Convergence of a Finite Volume Scheme for Nonlocal Conservation Laws in Several Space Dimensions

Aekta Aggarwal¹ Paola Goatin ² Rinaldo M. Colombo³

¹INRIA, Sophia Antipolis
²INRIA, Sophia Antipolis ³University of Brescia

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Introduction to the class of Nonlocal systems of conservation laws we aim at.
Outline

- Introduction to the class of Nonlocal systems of conservation laws we aim at.
- Construction of Lax-Friedrichs type of algorithms.
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- Application to Crowd Dynamics - Lanes Formation
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Introduction to the problem

System of Conservation Laws

\[ \partial_t U + \text{div}_x F(t, x, U) = 0 \]

\[ F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^N \rightarrow \mathbb{R}^{N \times d} \]

\[ t \quad x \quad U \quad \rightarrow \quad F(t, x, U) \]
Introduction to the problem

System of Conservation Laws

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Nonlocal systems

\[ \partial_t U + \text{div}_x F(t, x, U, \eta \ast U) = 0 \]

\[ F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^{N \times d} \]

\[ \eta \in (C^2 \cap W^{2, \infty})(\mathbb{R}^n; \mathbb{R}^{m \times N}) \text{ and need not have a compact support.} \]
Introduction to the problem

System of Conservation Laws

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Nonlocal systems

\[ \partial_t U + \text{div}_x F(t, x, U, \eta \ast U) = 0 \]

\[ F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^m \rightarrow \mathbb{R}^{N \times d} \]

- \( \eta \in (C^2 \cap W^{2,\infty})(\mathbb{R}^n; \mathbb{R}^{m \times N}) \) and need not have a compact support.
- Coupling – only due to nonlocal terms \( \eta \ast U \) with no coupling in \( U \).
Applications

- Crowd Dynamics: particularly suitable in describing the behavior of crowds, where each member moves according to her/his evaluation of the crowd density and its variations within her/his horizon.
- Vehicular Traffic
- Sedimentation
- Supply Chain Models
- Granular Matter
- Biological Applications
Existing Results

- **Kružkov Entropy Solutions**
  - Colombo, Herty, Mercier: ESAIM COCV, 2011

- **Measure Valued Solutions**
  - Crippa, Lécureux-Mercier: NODEA, 2011
  - Cristiani, Piccoli, Tosin: Multisc. Mod. & Simul., 2011
  - Piccoli, Tosin: ARMA, 2010
  - Piccoli, Rossi: ARMA, 2014

- **Convergence of Lax-Friedrichs type scheme, $d = 1, N = 1$:**
  - Betancourt, Bürger, Karlsen, Tory: Nonlinearity 2011
  - Blandin, Goatin: Preprint, 2014
  - Amorim, Colombo, Teixeira: To appear on ESAIM M2AN
Problem in Two Dimensions; Settings

Consider the $k$-th equation for $k = 1, \ldots, N$:

$$\partial_t U^k + \partial_x f^k(t, x, y, U^k, \eta \ast U) + \partial_y g^k(t, x, y, U^k, \vartheta \ast U) = 0$$
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$$

where

$$
U(t, x, y) = (U^1, \ldots, U^N)(t, x, y) \in \mathbb{R}^N,
$$
Problem in Two Dimensions; Settings

Consider the $k$-th equation for $k = 1, \ldots, N$:

$$\partial_t U^k + \partial_x f^k(t, x, y, U^k, \eta \ast U) + \partial_y g^k(t, x, y, U^k, \vartheta \ast U) = 0$$

where

$$U(t, x, y) = (U^1, \ldots, U^N)(t, x, y) \quad \in \mathbb{R}^N,$$

$$\eta(x, y) = \begin{bmatrix} \eta^{11} & \ldots & \eta^{1N} \\ \vdots & \ldots & \vdots \\ \eta^{m1} & \ldots & \eta^{mN} \end{bmatrix} (x, y) \quad \in \mathbb{R}^{m \times N},$$

$$\vartheta(x, y) = \begin{bmatrix} \vartheta^{11} & \ldots & \vartheta^{1N} \\ \vdots & \ldots & \vdots \\ \vartheta^{m1} & \ldots & \vartheta^{mN} \end{bmatrix} (x, y) \quad \in \mathbb{R}^{m \times N},$$

with $U^k(t, x, y), \vartheta^{\ell,k}(x, y), \nu^{\ell,k}(x, y) \in \mathbb{R}$. And
Definition of Convolutions

\[(\eta \ast U)(t, x, y) \in \mathbb{R}^m, (\vartheta \ast U)(t, x, y) \in \mathbb{R}^m, \text{ where for } \ell = 1, \ldots, m,\]

\[(\eta \ast U)_\ell(t, x, y) = \int \int_{\mathbb{R}^2} \sum_{k=1}^{N} \eta^{\ell,k}(x - x', y - y') \ U^k(t, x', y') \ dx' \ dy',\]

\[(\vartheta \ast U)_\ell(t, x, y) = \int \int_{\mathbb{R}^2} \sum_{k=1}^{N} \vartheta^{\ell,k}(x - x', y - y') \ U^k(t, x', y') \ dx' \ dy'.\]
Hypothesis

Let \( f^k = f^k(t, x, y, u, A) \), \( g^k = g^k(t, x, y, u, B) \) for \( k = 1, \ldots, N \).

(H0): Smoothness:

\[
f^k, g^k \in C^2(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^m; \mathbb{R});
\]
Hypothesis

Let $f^k = f^k(t, x, y, u, A)$, $g^k = g^k(t, x, y, u, B)$ for $k = 1, \ldots, N$.

(H0); Smoothness:

- $f^k, g^k \in C^2(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^m; \mathbb{R})$;
- $\partial_u f^k, \partial_u g^k \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^m; \mathbb{R})$;
Hypothesis

Let \( f^k = f^k(t, x, y, u, A), \ g^k = g^k(t, x, y, u, B) \) for \( k = 1, \ldots, N \).

(H0); Smoothness :

- \( f^k, g^k \in C^2(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^m; \mathbb{R}) \);
- \( \partial u f^k, \partial u g^k \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^m; \mathbb{R}) \);
- For all \( t \in \mathbb{R}_+, (x, y) \in \mathbb{R}^2 \) and \( A, B \in \mathbb{R}^m \),
  \( f^k(t, x, y, 0, A) = g^k(t, x, y, 0, B) = 0 \).
Introduction to the problem

Hypothesis

Let $f^k = f^k(t, x, y, u, A)$, $g^k = g^k(t, x, y, u, B)$ for $k = 1, \ldots, N$.

(H0); Smoothness:

- $f^k, g^k \in C^2(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^m; \mathbb{R})$;
- $\partial_u f^k, \partial_u g^k \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^m; \mathbb{R})$;
- For all $t \in \mathbb{R}_+$, $(x, y) \in \mathbb{R}^2$ and $A, B \in \mathbb{R}^m$,
  
  $f^k(t, x, y, 0, A) = g^k(t, x, y, 0, B) = 0$.

(H1); Uniform Bounds on Derivatives: There exists a constant $M > 0$:

\[
\max \left\{ \left| \partial_x f^k \right|, \left| \partial_y g^k \right|, \left\| \nabla_A f^k \right\|, \left\| \nabla_B g^k \right\| \right\} \leq M |u|,
\]

\[
\max \left\{ \left| \partial^2_{xx} f^k \right|, \left| \partial^2_{yy} g^k \right|, \left| \partial^2_{xy} f^k \right|, \left| \partial^2_{xy} g^k \right| \right\} \leq M |u|,
\]

\[
\max \left\{ \left\| \partial_x \nabla_A f^k \right\|, \left\| \partial_x \nabla_B g^k \right\| \right\} \leq M |u|,
\]

\[
\max \left\{ \left\| \partial_y \nabla_A f^k \right\|, \left\| \partial_y \nabla_B g^k \right\|, \left\| \nabla^2_{AA} f^k \right\|, \left\| \nabla^2_{BB} g^k \right\| \right\} \leq M |u|.
\]
Let $U_0 \in L^\infty(\mathbb{R}^2; \mathbb{R}^N)$. A map $U: [0, T] \to L^\infty(\mathbb{R}^2; \mathbb{R}^N)$ is a solution to (1) with initial datum $U_0$ if, for $k = 1, \ldots, N$, setting for all $w \in \mathbb{R}$

$$
\tilde{f}^k(t, x, y, w) = f^k(t, x, y, w, (\eta \ast U)(t, x, y)),
$$
$$
\tilde{g}^k(t, x, y, w) = g^k(t, x, y, w, (\vartheta \ast U)(t, x, y)),
$$

the map $U^k$ is a Kružkov solution to the conservation law

$$
\begin{cases}
\partial_t U^k + \partial_x \tilde{f}^k(t, x, y, U^k) + \partial_y \tilde{g}^k(t, x, y, U^k) = 0 \\
U^k(0, x, y) = U_0^k(x, y).
\end{cases}
$$

Above, by Kružkov solution we refer to the definition in Kružkov: Mat.Sb., 1970.
Construction of Algorithm

Numerical Grid: Cartesian Mesh

\[ x_i = i \Delta x, \quad x_{i+1/2} = \left( i + \frac{1}{2} \right) \Delta x, \quad i \in \mathbb{Z}; \]

\[ y_j = j \Delta y, \quad y_{j+1/2} = \left( j + \frac{1}{2} \right) \Delta y, \quad j \in \mathbb{Z}; \]
Construction of Numerical Algorithm

Choose a time step $\Delta t$ adapted according to CFL condition.

$$t^n = n \Delta t, \quad n \in \mathbb{Z} ;$$

$$\lambda_x = \Delta t / \Delta x ,$$

$$\lambda_y = \Delta t / \Delta y .$$

Let an initial datum $U_0 \in (L^\infty \cap BV)(\mathbb{R}^2; \mathbb{R}^N)$ be fixed. Define
Construction of Numerical Algorithm

Choose a time step $\Delta t$ adapted according to CFL condition.

$$t^n = n \Delta t, \quad n \in \mathbb{Z}; \quad \lambda_x = \Delta t / \Delta x, \quad \lambda_y = \Delta t / \Delta y.$$  

Let an initial datum $U_o \in (L^\infty \cap BV)(\mathbb{R}^2; \mathbb{R}^N)$ be fixed. Define

$$u_{ij}^{k,0} = \frac{1}{\Delta x \Delta y} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} u_o^k(x, y) \, dx \, dy \quad \text{for } i, j \in \mathbb{Z}.$$
Construction of Numerical Algorithm

Choose a time step $\Delta t$ adapted according to CFL condition.

$$t^n = n \Delta t, \ n \in \mathbb{Z} ; \quad \lambda_x = \Delta t / \Delta x , \quad \lambda_y = \Delta t / \Delta y .$$

Let an initial datum $U_o \in (L^\infty \cap BV)(\mathbb{R}^2; \mathbb{R}^N)$ be fixed. Define

$$u_{ij}^{k,0} = \frac{1}{\Delta x \Delta y} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} U_o^k(x, y) \, dx \, dy \quad \text{for } i, j \in \mathbb{Z} .$$

Define a piecewise constant approximate solution $u_{\Delta} \equiv (u_{\Delta}^1, \ldots, u_{\Delta}^N)$ to (1) by

$$u_{\Delta}^k(t, x, y) = u_{ij}^{k,n} \quad \text{for } \begin{cases} t \in [t^n, t^{n+1}], \\ x \in [x_{i-1/2}, x_{i+1/2}], \\ y \in [y_{j-1/2}, y_{j+1/2}], \end{cases} \quad \text{where } \begin{cases} n \in \mathbb{N} \\ i \in \mathbb{Z} \\ j \in \mathbb{Z} \\ k \in \{1, \ldots, n\} \end{cases}$$
through the 5-points algorithm based on dimensional splitting, (Crandall, Majda: Math. Comp., 1980)

\[ u_{ij}^{k,n+1/2} = u_{ij}^{k,n} - \lambda_x \left[ F_{i+1/2,j}^{k,n}(u_{ij}^{k,n}, u_{i+1,j}^{k,n}) - F_{i-1/2,j}^{k,n}(u_{i-1,j}^{k,n}, u_{ij}^{k,n}) \right] \]
Construction of Algorithm through the 5-points algorithm based on dimensional splitting, (Crandall, Majda: Math. Comp., 1980)

\[
\begin{align*}
    u_{ij}^{k,n+1/2} &= u_{ij}^{k,n} - \lambda_x \left[ F_{i+1/2,j}^{k,n}(u_{ij}^{k,n}, u_{i+1,j}^{k,n}) - F_{i-1/2,j}^{k,n}(u_{i-1,j}^{k,n}, u_{ij}^{k,n}) \right] \\
    u_{ij}^{k,n+1} &= u_{ij}^{k,n+1/2} - \lambda_y \left[ G_{i,j+1/2}^{k,n}(u_{ij}^{k,n+1/2}, u_{i,j+1}^{k,n+1/2}) - G_{i,j-1/2}^{k,n}(u_{i,j-1}^{k,n+1/2}, u_{ij}^{k,n+1/2}) \right]
\end{align*}
\]
through the 5-points algorithm based on dimensional splitting, (Crandall, Majda: Math. Comp., 1980)

\[ u_{ij}^{k,n+1/2} = u_{ij}^{k,n} - \lambda_x \left[ F_{i+1/2,j}^{k,n}(u_{ij}^{k,n}, u_{i+1,j}^{k,n}) - F_{i-1/2,j}^{k,n}(u_{i-1,j}^{k,n}, u_{ij}^{k,n}) \right] \]

\[ u_{ij}^{k,n+1} = u_{ij}^{k,n+1/2} - \lambda_y \left[ G_{i,j+1/2}^{k,n}(u_{ij}^{k,n+1/2}, u_{i,j+1}^{k,n+1/2}) - G_{i,j-1/2}^{k,n}(u_{i,j-1}^{k,n+1/2}, u_{ij}^{k,n+1/2}) \right] \]

where

\[ F_{i+1/2,j}^{k,n}(u, v) = \frac{f_{i+1/2,j}^{k,n}(u) + f_{i+1/2,j}^{k,n}(v)}{2} - \frac{\alpha(v - u)}{2 \lambda_x}, \]

\[ G_{i,j+1/2}^{k,n}(u, v) = \frac{g_{i,j+1/2}^{k,n+1/2}(u) + g_{i,j+1/2}^{k,n+1/2}(v)}{2} - \frac{\beta(v - u)}{2 \lambda_y}, \]

\[ f_{i+1/2,j}^{k,n}(u) = f^k(t^n, x_{i+1/2}, y_j, u, A_{i+1/2,j}^n) \]

\[ g_{i,j+1/2}^{k,n+1/2}(u) = g^k(t^{n+1/2}, x_i, y_{j+1/2}, u, B_{i,j+1/2}^{n+1/2}). \]

where for \( \alpha, \beta \) in \( [0, 2/3] \).
CFL condition

- \( \Delta t \) is chosen in order to satisfy the CFL condition

\[
\lambda_x \leq \frac{\min\{6\alpha, 4 - 6\alpha, 1\}}{1 + 6 \max_k \| \partial_u f^k \|_{L^\infty}}, \quad \lambda_y \leq \frac{\min\{6\beta, 4 - 6\beta, 1\}}{1 + 6 \max_k \| \partial_u g^k \|_{L^\infty}},
\]

with \( \Delta x, \Delta y \leq 1/(3M) \), where \( M \) is as in \((H1)\).
CFL condition

- $\Delta t$ is chosen in order to satisfy the CFL condition
  \[
  \lambda_x \leq \frac{\min\{6\alpha, 4 - 6\alpha, 1\}}{1 + 6 \max_k \| \partial_u f^k \|_{L_\infty}}, \quad \lambda_y \leq \frac{\min\{6\beta, 4 - 6\beta, 1\}}{1 + 6 \max_k \| \partial_u g^k \|_{L_\infty}}, \quad (1)
  \]
  with $\Delta x, \Delta y \leq 1/(3M)$, where $M$ is as in (H1).
- The convolution terms are computed through quadrature formulæ, i.e.,
  \[
  A^n_{i+1/2,j} = \Delta x \Delta y \left( \sum_{l,p \in \mathbb{Z}} \sum_{k=1}^N \eta_{i+1/2-l,j-p}^{s,k} u_{l+1/2,p}^{k,n} \right)_{1 \leq s \leq m},
  \]
  \[
  B^{n+1/2}_{i,j+1/2} = \Delta x \Delta y \left( \sum_{l,p \in \mathbb{Z}} \sum_{k=1}^N \vartheta_{i-l,j+1/2-p}^{s,k} u_{l,p+1/2}^{k,n+1/2} \right)_{1 \leq s \leq m}.
  \]
**CFL condition**

- $\Delta t$ is chosen in order to satisfy the CFL condition
  \[
  \lambda_x \leq \min\{6\alpha, 4 - 6\alpha, 1\} \frac{1}{1 + 6 \max_k \|\partial_u f^k\|_{L^\infty}}, \quad \lambda_y \leq \min\{6\beta, 4 - 6\beta, 1\} \frac{1}{1 + 6 \max_k \|\partial_u g^k\|_{L^\infty}},
  \]  
  (1)
  with $\Delta x, \Delta y \leq 1/(3M)$, where $M$ is as in (H1).
- The convolution terms are computed through quadrature formulæ, i.e.,
  \[
  A_{n,i+1/2,j+1/2} = \Delta x \Delta y \left( \sum_{l,p \in \mathbb{Z}} \sum_{k=1}^N \eta_{i+1/2-l,j-p}^{s,k} u_{k,n}^{l+1/2,p} \right)_{1 \leq s \leq m},
  \]
  \[
  B_{n,i+1/2,j+1/2} = \Delta x \Delta y \left( \sum_{l,p \in \mathbb{Z}} \sum_{k=1}^N \vartheta_{i-l,j+1/2-p}^{s,k} u_{k,n+1/2}^{l,p+1/2} \right)_{1 \leq s \leq m}.
  \]
Theorem 1

Let the smoothness and boundedness assumptions and CFL conditions hold. Fix $U_o \in (L^1 \cap L^\infty \cap BV)(\mathbb{R}^2; \mathbb{R}^N)$. Then, the algorithm defines a sequence of approximate solutions which converges, up to a subsequence, to a solution $U \in C^0 \left( \mathbb{R}^+; L^1(\mathbb{R}^2; \mathbb{R}^N) \right)$ and for all $k$ and for all positive $t$, we have

\[
\| U(t) \|_{L^\infty(\mathbb{R}^2;\mathbb{R}^N)} \leq e^{Ct} \| U_o \|_{L^\infty(\mathbb{R}^2;\mathbb{R}^N)} ,
\]

\[
\| U^k(t) \|_{L^1(\mathbb{R}^2;\mathbb{R})} = \| U^k_o \|_{L^1(\mathbb{R}^2;\mathbb{R})} ,
\]

\[
TV(U^k(t)) \leq e^{K_1 t} TV(U^k_o) + K_2 \left( e^{K_1 t} - 1 \right)
\]

\[
\| U(t + \tau) - U(t) \|_{L^1(\mathbb{R}^2;\mathbb{R}^N)} \leq C(t) \tau
\]
Idea of the Proof

Construct a sequence of piecewise approximate solutions $u_{\Delta}$ by the Lax-Friedrichs algorithm described above and obtain a strong BV estimate. Then the classical tools as in (Sanders: Math. Comp., 1983) or like in (Betancourt, Bürger, Karlsen, Tory: Nonlinearity 2011) allow to extract a convergent subsequence.

To prove this, we have the following Lemmas:
Throughout, let the smoothness, boundedness assumptions and CFL condition and $t \in \mathbb{R}_+$ and $(x, y) \in \mathbb{R}^2$. Then
Lemma 1 (Positivity)

\[ u^k_\Delta(t, x, y) \geq 0. \]
Lemma 1 (Positivity)

\[ u^k_\Delta(t, x, y) \geq 0. \]

\[ \Downarrow \]

Lemma 2 (\( L^1 \) bound)

\[ \left\| u^k_\Delta(t) \right\|_{L^1} = \left\| u^k_\Delta(0) \right\|_{L^1}. \]
Lemma 1 (Positivity)

\[ u^k_\Delta(t, x, y) \geq 0. \]

\[ \Downarrow \]

Lemma 2 (\( L^1 \) bound)

\[ \left\| u^k_\Delta(t) \right\|_{L^1} = \left\| u^k_\Delta(0) \right\|_{L^1}. \]

\[ \Downarrow \]

Lemma 3 (\( L^\infty \) bound)

\[ \left\| u_\Delta(t) \right\|_{L^\infty} \leq e^{C t} (1 + \left\| U_o \right\|_{L^1}) \left\| U_o \right\|_{L^\infty} \]

with \( C \) depending only on \( \eta, \vartheta, f^1, \ldots, f^N, g^1, \ldots, g^N \).
Lemma 4 (BV bound)

For all $n$, for all $t \in [t^n, t^{n+1}[\text{ and for all } k = 1, \ldots, N$

$$\sum_{ij} \left( |u_{i+1,j}^{k,n} - u_{ij}^{k,n}| \Delta y + |u_{i,j+1}^{k,n} - u_{ij}^{k,n}| \Delta x \right)$$

$$\leq e^{\mathcal{K}_1 t} \sum_{ij} \left( |u_{i+1,j}^{k,0} - u_{ij}^{k,0}| \Delta y + |u_{i,j+1}^{k,0} - u_{ij}^{k,0}| \Delta x \right) + \mathcal{K}_2 \left( e^{\mathcal{K}_1 t} - 1 \right)$$

with $\mathcal{K}_1$ and $\mathcal{K}_2$ depending only on $\|U_0\|_{L^1}, \eta, \vartheta, f^1, \ldots, f^N, g^1, \ldots, g^N$. 
Lemma 4 (BV bound)

For all $n$, for all $t \in [t^n, t^{n+1}]$ and for all $k = 1, \ldots, N$

$$
\sum_{ij} \left( |u_{i+1,j}^{k,n} - u_{ij}^{k,n}| \Delta y + |u_{i,j+1}^{k,n} - u_{ij}^{k,n}| \Delta x \right)
\leq e^{K_1 t} \sum_{ij} \left( |u_{i+1,j}^{k,0} - u_{ij}^{k,0}| \Delta y + |u_{i,j+1}^{k,0} - u_{ij}^{k,0}| \Delta x \right) + K_2 \left( e^{K_1 t} - 1 \right)
$$

with $K_1$ and $K_2$ depending only on $\|U_0\|_{L^1}, \eta, \vartheta, f^1, \ldots, f^N, g^1, \ldots, g^N$. 

Lemma 5 (Discrete entropy condition)

For all $k = 1, \ldots, N$ and for all $\kappa \in \mathbb{R}$ the discrete entropy inequality

$$
\left| u_{ij}^{k,n+1} - \kappa \right| - \left| u_{ij}^{k,n} - \kappa \right| + \\
\lambda_x \left( \Phi_{i+1/2,j}^{k,n,\kappa}(u_{ij}^{k,n}, u_{i+1,j}^{k,n}) - \Phi_{i-1/2,j}^{k,n,\kappa}(u_{i-1,j}^{k,n}, u_{ij}^{k,n}) \right) \\
+ \lambda_x \sgn(u_{ij}^{k,n+1/2} - \kappa) \left( f_{i+1/2,j}^{k,n}(\kappa) - f_{i-1/2,j}^{k,n}(\kappa) \right) \\
+ \lambda_y \left( \Gamma_{i,j+1/2}^{k,n,\kappa}(u_{ij}^{k,n+1/2}, u_{i,j+1}^{k,n+1/2}) - \Gamma_{i,j-1/2}^{k,n,\kappa}(u_{i,j-1}^{k,n+1/2}, u_{ij}^{k,n+1/2}) \right) \\
+ \lambda_y \sgn(u_{ij}^{k,n+1} - \kappa) \left( g_{i,j+1/2}^{k,n+1/2}(\kappa) - g_{i,j-1/2}^{k,n+1/2}(\kappa) \right) \leq 0.
$$

where the Kružkov numerical entropy flows (Amorim, Colombo, Teixeira: Preprint, 2013; Crandall, Majda, Math.Comp.1980) are given by:

$$
\Phi_{i+1/2,j}^{k,n,\kappa}(u_1, u_2) = f_{i+1/2,j}^{k,n}(u_1 \vee \kappa, u_2 \vee \kappa) - f_{i+1/2,j}^{k,n}(u_1 \wedge \kappa, u_2 \wedge \kappa),
$$

$$
\Gamma_{i,j+1/2}^{k,n,\kappa}(u_1, u_2) = g_{i,j+1/2}^{k,n+1/2}(u_1 \vee \kappa, u_2 \vee \kappa) - g_{i,j+1/2}^{k,n+1/2}(u_1 \wedge \kappa, u_2 \wedge \kappa).
$$
Lemma 6 ($L^1$ Lipschitz Continuity in time)

For any $n \in \mathbb{N}$ fixed, there exists a constant, for $k \in \{1, \ldots, N\}$ such that for all $n = 1, \ldots, n$, then

$$\| u_\Delta(\tau^{n+1}) - u_\Delta(\tau^n) \|_{L^1} \leq C \Delta t.$$ 

with $C$ depending on $\| U_0 \|_{L^1}$, $TV(U_0)$, $\tau^n$, $\lambda_x$, $\lambda_y$, $\alpha$, $\beta$ and on the functions $\eta$, $\varphi$, $f^k$ and $g^k$. 
Improvements from previous results

- The support of kernels need not be compact.
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Convergence of a finite volume scheme for $F(t, x, U), N = 1, d \geq 1$ with the assumption $\text{div}_x F = 0$ was given in Chinais-Hillairet: M2AN, 1999. This *Divergence Free* assumption is not needed for convergence.
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Convergence result for the Lax Friedrichs scheme and existence result for solutions for systems in multi dimensions for non-local conservation laws.

Ensures the existence of the solutions with flows more general than in the previous studies in the framework of crowd dynamics.
A Crowd Dynamics Sample Integration

Consider the crowd dynamics model introduced in (Colombo, Herty, Mercier: ESAIM COCV, 2011):

\[ \partial_t U + \nabla \cdot \left( U (1 - U) (1 - U * \mu) \vec{v} \right) = 0. \]

Lemma 7 (Fits into the Framework)

\[ N = 1, \quad m = 1, \quad f(t, x, y, U, A) = U (1 - U) (1 - A) v^1(x, y) \quad \eta = \mu \]
\[ m = 1, \quad g(t, x, y, U, B) = U (1 - U) (1 - B) v^2(x, y) \quad \vartheta = \mu. \]

Moreover, if \( v \in (C^2 \cap W^{2,\infty})(\mathbb{R}^2, \mathbb{R}^2) \) and \( \mu \in (C^2 \cap W^{2,\infty})(\mathbb{R}^2; \mathbb{R}) \), then the existence theorem applies to any initial datum in \( (L^1 \cap L^\infty \cap BV)(\mathbb{R}^2; [0, 1]) \).

The proof essentially relies on the invariance of the interval \([0, 1]\) for the density.
A Lax Friedrichs Scheme – Systems of Nonlocal Conservation Laws
\[ \Delta x = \Delta y = 0.0125, \Delta t \approx 9.62 \times 10^{-4}, \alpha = \beta = 0.3333 \]
Integration of *non-nonlocal* Analogue with standard Lax-Friedrichs scheme.

Integration of the Nonlocal analogue with $r = .4$
Numerical Integration

A Crowd Dynamics Sample Integration

<table>
<thead>
<tr>
<th>$h$</th>
<th>$|u_h - \frac{u_h}{2}|_{L^1}$</th>
<th>$\gamma$</th>
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<tr>
<td>0.05</td>
<td>0.6565</td>
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<tr>
<td>0.025</td>
<td>0.5389</td>
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<td>0.5572</td>
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<tr>
<td>0.00625</td>
<td>0.2623</td>
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</tbody>
</table>

Convergence rate $\gamma$

$$\gamma = \log_2 \left( \frac{\|u_h - \frac{u_h}{2}\|_{L^1}}{\|u_h - \frac{u_h}{4}\|_{L^1}} \right)$$
An open problem in the analytical theory is whether the formal convergence to its non–nonlocal analogue $\partial_t U + \text{div}_x F(t, x, U, U) = 0$ as $\eta$ tends to Dirac delta can be made rigorous.
An open problem in the analytical theory is whether the formal convergence to its non–nonlocal analogue \( \partial_t U + \text{div}_x F(t, x, U, U) = 0 \) as \( \eta \) tends to Dirac delta can be made rigorous.

Consider the classical Keyfitz-Kranzer system in 2 dimensions

\[
\begin{align*}
\partial_t U^1 + \partial_x \left( U^1 \varphi^1(U) \right) + \partial_y \left( U^1 \varphi^2(U) \right) &= 0 \\
\partial_t U^2 + \partial_x \left( U^2 \varphi^1(U) \right) + \partial_y \left( U^2 \varphi^2(U) \right) &= 0
\end{align*}
\]

- Ambrosio, Bouchut & De Lellis: Comm. PDE, 2004

with its non-local generalization

\[
\begin{align*}
\partial_t U^1 + \partial_x \left( U^1 \varphi_1(U * \mu) \right) + \partial_y \left( U^1 \varphi_2(U * \mu) \right) &= 0 \\
\partial_t U^2 + \partial_x \left( U^2 \varphi_1(U * \mu) \right) + \partial_y \left( U^2 \varphi_2(U * \mu) \right) &= 0
\end{align*}
\]
Lemma 8 (Fits into the framework)

\[
N = 2, \quad f(t, x, y, U, A) = U \varphi(A), \quad g(t, x, y, U, B) = U \varphi(B), \quad \eta = \vartheta = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}.
\]

Moreover, if \( \varphi_1, \varphi_2 \in (C^2 \cap W^{2,\infty})(\mathbb{R}^2; \mathbb{R}) \) and \( \mu \in (C^2 \cap W^{2,\infty})(\mathbb{R}^2; \mathbb{R}) \), then for any compactly supported initial datum \( (U_1^1, U_2^1) \in (L^1 \cap L^\infty \cap BV)(\mathbb{R}^2; \mathbb{R}^2_+) \), the existence theorem holds.
Take

\[ \varphi_1(A_1, A_2) = \sin(A_1^2 + A_2^2) \]
\[ \varphi_2(B_1, B_2) = \cos(B_1^2 + B_2^2) \]

\[ \tilde{\mu}(x, y) = \left( r^2 - (x^2 + y^2) \right)^3 \chi\{ (x, y): x^2 + y^2 \leq r^2 \} (x, y) \]

\[ \mu(x, y) = \frac{1}{\iint_{\mathbb{R}^2} \tilde{\mu}(x, y)} \tilde{\mu}(x, y) \]

so that \( \iint_{\mathbb{R}^2} \mu(x, y) \, dx \, dy = 1 \) with the initial data.
Integration of *non-nonlocal* Analogue with standard Lax-Friedrichs scheme.

Integration of the Nonlocal analogue with $r = .0125$
### Numerical Integration

**Non-Local Analogue to Local??**

<table>
<thead>
<tr>
<th>$t$</th>
<th>$r$</th>
<th>$u_1$</th>
<th>$u_2$</th>
</tr>
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<tr>
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<td>0.05</td>
<td>0.0375</td>
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<tr>
<td>0.10</td>
<td>0.2018</td>
<td>0.1641</td>
<td>0.1305</td>
</tr>
</tbody>
</table>

Difference in $L^1$ norms of solutions of Nonlocal and _non-nonlocal_ Analogue
Difference in $L^1$ norms of solutions of Nonlocal and non-nonlocal Analogue

These can not provide any proof of the convergence of the nonlocal to non-nonlocal problem, nevertheless they suggest that a positive answer may be possible.
Lane Formation

Consider the crowd dynamics model introduced in (Colombo, Mercier: AMS, 2011) for two populations trying to avoid each other and described by their densities $U^1, U^2$.

\[
\begin{align*}
\partial_t U^1 + \text{div} \left( c_1 U^1 (1 - U^1) \left( 1 - \varepsilon_1 \frac{U^1 \mu}{\sqrt{1 + \|U^1 \mu\|^2}} \right) \vec{v}^1(x, y) \right) &= 0, \\
\partial_t U^2 + \text{div} \left( c_2 U^2 (1 - U^2) \left( 1 - \varepsilon_1 \frac{U^2 \mu}{\sqrt{1 + \|U^2 \mu\|^2}} \right) \vec{v}^2(x, y) \right) &= 0,
\end{align*}
\]

\[\begin{align*}
\partial_t U^1 + \text{div} \left( -\varepsilon_2 \frac{\nabla U^2 \mu}{\sqrt{1 + \|\nabla U^2 \mu\|^2}} \right) &= 0, \\
\partial_t U^2 + \text{div} \left( -\varepsilon_2 \frac{\nabla U^1 \mu}{\sqrt{1 + \|\nabla U^1 \mu\|^2}} \right) &= 0.
\end{align*}\]
Lemma 9

System (2) fits into (1) setting, \( N = 2, m = 6 \) and

\[
\eta = \vartheta = \begin{bmatrix}
\mu & 0 \\
\mu_x & 0 \\
\mu_y & 0 \\
0 & \mu \\
0 & \mu_x \\
0 & \mu_y
\end{bmatrix},
\]

with \( A = U^1 \ast \eta, B = U^2 \ast \vartheta \in \mathbb{R}^6 \). Moreover, if \( \bar{v}^i \in (C^2 \cap W^{2,\infty})(\mathbb{R}^2; \mathbb{R}^2) \) and \( \mu \in (C^3 \cap W^{3,\infty})(\mathbb{R}^2; \mathbb{R}) \), then for any compactly supported initial datum \((U^1_o, U^2_o) \in (L^1 \cap L^\infty \cap BV)(\mathbb{R}^2; [0, 1]^2)\), problem (2) admits a solution defined for all \( t \in \mathbb{R}^+ \) and satisfying the estimates in Theorem 1.
Crossing of two moving crowds crossing each other radially
\[ \Delta x = \Delta y = 0.0125, \Delta t \approx 9.62 \times 10^{-4}, \alpha = \beta = 0.3333 \]
\Delta x = \Delta y = 0.0125, \Delta t \approx 9.62E - 4, \alpha = \beta = 0.3333, r = 20 \times 0.0125
People crossing vertically and horizontally

Vector Field $u_1$

Vector Field $u_2$
\[ \Delta x = \Delta y = 0.0125, \Delta t \approx 0.00125, \alpha = \beta = 0.3333, \ r = 10 \times 0.0125 \]
Crowd Dynamics

Measure Valued Non-Local Crowd Models
Piccoli, Tosin: ARMA, 2010
Crippa, Lécureux-Mercier: NODEA, 2011
Cristiani, Piccoli, Tosin: Multisc. Mod. & Simul., 2011
Piccoli, Rossi: ARMA, 2014

Vehicular Traffic/Supply Chain Models
Blandin, Goatin: Preprint, 2014
Colombo, Herty, Mercier: ESAIM COCV, 2011

Sedimentation
Betancourt, Bürger, Karlsen, Tory: Nonlinearity 2011

Granular Matter
Amadori, Shen: JHDE, 2012

Biological Applications
Perthame: Frontiers in Mathematics, 2007
THANK YOU!
Lemma 10 (Positivity)

\[ u^k_{\Delta}(t, x, y) \geq 0. \]
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\[ u^k_\Delta(t, x, y) \geq 0. \]

Write \( u^{k,n+1/2}_{ij} \) as a sum of
\[-\lambda_x \left( F^{k,n}_{i+1/2,j}(u^{k,n}_{ij}, u^{k,n}_{ij}) - F^{k,n}_{i-1/2,j}(u^{k,n}_{ij}, u^{k,n}_{ij}) \right) \]
and a convex combination of \( u^{k,n}_{ij}, u^{k,n}_{i-1,j} \) and \( u^{k,n}_{i+1,j}. \)
Lemma 10 (Positivity)

\[ u^k_\Delta(t, x, y) \geq 0. \]

- Write \( u_{ij}^{k,n+1/2} \) as a sum of
  \[ -\lambda_x \left( F_{i+1/2,j}^{k,n}(u_{ij}^{k,n}, u_{ij}^{k,n}) - F_{i-1/2,j}^{k,n}(u_{ij}^{k,n}, u_{ij}^{k,n}) \right) \] and a convex combination of \( u_{ij}^{k,n}, u_{i-1,j}^{k,n} \) and \( u_{i+1,j}^{k,n} \).
- Prove that coefficients of the convex combination are positive and less than 1/3 under (1).
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- Use Lagrange MVT and \((H1)\) to see that first term \( \geq -1/3\lambda_x u^{k,n}_{ij} \).
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Lemma 11 ($L^\infty$ bound)

$$\|u_\Delta(t)\|_{L^\infty} \leq e^{C t} (1 + \|U_0\|_{L^1}) \|U_0\|_{L^\infty}.$$  \hspace{1cm} (3)

with $C$ depending only on $\eta, \vartheta, f^1, \ldots, f^N, g^1, \ldots, g^N$

- Write $u_{ij}^{k, n+1/2}$ as a sum of
  $$-\lambda_x \left( F_{i+1/2, j}^{k, n} (u_{ij}^{k, n}, u_{ij}^{k, n}) - F_{i-1/2, j}^{k, n} (u_{ij}^{k, n}, u_{ij}^{k, n}) \right)$$
  and a convex combination of $u_{ij}^{k, n}, u_{i-1, j}^{k, n}$ and $u_{i+1, j}^{k, n}$.
- Use Lagrange MVT and (H1), to see that the modulus of the first term
  $$\leq \lambda_x M u_{ij}^{k, n} |\Delta x| \left( \|\partial_x \eta\|_{L^\infty} \|u_\Delta(t^n)\|_{L^1} + 1 \right).$$
Lemma 11 (L∞ bound)

\[ \| u_\Delta(t) \|_{L^\infty} \leq e^C t (1 + \| U_0 \|_{L^1}) \| U_0 \|_{L^\infty}. \]  

(3)

with C depending only on \( \eta, \vartheta, f^1, \ldots, f^N, g^1, \ldots, g^N \)

- Write \( u_{ij}^{k,n+1/2} \) as a sum of 
  \[ -\lambda_x \left( F_{i+1/2,j}^{k,n}(u_{ij}^{k,n}, u_{ij}^{k,n}) - F_{i-1/2,j}^{k,n}(u_{ij}^{k,n}, u_{ij}^{k,n}) \right) \]  
  and a convex combination of \( u_{ij}^{k,n}, u_{i-1,j}^{k,n} \) and \( u_{i+1,j}^{k,n} \).

- Use Lagrange MVT and (H1), to see that the modulus of the first term \( \leq \lambda_x M |u_{ij}^{k,n}| \Delta x \left( \| \partial_x \eta \|_{L^\infty} \| u_\Delta(t^n) \|_{L^1} + 1 \right) \).

- Use that coefficients are in convex combination, positive to bound the convex combination by \( \| u_\Delta(t) \|_{L^\infty} \) and use induction to get the result.
Lemma 12 (BV bound)

For all $n$, for all $t \in [t^n, t^{n+1}]$ and for all $k = 1, \ldots, N$

$$
\sum_{ij} \left( |u_{i+1,j}^{k,n} - u_{ij}^{k,n}| \Delta y + |u_{i,j+1}^{k,n} - u_{ij}^{k,n}| \Delta x \right)
\leq e^{K_1 t} \sum_{ij} \left( |u_{i+1,j}^{k,0} - u_{ij}^{k,0}| \Delta y + |u_{i,j+1}^{k,0} - u_{ij}^{k,0}| \Delta x \right) + K_2 \left( e^{K_1 t} - 1 \right).
$$

with $K_1$ and $K_2$ depending only on $\|U_o\|_{L^1}, \eta, \vartheta, f^1, \ldots, f^N, g^1, \ldots, g^N$. 
Lemma 12 (BV bound)

For all $n$, for all $t \in [t^n, t^{n+1}]$ and for all $k = 1, \ldots, N$

$$
\sum_{ij} \left( |u_{i+1,j}^{k,n} - u_{ij}^{k,n}| \Delta y + |u_{i,j+1}^{k,n} - u_{ij}^{k,n}| \Delta x \right) \\
\leq e^{K_1 t} \sum_{ij} \left( |u_{i+1,j}^{k,0} - u_{ij}^{k,0}| \Delta y + |u_{i,j+1}^{k,0} - u_{ij}^{k,0}| \Delta x \right) + K_2 \left( e^{K_1 t} - 1 \right).
$$

with $K_1$ and $K_2$ depending only on $\|U_o\|_{L^1}, \eta, \vartheta, f^1, \ldots, f^N, g^1, \ldots, g^N$.

- Obtain the bounds of
  $$
  \sum_{ij} \left( |u_{i+1,j}^{k,n+1/2} - u_{ij}^{k,n+1/2}| \Delta y + |u_{i,j+1}^{k,n+1/2} - u_{ij}^{k,n+1/2}| \Delta x \right)
  $$
  in terms of
  $$
  \left( |u_{i+1,j}^{k,n} - u_{ij}^{k,n}| \Delta y + |u_{i,j+1}^{k,n} - u_{ij}^{k,n}| \Delta x \right).
  $$
  Entirely analogous estimates at $t^{n+1}$ in terms of terms at $n + \frac{1}{2}$ lead to the result.
Lemma 12 (BV bound)

For all \( n \), for all \( t \in [t^n, t^{n+1}] \) and for all \( k = 1, \ldots, N \)

\[
\sum_{ij} \left( |u_{i+1,j}^{k,n} - u_{ij}^{k,n}| \Delta y + |u_{i,j+1}^{k,n} - u_{ij}^{k,n}| \Delta x \right)
\]

\[
\leq e^{K_1 t} \sum_{ij} \left( |u_{i+1,j}^{k,0} - u_{ij}^{k,0}| \Delta y + |u_{i,j+1}^{k,0} - u_{ij}^{k,0}| \Delta x \right) + K_2 \left( e^{K_1 t} - 1 \right). 
\]

with \( K_1 \) and \( K_2 \) depending only on \( \|U_o\|_{L^1}, \eta, \vartheta, f^1, \ldots, f^N, g^1, \ldots, g^N \).

- Obtain the bounds of
  \[
  \sum_{ij} \left( |u_{i+1,j}^{k,n+1/2} - u_{ij}^{k,n+1/2}| \Delta y + |u_{i,j+1}^{k,n+1/2} - u_{ij}^{k,n+1/2}| \Delta x \right) \text{ in terms of}
  \]
  \[
  \left( |u_{i+1,j}^{k,n} - u_{ij}^{k,n}| \Delta y + |u_{i,j+1}^{k,n} - u_{ij}^{k,n}| \Delta x \right). \text{ Entirely analogous}
  \]
  estimates at \( t^{n+1} \) in terms of terms at \( n + \frac{1}{2} \) lead to the result.
Write \( u_{i+1,j}^{k,n+1/2} - u_{i,j}^{k,n+1/2} = C_{ij}^{n} - \lambda x D_{ij}^{n} \) where
Write \( u_{i+1,j}^k - u_{ij}^k = C_{ij}^n - \lambda_x D_{ij}^n \) where

\[
C_{ij}^n \text{ is a convex combination of } (u_{i+1,j}^k - u_{ij}^k), (u_{i+2,j}^k - u_{i+1,j}^k) \text{ and } (u_{i,j}^k - u_{i-1,j}^k) \text{ with coefficients in } [0, 1/3] \text{ to get}
\]

\[
\sum_{ij} |C_{ij}^n| \leq \sum_{ij} |u_{i+1,j}^k - u_{ij}^k|.
\]
Write $u_{i+1,j}^{k,n+1/2} - u_{ij}^{k,n+1/2} = C_{ij}^n - \lambda_x D_{ij}^n$ where

$C_{ij}^n$ is a convex combination of $(u_{i+1,j}^{k,n} - u_{ij}^{k,n})$, $(u_{i+2,j}^{k,n} - u_{i+1,j}^{k,n})$ and $(u_{i,j}^{k,n} - u_{i-1,j}^{k,n})$ with coefficients in $[0, 1/3]$ to get

$$\sum_{ij} |C_{ij}^n| \leq \sum_{ij} |u_{i+1,j}^{k,n} - u_{ij}^{k,n}|.$$

$D_{i,j}$ is approximated using Lagrange MVT, (H1) and bounds on $A_{i+1/2,j}^n$ to obtain

$$\sum_{ij} \lambda_x |D_{ij}^n| \Delta y \leq \Delta t \left( K + K_1 \sum_{ij} |u_{i+1,j}^{k,n} - u_{ij}^{k,n}| \Delta y \right).$$

The sum of these two estimates gives the required estimate of

$$\sum_{ij} \left( |u_{i+1,j}^{k,n+1/2} - u_{ij}^{k,n+1/2}| \Delta y \right).$$

Analogous estimates for $\sum_{ij} \left( |u_{i,j+1}^{k,n+1/2} - u_{ij}^{k,n+1/2}| \Delta x \right)$ are proved.
Proof of Lemma 7. \( f \) and \( g \) are both of class \( C^2 \) in all their arguments. Let \( p \in C^3_c(\mathbb{R}; \mathbb{R}) \) be such that \( p(U) = U(1 - U) \) for all \( U \in [0, 1] \) and \( q \in C^3_c(\mathbb{R}; \mathbb{R}) \) be such that \( q(A) = 1 - A \) for all \( A \in [0, 1] \). Define

\[
\hat{f}(t, x, y, U, A) = p(U) q(A) v^1(x, y), \quad \hat{g}(t, x, y, U, B) = p(U) q(B) v^2(x, y).
\]

\( \hat{f} \) and \( \hat{g} \) both are of class \( C^2 \cap W^{3, \infty} \).

Then, Theorem 1 applies and yields a solution to

\[
\partial_t U + \nabla \cdot \left( p(U) q(U * \mu) v \right) = 0
\]

in the sense of definition of the solutions, \( u \) attains values inside \([0, 1]\), hence it also solves (2). \qed