Asymptotic-preserving methods: a new paradigm for multiscale differential problems

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Motivations

Many applications involve *multiscale differential problems*, like convection-diffusion systems

**Convection-Diffusion Equations**

\[
\partial_t U + \partial_x F(U) = \partial_x (K(U) \partial_x U), \quad x \in \mathbb{R},
\]

or hyperbolic balance laws with source terms of the form

**Conservation Laws with Relaxation**

\[
\partial_t U + \partial_x F(U) = \frac{1}{\varepsilon} R(U), \quad x \in \mathbb{R},
\]

where \( U = U(x, t) \in \mathbb{R}^N \), and \( \varepsilon > 0 \) is called relaxation parameter. Several kinetic equations have the same structure with \( U = F(U) = f(x, v, t) \geq 0, \ v \in \mathbb{R} \).

The numerical discretization of such systems may be challenging in presence of multiple scales, typically when the diffusion or the source terms are stiff.
Applications

- Gas dynamics (gas with chemical reactions, extended thermodynamics)
- Shallow water equations
- Granular gases
- Hydrodynamic models for semiconductors
- Traffic flow models
- Kinetic equations (Fokker-Planck, Boltzmann-like)
- Convection-diffusion-reaction equations (biology, environment)
Method of lines (MOL) approach

- Discretize all spatial operators
- Obtain a system of ODEs

\[ U' = F(U) + G(U) \]

with \( F \) non stiff and \( G \) a stiff term.
- Integrate ODEs system in time

Advantages

- Spatial discretization and time integration are treated separately
- Spatial discretization: easy to combine different schemes
- Time integration: free to choose suitable method (Runge-Kutta, multi-step, etc.)
Numerical approaches

**Fully explicit methods**
- Non stiff term: $\Delta t \leq \rho(\nabla u F) \Delta x$ (CFL condition)
- Stiff term: $\Delta t \leq D^{-1}(\Delta x)^2$ or $\Delta t \leq C\varepsilon$.
  Stability will require very small step-sizes for stiff sources, diffusion or relaxation terms ($\varepsilon$ small).

**Fully implicit methods**
- For problems with shocks or steep gradients, implicit methods are not much better than explicit ones (spurious shocks and wrong wave propagation speed when the CFL is violated).
- For convection discretizations with slope limiters, the implicit relations are hard (expensive) to solve even for linear problems.

▶ Thus it is desirable to develop schemes which are Implicit in $G(U)$ and Explicit in $F(U)$ (IMEX).
Explicit vs Implicit Convection

Buckley-Leverett equation $F(U) = \frac{U^2}{(U^2 + \frac{1}{3}(1-U)^2)}$ with second order space discretization (Van Leer limiter). Solution for explicit BDF2 (top) and implicit BDF2 (bottom) with $\Delta t/\Delta x = 1/8$ (blue), $1/4$ (magenta), $1/2$ (red) at $t = 0.25$.

$$U^{n+1} = \frac{4}{3}U^n - \frac{1}{3}U^{n-1} + \frac{4}{3}\Delta t F(U^n) - \frac{2}{3}\Delta t F(U^{n-1})$$

$$U^{n+1} = \frac{4}{3}U^n - \frac{1}{3}U^{n-1} + \frac{2}{3}\Delta t F(U^{n+1})$$
The combination of the implicit and explicit method should satisfy suitable order conditions. For linear multistep methods (LMM) if both methods are of order \( p \) then the IMEX scheme has order \( p \). For Runge-Kutta (RK) schemes we need to satisfy additional mixed compatibility conditions.

**Explicit method**
- The stability region should be the largest possible.
- Monotonicity requirements

\[
\|U^{n+1}\| \leq \|U^n\|, \quad \Delta t \leq \Delta t_*
\]

Strong Stability Preserving (SSP) property\(^1\).

**Implicit method**
- Stable for stiff systems, and good damping properties.
- The method should be Asymptotic Preserving (AP) namely it should be consistent with the model reduction that may occur in very stiff regimes \(^2\).

\(^1\)S.Gottlieb, C-W.Shu, E.Tadmor '01, R.Spiteri, S.Ruth, '02
\(^2\)S.Jin '99
Consider the **singly perturbed problem**\(^3\)

### Singularly perturbed problem

\[
P^\varepsilon : \begin{cases}
u'(t) &= f(u,v), \\
\varepsilon v'(t) &= g(u,v), & \varepsilon > 0.
\end{cases}
\]

As \(\varepsilon \to 0\) we get the index 1 **differential algebraic equation** (DAE)

\[
u'(t) = f(u,v), \quad 0 = g(u,v).
\]

Assuming that \(g(u,v) = 0 \iff v = E(u)\) we obtain

\[
P^0 : \quad u'(t) = f(u, E(u)).
\]

**Explicit methods**: restricted to \(\Delta t \sim \varepsilon\).

**Implicit methods**: require the numerical inversion of \(f(u,v)\) and \(g(u,v)\), and as \(\varepsilon \to 0\) must satisfy the algebraic condition \(g(u,v) = 0 \iff v = E(u)\).

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\(^3\) E. Hairer, C. Lubich, M. Roche ’89
In the diagram $P^\varepsilon$ is the original singular perturbation problem and $P^\varepsilon_{\Delta t}$ its numerical approximation characterized by a discretization parameter $\Delta t$. The *asymptotic-preserving (AP) property* corresponds to the request that $P^\varepsilon_{\Delta t}$ is a consistent discretization of $P^0$ as $\varepsilon \to 0$ independently of $\Delta t$. 

The AP diagram

\[
\begin{array}{ccc}
P^\varepsilon & \xrightarrow{\varepsilon \to 0} & P^0 \\
\uparrow & & \uparrow \\
\Delta t \to 0 & \mathrel{\quad} & \Delta t \to 0 \\
\downarrow & & \downarrow \\
P^\varepsilon_{\Delta t} & \xrightarrow{\varepsilon \to 0} & P^0_{\Delta t}
\end{array}
\]
Consider the case of the single hyperbolic equation

\[ U_t + \partial_x F(U) = 0. \]

- We can use any finite difference/volume or spectral method to approximate the spatial derivative, and use the standard (linear) stability analysis.
- In presence of shocks and discontinuities this stability analysis is not sufficient (nonlinear problems can develop discontinuous solutions in finite time even starting from a smooth solution).
- Build spatial discretizations which capture the shock structure and that satisfy some nonlinear stability properties. These methods include total variation diminishing (TVD) schemes and essentially non-oscillatory (ENO) or weighted ENO (WENO) schemes\(^4\).

\(^4\) A. Harten '87, T.Chan, X-D.Liu, S.Osher '94, G-S.Jang, C-W.Shu '95
We consider the system of stiff ODE’s

\[ U' = \mathcal{F}(U) + \mathcal{G}(U) \]

where \( \mathcal{F} \) is non stiff and \( \mathcal{G} \) is a stiff term.

**Splitting methods**

- Solve separately the advection problem and the stiff source problem

  \[ U' = \mathcal{F}(U), \quad t \in [0, T] \quad U' = \mathcal{G}(U), \quad t \in [0, T]. \]

- Although it is only first order accurate (even if the two steps are exact, unless the operators commute), it is very popular due to its simple concept and the freedom in choosing different solvers for advection and sources.

- Higher order splitting (ex. **Strang splitting**) can be constructed but may present a loss of accuracy when the source term is highly stiff.
IMEX Runge-Kutta methods

\[ U_i = U^n + \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij} F(t_0 + \tilde{c}_j \Delta t, U_j) + \Delta t \sum_{j=1}^{\nu} a_{ij} G(t_0 + c_j \Delta t, U_j), \]

\[ U^{n+1} = U^n + \Delta t \sum_{i=1}^{\nu} \tilde{w}_i F(t_0 + \tilde{c}_i \Delta t, U_i) + \Delta t \sum_{i=1}^{\nu} w_i G(t_0 + c_i \Delta t, U_i). \]

\[ \tilde{A} = (\tilde{a}_{ij}), \quad \tilde{a}_{ij} = 0, \quad j \geq i \] and \[ A = (a_{ij}): \nu \times \nu \] matrices.

The coefficient vectors are \[ \tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_\nu)^T, \quad \tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_\nu)^T, \]
\[ c = (c_1, \ldots, c_\nu)^T, \quad w = (w_1, \ldots, w_\nu)^T. \]

\[ \text{We restrict to diagonally implicit (DIRK) scheme, } a_{ij} = 0, \ j > i \] since they guarantee that \( F \) is evaluated explicitly.

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5U.Ascher, S.Ruth, R.Spiteri '97, L.P., G.Russo '00
Order conditions

- If $w_i = \tilde{w}_i$ and $c_i = \tilde{c}_i$ mixed conditions are automatically satisfied. This is not true for higher than third order accuracy.
- IMEX-RK schemes are a particular case of additive Runge-Kutta (ARK) methods. Higher order conditions can be derived using a generalization of Butcher 1-trees to 2-trees.
- The number of coupling conditions increase dramatically with the order of the schemes\(^6\).

<table>
<thead>
<tr>
<th>IMEX-RK Order</th>
<th>Number of coupling conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>General case</td>
</tr>
<tr>
<td>1</td>
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<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
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<tr>
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</tr>
<tr>
<td>6</td>
<td>1128</td>
</tr>
</tbody>
</table>

\(^6\)M.Carpenter, C.Kennedy, '03
Design of IMEX-RK

Start with a $p$-order explicit SSP method and find the $p$-order DIRK method that matches the order conditions with good damping properties (L-stability).

**Second order SSP IMEX-RK**

\[
\begin{align*}
U_1 &= U^n + \gamma \Delta t G(U_1) \\
U_2 &= U^n + \Delta t F(U^n) + (1 - 2\gamma) \Delta t G(U_1) + \gamma \Delta t G(U_2) \\
U^{n+1} &= U^n + \frac{1}{2} \Delta t (F(U^n) + F(U_1)) + \frac{1}{2} \Delta t (G(U_1) + G(U_2)),
\end{align*}
\]

with $\gamma = (1 - \sqrt{2})/2$.

**Third order SSP IMEX-RK**

\[
\begin{align*}
U_1 &= U^n + \gamma \Delta t G(U_1) \\
U_2 &= U^n + \Delta t F(U^n) + (1 - 2\gamma) \Delta t G(U_1) + \gamma \Delta t G(U_2) \\
U_3 &= U^n + \frac{1}{4} \Delta t (F(U^n) + F(U_1)) + (1/2 - \gamma) \Delta t G(U_1) + \gamma \Delta t G(U_3) \\
U^{n+1} &= U^n + \frac{1}{6} \Delta t (F(U^n) + F(U_1) + 4F(U_2)) + \frac{1}{6} \Delta t (G(U_1) + G(U_2) + 4G(U_3)),
\end{align*}
\]

with $\gamma = (1 - \sqrt{2})/2$. 
IMEX Linear Multistep

\[ U^{n+1} = \sum_{j=0}^{s-1} a_j U^{n-j} + \Delta t \sum_{j=0}^{s-1} \tilde{w}_j \mathcal{F}(U^{n-j}) + \Delta t \sum_{j=-1}^{s-1} w_j \mathcal{G}(U^{n-j}), \]

with starting values \( U^0, U^1, \ldots, U^n \).

- Order \( p = s \).
- Schemes are a direct combination of an explicit and an implicit method. No need of compatibility conditions.
- Stability constraints usually increase with the order of the schemes (rather severe for SSP methods). A-stable schemes have accuracy \( p \leq 2 \).
- Off course they need to be initialized and the starting values imply additional storage requirements.

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\[ ^7 \text{U.Ascher, S.Ruth, B.Wetton '95, W.Hundsdorfer, S.Ruth '07} \]
Design of IMEX-LMM

As for IMEX-RK we can start from an explicit SSP method and find the corresponding implicit method with good damping properties (A($\alpha$)-stability). Or we can start from an implicit method (BDF) and use the corresponding explicit scheme.

**Second order IMEX-BDF**

\[
U^{n+1} = \frac{4}{3}U^n - \frac{1}{3}U^{n-1} + \frac{4}{3}\Delta t F(U^n) - \frac{2}{3}\Delta t F(U^{n-1}) + \frac{2}{3}\Delta t G(U^{n+1}).
\]

**Third order SSP IMEX-LM**

\[
U^{n+1} = \frac{3909}{2048}U^n - \frac{1367}{1024}U^{n-1} + \frac{873}{2048}U^{n-2} \\
+ \frac{18463}{12288}\Delta t F(U^n) - \frac{1271}{768}\Delta t F(U^{n-1}) + \frac{8233}{12288}\Delta t F(U^{n-2}) \\
+ \frac{1089}{2048}\Delta t G(U^{n+1}) - \frac{1139}{12288}\Delta t G(U^n) - \frac{367}{6144}\Delta t G(U^{n-1}) + \frac{1699}{12288}\Delta t G(U^{n-2}).
\]
Consider the case of hyperbolic relaxation systems\(^8\)

### Hyperbolic system with relaxation (Full model)

\[
\frac{\partial}{\partial t} U + \frac{\partial}{\partial x} F(U) = \frac{1}{\varepsilon} R(U), \quad x \in \mathbb{R}.
\]

\(R : \mathbb{R}^N \rightarrow \mathbb{R}^N\) is a relaxation operator if there exists a \(n \times N\) matrix \(Q\) with \(\text{rank}(Q) = n < N\) s.t. \(QR(U) = 0\) \(\forall U \in \mathbb{R}^N\).

This gives \(n\) conserved quantities \(u = QU\) that uniquely determine a local equilibrium \(U = \mathcal{E}(u)\), s.t. \(R(\mathcal{E}(u)) = 0\), and satisfy

\[
\frac{\partial}{\partial t} (QU) + \frac{\partial}{\partial x} (QF(U)) = 0.
\]

As \(\varepsilon \rightarrow 0 \Rightarrow R(U) = 0 \Rightarrow U = \mathcal{E}(u) \Rightarrow\) (subcharacteristic condition on on \(f(u)\))

### Equilibrium system (Reduced model)

\[
\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} f(u) = 0, \quad f(u) = QF(\mathcal{E}(u)).
\]
A simple example

A simple prototype example of relaxation system is given by

\[
\begin{aligned}
\partial_t u + \partial_x v &= 0, \\
\partial_t v + \partial_x au &= -\frac{1}{\varepsilon}(v - f(u)),
\end{aligned}
\]

where \( u = u(x, t) \), \( v = v(x, t) \), \( (x, t) \in \mathbb{R} \times \mathbb{R}_+ \).

For small values of \( \varepsilon \) we get the local equilibrium

\[ v = f(u) \]

and (subcharacteristic condition \( a > f'(u)^2 \)) we obtain at \( O(\varepsilon) \)

\[ \partial_t u + \partial_x f(u) = \varepsilon \partial_x ((a - f'(u)^2)\partial_x u). \]

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\(^9\text{S.Jin, Z.Xin '95}\)
**Definition**

An IMEX scheme for an hyperbolic system with relaxation is asymptotic preserving (AP) if in the limit $\epsilon \to 0$ the scheme becomes a consistent discretization of the limit system of conservation laws. We use the notation $AP_k$ if the scheme is of order $k$ in the limit $\epsilon \to 0$.

We can prove\(^\text{10}\)

**Theorem**

*If* $\det A \neq 0$ *then in the limit* $\epsilon \to 0$, *the IMEX scheme applied to an hyperbolic system with relaxation becomes the explicit RK scheme characterized by* $(\tilde{A}, \tilde{w}, \tilde{c})$ *applied to the limit system of conservation laws.*

- The theorem guarantees that in the stiff limit the IMEX scheme becomes the explicit RK scheme applied to the equilibrium system.
- To satisfy $\det A \neq 0$ it is necessary that $c \neq \tilde{c}$. The corresponding scheme may be inaccurate if the initial condition is not “well prepared” (initial layer).

\(^\text{10}\) L.P., G.Russo, '04
To study the $A$-stability of a IMEX-RK scheme, one may consider the problem\textsuperscript{11}

**Test problem**

\[ u' = \lambda u + \mu u, \quad u(0) = 1, \quad \lambda, \mu \in \mathbb{C}. \]

This test problem characterizes the stability properties also for linear systems

\[ U' = AU + BU, \quad U(0) = U_0 \]

with $U \in \mathbb{R}^m$, and $A, B \in \mathbb{R}^{m \times m}$ if $A$ and $B$ are normal, commuting matrices. In general the two matrices do not share the same eigenvectors, and can not be diagonalized simultaneously. This makes the stability analysis for systems extremely difficult. Only available results are for the case $m = 2$.

\textsuperscript{11}L.P., G.Russo ‘00, L.P. G.Russo ‘08
Broadwell model

\[ \partial_t \rho + \partial_x m = 0, \]
\[ \partial_t m + \partial_x z = 0, \]
\[ \partial_t z + \partial_x m = \frac{1}{\varepsilon} (\rho^2 + m^2 - 2\rho z), \]

with \( \varepsilon \) is the mean free path. The dynamical variables \( \rho \) and \( m \) are the density and the momentum respectively, while \( z \) represents the flux of momentum.

In the relaxation limit \( \varepsilon \to 0 \) we obtain

\[ \partial_t \rho + \partial_x m = 0 \]
\[ \partial_t m + \frac{1}{2} \partial_x \left( \rho + \frac{m^2}{\rho} \right) = 0 \]

(1) Accuracy test for IMEX-RK schemes with smooth initial data and periodic b.c.
(2) Shock test for IMEX-RK schemes.
## Convergence rates

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Convergence rates for $\rho$</th>
<th>Convergence rates for $z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ARS</td>
<td>2.018553 1.513026 1.159427 1.165810 1.165483 1.165396 1.165387</td>
<td>1.950904 1.438073 1.114938 1.121625 1.121979 1.121974 1.121972</td>
</tr>
<tr>
<td>IMEX-SSP2</td>
<td>2.042282 2.054974 2.051493 2.053945 2.043920 2.042585 2.042449</td>
<td>2.027090 2.045509 1.965834 1.501554 1.309825 1.302670 1.302573</td>
</tr>
<tr>
<td>ARSF</td>
<td>2.044539 2.071481 2.007092 1.982191 2.042974 2.040301 2.040004</td>
<td>2.031510 2.174882 1.762384 1.596958 2.061907 2.040474 2.039201</td>
</tr>
<tr>
<td>IMEX-SSP2F</td>
<td>2.050203 2.064147 2.061288 2.065251 2.056496 2.05314 2.055192</td>
<td>2.036402 2.034042 2.038568 2.368099 2.127720 2.052993 2.051640</td>
</tr>
</tbody>
</table>

Convergence rates where extrapolation has been used in order to avoid degradation of accuracy.
Relative error for different second and third order IMEX-RK schemes for the Broadwell equations with $\epsilon = 1$. Left: no initial layer. Right: initial layer.
Convergence rates $\epsilon = 10^{-3}$

Relative error for different second and third order IMEX-RK schemes for the Broadwell equations with $\epsilon = 10^{-3}$. Left: no initial layer. Right: initial layer.
Convergence rates $\epsilon = 10^{-6}$

Relative error for different second and third order IMEX-RK schemes for the Broadwell equations with $\epsilon = 10^{-6}$. Left: no initial layer. Right: initial layer.
Numerical examples

Shock test $\epsilon = 1$

Numerical solution for second and third order SSP IMEX-RK schemes for the Broadwell equations with $\epsilon = 1$
Shock test $\epsilon = 10^{-3}$

Numerical solution for second and third order SSP IMEX-RK schemes for the Broadwell equations with $\epsilon = 10^{-3}$
**Shock test** $\epsilon = 10^{-6}$

Numerical solution for second and third order SSP IMEX-RK schemes for the Broadwell equations with $\epsilon = 10^{-6}$
Penalized IMEX Runge-Kutta for the Boltzmann equation

**Design principles for kinetic equations**

- AP schemes *avoiding the implicit solution* of the collision term $Q(f, f)$

**Boltzmann-like equations**

$$\partial_t f + \partial_x f = \frac{1}{\varepsilon} Q(f, f), \quad x \in \mathbb{R},$$

where $f = f(x, v, t)$ and $Q(f, f) = 0$ implies $f = M[f]$ the local Maxwellian.

- When $f \approx M[f]$ the collision operator is well approximated by its linear counterpart $Q(M, f)$ or directly by a BGK relaxation operator $\mu(M[f] - f)$.
- If we denote by $L_P(f)$ the linear approximating operator we can write

$$Q(f, f) = G(f) + L_P(f), \quad G(f) = Q(f, f) - L_P(f).$$

12 S. Jin, F. Filbet '11
In the sequel we assume $L_P(f) = \mu (M[f] - f)$, $\mu > 0$. The IMEX-RK scheme take the form:\[\text{Penalized IMEX-RK for Boltzmann}\]

$$F = f^n e + \Delta t \tilde{A} \left( \frac{1}{\varepsilon} G(F) - v \cdot \nabla_x F \right) + \frac{\mu \Delta t}{\varepsilon} A (M[F] - F)$$

$$f^{n+1} = f^n + \Delta t \tilde{w}^T \left( \frac{1}{\varepsilon} G(F) - v \cdot \nabla_x F \right) + \frac{\mu \Delta t}{\varepsilon} w^T (M[F] - F).$$

- Clearly the scheme being implicit only in the linear part, which can be easily inverted and computed, can be implemented explicitly.
- Note however that here the problem is stiff as a whole. The hope is that applying the same design principles we used for hyperbolic systems with relaxation we get an AP-scheme for the full Boltzmann model.

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\(^{13}\)G.Dimarco, L.P. '13
First let us point out that since the linear operator enjoys the same conservation property of the full Boltzmann operator we have the same associated *moment scheme* characterized by \((\tilde{A}, \tilde{w})\) of the explicit method

\[
\begin{align*}
\int_{\mathbb{R}^3} F \phi(v) \, dv &= \int_{\mathbb{R}^3} f^n e \phi(v) \, dv - \Delta t \tilde{A} \int_{\mathbb{R}^3} v \cdot \nabla x F \phi(v) \, dv \\
\int_{\mathbb{R}^3} f^{n+1} \phi(v) \, dv &= \int_{\mathbb{R}^3} f^n \phi(v) \, dv - \Delta t \tilde{w}^T \int_{\mathbb{R}^3} v \cdot \nabla x F \phi(v) \, dv.
\end{align*}
\]

Consider now an invertible matrix \(A\) and solve the IMEX scheme for \((M[F] - F)\)

\[
\Delta t (M[F] - F) = \frac{\varepsilon}{\mu} A^{-1} \left[ F - f^n e + \Delta t \tilde{A} \left( v \cdot \nabla x F - \frac{1}{\varepsilon} G(F) \right) \right]
\]

Again as \(\varepsilon \to 0\) we get

\[F^{(i)} = M[F^{(i)}], \quad i = 1, \ldots, \nu.\]

In fact \(\tilde{A}\) is lower triangular with \(\tilde{a}_{ii} = 0\) and we have a hierarchy of equations

\[G(F^{(i)}) = Q(F^{(i)}, F^{(i)}) - \mu (M[F^{(i)}] - F^{(i)}) = 0, \quad i = 1, \ldots, \nu.\]
Further requirements

As opposite to the case of hyperbolic systems with relaxation, now the last level still depends on $\varepsilon$. After some manipulations it reads

$$f^{n+1} = f^n (1 - w^T A^{-1} e) - \Delta t \tilde{w}^T \left( v \cdot \nabla_x F - \frac{1}{\varepsilon} G(F) \right)$$

$$+ \Delta t w^T A^{-1} \tilde{A} \left( v \cdot \nabla_x F - \frac{1}{\varepsilon} G(F) \right) + w^T A^{-1} F.$$

For small values of $\varepsilon$ the scheme turns out to be unstable since $f^{n+1}$ is not bounded. A remedy, is to consider globally stiffly accurate schemes for which

$$f^{n+1} = F(\nu),$$

and so as $\varepsilon \to 0$

$$F(\nu) = M[F(\nu)] \Rightarrow f^{n+1} = M[f^{n+1}].$$

For the Boltzmann case the stiffly accurate property is required to have a stable AP and asymptotically accurate scheme.
Mixing regimes problem

Collision term approximated by the *Fast Fourier-Galerkin method* \(^{14}\). Second and third order *WENO* is used in space \(^{15}\)

Knudsen number value for the mixed regime test with \(\varepsilon_0 = 10^{-4}\)

\[
\begin{align*}
\varepsilon &= \varepsilon_0 + \frac{1}{2}(\tanh(16 - 20x) + \tanh(-4 + 20x)), & x \leq 0.7 \\
\varepsilon &= \varepsilon_0, & x > 0.7
\end{align*}
\]

\(^{14}\) L.P., B.Perthame '96, C.Mouhot, L.P. '06

\(^{15}\) C-W.Shu '97
Mixing regimes: third order scheme

Density (left) and temperature (right) profiles for the mixing regime problem. Time $t = 0.5$, $N_x = 100$ using third order WENO. Reference solution computed using a third order Runge-Kutta. Here $\Delta t_{IMEX}/\Delta t_{RK} = 7$. 
Density (left) and temperature (right) profiles for the mixing regime problem at $t = 0.5$ for $x \in [0.7, 0.8]$. 
Conclusion

- IMEX schemes represent a powerful tool for the time discretization of partial differential equations where convection and stiff sources/diffusion are present.

- In principle they can be applied to any other PDE where there is the presence of multiple time-scales.

- However they are not a universal cure for all problems. It is not difficult to imagine a situation where a fully explicit (or implicit) method is preferable.

- The most critical case is the application to (nonlinear) PDEs where the stiff scales originate a model reduction. In such cases AP methods are essential in order to capture the correct physical behavior.

- The AP-property can be generalized by expanding the solution in powers of $\varepsilon$ and matching higher order terms in the expansion with a given accuracy $^{16}$.  

$^{16}$S.Boscarino, ’08, S.Boscarino, G.Russo ’09