Gauge Uniqueness of solutions to the Ginzburg-Landau system for small domains

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Energy functional:

\[ G_{\kappa d}(\psi, A) = \frac{1}{2} \int_D \left( |\nabla \psi - iA\psi|^2 + \frac{\kappa^2 d^2}{2} (1 - |\psi|^2)^2 + \frac{1}{d^2} |\text{curl} A - h_e e_3|^2 \right) dx, \]

Ginzburg-Landau system:

\[
\begin{cases}
(\nabla - iA)^2 \psi = \kappa^2 d^2 (1 - |\psi|^2) \psi & \text{in } D, \\
\text{curl(curl } A) = -d^2 \left( \frac{i}{2} (\psi^* \nabla \psi - \psi \nabla \psi^*) + A|\psi|^2 \right), \\
(\nabla - iA) \psi \cdot \nu = 0 & \text{on } \partial D, \\
\text{curl } A = h_e e_3.
\end{cases}
\]

Unknowns:

\[ \psi \in W^{1,2}(D; C) : D \to C \] order parameter

\[ A \in W^{1,2}(D; \mathbb{R}^2) : D \to \mathbb{R}^2 \] magnetic potential
• $\kappa > 0$ Ginzburg-Landau parameter

• $d > 0$ distinguished length-scale

• $\nu$ outward normal to $\partial D$

• $\text{curl } A = (\partial_{x_1} A_2 - \partial_{x_2} A_1) e_3$

• $h_e > 0$ intensity of applied magnetic field

solutions are invariant under gauge transformation:

$$
\psi' = \psi e^{i\chi}, \quad A' = A + \nabla \chi,
$$

for an arbitrary real-valued function $\chi \in W^{2, 2}(D)$

$\psi \neq 0$ material is in superconducting state

$\psi \equiv 0$ and $\text{curl } A = h_e e_3$ normal state

Well-known: $|\psi| \leq 1$
Coulomb Gauge:

\[
\begin{cases}
\text{div } A = 0 \quad \text{in } D, \\
A \cdot \nu = 0 \quad \text{on } \partial D,
\end{cases}
\]

gauge is fixed up to a multiplicative complex constant of modulus one.

Elliptic regularity theory implies that an equilibrium solution is classical:

\[\psi \in C^2(D; C), \quad A \in C^2(D; \mathbb{R}^2)\]

Particular normal state: \((0, h_e a_n)\) with

\[
\begin{cases}
\text{curl curl } a_n = 0 \quad \text{in } D \\
\text{div } a_n = 0 \quad \text{in } D, \\
\text{curl } a_n = e_3 \text{ and } a_n \cdot \nu = 0 \quad \text{on } \partial D,
\end{cases}
\]

With these choices:

\[||\text{curl curl}(A - h_e a_n)||_{L^2}\]

is a norm for \(A - h_e a_n\) in \(W^{2,2}(D, \mathbb{R}^2)\), and

\[||\text{curl}(A - h_e a_n)||_{L^2}\]

is a norm for \(A - h_e a_n\) in \(W^{1,2}(D, \mathbb{R}^2)\).
Goal: study gauge-uniqueness of solutions to (1) for small \( d \) and \( \kappa \) fixed

Important: uniqueness of solutions not only of global minimizers

What is known?


A-D: show that for any non-normal solution the order parameter is never zero

Pan: derives a result in the same spirit for global minimizers for three dimensions

A-D: claim for \( D \) small circle a uniqueness result: there is a critical field \( h_*(d, \kappa) \) with

\[
\lim_{d \to 0} \frac{h_*(d, \kappa)}{d} = 2\kappa \sqrt{2},
\]

such that if \( h_e > h_* \) the only solution to (1) are the normal states, otherwise the only non-normal solutions are never zero and they are unique up to a gauge

Pan: presents a result in the same spirit for general domains in 3-d but only for global minimizers
Main contribution:

(i) Given $D$ a bounded smooth simply connected domain, if $d$ is small enough, then superconducting solutions (meaning solutions for which $\psi$ is not identically equal to zero) to (1) exist, they are unique up to a gauge, and they are global minimizers.

(ii) Additionally, there is a field strength $h_*$, for which if $h_e < h_*$ non-normal solutions exist, they are global minimizers and they are unique up to a gauge, while if $h_e > h_*$ normal solutions are the only solutions. And,

\[
\frac{\kappa d}{\sqrt{\omega(D)}} \leq h_* < \frac{\kappa d}{\sqrt{\omega(D)}} + C \left( \kappa d^2 + (\kappa d)^{\frac{3}{2}} \right)
\]

where $\omega(D) = \frac{1}{|D|} \int_D |a_n|^2 \, dx$

Hence, \[
\lim_{d \to 0} \frac{h_*(d, \kappa)}{d} = \frac{\kappa}{\sqrt{\omega(D)}}
\]
**Strategy of proof:** Fix $\kappa > 0$.

Step 1- Standard upper bound on $h_e$: show that for $d$ small enough, if $h_e > C\kappa d$, the only solutions to (1) are the normal states.

Step 2- Use upper bound on $h_e$ to find (improved) $L^\infty$ estimate, for $d$ small enough:

(i) $|A|_\infty \leq C\kappa d$

(ii) $\left| \frac{\nabla \psi}{|\psi|_\infty} \right|_\infty \leq C\kappa^2 d^2$

(iii) $\left| \frac{|\psi|^2}{|\psi|_\infty^2} - 1 \right|_\infty \leq C\kappa^4 d^4$

(i)-(iii) imply that for $d$ small if s.c. solutions exist they have less energy than the normal solutions.

Step 3- Estimate (iii) implies: superconducting solutions are never zero (but they could be very close to zero) → one can work with a gauge where the order parameter is real-valued:

$$f = \psi e^{-i\theta}, \quad |\psi| = f, \quad Q = A - \nabla \theta$$

For this real-order parameter gauge a few interesting facts hold...
Step-4 uniqueness via ad-hoc energy expansion

Note: we don’t look only at global minimizers

Step-5 existence and estimate of $h_*$ via Step1 & Step2

Note: the limiting value of $h_*$ is suggested by the work of Pan
Step3- Real-order parameter solutions \((f, Q)\):

The Ginzburg-Landau system (1) yields

\[
\begin{cases}
- \text{div}(f \nabla f) + |\nabla f|^2 + f^2 |Q|^2 = \kappa^2 d^2 (1 - f^2) f^2 \\
\text{curl(curl } A\text{)} = -d^2 f^2 Q \quad \text{in } D \\
\text{div}(f^2 Q) = 0. \\
f \nabla f \cdot \nu = 0 \quad \text{on } \partial D \\
f^2 Q \cdot \nu = 0
\end{cases}
\] (2)

Given any \(g \in H^1(D, C)\), multiply first equation in (2) by \(g^2 / f^2\), integrate over \(D\), and use boundary condition:

\[
\int_D f \nabla f \cdot \nabla \left(\frac{g^2}{f}\right) + g^2 |Q|^2 - \kappa^2 d^2 (1 - f^2) g^2 \ dx = 0 \quad (3)
\]

Note: these hold for any real-valued solution
For small enough $d$, one can show:

$$|Q|_{\infty} \leq C\kappa d$$

$$Q \cdot \nu = 0 \quad \text{on } \partial D$$

$$\text{div } Q = -2 \frac{\nabla f}{f} \cdot Q$$

If $(f_1, Q_1)$ is another real-valued solution, set $q = Q_1 - Q$:

$q \cdot \nu = 0 \text{ on } \partial D \rightarrow L^2$-norm of $q$ is controlled by the curl and div norms $\rightarrow$ using Step-1:

$$\int_D |q|^2 \, dx \leq C \left\{ \int_D |\text{curl } q|^2 \, dx + \kappa^2 d^2 \int_D \left| \frac{f}{f_1} \nabla \left( \frac{f_1}{f} \right) \right|^2 \, dx \right\}$$

Also, since for $d$ small enough $\frac{f_1^2}{|f_1|_{\infty}^2} \geq \frac{1}{4}$:

$$\int_D f_1^2 |q|^2 \, dx \leq C \left\{ \int_D |\text{curl } q|^2 \, dx + \kappa^2 d^2 \int_D \left| f \nabla \left( \frac{f_1}{f} \right) \right|^2 \, dx \right\}$$
Step4- Ad-hoc energy expansion:

Rewrite the energy of \((\psi, A)\), in terms of the real-order parameter gauge \((f, Q)\):

\[
G_{kd}(f, Q) = \frac{1}{2} \int_D \left\{ |\nabla f|^2 + \frac{1}{d^2} |\text{curl } Q - h_e|^2 
+ \frac{k^2 d^2}{2} (1 - f^2)^2 + |f|^2 |Q|^2 \right\} dx
\]

Let \((\psi, A)\) and \((\psi_1, A_1)\) be two non-normal solutions, a direct computation, which uses identity (3), shows that their energies can be expressed using real-valued order parameter gauge:

\[
G_{kd}(f_1, Q_1) = G_{kd}(f, Q) + \frac{1}{2} \int_D \left\{ f^2 \left| \nabla \left( \frac{f_1 - f}{f} \right) \right|^2 
+ \frac{1}{d^2} |\text{curl } q|^2 + \frac{k^2 d^2}{2} (f_1^2 - f^2)^2 - f_1^2 |q|^2 
+ 2 f_1^2 Q_1 \cdot q - 2 f^2 Q \cdot q \right\} dx,
\]

where \(q = Q_1 - Q\)

Need control on the last two terms:

\[
\frac{1}{2} \int_D 2 \left( f_1^2 Q_1 \cdot q - f^2 Q \cdot q \right) dx
\]
But, for small $d$:

$$
\int_D 2 \left( f_1^2 Q_1 \cdot q - f^2 Q \cdot q \right) \, dx \geq -\int_D \frac{\kappa^2 d^2}{2} (f_1^2 - f^2)^2 \, dx
$$

$$
+ \int_D \left( \frac{\kappa d}{\sqrt{2}} (f_1^2 - f^2) + \frac{\sqrt{2}}{\kappa d} Q \cdot (A_1 - A) \right)^2 \, dx
$$

$$
- C \left\{ \int_D |\text{curl } q|^2 \, dx + \kappa^2 d^2 \int_D \left| f \nabla \left( \frac{f_1}{f} \right) \right|^2 \, dx \right\}
$$

Note:

due to the boundary condition for $Q, Q_1$ one has

$$
\int_D (f_1^2 Q_1 - f^2 Q) \cdot \nabla (\theta - \theta_1) \, dx = 0
$$

Splitting the magnetic potentials ($Q = A - \nabla \theta$) gives

$$
\int_D (f_1^2 Q_1 \cdot q - f^2 Q \cdot q) \, dx = \int_D (f_1^2 Q_1 - f^2 Q) \cdot (A_1 - A) \, dx
$$

Then recall

$$f_1 \leq 1 \text{ and } \text{curl}(Q_1 - Q) = \text{curl}(A_1 - A)$$

$$\rightarrow L^2\text{-norm of } A_1 - A \text{ can be controlled with } L^2\text{-norm of } \text{curl } q$$
Using the above bounds, in the energy expansion (4), we obtain:

\[ 2 G_{\kappa d}(\psi_1, A_1) \geq 2 G_{\kappa d}(\psi, A) \]

\[ + (1 - C \kappa^2 d^2) \int_D \left| f \nabla \left( \frac{f_1}{f} \right) \right|^2 \, dx \]

\[ + \int_D \left( \frac{\kappa d}{\sqrt{2}} (f_1^2 - f^2) + \frac{\sqrt{2}}{\kappa d} Q \cdot (A_1 - A) \right)^2 \, dx \]

\[ + \left( \frac{1}{d^2} - C \right) \int_D |\text{curl}\ q|^2 \, dx \]

Without loss of generality we can assume

\[ G_{\kappa d}(\psi_1, A_1) \leq G_{\kappa d}(\psi, A), \]

and, above inequality implies, for \( d \) small enough:

\[ \frac{1}{2} \int_D \left| f \nabla \left( \frac{f_1}{f} \right) \right|^2 \, dx + \frac{1}{2} \int_D |\text{curl}\ q|^2 \, dx \]

\[ + \int_D \left( \frac{\kappa d}{\sqrt{2}} (f_1^2 - f^2) + \frac{\sqrt{2}}{\kappa d} Q \cdot (A_1 - A) \right)^2 \, dx \leq 0 \]
Looking at
\[
\frac{1}{2} \int_D \left| f \nabla \left( \frac{f_1}{f} \right) \right|^2 \, dx + \frac{1}{2} \int_D |\text{curl } q|^2 \, dx
\]
\[
+ \int_D \left( \frac{\kappa d}{\sqrt{2}} (f_1^2 - f^2) + \frac{\sqrt{2}}{\kappa d} Q \cdot (A_1 - A) \right)^2 \, dx \leq 0,
\]
we have \(|\text{curl } q|_2 = 0\), that is \(A = A_1\) (Coulomb gauge). And, looking to the third integral: \(|\psi| = |\psi_1|\), that is \(\psi = \alpha(x)\psi_1\), for some smooth complex function \(\alpha(x)\) of modulus one

Using \(A = A_1\) and rewriting \(\psi = \alpha\psi_1\) in the 2nd equation of (1) gives \(\alpha^* \nabla \alpha - \alpha \nabla \alpha^* = 0\), which combined with \(\nabla |\alpha|^2 = 0\) implies \(2\alpha^* \nabla \alpha = 0\)

By our assumptions, \(\alpha\) is never zero, hence we conclude \(\nabla \alpha = 0\), that is \(\alpha = e^{i\eta}\) for some constant \(\eta\), in other words \((\psi, A)\) and \((\psi_1, A_1)\) are gauge equivalent.