Preconditioning semismooth Newton methods for optimal control problems with L^1- sparsity and control constraints

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Joint work with
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The Constrained Optimal Control Problem

$$\min \quad J(y,u) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} + \beta\|u\|_{L^1(\Omega)} \leftarrow \text{sparsity constr.}$$

$$\text{over } (y,u) \in H^1_0(\Omega) \times L^2(\Omega)$$

$$\text{s.t.} \quad \mathcal{L}y = u \quad \text{in } \Omega \qquad \leftarrow \text{state equation}$$

$$\text{and} \quad a \le u \le b \text{ a.e. in } \Omega \qquad \leftarrow \text{box constraints}$$

- u and y are the control and state variables
- $y_d \in L^2(\Omega)$ is the desired state, $\Omega \subseteq \mathbb{R}^d$ with $d = 2, 3$
- $\mathcal{L}$ is a second-order linear elliptic differential operator
- Control box constraints: $a, b \in L^2(\Omega)$ and $a < 0 < b$
- Parameters: L^2-norm term $\alpha > 0$ and L^1-norm term $\beta > 0$.

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Sparsity constraints in optimal control problems

Motivation

- Optimal control applications: provide information about the optimal location of control device and actuators [Stadler, COAP 2009] [Costa et al. Comput. Struct. 2007].

Main references:

- L^1-norm: Casas, Clacson, Kunish, Herzog, Stadler, Wachsmuth, 2009-2012
- Directional Sparsity $\|\cdot\|_{1,2}$ Herzog, Stadler and Wachsmuth SICON 2012.

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Box constraints: optimality conditions (simple case)

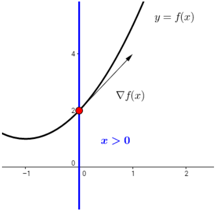
Box constrained optimization problem

$$\min f(x) \text{ s.t. } x \geq a$$

where $f : \mathbb{R}^n \to \mathbb{R}$.

The first order optimality conditions are

$$\begin{cases} \nabla f(x) = -\mu & \\ \mu = 0, & x > a \\ \mu < 0 & x = a \end{cases}$$



where the last complementarity conditions can be restated as

$$c\mu - \min(0, (x-a) + c\mu)) = 0, \quad c > 0.$$

μ is the Lagrange multiplier (component-wise conditions).

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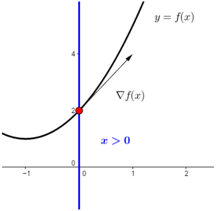

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L^1 sparsity constraint: optimality conditions (simple case)

ℓ_1 optimization problem and a constrained problem

$$\min_{x \in \mathbb{R}^n} f(x) + \beta \|x\|_1, \quad \Leftrightarrow \quad \min_x f(x) \text{ s.t. } \|x\|_1 \leq \hat{\beta},$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $\beta, \hat{\beta} \in \mathbb{R}^+$, $\|x\|_1 = \sum_{i=1}^n |x_i|$.

The first order optimality conditions are

$$\nabla f(x) = \begin{cases} = -\beta, & x > 0 \\ \in [-\beta, \beta], & x = 0 \\ = \beta, & x < 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \nabla f(x) = -\mu & \\ \mu = \beta, & x > 0 \\ |\mu| \leq \beta, & x = 0 \\ \mu = -\beta, & x < 0 \end{cases}$$

where the last complementarity conditions can be restated as

$$x - \max(0, x + c(\mu - \beta)) - \min(0, x + c(\mu + \beta)) = 0, \quad c > 0.$$

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$$x - \max(0, x + c(\mu - \beta)) - \min(0, x + c(\mu + \beta)) = 0, \quad c > 0.$$

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Optimality conditions

The KKT system [Stadler COAP 2009]

The solution $(\bar{y}, \bar{u}) \in H_0^1(\Omega) \times L^2(\Omega)$ of the optimal control problem is characterized by the existence of $(\bar{p}, \bar{\mu}) \in H_0^1(\Omega) \times L^2(\Omega)$ such that

$$
\begin{aligned}
&\mathcal{L}\bar{y} - \bar{u} = 0\\
&\mathcal{L}^*\bar{p} + \bar{y} - y_d = 0\\
&-\bar{p} + \alpha\bar{u} + \bar{\mu} = 0
\end{aligned}
$$

$$
\begin{aligned}
F(u,\mu) := \bar{u} &- \max(0, \bar{u} + c(\bar{\mu} - \beta)) - \min(0, \bar{u} + c(\bar{\mu} + \beta))\\
&+ \max(0, (\bar{u} - b) + c(\bar{\mu} - \beta)) + \min(0, (\bar{u} - a) + c(\bar{\mu} + \beta)) = 0 \text{ a.e. in } \Omega,
\end{aligned}
$$

with $c > 0$.

- The complementarity function F is nonlinear and semismooth $\Rightarrow$ SemiSmooth Newton's method (SSN) for the KKT system, i.e. a Newton's method where the Jacobian of the system is obtained using generalized derivatives.

The discretized (FE) optimality system (KKT)

Let $\Theta : \mathbb{R}^{4n} \to \mathbb{R}^{4n}$ and $x = (y, u, p, \mu) \in \mathbb{R}^{4n}$:

$$\Theta(x) = \begin{bmatrix} \Theta(x)^y \\ \Theta(x)^u \\ \Theta(x)^p \\ \Theta(x)^\mu \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} My + L^T p - My_d \\ \alpha Mu - \bar{M}^T p + M\mu \\ Ly - \bar{M}u - f \\ MF(u,\mu) \end{bmatrix} = 0$$

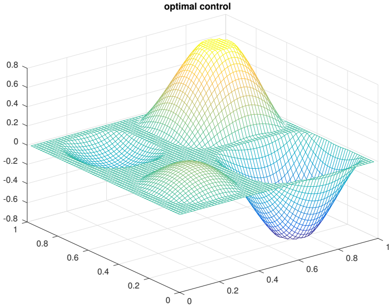
and the complementarity function $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is component-wise defined by

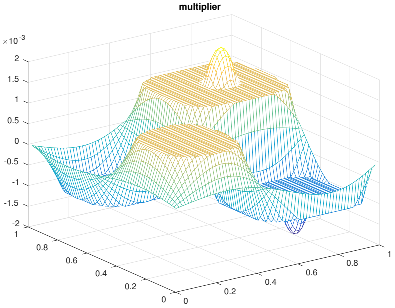
$$\begin{aligned} F(u,\mu) = \quad & u - \max(0, u + c(\mu - \beta)) - \min(0, u + c(\mu + \beta)) \\ & + \max(0, (u - b) + c(\mu - \beta)) + \min(0, (u - a) + c(\mu + \beta)), \;\; c > 0 \end{aligned}$$

- n: dimension of the discretized space.
- L: *stiffness matrix* (possibly unsym); M: *mass matrix* (lumped)
- $\bar{M}$: discretization of the control term within the PDE-constraint (unsym).

⋆ Since M is diag, componentwise complementarity conditions still hold with FE.

Illustration of the Active-Set (AS) interpretation





- $u_i = 0$ for $i \in \mathcal{A}_0$ (sparsity);
- $u_i = a$ for $i \in \mathcal{A}_a$ (lower bound);
- $u_i = b$ for $i \in \mathcal{A}_b$ (upper bound);

- $\mu_i = -\beta$ for $i \in \mathcal{I}_-$ (unc.);
- $\mu_i = \beta$ for $i \in \mathcal{I}_+$ (unc.).

- Sparse control is expected at the solution (e.g. for not so small values of β) $\Rightarrow$ large size of $\mathcal{A}_0$;

Active-Set approach definition

- Complementarity function:

$$
\begin{aligned}
F(u,\mu) = \quad & u - \max(0, u + c(\mu-\beta)) - \min(0, u + c(\mu+\beta)) \\
& + \max(0, (u-b) + c(\mu-\beta)) + \min(0, (u-a) + c(\mu+\beta))
\end{aligned}
$$

- Sets of **Active** $\mathcal{A} = \mathcal{A}_b \cup \mathcal{A}_a \cup \mathcal{A}_0$ and **inactive** $\mathcal{I} = \mathcal{I}_+ \cup \mathcal{I}_-$ constraints

$$
\begin{aligned}
\mathcal{A}_b &= \{i \mid c(\mu_i - \beta) + (u_i - b_i) > 0\} \\
\mathcal{A}_a &= \{i \mid c(\mu_i + \beta) + (a_i - u_i) < 0\} \\
\mathcal{A}_0 &= \{i \mid u_i + c(\mu_i + \beta) \geq 0\} \cup \{i \mid u_i + c(\mu_i - \beta) \leq 0\}
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$$\Downarrow$$

$$F(u,\mu) = \Pi_{\mathcal{A}_0} u + \Pi_{\mathcal{A}_b}(u-b) + \Pi_{\mathcal{A}_a}(u-a) - c(\Pi_{\mathcal{I}_+}(\mu-\beta) + \Pi_{\mathcal{I}_-}(\mu+\beta))$$

$\Pi_{\mathcal{C}}$ 0-1 diagonal matrix with 1's corresponding to $\mathcal{C}$.

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Generalized Jacobian:

$$F(u,\mu) = \underbrace{\Pi_{\mathcal{A}_0} u + \Pi_{\mathcal{A}_b}(u-b) + \Pi_{\mathcal{A}_a}(u-a)}_{u,\mathcal{A}} \underbrace{-c(\Pi_{\mathcal{I}_+}(\mu-\beta) + \Pi_{\mathcal{I}_-}(\mu+\beta))}_{\mu,\mathcal{I}} = 0$$

Generalized derivative: $F'(u,\mu) = \begin{bmatrix} \Pi_{\mathcal{A}} & -c\Pi_{\mathcal{I}} \end{bmatrix}$

SemiSmooth Newton's method (SSN)

Given the current iterate $x_k = (y_k, u_k, p_k, \mu_k)$, a SSN step is

$$\underbrace{\begin{bmatrix} M & 0 & L^T & 0 \\ 0 & \alpha M & -\bar{M}^T & M \\ L & -\bar{M} & 0 & 0 \\ 0 & \Pi_{\mathcal{A}_k} M & 0 & -c\Pi_{\mathcal{I}_k} M \end{bmatrix}}_{\Theta'(x_k)} \begin{bmatrix} \Delta y \\ \Delta u \\ \Delta p \\ \Delta \mu \end{bmatrix} = - \begin{bmatrix} \Theta_k^y \\ \Theta_k^u \\ \Theta_k^p \\ \Theta_k^\mu \end{bmatrix}.$$

$\Theta'(x_k)$ is unsymmetric and of dimension $(4n \times 4n)$.

From now on, $P_{\mathcal{C}}$ is the projection on the subspace defined by the set $\mathcal{C}$ and $\Pi_{\mathcal{C}} = P_{\mathcal{C}}^T P_{\mathcal{C}}$.

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Saddle-point symmetric formulations of the Newton's equation

- **Augmented form** 4×4 $(3n + |\mathcal{A}_k|) \times (3n + |\mathcal{A}_k|)$

$$\begin{bmatrix} M & 0 & L^T & 0 \\ 0 & \alpha M & -\bar{M}^T & M P_{\mathcal{A}_k}^T \\ L & -\bar{M} & 0 & 0 \\ 0 & P_{\mathcal{A}_k} M & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta u \\ \Delta p \\ (\Delta\mu)_{\mathcal{A}_k} \end{bmatrix} = - \begin{bmatrix} \Theta_k^y \\ \tilde{\Theta}_k^u \\ \Theta_k^p \\ \tilde{\Theta}_k^\mu \end{bmatrix} \quad \Leftrightarrow \quad J_k^{aug} \Delta x^{aug} = b_k^{aug}$$

with $(\mu_{k+1})_{(\mathcal{I}_+)_{k+1}} = \beta$ and $(\mu_{k+1})_{(\mathcal{I}_-)_{k+1}} = -\beta$.

- **Reduced form** 2×2 $(2n \times 2n)$

$$\begin{bmatrix} M & L^T \\ L & -\frac{1}{\alpha} \bar{M} M^{-1} \Pi_{\mathcal{I}_k} \bar{M}^T \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta p \end{bmatrix} = - \begin{bmatrix} \Theta_k^y \\ \hat{\Theta}_k^p \end{bmatrix} \quad \Leftrightarrow \quad J_k^{red} \Delta x^{red} = b_k^{red}$$

with

$$\begin{aligned} \Delta u &= \frac{1}{\alpha}(M^{-1}\bar{M}^T \Delta p - P_{\mathcal{A}_k}^T (\Delta\mu)_{\mathcal{A}_k} - M^{-1}\tilde{\Theta}_k^u), \\ (\Delta\mu)_{\mathcal{A}_k} &= P_{\mathcal{A}_k} M^{-1} \Pi_{\mathcal{A}_k} \bar{M}^T \Delta p - \alpha P_{\mathcal{A}_k} M^{-1} P_{\mathcal{A}_k}^T \left(\Theta_k^\mu + \frac{1}{\alpha} P_{\mathcal{A}_k} \tilde{\Theta}_k^u \right) \end{aligned}$$

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Preconditioning the sequence of Newton equations

$$J_k(x_k)\Delta x = b_k \qquad (*)$$

where J_k (J_k^{aug} or J_k^{red}) is a saddle point matrix.

- Assume that Krylov subspace methods are used to solve the large and sparse Newton equations $\Rightarrow$ preconditioning is mandatory.

Objective

Find effective optimal and robust preconditioners for the Newton equations for general box/sparsity constraints such that the number of iterations required to solve $(*)$ is low and (roughly) independent of the problem parameters:

- regularization parameters: α (L^2) and β (L^1);
- mesh size h

Some background

Preconditioner

Given a system $Ax = b$, find a matrix $P \approx A$ such that

$$P^{-1}Ax = P^{-1}b$$

is “easier” to solve than the original system by an iterative method $\Rightarrow$ $P^{-1}A$ has ‘better” spectral properties than A.

Saddle-point matrix and its block-triangular factorization:

$$H = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} = \begin{bmatrix} I & 0 \\ BA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & -S \end{bmatrix} \begin{bmatrix} I & A^{-1}B^T \\ 0 & I \end{bmatrix}$$

where $S = (C + BA^{-1}B^T)$ is the Schur complement of A in H.

- If $A \succeq 0$, B full rank, $C \succ 0$ $(S \succ 0) \Rightarrow \Lambda(H) = \Lambda(A) \cup \Lambda(-S)$ (indefinite)
- If $A \succeq 0$, B full rank, $C = 0 \Rightarrow S \succeq 0$;
  If $ker(A) \cap ker(B) = \emptyset \Rightarrow H$ is nonsingular.

Matching Schur complement approximations: the unconstrained case

$$J = \left[\begin{array}{cc|c} M & 0 & -L^T \\ 0 & \alpha M & M \\ \hline -L & M & 0 \end{array}\right]$$

- J does not depends on k (corresponds to $\mathcal{A}_k = \emptyset$).
- Schur complement:

$$S = \alpha L M^{-1} L^T + M = \underbrace{(\sqrt{\alpha} L + M) M^{-1} (\sqrt{\alpha} L + M)^T}_{\hat{S}} - 2\sqrt{\alpha} L$$

- Approximation: $S \approx (\sqrt{\alpha} L + M) M^{-1} (\sqrt{\alpha} L + M)^T$
- Optimal bounds:

$$\lambda(\hat{S}^{-1} S) \in \left[\frac{1}{2}, 1\right]$$

independence on α [Pearson and Wathen, NLAA 2012]

The active-set Schur complement of J^{aug}

Augmented form

$$J_k^{aug} = \left[\begin{array}{cc|cc} M & 0 & L^T & 0 \\ 0 & \alpha M & -\bar{M}^T & MP_{\mathcal{A}}^T \\ \hline L & -\bar{M} & 0 & 0 \\ 0 & P_{\mathcal{A}}M & 0 & 0 \end{array}\right] = \begin{bmatrix} J_{11} & J_{21}^T \\ J_{21} & 0 \end{bmatrix}$$

Matching strategy in the general case

$$S = J_{21}J_{11}^{-1}J_{21}^T = \frac{1}{\alpha}\begin{bmatrix} I & -\bar{M}\Pi_{\mathcal{A}}M^{-1}P_{\mathcal{A}}^T \\ 0 & I \end{bmatrix}\begin{bmatrix} \mathbb{S} & 0 \\ 0 & P_{\mathcal{A}}MP_{\mathcal{A}}^T \end{bmatrix}\begin{bmatrix} I & 0 \\ -P_{\mathcal{A}}M^{-1}\Pi_{\mathcal{A}}\bar{M}^T & I \end{bmatrix},$$

where $\mathbb{S}$ is the Schur complement of S,

$$\mathbb{S} = \alpha LM^{-1}L^T + \bar{M}(I - \Pi_{\mathcal{A}})M^{-1}\bar{M}^T$$

- If $\mathcal{A} = \emptyset$ and $\bar{M} = M$, $\mathbb{S} = \alpha LM^{-1}L^T + M$

From now on the index k is omitted.

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$$J_k^{aug} = \left[\begin{array}{cc|cc} M & 0 & L^T & 0 \\ 0 & \alpha M & -\bar{M}^T & MP_{\mathcal{A}}^T \\ \hline L & -\bar{M} & 0 & 0 \\ 0 & P_{\mathcal{A}} M & 0 & 0 \end{array}\right] = \begin{bmatrix} J_{11} & J_{21}^T \\ J_{21} & 0 \end{bmatrix}$$

Matching strategy in the general case

$$S = J_{21} J_{11}^{-1} J_{21}^T = \frac{1}{\alpha} \begin{bmatrix} I & -\bar{M}\Pi_{\mathcal{A}} M^{-1} P_{\mathcal{A}}^T \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbb{S} & 0 \\ 0 & P_{\mathcal{A}} M P_{\mathcal{A}}^T \end{bmatrix} \begin{bmatrix} I & 0 \\ -P_{\mathcal{A}} M^{-1} \Pi_{\mathcal{A}} \bar{M}^T & I \end{bmatrix},$$

where $\mathbb{S}$ is the Schur complement of S,

$$\mathbb{S} = \alpha L M^{-1} L^T + \bar{M}(I - \Pi_{\mathcal{A}}) M^{-1} \bar{M}^T$$

- If $\mathcal{A} = \emptyset$ and $\bar{M} = M$, $\mathbb{S} = \alpha L M^{-1} L^T + M$

From now on the index k is omitted.

Approximating the AS Schur complement

The active-set Schur complement of J^{aug}

$$S = \frac{1}{\alpha}\begin{bmatrix} I & -\bar{M}\Pi_{\mathcal{A}}M^{-1}P_{\mathcal{A}}^T \\ 0 & I \end{bmatrix}\begin{bmatrix} \mathbb{S} & 0 \\ 0 & P_{\mathcal{A}}MP_{\mathcal{A}}^T \end{bmatrix}\begin{bmatrix} I & 0 \\ -P_{\mathcal{A}}M^{-1}\Pi_{\mathcal{A}}\bar{M}^T & I \end{bmatrix},$$

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The active-set Schur complement approximation

$$\widehat{S} = \frac{1}{\alpha}\begin{bmatrix} I & -\bar{M}\Pi_{\mathcal{A}}M^{-1}P_{\mathcal{A}}^T \\ 0 & I \end{bmatrix}\begin{bmatrix} \widehat{\mathbb{S}} & 0 \\ 0 & P_{\mathcal{A}}MP_{\mathcal{A}}^T \end{bmatrix}\begin{bmatrix} I & 0 \\ -P_{\mathcal{A}}M^{-1}\Pi_{\mathcal{A}}\bar{M}^T & I \end{bmatrix} \approx S$$

with

$$\widehat{\mathbb{S}} := (\sqrt{\alpha}L + \bar{M}(I - \Pi_{\mathcal{A}}))M^{-1}(\sqrt{\alpha}L + \bar{M}(I - \Pi_{\mathcal{A}}))^T \quad \approx \quad \mathbb{S}$$

- Proposed for bound-constrained optimal control problems and $\bar{M} = M$ in [P., Simoncini, Tani, SISC 2015] .

Active-set block-diagonal and indefinite preconditioners

4×4 formulation

$$\mathcal{P}^{BD}_{aug} = \begin{bmatrix} J_{11} & 0 \\ 0 & \widehat{S} \end{bmatrix},$$

and

$$\mathcal{P}^{IP}_{aug} = \begin{bmatrix} I & 0 \\ J_{12}J_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} J_{11} & 0 \\ 0 & -\widehat{S} \end{bmatrix} \begin{bmatrix} I & J_{11}^{-1}J_{12}^T \\ 0 & I \end{bmatrix},$$

where

$$J_{11} = \mathrm{blkdiag}(M, \alpha M) \text{ and } J_{12} = \begin{bmatrix} L & -\bar{M} \\ 0 & P_{\mathcal{A}}M \end{bmatrix}.$$

and $\widehat{S} \approx S = J_{12}J_{11}^{-1}J_{12}^T$, i.e. the Schur complement of J^{aug}.

2×2 formulation

$$\mathcal{P}^{BD}_{red} = \begin{bmatrix} M & 0 \\ 0 & \frac{1}{\alpha}\widehat{\mathbb{S}} \end{bmatrix} \qquad \mathcal{P}^{IP}_{red} = \begin{bmatrix} I & 0 \\ LM^{-1} & I \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & -\frac{1}{\alpha}\widehat{\mathbb{S}} \end{bmatrix} \begin{bmatrix} I & M^{-1}L^T \\ 0 & I \end{bmatrix}$$

and $\frac{1}{\alpha}\widehat{\mathbb{S}} \approx \frac{1}{\alpha}\mathbb{S}$, i.e. the Schur complement of J^{red}.

Active-set block-diagonal and indefinite preconditioners

4×4 formulation

$$\mathcal{P}_{aug}^{BD} = \begin{bmatrix} J_{11} & 0 \\ 0 & \widehat{S} \end{bmatrix},$$

and

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where

$$J_{11} = \mathrm{blkdiag}(M, \alpha M) \;\text{ and }\; J_{12} = \begin{bmatrix} L & -\bar{M} \\ 0 & P_{\mathcal{A}} M \end{bmatrix}.$$

and $\widehat{S} \approx S = J_{12} J_{11}^{-1} J_{12}^T$, i.e. the Schur complement of J^{aug}.

2×2 formulation

$$\mathcal{P}_{red}^{BD} = \begin{bmatrix} M & 0 \\ 0 & \frac{1}{\alpha}\widehat{\mathbb{S}} \end{bmatrix} \qquad \mathcal{P}_{red}^{IP} = \begin{bmatrix} I & 0 \\ LM^{-1} & I \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & -\frac{1}{\alpha}\widehat{\mathbb{S}} \end{bmatrix} \begin{bmatrix} I & M^{-1}L^T \\ 0 & I \end{bmatrix}$$

and $\frac{1}{\alpha}\widehat{\mathbb{S}} \approx \frac{1}{\alpha}\mathbb{S}$, i.e. the Schur complement of J^{red}.

Properties of the active-set Schur complement approximation

- $\widehat{\mathbb{S}} = \mathbb{S} + \sqrt{\alpha}(LM^{-1}\,\Pi_{\mathcal{I}}\,\bar{M}^T + \bar{M}\,\Pi_{\mathcal{I}}\,M^{-1}L^T)$.
- If $\mathcal{A} = \{1,\ldots,n\}$ $(\mathcal{I} = \emptyset)$ $\quad\Rightarrow\quad \widehat{\mathbb{S}} = \mathbb{S} \quad\Rightarrow\quad S = \widehat{S}$
- If $\mathcal{A} = \emptyset$ (or unconstrained): $\lambda(\widehat{\mathbb{S}}^{-1}\mathbb{S}) \in \left[\frac{1}{2}, 1\right]$ (cf. [Pearson, Wathen,2012-2014])

Spectral properties of the approximation $\widehat{\mathbb{S}}$

- $\lambda \in \lambda(\widehat{\mathbb{S}}^{-1}\mathbb{S})$ satisfies

$$\frac{1}{2} \leq \lambda \leq \zeta^2 + (1+\zeta)^2$$

with

$$\zeta = \|M^{\frac{1}{2}}\left(\sqrt{\alpha}L + \bar{M}\Pi_{\mathcal{I}}\right)^{-1}\sqrt{\alpha}LM^{-\frac{1}{2}}\|.$$

$\star$ Moreover, if $L\bar{M}^T + \bar{M}L^T \succ 0$, then for $\alpha \to 0$, ζ is bounded by a constant independent of α.

[P., Simoncini, Tani, 2015, P., Simoncini, Stoll, 2017]

Spectral analysis of the preconditioner

Theorem (indefinite case)

Let λ be an eigenvalue of $(\mathcal{P}^{IP})^{-1}J$. Then

$$\lambda \in \{1\} \cup \left[\frac{1}{2}, \zeta^2 + (1+\zeta)^2\right].$$

- 4×4: there are at least $3n + |\mathcal{A}| - 2|\mathcal{I}|$ eigenvalues equal to 1.
- 2×2: there are at least $2n - 2|\mathcal{I}|$ eigenvalues equal to 1.

Theorem (block-diagonal case)

Let $\xi = \zeta^2 + (1+\zeta)^2$. Then $\lambda((\mathcal{P}^{BD})^{-1}J) \subset I^- \cup I^+$ with

- 4×4: $I^- = \left[\frac{1-\sqrt{1+4\xi}}{2}, \frac{1-\sqrt{3}}{2}\right]$ and $I^+ = \left[1, \frac{1+\sqrt{1+4\xi}}{2}\right]$.
- 2×2: $I^- = \left[\frac{-\xi+1-\sqrt{(\xi+1)^2+4\zeta^2}}{2}, \frac{1-\sqrt{3}}{2}\right]$ and $I^+ = \left[1, \frac{1+\sqrt{1+4\zeta^2}}{2}\right]$.

using [Perugia, Simoncini, 2000]

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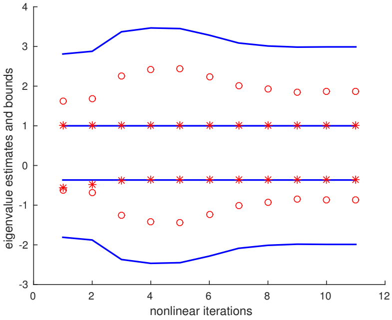
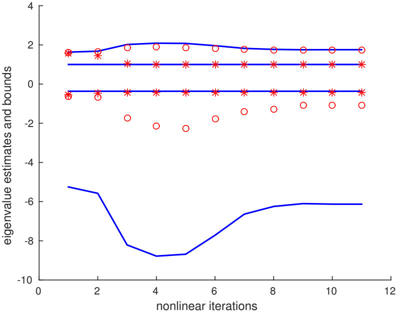
using [Perugia, Simoncini, 2000]

Bounds for the block-diagonal case



(a) $\lambda((\mathcal{P}_{aug}^{BD})^{-1} J^{aug})$



(b) $\lambda((\mathcal{P}_{red}^{BD})^{-1} J^{red})$

Figure: Eigenvalues of the preconditioned matrices and bounds provided in the Theorem versus number of nonlinear iterations (CD problem, $\alpha = 10^{-4}, \beta = 10^{-4}$).

Global Inexact Semismooth Newton's method

Algorithm [Martinez, Qi COAP 1985]

1. Starting x_0, parameters $\sigma \in (0,1]$, $\gamma \in (0,1)$.
2. Until convergence do
   - 2.1 Choose $\eta_k \in (0,1)$ (forcing term)
   - 2.2 Solve
   $$\Theta'(x_k)\Delta x = -\Theta(x_k) + r_k,$$
   with
   $$\|r_k\|_2 \leq \eta_k \|\Theta(x_k)\|_2 \qquad \text{(inexact condition)}$$
   - 2.3 $\rho \leftarrow 1$;
   - 2.4 While $\|\Theta(x_k + \rho\Delta x)\|_2^2 - \|\Theta(x_k)\|_2^2 > -2\sigma\gamma\rho\|\Theta(x_k)\|_2^2$
   $\rho \leftarrow \rho/2$.
3. $x_{k+1} \leftarrow x_k + \rho\Delta x$, $k \leftarrow k+1$.

- Fast local convergence of SSN [Stadler '09] $\Rightarrow$ globalization strategy is needed [Martinez and Qi '95].
- Inexactness adapted to the 4×4 and 2×2 case so that Newton's steps computed with the same accuracy.

Experiments: implementation issues

- Compared Matlab implementations:

SSN-GMRES-IP	Global SSN with linear solver GMRES + $\mathcal{P}^{IP}_{aug}/\mathcal{P}^{IP}_{red}$
SSN-MINRES-BD	Global SSN with linear solver MINRES + $\mathcal{P}^{BD}_{aug}/\mathcal{P}^{BD}_{red}$

- Preconditioners via AMG (HSL-MI20)
  - Application of $\widehat{\mathbb{S}}$: solving with $(\sqrt{\alpha}L + \bar{M}(I - \Pi_{\mathcal{A}_k}))$ (and its transpose)
- FEM matrices from the open source FE library deal.II
- Stopping test for the outer iteration:

$$\|\Theta(x_k)\| \leq 10^{-6}$$

- Relative residual for inner stopping criterion:
  - Exact case:

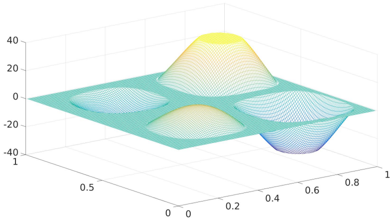
$$\eta_k = 10^{-10}$$

  - Inexact case:

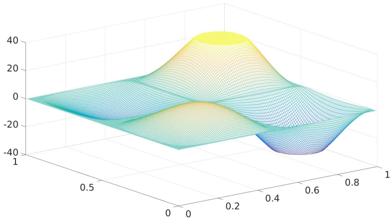
$$\eta_k = \min\left\{\chi\left(\|\Theta_{k+1}\|/\|\Theta_k\|\right)^2, \eta_{max}\right\}$$

with $\chi = 0.9, \eta_0 = \eta_{max} = \{10^{-1}, 10^{-2}, 10^{-4}, 10^{-10}\}$, [Eisenstat, Walker, SISC 1996].

Poisson state equation with FDs ($\mathcal{L} = -\Delta, M = \bar{M} = I$)



(a) $\beta = 10^{-3}$



(b) $\beta = 10^{-4}$

Figure: Control for two different values of the parameter β. Desired State $y_d = \sin(2\pi x)\sin(2\pi y)\exp(2x)/6$, $a = -30$, $b = 30$.

- 2D: $n = 2^{2\ell}$ with $\ell = 7, 8, 9 \Rightarrow n = 16384, 65536, 262144$;
- 3D: $n = 2^{3\ell}$ with $\ell = 4, 5, 6 \Rightarrow n = 4096, 32768, 262144$.

Poisson state equation with FDs (2D case, $\beta = 10^{-4}$)

			SSN-GMRES-IP			SSN-MINRES-BD					
4x4	ℓ	$\log_{10}\alpha$	LI	CPU	TCPU	LI	CPU	TCPU	%u=0	NLI	BT
	7	-2	11	2.0	4.1	25	6.0	12.1	3.5	2	0
		-4	16	2.9	14.5	34	7.8	39.2	8.6	5	0
		-6	27	6.2	69.1	61	14.0	154.9	35.6	11	9
	9	-2	12	30.9	61.8	27	103.0	206.0	4.7	2	0
		-4	17	28.6	**143.2**	37	93.9	**469.6**	9.7	5	0
		-6	28	60.2	**782.5**	68	190.8	**2481.5**	36.4	13	11
2x2											
	7	-2	11	2.0	4.0	24	4.8	9.6	3.5	2	0
		-4	16	2.8	14.1	35	6.2	31.3	8.6	5	0
		-6	26	5.3	58.9	65	11.2	124.0	35.6	11	9
	9	-2	12	26.0	52.0	25	95.2	190.5	4.7	2	0
		-4	17	25.9	**129.8**	38	74.4	**372.4**	9.7	5	0
		-6	28	50.6	**657.7**	71	154.4	**2007.6**	36.4	13	11

- LI/NLI: average number of Linear/number of NonLinear iterations
- CPU/TCPU: average/total CPU time
- BT: total number of Back-Tracking steps in the line-search strategy

Poisson state equation with FDs (3D case, $\beta = 10^{-4}$)

			SSN-GMRES-IP			SSN-MINRES-BD					
4x4	ℓ	$\log_{10}\alpha$	LI	CPU	TCPU	LI	CPU	TCPU	%u=0	NLI	BT
	4	-2	10	0.5	1.0	21	1.9	3.8	7.4	2	0
		-4	15	0.5	2.2	31	2.3	9.1	7.6	4	0
		-6	24	0.9	7.7	51	3.8	30.6	38.6	8	5
	6	-2	11	19.1	38.2	23	82.2	164.4	11.8	2	0
		-4	16	27.5	**110.1**	33	121.5	**486.2**	17.3	4	0
		-6	31	71.3	**570.5**	69	261.4	**2091.9**	46.9	8	2
2x2											
	4	-2	10	0.3	0.7	20	1.2	2.4	7.4	2	0
		-4	14	0.5	2.1	32	1.8	7.4	7.6	4	0
		-6	23	0.8	6.7	53	2.8	22.7	38.6	8	5
	6	-2	10	17.3	34.7	22	61.4	122.9	11.8	2	0
		-4	16	26.1	**104.5**	34	92.8	**371.5**	17.3	4	0
		-6	31	59.4	**475.8**	74	210.0	**1680.3**	46.9	8	2

- LI/NLI: average number of Linear/number of NonLinear iterations
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Convection-Diffusion state equation with FE (1)

$$\mathcal{L} = -\epsilon\Delta + w \cdot \nabla$$

- ⋆ $w = (2y(1-x^2), -2x(1-y^2)), a = -20, b = 20;$
- ⋆ $\bar{M}$ and L SUPG discretization (unsym).

$$n = 4225, \qquad \beta = 10^{-2}$$

SSN-GMRES-IP (2×2)

	$\epsilon = 1$		$\epsilon = 0.5$		$\epsilon = 0.1$	
$\log_{10}\alpha$	LI (NLI)	BT	LI (NLI)	BT	LI (NLI)	BT
-1	10(2)	0	13(4)	2	11(2)	0
-2	13(5)	0	17(13)	17	14(6)	3
-3	15(7)	4	23(19)	46	17(9)	8
-4	19(10)	11	27(23)	45	22(14)	29
-5	22(43)	183	35(39)	127	24(36)	156

- LI: average number of Linear Iterations
- NLI: total number of NonLinear Iterations
- BT: total number of Back-Tracking steps in the line-search strategy

Convection-Diffusion state equation with FE (2)

SSN-GMRES-IP

$n = 16641, \beta = 10^{-4}, \epsilon = 0.1$

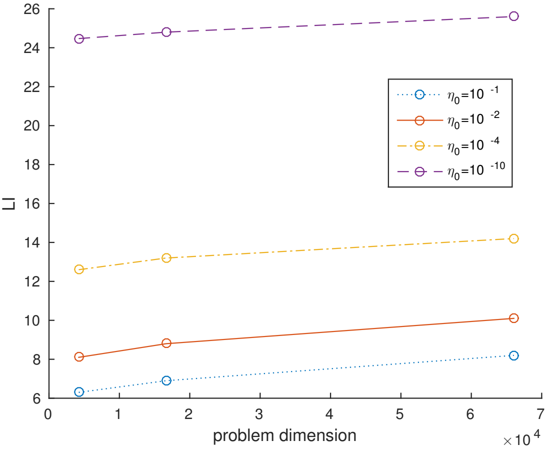
	4×4		2×2	
$\log_{10}\alpha$	LI (NLI)	TCPU	LI (NLI)	TCPU
-1	13(4)	21.5	13(4)	20.2
-3	23(15)	137.3	23(15)	123.6
-5	33(56)	970.6	31(56)	864.3

$n = 66049, \beta = 10^{-4}, \epsilon = 0.1$

	4×4		2×2	
$\log_{10}\alpha$	LI (NLI)	TCPU	LI (NLI)	TCPU
-1	13(4)	75.9	13(4)	70.8
-3	23(15)	**502.4**	23(15)	**406.0**
-5	31(95)	**5056.8**	30(95)	**4004.2**

- LS needed for convergence in 80% of the runs.
- LS slow for small α.

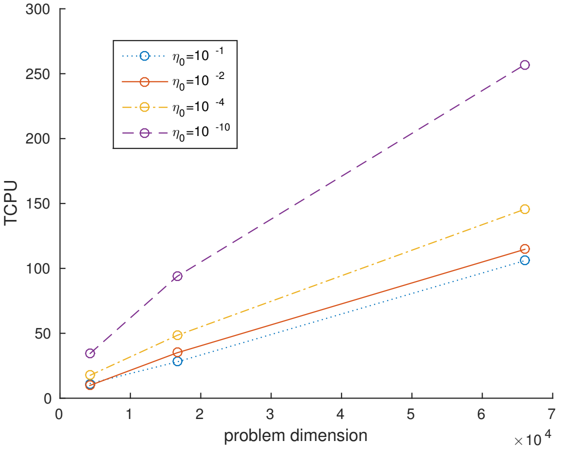
Inexact solution of the Newton's equation (2)



(a) LI versus n

Figure: Inexact Newton's method for different η_0 (CD, $\alpha = 10^{-3}, \beta = 10^{-2}, \epsilon = 1/10$).

Inexact solution of the Newton's equation (1)



(a) TCPU versus n

Figure: Inexact Newton's method for different η_0 (CD, $\alpha = 10^{-3}, \beta = 10^{-2}, \epsilon = 1/10$).

Conclusions

- General SemiSmooth Newton's algorithm for control problems with bound and sparsity constraints;
- Enhancement of the linear algebra phase by:
  - reduced 2×2 formulation of the Newton's equation;
  - preconditioners based on the active-set Schur complement approximation;
  - inexact framework;
- Spectral analysis of the Active-Set Schur complement approximation for general state equation with unsymmetric mass matrix (e.g. CD).
- Numerical experiments show good performance wrto different parameters

Current work

- use alternative 2nd-order methods to accelerate convergence for limit values of the parameters

Porcelli, Simoncini, Stoll, *Preconditioning PDE-constrained optimization with L^1-sparsity and control constraints*, Computers and Mathematics with Applications, 74:5 (2017), pp. 1059-1075.

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Thanks for your attention