Chorin's approaches revisited:
Particle Vortex Method vs Finite Volume Method

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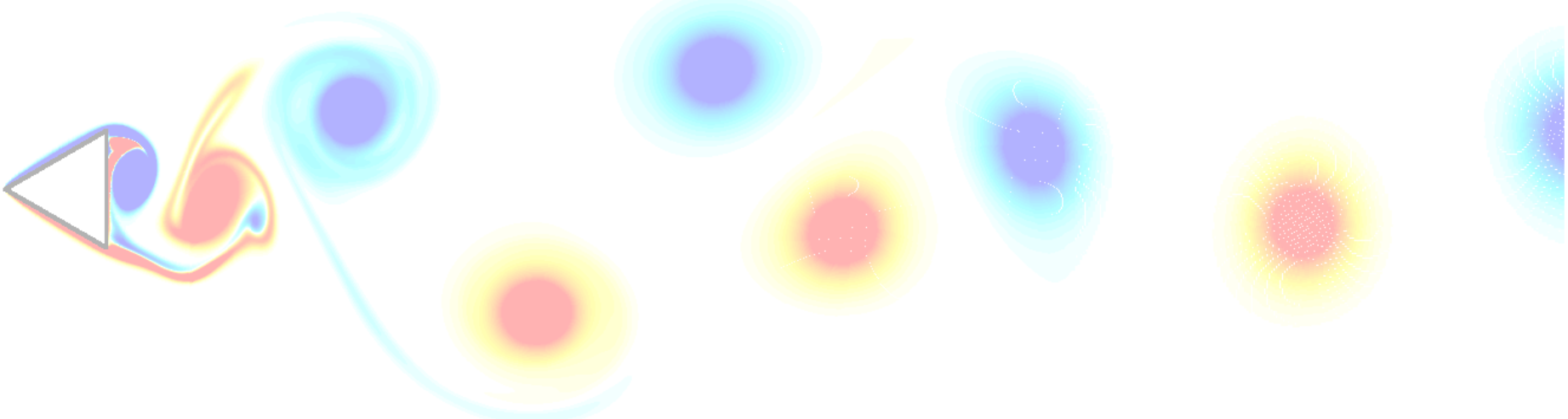
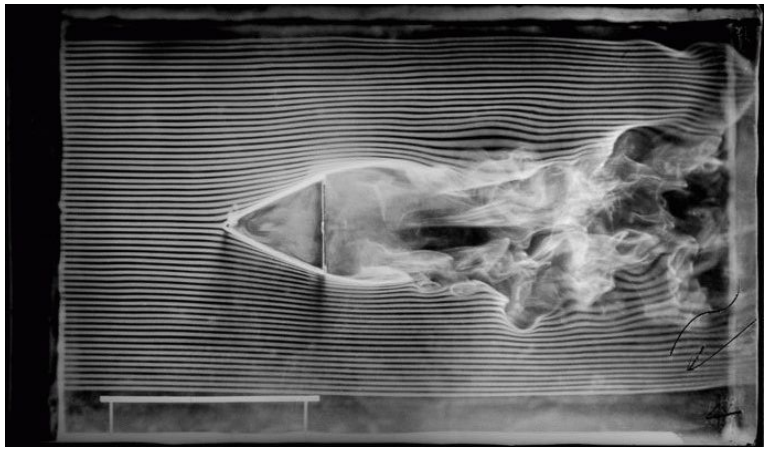


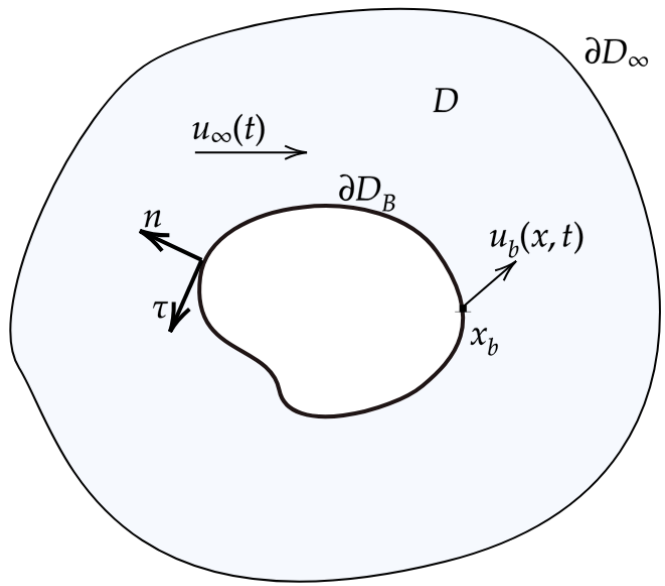
Outline

Introduction

Statement of the problem



$$\begin{cases} \dfrac{D\mathbf{u}}{Dt} = -\dfrac{\nabla p}{\rho_0} + \nu\nabla^2\mathbf{u} + \mathbf{f} & \mathbf{x} \in D \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$



$$\begin{cases} \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) & \mathbf{x} \in D, \\ \mathbf{u}(\mathbf{x}_b) = \mathbf{u}_b & \mathbf{x} \in \partial D_B, \\ \mathbf{u}(\mathbf{x},t) \to \mathbf{u}_\infty(t) & |\mathbf{x}| \to \infty, \end{cases}$$

Splitting approach and projection method for NSE

Solving NSE in $\omega - u$

Vorticity formulation of NS for incompressible fluid:

$$\left.\begin{array}{ll} \nabla \times \left(\dfrac{D\mathbf{u}}{Dt} = -\dfrac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u} \right) & \\ \omega = \nabla \times \mathbf{u} & \nabla \cdot \mathbf{u} = 0 \end{array}\right\} \quad \Rightarrow \frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega},$$

Solving NSE in $\omega - u$

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- in 2D the stretching term $\omega \cdot \nabla \mathbf{u}$ is zero

$$\begin{cases} \dfrac{D\omega}{Dt} = \nu \nabla^2 \omega, & on \quad D, \\ \omega(\mathbf{x}, 0) = \omega_0(\mathbf{x}), & \text{for all } \mathbf{x} \in D, \end{cases}$$

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- Use Operator Splitting to get the two sub–problems of advection and diffusion

$$ADVECTION \begin{cases} \dfrac{D\omega}{Dt} = 0, \\ \dfrac{D\mathbf{x}}{Dt} = \mathbf{u}(\mathbf{x}, t), \end{cases} \qquad \nabla^2 \mathbf{u} = -\nabla \times \omega$$

$$DIFFUSION \begin{cases} \partial_t \omega = \nu \nabla^2 \omega, \quad in \quad D \\ \nu \dfrac{\partial \omega}{\partial n} = -\dot{\gamma}, \quad on \quad \partial D_B \end{cases}$$

Alexandre Joel Chorin. "Numerical study of slightly viscous flow". In: *Journal of fluid mechanics* 57.4 (1973), pp. 785–796

Representation of potential flow through an IBEM ($\omega - u$)

$$\textit{ADVECTION} \begin{cases} \dfrac{D\omega}{Dt} = 0, \\ \dfrac{D\mathbf{x}}{Dt} = \mathbf{u}(\mathbf{x}, t), \end{cases} \qquad \nabla^2 \mathbf{u} = -\nabla \times \omega,$$

- Decompose velocity field using Helmholtz-Hodge decomposition

$$\mathbf{u}(\mathbf{x}, t) = \nabla\phi + \nabla \times \psi = \mathbf{u}_\infty(t) + \mathbf{u}_\phi(\mathbf{x}, t) + \mathbf{u}_\omega(\mathbf{x}, t),$$

- Assign the rotational part to $\mathbf{u}_\omega$ where

$$\mathbf{u}_\omega = \int_D K(\mathbf{x}, \mathbf{y})\omega(\mathbf{y})\, dy$$

- Represent irrotational part through an IBEM and use it to enforce the no-slip BCs

$$\mathbf{u}_\phi(\mathbf{x}, t) = -\int_{\partial D_B} \underbrace{\boxed{\sigma(\mathbf{y}, t)}}_{\text{Unknown source density}} \nabla G(\mathbf{x}, \mathbf{y})\, dl_y + \int_{\partial D_B} \underbrace{\boxed{\gamma(\mathbf{y}, t)}}_{\text{Unknown circulation density}} \mathbf{K}(\mathbf{x}, \mathbf{y})\, dl_y,$$

where $G(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \log(|\mathbf{x} - \mathbf{y}|)$, $K(\mathbf{x}, \mathbf{y}) = \nabla \times (G\mathbf{e}_3)$ are in free space

Representation of potential flow through an IBEM ($\omega - u$)

Evaluation of σ:

- Assign a constant density circulation

$$\gamma(\mathbf{y}, t) = \hat{\gamma}(t) \;=\; -\Gamma(t)\big/\mathcal{P}, \qquad \Gamma(t) = \int_D \omega(\mathbf{y}, t)\, d\mathbf{y}\,.$$

- this leads to

$$\mathbf{u}'_\phi(\mathbf{x}, t) \;= -\int_{\partial D_B} \sigma(\mathbf{y}, t) \nabla G(\mathbf{x}, \mathbf{y})\, dl_y \;+\; \hat{\gamma}(t) \int_{\partial D_B} \mathbf{K}(\mathbf{x}, \mathbf{y})\, dl_y\,,$$

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- Apply no through condition for potential flow to find σ

$$(\mathbf{u}'_\phi \cdot \mathbf{n})(s, t) = [\mathbf{u}_b(s, t) - \mathbf{u}_\infty(t)] \cdot \mathbf{n}(s, t) - \mathbf{u}_\omega(s, t) \cdot \mathbf{n}(s, t), \qquad s \in \partial D_B.$$

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- Discretize the boundary into N panels with constant strength $\hat{\gamma}$:

$$[\mathbf{A}]\{\sigma\} = [\mathbf{C}]\{\hat{\gamma}\} - \{\mathbf{u}_b \cdot \mathbf{n}\} + \{\mathbf{u}_\infty \cdot \mathbf{n}\} + \{\mathbf{u}_\omega \cdot \mathbf{n}\}$$

- Solve the $N \times N$ system to get σ using *LDU decomposition*

Representation of potential flow through an IBEM ($\omega - u$)

Evaluation of γ:

- Generation of circulation density to enforce no-slip BC

$$\gamma(s,t) = [\mathbf{u}_b(s,t) - \mathbf{u}_\infty(t)] \cdot \boldsymbol{\tau}(s,t) - [\mathbf{u}'_\phi(s,t) + \mathbf{u}_\omega(s,t)] \cdot \boldsymbol{\tau}(s,t), \quad s \in \partial D_B,$$

- The discretized form is

$$\{\gamma\} = [\mathbf{A}]\{\hat{\gamma}\} + [\mathbf{C}]\{\sigma\} - \{\mathbf{u}_b \cdot \mathbf{n}\} + \{\mathbf{u}_\infty \cdot \mathbf{n}\} + \{\mathbf{u}_\omega \cdot \mathbf{n}\}$$

γ is diffused during the diffusion step into the flow using the Neumann bc

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$$\mathbf{u}_\phi(\mathbf{x},t) = -\int_{\partial D_B} \sigma(\mathbf{y},t) \nabla G(\mathbf{x},\mathbf{y})\, dl_y + \int_{\partial D_B} \gamma(\mathbf{y},t) \mathbf{K}(\mathbf{x},\mathbf{y})\, dl_y,$$

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- Particles are advected during the next advection step with velocity that satisfies the no-slip BC

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}_\infty(t) + \int_{\partial D_B} \left[\mathbf{K}(\mathbf{x},\mathbf{y})\gamma(\mathbf{y}) - \mathbf{e_3} \times \mathbf{K}(\mathbf{x},\mathbf{y})\sigma(\mathbf{y})\right] dl_y + \int_D \mathbf{K}(\mathbf{x},\mathbf{y})\omega(\mathbf{y})\, d\mathbf{y}\,.$$

Evaluation of hydrodynamic loads ($\omega - u$)

- Total force: Drag and Lift coefficients

$$\mathbf{F}_B = \int_{\partial D_B} (-p\mathbf{n} + \mu\omega\boldsymbol{\tau}) \rightarrow \begin{cases} C_D = \dfrac{F_{B_x}}{\frac{1}{2}\rho u_\infty^2 S} \\ C_L = \dfrac{F_{B_y}}{\frac{1}{2}\rho u_\infty^2 S}, \end{cases}$$

- Torque around the center of mass $\mathbf{x}_{cm}$

$$\mathbf{T}_B = \int_{\partial D_B} (\mathbf{x} - \mathbf{x}_{cm}) \times (-p\mathbf{n} + \mu\omega\boldsymbol{\tau}),$$

- Pressure

$$\boxed{\frac{1}{\rho_0}\frac{\partial p}{\partial s} = \boldsymbol{\tau}(s) \cdot [\dot{\mathbf{u}}_\infty - \dot{\mathbf{u}}_b(s)] + \dot{\gamma}(s), \quad s \in \partial D_B.}$$

Solving NSE in $u - p$

- Apply the divergence operator on NSE

$$\frac{D}{Dt}(\nabla \cdot \mathbf{u}) - \omega^2 + \nabla \mathbf{u} : \nabla \mathbf{u} = -\frac{\nabla^2 p}{\rho_0} + \nu \nabla^2 (\nabla \cdot \mathbf{u}),$$

Method 1

- Use incompressibility condition to get a Pressure Poisson equation

$$\begin{cases} \nabla^2 p = \rho_0 \left(\omega^2 - \nabla \mathbf{u} : \nabla \mathbf{u} \right), \\ \\ \dfrac{\partial p}{\partial n} = -\mathbf{n} \cdot [\dot{\mathbf{u}}_b - \dot{\mathbf{u}}_\infty] + \nu \dfrac{\partial \omega}{\partial \tau}, \quad on \quad \partial D_B, \end{cases}$$

Main drawback

- The method does not guarantee divergence free flow field, because

$$\frac{D}{Dt}(\nabla \cdot \mathbf{u}) = \nu \nabla^2 (\nabla \cdot \mathbf{u}),$$

$$\nabla \cdot \mathbf{u}\Big|_{t=0} = 0,$$

- If incompressibility is not satisfied at $t = 0$ or on the boundary, then $\nabla \cdot \mathbf{u} \neq 0$ in the field

Solving NSE in $u - p$

Method 2

- Evaluate velocity from momentum equation ignoring viscous contributions and the divergence constraint

$$\mathbf{u}^* = \mathbf{u}^n + \nu \nabla^2 \mathbf{u}^n \, \Delta t.$$

- Decompose resulting velocity using Helmholtz - Hodge decomposition (HHD)

$$\mathbf{u}^* = \nabla \times \boldsymbol{\Psi} + \nabla \Phi.$$

- Assign the rotational part to the velocity at next level and the irrotational to the pressure gradient

$$\nabla \times \boldsymbol{\Psi} = \mathbf{u}^{n+1}, \qquad \nabla \Phi = \left(\frac{\nabla p}{\rho_0} \right)^{n+1} \Delta t.$$

- The velocity at next level is divergent free

$$\mathbf{u}^{n+1} = \mathbf{u}^* - \left(\frac{\nabla p}{\rho_0} \right)^{n+1} \Delta t$$

- Apply divergence operator

$$\nabla^2 p^{n+1} = \rho_0 \frac{(\nabla \cdot \mathbf{u}^*)}{\Delta t}.$$

Alexandre Joel Chorin. "Numerical solution of the Navier-Stokes equations". In: *Mathematics of computation* 22.104 (1968), pp. 745–762

Solving NSE in $u - p$

Method 3

- The divergence constraint is coupled to the momentum equation by introducing the pressure field

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial \tau} + \dfrac{D\mathbf{u}}{Dt} = -\dfrac{\nabla p}{\rho_0} + \nu \nabla^2 \mathbf{u}, \\ \dfrac{\partial p}{\partial \tau} + \beta \nabla \cdot \mathbf{u} = 0, \end{cases}$$

where β: pseudo-compressibility parameter, τ: pseudo-time, $\sqrt{\beta}$: pseudo-sound speed

- Use Dual Time Stepping
- At each iteration in real time we do sub-iterations in pseudo-time to enforce incompressibility
- Obtain a solution that satisfies $\nabla \cdot \mathbf{u}$ when steady state is reached in τ

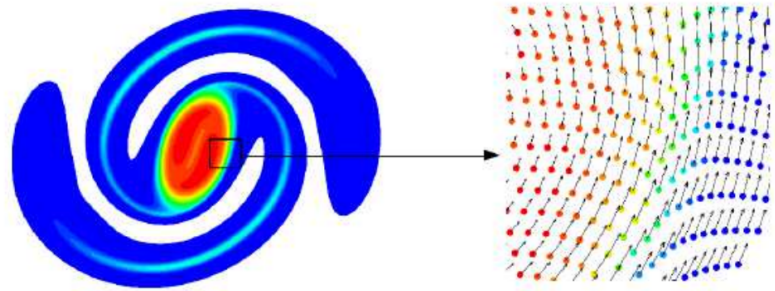
Alexandre Joel Chorin. "A numerical method for solving incompressible viscous flow problems". In: *Journal of computational physics* 2.1 (1967), pp. 12–26

Brief description on the numerical schemes adopted

Diffused Vortex Hydrodynamics: The advection step

- Discretization of vorticity field

$$\omega_\epsilon(\mathbf{x}, t) \approx \sum_{j=1}^{N_v} \Gamma_j(t)\delta_\epsilon(\mathbf{x} - \mathbf{x}_j(t))$$



  - Evaluation of velocity that satisfies no - slip condition

$$\mathbf{u}(\mathbf{x}_i, t) = \mathbf{u}_\infty(t) + \underbrace{\boxed{\sum_{k=1}^{N_b}[\mathbf{K}_\epsilon(\mathbf{x}_i, \mathbf{y}_k)\gamma_k(t) - \mathbf{e}_3 \times \mathbf{K}_\epsilon(\mathbf{x}_i, \mathbf{y}_k)\sigma_k(t)]\Delta s}}_{\mathbf{u}_\phi(t)} + \underbrace{\boxed{\sum_{j=1}^{N_v} \Gamma_j(t)\mathbf{K}_\epsilon(\mathbf{x}_i, \mathbf{y}_j)}}_{\mathbf{u}_\omega(t)}$$

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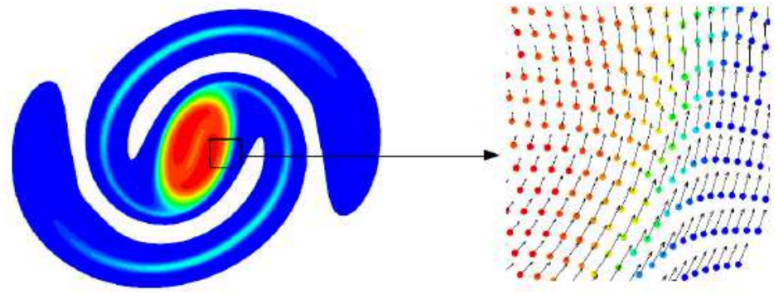

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- Advect particles using 4^{th} order Runge - Kutta

$$\begin{cases} \dfrac{d\mathbf{x}_i(t)}{dt} & = \mathbf{u}(\mathbf{x}_i, t) \\ \dfrac{D\Gamma_i(t)}{Dt} & = 0 \end{cases}$$

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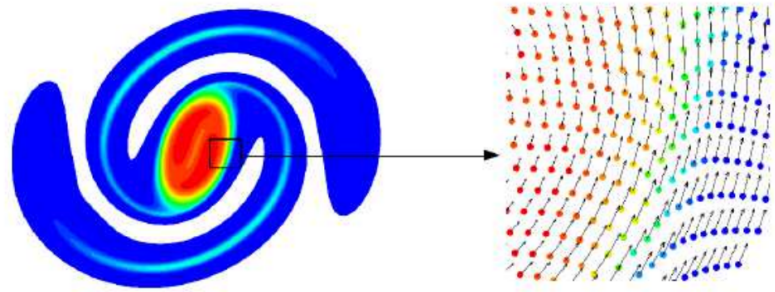

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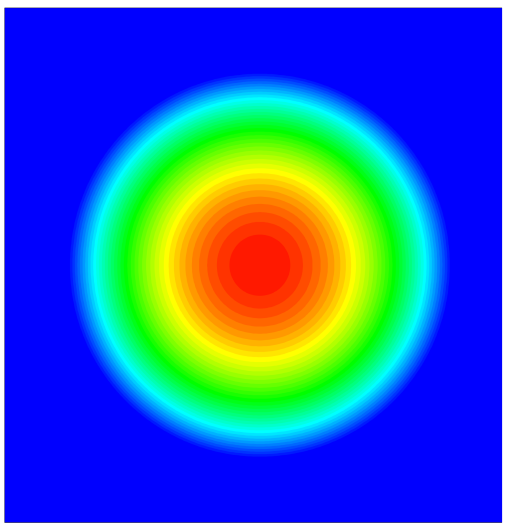
Advection Time step

$$\Delta t_a = K_a \frac{\Delta r_{min}}{U}$$

$K_a \sim O(10^{-1})$
Δr_{min}: size of the smallest particle
U: reference velocity

Diffused Vortex Hydrodynamics: Diffusion



- Diffusion is performed in deterministic way
- Truncate the exact solution of the heat equation within R_d (diffusive radius).
- Then *each particle* diffuses according to

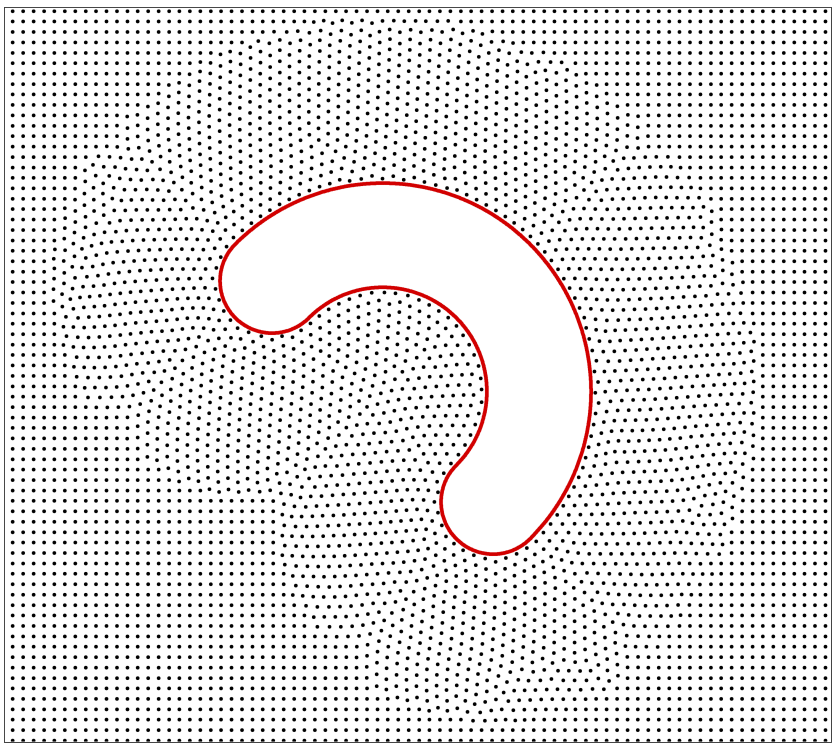
$$\omega(\mathbf{x}, t+\Delta t) = \begin{cases} \dfrac{\Gamma_i}{4\pi\nu\Delta t_d} \exp\left(-\dfrac{|\mathbf{x}-\mathbf{x}_i|^2}{4\nu\Delta t_d}\right) & |\mathbf{x}-\mathbf{x}_i| \leq R_d, \\ 0 & \text{otherwise}, \end{cases}$$

- The truncation introduces a relative error of circulation conservation:

$$\xi = exp\left(-\frac{R_d^2}{4\nu\Delta t_d}\right).$$

- To respect the conservation of circulation a redistribution procedure is performed

Packing Algorithm



- Generates a body fitted distribution of points around the boundary

$\Downarrow$

Allows to model flows around arbitrary geometries without the need of generating a mesh

- The distribution of particles is regular (all particles have a uniform mean distance)

$\Downarrow$

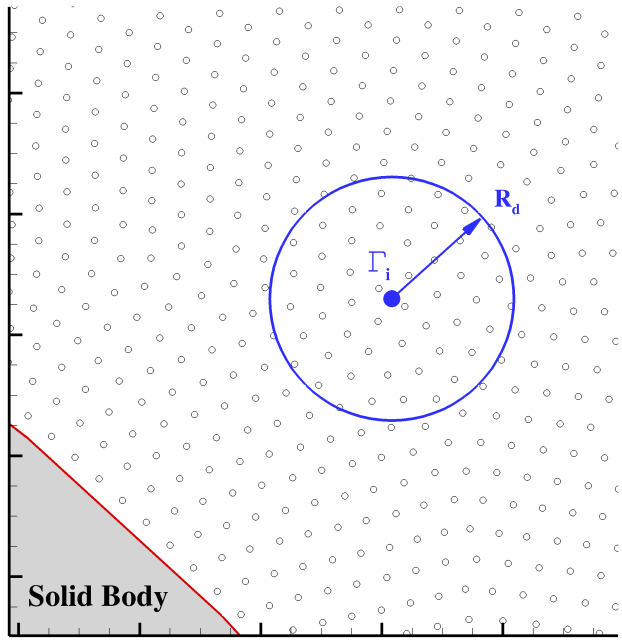
Can be used to address the problem of Lagrangian distortion

- We call this distribution *Regular Point Distribution (RPD)*

Andrea Colagrossi et al. "Particle packing algorithm for SPH schemes". In: *Computer Physics Communications* 183.8 (2012), pp. 1641–1653

Diffused Vortex Hydrodynamics: The Diffusion step

- Diffusion of a vortex is performed on a Regular Point Distribution (RPD)

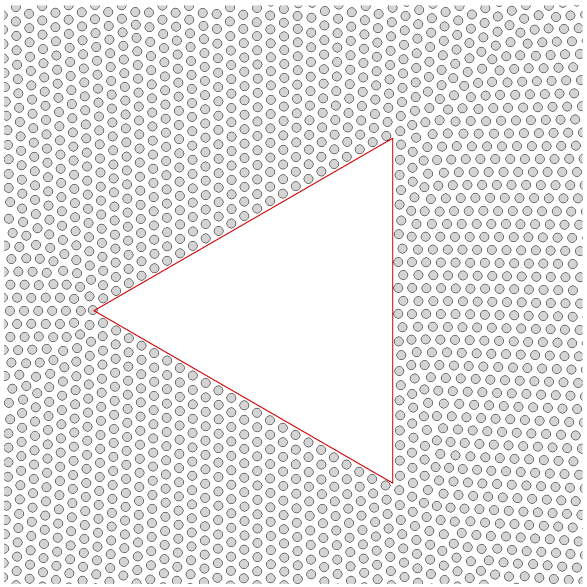


- Start with a single point vortex
- Find the points with $|x - x_0| \leq R_d$
- Use approx. solution of heat eq. to diffuse circulation over the nodes
- Conserve circulation with a redistribution procedure
- The nodes of the RPD become the new vortices

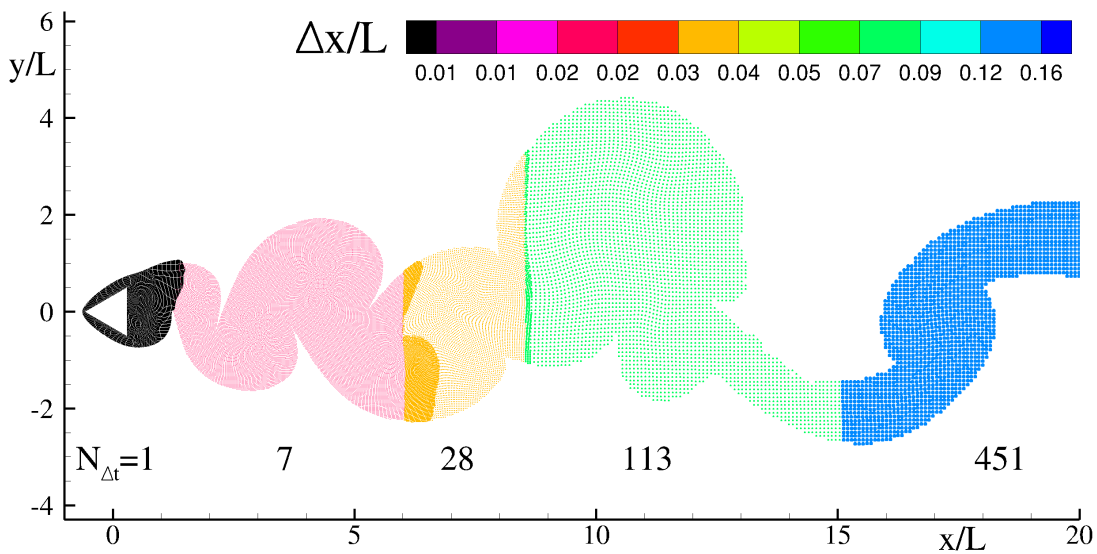
Diffusion Time step

$$\Delta t_d = 0.021 \frac{R_d^2}{\nu} = 0.34 \frac{\Delta r_j^2}{\nu}$$

Adaptive Particle Refinement



- Close to the body we have the finest resolution (small particles)
- Far from the body the resolution in decreased
- In this way we contain the growth of particles during diffusion



FVM: Discretization

- The computational domain D is split into subdomains, each discretized by a block-structured grid subdivided into $N_i \times N_j$ subdomains D_{ij}. The NSEs are written in integral form on each volume D_{ij}:

$$\begin{cases} \displaystyle\int_{\partial D_{ij}} \mathbf{u} \cdot \mathbf{n}\, dS = 0, \\ \displaystyle\int_{D_{ij}} \frac{\partial \mathbf{u}}{\partial t} dV + \int_{\partial D_{ij}} \mathbf{F} \cdot \mathbf{n}\, dS = 0, \end{cases}$$

$$\mathbf{F} = \mathbf{u} \otimes \mathbf{u} + \frac{p}{\rho_0}\mathbb{I} - \mathbb{T}$$

$$\mathbb{T} = \nu \left(\nabla \mathbf{u} + \nabla^T \mathbf{u} \right),$$

$\mathbb{I}$: unit tensor, $\mathbb{T}$: stress tensor.

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$$\mathbf{F} = \mathbf{u} \otimes \mathbf{u} + \frac{p}{\rho_0}\mathbb{I} - \mathbb{T}$$

$$\mathbb{T} = \nu \left(\nabla \mathbf{u} + \nabla^T \mathbf{u} \right),$$

$\mathbb{I}$: unit tensor, $\mathbb{T}$: stress tensor.

Setting:

$$\mathbf{u}_{ij} = \frac{1}{V_{ij}} \int_{D_{ij}} \mathbf{u}\, dV,$$

$$\mathbf{F}_l = \int_{S_l} \mathbf{F} \cdot \mathbf{n}\, dS,$$

$$\mathbf{U}_l = \int_{S_l} \mathbf{u} \cdot \mathbf{n}\, dS,$$

FVM: Discretization

- The computational domain D is split into subdomains, each discretized by a block-structured grid subdivided into $N_i \times N_j$ subdomains D_{ij}. The NSEs are written in integral form on each volume D_{ij}:

$$\begin{cases} \displaystyle\int_{\partial D_{ij}} \mathbf{u} \cdot \mathbf{n}\, dS = 0, \\ \displaystyle\int_{D_{ij}} \frac{\partial \mathbf{u}}{\partial t} dV + \int_{\partial D_{ij}} \mathbf{F} \cdot \mathbf{n}\, dS = 0, \end{cases}$$

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$$\mathbf{u}_{ij} = \frac{1}{V_{ij}} \int_{D_{ij}} \mathbf{u}\, dV,$$

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$$\mathbb{T} = \nu \left(\nabla \mathbf{u} + \nabla^T \mathbf{u} \right),$$

$\mathbb{I}$: unit tensor, $\mathbb{T}$: stress tensor.

$$\boxed{\begin{cases} \displaystyle\sum_{l \in L_{ij}} \mathbf{U}_l = 0, \\ \displaystyle V_{ij} \frac{d\mathbf{u}_{ij}}{dt} + \sum_{l \in L_{ij}} \mathbf{F}_l = 0, \end{cases}}$$

- L_{ij} is the subset of the cell interfaces for subdomain D_{ij}.

FVM: Pseudo-Compressibility

- Implicit Crank Nicolson in time

$$\begin{cases} \displaystyle\sum_{I \in L_{ij}} \mathbf{U}_I^{n+1} = 0, \\ \displaystyle V_{ij} \frac{\mathbf{v}_{ij}^{n+1} - \mathbf{v}_{ij}^{n}}{\Delta t} + \frac{1}{2} \left(\sum_{I \in L_{ij}} \mathbf{F}_I^{n+1} + \sum_{I \in L_{ij}} \mathbf{F}_I^{n} \right) = 0, \end{cases}$$

- Pseudo-Compressibility in pseudo-time

$$\begin{cases} \displaystyle\frac{\partial p^{n+1}}{\partial \tau} + \rho_0 \beta \left(\sum_{I \in L_{ij}} \mathbf{U}_I^{n+1} \right) = 0, \\ \displaystyle\frac{\partial \mathbf{v}^{n+1}}{\partial \tau} + \left[V_{ij} \frac{\mathbf{v}_{ij}^{n+1} - \mathbf{v}_{ij}^{n}}{\Delta t} + \frac{1}{2} \left(\sum_{I \in L_{ij}} \mathbf{F}_I^{n+1} + \sum_{I \in L_{ij}} \mathbf{F}_I^{n} \right) \right] = 0. \end{cases}$$

FVM: Grid structure

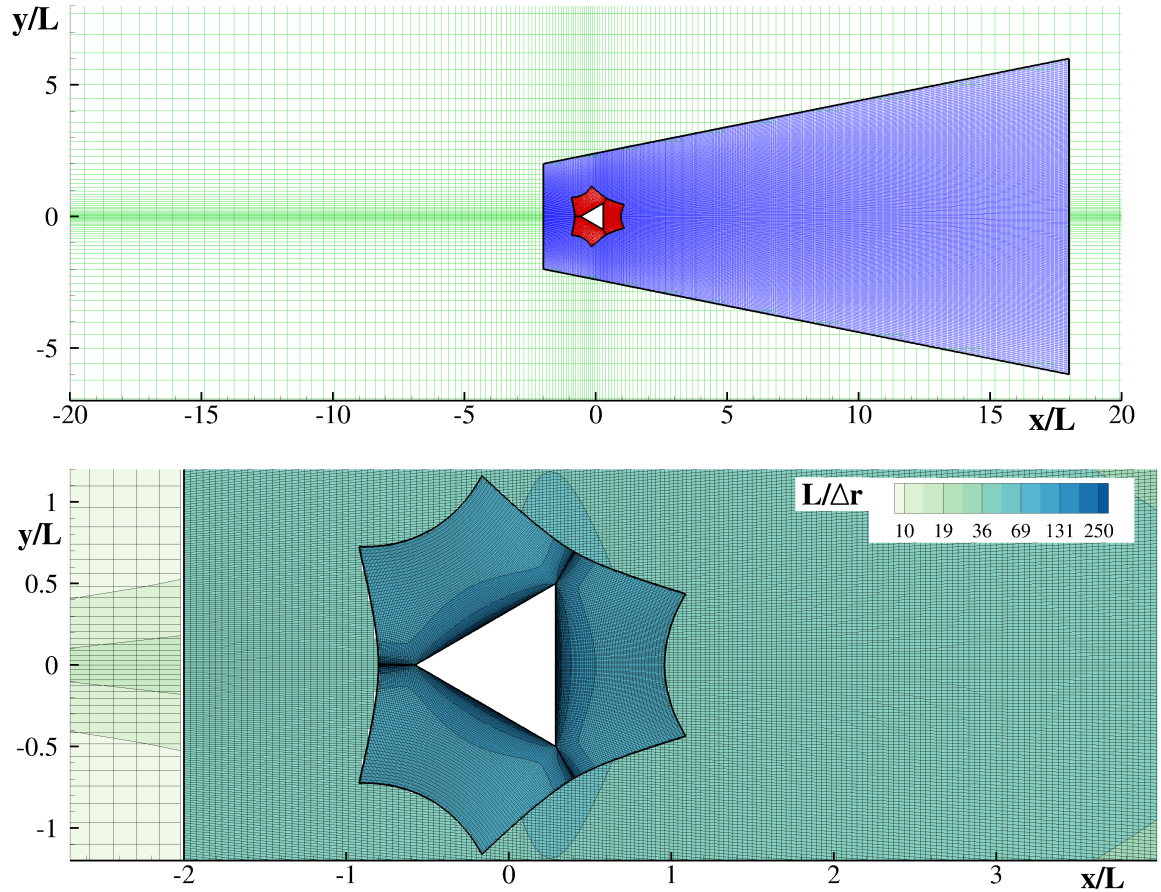


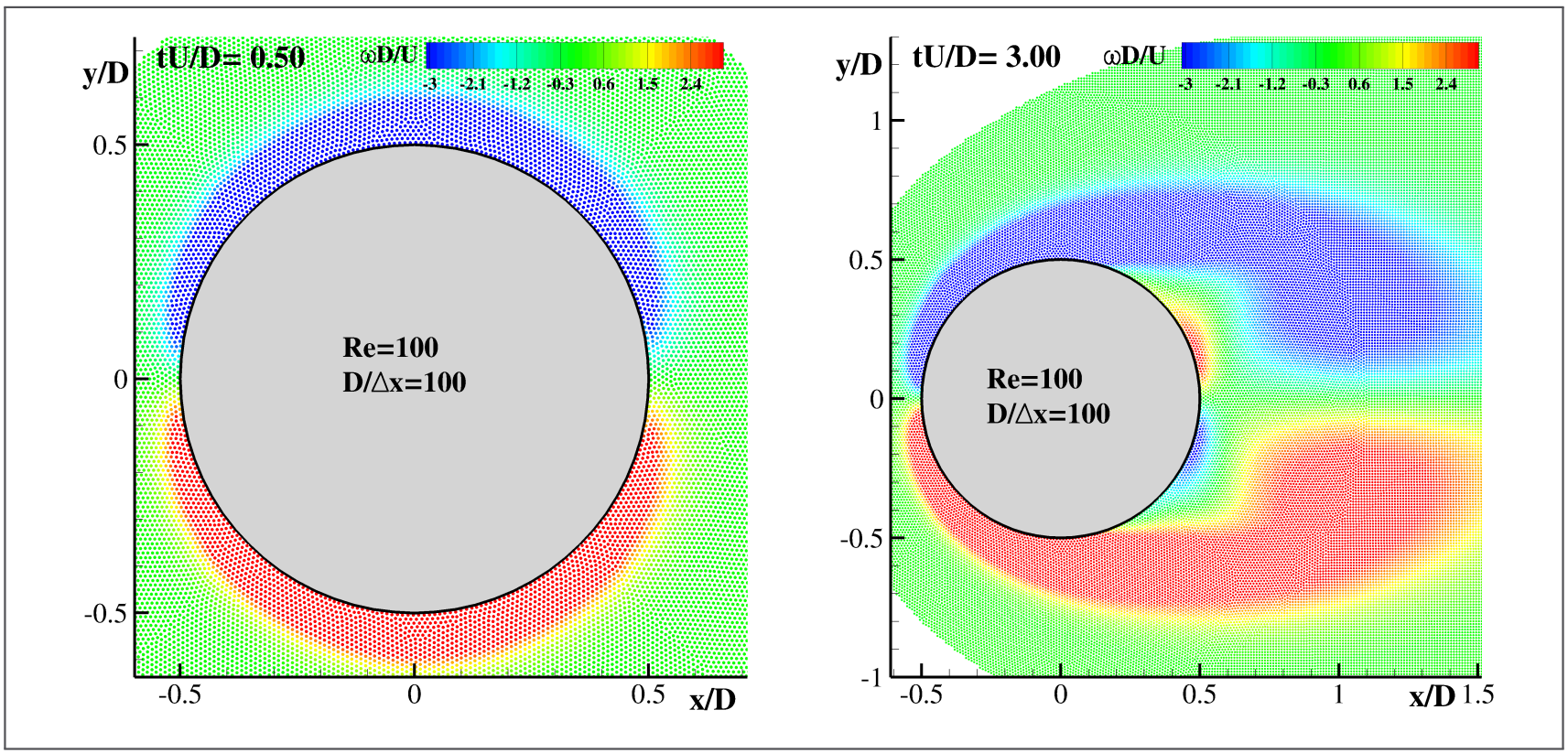
Figure: Discretization of the computational domain used for the FVM. Top: fluid domain from *-20L to 20L to enforce the radiation condition for the pressure;* the three chimera blocks in different colors. Bottom: view around the solid body; the color is representative of the local spatial resolution $L/\Delta r$, where Δr is the square root of the cell volume.

Comparison between FVM and DVH method through benchmark tests

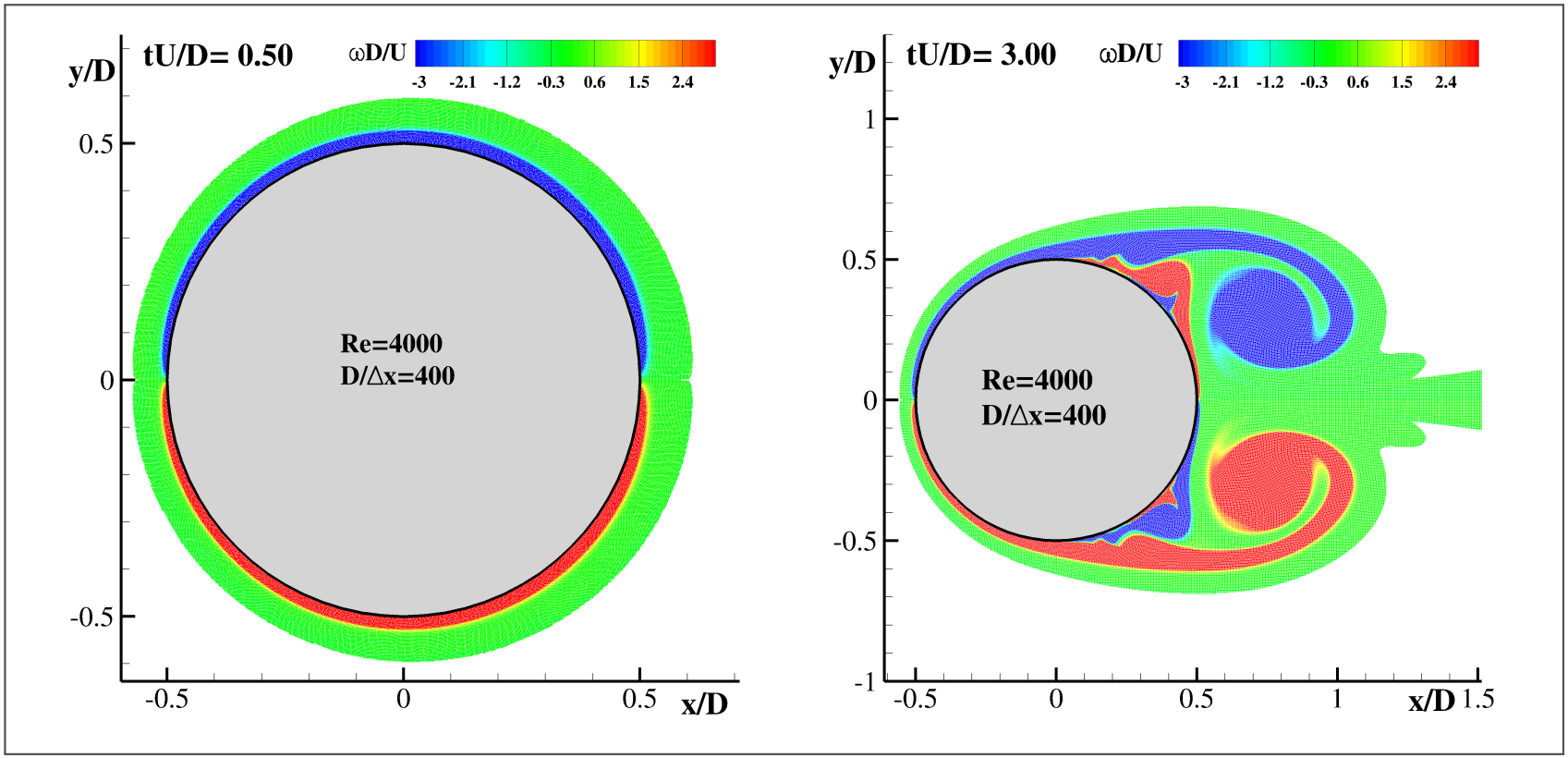
Flow past a circular cylinder in an unsteady current

$$u_\infty(t) = \begin{cases} \frac{U}{2}\left[1 - \cos\left(\frac{\pi\, t}{t_r}\right)\right] & t \le t_r \\ U & t > t_r \end{cases}$$

$$C_{Dp} = \frac{\int_{\partial D_B} -p\mathbf{n}\cdot\mathbf{e_1}\, ds}{\frac{1}{2}\rho U^2 D}, \qquad C_{Dv} = \frac{\int_{\partial D_B} \mu\omega\tau\cdot\mathbf{e_1}\, ds}{\frac{1}{2}\rho U^2 D}$$



Flow past a circular cylinder in an unsteady current



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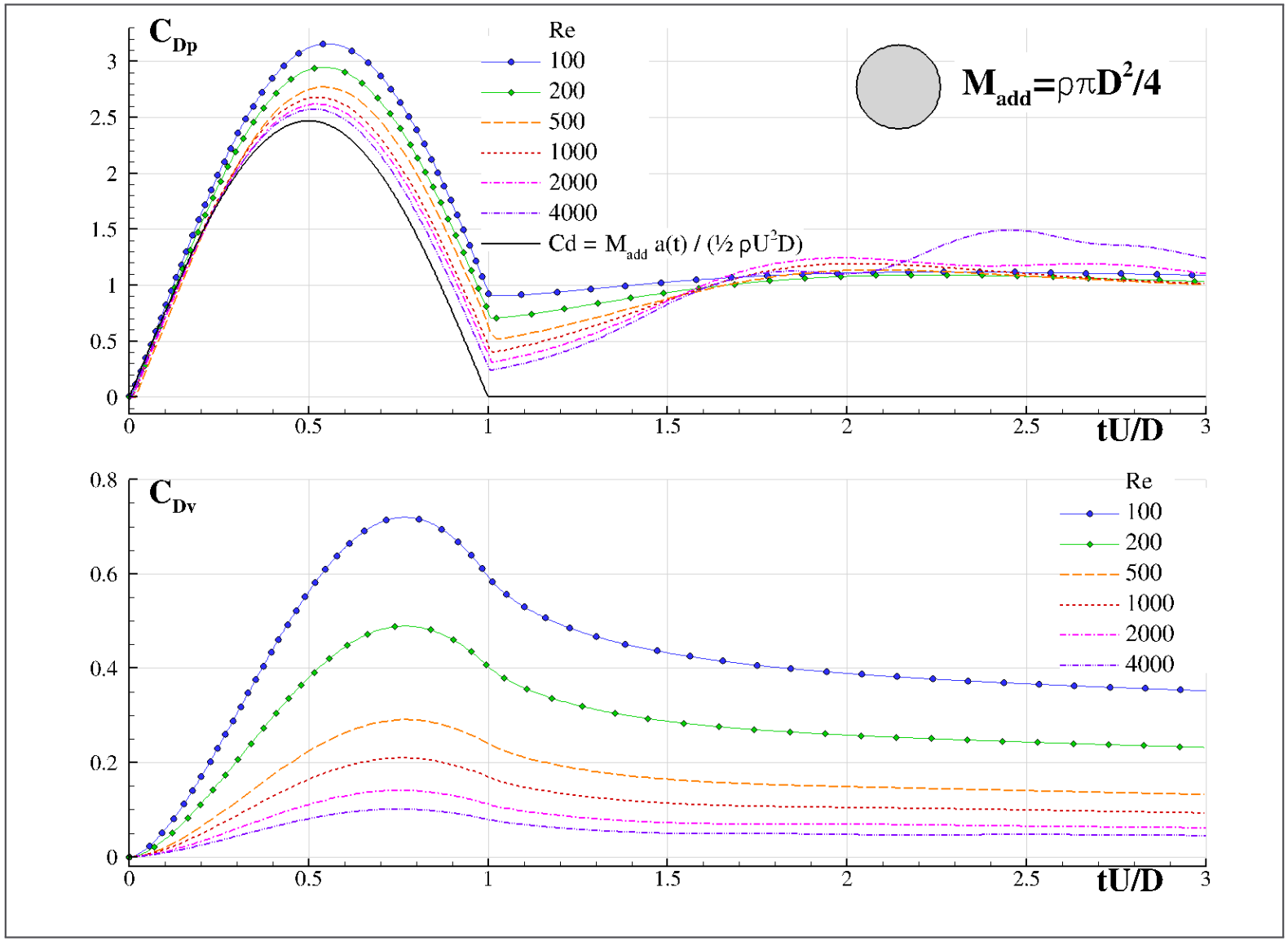
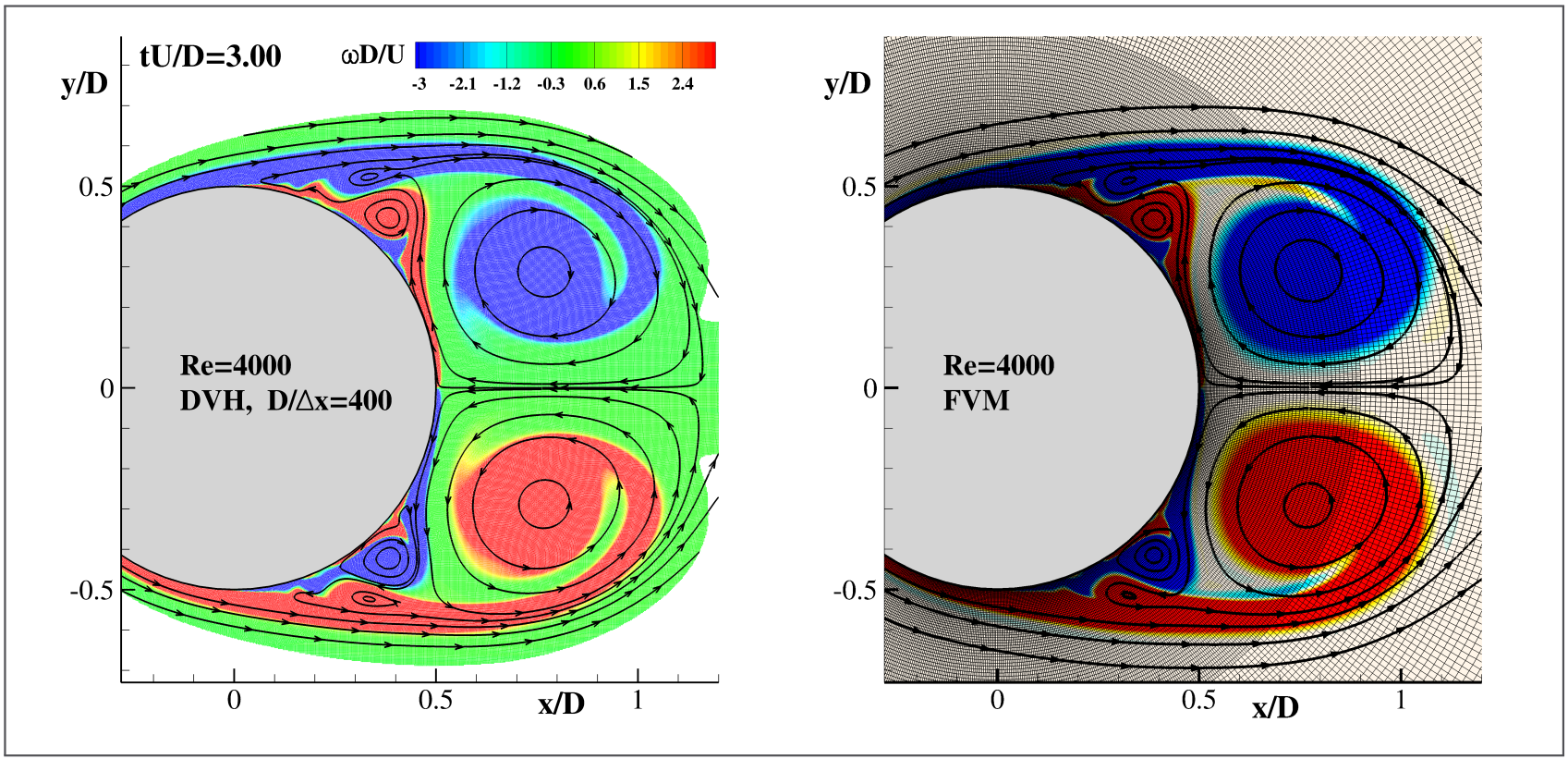


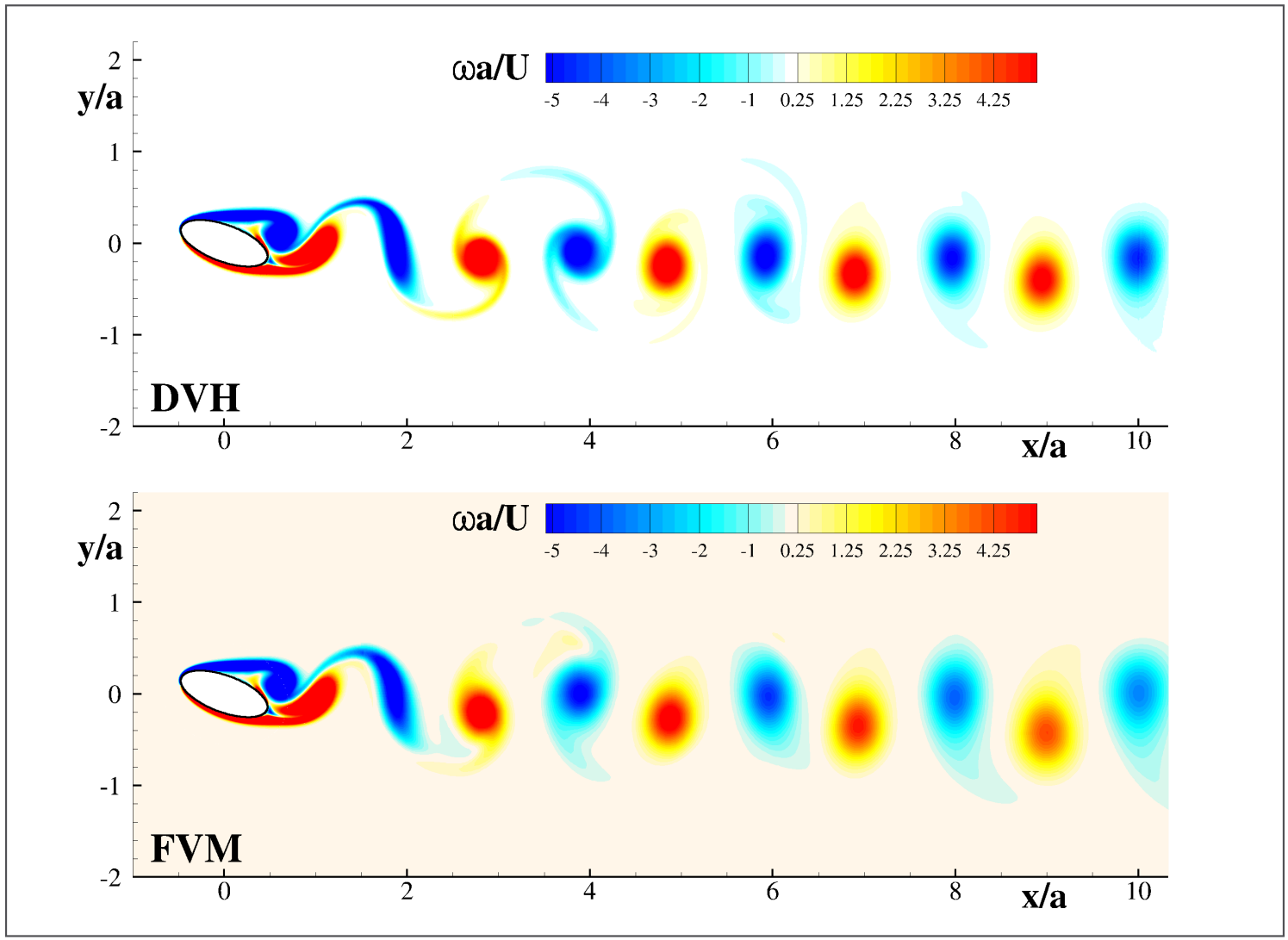
Flow past a circular cylinder in an unsteady current

Comparison with FVM for Re = 4000



Flow around an ellipse with incidence

Vorticity Fields, Re = 1000

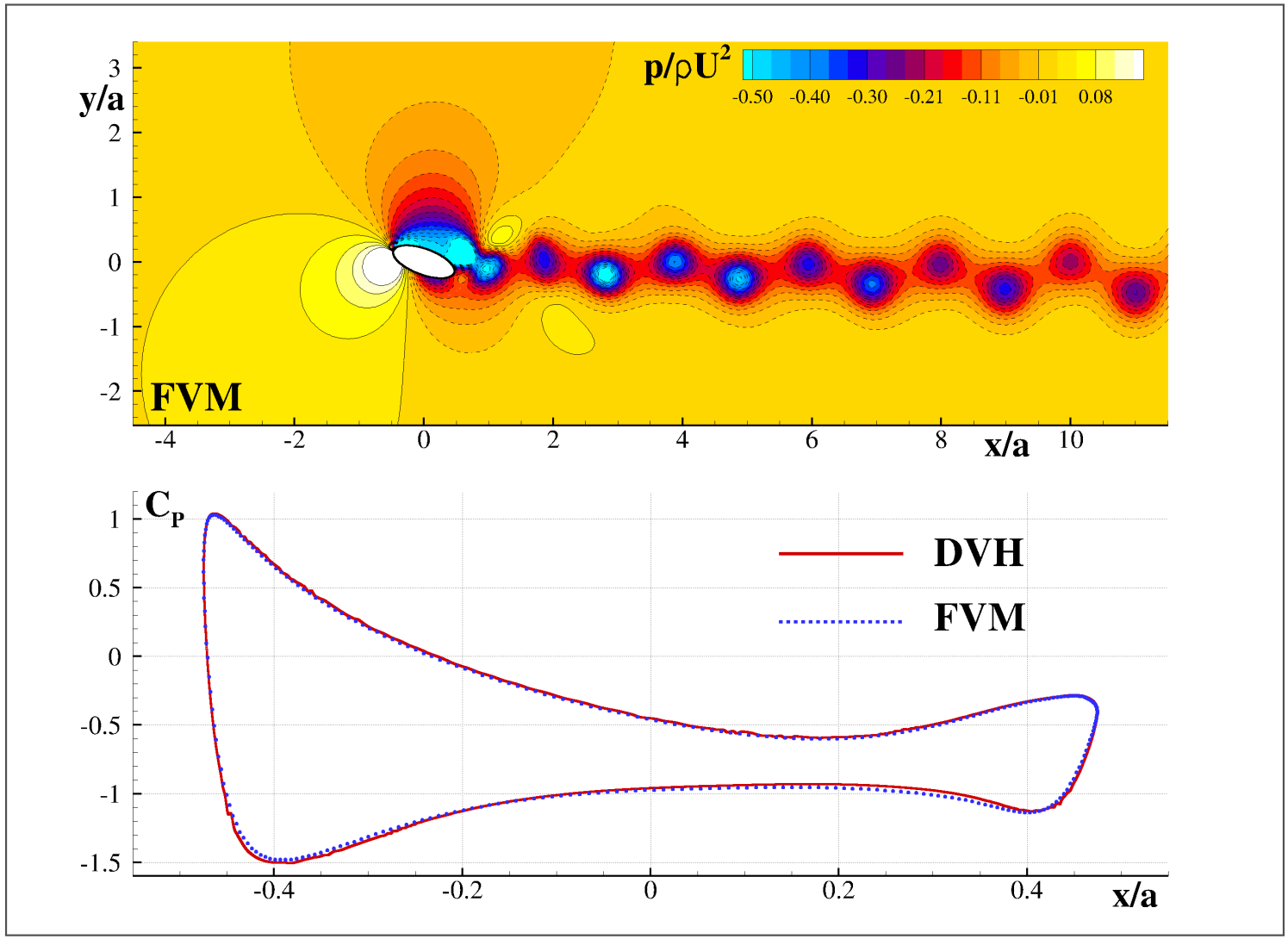


AoA: $\alpha = 20°$

Axis ratio : $b/a = 0.4$

Flow around an ellipse with incidence

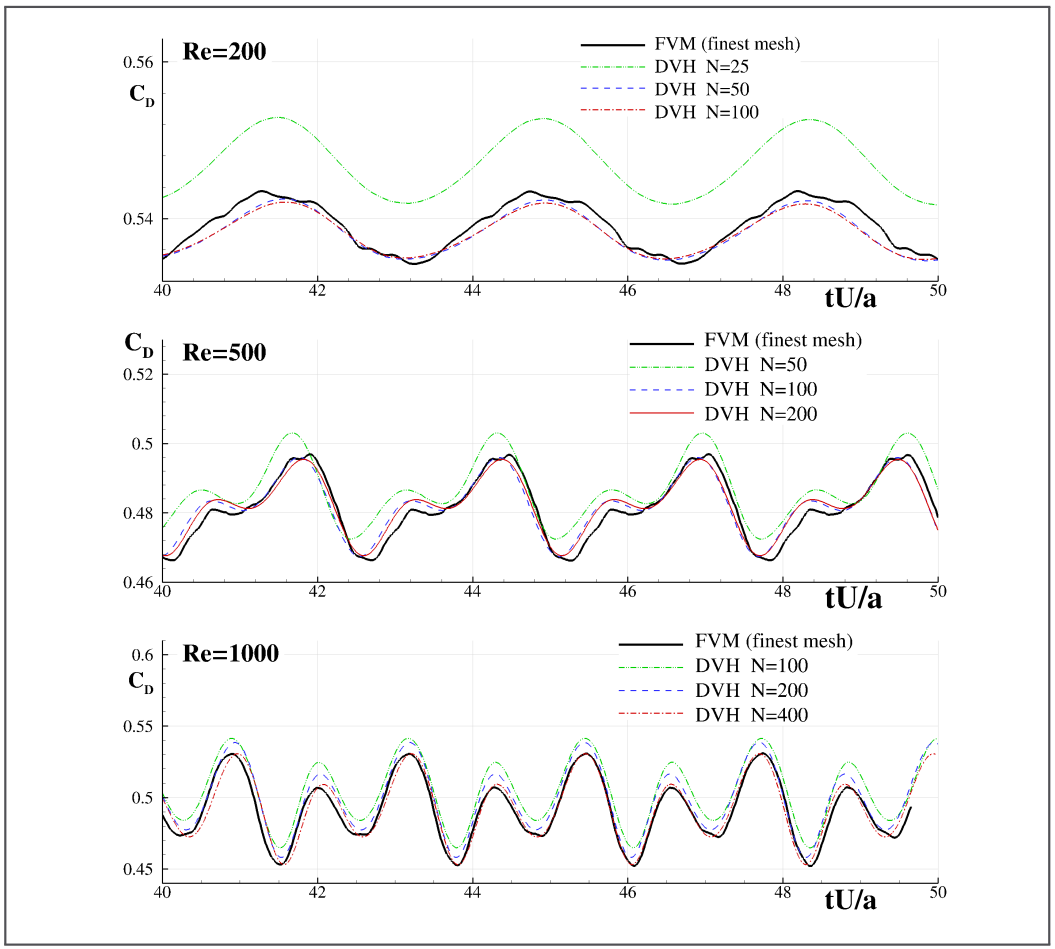
Comparison with FVM for Re = 1000, Pressure coeff.



Pressure coefficient: $C_p = \frac{(p - p_\infty)}{12\,\rho\,U^2}$

Flow around an ellipse with incidence

Comparison with FVM for Re = 200, 500, 1000. Convergence results



$$\epsilon_{21} = \int_{t_0}^{t_f} |f_2 - f_1| \, dt$$

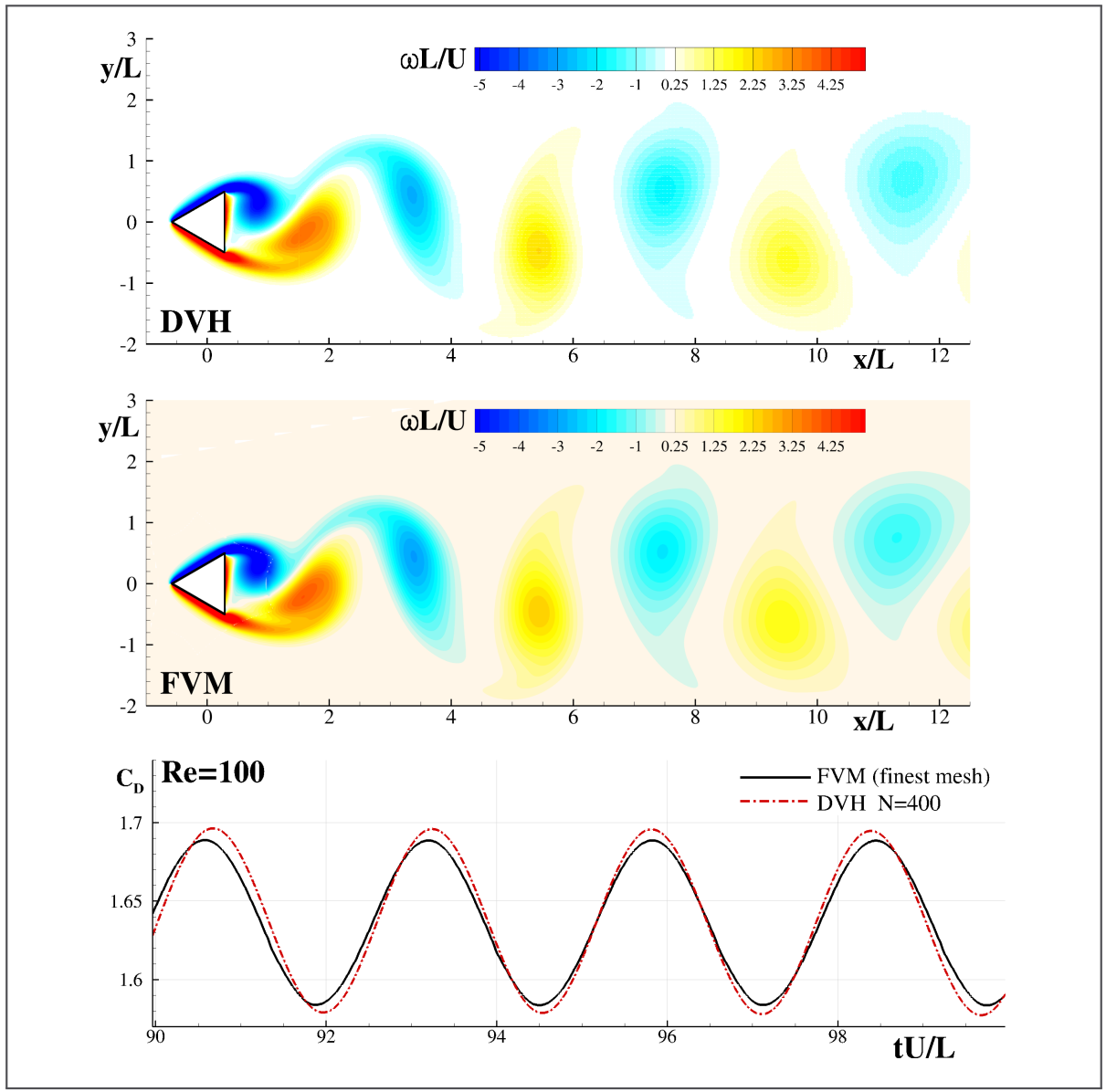
Convergence rate

$$C(C_D) = \log\left(\frac{\epsilon_{32}}{\epsilon_{21}}\right) \Big/ \log(2)$$

	$C(C_D)$	
Re	*DVH*	*FVM*
200	4.1	4.2
500	2.7	4.1
1000	2.0	2.4

Flow around a cylinder with triangular section Re=100

Vorticity fields and forces



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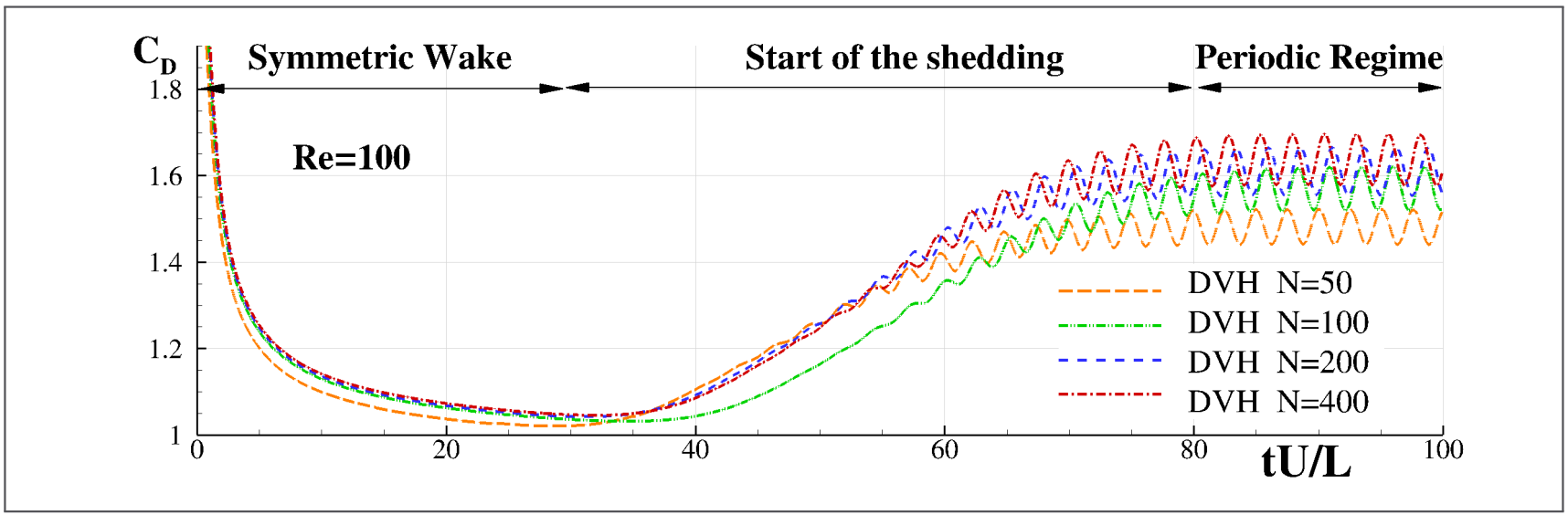
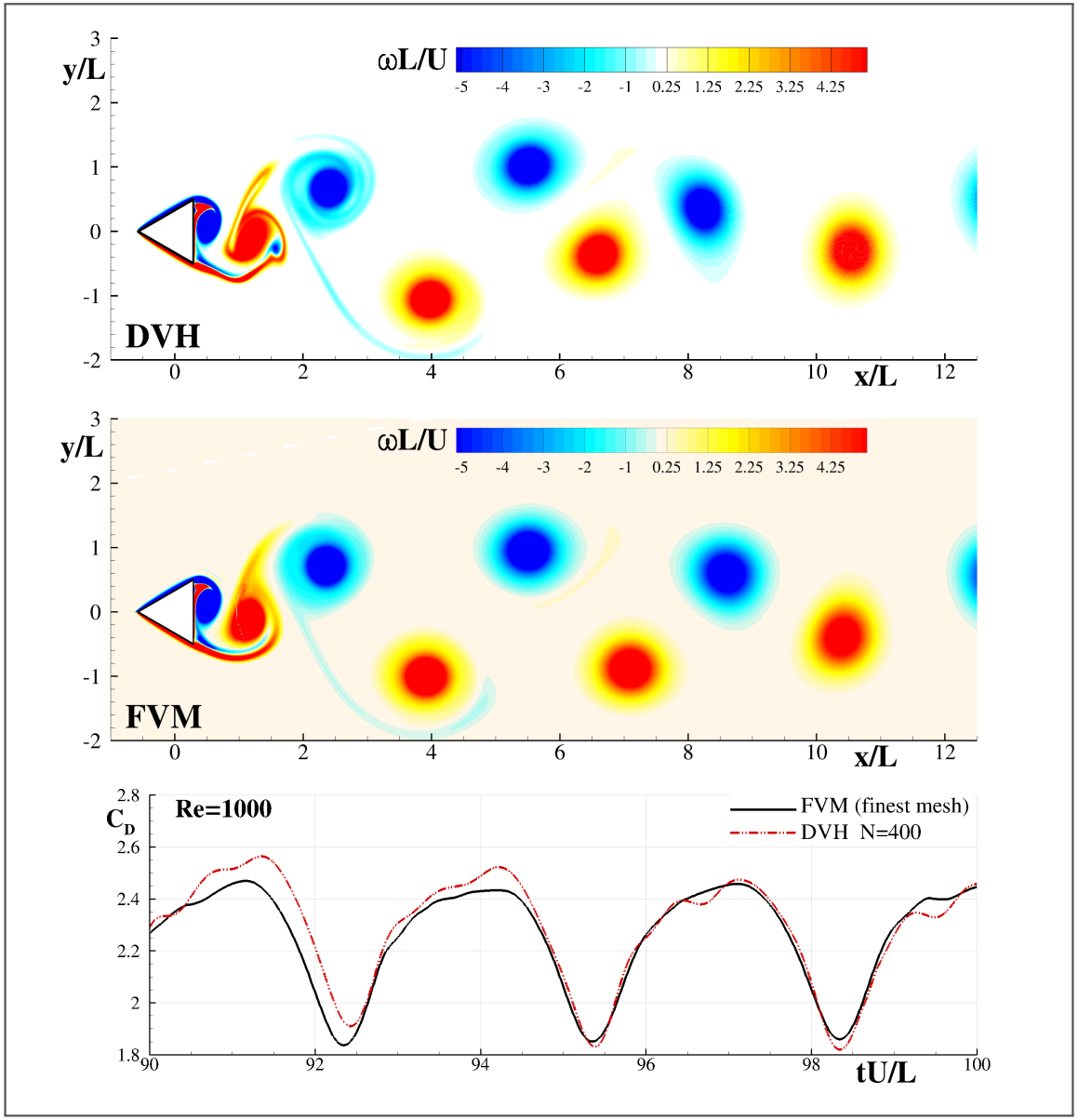


	$C(C_D)$	
	tU/L$\in [0,30]$	tU/L>30
DVH	2.0	1.0
FVM	2.8	1.0

Flow around a cylinder with triangular section Re=1000

Vorticity fields and forces



Conclusions

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1. Chorin's approaches for solving NSEs in $(\omega - u)$ and $(u - p)$ were presented
   - A meshless Particle Vortex Method based on operator splitting and a mesh-based FVM based on pseudo - compressibility
2. The numerical behaviour of DVH was investigated for three geometries
   - All test cases were unsteady characterized by flow separation and shear layer formation
3. First two geometries: the simulations are accurate, capturing the separation point and the flow characteristics for increasing Re. Third case: sharp edges freeze the separation point and boundary singularities induce strong, localized gradients.
4. For each test case the convergence rates were studied for increasing Re
   - For smooth boundaries the rate degrades with increasing Re but is always at least 2 for both solvers
   - For non-smooth boundary the rate is $O(1)$ for both solvers
5. Comparison with FVM - Pseudo Compressibility
   - Matching of the solutions in both local and global forces
   - Differences only for high Re for flow past a triangle, in the vortex positions in the far wake (different far wake resolution, transition to chaotic flow beahaviour)

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Thank you for your attention!