\
&\le 8 l_f^2 \Big[h_n^2 + h_n^2 \|\delta u_{j-1}^n\|_B^2 + h_n^2 \|\delta^2 u_{j-1}^n\|_B^2 + \|\delta^\alpha u_j^n - \delta^\alpha u_{j-1}^n\|_B^2 \\
&\qquad\qquad + \|\zeta^\beta u_j^n - \zeta^\beta u_{j-1}^n\|_B^2 \Big] \\
&\le 16 l_f^2 \Big[h_n^2 + h_n^2 \frac{C\beta^2 T^{2\beta-2}}{(\Gamma(1+\beta))^2} + h_n^2 \Big(1 + \frac{T^{2-2\alpha}}{(\Gamma(2-\alpha))^2}\Big) \|\delta^2 u_{j-1}^n\|_B^2 \\
&\quad + h_n^{2-2\alpha} \Big(1 + \frac{T^{2\beta}}{(\Gamma(1+\beta))^2} + \frac{1}{(\Gamma(2-\alpha))^2}\Big) \|\delta u_{j-1}^n\|_B^2 \Big]. \qquad (3.21)
\end{aligned}$$

Put $j = 1$ and $v = \delta^2 u_1^n$ in (3.13), then

$$\langle \delta^2 u_1^n, \delta^2 u_1^n \rangle_B + \langle u_1^n, \delta^2 u_1^n \rangle = \Big\langle f(t_1^n, u_0^n, \delta u_0^n, \delta^\alpha u_1^n, \zeta^\beta u_1^n), \delta^2 u_1^n \Big\rangle_B. \qquad (3.22)$$

Therefore,

$$\begin{aligned}
&\|\delta^2 u_1^n\|_B^2 + \langle \delta u_1^n, \delta u_1^n - U_1 \rangle \\
&\quad = \Big\langle f\Big(t_1^n, u_0^n, \delta u_0^n, \delta^\alpha u_1^n, \zeta^\beta u_1^n\Big), \delta^2 u_1^n \Big\rangle_B + \Big\langle \frac{d^2 U_0}{dx^2}, \delta^2 u_1 \Big\rangle_B. \qquad (3.23)
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|\delta^2 u_1\|_B^2 + \|\delta u_1\|^2 \\
&\le 16 l_f^2 \Big[T^2 + \Big(1 + \frac{T^{2\beta}}{(\Gamma(1+\beta))^2}\Big) \|U_0\|_B^2 + \Big(1 + \frac{T^{2-2\alpha}}{(\Gamma(2-\alpha))^2}\Big) \|U_1\|_B^2 \Big] \\
&\qquad\qquad + \|U_1\|^2 + 2\|f(0,0,0,0,0)\|_B^2 + \Big\|\frac{d^2 U_0}{dx^2}\Big\|_B^2. \qquad (3.24)
\end{aligned}$$

By (3.21), (3.24) and (3.20), for $j = 1, 2, \dots, n$,

$$
\begin{aligned}
&\|\delta^2 u_j^n\|_B^2 + \|\delta u_j^n\|^2 \\
&\le \frac{1}{h_n} 16 l_f^2 \sum_{i=2}^{j} \Big[h_n^2 + h_n^2 \frac{C\beta^2 T^{2\beta-2}}{(\Gamma(1+\beta))^2} + h_n^2 \Big(1 + \frac{T^{2-2\alpha}}{(\Gamma(2-\alpha))^2}\Big) \|\delta^2 u_{i-1}^n\|_B^2 \\
&\quad + h_n^{2-2\alpha} \Big(1 + \frac{T^{2\beta}}{(\Gamma(1+\beta))^2} + \frac{1}{(\Gamma(2-\alpha))^2}\Big) \|\delta u_{i-1}^n\|_B^2 \Big] + h_n \sum_{i=2}^{j} \|\delta^2 u_i^n\|_B^2 \\
&\quad + 16 l_f^2 \Big[T^2 + \Big(1 + \frac{T^{2\beta}}{(\Gamma(1+\beta))^2}\Big) \|U_0\|_B^2 + \Big(1 + \frac{T^{2-2\alpha}}{(\Gamma(2-\alpha))^2}\Big) \|U_1\|_B^2 \Big] \\
&\quad + \|U_1\|^2 + 2\|f(0,0,0,0,0)\|_B^2 + \Big\| \frac{d^2 U_0}{dx^2} \Big\|_B^2 \\
&\le 16 l_f^2 \Big[\Big(1 + \frac{T^{2\beta}}{(\Gamma(1+\beta))^2} + \frac{1}{(\Gamma(2-\alpha))^2}\Big) h_n^{1-2\alpha} \sum_{i=2}^{j} \|\delta u_{i-1}^n\|_B^2 \\
&\quad + \Big(1 + \frac{T^{2-2\alpha}}{(\Gamma(2-\alpha))^2}\Big) h_n^{1-2\alpha} \sum_{i=2}^{j} \|\delta^2 u_{i-1}\|_B^2 + T + \frac{C\beta^2 T^{2\beta-1}}{(\Gamma(1+\beta))^2} \Big] \\
&\quad + 16 l_f^2 \Big[T^2 + \Big(1 + \frac{T^{2\beta}}{(\Gamma(1+\beta))^2}\Big) \|U_0\|_B^2 + \Big(1 + \frac{T^{2-2\alpha}}{(\Gamma(2-\alpha))^2}\Big) \|U_1\|_B^2 \Big] \\
&\quad + h_n^{1-2\alpha} \sum_{i=2}^{j} \|\delta^2 u_i^n\|_B^2 + \|U_1\|^2 + 2\|f(0,0,0,0,0)\|_B^2 + \Big\| \frac{d^2 U_0}{dx^2} \Big\|_B^2 .
\end{aligned}
$$

Therefore,

$$
\|\delta u_j^n\|^2 + \|\delta^2 u_j^n\|_B^2 \le a + b h_n^{1-2\alpha} \sum_{i=1}^{j} (\|\delta u_i^n\|^2 + \|\delta^2 u_i^n\|_B^2),
$$

where

$$
\begin{aligned}
a &= \|U_1\|^2 + 16 l_f^2 \Big[T + T^2 + \frac{C\beta^2 T^{2\beta-1}}{(\Gamma(1+\beta))^2} + \Big(1 + \frac{T^{2\beta}}{(\Gamma(1+\beta))^2}\Big) \|U_0\|_B^2 \\
&\quad + \Big(1 + \frac{T^{2-2\alpha}}{(\Gamma(2-\alpha))^2}\Big) \|U_1\|_B^2 \Big] + 2\|f(0,0,0,0,0)\|_B^2 + \Big\| \frac{d^2 U_0}{dx^2} \Big\|_B^2, \\
b &= 1 + 16 l_f^2 \Big(1 + \frac{T^{2\beta}}{(\Gamma(1+\beta))^2} + \frac{1}{(\Gamma(2-\alpha))^2} + \frac{T^{2-2\alpha}}{(\Gamma(2-\alpha))^2}\Big).
\end{aligned}
$$

By Gronwall's inequality,

$$
\|\delta u_j^n\|^2 + \|\delta^2 u_j^n\|^2 \le \frac{a}{1 - b h_n^{1-2\alpha}} \exp\Big(\frac{b(j-1) h_n^{1-2\alpha}}{1 - b h_n^{1-2\alpha}} \Big).
$$

Since

$$\frac{a}{1-bh_n^{1-2\alpha}}\exp\Big(\frac{b(j-1)h_n^{1-2\alpha}}{1-bh_n^{1-2\alpha}}\Big) \le \frac{a}{1-bh_n^{1-2\alpha}}\exp\Big(\frac{bn^{2\alpha}T^{1-2\alpha}}{n^{1-2\alpha}-bT^{1-2\alpha}}\Big)$$
$$\le \frac{a}{1-bh_n^{1-2\alpha}}\exp\Big(\frac{bT^{1-2\alpha}}{n^{1-4\alpha}-bT^{1-2\alpha}n^{-2\alpha}}\Big)$$

and $0 < \alpha < \frac{1}{4}$, therefore, $\|\delta u_j^n\|^2 + \|\delta^2 u_j^n\|_B^2$ is uniformly bounded for all $n \in \mathbb{N}$ and $1 \le j \le n$. □

4. Convergence and error estimates

Define $Y_n(t)$, $Z_n(t)$, $\Phi_n(t)$, $\Psi_n(t)$, $\mathfrak{D}_n^\alpha(t)$ and $\mathcal{I}_n^\beta(t)$ by

$$Y_n(t) = \begin{cases} u_0^n & \text{if} \quad t=0, \\ u_{j-1}^n + (t-t_{j-1}^n)\delta u_j^n & \text{if} \quad t \in (t_{j-1}^n, t_j^n], \end{cases} \tag{4.1}$$

$$Z_n(t) = \begin{cases} \delta u_0^n & \text{if} \quad t=0, \\ \delta u_{j-1}^n + (t-t_{j-1}^n)\delta^2 u_j^n & \text{if} \quad t \in (t_{j-1}^n, t_j^n], \end{cases} \tag{4.2}$$

$$\Phi_n(t) = \begin{cases} u_0^n & \text{if} \quad t=0, \\ u_j^n & \text{if} \quad t \in (t_{j-i}^n, t_j^n], \end{cases} \tag{4.3}$$

$$\Psi_n(t) = \begin{cases} \delta u_0^n & \text{if} \quad t=0, \\ \delta u_j^n & \text{if} \quad t \in (t_{j-i}^n, t_j^n], \end{cases} \tag{4.4}$$

$$\mathfrak{D}_n^\alpha(t) = \begin{cases} 0 & \text{if} \quad t=0, \\ \delta^\alpha u_j^n & \text{if} \quad t \in (t_{j-i}^n, t_j^n], \end{cases} \tag{4.5}$$

and

$$\mathcal{I}_n^\beta(t) = \begin{cases} 0 & \text{if} \quad t=0, \\ \zeta^\beta u_j^n & \text{if} \quad t \in (t_{j-i}^n, t_j^n]. \end{cases} \tag{4.6}$$

For $t \in (t_{j-1}^n, t_j^n]$, $1 \le j \le n$, define $F_n(t)$ by

$$F_n(t) = f\Big(t_j^n, u_{j-1}^n, \delta u_{j-1}^n, \delta^\alpha u_j^n, \zeta^\beta u_j^n\Big). \tag{4.7}$$

Denote the left derivative by $\dfrac{d^-}{dt}$, then (3.1) can be written as

$$\frac{d^- Z_n}{dt}(t) - \frac{\partial^2 \Phi_n}{\partial x^2}(t) = F_n(t), \quad t \in (0,T]. \tag{4.8}$$

For $t \in (0,T]$, we have

$$-\int_0^t \frac{\partial^2 \Phi_n}{\partial x^2}(s)ds = U_1 - Z_n(t) + \int_0^t F_n(s)ds. \tag{4.9}$$

Lemma 4.1. *There exists* $u \in C(I_T; L^2(0,1))$ *with the properties*

$$\frac{du}{dt} \in L^\infty(I_T; S_B) \cap C(I_T; S_B), \quad \frac{d^2u}{dt^2} \in L^\infty(I_T; S_B)$$

such that

$$Y_n(t) \to u(t) \text{ in } C(I_T; L^2(0,1))$$

and

$$Z_n(t) \to \frac{du}{dt}(t) \text{ in } C(I_T; S_B).$$

Furthermore, $u(t)$ *and* $\frac{du}{dt}(t)$ *are Lipschitz continuous functions on* I_T.

Proof. By (4.8) and (3.12), we get

$$\begin{aligned}&\left\langle \frac{dZ_n}{dt} - \frac{dZ_m}{dt}, \Psi_n(t) - \Psi_m(t) \right\rangle_B + \langle \Phi_n(t) - \Phi_m(t), \Psi_n(t) - \Psi_m(t) \rangle \\ &= \langle F_n(t) - F_m(t), \Psi_n(t) - \Psi_m(t) \rangle_B.\end{aligned} \tag{4.10}$$

Consider,

$$\begin{aligned}&\left\langle \frac{dZ_n}{dt} - \frac{dZ_m}{dt}, \Psi_n(t) - \Psi_m(t) \right\rangle_B \\ &= \left\langle \frac{d}{dt}\Big(Z_n(t) - Z_m(t)\Big), \Psi_n(t) - Z_n(t) + Z_m(t) - \Psi_m(t) \right\rangle_B \\ &\quad + \frac{1}{2}\frac{d}{dt}\|Z_n(t) - Z_m(t)\|_B^2\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}&\langle \Phi_n(t) - \Phi_m(t), \Psi_n(t) - \Psi_m(t) \rangle \\ &= \left\langle \frac{d}{dt}\Big(Y_n(t) - Y_m(t)\Big), \Phi_n(t) - Y_n(t) + Y_m(t) - \Phi_m(t) \right\rangle \\ &\quad + \frac{1}{2}\frac{d}{dt}\|Y_n(t) - Y_m(t)\|^2.\end{aligned} \tag{4.12}$$

By (4.10), (4.11) and (4.12),

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|Z_n(t) - Z_m(t)\|_B^2 + \frac{1}{2}\frac{d}{dt}\|Y_n(t) - Y_m(t)\|^2 \\ &= \left\langle \frac{d}{dt}\Big(Z_n(t) - Z_m(t)\Big), Z_n(t) - \Psi_n(t) + \Psi_m(t) - Z_m(t) \right\rangle_B \\ &\quad + \left\langle \frac{d}{dt}\Big(Y_n(t) - Y_m(t)\Big), Y_n(t) - \Phi_n(t) + \Phi_m(t) - Y_m(t) \right\rangle \\ &\quad + \Big\langle F_n(t) - F_m(t), \Psi_n(t) - \Psi_m(t) \Big\rangle_B.\end{aligned} \tag{4.13}$$

For $t \in (t_{j-1}^n, t_j^n]$ and $t \in (t_{j-1}^m, t_j^m], 1 \le j \le n, 1 \le l \le m$, consider

$$\|u_{j-1}^n - u_{l-1}^m\|_B \le C(h_n + h_m) + \|Y_n(t) - Y_m(t)\|, \tag{4.14}$$

$$\|\delta u_{j-1}^n - \delta u_{l-1}^m\|_B \le C(h_n + h_m) + \|Z_n(t) - Z_m(t)\|_B, \tag{4.15}$$

$$\begin{aligned}\|\delta^\alpha u_j^n - \delta^\alpha u_l^m\|_B &= \Big\|\frac{(t_j^n)^{1-\alpha}}{\Gamma(2-\alpha)}\delta u_{j-1}^n - \frac{(t_l^m)^{1-\alpha}}{\Gamma(2-\alpha)}\delta u_{l-1}^m\Big\| \\ &\le \frac{1}{\Gamma(2-\alpha)}\Big[(t_j^n)^{1-\alpha}\|\delta u_{j-1}^n - \delta u_{l-1}^m\|_B \\ &\qquad + |(t_j^n)^{1-\alpha} - (t_l^m)^{1-\alpha}|\|\delta u_{l-1}^m\|_B\Big].\end{aligned}$$

Therefore, by (4.15) and Lemma 3.2,

$$\begin{aligned}\|\delta^\alpha u_j^n - \delta^\alpha u_l^m\|_B \le \frac{1}{\Gamma(2-\alpha)}\Big[&CT^{1-\alpha}(h_n+h_m) + T^{1-\alpha}\|Z_n(t) - Z_m(t)\|_B \\ &+C(h_n^{1-\alpha} + h_m^{1-\alpha})\Big].\end{aligned} \tag{4.16}$$

And,

$$\begin{aligned}\|\zeta^\beta u_j^n - \zeta^\beta u_l^m\| &= \Big\|\frac{(t_j^n)^\beta}{\Gamma(1+\beta)}u_{j-1}^n - \frac{(t_l^m)^\beta}{\Gamma(1+\beta)}u_{l-1}^m\Big\| \\ &\le \frac{1}{\Gamma(1+\beta)}\Big[(t_j^n)^\beta\|u_{j-1}^n - u_{l-1}^m\|_B + |(t_j^n)^\beta - (t_l^m)^\beta|\|u_{l-1}^m\|_B\Big].\end{aligned}$$

Therefore, by (4.14) and Lemma 3.2,

$$\begin{aligned}\|\zeta^\beta u_j^n - \zeta^\beta u_l^m\| \le \frac{1}{\Gamma(1+\beta)}\Big[&CT^\beta(h_n+h_m) + T^\beta\|Y_n(t) - Y_m(t)\| \\ &+ C\beta(h_n+h_m)T^{\beta-1}\Big].\end{aligned} \tag{4.17}$$

And also,

$$\begin{aligned}\|\Psi_n(t) - \Psi_m(t)\|_B &\le \|\Psi_n(t) - Z_n(t)\|_B + \|Z_n(t) - Z_m(t)\|_B \\ &\quad + \|Z_m(t) - \Psi_m(t)\|_B \\ &\le 2C(h_n+h_m) + \|Z_n(t) - Z_m(t)\|_B,\end{aligned} \tag{4.18}$$

$$\begin{aligned}&\Big\langle \frac{d}{dt}\Big(Z_n(t) - Z_m(t)\Big), Z_n(t) - \Psi_n(t) + \Psi_m(t) - Z_m(t)\Big\rangle_B \\ &\le \Big(\Big\|\frac{dZ_n}{dt}\Big\|_B + \Big\|\frac{dZ_m}{dt}\Big\|_B\Big)\Big(\|Z_n(t) - \Psi_n(t)\|_B + \|Z_m(t) - \Psi_m(t)\|_B\Big) \\ &\le 4C^2(h_n+h_m),\end{aligned} \tag{4.19}$$

$$\begin{aligned}&\Big\langle \frac{d}{dt}\Big(Y_n(t) - Y_m(t)\Big), Y_n(t) - \Phi_n(t) + \Phi_m(t) - Y_m(t)\Big\rangle \\ &\le \Big(\Big\|\frac{dY_n}{dt}\Big\| + \Big\|\frac{dY_m}{dt}\Big\|\Big)\Big(\|Y_n(t) - \Phi_n(t)\| + \|Y_nm(t) - \Phi_m(t)\|\Big) \\ &\le 4C^2(h_n+h_m).\end{aligned} \tag{4.20}$$

By (4.14), (4.15), (4.16) and (4.17), we get

$$
\begin{aligned}
&\|F_n(t)-F_m(t)\|_B\\
&=\Big\|f\Big(t_j^n,u_{j-1}^n,\delta u_{j-1}^n,\delta^\alpha u_j^n,\zeta^\beta u_j^n\Big)-f\Big(t_l^m,u_{l-1}^m,\delta u_{l-1}^m,\delta^\alpha u_l^m,\zeta^\beta u_l^m\Big)\Big\|_B\\
&\le l_f\Big[|t_j^n-t_l^m|+\|u_{j-1}^n-u_{l-1}^m\|_B+\|\delta u_{j-1}^n-\delta u_{l-1}^m\|_B+\Big\|\delta^\alpha u_j^n-\delta^\alpha u_l^m\Big\|_B\\
&\quad+\Big\|\zeta^\beta u_j^n-\zeta^\beta u_l^m\Big\|_B\Big]\\
&\le l_f\Big[(h_n+h_m)\Big(1+2C+\frac{CT^{1-\alpha}}{\Gamma(2-\alpha)}+\frac{CT^\beta}{\Gamma(1+\beta)}+\frac{C\beta T^{\beta-1}}{\Gamma(1+\beta)}\Big)\\
&\quad+\frac{C(h_n^{1-\alpha}+h_m^{1-\alpha})}{\Gamma(2-\alpha)}+\Big(1+\frac{T^\beta}{\Gamma(1+\beta)}\Big)\|Y_n(t)-Y_m(t)\|\\
&\quad+\Big(1+\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\Big)\|Z_n(t)-Z_m(t)\|_B\Big].
\end{aligned}
$$

Therefore, by (4.18),

$$
\begin{aligned}
&\|F_n(t)-F_m(t)\|_B\|\Psi_n(t)-\Psi_m(t)\|_B\\
&\le l_f\Big[2C(h_n+h_m)(h_n+h_m+1)\Big(1+2C+\frac{CT^{1-\alpha}}{\Gamma(2-\alpha)}+\frac{CT^\beta}{\Gamma(1+\beta)}+\frac{C\beta T^{\beta-1}}{\Gamma(1+\beta)}\Big)\\
&+4C^2(h_n+h_m)\Big(2+\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}+\frac{T^\beta}{\Gamma(1+\beta)}\Big)\\
&+\frac{2C^2}{\Gamma(1-\alpha)}(h_n^{1-\alpha}+h_m^{1-\alpha})(h_n+h_m+1)\Big]\\
&+l_f\Big(\frac{3}{2}+\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}+\frac{T^\beta}{2\Gamma(1+\beta)}\Big)\Big(\|Y_n(t)-Y_m(t)\|^2\\
&\qquad\qquad+\|Z_n(t)-Z_m(t)\|_B^2\Big).
\end{aligned}
\tag{4.21}
$$

By (4.13), (4.19), (4.20) and (4.21),

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|Y_n(t)-Y_m(t)\|^2+\|Z_n(t)-Z_m(t)\|_B^2\Big)\\
&\quad\le K_{mn}+K\Big(\|Y_n(t)-Y_m(t)\|^2+\|Z_n(t)-Z_m(t)\|_B^2\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
K_{mn}&=2l_f\Big[2C(h_n+h_m)(h_n+h_m+1)\Big(1+2C+\frac{CT^{1-\alpha}}{\Gamma(2-\alpha)}+\frac{CT^\beta}{\Gamma(1+\beta)}\\
&\quad+\frac{C\beta T^{\beta-1}}{\Gamma(1+\beta)}\Big)+4C^2(h_n+h_m)\Big(2+\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}+\frac{T^\beta}{\Gamma(1+\beta)}\Big)\\
&\quad+\frac{2C^2}{\Gamma(1-\alpha)}(h_n^{1-\alpha}+h_m^{1-\alpha})(h_n+h_m+1)\Big]+16C^2(h_n+h_m),\\
K&=l_f\Big(3+\frac{2T^{1-\alpha}}{\Gamma(2-\alpha)}+\frac{T^\beta}{\Gamma(1+\beta)}\Big).
\end{aligned}
$$

Therefore,

$$\begin{aligned}&\|Y_n(t)-Y_m(t)\|^2+\|Z_n(t)-Z_m(t)\|_B^2\\&\quad\le TK_{mn}+K\int_0^t(\|Z_n(t)-Z_m(t)\|_B^2+\|Y_n(t)-Y_m(t)\|^2)dt.\end{aligned}$$

By Gronwall's inequality,

$$\|Y_n(t)-Y_m(t)\|^2+\|Z_n(t)-Z_m(t)\|_B^2\le TK_{mn}e^{Kt}\le TK_{mn}e^{KT} \tag{4.22}$$

for all $t\in I_T$. Thus, there exist $u,v\in C(I_T;L^2(0,1))$ such that $Y_n\to u$ in $C(I_T;L^2(0,1))$ and $Z_n\to v$ in $C(I_T;S_B)$ as $n\to\infty$. Since $\|Y_n(t_1)-Y_n(t_2)\|\le C|t_1-t_2|$ and $\|Z_n(t_1)-Z_n(t_2)\|_B\le C|t_1-t_2|$, $u(t)$ and $\frac{du}{dt}(t)$ are Lipschitz continuous functions on I_T. Thus, $\frac{du}{dt}\in L^\infty(I_T;S_B)$ and $\frac{dv}{dt}\in L^\infty(I_T;S_B)$. Since $\|Z_n(t)-\Psi_n(t)\|_B\le Ch_n$, $\Psi_n(t)\to v(t)$ in $C(I_T;S_B)$ as $n\to\infty$. For $w\in S_B$,

$$\begin{aligned}\langle Y_n(t),w\rangle_B&=\int_0^t\left\langle\frac{dY_n}{dt}(s),w\right\rangle_B ds+\langle u_0^n,w\rangle_B\\&=\int_0^t\langle\Psi_n(s),w\rangle_B ds+\langle U_0,w\rangle_B.\end{aligned}$$

Therefore,

$$\langle u(t),w\rangle_B=\int_0^t\langle v(s),w\rangle_B ds+\langle U_0,w\rangle_B.$$

Hence, $\frac{du}{dt}(t)=v(t)$ a.e. on I_T, and $\frac{d^2u}{dt^2}\in L^\infty(I_T;S_B)$. □

Now, we provide some results for the limit function u.

Lemma 4.2. *$\mathfrak{D}_n^\alpha(t)\to D_t^\alpha u$ in $C(I_T;S_B)$, and $\mathcal{I}_n^\beta(t)\to I_t^\beta u$ in $C(I_T;L^2(0,1))$.*

Proof. For $j=1,2,\dots,n$, $n\in\mathbb{N}$, let $t\in(t_{j-1}^n,t_j^n]$. Then,

$$\begin{aligned}\|u_{j-1}^n-u(t)\|&\le\|u_{j-1}^n-Y_n(t)\|+\|Y_n(t)-u(t)\|\\&\le Ch_n+\|Y_n(t)-u(t)\|,\end{aligned}\tag{4.23}$$

$$\begin{aligned}\left\|\delta u_{j-1}^n-\frac{du}{dt}\right\|_B&\le\|\delta u_{j-1}^n-Z_n(t)\|_B+\left\|Z_n(t)-\frac{du}{dt}\right\|_B\\&\le Ch_n+\left\|Z_n(t)-\frac{du}{dt}\right\|_B,\end{aligned}\tag{4.24}$$

$$
\begin{aligned}
&\|\mathfrak{D}_n^\alpha(t) - D_t^\alpha u\|_B \\
&= \frac{1}{\Gamma(1-\alpha)}\Big\| \int_0^{t_j^n} \frac{\delta u_{j-1}^n}{(t_j^n - s)^\alpha} ds - \int_0^t \frac{\frac{du}{ds}}{(t-s)^\alpha} ds \Big\|_B \\
&\leq \frac{1}{\Gamma(1-\alpha)}\Big\| \int_0^{t} \frac{\delta u_{j-1}^n}{(t_j^n - s)^\alpha} ds - \int_0^t \frac{\frac{du}{ds}}{(t-s)^\alpha} ds \Big\|_B \\
&\qquad\qquad\qquad\qquad + \frac{1}{\Gamma(1-\alpha)}\Big\| \int_t^{t_j^n} \frac{\delta u_{j-1}^n}{(t_j^n - s)^\alpha} ds \Big\|_B \\
&\leq \frac{1}{\Gamma(1-\alpha)} \int_0^t \Big(\frac{1}{(t_j^n - s)^\alpha} - \frac{1}{(t-s)^\alpha} \Big) \|\delta u_{j-1}^n\|_B ds \\
&\quad + \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\|\delta u_{j-1}^n - \frac{du}{ds}\|_B}{(t-s)^\alpha} ds \\
&\quad\ + \frac{1}{\Gamma(1-\alpha)} \int_t^{t_j^n} \frac{\|\delta u_{j-1}^n\|_B}{(t_j^n - s)^\alpha} ds.
\end{aligned} \tag{4.25}
$$

By Lemmas 3.1 and 3.4, and by (4.24), we get

$$
\begin{aligned}
&\|\mathfrak{D}_n^\alpha(t) - D_t^\alpha u\|_B \\
&\leq \frac{Ch_n^{1-\alpha}}{\Gamma(1-\alpha)} + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \Big(Ch_n + \sup_{s\in[0,T]} \Big\| Z_n(s) - \frac{du}{ds} \Big\|_B \Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \frac{Ch_n^{1-\alpha}}{\Gamma(2-\alpha)}.
\end{aligned} \tag{4.26}
$$

Therefore, by Lemma 4.1, $\mathfrak{D}_n^\alpha(t) \to D_t^\alpha u$ in $C(I_T; S_B)$.

Now, consider

$$
\begin{aligned}
&\|\mathcal{I}_n^\beta(t) - I_t^\beta u\| \\
&= \frac{1}{\Gamma(\beta)}\Big\| \int_0^{t_j} \frac{u_{j-1}^n}{(t_j^n - s)^{1-\beta}} ds - \int_0^t \frac{u(s)}{(t-s)^{1-\beta}} ds \Big\| \\
&\leq \frac{1}{\Gamma(\beta)}\Big\| \int_0^{t} \frac{u_{j-1}^n}{(t_j^n - s)^{1-\beta}} ds - \int_0^t \frac{u(s)}{(t-s)^{1-\beta}} ds \Big\| \\
&\qquad\qquad + \frac{1}{\Gamma(\beta)}\Big\| \int_t^{t_j^n} \frac{u_{j-1}^n}{(t_j^n - s)^{1-\beta}} ds \Big\| \\
&\leq \frac{1}{\Gamma(\beta)} \int_0^t \Big(\frac{1}{(t_j^n - s)^{1-\beta}} - \frac{1}{(t-s)^{1-\beta}} \Big) \|u_{j-1}^n\| ds \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^t \frac{\|u_{j-1}^n - u(s)\|}{(t-s)^{1-\beta}} ds + \frac{1}{\Gamma(\beta)} \int_t^{t_j^n} \frac{\|u_{j-1}^n\|}{(t_j^n - s)^{1-\beta}} ds.
\end{aligned} \tag{4.27}
$$

By Lemmas 3.1 and 3.3, and by (4.23), we get

$$\begin{aligned}&\|\mathcal{I}_n^{\beta}(t)-I_t^{\beta}u\|\\&\leq \frac{Ch_nT^{\beta-1}}{\Gamma(\beta)}+\frac{T^{\beta}}{\Gamma(1+\beta)}\Big(Ch_n+\sup_{s\in[0,T]}\|Y_n(s)-u(s)\|\Big)\\&\qquad\qquad\qquad\qquad\qquad\qquad+\frac{Ch_n^{\beta}}{\Gamma(1+\beta)}.\end{aligned}\tag{4.28}$$

Therefore, by Lemma 4.1, $\mathcal{I}_n^{\beta}(t)\to I_t^{\beta}u$ in $C(I_T;L^2(0,1))$. □

Lemma 4.3. *The errors* $\|Y_n(t)-u(t)\|$, $\left\|Z_n(t)-\frac{du}{dt}\right\|_B$, $\|u_j^n-u(t_j^n)\|$, $\|\mathfrak{D}_n^{\alpha}(t)-D_t^{\alpha}u\|_B$ *and* $\|\mathcal{I}_n^{\beta}(t)-I_t^{\beta}u\|$ *satisfy the following estimates:*

(i) $\|Y_n(t)-u(t)\|+\left\|Z_n(t)-\frac{du}{dt}\right\|_B\leq\frac{M_1}{\sqrt{n}}$ *for all* $t\in I_T$ *and* $n\in\mathbb{N}$, *and for some* $M_1>0$,

(ii) $\|u_j^n-u(t_j^n)\|\leq\frac{M_2}{\sqrt{n}}$ *for all* $j=1,2,\dots,n$, $n\in\mathbb{N}$ *and for some* $M_2>0$. *Moreover,* $\|u_j^n-u(t)\|\leq\frac{M_2}{\sqrt{n}}$ *for all* $t\in(t_{j-1}^n,t_j^n]$,

(iii) $\|\mathfrak{D}_n^{\alpha}(t)-D_t^{\alpha}u\|_B\leq\frac{M_3}{\sqrt{n}}$ *for all* $t\in I_T$ *and* $n\in\mathbb{N}$, *and for some* $M_3>0$,

(iv) $\|\mathcal{I}_n^{\beta}(t)-I_t^{\beta}u\|\leq\frac{M_4}{\sqrt{n}}$ *for all* $t\in I_T$ *and* $n\in\mathbb{N}$, *and for some* $M_4>0$.

Proof. Apply limit $m\to\infty$ in (4.22), then

$$\|Y_n(t)-u(t)\|+\left\|Z_n(t)-\frac{du}{dt}\right\|_B\leq\frac{M_1}{\sqrt{n}}\tag{4.29}$$

for all $n\in\mathbb{N}$ and $t\in I_T$, and for some $M_1>0$. Hence, $\|Y_n(t)-u(t)\|\leq\frac{M_1}{\sqrt{n}}$ for all $t\in I_T$ and $n\in\mathbb{N}$.

For $j=1,2,\dots,n$, $n\in\mathbb{N}$, if $t\in(t_{j-1}^n,t_j^n]$, then, by (4.23) and (4.29),

$$\|u_{j-1}^n-u(t)\|\leq\frac{CT}{n}+\frac{M_1}{\sqrt{n}}\leq\frac{CT+M_1}{\sqrt{n}}.\tag{4.30}$$

Therefore, by Lemma 3.4,

$$\begin{aligned}\|u_j^n-u(t)\|&\leq\|u_j^n-u_{j-1}^n\|+\|u_{j-1}^n-u(t)\|\\&\leq Ch_n+\|u_{j-1}^n-u(t)\|\\&\leq\frac{2CT+M_1}{\sqrt{n}}.\end{aligned}$$

Thus, $\|u_j^n-u(t)\|\leq\frac{M_2}{\sqrt{n}}$ for all $t\in[t_{j-1}^n,t_j^n]$, $j=1,2,\dots,n$, $n\in\mathbb{N}$, where $M_2=2CT+M_1$. In particular, $\|u_j^n-u(t_j^n)\|\leq\frac{M_2}{\sqrt{n}}$.

Let $t \in I_T$. Then, by (4.26) and (4.29),

$$\begin{aligned}\|\mathfrak{D}_n^\alpha(t) - D_t^\alpha u\|_B &\le \frac{Ch_n^{1-\alpha}}{\Gamma(1-\alpha)} + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\Big(Ch_n + \frac{M_1}{\sqrt{n}}\Big) + \frac{Ch_n^{1-\alpha}}{\Gamma(2-\alpha)}\\ &\le \frac{M_3}{\sqrt{n}},\end{aligned}$$

where $M_3 = \dfrac{[M_1 + C(T+2)]T^{1-\alpha}}{\Gamma(2-\alpha)}$.

Now, by (4.28) and (4.29),

$$\begin{aligned}\|\mathcal{I}_n^\beta(t) - I_t^\beta u\| &\le \frac{Ch_nT^{\beta-1}}{\Gamma(\beta)} + \frac{T^\beta}{\Gamma(1+\beta)}\Big(Ch_n + \frac{M_1}{\sqrt{n}}\Big) + \frac{Ch_n^\beta}{\Gamma(1+\beta)}\\ &\le \frac{M_4}{\sqrt{n}},\end{aligned}$$

where $M_4 = \dfrac{[M_1 + C(T+\beta+1)]T^\beta}{\Gamma(1+\beta)}$. □

5. Main results

In this section, we will prove that the function $u \in C(I_T; L^2(0,1))$ which is obtained in Lemma 4.1 is a strong solution of (1.1)–(1.3) on I_T, and also discuss continuous dependence of u upon U_0, U_1 and f.

Theorem 5.1. *Suppose that (H1)–(H3) are satisfied.*
If $Tl_f\Big(\dfrac{5}{2} + \dfrac{T^\beta}{2\Gamma(1+\beta)} + \dfrac{2T^{1-\alpha}}{\Gamma(2-\alpha)}\Big) < 1$, *then* (1.1)–(1.3) *has a unique strong solution on* I_T.

Proof. For $j = 1, 2, \dots, n$, $n \in \mathbb{N}$, let $t \in (t_{j-1}^n, t_j^n]$. Then,

$$\begin{aligned}&\Big\|F_n(t) - f\Big(t, u(t), \frac{du}{dt}, D_t^\alpha u, I_t^\beta u\Big)\Big\|_B\\ &\quad= \Big\|f\Big(t_j^n, u_{j-1}^n, \delta u_{j-1}^n, \delta^\alpha u_j^n, \zeta^\beta u_j^n\Big) - f\Big(t, u(t), \frac{du}{dt}, D_t^\alpha u, I_t^\beta u\Big)\Big\|_B\\ &\quad\le l_f\Big(|t - t_j^n| + \|u_{j-1}^n - u(t)\|_B + \Big\|\delta u_{j-1}^n - \frac{du}{dt}\Big\|_B + \Big\|\delta^\alpha u_j^n - D_t^\alpha u\Big\|_B\\ &\qquad\qquad + \Big\|\zeta^\beta u_j^n - I_t^\beta u\Big\|_B\Big)\\ &\quad\le l_f\Big(h_n + \|u_{j-1}^n - Y_n(t)\|_B + \|Y_n(t) - u(t)\|_B + \Big\|\delta u_{j-1}^n - Z_n(t)\Big\|_B\\ &\qquad + \Big\|Z_n(t) - \frac{du}{dt}\Big\|_B + \Big\|\mathfrak{D}_n^\alpha(t) - D_t^\alpha u\Big\|_B + \Big\|\mathcal{I}_n^\beta(t) - I_t^\beta u\Big\|_B\Big)\\ &\quad\le l_f\Big[h_n(1+2C) + \|Y_n(t) - u(t)\| + \Big\|Z_n(t) - \frac{du}{dt}\Big\|_B + \Big\|\mathfrak{D}_n^\alpha(t) - D_t^\alpha u\Big\|_B\\ &\qquad + \Big\|\mathcal{I}_n^\beta(t) - I_t^\beta u\Big\|\Big].\end{aligned} \tag{5.1}$$

By Lemmas 4.1 and 4.2,

$$F_n(t) \to f\Big(t, u(t), \frac{du}{dt}, D_t^\alpha u, I_t^\beta u\Big)$$

as $n \to \infty$ uniformly on I_T.

Apply limit as $n \to \infty$ in (4.9), then

$$-\int_0^t \frac{\partial^2 u}{\partial x^2}(s)ds = U_1 - \frac{du}{dt}(t) + \int_0^t f\Big(s, u(s), \frac{du}{ds}, D_s^\alpha u, I_s^\beta u\Big)ds. \tag{5.2}$$

This implies that

$$\frac{\partial^2 u(t)}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f\Big(t, u(t), \frac{du}{dt}, D_t^\alpha u, I_t^\beta u\Big)$$

a.e. $t \in I_T$. Therefore, $u(t)(x) = u(t, x)$ is a strong solution of (1.1)–(1.3). To prove uniqueness, consider that u_1, u_2 are two strong solutions of (1.1)–(1.3). If $\omega = u_1 - u_2$, then for any $v \in W$,

$$\begin{aligned}\Big\langle \frac{d^2\omega}{dt^2}, v\Big\rangle_B + \langle \omega, v\rangle = \Big\langle f\Big(t, u_1(t), \frac{du_1}{dt}, D_t^\alpha u_1, I_t^\beta u_1\Big) \\ -f\Big(t, u_2(t), \frac{du_2}{dt}, D_t^\alpha u_2, I_t^\beta u_2\Big), v\Big\rangle_B.\end{aligned} \tag{5.3}$$

Put $v = \frac{d\omega}{dt}$, then

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big\|\frac{d\omega}{dt}\Big\|_B^2 + \frac{1}{2}\frac{d}{dt}\|\omega(t)\|^2 \\ &\leq \Big\|f\Big(t, u_1(t), \frac{du_1}{dt}, D_t^\alpha u_1, I_t^\beta u_2\Big) - f\Big(t, u_2(t), \frac{du_2}{ds}, D_t^\alpha u_2, I_t^\beta u_2\Big)\Big\|_B \Big\|\frac{d\omega}{dt}\Big\|_B \\ &\leq l_f\Big[\sup_{t\in I_T}\|\omega(t)\|_B + \sup_{t\in I_T}\Big\|\frac{d\omega}{dt}\Big\|_B + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{t\in I_T}\Big\|\frac{d\omega}{dt}\Big\|_B \\ &\qquad + \frac{T^\beta}{\Gamma(1+\beta)}\sup_{t\in I_T}\|\omega(t)\|_B\Big]\sup_{t\in I_T}\Big\|\frac{d\omega}{dt}\Big\|_B \\ &\leq l_f\Big(\frac{5}{4} + \frac{T^\beta}{4\Gamma(1+\beta)} + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\Big)\Big(\sup_{t\in I_T}\|\omega(t)\|^2 + \sup_{t\in I_T}\Big\|\frac{d\omega}{dt}\Big\|_B^2\Big).\end{aligned} \tag{5.4}$$

Therefore, for $t \in I_T$,

$$\begin{aligned}&\|\omega(t)\|^2 + \Big\|\frac{d\omega}{dt}\Big\|_B^2 \\ &\leq l_f T\Big(\frac{5}{2} + \frac{T^\beta}{2\Gamma(1+\beta)} + \frac{2T^{1-\alpha}}{\Gamma(2-\alpha)}\Big)\Big(\sup_{t\in I_T}\|\omega(t)\|^2 + \sup_{t\in I_T}\Big\|\frac{d\omega}{dt}\Big\|_B^2\Big).\end{aligned} \tag{5.5}$$

If $l_f T\Big(\frac{5}{2} + \frac{T^\beta}{2\Gamma(1+\beta)} + \frac{2T^{1-\alpha}}{\Gamma(2-\alpha)}\Big) < 1$, then

$$\sup_{t\in I_T}\Big\|\frac{d\omega}{dt}\Big\|_B^2 + \sup_{t\in I_T}\|\omega(t)\|^2 = 0.$$

Therefore, $\sup_{t\in I_T}\|\omega(t)\| = 0$ and $\sup_{t\in[0,T]}\Big\|\frac{d\omega}{dt}\Big\|_B = 0$.

Thus $u_1(t) = u_2(t)$ for all $t \in I_T$. □

Now, we provide a result of continuous dependence of solution u upon U_0, U_1 and f.

Theorem 5.2. *Consider that u and v are two strong solutions of (1.1)–(1.3) corresponding to (U_0, U_1, f^*) and $(V_0, V_1, f^{**},)$ respectively. Assume that (U_0, U_1, f^*) and (V_0, V_1, f^{**}) satisfy (H1)–(H3), then the following inequality*

$$\begin{aligned}&\|u(t) - v(t)\|^2 + \Big\|\frac{du}{dt} - \frac{dv}{dt}\Big\|_B^2\\ &\quad\le \Big(\|U_0 - V_0\|^2 + \frac{1}{2}\|U_1 - V_1\|^2 + \int_0^T \|f_u^*(s) - f_v^{**}(s)\|_B^2 ds\Big)e^t\end{aligned}$$

holds for all $t \in I_T$, where

$$f_u^*(t) = f^*\Big(t, u(t), \frac{du}{dt}, D_t^\alpha u, I_t^\beta u\Big),$$
$$f_v^{**}(t) = f^{**}\Big(t, v(t), \frac{dv}{dt}, D_t^\alpha v, I_t^\beta v\Big).$$

Proof. Let $w = u - v$, then by (1.1), we have

$$\Big\langle \frac{d^2w}{dt^2}, \frac{dw}{dt}\Big\rangle_B + \Big\langle w, \frac{dw}{dt}\Big\rangle = \Big\langle f_u^*(t) - f_v^{**}(t), \frac{dw}{dt}\Big\rangle_B, \tag{5.6}$$

that is,

$$\frac{d}{dt}\Big(\Big\|\frac{dw}{dt}\Big\|_B^2 + \|w(t)\|^2\Big) \le 2\|f_u^*(t) - f_v^{**}(t)\|_B\Big\|\frac{dw}{dt}\Big\|_B. \tag{5.7}$$

Therefore, for $t \in I_T$,

$$
\begin{aligned}
&\|u(t)-v(t)\|^2+\Big\|\frac{du}{dt}-\frac{dv}{dt}\Big\|_B^2\\
&\quad\le \|U_0-V_0\|^2+\|U_1-V_1\|_B^2+2\int_0^t\Big(\|f_u^*(s)-f_v^{**}(s)\|_B\Big\|\frac{du}{ds}-\frac{dv}{ds}\Big\|_B\Big)ds\\
&\quad\le \|U_0-V_0\|^2+\|U_1-V_1\|_B^2+\int_0^t\|f_u^*(s)-f_v^{**}(s)\|_B^2ds\\
&\qquad\qquad+\int_0^t\Big\|\frac{du}{ds}-\frac{dv}{ds}\Big\|_B^2ds\\
&\quad\le \|U_0-V_0\|^2+\|U_1-V_1\|_B^2+\int_0^t\|f_u^*(s)-f_v^{**}(s)\|_B^2ds\\
&\qquad\qquad+\int_0^t\Big(\|u(s)-v(s)\|^2+\Big\|\frac{du}{ds}-\frac{dv}{ds}\Big\|_B^2\Big)ds.
\end{aligned}
$$

By Gronwall's inequality,

$$
\begin{aligned}
&\|u(t)-v(t)\|^2+\Big\|\frac{du}{dt}-\frac{dv}{dt}\Big\|_B^2\\
&\quad\le\Big(\|U_0-V_0\|^2+\|U_1-V_1\|_B^2+\int_0^T\|f_u^*(s)-f_v^{**}(s)\|_B^2ds\Big)e^t\\
&\quad\le\Big(\|U_0-V_0\|^2+\frac{1}{2}\|U_1-V_1\|^2+\int_0^T\|f_u^*(s)-f_v^{**}(s)\|_B^2ds\Big)e^t,
\end{aligned}
$$

which gives the required result. □

6. Applications

Example 6.1. Consider the following hyperbolic partial fractional differential equation with fractional integral forcing function

$$
\frac{\partial^2 u}{\partial t^2}+a\frac{\partial u}{\partial t}+b\frac{\partial^{\frac{1}{5}}u}{\partial t^{\frac{1}{5}}}+cu(t,x)-\frac{\partial^2 u}{\partial x^2}=g\Big(t,I_t^{\frac{3}{2}}u\Big),\ t\in(0,T],\ x\in(0,1), \tag{6.1}
$$

subject to the initial conditions

$$
u(0,x)=6x^2-6x+1;\quad\text{and}\quad\frac{\partial u}{\partial t}(0,x)=10x^3-12x^2+3x\quad\text{on }(0,1), \tag{6.2}
$$

and the integral conditions

$$
\int_0^1 u(t,x)dx=\int_0^1 xu(t,x)dx=0,\quad t\in[0,T], \tag{6.3}
$$

where a, b, c are nonzero real numbers and T is a positive real number and g : $[0,T]\times B(0,1)$ satisfies the condition that $\exists\, l_g>0$ such that

$$
\|g(t_1,u_1)-g(t_2,u_2)\|_B\le l_g(|t_1-t_2|+\|u_1-u_2\|_B)
$$

for all $t_1, t_2 \in [0,T]$ and for all $u_1, u_2 \in B(0,1)$. Here, $f\Big(t,u,w,y,z\Big) = g\Big(t,z\Big) - aw - by - cu$, $U_0(x) = 6x^2 - 6x + 1$, and $U_1(x) = 10x^3 - 12x^2 + 3x$. Clearly,

$$\begin{aligned}
&\int_0^1 U_0(x)dx = \int_0^1 (6x^2 - 6x + 1)dx = 0,\\
&\int_0^1 xU_0(x)dx = \int_0^1 (6x^3 - 6x^2 + x)dx = 0, &(6.4)\\
&\int_0^1 U_1(x)dx = \int_0^1 10x^3 - 12x^2 + 3xdx = 0,\\
&\int_0^1 xU_1(x)dx = \int_0^1 (10x^4 - 12x^3 + 3x^2)xdx = 0. &(6.5)
\end{aligned}$$

Let $\lambda = \max\{l_g, |a|, |b|, |c|\}$. Then,

$$\begin{aligned}
&\Big|\int_0^x (f(t_1,u_1,w_1,y_1,z_1) - f(t_2,u_2,w_2,y_2,z_2)(s)ds\Big|\\
&\quad\le \int_0^x |f(t_1,u_1,w_1,y_1,z_1) - f(t_2,u_2,w_2,y_2,z_2))(s)|ds\\
&\quad\le \int_0^x (l_g|t_1 - t_2| + |c||u_1 - u_2| + |a||w_1 - w_2| + |b||y_1 - y_2|\\
&\qquad + l_g|z_1 - z_2|)(s)ds\\
&\quad\le \lambda\Big[x|t_1 - t_2| + \int_0^x |u_1 - u_2|(s)ds + \int_0^x |w_1 - w_2|(s)ds + \int_0^x |y_1 - y_2|\\
&\qquad + \int_0^x |z_1 - z_2|(s)ds\Big].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Big(\int_0^x (f(t_1,u_1,w_1,y_1,z_1) - f(t_2,u_2,w_2,y_2,z_2))(s)ds\Big)^2\\
&\quad\le 4\lambda^2\Big[x^2|t_1 - t_2|^2 + \Big(\int_0^x |u_1 - u_2|(s)ds\Big)^2 + \Big(\int_0^x |w_1 - w_2|(s)ds\Big)^2\\
&\qquad + \Big(\int_0^x |y_1 - y_2|(s)ds\Big)^2 + \Big(\int_0^x |z_1 - z_2|(s)ds\Big)^2\Big]\\
&\quad\le 8\lambda^2\Big[x^2|t_1 - t_2|^2 + \Big(\xi_x(|u_1 - u_2|)(x)\Big)^2 + \Big(\xi_x(|w_1 - w_2|)(x)\Big)^2\\
&\qquad + \Big(\xi_x(|y_1 - y_2|)(x)\Big)^2 + \Big(\xi_x(|z_1 - z_2|)(x)\Big)^2\Big]. &(6.6)
\end{aligned}$$

Since

$$\begin{aligned}
&\|f(t_1,u_1,w_1,y_1,z_1)-f(t_2,u_2,w_2,y_2,z_2)\|_B^2\\
&=\int_0^1\Big(\xi_x(f(t_1,u_1,w_1,y_1,z_1)-f(t_2,u_2,w_2,y_2,z_2))(x)\Big)^2dx\\
&=\int_0^1\Big(\int_0^x(f(t_1,u_1,w_1,y_1,z_1)-f(t_2,u_2,w_2,y_2,z_2))(s)ds\Big)^2dx,
\end{aligned}$$

therefore, by (6.6),

$$\begin{aligned}
&\|f(t_1,u_1,w_1,y_1,z_1)-f(t_2,u_2,w_2,y_2,z_2)\|_B^2\\
&\le 8\lambda^2\Big[\frac{1}{3}|t_1-t_2|^2+\int_0^1\Big(\xi_x(|u_1-u_2|)(x)\Big)^2dx+\int_0^1\Big(\xi_x(|w_1-w_2|)(x)\Big)^2dx\\
&+\int_0^1\Big(\xi_x(|y_1-y_2|)(x)\Big)^2dx+\int_0^1\Big(\xi_x(|z_1-z_2|)(x)\Big)^2dx\Big]\\
&\le 8\lambda^2\Big(|t_1-t_2|^2+\|u_1-u_2\|_B^2+\|w_1-w_2\|_B^2+\|y_1-y_2\|_B^2+\|z_1-z_2\|_B^2\Big)
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|f(t_1,u_1,w_1,y_1,z_1)-f(t_2,u_2,w_2,y_2,z_2)\|_B\\
&\qquad\le 2\lambda(|t_1-t_2|+\|u_1-u_2\|_B+\|w_1-w_2\|_B+\|y_1-y_2\|_B\\
&\qquad+\|z_1-z_2\|_B). \qquad (6.7)
\end{aligned}$$

Since (H1)–(H3) are satisfied, by Theorem 5.1, if $2\lambda T\Big(\frac{5}{2}+\frac{2T\sqrt{T}}{3\sqrt{\pi}}+\frac{5T^{\frac{4}{5}}}{2\Gamma(\frac{4}{5})}\Big)<1$, then (6.1)–(6.3) has a unique solution.

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