Some high order schemes for parabolic Hamilton-Jacobi-Bellman equations

Athena Picarelli
joint work with O. Bokanowski and C. Reisinger

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UNIVERSITÀ di VERONA

Introduction

We consider second order Hamilton-Jacobi-Bellman (HJB) equations:

$$\begin{cases} v_t + \sup_{a\in\Lambda} \mathcal{L}^a(t,x,v,D_x v, D_x^2 v) = 0 & x \in \mathbb{R}^d, t \in (0,T) \\ v(0,x) = \psi(x) & x \in \mathbb{R}^d, \end{cases}$$

where $\mathcal{L}^a : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathscr{S}^{d\times d} \to \mathbb{R}$ takes the form

$$\mathcal{L}^a(t,x,r,p,Q) = \left\{ -b(t,x,a)\cdot p - \frac{1}{2} Tr[\sigma\sigma^T(t,x,a)Q] - f(t,x,a)r + \ell(t,x,a) \right\}.$$

- $\Lambda \subset \mathbb{R}^m$ (set of control values): compact set;
- $T > 0$: terminal time;
- Existence and uniqueness of a viscosity solution of the HJB equation.

Introduction

Lions ('83), Fleming-Soner ('93), Yong-Zhou ('99), et al.:
v is the value function of a stochastic optimal control problem

$$v(t,x) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\Big[\int_t^T e^{F(t,s,x,\alpha)} \ell(s, X^\alpha_{t,x}(s), \alpha(s))\, ds + e^{F(t,T,x,\alpha)} \psi(X^\alpha_{t,x}(T)) \Big]$$

where $F(t,\cdot,x,\alpha) = \int_t^\cdot f(s, X^\alpha_{t,x}(s), \alpha(s))\, ds$ and $X^\alpha_{t,x}(\cdot)$ solves

$$\begin{cases} dX(s) = b(s, X(s), \alpha(s))\, ds + \sigma(s, X(s), \alpha(s))\, dW(s) \\ X(t) = x. \end{cases}$$

Example 1: Mean-variance asset allocation

Wealth equation (α control: proportion of wealth invested in risky asset)

$$dX(s) = [(r + \alpha\xi\sigma)X(s) + c]\, ds + \alpha\sigma X(s)\, dW(s),$$

with contribution rate c, $\xi = (b - r)\sigma^{-1}$. Mean-variance asset allocation problem:

$$\sup_{\alpha\in\mathcal{A}} \Big(\mathbb{E}[X^{\alpha}_{t,x}(T)] - \lambda Var[X^{\alpha}_{t,x}(T)] \Big).$$

This problem can be embedded in the following (Zhou and Li ('00)):

$$v(t,x) = \inf_{\alpha\in\mathcal{A}} \mathbb{E}\Big[\big(X^{\alpha}_{t,x}(T) - \frac{\gamma}{2}\big)^2 \Big],$$

associated with the following HJB equation

$$\begin{cases} v_t + \sup_{a\in\Lambda}\big\{ -\frac{1}{2}(\sigma a x)^2 v_{xx} - (c + x(r + a\sigma\xi))v_x \big\} &= 0, \\ v(0,x) &= \big(x - \frac{\gamma}{2}\big)^2. \end{cases}$$

Example 2: Uncertain volatility model

An underlying asset is assumed to follow

$$dX(s) = rX(s)\,dt + \sigma_s X(s)\,dW(s),$$

where we only assume $\sigma_s \in [\sigma_{\min}, \sigma_{\max}]$, an "uncertain" volatility.

The lowest arbitrage-free price is given by

$$v(t,x) = \inf_{\sigma} \mathbb{E}\left[e^{r(T-t)}\psi(X^{\sigma}_{t,x}(T))\right]$$

associated to the following HJB equation

$$\begin{cases} v_t + \sup_{\sigma\in\{\sigma_{\min},\sigma_{\max}\}}\left(-\frac{\sigma^2}{2}x^2 v_{xx}\right) - rxv_x + rv &= 0, \\ v(0,x) &= \psi(x), \end{cases}$$

where ψ is a European option payoff, and similar for the upper bound.

Approximation schemes

Consider $d = 1$.
Let us consider a space and time discretization $h, \tau > 0$:

$$t_k = k\tau \qquad x_i = ih \qquad k = 0, \dots, N \quad i \in \mathbb{I}.$$

$$u_i^k \sim v(t_k, x_i).$$

Let u be a numerical approximation of v defined by a scheme

$$\begin{aligned} \mathscr{S}(t_{k+1}, x_i, u_i^{k+1}, [u])_i &= 0 && k = 0, \dots, N-1, \quad i \in \mathbb{I} \\ u_i^0 &= \psi(x_i) && i \in \mathbb{I} \end{aligned}$$

Scheme in "explicit form":

$$\begin{cases} u_i^{k+1} = S(t_{k+1}, x_i, [\dots, u^{k-1}, u^k])_i & k = 0, \dots, N-1, \ i \in \mathbb{I} \\ u_i^0 \quad = \psi(x_i) & i \in \mathbb{I}. \end{cases}$$

If $\mathscr{S}$ one step explicit: $\mathscr{S}(t_{k+1}, x_i, u_i^{k+1}, u^k)_i = \frac{(u^{k+1} - S(t_{k+1}, x_i, u^k))_i}{\tau}$.

Key properties of numerical schemes

- Stability: (τ, h) the scheme admits a bounded solution;
- Consistency: For any smooth function φ, there exists a function $\mathcal{E}$ such that $\mathcal{E}(\tau, h) \to 0$ as $(\tau, h) \to 0$ and

$$\left| \mathscr{S}(t, x, \varphi(t, x), [\varphi]) - \left(\varphi_t + \sup_{a \in \Lambda} \mathcal{L}^a(t, x, \varphi, \varphi_x, \varphi_{xx}) \right) \right| \leq \mathcal{E}(\tau, h).$$

  $\mathcal{E}(\tau, h)$: consistency error;
  $\mathcal{E}(\tau, h) \leq C(\tau^\rho + h^\rho)$: scheme of order ρ;
- Monotonicity: For any bounded functions ϕ and ψ such that $\phi \leq \psi$:

$$\mathscr{S}(t, x, r, [\phi]) \geq \mathscr{S}(t, x, r, [\psi]).$$

Theorem (Barles-Souganidis ('91))

If the scheme $\mathscr{S}$ is stable, consistent and monotone, then its solution u converges to the unique viscosity solution of the HJB equation.

Finite difference schemes

For simplicity set $b = \ell = 0$:

$$v_t + \sup_{a\in\Lambda}\Big(-\tfrac{1}{2}\sigma^2(t,x,a)D^2 v - f(t,x,a)v\Big) = 0.$$

Then for $\theta \in [0,1]$ define a scheme

$$\frac{u_i^{k+1} - u_i^k}{\tau} + \sup_{a\in\Lambda}\Big(\theta\Big(-\tfrac{1}{2}\sigma^2(t_{k+1},x_i,a)D^2 u_i^{k+1} + f(t_{k+1},x_i,a)u_i^{k+1}\Big)$$
$$+(1-\theta)\Big(-\tfrac{1}{2}\sigma^2(t_k,x_i,a)D^2 u_i^k + f(t_n,x_i,a)u_i^k\Big)\Big) \;=\; 0,$$

where

$$D^2 u_i := \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}.$$

Typically: $\theta = 1/2$ (CN), $\theta = 0$ (EE), $\theta = 1$ (IE).

The Crank-Nicolson scheme

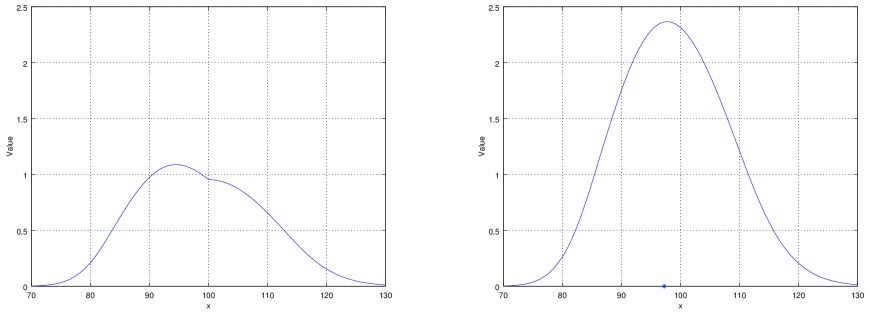
The IE scheme is: consistent ($\rho = 1$), unconditionally stable and monotone.

The CN scheme is:

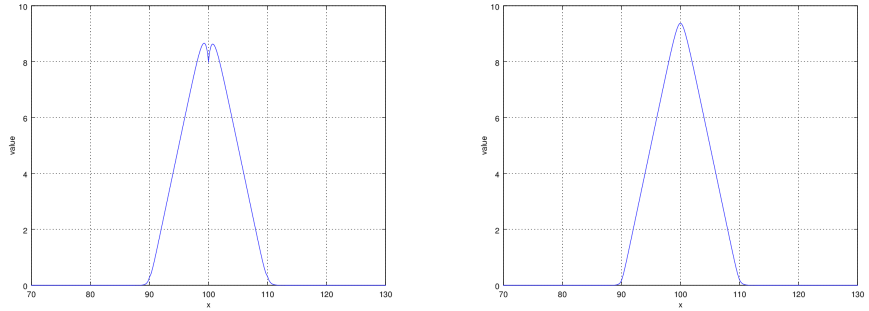
- consistent of order $\rho = 2$ in both τ and h;
- unconditionally stable, i.e. for all τ and h;
- monotone only under a CFL condition, a restriction on the timestep $\tau \leq Ch^2$.

Forsyth et al. ('01): butterfly options with UV

- Without CFL condition the CN scheme may converge to a wrong solution;
- Rannacher timestepping (fully implicit scheme for the first iterations) shows convergence to the exact solution;
- No convergence proof since the scheme is non monotone.



Value function $t = t_N = 0.1$



Value function $t = t_1$

Filtered schemes

Theorem (Godunov ('59))

Monotone linear schemes have order of consistency at most $\rho = 1$.

Idea of filtered schemes: Consider "almost-monotone" schemes obtained as a perturbation of high order schemes and for which convergence to the viscosity solution can be proved.

References:

- Froese-Oberman ('13): Monge-Ampère equation;
- Oberman-Salvador ('15): first order HJ equations;
- Bokanowski-Falcone-Sahu ('15): first order time dependent HJ eq.

Aim of this work:

- Generalise to second order time dependent HJB equations;
- Generalise to implicit finite difference schemes;
- Tests on some control problems arising in financial applications.

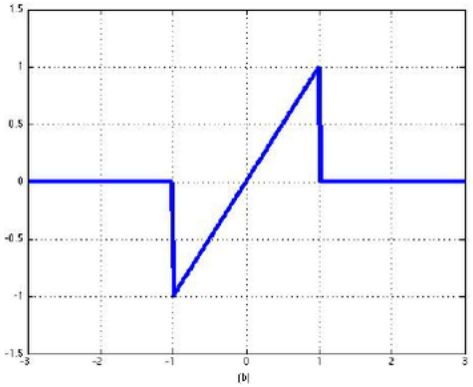
Filtered schemes for second order HJB equations

To define the filtered scheme, the following three ingredients are needed:

- A monotone scheme: $u^{k+1} = S_M(u^k)$;
- A higher order ($\rho > 1$) scheme: $u^{k+1} = S_H([\ldots, u^{k-1}, u^k])$;
- A filter function F.

We switch between the schemes using the following filter:

$$F(x) := \begin{cases} x & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$



The filtered scheme is defined by

$$S_F := S_M + \varepsilon\tau \; F\left(\frac{S_H - S_M}{\varepsilon\tau}\right).$$

The monotone scheme

We assume that $\mathbb{I} = \{1, \dots, J\}$ and consider a monotone, consistent and stable scheme $\mathscr{S}_M$ ($\Rightarrow$ convergent) of the following form:

$$\mathscr{S}_M(t_k, x_i, u_i^k, u^{k-1})_i \equiv \frac{1}{\tau} \sup_{a \in \Lambda} \left\{ \mathbb{M}^{a,k} u^k - G^{a,k-1}(u^{k-1}) \right\}_i$$

with $\mathbb{M}^{a,k} \in \mathbb{R}^{J \times J}$ and $G^{a,k-1}(\cdot) \in \mathbb{R}^J$, such that

$$\mathcal{E}_M(\tau, h) \leq C_M(\tau + h).$$

Under suitable assumptions on $\mathbb{M}$:

$$\begin{cases} u_i^k = S_M(u^{k-1})_i & k = 1, \dots, N, \quad i \in \mathbb{I} \\ u_i^0 \quad = \psi(x_i) & \qquad\qquad\qquad\quad i \in \mathbb{I} \end{cases}$$

Examples: Implicit Euler, Semi-Lagrangian;

The high order scheme

Consider a higher order scheme:

$$\begin{cases} u_i^k = S_H([\ldots, u^{k-2}, u^{k-1}])_i & k = 1, \ldots, N, \quad i \in \mathbb{I} \\ u_i^0 \quad = \psi(x_i) & \qquad\qquad\qquad\qquad\quad i \in \mathbb{I} \end{cases}$$

such that

$$\mathcal{E}_H(\tau, h) \leq C_H(\tau^p + h^q)$$

for some $p, q > 1$.

Examples ($p = q = 2$): CN, BDF2 :

$$\frac{3u_i^k - 4u_i^{k-1} + u_i^{k-2}}{2\tau} + \sup_{a\in\Lambda}\Big\{ -\tfrac{1}{2}\sigma^2(t_k, x_i, a)D^2 u_i^k + b^+(t_k, x_i, a)D^{1,-}u_i^k$$

$$+ b^-(t_k, x_i, a)D^{1,+}u_i^k + f(t_k, x_i, a)u_i^k - \ell(t_k, x_i, a)\Big\} = 0,$$

where

$$D^{1,-}u_i := \frac{3u_i - 4u_{i-1} + u_{i-2}}{2h} \quad \text{and} \quad D^{1,+}u_i := \frac{3u_i - 4u_{i+1} + u_{i+2}}{2h}.$$

Main results

Theorem

The solution of the filtered scheme converges locally uniformly to the unique viscosity solution of the HJB equation as $(\tau, h, \varepsilon) \to 0$.

Theorem

Let the solution v *be smooth in* $\mathcal{O}$. *If* S_F *is stable, there exists* $C_0 > 0$ *such that if* $\varepsilon = C_0(\tau + h)$ *and* τ, h *are small enough then*

$$S_F(v) = S_H(v) \quad \text{and} \quad \mathcal{E}_F(\tau, h) = \mathcal{E}_H(\tau, h) \qquad \text{in } \mathcal{O}.$$

Proof.

One has to prove that $\frac{|S_H([\ldots, v^{n-1}, v^n]) - S_M(v^n)|}{\varepsilon\tau} \leq 1$. For simplicity let us consider explicit schemes. From consistency of the schemes there is $C_M > 0$ such that:

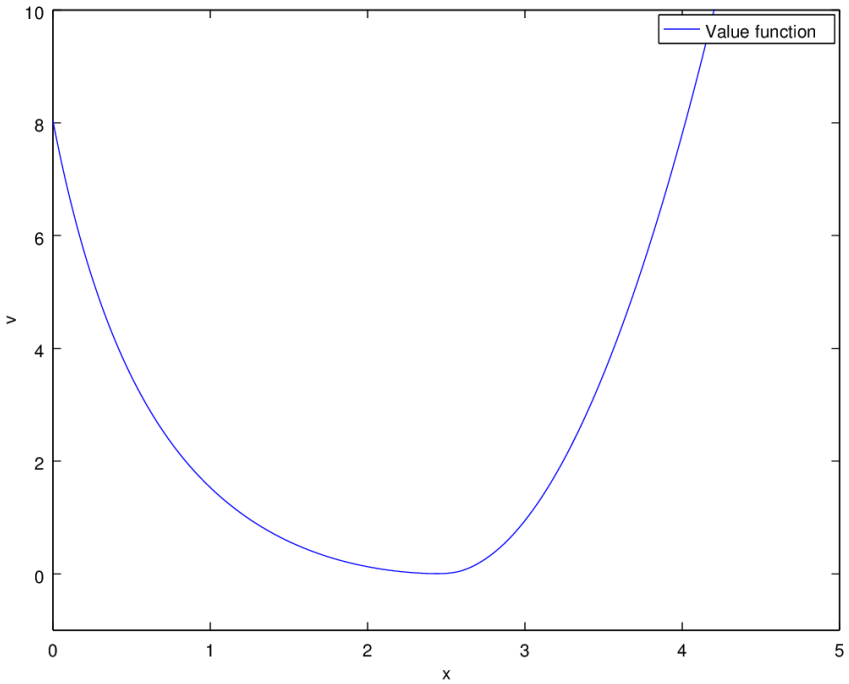
$$\frac{|S_M(v^n) - S_H([v])|}{\varepsilon\tau} \leq \frac{\mathcal{E}_M(\tau, h) + \mathcal{E}_H(\tau, h)}{\varepsilon} \leq \frac{C_M(\tau + h) + o(\tau + h)}{\varepsilon}.$$

The result follows for τ, h small enough and $C_0 > C_M$ big enough .

Test 1: Mean-variance asset allocation problem

Recall the HJB equation

$$\begin{cases} v_t + \sup\limits_{a\in\Lambda}\Big\{ -\frac{1}{2}(\sigma a x)^2 v_{xx} - (c + x(r + a\sigma\xi))v_x \Big\} &= 0, \\ v(0,x) &= \left(x - \frac{\gamma}{2}\right)^2. \end{cases}$$

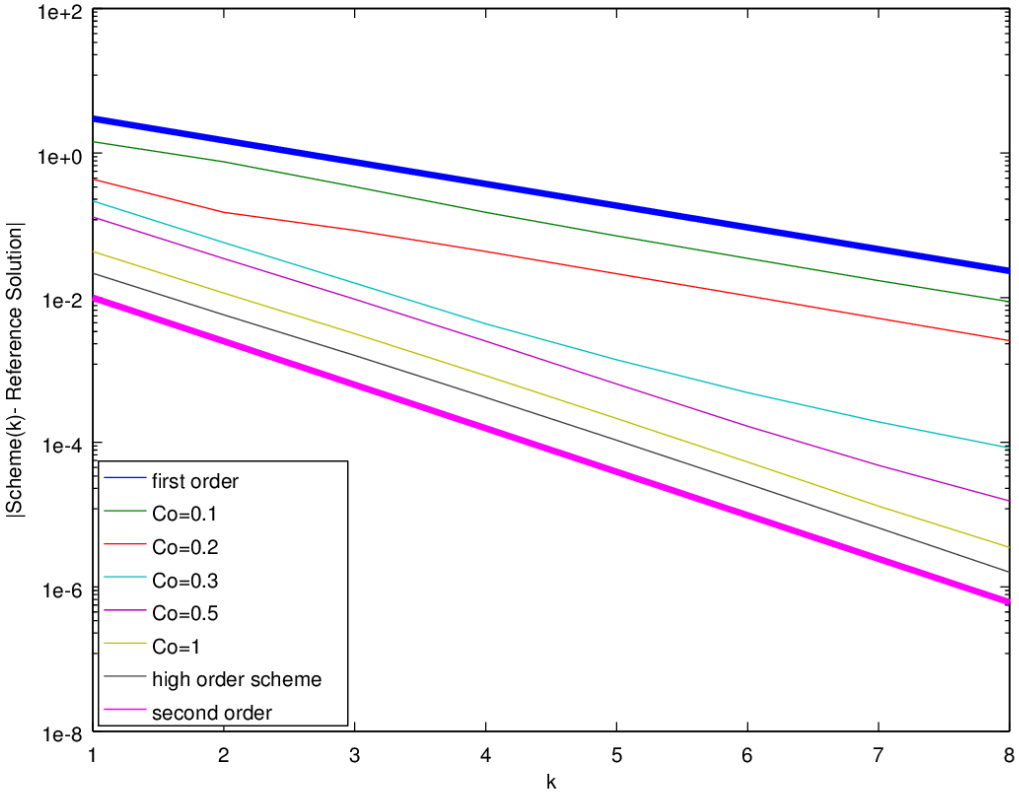


Test 1: Mean-variance asset allocation problem

Monotone scheme: Implicit Euler scheme. One has:

$$C_M \sim 35.$$

High order scheme: BDF2, Crank-Nicolson



Test 2: a two dimensional example

Consider the following HJB equation in $[0, T] \times ([-\pi, \pi] \times [-\pi, \pi])$:

$$\begin{cases} v_t - \inf_{a \in \Lambda} \left(\frac{1}{2} Tr[\sigma\sigma^T(a) D_x^2 v] + \ell(t, x, a) \right) & = & 0, \\ v(0, x) & = & 2 \sin x_1 \sin x_2 \end{cases}$$

with

$$\Lambda = \{a \in \mathbb{R}^2 : a_1^2 + a_2^2 = 1\}, \qquad \sigma(a) = \sqrt{2} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

$$\ell(t, x, a) = (1 - t) \sin x_1 \sin x_2 + (2 - t)(a_1^2 \cos x_1 + a_2^2 \cos x_2).$$

This problem has exact solution:

$$v(t, x) = (2 - t) \sin x_1 \sin x_2.$$

Test 2: a two dimensional example

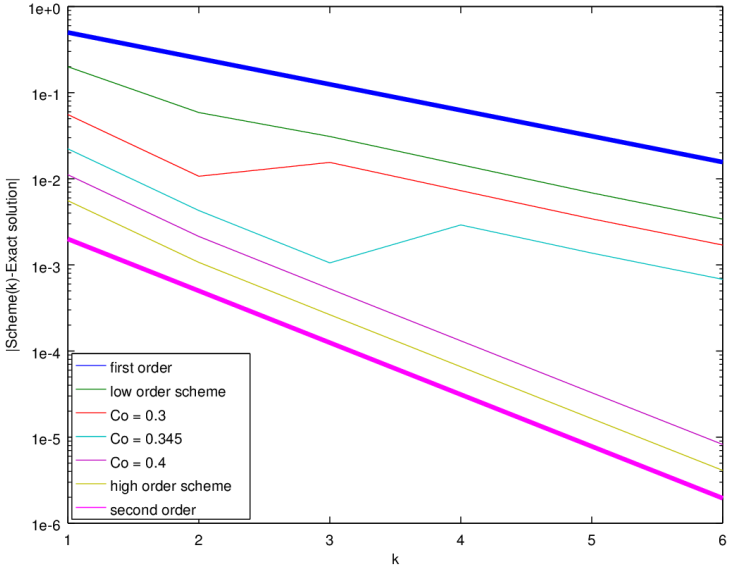
Monotone scheme: Explicit semi-Lagrangian scheme. One has:

$$C_M \sim \frac{20}{3}.$$

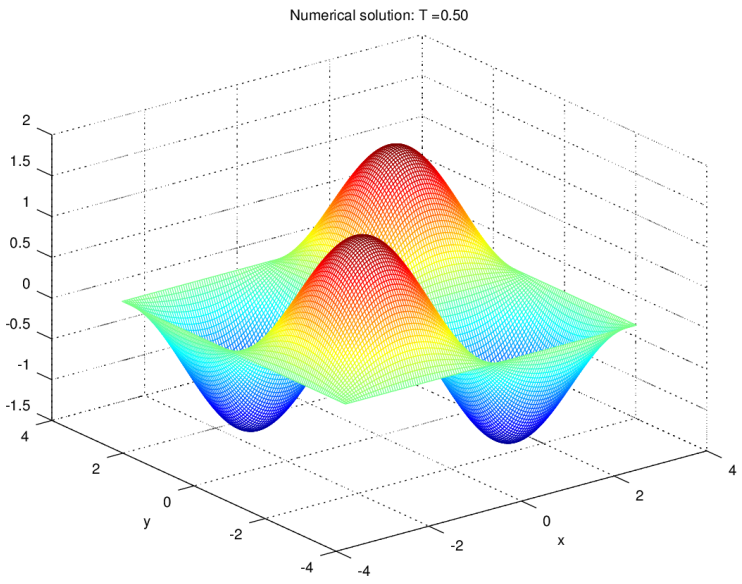
High order scheme: Implicit Euler timestepping and

$$D^2_{xx} v_{i,j} := \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{{h_x}^2}, \qquad D^2_{yy} v_j := \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{{h_y}^2}$$

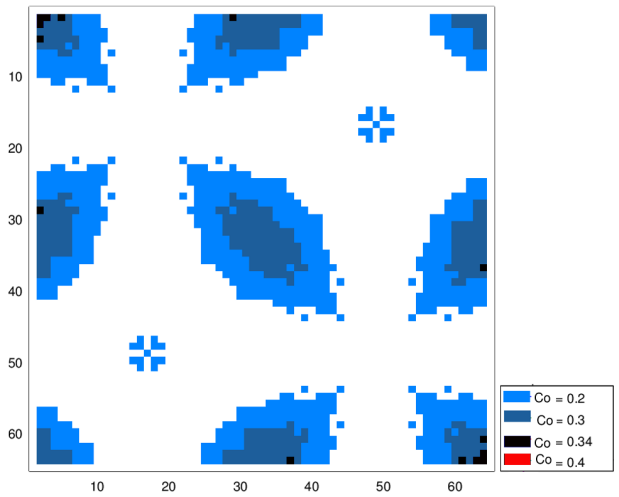
$$D^2_{xy} v_{i,j} := \frac{v_{i+1,j+1} - v_{i+1,j-1} + v_{i-1,j-1} - v_{i-1,j+1}}{4h_x h_x}.$$



Test 2: a two dimensional example



Value function at $T = 0.5$



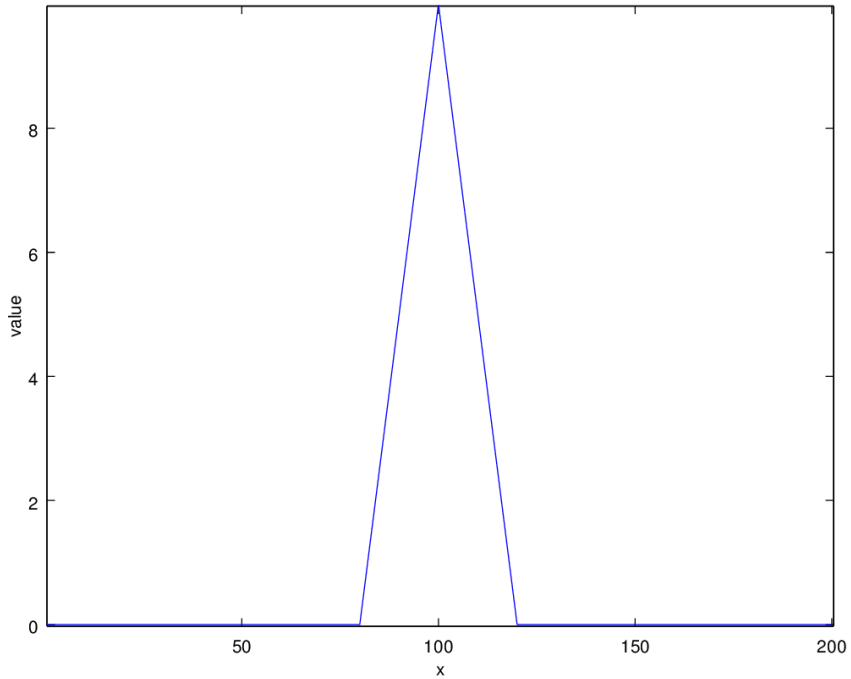
Activity of the filter

Test 3: butterfly option with UV

Recall the HJB equation:

$$\begin{cases} v_t + \sup_{\sigma \in \{\sigma_{\min}, \sigma_{\max}\}} \left(-\frac{\sigma^2}{2} x^2 v_{xx} \right) - rxv_x + rv & = & 0, \\ v(0,x) & = & \psi(x) \end{cases}$$

where we use a "butterfly" payoff ψ of mixed convexity:

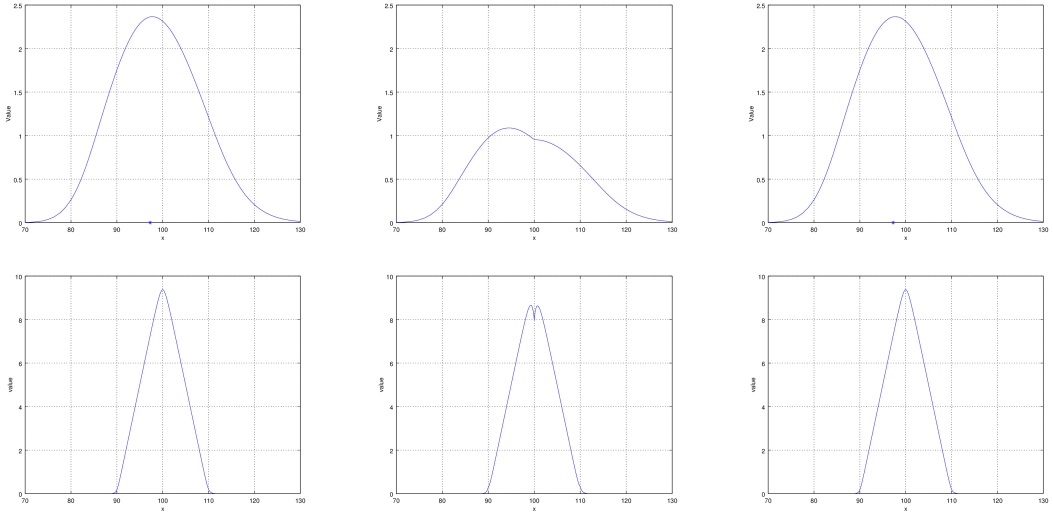


Test 3: butterfly option with UV

Pooley-Forsyth-Vetzal ('01): in absence of the CFL, the CN scheme may converge to a wrong solution.
Bokanowski-AP-Reisinger ('17): the BDF2 scheme shows good performances.

Value function
$t = t_N = 0.1$:

Value function
$t = t_1$:

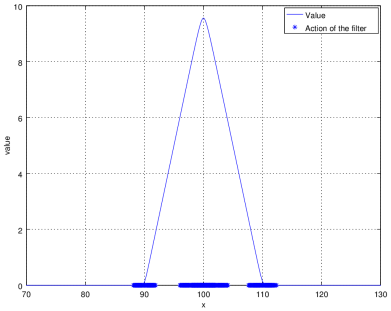


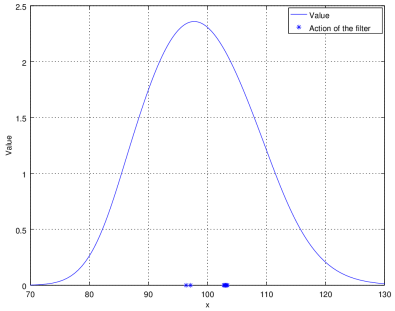
Monotone scheme | Crank-Nicolson (no CFL) | BDF2

Test 3: butterfly option with UV

Applying the filter to the Crank-Nicholson scheme:

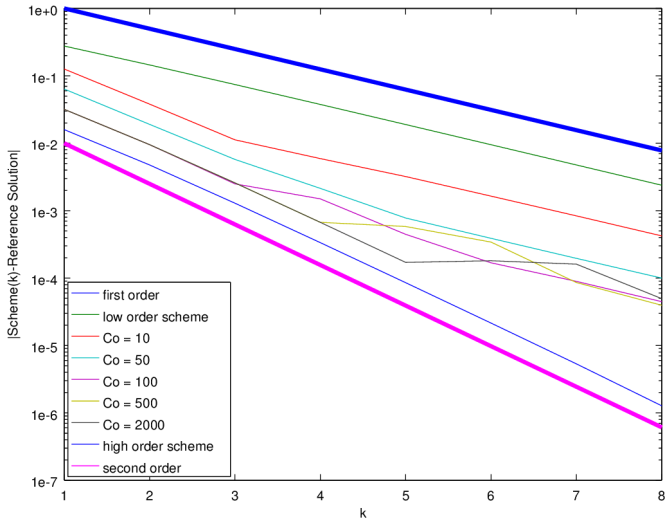
- The filter becomes active to correct the scheme;
- the order of convergence is not more than one in the relevant region.





Filtered CN	BDF	Rannacher
1.25	0.72	1.28
0.85	2.34	1.86
0.90	1.70	1.88
1.10	1.82	1.94
0.94	1.95	1.98
1.10	1.98	1.99
0.92	1.83	1.96

Applying the filter on Rannacher or BDF2:
The schemes converges, but the unboundedness of the derivatives prevents second order convergence.



Conclusions

- We have defined filtered schemes for second order time dependent HJB equations;
- We have showed convergence and high order behavior for smooth solutions;

...moreover...

Motivated by the results in our numerical tests, we have studied a second order (non monotone) BDF2 scheme (mainly in one dimension):

- Stability proof in $|\cdot|_{H_1}$ and $\|\cdot\|_2$ for the BDF2 scheme;
- Error estimates for piecewise smooth solutions.

Future directions of work:

- Stability proof for BDF2 scheme in L^∞ norm;
- Multiple dimensions.

O. Bokanowski, AP, C. Reisinger.**High-order filtered schemes for time-dependent second order HJB equations**. ESAIM: M2AN. (2017).

O. Bokanowski, AP, C. Reisinger. **Stability and convergence properties for a BDF2-type scheme for parabolic Hamilton-Jacobi-Bellman equations**. https://arxiv.org/abs/1802.07146.