A space-time Trefftz method for the second order wave equation

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Outline of the talk

Outline

1. Motivation
2. An interior-penalty space-time dG method
3. Damped wave equation
4. Numerical results

Acoustic wave equation

Find $u(t) \in H_0^1(\Omega)$, $t \in [0, T]$, s.t.

$$\begin{aligned} (\ddot{u}, v)_{L^2(\Omega)} + (a \nabla u, \nabla v)_{L^2(\Omega)} &= 0 && \text{for all } v \in H_0^1(\Omega), \\ u(x,0) = u_0(x),\ \dot{u}(x,0) &= v_0(x), && \text{in } \Omega. \end{aligned}$$

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$$u(x, 0) = u_0(x), \; \dot{u}(x, 0) = v_0(x), \quad \text{in } \Omega.$$

- Initial data $u_0 \in H_0^1(\Omega)$, $v_0 \in L^2(\Omega)$.
- $a(x)$ piecewise constant $0 < c_a < a(x) < C_a$.
- Unique solution exists with

$$u \in L^2([0, T]; H_0^1(\Omega)), \; \dot{u} \in L^2([0, T]; L^2(\Omega)), \; \ddot{u} \in L^2([0, T]; H^{-1}(\Omega))$$

$$u \in C([0, T]; H_0^1(\Omega)), \quad \dot{u} \in C([0, T]; L^2(\Omega)).$$

- For smooth enough solution we have the transmission conditions

$$u_j = u_k, \quad a_j \partial_{\mathbf{n}} u_j = a_k \partial_{\mathbf{n}} u_k, \qquad \text{on } \partial\Omega_j \cap \partial\Omega_k, u_j = u|_{\Omega_j}, u_k = u|_{\Omega_k},$$

where Ω_j and Ω_k are subsets of Ω with $a \equiv a_k$ in Ω_k and $a \equiv a_j$ in Ω_j.

How to discretize the wave equation?

The usual approach:

- Construct a spatial mesh and a corresponding spatially discrete space: locally polynomial, continuous or discontinuous across the boundaries of the spatial elements (the spatial skeleton).
- Finite difference approximation in time.
- Solution computed by time-stepping.

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In this talk (Trefftz):

- Construct a space-time mesh and corresponding fully discrete space:
  - In each space time element exact (polynomial or non-polynomial) exact solution of the wave equation.
  - Discontinous across the space-time skeleton.
- Solve either by time-stepping or as a large system.

Frequency domain motivation

Frequency domain

Take cue from frequency domain

$$\hat{u}(\mathbf{x}) \approx \sum_{j=1}^{k} f_j e^{i\omega \mathbf{x}\cdot\mathbf{a}_j},$$

where $\mathbf{a}_j$ are directions, $|\mathbf{a}_j| = 1$.

Motivation $-\Delta\hat{u} - \omega^2\hat{u} = 0$:

For large ω, minimize the number of degrees of freedom per wavelength.

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$$\hat{u}(\mathbf{x},\omega) \approx \sum_{j=1}^{k} \hat{f}_j(\omega) e^{i\omega \mathbf{x}\cdot\mathbf{a}_j},$$

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Time-domain

Time-domain equivalent

$$u(\mathbf{x},t) \approx \sum_{j=1}^{k} f_j(t - \mathbf{x}\cdot\mathbf{a}_j)$$

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Time-domain

Time-domain equivalent

$$\begin{aligned} u(\mathbf{x},t) &\approx \sum_{j=1}^{k} f_j(t - \mathbf{x}\cdot\mathbf{a}_j) \\ &\approx \sum_{j=1}^{k}\sum_{\ell=0}^{p} \alpha_{j,\ell}(t - \mathbf{x}\cdot\mathbf{a}_j)^{\ell}. \end{aligned}$$

(A bit of) Literature on Trefftz methods for waves

Plenty of literature in the frequency domain

- O. Cessenat and B. Després, *Application of an ultra weak variational formulation of elliptic PDEs to the two-dimensional Helmholtz equation*, SIAM J. Numer. Anal., (1998)
- R. Hiptmair, A. Moiola, I. Perugia, *A survey of Trefftz methods for the Helmholtz equation*. Springer Lect. Notes Comput. Sci. Eng., 2016, pp. 237-278.

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Fewer in time-domain

- S. Petersen, C. Farhat, and R. Tezaur, *A space-time discontinuous Galerkin method for the solution of the wave equation in the time domain*, Internat. J. Numer. Methods Engrg. (2009)
- F. Kretzschmar, A. Moiola, I. Perugia, S. M. Schnepp, *A priori error analysis of space-time Trefftz discontinuous Galerkin methods for wave problems*, IMA J. Numer. Anal., 36(4) 2016, pp. 1599-1635.
- L. Banjai, E. Georgoulis, O Lijoka, *A Trefftz polynomial space-time discontinuous Galerkin method for the second order wave equation*, SIAM J. Numer. Anal. 55-1 (2017), pp. 63–86.

Outline

DG setting

- Time discretization $0 = t_0 < t_1 < \cdots < t_N = T$, $I_n = [t_n, t_{n+1}]$; $\tau_n = t_{n+1} - t_n$.
- Spatial-mesh $\mathcal{T}_n$ of Ω consisting of open simplices such that $\Omega = \cup_{K \in \mathcal{T}_n} \overline{K}$. *In each simplex K, $a(x)$ is constant.*
- Space-time slabs $\mathcal{T}_n \times I_n$, h-space-time meshwidth.
- The skeleton of the space mesh denoted Γ_n and $\hat{\Gamma}_n := \Gamma_{n-1} \cup \Gamma_n$.
- Usual jump and average definitions ($e = K^+ \cap K^- \in \Gamma_{\text{int}}$)

$$\{u\}|_e = \frac{1}{2}(u^+ + u^-), \quad \{\mathbf{v}\}|_e = \frac{1}{2}(\mathbf{v}^+ + \mathbf{v}^-),$$
$$[u]|_e = u^+\mathbf{n}^+ + u^-\mathbf{n}^-, \quad [\mathbf{v}]|_e = \mathbf{v}^+ \cdot \mathbf{n}^+ + \mathbf{v}^- \cdot \mathbf{n}^-,$$

and if $e \in K^+ \cap \partial\Omega$,

$$\{\mathbf{v}\}|_e = \mathbf{v}^+, \quad [u]|_e = u^+\mathbf{n}^+$$

Also

$$[\![u(t_n)]\!] = u(t_n^+) - u(t_n^-), \qquad [\![u(t_0)]\!] = u(t_0^+).$$

Local Trefftz spaces

The space of piecewise polynomials on the time-space mesh denoted

$$S_n^{h,p} := \left\{ u \in L^2(\Omega \times I_n) \ : \ u|_{K \times I_n} \in \mathcal{P}_p(\mathbb{R}^{d+1}),\ K \in \mathcal{T}_n \right\},$$

Let $S_{n,\mathrm{Trefftz}}^{h,p} \subset S_n^{h,p}$ with

$$\ddot{v}(t,x) - a\Delta v(t,x) = 0, \qquad t \in I_n, x \in K, \text{ for any } v \in S_{n,\mathrm{Trefftz}}^{h,p}.$$

The dimensions of the spaces $S_n^{h,p}$ and $S_{n,\mathrm{Trefftz}}^{h,p}$ for spatial dimension d are

	1D	2D	3D
poly	$\frac{1}{2}(p+1)(p+2)$	$\frac{1}{6}(p+1)(p+2)(p+3)$	$2\mathrm{D} \times \frac{1}{4}(p+4)$
Trefftz	$2p+1$	$(p+1)^2$	$\frac{1}{6}(p+1)(p+2)(2p+3)$

- We expect the approximation properties of solutions of the wave equation to be the same for the two spaces of different dimension.

Constructing the polynomial spaces

- Choose directions ξ_j :

$$\left(t - \frac{1}{\sqrt{a}} x \cdot \xi_j\right)^{\ell}, \qquad \ell = 0, 1, \ldots, p.$$

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- Alternatively propagate polynomial initial condition:

$$u(0) = x^{\alpha_k}, \qquad \dot{u}(0) = 0,$$

and

$$u(0) = 0, \qquad \dot{u}(0) = x^{\beta_k},$$

with $|\alpha_k| \leq p$ and $|\beta_k| \leq p - 1$, α_k, β_k multi-indices.

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with $|\alpha_k| \le p$ and $|\beta_k| \le p - 1$, α_k, β_k multi-indices.

- Important observation: trunctation of the Taylor expansion of exact solution is a polynomial solution of the wave equation.

The space on $\Omega \times [0, T]$ is then defined as

$$V^{h,p}_{\text{Trefftz}} = \{u \in L^2(\Omega \times [0, T]) \;:\; u|_{\Omega \times I_n} \in S^{h,p}_{n,\text{Trefftz}},\; n = 0, 1 \ldots, N\}.$$

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(Abuse of) Notation:

- $u^h \in V^{h,p}_{\text{Trefftz}}$-discrete function on $\Omega \times [0, T]$
- $u^n \in S^{h,p}_{n,\text{Trefftz}}$, restriction of u on $\Omega \times I_n$.
- u_{ex}– the exact solution.

An interior penalty dG method

We start with

$$\int_{t_n}^{t_{n+1}} \left[\int_\Omega \ddot{u}\dot{v} + a\nabla u \cdot \nabla \dot{v} dx - \int_\Gamma \{a\nabla u\} \cdot [\dot{v}]\, ds - \int_\Gamma [u] \cdot \{a\nabla \dot{v}\}\, ds \right.$$
$$\left. + \sigma_0 \int_\Gamma [u] \cdot [\dot{v}]\, ds \right] dt = 0.$$

Testing with $v = u$ gives

$$\int_{t_n}^{t_{n+1}} \frac{d}{dt} E(t, u) dt = 0,$$

where the energy is given by

$$E(t, u) = \tfrac{1}{2} \|\dot{u}(t)\|_\Omega^2 + \tfrac{1}{2} \|\sqrt{a} \nabla u(t)\|_\Omega^2 + \tfrac{1}{2} \|\sqrt{\sigma_0}\, [u(t)]\, \|_\Gamma^2 - \int_\Gamma \{a\nabla u\} \cdot [u]\, ds.$$

Discrete inverse inequality in space and usual choice of penalty parameter gives $E(t, u) \geq 0$.

Jumps in time

Summing over n gives

$$E(t_N^-) - E(t_0^+) - \sum_{n=1}^{N-1} [\![E(t_n)]\!] = 0.$$

To give a sign to the extra terms (Hughes, Hulbert '88):

$$\frac{1}{2}[\![(\dot{u}(t_n), \dot{u}(t_n))_{L^2(\Omega)}]\!] - ([\![\dot{u}(t_n)]\!], \dot{u}(t_n^+))_{L^2(\Omega)} = \frac{1}{2}([\![\dot{u}(t_n)]\!], [\![\dot{u}(t_n)]\!])_{L^2(\Omega)}.$$

- Do this for all the terms, including the stabilization.
- Obtain a dissipative method.

Space-time dG formulation

$$
\begin{aligned}
a(u,v) := \sum_{n=0}^{N-1} &(\ddot{u},\dot{v})_{\Omega\times I_n} + (\llbracket \dot{u}(t_n)\rrbracket, \dot{v}(t_n^+))_\Omega \\
&+ (a\nabla u, \nabla\dot{v})_{\Omega\times I_n} + (\llbracket a\nabla u(t_n)\rrbracket, \nabla v(t_n^+))_\Omega \\
&- (\{a\nabla u\}, [\dot{v}])_{\Gamma_n\times I_n} - (\llbracket\{a\nabla u(t_n)\}\rrbracket, [v(t_n^+)])_{\hat{\Gamma}_n} \\
&- ([u], \{a\nabla\dot{v}\})_{\Gamma_n\times I_n} - (\llbracket[u(t_n)]\rrbracket, \{a\nabla v(t_n^+)\})_{\hat{\Gamma}_n} \\
&+ (\sigma_0[u], [\dot{v}])_{\Gamma_n\times I_n} + (\sigma_0\llbracket[u(t_n)]\rrbracket, [v(t_n^+)])_{\hat{\Gamma}_n} \\
&+ (\sigma_1[u], [v])_{\Gamma_n\times I_n} + (\sigma_2[a\nabla u], [a\nabla v])_{\Gamma_n\times I_n}
\end{aligned}
$$

and

$$
\begin{aligned}
b^{\text{init}}(v) := (v_0, \dot{v}(t_0^+))_\Omega + (a\nabla u_0, \nabla v(t_0^+))_\Omega - (\{a\nabla u_0\}, [v(t_0^+)])_{\Gamma_0} \\
- ([u_0], \{a\nabla v(t_0^+)\})_{\Gamma_0} + (\sigma_0[u_0], [v(t_0^+)])_{\Gamma_0}.
\end{aligned}
$$

Find $u^h \in V^{h,p}_{\text{Trefftz}}(\Omega\times[0,T])$ such that

$$
a(u^h, v) = b^{\text{init}}(v), \qquad \forall v \in V^{h,p}_{\text{Trefftz}}(\Omega\times[0,T]).
$$

Time-space dG as a time-stepping method

$$
\begin{aligned}
a_n(u,v) := \ & (\ddot{u},\dot{v})_{\Omega\times I_n} + (\dot{u}(t_n^+),\dot{v}(t_n^+))_{\Omega} \\
& + (a\nabla u,\nabla\dot{v})_{\Omega\times I_n} + (a\nabla u(t_n^+),\nabla v(t_n^+))_{\Omega} \\
& - (\{a\nabla u\},[\dot{v}])_{\Gamma_n\times I_n} - (\{a\nabla u(t_n^+)\},[v(t_n^+)])_{\Gamma_n} \\
& - ([u],\{a\nabla\dot{v}\})_{\Gamma_n\times I_n} - ([u(t_n^+)],\{a\nabla v(t_n^+)\})_{\Gamma_n} \\
& + (\sigma_0[u],[\dot{v}])_{\Gamma_n\times I_n} + (\sigma_0[u(t_n^+)],[v(t_n^+)])_{\Gamma_n} \\
& + (\sigma_1[u],[v])_{\Gamma_n\times I_n} + (\sigma_2[a\nabla u],[a\nabla v])_{\Gamma_n\times I_n},
\end{aligned}
$$

$$
\begin{aligned}
b_n(u,v) := \ & (\dot{u}(t_n^-),\dot{v}(t_n^+))_{\Omega} + (a\nabla u(t_n^-),\nabla v(t_n^+))_{\Omega} - (\{a\nabla u(t_n^-)\},[v(t_n^+)])_{\Gamma} \\
& - ([u(t_n^-)],\{a\nabla v(t_n^+)\})_{\Gamma_{n-1}} + (\sigma_0[u(t_n^-)],[v(t_n^+)])_{\hat{\Gamma}_n},
\end{aligned}
$$

Find $u^n \in S^{h,p}_{n,\mathrm{Trefftz}}$ such that

$$
a_n(u^n,v) = b_n(u^{n-1},v), \qquad \forall v \in S^{h,p}_{n,\mathrm{Trefftz}}.
$$

Consistency and stability

Theorem

The following statements hold:

1. The method is consistent for a sufficiently smooth solution u.
2. There exists a choice of $\sigma_0 \sim h^{-1}$, such that for any $v \in S^{h,p}_{n,\mathrm{Trefftz}}$ and $t \in I_n$ the energy is bounded below as

$$E(t,v) \geq \frac{1}{2}\|\dot{v}(t)\|^2_{L^2(\Omega)} + \frac{1}{4}\|\sqrt{a}\nabla v(t)\|^2_{L^2(\Omega)}.$$

3. Let $u^h \in V^{h,p}_{\mathrm{Trefftz}}$ discrete solution. Then

$$E(t_N^-, u^h) \leq E(t_1^-, u^h).$$

$a(\cdot,\cdot)^{1/2} := |||\cdot|||$ - a norm on $V^{h,p}_{\text{Trefftz}}$

$$
\begin{aligned}
a(w,w) = E_h(t_N^-, w) + E_h(t_0^+, w) + \sum_{n=1}^{N-1} \Big(& \tfrac{1}{2} \| [\![\dot{w}(t_n)]\!] \|_\Omega^2 + \tfrac{1}{2} \| \sqrt{a} [\![\nabla w(t_n)]\!] \|_\Omega^2 \\
& - ([\![\{ a \nabla w(t_n) \}]\!], [\![[w(t_n)]]\!])_{\hat{\Gamma}_n} + \tfrac{1}{2} \| [\![\sqrt{\sigma_0} [w(t_n)]]\!] \|_{\hat{\Gamma}_n}^2 \Big) \\
& + \sum_{n=0}^{N-1} \Big(\| \sqrt{\sigma_1} [w] \|_{\Gamma_n \times I_n}^2 + \| \sqrt{\sigma_2} [a \nabla w] \|_{\Gamma_n \times I_n}^2 \Big).
\end{aligned}
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$$a(w,w) = E_h(t_N^-, w) + E_h(t_0^+, w) + \sum_{n=1}^{N-1} \Big(\tfrac{1}{2} \| [\![\dot{w}(t_n)]\!] \|^2_\Omega + \tfrac{1}{2} \| \sqrt{a} [\![\nabla w(t_n)]\!] \|^2_\Omega$$
$$- ([\![\{ a \nabla w(t_n) \}]\!], [\![[w(t_n)]]\!])_{\hat{\Gamma}_n} + \tfrac{1}{2} \| [\![\sqrt{\sigma_0} [w(t_n)]]\!] \|^2_{\hat{\Gamma}_n} \Big)$$
$$+ \sum_{n=0}^{N-1} \Big(\| \sqrt{\sigma_1} [w] \|^2_{\Gamma_n \times I_n} + \| \sqrt{\sigma_2} [a \nabla w] \|^2_{\Gamma_n \times I_n} \Big).$$

Theorem

- $a(v,v)^{1/2} = |||v||| = 0 \implies v = 0$, for $v \in V^{h,p}_{\text{Trefftz}}$.
- Hence, the time-space dG method

$$a(u,v) = b^{\text{init}}(v), \qquad \forall v \in V^{h,p}_{\text{Trefftz}}$$

has a unique solution in $V^{h,p}_{\text{Trefftz}}$.

- The proof is by noticing that if $|||v||| = 0$ then v is a smooth solution of the wave equation, uniquely determined by the initial condition.

Convergence analysis

If we prove continuity of $a(\cdot,\cdot)$

$$|a(u,v)| \leq C_\star \|u\|_\star \|v\|, \quad \forall u \in \text{cont. sol.} + V^{h,p}_{\text{Trefftz}}, v \in V^{h,p}_{\text{Trefftz}},$$

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$$|a(u,v)| \leq C_\star |||u|||_\star |||v|||, \quad \forall u \in \text{cont. sol.} + V^{h,p}_{\text{Trefftz}}, v \in V^{h,p}_{\text{Trefftz}},$$

we can use Galerkin orthogonality to show, for any $v \in V^{h,p}_{\text{Trefftz}}$

$$\begin{aligned} |||u^h - v|||^2 &= a(u^h - v, u^h - v) \\ &= a(u_{ex} - v, u^h - v) \\ &\leq C_\star \, |||u_{ex} - v|||_\star |||u^h - v||| \end{aligned}$$

and hence we have quasi-optimality

$$\begin{aligned} |||u^h - u_{ex}||| &\leq \inf_{v \in V^{h,p}_{\text{Trefftz}}} |||u^h - v||| + |||v - u_{ex}||| \\ &\leq \inf_{v \in V^{h,p}_{\text{Trefftz}}} |||v - u_{ex}||| + C_\star |||v - u_{ex}|||_\star \end{aligned}$$

Integrating by parts a few times (this is how to implement the method)

$$a(w,v) = \sum_{n=0}^{N-1}\Big((\{a\nabla \dot{w}\},[v])_{\Gamma_n\times I_n} - (\sigma_0[\dot{w}],[v])_{\Gamma_n\times I_n} - (\{\dot{w}\},[a\nabla v])_{\Gamma_n^{\mathrm{int}}\times I_n}$$

$$+ (\sigma_1[w],[v])_{\Gamma_n\times I_n} + (\sigma_2[a\nabla w],[a\nabla v])_{\Gamma_n\times I_n}\Big)$$

$$- \sum_{n=1}^{N}\Big((\dot{w}(t_n^-),[\![\dot{v}(t_n)]\!])_\Omega + (a\nabla w(t_n^-),[\![\nabla v(t_n)]\!])_\Omega$$

$$- (\{a\nabla w(t_n^-)\},[\![[v(t_n)]]\!])_{\Gamma_n} - ([w(t_n^-)],[\![\{a\nabla v(t_n)\}]\!])_{\Gamma_n}$$

$$+ (\sigma_0[w(t_n^-)],[\![[v(t_n)]]\!])_{\Gamma_n}\Big).$$

Recall

$$|\!|\!|v|\!|\!|^2 = E(t_N^-,v) + E(t_0^+,v) + \sum_{n=1}^{N-1}\Big(\tfrac{1}{2}\|[\![\dot{v}(t_n)]\!]\|_\Omega^2 + \tfrac{1}{2}\|[\![\nabla v(t_n)]\!]\|_\Omega^2$$

$$+ ([\![\{\nabla v(t_n)\}]\!],[\![[v(t_n)]]\!])_{\hat{\Gamma}_n} + \tfrac{1}{2}\|[\![\sqrt{\sigma_0}[v(t_n)]]\!]\|_{\hat{\Gamma}_n}^2\Big)$$

$$+ \sum_{n=0}^{N-1}\Big(\|\sqrt{\sigma_1}[v]\|_{\Gamma\times I_n}^2 + \|\sqrt{\sigma_2}[\nabla v]\|_{\Gamma\times I_n}^2\Big),$$

Integrating by parts a few times (this is how to implement the method)

$$a(w,v) = \sum_{n=0}^{N-1} \Big((\{a\nabla \dot{w}\}, [v])_{\Gamma_n \times I_n} - (\sigma_0 [\dot{w}], [v])_{\Gamma_n \times I_n} - (\{\dot{w}\}, [a\nabla v])_{\Gamma_n^{\mathrm{int}} \times I_n}$$

$$+ (\sigma_1 [w], [v])_{\Gamma_n \times I_n} + (\sigma_2 [a\nabla w], [a\nabla v])_{\Gamma_n \times I_n} \Big)$$

$$- \sum_{n=1}^{N} \Big((\dot{w}(t_n^-), [\![\dot{v}(t_n)]\!])_\Omega + (a\nabla w(t_n^-), [\![\nabla v(t_n)]\!])_\Omega$$

$$- (\{a\nabla w(t_n^-)\}, [\![[v(t_n)]]\!])_{\Gamma_n} - ([w(t_n^-)], [\![\{a\nabla v(t_n)\}]\!])_{\Gamma_n}$$

$$+ (\sigma_0 [w(t_n^-)], [\![[v(t_n)]]\!])_{\Gamma_n} \Big).$$

Hence define,

$$|\!|\!| w |\!|\!|_\star^2 = \frac{1}{2} \sum_{n=1}^{N} \Big(\|\dot{w}(t_n^-)\|_\Omega^2 + \|\sqrt{a}\nabla w(t_n^-)\|_\Omega^2 + \|\sqrt{\sigma_0}\, [w(t_n^-)]\,\|_{\Gamma_n}^2 + \|\sigma_0^{-1/2} \{a\nabla w($$

$$+ \sum_{n=0}^{N-1} \Big(\|\sqrt{\sigma_1}\, [w]\|_{\Gamma_n \times I_n}^2 + \|\sqrt{\sigma_2}\, [a\nabla w]\|_{\Gamma_n \times I_n}^2 + \|\sigma_2^{-1/2} \{\dot{w}\}\|_{\Gamma_n^{\mathrm{int}} \times I_n}^2$$

$$+ \|\sigma_1^{-1/2} \{a\nabla \dot{w}\}\|_{\Gamma_n \times I_n}^2 + \|\sigma_0 \sigma_1^{-1/2} [\dot{w}]\|_{\Gamma_n \times I_n}^2 \Big).$$

The choice of stabilization parameters and convergence

Let $\tau_n = t_{n+1} - t_n$, $h = \mathrm{diam}(K)$, $(x,t) \in K \times (t_n, t_{n+1}), K \in \mathcal{T}_n$.

- $\mathrm{diam}(K)/\rho_K \leq c_{\mathcal{T}}, \quad \forall K \in \mathcal{T}_n$, $n = 0, 1, \ldots, N-1$, where ρ_K is the radius of the inscribed circle of K.
- Assume space-time elements star-shaped with respect to a ball.
- Choice of parameters
  - $\sigma_0 = p^2 c_{\mathcal{T}} C_a^2 C_{\mathrm{inv}} (c_a h)^{-1}$,
  - $\sigma_1 = C_a p^3 (h\tau_n)^{-1}$
  - $\sigma_2 = h(C_a \tau_n)^{-1}$.

Theorem

For sufficiently smooth solution and $h \sim \tau$

$$|||u^h - u_{\mathrm{ex}}||| = O(h^{p-1/2}).$$

Proof uses truncated Taylor expansion.

Error estimate in mesh independent norm

Using a Gronwall argument we can show for $v \in S^{h,p}_{n,\text{Trefftz}}$

$$\|\dot{v}\|^2_{\Omega\times I_n} + \|\sqrt{a}\nabla v\|^2_{\Omega\times I_n} \le \tau_n e^{\widetilde{C}(t_{n+1}-t_n)/\underline{h}_n}\left(\|\dot{v}(t^-_{n+1})\|^2_\Omega + \|\sqrt{a}\nabla v(t^-_{n+1})\|^2_\Omega\right),$$

$\underline{h}_n := \min_{x\in\Omega} h(x,t),\ t \in I_n$. Let $\tau = \max \tau_n$ and $\tilde{h} = \min \underline{h}_n$. Then

$$\|\dot{V}\|^2_{\Omega\times(0,T)} + \|\sqrt{a}\nabla V\|^2_{\Omega\times(0,T)} \le C\tau e^{\widetilde{C}\tau/\tilde{h}} |\!|\!| V |\!|\!|^2_\star, \quad \forall V \in V^{h,p}_{\text{Trefftz}}.$$

Proposition

$$\|\dot{u}^h - \dot{u}_{\text{ex}}\|^2_{\Omega\times[0,T]} + \|\nabla u^h - \nabla u_{\text{ex}}\|^2_{\Omega\times[0,T]}$$
$$\le C \inf_{V\in V^{h,p}_{\text{Trefftz}}} \Big(\tau e^{\widetilde{C}\tau/\tilde{h}} |\!|\!| V - u_{\text{ex}} |\!|\!|^2_\star$$
$$+ \|\dot{V} - \dot{u}_{\text{ex}}\|^2_{\Omega\times[0,T]} + \|\sqrt{a}\nabla(V - u_{\text{ex}})\|^2_{\Omega\times[0,T]}\Big).$$

Hence in mesh independent energy norm we expect error $O(h^p)$.

Outline

Wave equation with damping

Damped wave equation:

$$\ddot{u} + \alpha \dot{u} - \Delta u = 0.$$

- The extra term decreases the energy:

$$\frac{d}{dt} E(t) = -\alpha \|\dot{u}\|^2,$$

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- However: truncations of the Taylor expansion are no longer solutions!
- Grysa, Maciag, Adamczyk-Krasa '14 consider solutions of the form

$$e^{-\alpha t} p_1(x,t) + p_2(x,t)$$

  with p_1 and p_2 polynomial in x and t.

Solution formula in 1D

- Instead, use basis functions obtained by propagating polynomial initial data:

$$u(x,0) = u_0(x) = x^{\alpha_j},\ \dot{u}(x,0) = v_0(x) = 0,$$

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  with

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- In 1D solution is then given by the d'Alambert-like formula

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[u_0(x-t) + u_0(x+t)\right] e^{-\alpha t/2} \\ &\quad + \frac{\alpha}{4} e^{-\alpha t/2} \int_{x-t}^{x+t} u_0(s) \left\{ I_0\left(\rho(s)\tfrac{\alpha}{2}\right) + \frac{t}{\rho(s)} I_1\left(\rho(s)\tfrac{\alpha}{2}\right) \right\} \mathrm{ds} \\ &\quad + \frac{1}{2} e^{-\alpha t/2} \int_{x-t}^{x+t} v_0(s) I_0\left(\rho(s)\tfrac{\alpha}{2}\right) \mathrm{ds}, \quad \rho(s;x,t) = \sqrt{t^2 - (x-s)^2}. \end{aligned}$$

Solution formula ctd.

- Rearranging (the last term)

$$\frac{1}{2} t e^{-\alpha t/2} \int_0^1 \left[v_0(x+st) + v_0(x-st) \right] I_0\left(\frac{\alpha t}{2} \sqrt{1-s^2} \right) ds.$$

- For example for $v_0(x) = x^2$

$$x^2 t e^{-\alpha t/2} \int_0^1 I_0\left(\frac{\alpha t}{2} \sqrt{1-s^2} \right) ds + t^3 e^{-\alpha t/2} \int_0^1 s^2 I_0\left(\frac{\alpha t}{2} \sqrt{1-s^2} \right) ds.$$

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- Corresponding term in 3D

$$t e^{-\alpha t/2} \int\limits_0^1 \fint_{\partial B(0,1)} \left[v_0(x+tsz) + Dv_0(x+tsz) \right] \cdot z dS_z \, I_0 \left(\tfrac{\alpha t}{2} \sqrt{1-s^2} \right) ds$$

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- Hence need efficient representation of functions of the type

$$\varrho_j(t) = \int_0^1 s^j I_0\left(\tfrac{\alpha t}{2}\sqrt{1-s^2}\right) ds$$

Damped wave equation: Efficient implementation

Need efficient representation of functions of the type

$$\varrho_j(t) = \int_0^1 s^j I_0\left(\frac{\alpha t}{2}\sqrt{1-s^2}\right) ds$$

- In Matlab: Chebfun works well.
- Note computations are done on a small space-time element, i.e. $t \in (0, h)$ for a small $h > 0$.
- Can use truncated Taylor expansions of analytic functions I_0 and I_1 with efficiency increasing for decreasing h.
- We expect similar effect for any lower order terms.

Changes to the analysis

- Polynomial in space $\implies$ the same discrete inverse inequalities used + the extra term decreases energy $\implies$ choice of parameters, stability and quasi-optimality proof identical.
- Approximation properties of the discrete space (in 1D):
  - Away from the boundary, in each space-time element $K \times (t^-, t^+)$ project solution to K and neighbouring elements at time t^- to polynomials and propagate.
  - At boundary, extend exact solution anti-symmetrically and again project and propagate.

Outline

One dimensional setting

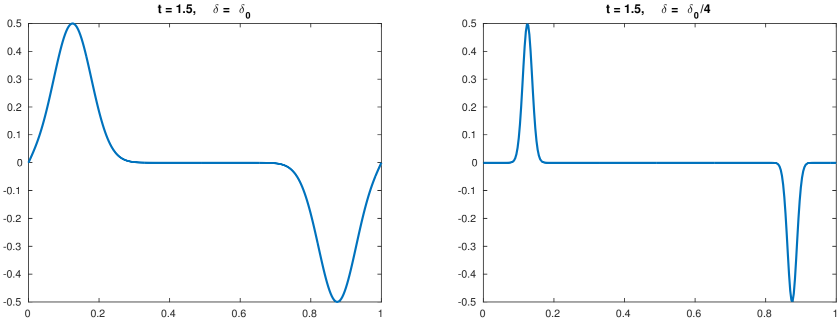
Simple 1D setting:

$$\Omega = (0,1), \qquad a \equiv 1.$$

Initial data

$$u_0 = e^{-\left(\frac{x-5/8}{\delta}\right)^2}, \quad v_0 = 0, \quad \delta \le \delta_0 = 7.5 \times 10^{-2}.$$

Interested in having few degrees of freedom for decreasing $\delta \le \delta_0$.



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- Energy of exact solution

$$\text{exact energy} = \tfrac{1}{2}\|u_x(x,0)\|_\Omega^2 \approx 2\delta^{-1}\int_{-\infty}^{\infty} y^2 e^{-2y^2}\,dy = \delta^{-1}\frac{\sqrt{\pi}}{2\sqrt{2}}.$$

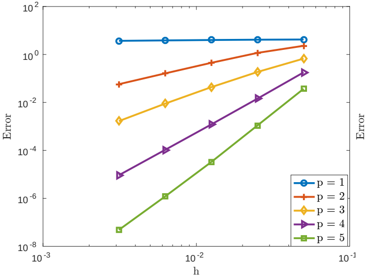
- We compare with full polynomial space.
- Note for polynomial order p

$$2p+1 \text{ Trefftz} \qquad \tfrac{1}{2}(p+1)(p+2)\text{full polynomial space.}$$

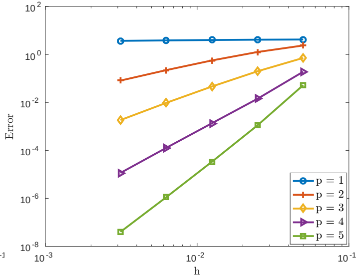
- In all 1D experiments square space-time elements.

Error in dG-norm, $\delta = \delta_0$, $T = 1/4$:

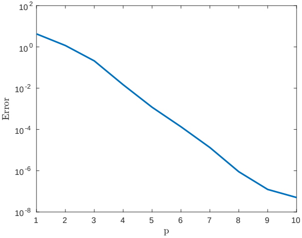
Trefftz poly



Full poly



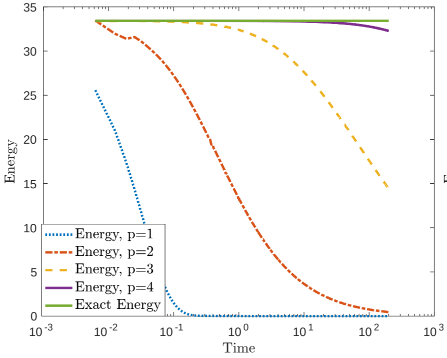
Trefftz p-convergence (fixed h in space and time):



Energy conservation

For $\delta = \delta_0/4$:

Trefftz poly



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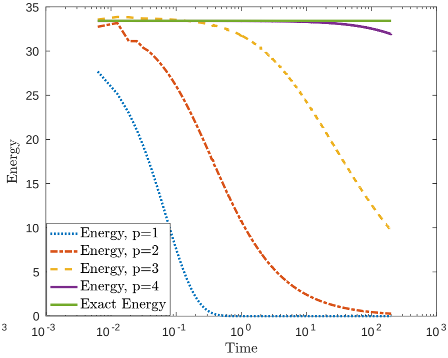
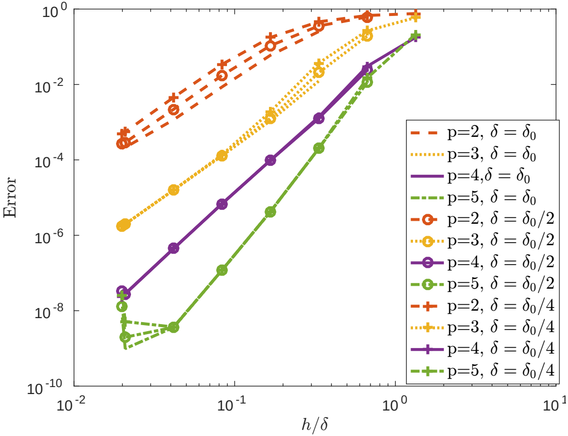


Relative error for decreasing δ

$$\text{error}_\delta = \left(\frac{\delta}{2} \|\dot{u}(\cdot, T) - \dot{u}_h(\cdot, T^-)\|^2_\Omega + \frac{\delta}{2} \|\nabla u(\cdot, T) - \nabla u_h(\cdot, T^-)\|^2_\Omega \right)^{1/2}$$



2D experiment

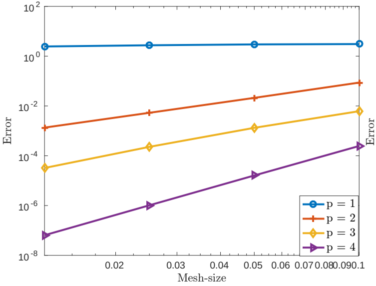
On square $[0,1]^2$ with exact solution

$$u(x,y,t) = \cos(\sqrt{2}\pi t)\sin \pi x \sin \pi y.$$

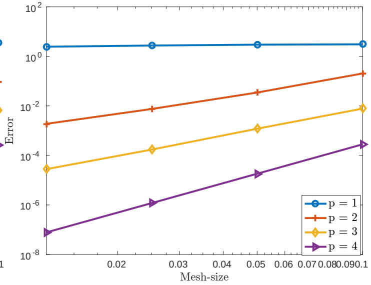
Energy of error at final time:

$$\text{error} = \left(\frac{1}{2}\|\dot{u}(\cdot,T) - \dot{u}_h(\cdot,T^-)\|_\Omega^2 + \frac{1}{2}\|\nabla u(\cdot,T) - \nabla u_h(\cdot,T^-)\|_\Omega^2\right)^{1/2}.$$

Trefftz poly



Full poly



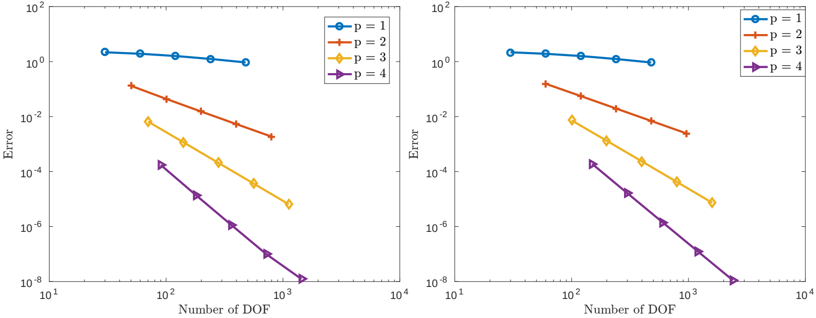
Damped wave equation in 1D

Error in dG norm. Exact solution on $\Omega = (0, 1)$:

$$u(x,t) = e^{(-\alpha t/2)} \sin(\pi x) \left[\cos \sqrt{\pi^2 - \frac{\alpha^2}{4}}\, t + \frac{\alpha}{2\sqrt{\pi^2 - \alpha^2/4}} \sin \sqrt{\pi^2 - \frac{\alpha^2}{4}}\, t \right].$$

Trefftz poly

Full poly



Conclusions

A space-time interior penalty dG method for the acoustic wave equation in second order form:

- Can be considered either as implicit time-stepping method or a large space-time system.
- Allows Trefftz basis functions, polynomial in space.
- Fewer degrees of freedom than full polynomial method and requires integration only over the space-time skeleton.
- Stability and convergence analysis available.
- Also practical for damped wave equation.

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To do:

- A posteriori error analysis
- Adaptivity in h, p, wave directions.
- Tent-pitching meshes allow quasi explicit time-stepping.
- p-analysis in higher dimension.
- Applications.