Mimetic Finite Difference methods
An introduction

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Some cronology...


- Shashkov-Steinberg, JCP 1995 (support-operator methods)
- (Hyman)-Shashkov-Steinberg, JCP 1996-(7) (rough diffusion)
- Hyman-Shashkov, SINUM 1999 (Maxwell)
- Campbell-Shashkov, JCP 2001 (gas dynamics)

**MORE RECENTLY (2004–2008)**

- Kuznetsov-Lipnikov-Shashkov, Comp. Geos. 2004 (polygons)
- Brezzi-Lipnikov-Shashkov, SINUM 2005 (error analysis)
- Brezzi-Lipnikov-Simoncini, $M^3$AS 2005 (a new family of MFD)
- Brezzi-Lipnikov-Shashkov, $M^3$AS 2006 (curved faces)
Some cronology (continued)

MORE RECENTLY (2004–2008) – continued

- Beirao Da Veiga, NM 2007 (residual error estimator)
- C.-Manzini, CMAME 2008 (post-processing)
- C.-Manzini-Russo, SINUM accepted (convection-reaction-diffusion)

IN PROGRESS

- Convection-dominated diffusion
- Higher-order MFD
- Nodal MFD
- Mimetic curl operator
- Mimetic discretization of Stokes
Features

RECENTLY

- Generalisation of finite differences to hexahedral meshes.
- Discrete differential operators defined so as to mimic the properties of the underlying continuum operators (e.g. vector calculus identities, conservation laws, solution symmetries)
- Applied to wide range of problems.

MORE RECENTLY

- Generalisation of (low order) Mixed Finite Elements/finite volumes to general polyhedral meshes. Constructs a family of methods.
- Discrete differential operators defined so as to mimic the properties of the underlying continuum operators. May gain from extra freedom given by method construction
- Currently limited to linear diffusion problems
Compatible Spatial Discretizations

We define compatible spatial discretizations as those that inherit or mimic fundamental properties of the PDE such as topology, conservation, symmetries, and positivity structures and maximum principles. (IMA "Hot Topics" Workshop Compatible Spatial Discretizations for Partial Differential Equations, May 11-15, 2004)

Advantages:
- Conserve crucial features of physical, geometrical, and mathematical model:
  - Conservation laws
  - Symmetry
  - Positivity and monotonicity
  - Duality properties of differential operators
- Provide reliability and accuracy
The following can be cast as compatible spatial discretizations:
mixed finite element methods, **mimetic finite differences**, support operator methods, control volume methods, discrete differential forms, Whitney forms, conservative differencing, discrete Hodge operators, discrete Helmholtz decomposition, finite integration techniques, staggered grid and dual grid methods, etc.

**Support operator method:**

- Most PDEs are written in terms of invariant differential operators (div, grad, and curl)
- Define discrete analogues of these invariant operators that satisfy exactly the discrete analogs of the identities satisfied by the continuum operators (e.g. Gauss’, Stokes’, Hodge orthogonal decomposition).
Linear diffusion in mixed form

Consider the linear diffusion equation bvp

\[-\text{div}(K \text{grad } p) = b \quad \text{in} \quad \Omega \subset \mathbb{R}^{2,3}\]

\[p = 0 \quad \text{on} \quad \partial \Omega\]

with $K$ strongly elliptic full symmetric tensor.

Mixed formulation:

\[\overline{F} = -K \text{grad } p \quad \text{(Constitutive Equation)}\]

\[\text{div } \overline{F} = b \quad \text{(Conservation Equation)}\]

We also introduce the flux operator

\[Gp = -K \text{grad } p.\]
Linear diffusion in mixed form

The operators $\text{div}$ and $\text{grad}$ satisfy the integral identity

$$
\int_{\Omega} p \, \text{div} F \, dV + \int_{\Omega} F \cdot \text{grad} p \, dV = 0
$$

which we rewrite:

$$
\int_{\Omega} p \, \text{div} F \, dV - \int_{\Omega} K^{-1} F \cdot G p \, dV = 0 \quad (1)
$$

Introducing the inner-products:

$$
(p, q)_W = \int_{\Omega} p \, q \, dV
$$

$$
(F, G)_V = \int_{\Omega} K^{-1} F \cdot G \, dV
$$

the identity (1) becomes:

$$
(\text{div} F, p)_W - (F, G p)_V = 0
$$

expressing the fact that divergence and flux are adjoint: $G = \text{grad}^*$
Linear diffusion in mixed form

Consider the linear diffusion equation bvp

\[-\text{div}(K \text{grad } p) = b \quad \text{in } \Omega \subset \mathbb{R}^{2,3}\]

\[p = 0 \quad \text{on } \partial\Omega\]

with \( K \) strongly elliptic full symmetric tensor.

Mixed formulation:

\[
\bar{F} = -K \text{grad } p \quad \text{(Constitutive Equation)}
\]

\[
\text{div } \bar{F} = b \quad \text{(Conservation Equation)}
\]

Mixed variational formulation: find \((p, \bar{F}) \in W \times V\) s.t.

\[
(K^{-1} \bar{F}, \bar{G}) - (p, \text{div } \bar{G}) = 0 \quad \forall \bar{G} \in V
\]

\[
(\text{div } \bar{F}, q) = (b, q) \quad \forall q \in W
\]

\[W = L^2(\Omega), \ V = H(\text{div}; \Omega)\]
Mixed Finite Element RT0-P0 Discretisation

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Mixed Finite Element RT0-P0 Discretisation

Let $\Omega_h$ be a triangularization of $\Omega \subset \mathbb{R}^2$. For $E \in \Omega_h$, let

\[
P_0(E) = \text{Polynomials of degree zero on } E
\]
\[
RT_0(E) = \{ \bar{a} + b \bar{x} : \bar{a} \in (P_0(E))^2, b \in P_0(E) \}
\]

from which we define the conforming FE spaces:

\[
W_h = \{ q_h : q_h|_E \in P_0(E), \forall E \in \Omega_h \}
\]
\[
V_h = \{ \bar{G}_h : \bar{G}_h|_E \in RT_0(E) \text{ and } \bar{G}_h \cdot \bar{n} \text{ cont. on } \partial \Omega_h \}
\]

MFE RT0-P0 discretization: find $(p_h, \bar{F}_h) \in W_h \times V_h$ s.t.

\[
(K^{-1}\bar{F}_h, \bar{G}_h) - (p_h, \text{div}\bar{G}_h) = 0 \quad \forall \bar{G}_h \in V_h
\]
\[
(\text{div}\bar{F}_h, q_h) = (b, q_h) \quad \forall q_h \in W_h
\]
RT0-P0 nodal variables

Finite Element Nodal variables

\( (E, P_0, N_0) \)

\[ N_0(q_h) = q_E \]

\[ q_E := \frac{1}{|E|} \int_E q \]

\( (E, RT_0, N_{RT}) \)

\[ N_{RT}(\overline{G}_h) = (G^e_E)^e = G_E \]

\[ G^e_E := \frac{1}{|e|} \int_e \overline{G} \cdot \overline{n}^e_E \]

Lifting

\[ \mathcal{L}^E_0 q_E \in P_0 \]

\[ \mathcal{L}^E_0 q_E \equiv q_E \]

\[ \mathcal{L}^E_{RT} G_E \in RT_0 \]

\[ \mathcal{L}^E_{RT} G_E |_e \cdot \overline{n}^e_E = G^e_E \]

Global FE space Global nodal spaces Global lifting

\( W_h \)

\[ Q_h := \{ q = (q_E)_{E \in \Omega_h} \} \]

\( V_h \)

\[ X_h := \begin{cases} G = (G_E)_{E \in \Omega_h} : \\ G^e_{E_-} + G^e_{E_+} = 0 & \forall e \in \partial \Omega_h \end{cases} \]

\[ \mathcal{L}_{RT} : X_h \rightarrow V_h. \]

Where \( E_- \) and \( E_+ \) are the two elements sharing the edge.
Discrete divergence

Let $\mathcal{L}_{RT} \mathbf{G} = \mathbf{G}_h$. Then,

$$|E| \text{ div } \mathbf{G}_h|_E = \int_E \text{ div } \mathbf{G}_h \overset{\text{Gauss}}{=} \int_{\partial E} \mathbf{G} \cdot \mathbf{n}_E = \sum_{e \in \partial E} |e| G^e_E$$

$$\Rightarrow \text{ div } \mathcal{L}_{RT}^E \mathbf{G} = \text{ div } \mathbf{G}_h|_E = \frac{1}{|E|} \sum_{e \in \partial E} |e| G^e_E =: \text{ div}_h \mathbf{G}_E$$

We may now define the discrete divergence operator

$$\text{ div}_h : X_h \to Q_h$$

$$\mathbf{G} \to \text{ div}_h \mathbf{G} = (\text{ div}_h \mathbf{G}_E)_E$$

satisfying $\text{ div } \mathcal{L}_{RT} \mathbf{G} = \text{ div}_h \mathbf{G}$.

$$\Rightarrow (p_h, \text{ div } \mathbf{G}_h) = \sum_E |E| p_E \text{ div}_h \mathbf{G}_E = [p, \text{ div}_h \mathbf{G}]_{Q_h}$$
Scalar product in $X_h$

$$(K^{-1} F_h, G_h) = \int_{\Omega} K^{-1} \mathcal{L}_{RT} F \cdot \mathcal{L}_{RT} G$$

$$= \sum_E \int_E K^{-1} \mathcal{L}_{RT}^E F_E \cdot \mathcal{L}_{RT}^E G_E$$

$$= \sum_E [F_E, G_E]_E =: [F, G]_{X_h}$$

MFE in terms of the spaces $Q_h$ and $X_h$: find $(p, F) \in Q_h \times X_h$:

$$[F, G]_{X_h} - [p, \text{div}_h G]_{Q_h} = 0 \quad \forall G \in X_h$$

$$[\text{div}_h F, q]_{Q_h} = [b, q]_{Q_h} \quad \forall q \in Q_h.$$ 

with $b = b^I$, $I$ = interpolation operator defined for $q \in V$ by

$$(q^I)_E := \frac{1}{|E|} \int_E q \quad \forall E \in \Omega_h$$
Mimetic finite difference discretization

$\Omega_h$ partition of $\Omega$ into polygonal (polyhedral) elements.

Why polyhedral meshes?
- They naturally arise in the treatment of complex solution domains and heterogeneous materials (e.g. reservoir models)
- They facilitate adaptive mesh refinement/de-refinement
Define the discrete pressure space \( Q_h \) and flux space \( X_h \).

\[
\begin{align*}
    \mathbf{q} \in Q_h & \quad \Rightarrow \quad \mathbf{q} = (q_E)_{E \in \Omega_h} \quad (\Rightarrow \text{piecewise constant}) \\
    \mathbf{G} \in X_h & \quad \Rightarrow \quad \mathbf{G} = (G_E^e)_{E \in \Omega_h}^e \quad (\Rightarrow \text{constant normal component}) \\
    & \quad e \in \partial E_\pm \cap \partial E_\pm \Rightarrow F_{E_-}^e + F_{E_+}^e = 0
\end{align*}
\]

We have the interpolation operators:

\[
\begin{align*}
    \text{For } \mathbf{q} \in \mathcal{W}, \quad & (\mathbf{q}^I)_E = \frac{1}{|E|} \int_E \mathbf{q} \quad \forall E \in \Omega_h, \\
    \text{For } \mathbf{G} \in \mathcal{V}, \quad & (\mathbf{G}^I)^e_E = \frac{1}{|e|} \int_{e} \mathbf{G} \cdot \mathbf{n}_E^e \quad \forall E \in \Omega_h \quad \forall e \in \partial E
\end{align*}
\]

And the discrete divergence operator:

\[
\begin{align*}
    \text{div}_h : X_h & \rightarrow Q_h \\
    \mathbf{G} & \rightarrow \text{div}_h \mathbf{G} = (\text{div}_h \mathbf{G}_E)_E \\
    \text{div}_h \mathbf{G}_E & = \frac{1}{|E|} \sum_{e \in \partial E} |e| G_E^e
\end{align*}
\]
Discretisation of the conservation equation

Let $\mathbf{b} = b^I$. The mimetic discrete conservation equation reads:

$$\text{div}_h \mathbf{F} = \mathbf{b}$$

i.e.

$$\frac{1}{|E|} \sum_{e \in \partial E} |e| \mathbf{F}_E^e = (\mathbf{b})_E \quad \forall E$$

Equivalently,

$$[\text{div}_h \mathbf{F}, \mathbf{q}]_{Q_h} = [\mathbf{b}, \mathbf{q}]_{Q_h} \quad \forall \mathbf{q} \in Q_h$$

where the scalar product for the pressure space $Q_h$ is given by

$$[\mathbf{p}, \mathbf{q}]_{Q_h} = \sum_{E \in \Omega_h} |E| \mathbf{p}_E \mathbf{q}_E$$

This MFD discrete conservation equation (2) coincides with the MFE RT0-P0 and also with many finite volume discretisation of the conservation equation.
Finite volume discretisation of the constitutive equation

Directly discretize the constitutive equation defining the flux:

\[ \bar{F} = -K \nabla p \quad (\text{with } K = k \mathbf{I}) \]

\[ -\mathbf{n}_{AB} \cdot \bar{F} \approx k \left( \frac{p_K - p_L}{|K - L|} \right) \]

\[ -\bar{F} \approx k \left( \frac{p_K - p_L}{|K - L|} \frac{\mathbf{n}_{AB}}{\gamma_{AB}} + \frac{p_B - p_A}{|B - A|} \frac{\mathbf{n}_{KL}}{\gamma_{KL}} \right) \]
Mimetic discretisation of the constitutive equation

Introduce a scalar product for the flux space $X_h$

$$[F, G]_{X_h} = \sum_{E \in \Omega_h} [F, G]_E. \quad ([\cdot, \cdot]_E \text{ to be defined!}).$$

Moreover, define a discrete flux operator $G_h : Q_h \rightarrow X_h$ as the adjoint of $\text{div}_h$:

$$[G, G_h q]_{X_h} = [q, \text{div}_h G]_{Q_h}, \quad \forall q \in Q_h \quad \forall G \in X_h.$$

This definition naturally establishes a discrete Green formula with respect to the discrete scalar products. We say that the discrete operators mimic the continuous ones.
Discretisation of the constitutive equation (cont.)

The mimetic discretisation of the constitutive equation reads:

\[ F = G_h p \]

or, equivalently,

\[ [F, G]_{X_h} = [G_h p, G]_{X_h} = [p, \text{div}_h G]_{Q_h} \quad \forall G \in X_h \]

The (family of) Mimetic Finite Difference (MFD) schemes:

\[ [F, G]_{X_h} - [p, \text{div}_h G]_{Q_h} = 0 \quad \forall G \in X_h \]
\[ [\text{div}_h F, q]_{Q_h} = [b, q]_{Q_h} \quad \forall q \in Q_h \]

or, equivalently,

\[
\begin{bmatrix}
A & B^t \\
B & 0
\end{bmatrix}
\begin{bmatrix}
F \\
p
\end{bmatrix} =
\begin{bmatrix}
0 \\
b
\end{bmatrix}
\]
MFE and MFD

As we already saw, if $\Omega_h$ is made of triangles (tetrahedrons),

- The MFD conservation equation discretisation is equivalent to RT0-P0
- If we define the $X_h$-scalar product to be

$$\left[F, G\right]_{X_h} = \int_{\Omega} K^{-1} \mathcal{L}_{RT} F \cdot \mathcal{L}_{RT} G$$

the constitutive equation discretisation is equivalent.

...and the two methods coincide!

Which properties define $\mathcal{L}_{RT}$? Locally, on each $E \in \Omega_h$,

1. $\mathcal{L}_{RT}^E G|_e \cdot n_e^E = G_E^e \quad \forall e \in \partial E$,
2. $\text{div} \mathcal{L}_{RT}^E G = \text{div}_h G|_E$
3. $\mathcal{L}_{RT}^E \overline{G} = \overline{G}$ if $\overline{G}$ is constant
The flux scalar product: $P_0$-compatible liftings

**IDEA:** any *reasonable* lifting can be used to define a scalar product yielding a *reasonable* MFD formulation.

Set $X_E = X_h|_E$. We define *$P_0$-compatible lifting* a linear map $\mathcal{L}^E : X_E \to L^2(E)$ such that for all $G \in X_E$,

1. $\mathcal{L}^E G|_e \cdot n^e_E = G^e_E$, $\forall e \in \partial E$,
2. $\text{div}\mathcal{L}^E G = \text{div}_h G|_E$,

and, for all constant vector $\overline{C}$,

3. $\mathcal{L}^E \overline{C}^I = \overline{C}$.

The following defines a MFD scalar product for the flux space:

$$[F, G]_E = \int_{\Omega} K^{-1} \mathcal{L}^E F \cdot \mathcal{L}^E G \, dS.$$
The flux scalar product: $\mathcal{P}_0$-compatible liftings

Assume that $K$ is constant on $E$.

A crucial property. For any linear function $q^1$ and $G \in X_E$

\[
[(K \nabla q^1)^I, G]_E = \int_E K^{-1} L^E (K \nabla q^1)^I \cdot L^E G \, dS
\]

\[
\overset{3}{=} \int_E \nabla q^1 \cdot L^E G \, dS
\]

\[
\overset{\text{Green}}{=} - \int_E q^1 \text{div} L^E G \, dS + \int_{\partial E} q^1 (L^E G \cdot n_{\text{ext}}) \, dl
\]

\[
\overset{2,1}{=} - \int_E q^1 \text{div}_h G \, dS + \sum_{e \in \partial E} G^e E \int_e q^1 \, dl
\]

(local consistency)

- the scalar product is independent from the lifting if one of the factors is the interpolant of a constant vector field.

- the scalar product is exact on the interpolant of constant vector fields:

\[
[(e_i)^I, (e_j)^I]_E = (K^{-1})_{i,j} |E|
\]
A general locally consistent scalar product

**IDEA:** Use local consistency to construct a scalar product without a lifting!

Let $k_E$ be the number of edges (faces) of $E$. Set:

$$[F, G]_E = \sum_{s,t=1}^{k_E} M_E^{s,t} F^{e_s}_E G^{e_t}_E$$

- Imposing local consistency and symmetry we get a set of positive semi-definite matrices

- Imposing stability: exists $s_*, s^* > 0$ independent of $h$ s.t.

$$s_* \sum_{i=1}^{k_E} (G^{e_i}_E)^2 |E| \leq [G, G]_E \leq s^* \sum_{i=1}^{k_E} (G^{e_i}_E)^2 |E| \quad \forall G \in X_E.$$

We restrict to a set of acceptable (symmetric & pos. def.) matrices defining a family of reasonable (e.g. exact on constants) scalar products.
A family of consistent and stable MFD scalar products

Imposing:

- **local consistency** ⇒ $M_E^0$ symmetric & semidef.
- **stability** ⇒ $M_E = M_E^0 + CUC^t$ symmetric & pos. def., where $C$ is a given (computable) full-rank $k_E \times (k_E - d)$ matrix, $U$ is any symmetric & pos. def. $(k_E - d) \times (k_E - d)$ matrix

**Proposition**: Under a reasonable assumption on the minimum eigenvalue of $U$, the matrix $M_E$ is induced by a lifting.

In practice (so far) we use:

$$U = u \left( |E| \text{Trace}(K^{-1}) \right) I_{k_E-d}, \quad u \in \mathbb{R}$$

and the proposition requires that $u$ is sufficiently large.
MFD error in function of $u$

Quadrilaterals 20x20

BLS Example 1

Andrea Cangiani (IAC–CNR) mimetic finite difference methods
Example. If $\Omega_h$ is made of triangles,

$$M_E = M^0_E + u_E \left| e_i \right| \left| e_j \right|_{i,j}.$$

with $M^0_E$ symmetric and positive semidefinite.

$$u_E = \sum_{i=1}^{kE} \left( x_{e_i} - x_B \right)^T K^{-1} \left( x_{e_i} - x_B \right) \frac{12}{|E|}$$

Example Rectangles $\rightarrow$ central finite differences
Assumptions on the partition

- The number of edges per face and the number of faces per element is uniformly bounded.
- The length $l$ of each edge is such that $l \leq Ch$ (shape regularity).
- Every element is uniformly strictly starshaped.
- (if $\Omega \in \mathbb{R}^3$) Every face is uniformly strictly starshaped.
Error estimates

If
- \( \Omega \) is a convex polyhedron (polygon) with Lipschitz continuous boundary,
- \( \Omega_h \) is a partition satisfying the previous slide assumptions,
- the scalar product satisfies local consistency and stability,
- \( p \in H^2(\Omega) \)

Then,
\[
\| p^I - p \|_{Q_h} + \| F^I - F \|_{X_h} \leq C h,
\]
\[
\Rightarrow \| p - \mathcal{L}_0 p \|_{L^2} \leq C h
\]

where \( \| \cdot \|_{Q_h}^2 = [\cdot, \cdot]_{Q_h} \) and \( \| \cdot \|_{X_h}^2 = [\cdot, \cdot]_{X_h} \).

If, moreover,
- The scalar product is induced by (some) lifting.

Then,
\[
\| p^I - p \|_{Q_h} \leq C h^2 \quad \text{(pressure super convergence)}
\]