A new multiscale model for traffic flow

Emiliano Cristiani and Elisa Iacomini

Seminario di Modellistica Differenziale Numerica

December 11, 2018

Plan

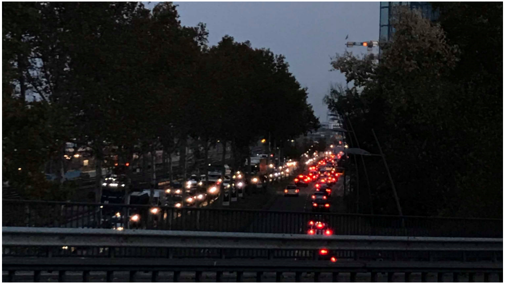
Plan

What is traffic flow?

Definition

Traffic flow is the study of interactions between

- travellers (pedestrians, cyclists, riders drivers and their vehicles)
- infrastructure (highways, signage, and traffic control devices).



Why is it so important?

Aims

Understanding and developing an optimal transport network with efficient movement of traffic and minimal traffic congestion problems in order to:

- improve the drivers' safety,
- reduce the travel time,
- reduce the fuel consumption,
- reduce the pollution related to the heavy traffic.

Why is it so important?

Aims

Understanding and developing an optimal transport network with efficient movement of traffic and minimal traffic congestion problems in order to:

- improve the drivers' safety,
- reduce the travel time,
- reduce the fuel consumption,
- reduce the pollution related to the heavy traffic.

Why is it so important?

Aims

Understanding and developing an optimal transport network with efficient movement of traffic and minimal traffic congestion problems in order to:

- improve the drivers' safety,
- reduce the travel time,
- reduce the fuel consumption,
- reduce the pollution related to the heavy traffic.

Why is it so important?

Aims

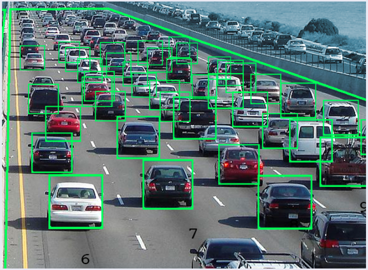
Understanding and developing an optimal transport network with efficient movement of traffic and minimal traffic congestion problems in order to:

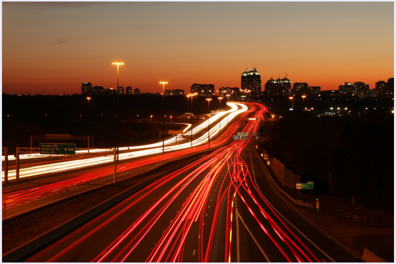
- improve the drivers' safety,
- reduce the travel time,
- reduce the fuel consumption,
- reduce the pollution related to the heavy traffic.

Framework

Mathematical modeling of traffic flow on a **single road**, by means of both

- a microscopic (agent-based) follow-the-leader model based on a system of ODEs
- a MACROSCOPIC (fluid-dynamic) model based on conservation laws





Introduction

Relevance

Multiscale models are useful for exploiting in a unique setting both Eulerian (i.e., flux-based) and Lagrangian (i.e., GPS) real traffic data.

Microscopic follow-the-leader models

Hypothesis:

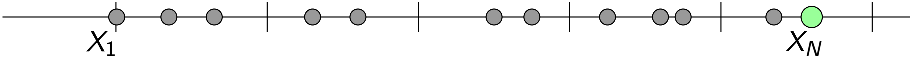
N cars on a infinite road, overtaking not possible

$X_k(t)$ position of car k at time t

$V_k(t)$ velocity of car k at time t

$A_k(t)$ acceleration of car k at time t

$$X_1 < X_2 < \ldots < X_N$$



Remark

Note that the N-th car (the leader) needs a special dynamic because has no one in front of him.

Microscopic follow-the-leader models

First-order

$$\begin{cases} \dot{X}_k(t) = V(X_k(t), X_{k+1}(t)), \quad k < N \\ \dot{X}_N(t) = V_{\max} \end{cases}$$

Main feature: Assume that accelerations are instantaneus and traffic conditions are always at the equilibrium.

Second-order

$$\begin{cases} \dot{X}_k(t) = V_k(t), \quad k \leq N \\ \dot{V}_k(t) = A(X_k(t), X_{k+1}(t), V_k(t), V_{k+1}(t)), \quad k < N \\ \dot{V}_N(t) = 0 \end{cases}$$

Main feature: Consider bounded accelerations, so it's closer to real dynamics of drivers. They are also able to reproduce some traffic phenomena like Stop & Go Waves.

Microscopic follow-the-leader models

First-order

$$\begin{cases} \dot{X}_k(t) = V(X_k(t), X_{k+1}(t)), & k < N \\ \dot{X}_N(t) = V_{\max} \end{cases}$$

Main feature: Assume that accelerations are instantaneus and traffic conditions are always at the equilibrium.

Second-order

$$\begin{cases} \dot{X}_k(t) = V_k(t), & k \leq N \\ \dot{V}_k(t) = A(X_k(t), X_{k+1}(t), V_k(t), V_{k+1}(t)), & k < N \\ \dot{V}_N(t) = 0 \end{cases}$$

Main feature: Consider bounded accelerations, so it's closer to real dynamics of drivers. They are also able to reproduce some traffic phenomena like Stop & Go Waves.

Possible choices for the acceleration term

ARZ limit model

$$A(X_k, X_{k+1}, V_k, V_{k+1}) = V_{\text{ref}}\, \frac{V_{k+1} - V_k}{X_{k+1} - X_k} + \frac{V_{des}(X_{k+1} - X_k) - V_k}{\tau}$$

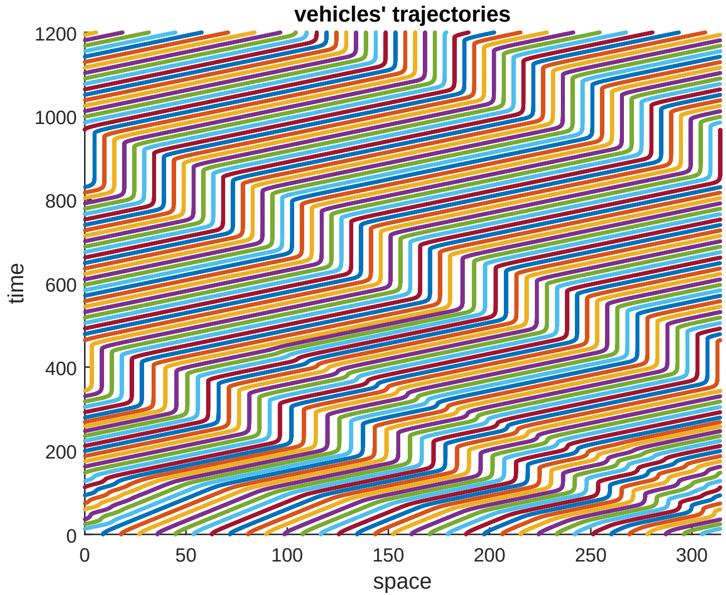
Simplified Zhao-Zhang Model

$$A(X_k, X_{k+1}, V_k, V_{k+1}) = \frac{v^{ZZ}(X_{k+1} - X_K) - V}{\tau}$$

$$v^{ZZ}(\Delta) := \begin{cases} 0, & \Delta \leq \Delta_{\text{min}}, \\ \alpha(\Delta - \Delta_{\text{min}}), & \Delta_{\text{min}} \leq \Delta \leq \Delta_{\text{min}} + V_{\text{max}}/\alpha, \\ V_{\text{max}}, & \Delta \geq \Delta_{\text{min}} + V_{\text{max}}/\alpha. \end{cases}$$

Y. Zhao and H. M. Zhang, *A unified follow-the-leader model for vehicle, bicycle and pedestrian traffic*, Transportation Res. Part B, 105 (2017), 315–327.

Stop & go waves with the Zhao-Zhang model



Resume on Microscopic approach

Advantages

- ✓ is the natural way to describe traffic flow
- ✓ is easy to implement
- ✓ is really accurate
- ✓ is able to easily describe second order effect

But...

- × requires a lot of time
- × is expensive in terms of memory
- × is impossible to use on a large network

Resume on Microscopic approach

Advantages

- ✓ is the natural way to describe traffic flow
- ✓ is easy to implement
- ✓ is really accurate
- ✓ is able to easily describe second order effect

But...

- x requires a lot of time
- x is expensive in terms of memory
- x is impossible to use on a large network

Macroscopic models

$\rho(x,t)$ density of cars at point x and time t

$v(x,t)$ velocity of cars at point x and time t

$f(x,t) = \rho(x,t)v(x,t)$ flux of cars at point x and time t

Definition

The fundamental diagram establishes the relationship between the flux and the density of vehicles, i.e. $\{(\rho(x,t), f(x,t)) \ : \ x \in \mathbb{R}, \ t > 0\}$

Macroscopic models

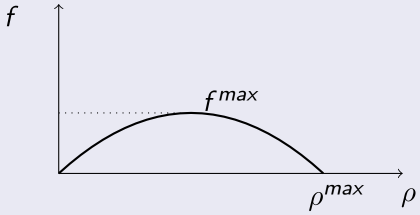
$\rho(x,t)$ density of cars at point x and time t

$v(x,t)$ velocity of cars at point x and time t

$f(x,t) = \rho(x,t)v(x,t)$ flux of cars at point x and time t

Definition

The fundamental diagram establishes the relationship between the flux and the density of vehicles, i.e. $\{(\rho(x,t), f(x,t)) \ : \ x \in \mathbb{R}, \ t > 0\}$



Macroscopic models

First-order (LWR model)

$$\partial_t \rho + \partial_x(\rho v(\rho)) = 0, \qquad x \in \mathbb{R},\ t > 0$$

Typically $v(\rho) = V_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$

Second-order (e.g., ARZ model)

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, & x \in \mathbb{R},\ t > 0 \\ \partial_t v + v\partial_x v = a(\rho, v), & x \in \mathbb{R},\ t > 0 \end{cases}$$

Main feature: More realistic but it requires complex implementation.

Macroscopic models

First-order (LWR model)

$$\partial_t \rho + \partial_x(\rho v(\rho)) = 0, \qquad x \in \mathbb{R},\ t > 0$$

Typically $v(\rho) = V_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$

Second-order (e.g., ARZ model)

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, & x \in \mathbb{R},\ t > 0 \\ \partial_t v + v\partial_x v = a(\rho, v), & x \in \mathbb{R},\ t > 0 \end{cases}$$

Main feature: More realistic but it requires complex implementation.

Resume on Macroscopic approach

Advantages

- ✓ is very cheap in terms of memory and time
- ✓ can be used also on large networks

But...

- x First order model is not able to reproduce real traffic phenomena like Stop &Go waves
- x the second order model is very difficult to implement

Resume on Macroscopic approach

Advantages

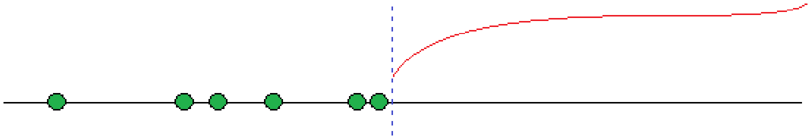
- ✓ is very cheap in terms of memory and time
- ✓ can be used also on large networks

But...

- x First order model is not able to reproduce real traffic phenomena like Stop &Go waves
- x the second order model is very difficult to implement

Plan

Existing multi-scale models (with fixed or mobile interface)



- First-order FtL + LWR[a]
- Second-order FtL + LWR[b]
- Second-order FtL + Phase-Transition model[c]
- Second-order FtL + ARZ[d]

[a] R. M. Colombo and F. Marcellini, *A mixed ODE-PDEmodel for vehicular traffic*, Math. Meth. Appl. Sci., 38 (2015), 1292–1302.

[b] M. Garavello and B. Piccoli, *Boundary coupling of microscopic and first order macroscopic traffic model*, Nonlinear Differ. Equ. Appl., 24:43 (2017).

[c] M. Garavello and B. Piccoli, *Coupling of microscopic and phase transition models at boundary*, Netw. Heterog. Media, 8 (2013), 649–661.

[d] C. Lattanzio and B. Piccoli, *Coupling of microscopic and macroscopic traffic models at boundaries*, Math. Models Methods Appl. Sci., 20 (2010), 2349–2370.

The new multiscale approach

Main feature

The model we propose is characterized by the fact that no interface (either fixed or mobile) it explicitly defined.
The **macroscopic model is always and everywhere alive, while the microscopic model is activated only where and when it is needed**.
The microscopic model corrects (in full or in part) the macroscopic one.

This procedure is expected to be advantageous if **one couples an easy-to-use first-order macroscopic model with a more realistic but still easy-to-use second-order microscopic model** (used only in small parts of the road).

The multiscale model with complete information

A SIMPLE CASE:

Both micro and macro models always and everywhere active

$$
\begin{cases}
\partial_t \displaystyle\int_a^b \rho(x,t)dx = \theta\big(f(a,t) - f(b,t)\big) + \\
\qquad + (1-\theta)\left(\displaystyle\sum_{k=1}^{N} \ell\delta_a(X_k(t)) - \displaystyle\sum_{k=1}^{N} \ell\delta_b(X_k(t))\right), \quad \forall a,b \in \mathbb{R} \\
\dot{X}_k(t) = V_k(t), \quad k \leq N, \\
\dot{V}_k(t) = A(X_k(t), X_{k+1}(t), V_k(t), V_{k+1}(t)), \quad k < N, \\
\dot{V}_N(t) = 0,
\end{cases}
$$

$\theta \in [0,1]$, $f = \rho v$, $x \mapsto \delta_{x_0}(x)$ Dirac delta function centred in x_0, ∂_t in distributional sense.

E. Cristiani, B. Piccoli, A. Tosin, *Multiscale modeling of granular flows with application to crowd dynamics*, Multiscale Model. Simul., 9 (2011), 155–182.

The multiscale model with complete information

A SIMPLE CASE:

Both micro and macro models always and everywhere active

$$
\begin{cases}
\partial_t \displaystyle\int_a^b \rho(x,t)dx = \theta\big(f(a,t) - f(b,t)\big) + \\
\qquad + (1-\theta)\left(\displaystyle\sum_{k=1}^{N} \ell\delta_a(X_k(t)) - \sum_{k=1}^{N} \ell\delta_b(X_k(t))\right), \quad \forall a,b \in \mathbb{R} \\
\dot{X}_k(t) = V_k(t), \quad k \leq N, \\
\dot{V}_k(t) = A(X_k(t), X_{k+1}(t), V_k(t), V_{k+1}(t)), \quad k < N, \\
\dot{V}_N(t) = 0,
\end{cases}
$$

$\theta \in [0,1]$, $f = \rho v$, $x \mapsto \delta_{x_0}(x)$ Dirac delta function centred in x_0, ∂_t in distributional sense.

E. Cristiani, B. Piccoli, A. Tosin, *Multiscale modeling of granular flows with application to crowd dynamics*, Multiscale Model. Simul., 9 (2011), 155–182.

The multiscale model with complete information

A SIMPLE CASE:

Both micro and macro models always and everywhere active

$$
\begin{cases}
\partial_t \displaystyle\int_a^b \rho(x,t)dx = \theta\big(f(a,t) - f(b,t)\big) + \\
\qquad + (1-\theta)\left(\displaystyle\sum_{k=1}^N \ell\delta_a(X_k(t)) - \sum_{k=1}^N \ell\delta_b(X_k(t))\right), \quad \forall a,b \in \mathbb{R} \\
\dot{X}_k(t) = V_k(t), \quad k \le N, \\
\dot{V}_k(t) = A(X_k(t), X_{k+1}(t), V_k(t), V_{k+1}(t)), \quad k < N, \\
\dot{V}_N(t) = 0,
\end{cases}
$$

$\theta \in [0,1]$, $f = \rho v$, $x \mapsto \delta_{x_0}(x)$ Dirac delta function centred in x_0, ∂_t in distributional sense.

E. Cristiani, B. Piccoli, A. Tosin, *Multiscale modeling of granular flows with application to crowd dynamics*, Multiscale Model. Simul., 9 (2011), 155–182.

The multiscale model with complete information

A SIMPLE CASE:

Both micro and macro models always and everywhere active

$$
\begin{cases}
\partial_t \int_a^b \rho(x,t)dx = \theta\big(f(a,t) - f(b,t)\big) + \\
\qquad + (1-\theta)\left(\sum_{k=1}^{N} \ell\delta_a(X_k(t)) - \sum_{k=1}^{N} \ell\delta_b(X_k(t))\right), \quad \forall a, b \in \mathbb{R} \\
\dot{X}_k(t) = V_k(t), \quad k \leq N, \\
\dot{V}_k(t) = A(X_k(t), X_{k+1}(t), V_k(t), V_{k+1}(t)), \quad k < N, \\
\dot{V}_N(t) = 0,
\end{cases}
$$

$\theta \in [0,1]$, $f = \rho v$, $x \mapsto \delta_{x_0}(x)$ Dirac delta function centred in x_0, ∂_t in distributional sense.

E. Cristiani, B. Piccoli, A. Tosin, *Multiscale modeling of granular flows with application to crowd dynamics*, Multiscale Model. Simul., 9 (2011), 155–182.

The multiscale model with complete information

A SIMPLE CASE:

Both micro and macro models always and everywhere active

$$
\begin{cases}
\partial_t \displaystyle\int_a^b \rho(x,t)dx = \theta\big(f(a,t)-f(b,t)\big)+ \\
\qquad + (1-\theta)\left(\displaystyle\sum_{k=1}^{N} \ell\delta_a(X_k(t)) - \sum_{k=1}^{N} \ell\delta_b(X_k(t))\right), \quad \forall a,b \in \mathbb{R} \\
\dot{X}_k(t) = V_k(t), \quad k \leq N, \\
\dot{V}_k(t) = A(X_k(t), X_{k+1}(t), V_k(t), V_{k+1}(t)), \quad k < N, \\
\dot{V}_N(t) = 0,
\end{cases}
$$

$\theta \in [0,1]$, $f = \rho v$, $x \mapsto \delta_{x_0}(x)$ Dirac delta function centred in x_0, ∂_t in distributional sense.

E. Cristiani, B. Piccoli, A. Tosin, *Multiscale modeling of granular flows with application to crowd dynamics*, Multiscale Model. Simul., 9 (2011), 155–182.

The multiscale model with complete information: numerical approximation

$$
\begin{cases}
\rho_j^{m+1} = \rho_j^m + \theta\lambda\Big(\mathcal{G}(\rho_{j-1}^m, \rho_j^m) - \mathcal{G}(\rho_j^m, \rho_{j+1}^m)\Big) + (1-\theta)\lambda\left(\mathcal{F}_{j-\frac{1}{2}}^m - \mathcal{F}_{j+\frac{1}{2}}^m\right), \\
X_k^{m+1} = X_k^m + \Delta t V_k^m, \quad k \leq N, \\
V_k^{m+1} = V_k^m + \Delta t A(X_k^m, X_{k+1}^m, V_k^m, V_{k+1}^m), \quad k < N, \\
V_N^{m+1} = V_N^m,
\end{cases}
$$

with the classical *Godunov's numerical flux*

$$
\mathcal{G}(\rho^-, \rho^+) := \begin{cases}
\min\{f(\rho^-), f(\rho^+)\}, & \text{if } \rho^- \leq \rho^+ \\
f(\rho^-), & \text{if } \rho^- > \rho^+ \text{ and } \rho^- < \sigma \\
f(\sigma), & \text{if } \rho^- > \rho^+ \text{ and } \rho^- \geq \sigma \geq \rho^+ \\
f(\rho^+), & \text{if } \rho^- > \rho^+ \text{ and } \rho^+ > \sigma
\end{cases}
$$

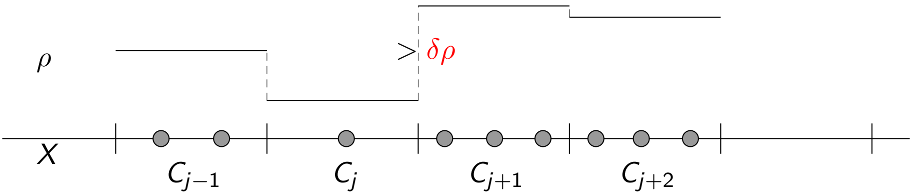
(where $\sigma := \arg \max_{\rho \in [0, \rho_{\max}]} f(\rho)$), and the *microscopic flux*

$$
\mathcal{F}_{j\pm\frac{1}{2}}^m := \frac{\ell}{\Delta t} \text{Card}\left\{k : x_k^m < x_{j\pm\frac{1}{2}} \leq x_k^{m+1}\right\}.
$$

The multiscale algorithm: Step 1

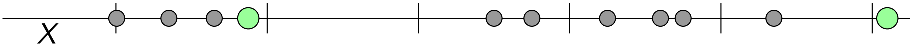
Moving from (ρ^m, X^m, V^m) to $(\rho^{m+1}, X^{m+1}, V^{m+1})$

1. Activation of cars. For all j, if $|\rho^m_{j+1} - \rho^m_j| > \delta\rho$, put new cars in cell C_i (unless the cell is already occupied), for $i \in \{j-1, j, j+1, j+2\}$.
The number of cars to put in the cell C_i is proportional to ρ^m_i and cars are initially equispaced in the cell. Their velocity is set to $v^{\text{equilibrium}}(\rho^m_i)$ (the corresponding macroscopic velocity at equilibrium).



The multiscale algorithm: Step 2

2. Labeling. Find $\textsc{next}(k)$ for all k. The rightmost car is labeled as leader ($\textsc{next} = 0$).
Also, all cars h such that $|X^m_{\textsc{next}(h)} - X^m_h| > \Delta x$ are also labeled as leader (every time a car has a free space of length $\geq \Delta x$ in front of it, its dynamics ceases to be dependent on the vehicle in front).

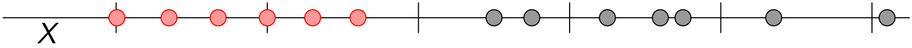


The multiscale algorithm: Step 3

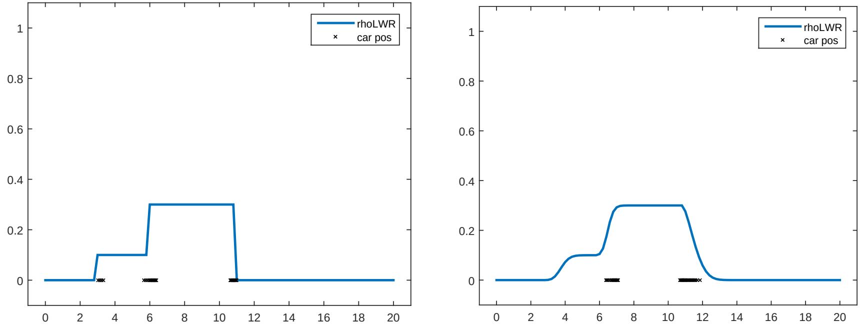
3. Deactivation of cars. Remove all followers k which are active since more than δt units of time and such that

$$\left| V_k - v^{\text{equilibrium}} \left(X_k^m, X_{\mathrm{NEXT}(k)}^m \right) \right| < \delta V$$

Note that, without the first condition new cars would immediately deactivated since their velocity is initially at equilibrium. In this way, instead, vehicles have enough time to fully exploit their second-order dynamics.
After that, if and when they get close to the equilibrium velocity again, they are deactivated.



Numerical test: activation and deactivation



The multiscale algorithm: Step 4

4. Update cars' positions and velocities. We run the microscopic second-order model

$$\begin{cases} X_k^{m+1} = X_k^m + \Delta t V_k^m, \quad \forall k, \\ V_k^{m+1} = V_k^m + \Delta t A(X_k^m, X_{\text{NEXT}(k)}^m, V_k^m, V_{\text{NEXT}(k)}^m), \quad \text{if NEXT}(k) > 0, \\ V_k^{m+1} = v^{\text{equilibrium}}(\rho_{j_k+1}^m), \quad \text{if NEXT}(k) = 0, \end{cases}$$

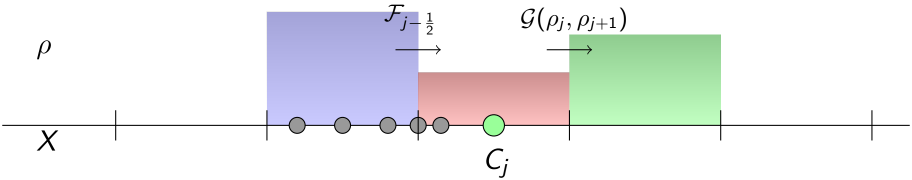
where j_k is the cell occupied by the vehicle k.
The velocity of a leader is that of macroscopic cars located in the cell in front of it.

The multiscale algorithm: Step 5

5. Update cars' density. We run the multiscale model, which reads as follows (for $\theta = 0$)

$$\Gamma_j^m := \text{number of particles in cell } j \text{ at time step } m$$

$$\rho_j^{m+1} = \rho_j^m + \lambda \begin{cases} \mathcal{F}^m_{j-\frac{1}{2}} - \mathcal{F}^m_{j+\frac{1}{2}} & \text{if } \Gamma^m_{j-1},\ \Gamma^m_j,\ \Gamma^m_{j+1} > 0 \\ \mathcal{F}^m_{j-\frac{1}{2}} - \mathcal{G}(\rho^m_j, \rho^m_{j+1}) & \text{if } \Gamma^m_{j-1},\ \Gamma^m_j > 0 \ \&\ \Gamma^m_{j+1} = 0 \\ \mathcal{G}(\rho^m_{j-1}, \rho^m_j) - \mathcal{F}_{j+\frac{1}{2}} & \text{if } \Gamma^m_{j-1} = 0 \ \&\ \Gamma^m_j,\ \Gamma^m_{j+1} > 0 \\ G(\rho^m_{j-1}, \rho^m_j) - G(\rho^m_j, \rho^m_{j+1}) & \text{otherwise.} \end{cases}$$



Mass conservation: $\Delta x \sum_j \rho_j^n = \Delta x \sum_j \rho_j^0 \quad \forall n$

Proof.

Denote by $\mathcal{M}_{n,j}^+$ ($\mathcal{M}_{n,j}^-$) the mass gained (lost) by a generic cell C_j in one time step $n \to n+1$.

$$\Delta x \rho_j^{n+1} = \Delta x \rho_j^n + \mathcal{M}_{n,j}^+ - \mathcal{M}_{n,j}^- \qquad \forall j,\ \forall n,$$

and then let us simply prove that

$$\mathcal{M}_{n,j}^- = \mathcal{M}_{n,j+1}^+ \qquad \forall j,\ \forall n.$$

But looking at the numerical scheme, we can conclude that:

$$\mathcal{M}_{n,j}^- = \mathcal{M}_{n,j+1}^+ = \Delta t \cdot \begin{cases} \mathcal{F}_{j+\frac{1}{2}}^n, & \text{if } \Gamma_j^n,\ \Gamma_{j+1}^n > 0, \\ \mathcal{G}(\rho_j^n, \rho_{j+1}^n), & \text{otherwise.} \end{cases}$$

$\square$

Plan

Numerical tests

In the following numerical test we consider

Microscopic version of the ARZ model (micro, FtL, second-order)

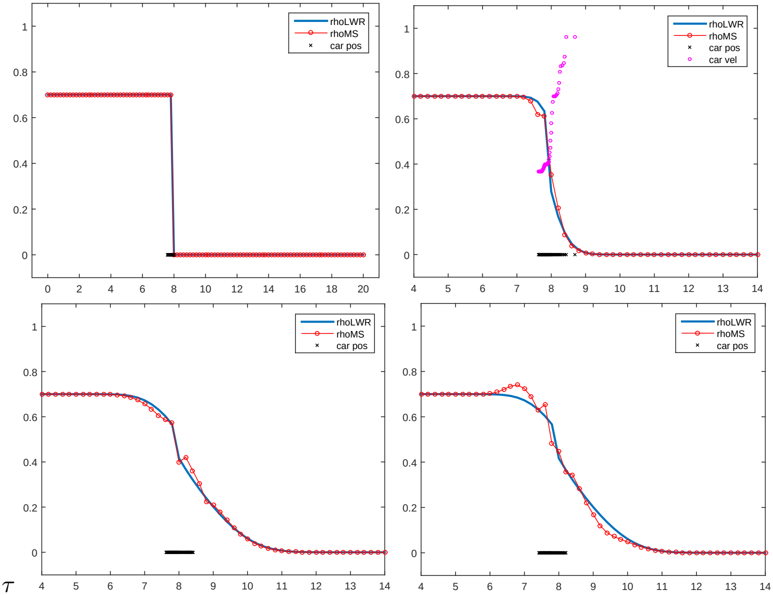
$$A(X_k, X_{k+1}, V_k, V_{k+1}) = V_{\text{ref}} \, \frac{V_{k+1} - V_k}{X_{k+1} - X_k} + \frac{V_{des}(X_{k+1} - X_k) - V_k}{\tau}$$

$+$

LWR model (macro, first-order)

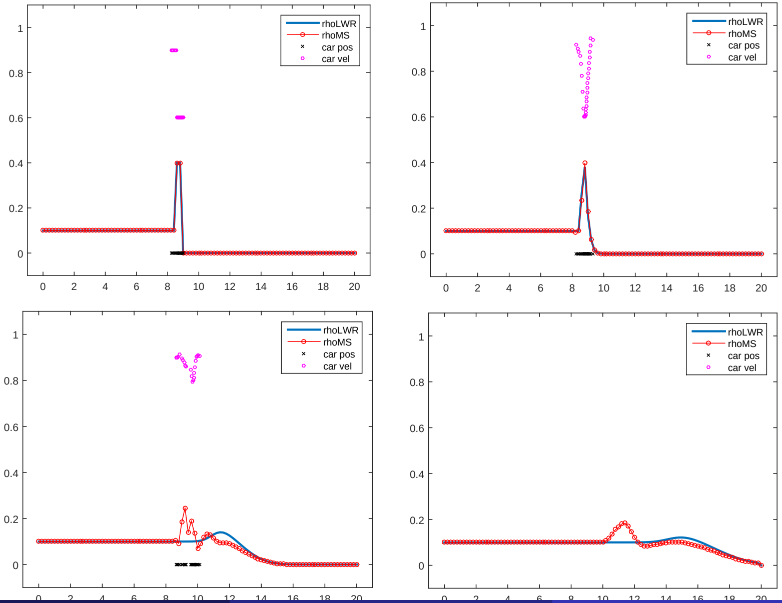
A. Aw, A. Klar, T. Materne, M. Rascle, *Derivation of continuum traffic flow models from microscopic follow-the-leader models*, SIAM Journal on Applied Mathematics 63.1 (2002), 259–278.

Test 1: effect of τ (reactivity^{-1} of drivers)

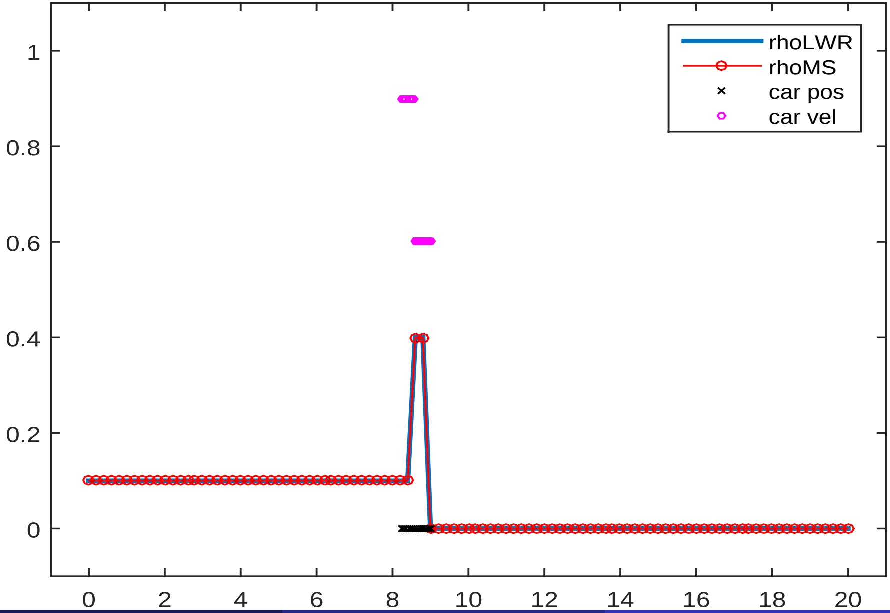


small τ large τ

Test 2: self-sustaining perturbation



Test 2: self-sustaining perturbation



Numerical tests

In the following numerical test we consider

Minimalized Zhao & Zhang's model (micro, second-order)

$$A(X_k, X_{k+1}, V_k, V_{k+1}) = \frac{v^{ZZ}(X_{k+1} - X_K) - V}{\tau}$$

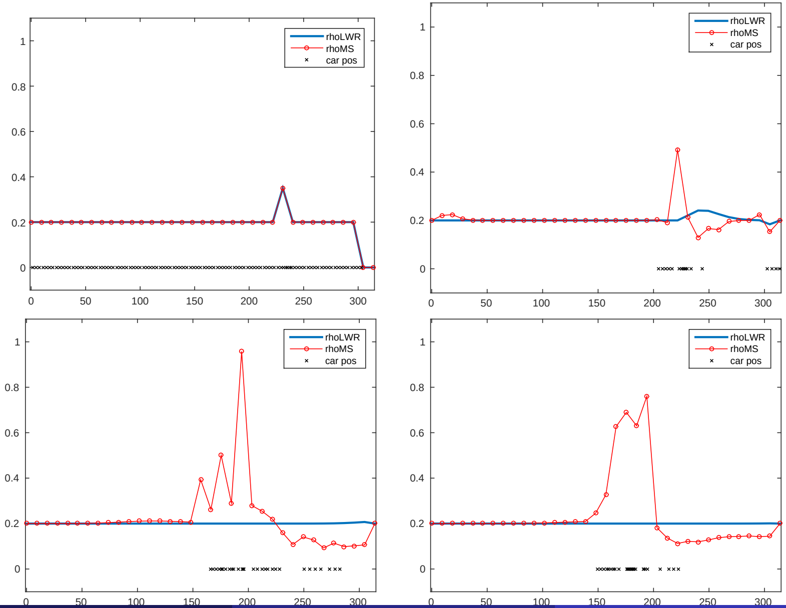
$$v^{ZZ}(\Delta) := \begin{cases} 0, & \Delta \leq \Delta_{\min}, \\ \alpha(\Delta - \Delta_{\min}), & \Delta_{\min} \leq \Delta \leq \Delta_{\min} + V_{\max}/\alpha, \\ V_{\max}, & \Delta \geq \Delta_{\min} + V_{\max}/\alpha. \end{cases}$$

$+$

LWR model (macro, first-order)

Y. Zhao and H. M. Zhang, *A unified follow-the-leader model for vehicle, bicycle and pedestrian traffic*, Transportation Res. Part B, 105 (2017), 315–327.

Test 3: Reproducing stop & go waves



Plan

Conclusions

- we get an easy-to-use algorithm able to reproduce second order effect, avoiding complex implementations;
- we save memory and time tracking the vehicles only where and when it is needed;
- we avoid any interface between the two scales.

Future Perspective

- studying the model analytically;
- studying the impact of the parameters and making them depending on the variables of the system;
- making a deeper analysis of the second order effects, like the stop & go waves.

References

- E. Cristiani, E. Iacomini, *An interface-free multi-scale multi-order model for traffic flow*, submitted.
- E. Cristiani, *Blending Brownian motion and heat equation*, J. Coupled Syst. Multiscale Dyn., 3 (2015), 351–356.
- E. Cristiani, B. Piccoli, and A. Tosin, *Multiscale modeling of granular flows with application to crowd dynamics*, Multiscale Model. Simul., 9 (2011), 155–182.
- Y. Zhao and H. M. Zhang, *A unified follow-the-leader model for vehicle, bicycle and pedestrian traffic*, Transportation Res. Part B, 105 (2017), 315–327.

...Thank you for your attention!