Deformation theory via differential graded Lie algebras

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This paper concerns the basic philosophy that, over a field of characteristic 0, every deformation problem is governed by a differential graded Lie algebra (DGLA) via solutions of Maurer-Cartan equation modulo gauge action. The classical approach (Grothendieck-Mumford-Schlessinger) to infinitesimal deformation theory is described by the procedure (see e.g. [2])

Deformation problem \( \leadsto \) Deformation functor

The above picture is rather easy and suffices for many applications; it is however clear that in this way we forget information which can be useful. The other classical approach, which consider categories fibred in groupoids instead of deformation functors, is not much better. A possible and useful way to preserve information is to consider a factorization

Deformation problem \( \leadsto \) DGLA \( \leadsto \) Deformation functor

where by DGLA we mean a differential graded Lie Algebra depending from the data of the deformation problem and the arrow DGLA \( \leadsto \) Deformation functor is a well defined, functorial procedure explained in Section 3 of this notes. Moreover we prove that every quasiisomorphism of differential graded Lie algebras induces an isomorphism of deformation functors.

Given a deformation problem, in general it is not an easy task to find a factorization as above; some general technics of this “art” (cf. [12, p. 5]), will be discussed elsewhere. Here we only point out that in general the correct DGLA is only defined up to quasiisomorphism and then this note represents the necessary background for the whole theory. Although the interpretation of deformation problems in terms of solutions of Maurer-Cartan equation is very useful on its own, in many situation it is unavoidable to recognize that the category of DGLA is too rigid for a “good” theory. The appropriate way of extending this category will be the introduction of homotopy Lie algebras and \( L_\infty \)-algebras; these new objects will be described in next lectures.

The results of this paper are, more or less, known to experts; if some originality is present in this notes then it is only contained in the proofs.

1 Differential graded Lie algebras and the Maurer-Cartan equation

Unless otherwise specified we shall follow the notation of [11].

\( \mathbb{K} \) is a field of characteristic 0.

\( \text{Graded}^\mathbb{K} \) is the category of \( \mathbb{Z} \)-graded vector space over \( \mathbb{K} \).

\( [n] : \text{Graded}^\mathbb{K} \rightarrow \text{Graded}^\mathbb{K} \) is the shift operator, \( V[n]^i = V^{i+n} \); for example \( \mathbb{K}[1]^{-1} = \mathbb{K} \), \( \mathbb{K}[1]^i = 0 \) for \( i \neq -1 \).
If $V$ is a $\mathbb{Z}$-graded vector space and $v \in V$ is a homogeneous element we write $\overline{v} \in \mathbb{Z}/2\mathbb{Z}$ for the class modulo $2$ of the degree of $v$. 

**Definition 1.1.** A Differential graded Lie algebra (DGLA for short) $(L, [\cdot, \cdot], d)$ is the data of a $\mathbb{Z}$-graded vector space $L = \oplus_{i \in \mathbb{Z}} L^i$ together a bilinear bracket $[\cdot, \cdot]: L \times L \to L$ and a linear map $d: L \to L$ satisfying the following condition:

1. $[\cdot, \cdot]$ is homogeneous skewsymmetric; this means $[L^i, L^j] \subset L^{i+j}$ and $[a, b] + (-1)^{\overline{a}\overline{b}}[b, a] = 0$ for every $a, b$ homogeneous.

2. Every $a, b, c$ homogeneous satisfy the Jacobi identity $[[a, b], c] = [[a, b], c] + (-1)^{\overline{a}\overline{b}}[a, [b, c]]$.

3. $d(L^i) \subset L^{i+1}$, $d \circ d = 0$ and $d[a, b] = [da, b] + (-1)^{\overline{a}\overline{b}}[a, db]$. The map $d$ is called the differential of $L$.

Note that $L^0$ and $L^{\text{even}} = \oplus L^{2n}$ are Lie algebras in the usual sense.

**Definition 1.2.** A linear map $f: L \to L$ is called a derivation of degree $n$ if $f(L^i) \subset L^{i+n}$ and satisfies the graded Leibniz rule $f([a, b]) = [[f(a), b] + (-1)^{\overline{a}\overline{b}}[a, f(b)]$.

We note that if $a \in L^i$ then $ad(a): L \to L, ad(a)(b) = [a, b]$, is a derivation of degree $i$ and $d$ is a derivation of degree $1$.

By following the standard notation we denote by $Z^i(L) = \ker(d: L^i \to L^{i+1}), B^i(L) = \text{image}(d: L^{i+1} \to L^i), H^i(L) = Z^i(L)/B^i(L)$.

**Definition 1.3.** The Maurer-Cartan equation (also called the deformation equation) of a DGLA $L$ is

$$da + \frac{1}{2}[a, a] = 0, \quad a \in L^1.$$ 

There is also an obvious notion of morphisms of DGLA’s; every morphism of DGLA induces a morphism between cohomology groups. It is also evident that morphisms of DGLA preserves solutions of the Maurer-Cartan equation.

A quasiisomorphism is a morphism inducing isomorphisms in cohomology. Two DGLA’s are quasiisomorphic if they are equivalent under the equivalence relation generated by quasiisomorphisms.

The cohomology of a DGLA is itself a differential graded Lie algebra with the induced bracket and zero differential:

**Definition 1.4.** A DGLA $L$ is called Formal if it is quasiisomorphic to its cohomology DGLA $H^*(L)$.

**Exercise:** Let $D: L \to L$ be a derivation, then the kernel of $D$ is a graded Lie subalgebra.

**Exercise:** Let $L = \oplus L^i$ be a DGLA and $a \in L^i$. Prove that:

1. If $i$ is even then $[a, a] = 0$.

2. If $i$ is odd then $[a, [a, b]] = \frac{1}{2}[a, [a, b]]$ for every $b \in L$ and $[[a, a], a] = 0$. 

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In nature there exists several examples of DGLA, most of which govern deformation problems via Maurer-Cartan equation; here we present some of the most interesting ones.

**Example 1.5.** Every Lie algebra is a DGLA concentrated in degree 0.

**Example 1.6.** Let $A = \oplus A^i$ be an associative graded-commutative $\mathbb{K}$-algebra (this means that $ab = (-1)^{\sigma(a,b)}ba$ for $a, b$ homogeneous) and $L = \oplus L^i$ a DGLA. Then $L \otimes_K A$ has a natural structure of DGLA by setting:

$$(L \otimes_K A)^n = \oplus_i (L^i \otimes_K A^{n-i}), \quad d(x \otimes a) = dx \otimes a, \quad [x \otimes a, y \otimes b] = (-1)^{\sigma(x,y)}[x,y] \otimes ab.$$

In the next examples, when $M$ is a complex variety and $E$ a holomorphic vector bundle on $M$, we denote by $A^{p,q}(E)$ the sheaf of $C^\infty(p,q)$-differential forms with values in $E$.

**Example 1.7.** Let $E$ be a holomorphic vector bundle on a complex variety $M$. We have a DGLA $L = \oplus \Gamma(M, A^{0,p}(\text{End}(E)))[-p]$ with the Dolbeault differential and the natural bracket. More precisely if $e, g$ are local holomorphic sections of $\text{End}(E)$ and $\phi, \psi$ differential forms we define $d\phi = (\overline{\partial}\phi)e$, $[\phi\cdot e, \psi\cdot g] = \phi \wedge \psi[e, g]$.

**Example 1.9.** Let $T_M$ be the holomorphic tangent bundle of a complex variety $M$. The Kodaira-Spencer DGLA is defined as $KS(M) = \oplus \Gamma(M, A^{0,p}(T_M))[-p]$ with the Dolbeault differential; if $\xi_1, \ldots, \xi_n$ are local holomorphic coordinates we have $[\phi d\xi_i, \psi d\xi_j] = [\phi, \psi] d\xi_i \wedge d\xi_j$ for $\phi, \psi \in A^{0,0}(T_M)$, $I, J \in \mathbb{N} \{1, \ldots, n\}$.

**Example 1.10.** Let $\mathbb{K} = \mathbb{R}$ and $X$ be a smooth differentiable variety. The algebra of polyvector fields is given by

$$T_{\text{poly}}(X) = \oplus \Gamma(X, \bigwedge^{n+1} T_X)[-n], \quad n \geq 1$$

with zero differential and the Schouten-Nijenhuis bracket defined in the following way (cf. 1.12):

For every open subset $U \subset X$, every function $h \in T_{\text{poly}}^{-1}(U) = C^\infty(U)$ and every vector fields $\xi_0, \ldots, \xi_n, \zeta_0, \ldots, \zeta_m \in \Gamma(U, T_U)$ we set

$$[\xi_0 \wedge \cdots \wedge \xi_n, h] = \sum_{i=0}^n (-1)^{n-i} \xi_i(h)\xi_0 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_n$$

$$[\xi_0 \wedge \cdots \wedge \xi_n, \zeta_0 \wedge \cdots \wedge \zeta_m] = \sum_{i=0}^n \sum_{j=0}^m (-1)^{i+j}[\xi_i, \zeta_j] \wedge \xi_0 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_n \wedge \zeta_0 \wedge \cdots \wedge \hat{\zeta}_j \wedge \cdots \wedge \zeta_m.$$

**Example 1.11.** Let $A$ be an associative $\mathbb{K}$-algebra, the DGLA of Hochschild cochains is defined by

$$G = \oplus \text{Hom}_{\mathbb{K}}(A^{\otimes(n+1)}, A)[-n]$$

where by $\text{Hom}_{\mathbb{K}}$ we mean homomorphisms of $\mathbb{K}$-vector spaces. The differential is the usual differential of Hochschild cohomology: $\phi \in G^n$

$$(d\phi)(a_0 \otimes \cdots \otimes a_n) = a_0 \phi(a_0 \otimes \cdots \otimes a_{n+1}) +$$

$$+ (-1)^n \phi(a_0 \otimes \cdots \otimes a_n)a_{n+1} + \sum_{i=0}^n (-1)^i \phi(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1})$$

The bracket is the Gerstenhaber one:

$$[\phi, \psi] = \phi \circ \psi - (-1)^{\sigma(\phi,\psi)} \psi \circ \phi$$

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where the (non-associative) product ∘ is defined, for φ ∈ G^n and ψ ∈ G^m, by the formula
\[ \phi \circ \psi(a_0 \otimes \cdots \otimes a_{n+m}) = \sum_{i=0}^n (-1)^{im} \phi(a_0 \otimes \cdots \otimes a_{i-1} \otimes \psi(a_i \otimes \cdots \otimes a_{i+m}) \otimes a_{i+m+1} \otimes \cdots \otimes a_{n+m}). \]

**Example 1.12.** Let A be a commutative K-algebra and V ⊂ Der_K(A, A) an A-submodule such that [V, V] ⊂ V.
We can define a DGLA \( L = \oplus_{n \geq 1} L^n \), where
\[ L^{-1} = A, \quad L^0 = V, \ldots, L^n = \bigwedge_{A}^{n+1} V, \ldots \]
with zero differential and the bracket uniquely characterized by the properties:

1. \([,] : L^0 \times L^0 \to L^0\) is the usual bracket on \( V \).
2. If \( a \in L^0, f \in L^{-1} = A \) then \([a, f] = a(f) \in A\).
3. For every \( a \in L^n, b \in L^m, c \in L^h \)
\[ [a, b \wedge c] = [a, b] \wedge c + (-1)^{(n+1)m} b \wedge [a, c], \quad [a \wedge b, c] = a \wedge [b, c] + (-1)^{(m+1)n}[a, c] \wedge b. \]

**Exercise:** Prove that the bracket in 1.12 is well defined. (The unicity is obvious, the existence is easy when \( V \) is free; in the general case it is convenient to think \( L^n \) as the quotient of the K-vector space generated by \( \xi_0 \wedge \cdots \wedge \xi_n \), with \( \xi_i \in V \), by the subspace generated by the skew-symmetric \( A \)-multilinear relation).

Prove moreover that for every \( h \in L^{-1}, \xi_0, \ldots, \xi_n, \zeta_0, \ldots, \zeta_m \in L^0 \) we have (cf. 1.10)
\[ [\xi_0 \wedge \cdots \wedge \xi_n, h] = \sum_{i=0}^n (-1)^{n-i} \xi_i(h) \xi_0 \wedge \cdots \wedge \hat{\xi_i} \wedge \cdots \wedge \xi_n \]
\[ [\xi_0 \wedge \cdots \wedge \xi_n, \zeta_0 \wedge \cdots \wedge \zeta_m] = \sum_{i=0}^n \sum_{j=0}^m (-1)^{i+j} [\xi_i, \zeta_j] \xi_0 \wedge \cdots \wedge \hat{\xi_i} \wedge \cdots \wedge \xi_n \wedge \zeta_0 \wedge \cdots \wedge \hat{\zeta_j} \wedge \cdots \wedge \zeta_m. \]

For a better understanding of some of next topics it is useful to consider the following functorial construction. Given a DGLA \((L, [,], d)\) we can construct a new DGLA \((L_d, [,], d_d)\) by setting \( L_d^i = L^i \) for every \( i \neq 1 \), \( L_d^1 = L^1 \oplus \mathbb{R}d \) with the bracket defined by extending by bilinearity the relation \([a + vd, b + wd]_d = [a, b] + vd(b) + (-1)^{b(a)}wd(a)\) and differential \( d_d(a + vd) = [d, a + vd]_d = d(a) \).

The natural inclusion \( L \subset L_d \) is a morphism of DGLA; in the manage of Maurer-Cartan equation it is convenient to consider the affine embedding \( \phi : L \to L_d, \phi(a) = a + d \).

For \( a \in L^1 \) we have
\[ d(a) + \frac{1}{2} [a, a] = 0 \iff [\phi(a), \phi(a)]_d = 0 \]

**Example 1.13.** (Small variations of almost complex structures).

Let \( M \) be a compact complex variety, \( A^{p,q} \) the vector bundle of differential forms of type \((p, q)\), \( A^{p,q} \) the fine sheaf of its smooth sections and \( KS(M) = \oplus \Gamma(M, A^{0,p}(T_M))[-p] \) the Kodaira-Spencer algebra of \( M \). Note that \( \Gamma(M, A^{0,p}(T_M)) \) is exactly the set of morphisms of vector bundles \( A^{1,0} \to A^{0,p} \). The choice of a hermitian metric on \( M \) induce a structure of pre-Hilbert space to \( \Gamma(M, A^{0,p}(T_M)) \). Given a sufficiently small section \( a \in \Gamma(M, A^{0,1}(T_M)) \) we can consider the perturbed differential \( \overline{\partial}_a = \overline{\partial} + a \partial : A^{0,p} \to A^{0,p+1} \); more precisely for \( U \subset M \) open subset with holomorphic coordinates \( z_1, \ldots, z_n \) and \( f \in C^\infty(U) \) we have
\[ \bar{\partial}_\alpha (fd\zeta_t) = (\bar{\partial} f + \alpha(\partial f)) \wedge d\zeta_t. \]

To \( \alpha \) we can associate also a direct sum decomposition

\[ A^{\alpha, 0} \oplus A^{\alpha, 1} = A^{1, 0} \oplus A^{0, 1} \]

where \( A^{\alpha, 0} \) is the graph of \(-\alpha\) and \( A^{\alpha, 1} = A^{\alpha, 1}_{KS} \). Therefore we can consider \( \alpha \) as a small variation of the almost complex structure, as easy computation (exercise) show that:

1. The sheaf of almost holomorphic functions of the almost complex structure \( A^{\alpha, j} \) is exactly the kernel of \( \bar{\partial}_\alpha : A^{0, 0} \to A^{0, 1} \).

2. \( \alpha \) satisfies the Maurer-Cartan equation in the Kodaira-Spencer algebra if and only if \( \bar{\partial}_\alpha^2 = 0 \). Therefore we can consider \( \alpha \) as a small variation of the almost complex structure, as easy computation (exercise) show that:

3. By Newlander-Nirenberg theorem the almost complex structure \( A^{\alpha, j} \) is integrable if and only if \( \alpha \) satisfies the Maurer-Cartan equation in \( KS(M) \).

Let’s now introduce the notion of gauge action on a DGLA:

There exists a functor \( \text{exp} \) from the category of nilpotent Lie algebras (i.e. with descending central series definitively \( = 0 \)) to the category of groups. For every nilpotent Lie algebra \( N \) there exists a natural bijection \( e: N \to \text{exp}(N) \) satisfying the following properties:

1. \( e^a e^b = e^{a+b} \) if \([a, b] = 0\). More generally \( e^a e^b = e^{(a+b)} \) where \( a \times b \in N \) is given by the Campbell-Baker-Hausdorff formula (cf \([1], [9], [10]\)).

2. For every vector space \( V \) and every homomorphism of Lie algebras \( \rho: N \to \text{End}(V) \) such that \( \rho(N) \) is a nilpotent subalgebra, the morphism

\[ \text{exp}(\rho): \text{exp}(N) \to \text{Aut}(V), \quad \text{exp}(\rho)(e^a) = e^{\rho(a)}, \]

is a homomorphism of groups (here \( e^{\rho(a)} \) denotes the usual exponential of endomorphisms).

3. If \( \rho: N \to \text{End}(V) = P \) is a representation of \( N \) as above and \( \text{ad}_\rho: N \to \text{End}(P) \) is the adjoint representation, then for every \( a \in N, f \in \text{End}(V) \)

\[ \text{exp}(\text{ad}_\rho)(e^a) f = e^{\rho(a)} \circ f \circ e^{-\rho(a)}. \]

**Exercise:** Prove the above items.

**Lemma 1.14.** Let \( V, W \) be vector spaces and \( \rho: N \to \text{End}(V), \eta: N \to \text{End}(W) \) representation of a nilpotent Lie algebra \( N \).

Let \( b: V \times V \to W \) be a bilinear symmetric form such that \( 2b(v, \rho(n)v) = \eta(n)b(v, v) \) for every \( n \in N, v \in V \) and \( q(v) = b(v, v) \) the associated quadratic form.

Then the cone \( Z = q^{-1}(0) \) is invariant under the exponential action \( \text{exp}(\rho) \).

**Proof.** Let \( n \in N \) be a fixed element, for every \( v \in V \) define the polynomial function \( F_v: \mathbb{K} \to W \) by

\[ F_v(t) = \exp(\eta(-tn)) q(\exp(\rho(tn))v) \]

For every \( s, t \in \mathbb{K} \), if \( u = \exp(\rho(sn))v \) then

\[ F_v(t + s) = e^{\eta(-sn)} F_w(t), \quad \frac{\partial F_v}{\partial t}(0) = -\eta(n) q(v) + 2b(v, \rho(n)v) = 0 \]

\[ \frac{\partial F_v}{\partial t}(s) = e^{\eta(-sn)} \frac{\partial F_w}{\partial t}(0) = 0 \]

As the field \( \mathbb{K} \) has characteristic \( 0 \) every function \( F_v \) is constant, in particular for every \( \exp(\eta) \) invariant subspace \( T \subset W \) the space \( q^{-1}(T) \) is \( \exp(\rho) \) invariant. \( \square \)
Corollary 1.15. Let \( L \) be a nilpotent DGLA, then the quadratic cone
\[
Z = \{ a \in L^1 \mid [v, v] = 0 \}
\]
is stable under the exponential of the adjoint action of \( L^0 \).

Proof. By Jacobi identity \([v, [a, v]] = -[v, [v, a]] = \frac{1}{2} [a, [v, v]] = 0\) and we can apply 1.14. □

Exercise: In the notation of 1.14, if \( K \) is algebraically closed and \( q^{-1}(0) \) is not a double plane then 1.14 holds under the weaker assumption \( b(v, \rho(n)v) = 0 \) for every \( n \in N, v \in Z \).

Again if \( L \) is a DGLA with \( L^0 \) nilpotent, the adjoint action of \( L^0 \) over \( L^1 \) preserves the cone \([v, v], d = 0\) and the affine hyperplane \( \{v + d \mid v \in L^1\}\); this gives an action of \( \exp(L^0) \) over \( L^1 \) preserving the solution of the Maurer-Cartan equation. The infinitesimal generator of this action is given by \( L^0 \ni a \mapsto [a, v + d], d = [a, v] - d(a) \in T_v L^1 \).

Remark. It is often convenient to think the elements of \( L_d \) as operators on a \( Z \)-graded vector space \( V = \oplus V^i \), this means that the map \( \rho: L_d \to \text{End}(V) = P \) is a morphism of DGLA.

For example the elements of the extended Kodaira-Spencer algebra
\[
KS(M)_d = \oplus \Gamma(M, A^0, p(T_M))[[-p]] \oplus \mathcal{C}F
\]
act in a natural way on the graded vector space \( V = \oplus \Gamma(M, A^0, p)[[-q]] \).

If we call \( \text{ad}_p: L_d \to \text{End}(P) \) and \( \text{Ad}_p: GL(L_d) \to GL(P) \) the the adjoint actions
\[
\text{ad}_p(a)B = [\rho(a), B] = \rho(a)B - B\rho(a), \quad \text{Ad}_p(e^a)B = e^{\rho(a)}Be^{-\rho(a)}
\]

by the properties of \( \exp \) we have \( \exp(\text{ad}_p(a)) = \text{Ad}_p(\exp(a)) \).

It is natural to consider two operator \( A, B \in \text{End}(V) \) gauge equivalent if there exists \( a \in L^0 \) such that \( A = e^{\rho(a)}Be^{-\rho(a)} \).

Therefore if \( x, y \in L^1_d \) are gauge equivalent then \( \rho(x) \) is gauge equivalent to \( \rho(y) \), independently from the particular representation.

Example 1.16. A pair \( (L, d) \) is a Polarized graded algebra (PGA) if \( L = \oplus L^i, i \in \mathbb{Z} \), is a graded associative \( K \)-algebra, \( d \in L^1 \) and \( d^2 = 0 \).

The typical example is, for a fixed differential graded vector space \( V, d: V \to V[1] \), the algebra \( \text{End}(V) = \oplus L^i \), where \( L^1 \) is the space of morphisms \( V \to V[1] \) in the category \( \text{Graded}(K) \).

Every PGA admits natural structures of DGA (differential graded algebra) with differential \( \delta(a) = da - (-1)^p ad \) and DGLA with bracket \([a, b] = ab - (-1)^p ba\).

If \( L^0 \) is a nilpotent \( K \)-algebra then we can define the gauge action over \( L^1 \) in a characteristic free way. First we define a group structure on \( L^0 \) by setting \( g \ast h = g + h + gh \) and the gauge action
\[
g \ast v = v + \sum_{i=0}^{\infty} (-1)^i [g, d + v] g^i = v + \sum_{i=0}^{\infty} (-1)^i ([g, v] - \delta(g)) g^i.
\]

Exercise: Prove that in characteristic 0 the action of 1.16 is equivalent to the usual gauge action.

Exercise: Let \( L \) be a DGLA with differential \( d \), then the universal enveloping algebra of \( L_d \) is a polarized graded algebra.

In some cases (as in [11, p. 9]) it is useful to describe a DGLA as a suitable subset of a concrete PGA.
2 Quickstart guide to functors of Artin rings

In this section $\mathbb{K}$ is a fixed field of arbitrary characteristic. We denote by:
$\mathcal{Art}$ the category of local complete Noetherian rings with residue field $\mathbb{K}$.
$\mathcal{Art} \subset \hat{\mathcal{Art}}$ the full subcategory of Artinian rings.
For a given $S \in \mathcal{Art}$, $\mathcal{Art}_S$ is the category of Artinian $S$-algebras with residue field $\mathbb{K}$.
$\hat{\mathcal{Art}}_S$ the category of local complete Noetherian $S$-algebras with residue field $\mathbb{K}$.
$\mathcal{Set}$ the category of sets (in a fixed universe). $*$ is a fixed 1-point set.
$\mathcal{Grp}$ the category of groups. ($\emptyset$ is not a group)

**Definition 2.1.** A functor of Artin rings is a covariant functor $F: \mathcal{Art}_S \to \mathcal{Set}$, $S \in \hat{\mathcal{Art}}$ such that $F(\mathbb{K}) = *$.

The main interest to functors of Artin rings comes from deformation theory and moduli problems; from this point of view the notion of prorepresentability is one of the most important.

Given $R \in \hat{\mathcal{Art}}_S$ we define a functor $h_R: \mathcal{Art}_S \to \mathcal{Set}$ by setting $h_R(A)$ as the set of $S$-algebra homomorphisms $R \to A$.

**Definition 2.2.** A functor $F: \mathcal{Art}_S \to \mathcal{Set}$ is prorepresentable if it is isomorphic to $h_R$ for some $R \in \hat{\mathcal{Art}}_S$.

The functors of Artin rings $F: \mathcal{Art}_S \to \mathcal{Set}$, with their natural transformation form a category denoted by $\mathcal{Fun}_S$.
We left as an exercise to prove that the Yoneda functor $\hat{\mathcal{Art}}_0 \to \mathcal{Fun}_S$, $R \to h_R$, is fully faithful.

A necessary condition for a functor $F$ to be prorepresentable is homogeneity. We first note that on $\mathcal{Art}_S$ there exist fibre products

$$B \times_A C \rightarrow C \rightarrow A \quad (1)$$

Applying a functor $F \in \mathcal{Fun}_S$ to the cartesian diagram (1) we get a map

$$\eta: F(B \times_A C) \rightarrow F(B) \times_{F(A)} F(C)$$

**Definition 2.3.** The functor $F$ is homogeneous if $\eta$ is an isomorphism whenever $B \to A$ is surjective.

Since the diagram (1) is cartesian, every prorepresentable functor is homogeneous.

**Example 2.4.** Other examples of homogeneous functors are:

2.4.0) The trivial functor $F(A) = *$ For every $A \in \mathcal{Art}_S$.

2.4.1) Let $M$ be a flat $S$-module, define $\hat{M}: \mathcal{Art}_S \to \mathcal{Set}$ as $\hat{M}(A) = M \otimes_S m_A$, where $m_A \subset A$ is the maximal ideal.

2.4.2) Assume $\text{char} \mathbb{K} = 0$ and let $L^0$ be a Lie Algebra over $\mathbb{K}$. We can define a group functor $\exp(L^0): \mathcal{Art}_S \to \mathcal{Grp}$ by setting $\exp(L^0)(A) = \exp(L^0 \otimes m_A)$ the exponential of the nilpotent Lie algebra $L^0 \otimes m_A$.

2.4.3) Let $X \to \text{Spec}(S)$ be a flat scheme over $S$, we can define $\text{Aut}(X/S): \mathcal{Art}_S \to \mathcal{Grp}$ by setting $\text{Aut}(X/S)(A)$ as the group of automorphisms of $X_A = X \times_{\text{Spec}(S)} \text{Spec}(A)$, commuting with the projection $X_A \to \text{Spec}(A)$ which are the identity on $X_\mathbb{K}$. 
Proposition 2.6 (Schlessinger, [13]). Let $F \to S$ by the maximal ideal, and in particular of square zero (e.g., the algebra $K$-algebra given by $\mathbb{K}[t]/(t^2)$ with the trivial structure of $S$-algebra given by $S \to \mathbb{K} \to \mathbb{K}[\epsilon]$). More generally by $\epsilon$ and $\epsilon_i$ we will always mean indeterminates annihilated by the maximal ideal, and in particular of square zero (e.g., the algebra $\mathbb{K}[\epsilon]$ has dimension 2 and $\mathbb{K}[\epsilon_1, \epsilon_2]$ has dimension 3 as a $\mathbb{K}$-vector space).

Definition 2.7. A functor $F$ is called a deformation functor if:

1. $\eta$ is surjective whenever $B \to A$ is surjective.
2. $\eta$ is an isomorphism whenever $A = \mathbb{K}$.

The name comes from the fact that most functors arising in deformation theory are deformation functors.

Exercise: Let $X$ be a scheme over $\mathbb{K}$, the deformation functor of $X$ is defined as $\text{Def}_X: \text{Art}_K \to \text{Set}$, where $\text{Def}_X(A)$ is the set of isomorphism classes of commutative cartesian diagrams

$$
X \xrightarrow{i} X_A
\downarrow p
\text{Spec}(\mathbb{K}) \longrightarrow \text{Spec}(A).
$$

with $i$ closed embedding and $p_A$ flat morphism. Prove that $\text{Def}_X$ is a deformation functor.

We denote $\mathbb{K}[\epsilon] = \mathbb{K} \oplus \mathbb{K} \epsilon = \mathbb{K}[t]/(t^2)$ with the trivial structure of $S$-algebra given by $S \to \mathbb{K} \to \mathbb{K}[\epsilon]$. More generally by $\epsilon$ and $\epsilon_i$, we will always mean indeterminates annihilated by the maximal ideal, and in particular of square zero (e.g., the algebra $\mathbb{K}[\epsilon]$ has dimension 2 and $\mathbb{K}[\epsilon_1, \epsilon_2]$ has dimension 3 as a $\mathbb{K}$-vector space).

Proposition 2.6 (Schlessinger, [13]). Let $F$ be a deformation functor, the set $t_F = F(\mathbb{K}[\epsilon])$ has a natural structure of $\mathbb{K}$-vector space. If $\phi: F \to G$ is a morphism of deformation functors the map $\phi: t_F \to t_G$ is linear.

Proof. Let $\alpha \in \mathbb{K}$, the scalar multiplication by $\alpha$ is induced by the morphism in $\text{Art}_S$, $\hat{\alpha}: \mathbb{K}[\epsilon] \to \mathbb{K}[\epsilon]$ given by $\hat{\alpha}(a + b\epsilon) = a + \alpha b$. Since $t_F \times t_F = F(\mathbb{K}[\epsilon_1] \times \mathbb{K}[\epsilon_2])$ the sum is induced by the map $\mathbb{K}[\epsilon_1] \times \mathbb{K}[\epsilon_2] = \mathbb{K}[\epsilon_1, \epsilon_2] \to \mathbb{K}[\epsilon]$ defined by $a + b\epsilon_1 + c\epsilon_2 \mapsto a + (b+c)\epsilon$. \qed

Exercise: Prove that if $R \in \text{Art}_S$, $F = h_R$ then $t_F$ is isomorphic to the Zariski tangent space of $\text{Spec}(R)$; more generally if $R \in \text{Art}_S$ then the tangent space of $h_R \in \text{Fun}_S$ is equal to the $\mathbb{K}$-dual of $m_R/(m_R^2 + m_S R)$.

Exercise: Let $A \in \text{Art}_S$, $F \in \text{Fun}_S$. Prove that there exists a natural bijection between $F(A)$ and morphisms $\text{Mor}(h_A, F)$.

Definition 2.7. A morphism $\phi: F \to G$ in the category $\text{Fun}_S$ is called:

1. unramified if $\phi: t_F \to t_G$ is injective.
2. smooth if for every surjection $B \to A$ in $\text{Art}_S$ the map $F(B) \to G(B) \times_{G(A)} F(A)$ is also surjective.
3. étale if it is smooth and unramified.

Exercise: If $\phi: F \to G$ is smooth then $F(A) \to G(A)$ is surjective for every $A$.

Exercise: Let $\phi: F \to G$, $\psi: H \to G$ be morphisms of deformation functors. If $\phi$ is smooth and $H$ is prorepresentable then there exists a morphism $\tau: H \to F$ such that $\psi \circ \tau = \phi$.
**Definition 2.8.** A functor \( F \) is smooth if the morphism \( F \to \ast \) is smooth, i.e. if \( F(A) \to F(B) \) is surjective for every surjective morphism of \( \mathbb{S} \)-algebras \( A \to B \).

**Lemma 2.9.** A prorepresentable functor \( h_R, R \in \text{Art}_S \), is smooth if and only if \( R = S[[x_1, \ldots, x_n]] \).

**Proof.** Exercise, cf. [13].

**Exercise:** Let \( X \) be a scheme over \( \mathbb{K} \). Prove that if for every \( A \in \text{Art}_\mathbb{K} \) and every deformation \( X_A \to \text{Spec}(A) \) the functor \( \text{Aut}(X_A) \) is smooth then \( \text{Def}_X \) is homogeneous.

**Lemma 2.10.** Let \( \phi: F \to G \) be an unramified morphism of deformation functors. If \( G \) is homogeneous then \( \phi: F(A) \to G(A) \) is injective for every \( A \).

**Proof.** We prove by induction on the length of \( A \) that \( F(A) \to G(A) \) is injective. Let \( e \in A \) be an element such that \( em_A = 0 \) and \( B = A/e \); by induction \( F(B) \to G(B) \) is injective. We note that

\[
A \times_{\mathbb{K}} \mathbb{K}[e] \to A \times_B A, \quad (a, a + e) \mapsto (a, a + e)
\]

is an isomorphism of \( \mathbb{S} \)-algebras, in particular for every deformation (resp. homogeneous) functor \( F \) there exists a natural surjective (resp. bijective) map

\[
v: F(A) \times t_F \to F(A) \times_{F(B)} F(A)
\]

such that \( v(F(A) \times \{0\}) = \Delta \) is the diagonal. Let \( \xi, \eta \in F(A) \) such that \( \phi(\xi) = \phi(\eta) \); since \( \phi \) is injective over \( F(B) \) the pair \((\xi, \eta)\) belongs to the fibred product \( F(A) \times F(B) \to F(A) \) and there exists \( h \in t_F \) such that \( v(\xi, h) = (\xi, \eta) \). Therefore \( v(\phi(\xi), \phi(h)) = (\phi(\xi), \phi(\eta)) \in \Delta \) and since \( G \) is homogeneous \( \phi(h) = 0 \). By the definition of unramified morphism \( \phi: t_F \to t_G \) is injective and then \( h = 0, \xi = \eta \).

**Corollary 2.11.** Let \( \phi: F \to G \) be an étale morphism of deformation functors. If \( G \) is homogeneous then \( \phi \) is an isomorphism.

**Proof.** Evident.

Let \( \sim \) the equivalence relation on the category of deformation functors generated by the étale morphisms. It is an easy consequence (left as exercise) of the above results that \( h_R \sim h_T \) if and only if \( R, T \) are isomorphic \( \mathbb{S} \)-algebras.

By a small extension \( e \) in \( \text{Art}_S \) we mean an exact sequence

\[
e: \quad 0 \to M \to B \to A \to 0
\]

where \( \phi \) is a morphism in \( \text{Art}_S \) and \( M \) is an ideal of \( B \) annihilated by the maximal ideal \( m_B \). In particular \( M \) is a finite dimensional vector space over \( B/m_B = \mathbb{K} \).

**Exercise:** Let \( 0 \to M \to B \to A \to 0 \) be a small extension and \( F \) a deformation functor. Then there exists a natural transitive action of \( F(\mathbb{K} \oplus M) \) on every nonempty fibre of \( F(A) \to F(B) \). (Hint: look at the proof of 2.10).

**Definition 2.12.** Let \( F \) be a functor of Artin rings; an obstruction theory \((V, v_e)\) for \( F \) is the data of a \( \mathbb{K} \)-vector space \( V \), called obstruction space, and for every small extension in \( \text{Art}_S \)

\[
e: \quad 0 \to M \to B \to A \to 0
\]

of an obstruction map \( v_e: F(A) \to V \otimes_{\mathbb{K}} M \) satisfying the following properties:
1. If $\xi \in F(A)$ can be lifted to $F(B)$ then $v_e(a) = 0$.
2. (base change) For every morphism $\alpha : e_1 \to e_2$ of small extension, i.e. for every commutative diagram

\[
e_1 : \begin{array}{ccc}
0 & \to & M_1 \\
\downarrow & & \downarrow \\
0 & \to & M_2
\end{array} \quad \begin{array}{ccc}
B_1 & \to & A_1 \\
\downarrow & & \downarrow \\
\tilde{B}_2 & \to & \tilde{A}_2
\end{array} \quad \begin{array}{ccc}
\alpha_M & & \alpha_B \\
\downarrow & & \downarrow \\
\alpha_A & & \alpha_A
\end{array}
\]

we have $v_{e_2}(\alpha_A(a)) = (Id_V \otimes \alpha_M)(v_{e_1}(a))$ for every $a \in F(A_1)$.

**Exercise:** If $F$ is smooth then all the obstruction maps are trivial.

**Definition 2.13.** An obstruction theory $(V, v_e)$ for $F$ is called complete if the converse of item i) in 2.12 holds; i.e. the lifting exists if and only if the obstruction vanish.

Clearly if $F$ admits a complete obstruction theory then it admits infinitely ones; it is in fact sufficient to embed $V$ in a bigger vector space. One of the main interest is to look for the “smallest” complete obstruction theory.

**Definition 2.14.** A morphism of obstruction theories $(V, v_e) \to (W, w_e)$ is a linear map $\theta : V \to W$ such that $w_e = \theta v_e$ for every small extension $e$.

An obstruction theory $(O_F, ob_e)$ for $F$ is called universal if for every obstruction theory $(V, v_e)$ there exist an unique morphism $(O_F, ob_e) \to (V, v_e)$.

If a universal obstruction theory exists then it is unique up to isomorphisms. An important result is

**Theorem 2.15.** Let $F$ be a deformation functor, then there exists an universal obstruction theory $(O_F, ob_e)$ for $F$. Moreover the universal obstruction theory is complete and every element of the vector space $O_F$ is of the form $ob_e(\xi)$ for some principal extension $e: 0 \to K \to B \to A \to 0$

and some $\xi \in F(A)$.

**Proof.** This is quite long and not easy. The interested reader can found a proof in [5].

We note that if $F$ is not a deformation functor then in general $F$ doesn’t have any complete obstruction theory even if $F$ satisfies Schlessinger’s conditions H1, H2, H3 of [13].

**Example 2.16.** (The primary obstruction map, $\text{char} K \neq 2$). Let $(V, v_e)$ be a complete obstruction theory for a deformation functor $F$ and let $[,] : t_F \times t_F \to V$ be the obstruction map associated to the small extension

\[
0 \to K \to K[x, y]/(x^2, y^2) \to K[x, y]/(x^2, xy, y^2) \to 0
\]

(Note that $K[x, y]/(x^2, xy, y^2) = K[x]/(x^2) \times K[y]/(y^2)$). Then $[,]$ is a symmetric bilinear map (Exercise).

The substitution $\alpha(t) = x + y$ gives a morphism of small extensions

\[
\begin{array}{c c c c c}
0 & \to & K & \xrightarrow{t^2} & K[t]/(t^3) \\
0 & \to & K & \xrightarrow{xy} & K[x, y]/(x, y^2)
\end{array} \quad \begin{array}{c c c c c}
\alpha & & \alpha \\
\alpha & & \alpha
\end{array} \quad \begin{array}{c c c c c}
K[t]/(t^2) & \to & 0 \\
K[x, y]/(x, y^2) & \to & 0
\end{array}
\]

From this and base change axiom it follows that the obstruction of lifting $\xi \in t_F$ to $F(K[t]/(t^3))$ is equal to $\frac{1}{2}[\xi, \xi]$ “the quadratic part of Maurer-Cartan equation”!!
Let $\phi: F \to G$ be a morphism of deformation functors and $(V, v_e), (W, w_e)$ obstruction theories for $F$ and $G$ respectively; a linear map $\phi^\flat: V \to W$ is compatible with $\phi$ if $w_e \phi = \phi^\flat v_e$ for every small extension $e$.

**Proposition 2.17** (Standard smoothness criterion). Let $\nu: F \to G$ be a morphism of deformation functors and $(V, v_e) \stackrel{\nu}{\to} (W, w_e)$ a compatible morphism between obstruction theories. If $(V, v_e)$ is complete, $V \twoheadrightarrow W$ injective and $t_F \to t_G$ surjective then $\phi$ is smooth.

**Proof.** Let $e: 0 \to \mathbb{K} \to B \to A \to 0$ be a small extension and let $(a, b') \in F(A) \times_{G(A)} G(B)$; let $a' \in G(A)$ be the common image of $a$ and $b'$. Then $v_e(a') = 0$, as $a'$ lifts to $G(B)$, hence $v_e(a) = 0$ by injectivity of $\nu'$. Therefore $a$ lifts to some $b \in F(B)$. In general $b'' = \nu(b)$ is not equal to $b'$. However, $(b'', b') \in G(B) \times_{G(A)} G(B)$ and therefore $b''$ differs from $b'$ by the action of an element $v \in t_G (v$ need not be unique). As $t_F \to t_G$ is surjective, $v$ lifts to a $w \in t_F$; acting with $w$ on $b$ produces a lifting of $a$ which maps to $b'$, as required. \qed

The composition $(V, v_e \phi)$ is an obstruction theory for $F$ and therefore there exists a unique compatible map $O_F \to V$. In particular this happens for the universal obstruction of $G$. Therefore every morphism of deformation functors $F \to G$ induces linear morphisms both in tangent $t_F \to t_G$ and obstruction spaces $O_F \to O_G$. As an immediate consequence of 2.17 we have

**Proposition 2.18.** A morphism of deformation functors $\nu: F \to G$ is smooth if and only if $t_F \to t_G$ is surjective and $o(\nu): O_F \to O_G$ is injective. In particular $F$ is smooth if and only if $O_F = 0$.

**Proof.** One implication is contained in 2.17. On the other side, if the morphism is smooth then $t_F \to t_G$ is surjective; since every $x \in O_F$ is the obstruction to lifting some element and $(O_G, \phi_x)$ is complete we have $o(\nu)(x) = 0$ if and only if $x = 0$. \qed

**Exercise:** Let $F \stackrel{\phi}{\to} G \stackrel{\psi}{\to} H$ be morphisms of deformation functors. Prove that:

1. If $\phi, \psi$ are smooth then the composition $\psi \phi$ is smooth.
2. If $\psi \phi, \phi$ are smooth then $\psi$ is smooth.
3. If $\psi \phi$ is smooth and $t_F \to t_G$ is surjective then $\phi$ is smooth.

**Exercise:** If $\phi: F \to G$ is smooth then $o(\phi): O_F \to O_G$ is an isomorphism.

In most concrete cases it is very difficult to calculate the universal obstruction space, while it is easy to describe complete obstruction theories and compatible morphism between obstruction spaces.

Consider now the following situation. $F: \text{Art}_S \to \text{Set}$ a deformation functor, $G: \text{Art}_S \to \text{Grp}$ a group functor of Artin rings which is a smooth deformation functor. (A theorem in [5] asserts that smoothness is automatic if $S$ is a field of characteristic 0).

We assume that $G$ acts on $F$; this means that for every $A \in \text{Art}_S$ there exists an action $G(A) \times F(A) \to F(A)$, all these actions must be compatible with morphisms in $\text{Art}_S$.

In particular there exists an action $t_G \times t_F \to t_F$, denote by $\nu: t_G \to t_F$ the map $h \mapsto h * 0$.

**Lemma 2.19.** The map $\nu$ is linear and $t_G$ acts on $t_F$ by translations, $h * v = \nu(h) + v$.

**Proof.** Since the vector space structure on $t_F$ and $t_G$ is defined functorially by using morphisms in $\text{Art}_S$, it is easy to see that for every $a, b \in t_F$, $g, h \in t_G$, $t \in \mathbb{K}$ we have $(g + h) * (a + b) = (g \ast a) + (h \ast b), t(g \ast a) = (tg) \ast (ta)$. Setting $a = b = 0$ we get the linearity of $\nu$ and setting $a = 0, h = 0$ we have $g \ast b = (g \ast 0) + (0 \ast b) = \nu(g) + b$. \qed
Lemma 2.20. In the notation above the quotient functor $D = F/G$ is a deformation functor, $t_D = \text{Coker} \nu$, the projection $F \to D$ is smooth and for every obstruction theory $(V, v)$ of $F$ the group functor $G$ acts trivially on the obstruction maps $v_c$. In particular the natural map $O_F \to O_D$ is an isomorphism.

Proof. We left as exercise the (very easy) proof that $D$ is a deformation functor and that $F \to D$ is a smooth morphism. The statement about obstruction follow easily from 2.15 and 2.18. Since this result will be fundamental in Section 3, in order to make this lecture self-contained we give here an alternative proof which do not use the existence of $O_F, O_D$.

Let $a \in F(A), g \in G(A)$ and $e : 0 \to K \xrightarrow{j} B \xrightarrow{A} 0$ be a small extension. We need to prove that $v_c(a) = v_c(g * a)$.

Let $\pi_i : A \times_k A \to A, i = 1, 2$ be the projections, $\delta : A \to A \times_k A$ be the diagonal and

$$\nabla e : 0 \to K^{(i, j)} \xrightarrow{(i, j)} A \times_k A \xrightarrow{0} 0$$

where $C$ is the quotient of $B \times_k B$ by the ideal generated by $(j, j)$.

Let $c = \delta(a) \in F(A \times_k A)$ and let $\tau \in G(A \times_k A)$ such that $\pi_1(\tau) = 1, \pi_2(\tau) = g$, as $\delta$ lifts to a morphism $A \to C$ and $G$ is smooth we have that $c$ and $\tau$ lift to $F(C)$ and $G(C)$ respectively and therefore also $\tau * c$ lifts to $F(C)$. Since $\pi_1(\tau * c) = \pi_1(\tau) * \pi_1(c) = a$, $\pi_2(\tau * c) = \pi_2(\tau) * \pi_2(c) = g * a$.

On the other hand it is an easy consequence of the base change axiom that $0 = v_c(\tau * c) = v_c(\pi_1(\tau * c)) - v_c(\pi_2(\tau * c)) = v_c(a) - v_c(g * a)$. □

Let now $R \in \text{Art}_S$ e $a, b \in F(R)$, we define a functor $\text{Iso}(a, b) : \text{Art}_R \to \text{Set}$ by setting

$$\text{Iso}(a, b)(R) = \{ g \in G(A) | g * f(a) = f(b) \}.$$  

Proposition 2.21. If $F$ is homogeneous then $\text{Iso}(a, b)$ is a deformation functor with tangent space $\ker \nu$ and complete obstruction space $\text{Coker} \nu = t_D$.

Proof. For simplicity of notation let’s denote $H = \text{Iso}(a, b)$. Assume it is given a commutative diagram in $\text{Art}_K$

$$\begin{align*}
S & \xrightarrow{\alpha} A \\
\downarrow{\beta} & \downarrow{\delta} \\
B & \xrightarrow{\gamma} C.
\end{align*}$$

with $\gamma$ surjective and let $(g_1, g_2) \in H(B) \times_{H(C)} H(A)$; since $G$ is a deformation functor there exists $g \in G(B \times_k A)$ which lifts $(g_1, g_2)$. Let $a', b'$ be the images of $a, b$ on $F(B \times_k A)$, the elements $g * a'$ and $b'$ have the same images in $F(A), F((B)$, since $F$ is homogeneous it follows that $g \in H(B \times_k A)$; this proves that $H$ is a deformation functor.

It is a trivial consequence of Lemma 2.19 that $t_H = \ker \nu$.

Let consider a small extension in $\text{Art}_S$

$$0 \to K \xrightarrow{e} A \xrightarrow{A} 0$$

and $g \in H(B)$, we want to determine the obstruction to lifting $g$ to $H(A)$, by assumption $G$ is smooth and then there exists $g' \in G(A)$ lifting $g$. Let $a', b' \in F(A)$ be the images of $a, b$, then

$$(g' * a', b') \in F(A) \times_{F(B)} F(A) = F(A \times_B A) = F(A) \times t_F$$

It is easy to see that, changing $g'$ inside the liftings of $g$, the projection over $t_F$ change by an element of the image of $\nu$, therefore the natural projection of the pair $(g' * a', b')$ into $\text{Coker} \nu$ gives a complete obstruction (some details are left to the reader). □
The above construction of the obstruction theory for \( Iso(a, b) \) is natural, in particular if \( G' \) is a smooth group functors acting on a homogeneous functors \( F' \) and \( G \to G' \), \( \phi : F \to F' \) are compatible morphisms then the induced morphism \( \text{Coker} \nu \to \text{Coker} \nu' \) is a morphism of obstruction spaces compatible with the morphisms \( Iso(a, b) \to Iso(\phi(a), \phi(b)) \). We have therefore the following corollary.

**Corollary 2.22.** In the above situation if \( \text{ker} \nu \to \text{ker} \nu' \) is surjective and \( \text{Coker} \nu \to \text{Coker} \nu' \) is injective then \( F/G \to F'/G' \) is injective.

**Proof.** (cf. [13]) We need to prove that, given \( a, b \in F(A) \), if \( Iso(\phi(a), \phi(b))(A) \neq \emptyset \) then \( Iso(a, b)(A) \neq \emptyset \), by standard criterion of smoothness the morphism \( Iso(a, b) \to Iso(\phi(a), \phi(b)) \) is smooth, in particular \( Iso(a, b)(A) \to Iso(\phi(a), \phi(b))(A) \) is surjective. \( \square \)

**Remark.** The hypothesis \( F \) homogeneous in Proposition 2.21 can be weakened by taking \( F \) deformation functor and assuming also the existence of a complete relative obstruction theory for the morphism \( G \to F, g \mapsto g * F(\mathbb{K}) \), see [5]. This last condition seems usually verified in concrete cases, we only know counterexamples in positive characteristic.

**Exercise:** In the same notation of Corollary 2.22, if in addition \( G' \) is homogeneous and \( \text{ker} \nu \to \text{ker} \nu' \) is bijective then \( G \times F \to G' \times F' \) is a fully faithful morphism of groupoids.

### 3 Deformation functors associated to a DGLA

Let \( L = \oplus L^i \) be a DGLA over a field \( \mathbb{K} \) of characteristic 0, we can define the following three functors:

1. The Gauge functor. \( G_L : \text{Art}_\mathbb{K} \to \text{Group} \), defined by \( G_L(A) = \exp(L^0 \otimes m_A) \). It is immediate to see that \( G_L \) is smooth and homogeneous.

2. The Maurer-Cartan functor. \( MC_L : \text{Art}_\mathbb{K} \to \text{Set} \) defined by

   \[
   MC_L(A) = \{ x \in L^1 \otimes m_A \mid dx + \frac{1}{2}[x, x] = 0 \}.
   \]

   \( MC_L \) is a homogeneous functor. If \( L \) is abelian then \( MC_L \) is smooth.

3. Since \( L \otimes m_A \) is a DGLA with \( (L \otimes m_A)^0 \) nilpotent, by the result of Section 1 there exists an action of the group functor \( G_L \) over \( MC_L \), we call \( \text{Def}_{L} = MC_L/G_L \) the corresponding quotient. \( \text{Def}_{L} \) is called the deformation functor associated to the DGLA \( L \).

   In general \( \text{Def}_{L} \) is not homogeneous but it is only a deformation functor.

   It is also evident that every morphism \( \alpha : L \to N \) of DGLA induces morphisms of functors \( G_L \to G_N, MC_L \to MC_N \). These morphisms are compatible with the gauge action and therefore induce a morphism between the deformation functors \( \text{Def}_{L} \to \text{Def}_{N} \).

We are now ready to compute tangent and complete obstructions for the above functors.

a) \( G_L \) is smooth, its tangent space is \( G_L(\mathbb{K}[e]) = L^0 \otimes \mathbb{K}e \), where \( e^2 = 0 \).

b) The tangent space of \( MC_L \) is

\[
\text{t}_{MC_L} = \{ x \in L^1 \otimes \mathbb{K}e \mid dx + \frac{1}{2}[x, x] = 0 \} = Z^1(L) \otimes \mathbb{K}e
\]
If \( b \in L^0 \otimes \mathbb{K}e \) and \( x \in Z^1 \otimes \mathbb{K}e \) we have \( \exp(b)(x + d) = \exp(ad(b))(x + d) = x + d + [b, x + d] + db \) and therefore the action \( t_{G_L} \times t_{MC_L} \rightarrow t_{MC_L} \) it is given by \( (b, x) \mapsto x + db \).

Let’s consider a small extension in \( \text{Art}_K \)

\[ e: \quad 0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0 \]

and let \( x \in MC_L(B) = \{ x \in L^1 \otimes m_B \mid dx + \frac{1}{2}[x, x] = 0 \} \); we define an obstruction \( v_e(x) \in H^2(L \otimes J) = H^2(L) \otimes J \) in the following way:

First take a lifting \( \tilde{x} \in L^1 \otimes m_A \) of \( x \) and consider \( h = \tilde{d}x + \frac{1}{2}[\tilde{x}, \tilde{x}] \in L^2 \otimes J \); we have

\[ d\tilde{h} = \tilde{d}^2 \tilde{x} + [\tilde{d} \tilde{x}, \tilde{x}] = [h, \tilde{x}] - \frac{1}{2}[[\tilde{x}, \tilde{x}], \tilde{x}] \]

Since \( [L^2 \otimes J, L^1 \otimes m_A] = 0 \) we have \([h, \tilde{x}] = 0\), by Jacobi identity \([[\tilde{x}, \tilde{x}], \tilde{x}] = 0\) and then \( d\tilde{h} = 0\).

We define \( v_e(x) \) as the class of \( h \) in \( H^2(L \otimes J) = H^2(L) \otimes J \); the first thing to prove is that \( v_e(x) \) is independent from the choice of the lifting \( \tilde{x} \); every other lifting is of the form \( y = \tilde{x} + z, z \in L^1 \otimes J \) and then

\[ d\tilde{y} = \frac{1}{2}[y, y] = h + dz. \]

It is evident from the above computation that:

1. \((H^2, v_e)\) is a complete obstruction theory for the functor \( MC_L \).

2. If \( \phi: L \rightarrow M \) is a morphism of DGLA then the linear map \( H^2(\phi): H^2(L) \rightarrow H^2(M) \) is a morphism of obstruction spaces compatible with the morphism \( \phi: MC_L \rightarrow MC_N \).

Let’s compute the primary obstruction map \( v_2: Z^1(L) = t_{MC_L} \rightarrow H^2(L) \) associated to the small extension

\[ 0 \rightarrow K[t]/(t^2) \rightarrow K[t]/(t^3) \rightarrow 0 \]

Let \( x \in MC_L(\mathbb{K}[t]/(t^3)) = Z^1(L) \otimes \mathbb{K}t \) and let \( \tilde{x} \) be a lifting of \( x \) to \( L^1 \otimes (t)/(t^3) \), we can choose \( \tilde{x} \in Z^1(L) \otimes (t)/(t^3) \) and therefore the primary obstruction \( v_2(x) \) is the class of \( h = \tilde{d}x + \frac{1}{2}[\tilde{x}, \tilde{x}] = \frac{1}{2}[[\tilde{x}, \tilde{x}], \tilde{x}] \) into \( H^2(L) \otimes \mathbb{K}t^2 \).

**Exercise:** Assume either \( H^2(L) = 0 \) or \( [L^1, L^1] \subset B^2 \), then \( MC_L \) is smooth. If \( MC_L \) is smooth then \([Z^1, Z^1] \subset B^2\).

**Exercise:** If \([Z^1, Z^1] = 0 \) then \( MC_L \) is smooth.

We shall prove later that if \( L \) is formal then \( MC_L \) is smooth if and only if \([Z^1, Z^1] \subset B^2\).

c) The tangent space of \( Def_L \) is simply the quotient of \( Z^1(L) \) under the translation action of \( L^0 \) defined by the map \( d: L^0 \rightarrow Z^1(L) \) and then \( t_{Def_L} = H^1(L) \).

The projection \( MC_L \rightarrow Def_L \) is smooth and then induces an isomorphism between universal obstruction theories; by 2.20 we can define naturally a complete obstruction theory \((H^2(L), o_e)\) by setting \( o_e(x) = v_e(x') \) for every small extension \( e \) as above, \( x \in Def_L(B) \) and \( x' \in MC_L(B) \) a lifting of \( x \).

In particular the primary obstruction map \( o_2: H^1(L) \rightarrow H^2(L) \) is equal to \( o_2(x) = \frac{1}{2}[x, x] \).
Corollary 3.2. Let \( \phi: L \to M \) be a morphism of DGLA and denote by \( H^1(\phi): H^1(L) \to H^1(N) \) the induced maps in cohomology.

1. If \( H^1(\phi) \) is bijective and \( H^2(\phi) \) injective then the morphism \( \text{Def}_L \to \text{Def}_N \) is étale.

2. If, in addition to 1), the map \( H^0(L) \to H^0(N) \) is surjective then \( \text{Def}_L \to \text{Def}_N \) is an isomorphism.

Proof. In case 1) the morphism \( \text{Def}_L \to \text{Def}_M \) is bijective on tangent spaces and injective on obstruction spaces, by the standard smoothness criterion it is étale.

In case 2), since étale morphisms are surjective, it is sufficient to prove that, for every \( S \in \mathcal{A}rt_k \) the map \( \text{Def}_L(S) \to \text{Def}_N(S) \) is injective. Let \( a, b \in MC_L(S) \), as in Section 2 we define the functor

\[
\text{Iso}(a,b): \mathcal{A}rt_S \to \text{Set}, \quad \text{Iso}(a,b)(S, \eta \to A) = \{ g \in \exp(L \otimes m_A) \mid g \ast \eta(a) = \eta(b) \}
\]

Since \( MC_L \) is homogeneous, \( \text{Iso}(a,b) \) is a deformation functor with tangent space \( Z^0(L) \) and complete obstruction space \( H^1(L) \).

Let \( K_a: \mathcal{A}rt_S \to \text{Grp} \) the group functor defined by

\[
K_a(S, \eta \to A) = \exp(\{ [\eta(a), b] + db \mid b \in L^{-1} \otimes m_A \}) \subset \exp(L^0 \otimes m_A)
\]

The above definition makes sense since it is easy to see that \( \{ [\eta(a), b] + db \mid b \in L^{-1} \otimes m_A \} \) is a Lie subalgebra of \( L^0 \otimes m_A \). Since \( L^{-1} \otimes m_A \to L^{-1} \otimes m_B \) is surjective for every surjective \( A \to B \), the functor \( K_a \) is smooth. Moreover \( K_a \) is a subfunctor of \( \text{Iso}(a,a) \) and therefore acts by right multiplication on \( \text{Iso}(a,b) \). Again by the result of Section 2 the quotient functor \( \text{Iso}(a,b)/K_a \) is a deformation functor with tangent space \( H^0(L) \) and complete obstruction space \( H^1(L) \).

Now the proof follows as in 2.22; the morphism \( \text{Iso}(a,b)/K_a \to \text{Iso}(\phi(a), \phi(b))/K_{\phi(a)} \) is smooth and in particular surjective. \( \square \)

Corollary 3.2. Let \( L \to N \) be a quasiisomorphism of DGLA. Then the induced morphism \( \text{Def}_L \to \text{Def}_N \) is an isomorphism.

Proof. Evident. \( \square \)

Exercise: Let \( L \) be a formal DGLA, then \( \text{Def}_L \) is smooth if and only if the quadratic map \( [\cdot, \cdot] : H^1 \times H^1 \to H^2 \) is zero.

Example 3.3. Let \( N^1 \oplus B^1 = L^1 \) be a vector space decomposition and consider the DGLA \( N = \oplus N^i \) where

\[
\begin{align*}
N^i &= 0 & \text{for } i \leq 0 \\
N^1 &= N^1 \\
N^i &= L^1 & \text{for } i \geq 2
\end{align*}
\]

with bracket and differential induced from \( L \). Then the natural inclusion \( N \to L \) gives isomorphisms \( H^i(N) \to H^i(L) \) for every \( i \geq 1 \). In particular the morphism \( \text{Def}_N \to \text{Def}_L \) is étale; note that \( \text{Def}_N = MC_N \) is homogeneous.

Exercise: Let \( z \in MC_L(S), S \in \mathcal{A}rt_k \) and consider the functor \( P_z: \mathcal{A}rt_S \to \text{Set} \) given by

\[
P_z(A) = \{ x \in L^1 \otimes m_A \mid dx + [z,x] = 0 \}.
\]

Compute tangent and obstruction spaces of \( P_z \).
Remark. The abstract $T^1$-lifting theorem (cf. [6]) implies that if the functor $P_z$ is smooth (informally this means that the linear operator $d + ad(z)$ has constant rank) for every $n > 0$ and every $z \in MC_L(\mathbb{K}[t]/(t^n))$ then $MC_L$ is also smooth.

**Corollary 3.4.** Let $L$ be a DGLA; if $H^0(L) = 0$ then $Def_L$ is homogeneous.

**Proof.** It is sufficient to apply 3.1 at the Example 3.3. □

**Example 3.5.** (The simplest example of deformation problem governed by a DGLA)
Let $V$ be a $\mathbb{K}$-vector space and $J: V \to V$ linear such that $J^2 = -I$ (if $\mathbb{K} = \mathbb{R}$ then $J$ is a complex structure on $V$). Let $M = \text{Hom}(V, V)$ and $d, \delta: M \to M$ defined by

$$d(A) = JA + AJ, \quad \delta(A) = JA - AJ, \quad A \in M$$

Since $\text{char}\mathbb{K} \neq 2$ it is easy to prove that $\ker d = \text{Image}(\delta)$, $\text{Image}(d) = \ker \delta$.

Consider the DGLA $L = \oplus_{n \geq 1} M[-n]$ with differentials

$$0 \to M[-1] \xrightarrow{d} M[-2] \xrightarrow{\delta} M[-3] \xrightarrow{d} M[-4] \xrightarrow{\delta} \ldots$$

and the bracket defined by

$$[A, B] = AB - (-1)^{\mathbb{T}A} BA$$

Clearly $H^i(L) = 0$ for every $i \neq 1$ and then the associated functor $Def_L = MC_L$ is smooth homogeneous with tangent space $\ker d$.

The elements of $MC_L(A)$ are in natural bijection with the deformation of the solution of $J^2 = -I$ over the Artinian ring $A$; in fact given $H \in M \otimes m_A$ we have $H \in MC_L(A)$ if and only if $(J + H)^2 = -I$. Since the gauge group is trivial, this example makes sense over arbitrary fields of $\text{char}\mathbb{K} \neq 2$.

## 4 The Kuranishi map and the Kuranishi functor

By a well known theorem of Schlessinger [13], if $F: Art_\mathbb{K} \to Set$ is a deformation functor with finite dimensional tangent vector space then there exists a prorepresentable functor $h_R$ and an étale morphism $h_R \to F$.

In particular Schlessinger theorem applies to the functor $Def_L$ for every DGLA $L$ such that $H^1(L)$ is finite dimensional. In this section we give a explicit construction of a prorepresentable functor $h_R$ and of an étale map $h_R \to Def_L$ by introducing Kuranishi maps.

Let $L = \oplus L^i$ be a DGLA, for every $i \in \mathbb{Z}$ choose direct sum decomposition

$$Z^i = B^i \oplus \mathcal{H}^i, \quad L^i = Z^i \oplus C^i$$

Let $\delta: L^{i+1} \to L^i$ be the linear map composition of

1. The projection $L^{i+1} \to B^{i+1}$ of kernel $C^{i+1} \oplus \mathcal{H}^{i+1}$.
2. $d^{-1}: B^{i+1} \to C^i$, the inverse of $d$.
3. The inclusion $C^i \to L^i$.

We note that $x \in \mathcal{H}^i$ if and only if $dx = \delta x = 0$ and $d\delta + \delta d = Id - H$, where $H: L^1 \to \mathcal{H}^i$ is the projection of kernel $B^i \oplus C^i$.

Using the same notation of 2.4.1, for a vector space $V$ we call $\hat{V}: Art_\mathbb{K} \to Set$ the homogeneous functor $\hat{V}(A) = V \otimes m_A$. 

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Definition 4.1. The Kuranishi map $F : \hat{L}^1 \to \hat{L}^1$ is the morphism of functors given by

$$x \in L^1 \otimes m_A, \quad F(x) = x + \frac{1}{2} \delta_x = x \in L^1 \otimes m_A.$$  

Lemma 4.2. The Kuranishi map $F$ is an isomorphism of functors.

Proof. $F$ is clearly a morphism of functors, since $\hat{L}^1$ is homogeneous, by 2.11 it is sufficient to prove that $F$ is étale.

By construction $F$ is the identity on the tangent space of $\hat{L}^1$. Since $\hat{L}^1$ is smooth, the trivial obstruction theory $(0, v_c)$ is complete; by standard smoothness criterion 2.17 $F$ is étale. \hfill \square

Definition 4.3. The Kuranishi functor $Kur : \text{Art}_\mathbb{K} \to \text{Set}$ is defined by

$$Kur(A) = \{ x \in \mathcal{H}^1 \otimes m_A \mid H([F^{-1}(x), F^{-1}(x)]) = 0 \}.$$  

In other words $Kur$ is the kernel the morphism of homogeneous functors $q ; \hat{H}^1 \to \hat{H}^2$ induced by the map of formal pointed schemes $q ; \mathcal{H}^1 \to \mathcal{H}^2$ which is composition of: the inclusion $\mathcal{H}^1 \to L^1$, the inverse of $F$, the bracket $[,] : L^1 \to L^2$ and the projection $L^2 \to \mathcal{H}^2$.

Lemma 4.4. $Kur$ is homogeneous; if moreover the dimension of $H^1(L)$ is finite then $Kur = h_R$ is prorepresented by a $\mathbb{K}$-algebra $R \in \text{Art}_\mathbb{K}$ such that $\text{Spec}(R) = (q^{-1}(0), 0)$ as formal scheme.

Proof. Exercise (easy). \hfill \square

Lemma 4.5. Let $x \in \hat{L}^1(A), y \in \hat{L}^1(A), c \in \mathbb{K}$ such that $y = c\delta[y, x]$, then $y = 0$.

Proof. Exercise (easy). \hfill \square

Clearly $MC_L$ and $Kur$ can be considered as subfunctor of $\hat{L}^1$.

Proposition 4.6. The isomorphism $F : \hat{L}^1 \to \hat{L}^1$ induces an isomorphism

$$F : MC_L \cap (C^1 \otimes \mathcal{H}^1) \to Kur.$$  

Proof. If $x \in MC_L(A) \cap (C^1 \otimes \mathcal{H}^1) \otimes m_A$, then $dx + \frac{1}{2} [x, x] = 0$, $\delta x = 0$ and $H([x, x]) = H(-2dx) = 0$. Moreover we have:

$$[x, x] = [x, x] - H([x, x]) = d\delta_x + \delta dx = d\delta_x,$$

$$\delta F(x) = \delta x + \frac{1}{2} \delta^2 [x, x] = \delta x,$$

$$dF(x) = dx + \frac{1}{2} d\delta [x, x] = dx + \frac{1}{2} [x, x] = 0.$$  

This prove that $F(x) \in Kur(A)$. Conversely if $x \in L^1 \otimes m_A$ and $F(x) \in Kur(A)$ then $\delta x = \delta F(x) = 0$, $H([x, x]) = 0$. $dx + \frac{1}{2} [x, x] = dx + \frac{1}{2} (d\delta x + d\delta x) = d\delta x + \frac{1}{2} d\delta [x, x] = \frac{1}{2} d\delta [x, x]$. Therefore it is sufficient to prove that $\delta d[x, x] = 0$;

$$\delta d[x, x] = 2\delta dx + \delta[x, x] = -\delta[d\delta x, x] = -\delta [x, x] - H([x, x]) - \delta d(x, x)$$  

Since $H([x, x]) = 0$ and by Jacoby $[x, x] = 0$ we have

$$\delta d[x, x] = \delta d[x, x].$$  

and the proof follows from Lemma 4.5. \hfill \square
Proposition 4.6 says that, under the Kuranishi map $F$, the Kuranishi functor $Kur$ is isomorphic to $MC_N$, where $N = \oplus N^i$ is the DGLA defined by

\[
\begin{align*}
N^i &= 0 & \text{for } i \leq 0 \\
N^1 &= C^1 \oplus H^1 \\
N^i &= L^i & \text{for } i \geq 2
\end{align*}
\]

By Example 3.3 the morphism $MC_N \to Df_L$ is étale. In conclusion what we have proved is the following

**Theorem 4.7.** For every DGLA $L$ the morphism $Kur \xrightarrow{F^{-1}} MC_L \to Df_L$ is étale.

**Exercise:** In the notation of 3.3, if $H^1(L)$ is finite dimensional then the isomorphism class of $MC_N$ is independent from the choice of the complement $N^1$. (This is always true but a proof of this fact without assumptions on $H^1(L)$ requires a nontrivial application of the factorization theorem [5, 6.2]).

## 5 Homotopy equivalence versus gauge equivalence

We introduce here a new equivalence relation in the set of solutions of the Maurer-Cartan equation in a differential graded Lie algebra. Let $(L, [\cdot, \cdot], \delta), L = \oplus L^i$, be a fixed DGLA and let $\mathbb{K}[t, dt] = \mathbb{K}[t] + \mathbb{K}[t] dt$ be the differential graded algebra with $t$ of degree 0, $dt$ of degree 1, endowed with the expected differential $d(p(t) + q(t) dt) = \frac{dp(t)}{dt} dt$.

We denote by $\Omega = L \otimes_{\mathbb{K}} \mathbb{K}[t, dt] = \oplus \Omega^i$; it is a DGLA with $\Omega^i = L^i[t] \oplus L^{i-1}[t] dt$, differential $\delta_{\Omega}(a(t) + b(t) dt) = \delta a(t) + (-1)^{\gamma} \partial a(t) dt + \delta b(t) dt$ and bracket

\[
[a(t) + b(t) dt, p(t) + q(t) dt] = [a(t), b(t)] + [a(t), q(t)] dt + (-1)^{\gamma} \partial a(t) dt.
\]

For every $s \in \mathbb{K}$ there exists an evaluation morphism of DGLA $v_s: \Omega \to L, v_s(t) = s, v_s(dt) = 0$. In particular we have morphisms of functors of Artin rings $v_s: MC_\Omega \to MC_L$ which are left inverses of the natural inclusion $MC_L \subset MC_\Omega$.

**Lemma 5.1.** The morphism $v_s: MC_\Omega \to MC_L$ is smooth for every $s \in \mathbb{K}$.

**Proof.** We first give a more explicit description of the solutions of the Maurer-Cartan equation in $\Omega$. Take $\omega \in \Omega^1, \omega = a(t) + b(t) dt$, we have

\[
\delta_{\Omega} \omega + \frac{1}{2} [\omega, \omega] = \delta a(t) + \frac{1}{2} [a(t), a(t)] + \left( -\frac{\partial a(t)}{\partial t} + \delta b(t) + [a(t), b(t)] \right) dt
\]

and then $\omega \in MC_\Omega$ if and only if

1. $a(t) \in MC_L$ for every $t \in \mathbb{K}$, and
2. $\frac{\partial a(t)}{\partial t} = \delta b(t) + [a(t), b(t)]$.

Let $0 \to J \to A \xrightarrow{\pi} B \to 0$ be a small extension of Artin rings, $\omega_B = a_B(t) + b_B(t) dt \in MC_\Omega(B)$ and $a(s) \in MC_L$ such that $\pi(a_A(s)) = a_B(s)$. Let $\tilde{a}_A(t), b_A(t)$ be liftings of $a_B(t), b_B(t)$ in $L[t] \otimes m_A$ such that $\tilde{a}_A(s) = a_A(s)$; since $Jm_A = 0$ the bracket $[\tilde{a}_A(t), b_A(t)]$ is independent from the choice of the liftings. Setting

\[
\gamma(t) = \delta b_A(t) + [\tilde{a}_A(t), b_A(t)] - \frac{\partial \tilde{a}_A(t)}{\partial t} \in L^1[t] \otimes J
\]

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and 

\[ a_A(t) = \tilde{a}_A(t) + \int_s^t \gamma(\tau) d\tau \]

it is evident that \( \omega_A = a_A(t) + b_A(t)dt \) satisfies the differential equation \( ii \) and \( v_s(\omega_A) = a_A(s) \). Moreover \( \delta a_A(t) + \frac{1}{2}[a_A(t),a_A(t)] \in L^2[t] \otimes J \) and then

\[
\frac{d}{dt}(\delta a_A(t) + \frac{1}{2}[a_A(t),a_A(t)]) = \delta \frac{da_A(t)}{dt} + \left[ \frac{da_A(t)}{dt},a_A(t) \right] = \\
= \delta[a_A(t),b_A(t)] + [\delta b_A(t) + [a_A(t),b_A(t)],a_A(t)] = \\
= \left[ -\frac{1}{2}[a_A(t),a_A(t)],b_A(t) \right] + [[a_A(t),b_A(t)],a_A(t)] = 0
\]

By assumption \( a_A(s) \in MC_L \) and then \( a_A(t) \in MC_L \) for every \( t \in \mathbb{K} \). \( \square \)

**Definition 5.2.** We shall say that \( x \in MC_L(A) \) is homotopic to \( y \in MC_L(A) \) if there exists \( \omega \in MC_{\Omega} \) such that \( x = v_0(\omega), y = v_1(\omega) \); it is easy to see that the relation “\( x \) is homotopic to \( y \)” is reflexive and symmetric (cf. exercise below). The equivalence relation generated is called homotopy equivalence.

**Exercise:** There exists a natural action of the group of affine isomorphisms of \( \mathbb{K} \) on \( MC_{\Omega} \). Let’s momentarily denote by \( F_L : Art_{\mathbb{K}} \to Set \) the quotient functor of \( MC_L \) by the homotopy equivalence (at the end of the trip \( F_L \) will be equal to \( Def_L \)).

**Proposition 5.3.** \( F_L \) is a deformation functor and the natural projection \( MC_L \to F_L \) is smooth.

**Proof.** Easy exercise (Hint: use that \( v_0, v_1 \) are smooth left inverses of the natural inclusion \( MC_L \subset MC_{\Omega} \)). \( \square \)

**Lemma 5.4.** The tangent space of \( F_L \) is the quotient of \( t_{MC_L} \) by the image of the linear map \( v_1 - v_0 : t_{MC_{\Omega}} \to t_{MC_L} \).

**Proof.** It is evident that if \( x \in t_{MC_L} \) is homotopy equivalent to \( y \) then \( x - y \) belongs to the image of \( v_1 - v_0 \). Conversely let \( x, \omega \in t_{MC_L}, h = v_1(\omega) - v_0(\omega) \) and \( y = x + y \). We can write \( x = v_0(x - v_0(\omega) + \omega), y = v_1(x - v_0(\omega) + \omega) \) and therefore \( x \) is homotopy equivalent to \( y \). \( \square \)

We note that \( \omega = a(t) + b(t)dt \in t_{MC_L} \) if and only if \( \delta a(t) = 0 \) and \( \left. \frac{da(t)}{dt} \right|_0 = \delta b(t) \); in particular

\[ v_1(\omega) - v_0(\omega) = a(1) - a(0) = \int_0^1 \delta b(t) dt \in B^1(L) \]

On the other hand for every \( b \in L^0, \delta(b)t + b dt \in t_{MC_{\Omega}} \) and therefore the image of \( v_1 - v_0 \) is exactly \( B^1(L) \).

**Theorem 5.5.** \( x \in MC_L(A) \) is homotopic to \( y \in MC_L(A) \) if and only if \( x \) is gauge equivalent to \( y \).
Before proving Theorem 5.5 we need some preliminary results about the exponential map:
let $N$ be a nilpotent Lie algebra and let $a \to e^a$ be its exponential map $N \to \exp(N)$. Given a polynomial $p(t) \in N[t]$ we have
\[ e^{p(t+h)} e^{-p(t)} = e^{h(p'(t)+\gamma_p(t)) + h^2 \eta_p(t,h)} \]
where $p'(t) = \frac{\partial p(t)}{\partial t}$ and $\gamma_p, \eta_p$ are polynomials determined exactly by Taylor expansion
\[ p(t + h) = p(t) + hp'(t) + h^2 \cdots \]
and Campbell-Baker-Hausdorff formula. For later use we point out that $\gamma_p(t)$ is a linear combination with rational coefficients of terms of the following types:
\[ ad(p(t))^{\alpha_1} ad(p'(t))^{\beta_1} \cdots ad(p(t))^{\alpha_n} ad(p'(t))^{\beta_n} Q \]
with $Q = p(t), p'(t)$ and $\sum \alpha_i + \sum \beta_i > 0$.

**Lemma 5.6.** In the notation above, for every $b(t) \in N[t]$ there exists an unique polynomial $p(t) \in N[t]$ such that $p(0) = 0$ and $p'(t) + \gamma_p(t) = b(t)$.

**Proof.** Let $N = N^{(1)} \supset N^{(2)} \supset \cdots \supset N^{(m)} = 0$ be the descending central series, $N^{(i+1)} = [N^{(i)}, N^{(i)}]$, and denote $N_i = N/N^{(i+1)}$. By construction $N_{i+1} \to N_i$ is a central extension for every $i \geq 0$.

The lemma is trivially true over $N_0$: assume that $p_i(t) \in N_i(t)$ is the unique solution of the differential equation $p_i'(t) + \gamma_i(t) = b(t)$ ($\gamma_i = \gamma_{p_i}$) on the vector space $N_i[t]$.

Let $\tilde{p}_{i+1}(t) \in N_{i+1}$ be a lifting of $p_i(t)$ such that $\tilde{p}_{i+1}(0) = 0$, as the kernel $K_{i+1}$ of the projection $N_{i+1} \to N_i$ is contained in the centre the polynomial $\gamma_{i+1} = \gamma_{\tilde{p}_{i+1}}$ does not depend from the choice of the lifting.

We introduce the polynomials
\[ \chi(t) = b(t) - \tilde{p}_{i+1}'(t) - \gamma_{i+1}(t) \in K_{i+1}[t] \]
\[ p_{i+1}(t) = \tilde{p}_{i+1}(t) + \int_0^t \chi(\tau) d\tau \in N_{i+1}[t] \]
It is clear that $p_{i+1}(t)$ is the unique solution of the differential equation. \hfill \Box

If $N \times V \overset{\cdot}{\to} V$ is a representation of $N$ into a vector space $V$, for every $v \in V$ we have
\[ \frac{d}{dt}(e^{p(t)}v) = [p'(t) + \gamma_p(t), e^{p(t)}v] \]
As an immediate consequence of 5.6 we have

**Proposition 5.7.** Let $N$ be a nilpotent Lie algebra, for every $b(t) \in N_i[t]$ there exists unique $p(t) \in N_i[t]$ such that $p(0) = 0$ and for every representation $N \times V \overset{\cdot}{\to} V$, for every $v \in V$ holds
\[ \frac{d}{dt}(e^{p(t)}v) + [b(t), e^{p(t)}v] = 0. \]

**Proof.** Evident. \hfill \Box

**Proof of 5.5.** The gauge equivalence and the notion of homotopy extends naturally to the space $V = (L^1 \otimes m_A) \oplus \mathbb{K}d$:

1. i) $x, y \in V$ are gauge equivalent if and only if there exists $p \in L^0 \otimes m_A$ such that $y = e^p x$.
2. ii) $x \in V$ is homotopic to $y \in V$ if and only if there exists $a(t) \in V[t], b(t) \in L^0 \otimes m_A[t]$ such that $a(0) = x, a(1) = y$ and $\frac{da(t)}{dt} + [b(t), a(t)] = 0$.  

\[ 20 \]
Now 5.5 follows from 5.7.

Remark. The notion of homotopy equivalence extends naturally to $L_{\infty}$-algebras see [4], while the gauge action requires the structure of DGLA; this motivates this section and the redundancy of proofs given here.

References


