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FORWARD-BACKWARD PARABOLIC EQUATIONS

Ph.D. Thesis in Mathematics

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Rome, October 2013

Contents

Introduction

In this thesis we address initial-boundary value problems (IBVPs) for the *forward-backward* quasi-linear parabolic equation

$$u_t = \Delta\varphi(u)\,, \tag{1}$$

where the nonlinear function φ is assumed to be *nonmonotonic*. It is well known that such problems are well-posed if the function φ is monotonically increasing (*e.g.*, see [LSU]). However, if φ is a nonmonotonic function, equation (1) is well-posed forward in time in regions where $\varphi'(u) > 0$, whereas it is ill-posed where $\varphi'(u) < 0$ (such regions are commonly referred to as *stable*, respectively *unstable phases*). We also study initial-boundary value problems for the related forward-backward equation

$$u_t = \nabla \cdot [\varphi(\nabla u)]\,. \tag{2}$$

Mainly two classes of nonlinearities φ have been considered in the literature:
(i) *cubic-like* φ, namely

$$(AS_1) \qquad \begin{cases} \varphi \to \infty \text{ as } s \to \infty, \\ \varphi'(s) > 0 \text{ if } s < b \text{ and } s > c, \\ \varphi'(s) < 0 \text{ if } b < s < c, \\ \varphi''(b) \neq 0, \varphi''(c) \neq 0, \\ \varphi(c) < \varphi(b)\,; \end{cases}$$

(ii) φ *of Perona-Malik type*, namely

$$(AS_2) \qquad \begin{cases} \varphi(s) > 0 \text{ if } s > 0,\ \varphi(s) = -\varphi(-s), \\ \varphi(s) \to 0 \text{ as } s \to \infty, \\ \varphi'(s) > 0 \text{ if } 0 \leq s < 1,\ \varphi'(s) < 0 \text{ if } s > 1\,. \end{cases}$$

Forward-backward problems with nonlinearities φ of either type arise in many applications of physical and biological interest, as discussed in the following section.

Motivations

It is well known that equation (1) under assumption (AS_1) arises in the theory of phase transitions, the unknown u representing the *phase field* [BS]. In this case the half-lines $(-\infty, b)$ and (c, ∞) correspond to stable phases and the interval (b, c) to an unstable phase, and equation (1) describes the dynamics of transition between different phases.

Equation (1) with assumption (AS_2) arises, in particular,
(i) when constructing the master equation of continuous-time and discrete-space random walk to describe a continuum model for movements of biological organisms [HPO];
(ii) in the diffusion approximation to a discrete model for aggregating populations [Pa].

In case (i) a typical choice of φ is

$$\varphi(u) = \frac{u}{k+u^2}\,, \qquad \text{or} \qquad \varphi(u) = u \exp(-u) \qquad (k > 0)\,,$$

whereas in case (ii) a typical choice is

$$\varphi(u) = u \exp(-u)\,.$$

A major motivation for equation (2) comes from the context of image processing, where the celebrated Perona-Malik edge enhancement model was introduced in 1990 (see [PM]). The *anisotropic diffusion* or *Perona-Malik diffusion* is a technique aiming at reducing image noise without removing some important contents of image, typically edges or lines. This leads to the *Perona-Malik equation*

$$w_t = \nabla \cdot [\sigma(|\nabla w|)\nabla w]\,,$$

where typically

$$\sigma(s) = \frac{1}{1+s^2} \qquad \text{or} \qquad \sigma(s) = \exp(-s)\,.$$

Observe that in the one-dimensional case the above equation reads

$$w_t = [\varphi(w_x)]_x\,. \tag{3}$$

Formally, differentiating (3) with respect to x and setting $u := w_x$ we obtain equation (1) in one space dimension, namely

$$u_t = [\varphi(u)]_{xx}\,, \tag{4}$$

where $\varphi(s) = s\sigma(s)$ satisfies assumption (AS_2).

Equation (3) independently arises as a mathematical model for heat and mass transfer in a stably stratified turbulent shear flow [BBDPU], or as a mathematical model for the formation of layers of constant temperature or salinity in the ocean [BBDU]. In fact, the temperature $w \geq 0$ satisfies the equation

$$w_t = [kw_x]_x \,,$$

where the effective temperature (or mass diffusivity) k only depends on the temperature gradient and decreases very quickly. Hence a reasonable analytical form is again

$$k = \sigma(w_x) := \frac{A}{B + w_x^2} \qquad (A, B > 0)\,.$$

Finally, observe that equation (3) can be regarded as the formal L^2-gradient system associated with a *nonconvex* energy density ψ in one space dimension (in this case $\varphi = \psi'$); for instance, $\psi(s) = log(1 + s^2)$ holds for the Perona-Malik equation, or the double well potential $\psi(s) = (1 - s^2)^2$ for a cubic nonlinearity. Thus the dynamics described by equation (3) in one space dimension is relevant to various settings, where nonconvex functionals arise (*e.g.*, see [BFG, Mü] for motivations in nonlinear elasticity).

How to regularize?

It has been already observed that the lack of forward parabolicity in equation (1) (under either assumption (AS_1)-(AS_2)) gives rise to ill-posed problems. Therefore, both development of singularities and lack of regularity can be expected whenever the initial data u_0 take values in the unstable phase.

In fact, the existence of solutions to the Neumann IBVP for equation (3) has been proven if the derivative of the initial data function w_0 takes values in the stable phase [KK], whereas for large values of $|w_0'|$ no global C^1-solution exists ([G, K]). This shows that even local existence of solutions (in some suitable functional space) to the initial-boundary value problem for equations (1) or (2) is a nontrivial problem (in this connection, see the numerical experiments in [BFG, FGP, SSW, BNPT]; let us also mention that the existence of solutions to the Neumann IBVP for equation (1) has been proven, if u_0 takes values in the stable phase [HPO]).

Concerning uniqueness of the forward-backward problems, the situation is even more complicated. In [Hö] the Neumann IBVP for (2) is shown to have infinitely many weak L^2-solutions, if φ is a nonmonotonic piecewise linear function satisfying the condition $s\varphi(s) \geq Cs^2$ for some constant $C > 0$. In [Zh], the existence of infinitely many weak $W^{1,\infty}$-solutions for equation (3) under assumption (AS_2) was proven.

Since IBVPs for equations (1), (2) are in general ill-posed, it is natural to investigate suitable regularizations (often suggested by modelling considerations) which make them well-posed. Therefore, a general strategy is first to study the regularized problem, then to study its *vanishing viscosity limit* as some "small" regularizing parameter $\epsilon > 0$ tends to zero. The underlying, widely accepted idea is that limiting points of the family of solutions of the regularized problem give rise to suitably defined solutions of the original ill-posed problem. Formally, this amounts to add some higher order terms to equation (1) or (2), thus obtaining

$$u_t = \Delta\varphi(u) + \epsilon\mathcal{F}(u) ,$$

respectively

$$u_t = \nabla \cdot [\varphi(\nabla u)] + \epsilon\mathcal{F}(u) ,$$

where $\mathcal{F}$ is a physically meaningful higher order differential operator.

Mainly two classes of additional terms $\epsilon\mathcal{F}(u)$ have been used in the literature ($\epsilon > 0$ being a small parameter):

(i) $\epsilon\Delta[\psi(u)]_t$, with $\psi' > 0$, leading to third-order *pseudo-parabolic equations* [BBDU, BST1, BST2, EP, MTT, NP, Pl1, Pl2, Pl3, S, STe, ST1, ST2, ST3]. If $\psi(u) = u$, this regularization is called *Sobolev regularization*:

$$u_t = \Delta[\varphi(u) + \epsilon u_t] \, ; \tag{5}$$

(ii) $-\epsilon\Delta^2 u$, leading to fourth-order *Cahn-Hilliard type equations* (see [BFG, Sl, Pl4] and references therein):

$$u_t = \Delta[\varphi(u) - \epsilon\Delta u] \, . \tag{6}$$

Equation (6) was introduced by Cahn and Hilliard [C, CH] with φ satisfying assumption (AS_1) for continuum models of spinoidal decomposition; it also arises when describing isothermal phase separation of binary mixture (see [NP]). As shown in [EZ], the Neumann IBVP for equation (6) under assumption (AS_1) has a unique global solution (see the following section sub (γ) and Chapter 3 for analogous results under assumption (AS_2)). On the other hand, both Dirichlet and Neumann IBVPs for

$$u_t = \nabla \cdot [\varphi(\nabla u)] - \epsilon\Delta^2 u \, , \tag{7}$$

with φ satisfying suitable growth assumptions, were studied by Slemrod [Sl] (see [BFG] in the case of assumption (AS_2)). By the theory of *Young measures* (*e.g.*, see [V1, V2]) it was proven that some subsequence of the family of approximating solutions to the above regularized problems converges in a suitable topology to a *Young measure-valued solution* of the corresponding ill-posed IBVP for equation (2). Under different growth assumptions on φ, the properties of the limiting Young measure have been investigated in [Pl4].

The above *Sobolev equation* (5) was widely investigated in the literature; the term $\epsilon\Delta u_t$ can be interpreted by taking viscous relaxation effects into account (see [NP, BFJ]). The Neumann IBVP for equation (5) was studied in [NP] under assumption (AS_1) and in [Pa] under assumption (AS_2). In both cases the global existence and uniqueness of a solution u^ϵ are proven to hold in $L^\infty(Q_T)$ ($Q_T := \Omega \times (0,T)$) for any $\epsilon > 0$. Moreover, these solutions satisfy a class of *viscous entropy inequalities*, this term being suggested by a formal analogy with the entropy inequalities for *viscous conservation laws* (see [E2, MTT1, Se]). It is well-known that such entropy inequalities carry over to weak solutions of the Cauchy problem for the *first order hyperbolic conservation laws* in the vanishing viscosity limit $\epsilon \to 0$ (*e.g.*, see [Se]). Therefore, it is natural to wonder whether in the limit $\epsilon \to 0$ it is possible to prove existence and uniqueness of suitably defined *weak entropy solutions* for the original equation (1) or (2).

An exhaustive answer to the above question was given by Plotnikov (see [Pl1]) for the case of a cubic-like φ. It turns out that the family $\{u^\epsilon\}$ of solutions to the Neumann IBVP for equation (5) is uniformly bounded in the L^∞-norm, and the limiting points (u, v) of the families $\{u^\epsilon\}$, $\{\varphi(u^\epsilon)\}$ satisfy in the weak sense the limiting equation

$$u_t = \Delta v \quad \text{in } \mathcal{D}'(Q_T) \tag{8}$$

with initial datum $u_0 \in L^\infty(\Omega)$ and Neumann boundary conditions. If we had $v = \varphi(u)$, equation (8) would give a weak solution of the Neumann IBVP for equation (1). However, no such conclusion can be drawn, due to the nonmonotonic character of φ.

In this connection, Plotnikov showed that the couple (u, v) is a measure-valued solution in the sense of Young measures to equation (1) (see [Pl1]). With respect to the results in [Sl] for the Cahn-Hilliard regularization, a major issue here is the study of the family $\{\tau^\epsilon\}$ of Young measures associated to the approximate solution u^ϵ, and the *characterization* of the *disintegration* $\tau_{(x,t)}$ of Young measure τ obtained as the *narrow limit* of such measures were given (see [E1, GMS, V1, V2]). More precisely, it is proven that the disintegration $\tau_{(x,t)}$ is an atomic measure given by the *superposition of three Dirac masses* concentrated on the branches s_0, s_1, s_2 of the equation $v = \varphi(u)$. Hence the solution u has the following representation:

$$u = \sum_{i=0}^{2} \lambda_i s_i(v)\,, \tag{9}$$

for some positive coefficients $\lambda_i \in L^\infty(Q_T)$ such that $\sum\limits_{i=0}^{2} \lambda_i = 1$ (see [E2, GMS, V1, V2]). Equality (9) can be explained by saying that the function u takes the fraction λ_i of its value at (x,t) on the branch $s_i(v)$ of the graph of φ. Then the coefficients λ_i can be regarded as *phase fractions*, and u itself as a *superposition* of *different phases.*

In addition, the couple (u, v) satisfies a class of suitable limiting entropy inequalities. Therefore, the above limiting is said to be a *weak entropy Young measure-valued solution* of the IBVP associated to equation (1). Let us mention that the long-time behaviour of weak entropy measure-valued solutions (u, v) was addressed in [ST1].

Using similar ideas, the IBVP for equation (1) under assumption (AS_2) was investigated in [S]. Let u^ϵ be the solution of the Neumann IBVP for the regularized equation (5) in any cylinder Q_T with $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$. The main difference with respect to [Pl1] is that only a uniform bound of the family $\{u^\epsilon\}$ in the L^1-norm, instead of the L^∞-norm as in [Pl1], can be proven in this case (by taking advantage of positivity and conservation of mass). Accordingly, the limiting point $\bar{u}$ of some subsequence $\{u^{\epsilon_k}\}$ of the family $\{u^\epsilon\}$ belongs the space of *positive Radon measures* $\mathcal{M}^+(Q_T)$. By the idea of *biting convergence* and the general properties of the narrow convergence for Young measures (*e.g.*, see [GMS, V1, V2]),

(i) the representation

$$\bar{u} = u + \mu$$

is proven, where $\mu \in \mathcal{M}^+(Q_T)$ and $u \in L^1(Q_T)$, $u \geq 0$;

(ii) it is shown that u is a superposition of the stable branch s_1 and the unstable branch s_2 associated to the graph of φ - namely, the following analogue of equality (9) holds:

$$u = \begin{cases} \lambda s_1(v) + (1 - \lambda) s_2(v) & \text{if } v > 0 \\ 0 & \text{if } v = 0 \end{cases}$$

for some $\lambda \in L^\infty(Q_T)$ such that $0 \leq \lambda \leq 1$; here $v \geq 0$ is the limit of the family $\{\varphi(u^{\epsilon_k})\}$ in the weak* topology of $L^\infty(Q_T)$. Accordingly, in this case the analogue of the limiting equation (8) as $\epsilon_k \to 0$ is proven to be

$$(u + \mu)_t = \Delta v \quad \text{in } \mathcal{D}'(Q_T)\,.$$

Finally, let us mention the *pseudo-parabolic regularization* $\epsilon\mathcal{F}(u) = \epsilon\Delta[\psi(u)]_t$. In [BBDU] the regularized equation

$$z_t = [\varphi(z_x)]_x + \epsilon[\psi(z_x)]_{xt}\,, \tag{10}$$

with φ satisfying assumption (AS_2) and $\psi(s) \to \gamma$ as $s \to \infty$, has been proposed by taking time delay effects into account. The well-posedness of the Neumann IBVP in any cylinder Q_T for the above *degenerate pseudo-parabolic approximation* of equation (3) has been studied in [BBDU]. The main feature of the solution $z^\epsilon \in BV(Q_T)$ is the possibility of *formation of discontinuities in finite time*, even for smooth initial data. Moreover, at any fixed point x_0 the discontinuity jump $z^\epsilon(x_0^+) - z^\epsilon(x_0^-)$ is nondecreasing in time.

If $N = 1$, by the usual transformation $u := z_x$ (see equations (3)-(4)), formally differentiating equation (10) with respect to x we obtain the degenerate pseudo-parabolic regularization of equation (4):

$$u_t = [\varphi(u)]_{xx} + \epsilon[\psi(u)]_{xxt}\,. \tag{11}$$

Existence and uniqueness of a suitably defined positive solution of the Dirichlet IBVP for equation (11) has been proven in a space of Radon measures under assumption (AS_2) (see [ST2]), whereas the vanishing viscosity limit as $\epsilon \to 0$ of such a solution has been addressed in [ST3].

Outline of results

Within the above general framework, the present thesis addresses three main points, as outlined below. Each point corresponds to a submitted paper [BuST1, BuST2, BuTo].

(α) In Chapter 1 we address the initial-boundary value problem

$$(P) \qquad \begin{cases} u_t = \nabla \cdot [\varphi(\nabla u)] & \text{in } Q_T := \Omega \times (0,T) \\ u = 0 & \text{in } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\}\,. \end{cases}$$

Here $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ if $N \geq 2$, $T \in (0, \infty]$ and the dot " $\cdot$ " denotes the scalar product in $\mathbb{R}^N$.

If $N = 1$, about the function $\varphi : \mathbb{R} \to \mathbb{R}$ which appears in problem (P) we shall assume the following:

$$(A_1) \qquad \begin{cases} \text{for any } R > 0 \text{ there exists } L_R > 0 \text{ such that} \\ |\varphi(\xi_1) - \varphi(\xi_2)| \leq L_R\,|\xi_1 - \xi_2| \text{ for any } \xi_1, \xi_2 \in B_R\,, \end{cases}$$

where B_R denotes the ball centered at 0 of radius R in $\mathbb{R}^N$;

$$(A_2) \qquad \begin{cases} \text{there exist } \xi_0 > 0,\ p \in (1,\infty) \text{ and } C_1 > 0 \text{ such that} \\ C_1|\xi|^{p-1} \leq |\varphi(\xi)| \text{ for any } |\xi| > \xi_0\,; \end{cases}$$

$$(A_3) \qquad \varphi(\xi)\xi \geq 0 \text{ for any } \xi \in \mathbb{R}\,.$$

If $N \geq 2$, concerning the map $\varphi : \mathbb{R}^N \to \mathbb{R}^N$, $\varphi \equiv (\varphi_1, \ldots, \varphi_N)$, the following assumptions will be made:

$$(H_1) \qquad \begin{cases} \text{there exists } L > 0 \text{ such that} \\ |\varphi(\xi_1) - \varphi(\xi_2)| \leq L\,|\xi_1 - \xi_2| \text{ for any } \xi_1, \xi_2 \in \mathbb{R}^N\,; \end{cases}$$

$$(H_2) \qquad \begin{cases} \text{there exist } \xi_0 > 0, \ \ p \in (1,2] \text{ and } C_0 > 0 \text{ such that} \\ |\varphi(\xi)| \leq C_0 \left(1 + |\xi|^{p-1}\right) \ \text{ for any } \ |\xi| > \xi_0 \,; \end{cases}$$

$$(H_3) \qquad \text{there exists } \Phi \in C^1(\mathbb{R}^N) \text{ such that } \varphi = \nabla\Phi \,;$$

$$(H_4) \qquad \begin{cases} \text{there exist } \xi_0 > 0, \ \ q \in (1,2] \text{ and } C_1, \, C_2 > 0 \text{ such that} \\ C_1|\xi|^q \leq \Phi(\xi) \ \text{ for any } \ |\xi| > \xi_0 \,; \end{cases}$$

$$(H_5) \qquad \varphi(\xi) \cdot \xi \geq 0 \ \text{ for any } \ \xi \in \mathbb{R}^N \,.$$

Observe that for $N \geq 2$ we assume global Lipschitz continuity of φ, instead of local Lipschitz continuity as in the case $N = 1$; this implies the stronger restriction $p \in (1,2]$ (instead of $p \in (1,\infty)$ as for $N = 1$) on the allowed values of p. Observe also that by (H_2)-(H_4) there holds

$$C_1|\xi|^q \leq \Phi(\xi) \leq C_3|\xi|^p \ \text{ for any } \ |\xi| > \xi_0 \,,$$

for some constant $C_3 > 0$. This implies the compatibility condition $q \leq p$ (the choice $q = p$ is always made). Concerning the initial data function u_0,

- if $N = 1$, we assume $u_0 \in W_0^{1,\infty}(\Omega)$ (by abuse of notation, we set hereafter $W_0^{1,\infty}(\Omega) := C_0(\bar{\Omega}) \cap W^{1,\infty}(\Omega)$);
- if $N \geq 2$, we assume $u_0 \in W_0^{1,p}(\Omega)$ $(p \in (1,2])$.

We study problem (P) by using the Sobolev regularization. Namely, we consider for any $\epsilon > 0$ the initial-boundary value problem

$$(P_\epsilon) \qquad \begin{cases} u_t = \nabla \cdot [\varphi(\nabla u)] + \epsilon \Delta u_t & \text{in } Q_T \\ u = 0 & \text{in } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\} \,. \end{cases}$$

The well-posedness of problem (P_ϵ) is established by Theorem 1.3.1. Then we study the "vanishing viscosity limit" as $\epsilon \to 0$ of the family $\{u_\epsilon\}$ of solutions of the approximating problems (P_ϵ). In doing so, we make use of the following estimates (see Propositions 1.3.2−1.3.3):

$$\|u_\epsilon\|_{L^\infty(\mathbb{R}_+;W_0^{1,\infty}(\Omega))} + \|u_{\epsilon t}\|_{L^2(Q_\infty)} + \sqrt{\epsilon}\,\|u_{\epsilon xt}\|_{L^2(Q_\infty)} \leq C\,\|u_0\|_{W_0^{1,\infty}(\Omega)}$$

if $N = 1$, and

$$\|u_\epsilon\|_{L^\infty(\mathbb{R}_+;W_0^{1,p}(\Omega))} + \|u_{\epsilon t}\|_{L^2(Q_\infty)} + \sqrt{\epsilon}\,\|\nabla u_{\epsilon t}\|_{L^2(Q_\infty)} \leq C\,\|u_0\|_{W_0^{1,p}(\Omega)}$$

if $N \geq 2$ $(p \in (1,2])$. The existence of a Young measure solution of the original problem (P) (see Definition 1.3.1) is stated in Theorem 1.3.6. Since the flux φ need not satisfy any monotonicity condition, this notion of solution is appropriate for problem (P). In fact, solutions of this kind of problem (P) were constructed in [D] by a discretization technique, and in [Sl] by taking the limit as $\epsilon \to 0$ (in a suitable sense) of solutions to the corresponding problem for the Cahn-Hilliard equation (7). The asymptotic behaviour for large time of such solutions was also investigated in [D, Sl]. However, the assumptions on the growth rate of φ made in [D, Sl] are more restrictive than (A_2) and (H_2),(H_4) (in fact, they correspond to $q = p = 2$ in [D], and to $p \in [2, 3)$ or $q = p = 2$ in [Sl].)

With respect to the above papers, one advantage of using the Sobolev regularization is that it allows us to characterize in the case $N = 1$ the limiting Young measure τ mentioned in Definition 1.3.1, proving that its disintegration is a linear combination of Dirac measures with support on the branches of the graph of φ (see Theorem 1.3.7):

$$\tau_{x,t} = \sum_{l=0}^{n} c_l(x,t)\sigma(\cdot - s_l(w(x,t))) + \sum_{k=0}^{n} d_k(x,t)\sigma(\cdot - t_k(w(x,t)))\,. \qquad (12)$$

In the above equality σ denotes the Dirac measure and w is the limiting function of a sequence $\{w_{\epsilon_k}\} \equiv \{u_{\epsilon x}\}$, $\{u_\epsilon\}$ denoting the solution of the regularized problem (P_ϵ) with $N = 1$, which is shown to converge in $L^2(\mathbb{R}_+; H^1(\Omega))$. The main ingredients of the proof of (12) are equality (1.5.57) in Proposition 1.5.3 and assumption (C), concerning the linear independence of the branches of the graph of φ. Recall that a similar characterization was first proven in [Pl1] for the particular case of a cubic φ, and in [S] for φ of Perona-Malik type.

We also study the asymptotic behaviour as $t \to \infty$ of Young measure solutions of (P), using compactness and ω-limit set techniques as in [D, Sl]. To this purpose, a major issue is proving the *tightness* of the sequence $\{\tau_n\}$ of time translates of the limiting Young measure τ (see Lemma 1.6.2).

(β) In Chapter 2 we study the *viscous Cahn-Hilliard equation*, written in the form

$$(1-\beta)u_t = \Delta[\varphi(u) - \alpha\Delta u + \beta u_t] \qquad (\alpha, \beta > 0)\,, \qquad (13)$$

with both Neumann and Dirichlet boundary conditions. Equation (13) has been derived by several authors using different physical considerations (in particular, see [G, JF, N]). It is worth mentioning the wide literature concerning both the relationship between the viscous Cahn-Hilliard equation and *phase field models*, and generalized versions of the equation suggested in [G] (*e.g.*, see [R] and references therein).

Formally, when $\beta = 0$ equation (13) gives the Cahn-Hilliard equation,

$$u_t = \Delta[\varphi(u) - \alpha\Delta u]\,, \qquad (14)$$

whereas for $\beta = 1$ it reduces to the *Allen-Cahn equation*,

$$u_t = \alpha\Delta u - \varphi(u)\,. \tag{15}$$

It is natural to wonder whether the above formal arguments can be given a sound analytical meaning, proving that the *singular limit* of solutions of equation (13) (complemented with suitable initial and boundary conditions), as either $\beta \to 0^+$ or $\beta \to 1^-$, obtains a solution (of the corresponding problem) for equation (14), respectively (15). If so, equation (14) can be regarded as a limiting case of a more complete physical model, thus motivating the use of regularization (*ii*) of the previous section. Observe that also the Sobolev regularization (see (*i*) of the previous section) can be regarded as the formal limit of (13) as $\alpha \to 0^+$.

In the light of the above considerations, we first investigate the singular limits as $\beta \to 0^+$ or $\beta \to 1^-$ (for fixed $\alpha > 0$) of solutions to the initial-boundary value problem

$$(PD)\qquad \begin{cases} (1-\beta)u_t = \Delta[\varphi(u) - \alpha\Delta u + \beta u_t] & \text{in } \Omega\times(0,T) =: Q \\ u = \Delta u = 0 & \text{on } \partial\Omega\times(0,T) \\ u = u_0 & \text{in } \Omega\times\{0\}\,, \end{cases}$$

where $\alpha \in (0,\infty)$, $\beta \in (0,1)$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ if $N \geq 2$, $T > 0$ and the initial data $u_0 \in H^1_0(\Omega)$. Concerning the function $\varphi : \mathbb{R} \to \mathbb{R}$, the following assumptions are used:

$$(H_0)\qquad \varphi \in W^{1,\infty}_{loc}(\mathbb{R})\,, \quad \varphi(u)u \geq 0 \text{ for any } u \in \mathbb{R}\,;$$

(H_1) there exists $K > 0$ such that

$$|\varphi'(u)| \leq K(1 + |u|^{q-1})$$

for some $q \in (1,\infty)$ if $N = 1, 2$, or $q \in \left(1, \frac{N+2}{N-2}\right]$ if $N \geq 3$.

Observe that by assumption (H_0) the function φ is locally Lipschitz continuous and there holds $\varphi(0) = 0$. As expected, we prove convergence in a suitable sense to solutions of the problem for the Cahn-Hilliard equation:

$$(CH)\qquad \begin{cases} u_t = \Delta[\varphi(u) - \alpha\Delta u] & \text{in } Q \\ u = \Delta u = 0 & \text{on } \partial\Omega\times(0,T) \\ u = u_0 & \text{in } \Omega\times\{0\}\,, \end{cases}$$

as $\beta \to 0^+$, or respectively of the problem for the Allen-Cahn equation:

$$(AC)\qquad \begin{cases} u_t = \alpha\Delta u - \varphi(u) & \text{in } Q \\ u = 0 & \text{on } \partial\Omega\times(0,T) \\ u = u_0 & \text{in } \Omega\times\{0\}\,, \end{cases}$$

as $\beta \to 1^-$ (see Theorems 2.2.4-2.2.5).

Further, we study the limit of solutions of problem (PD) as $\alpha \to 0^+$ (for fixed $\beta \in (0,1)$), proving convergence to solutions of the problem for the Sobolev equation:

$$(S) \qquad \begin{cases} (1-\beta)u_t = \Delta[\varphi(u) + \beta u_t] & \text{in } Q \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\} \end{cases}$$

under the following additional assumptions on the function φ (see Theorem 2.2.6):

$$(H_2) \qquad \varphi \in \mathrm{Lip}(\mathbb{R}), \quad \varphi(u)u \geq 0\,,$$

$$(H_3) \qquad \text{there exists } s_0 > 0 \text{ such that } \varphi'(u) > 0 \text{ if } |u| \geq s_0\,.$$

Finally, we study the limit of solutions of problem (CH) as $\alpha \to 0^+$, proving the existence of a triple (u, v, μ) - where u, v are functions and μ is a *finite Radon measure* on Q - which satisfies the weak limiting equality

$$\iint_Q u\zeta_t \, dxdt + \int_0^T < \mu(\cdot,t), \zeta_t(\cdot,t) >_\Omega dt = \int\int_Q \nabla v \cdot \nabla \zeta \, dxdt - \int_\Omega u_0(x)\zeta(x,0)\, dx$$

for every $\zeta \in C^1([0,T]; C^1_c(\Omega))$ such that $\zeta(\cdot,T) = 0$ in Ω (see Theorem 2.2.7). We cannot maintain that this triple is in some sense a solution of the limiting problem

$$\begin{cases} u_t = \Delta\varphi(u) & \text{in } Q \\ u = \Delta u = 0 & \text{on } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\}\,, \end{cases}$$

since the relation between v and the function $\varphi(u)$, even in the sense of *Young measures*, is unclear.

This point was addressed in [Pl4], taking advantage of the cubic-like growth of φ at infinity, which gives rise to better estimates of the family $\{u_\alpha\}$ of solutions of (CH); at the same time, this growth prevented the appearance of a Radon measure in the solution. Instead in the present case, if the antiderivative of φ grows linearly at infinity (see Chapter 2, assumption (H_4)) we only have L^1-estimates of the family $\{u_\alpha\}$, which are compatible with the need of a Radon measure to describe solutions of the problem (in this connection, see [BBDU, PST, S, ST3]). Similar and more enhanced phenomena can be expected, if φ either has a sublinear growth, or vanishes at infinity, pointing out this behaviour as a major feature for the understanding of the problem.

Our approach is based on a detailed analysis of solutions of problem (PD), which relies on an *approximation method* already used in similar cases [BBDU, BST1, BST2, PST, S, ST2, ST3]. More precisely,
(i) we approximate the initial data $u_0 \in H^1_0(\Omega)$ by a sequence $\{u_{0n}\} \subseteq C^\infty_0(\Omega)$ such that $u_{0n} \to u_0$ in $H^1_0(\Omega)$, and $\|u_{0n}\|_{H^1_0(\Omega)} \le \|u_0\|_{H^1_0(\Omega)}$;
(ii) we also approximate the nonlinearity φ by the following sequence of functions $\{\varphi_n\}$,

$$\varphi_n(u) := \begin{cases} \varphi(u) & \text{if } |u| \le n \\ \varphi(n) + K(u-n) & \text{if } u > n \\ \varphi(-n) + K(u+n) & \text{if } u < -n \end{cases} \qquad (u \in \mathbb{R}),$$

where $n \in \mathbb{N}$ and $K > 0$ is the constant in assumption (H_1). It is easily seen that for every $n \in \mathbb{N}$ the function φ_n satisfies assumption (H_2), and $\varphi_n(u) \to \varphi(u)$ for any $u \in \mathbb{R}$ as $n \to \infty$. By standard semigroup theory, there exists a unique solution u_n of the approximating problem obtained from problem (PD) replacing φ by φ_n and u_0 by u_{0n} (see Theorem 2.2.1). A priori estimates of the sequence $\{u_n\}$ allow us to obtain an existence result for problem (PD), which improves on the available results for the viscous Cahn-Hilliard equation ([CD, ES]) (see Theorem 2.2.2). Moreover, the estimates needed to study the singular limits are obtained in a natural way in Theorems 2.2.2 and 2.2.3.

Finally, analogous results are proven for the companion Neumann IBVP with $u_0 \in H^1(\Omega)$ (see Theorems 2.6.3, 2.6.4, 2.6.5 and 2.6.6). The novel features with respect to the Dirichlet IBVP are:
(a) when $\beta \to 1^-$ we prove convergence to solutions of the *nonlocal* Allen-Cahn equation (already investigated in [RS]):

$$\begin{cases} u_t = \alpha\Delta u - \varphi(u) + \dfrac{1}{|\Omega|}\int_\Omega \varphi(u)dx & \text{in } Q_T \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\}; \end{cases}$$

(b) by taking advantage of the conservation of mass

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx \text{ for any } t \ge 0,$$

the assumptions on the behaviour at infinity of the function φ can be weakened (see Chapter 2, assumption (H_5)).

(γ) In the last chapter we study the problem

$$(P_T) \qquad \begin{cases} u_t = \Delta[\varphi(u) - \epsilon\Delta u] & \text{in } Q_T := \Omega \times (0,T) \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial \Delta u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\}, \end{cases}$$

where $T > 0$, $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, and $\frac{\partial}{\partial\nu}$ denotes the outer normal derivative at $\partial\Omega$. We are interested in nonlinearities φ of the following types:

$$\varphi(u) = \frac{u}{1+u^2}\,, \quad \varphi(u) = u\exp(-u)\,.$$

Concerning the function $\varphi \in C^3(\mathbb{R})$ and the initial data function u_0, the following assumptions are used:

$$(A_1) \qquad \varphi' \in L^\infty(\mathbb{R})\,;$$

$$(A_2) \qquad \varphi'' \in L^\infty(\mathbb{R})\,;$$

$$(A_3) \qquad \varphi''' \in L^\infty(\mathbb{R})\,;$$

$$(A_4) \qquad s\varphi(s) \geq 0 \quad \text{for any} \;\; s \in \mathbb{R}\,,$$

and the initial data

$$u_0 \in H^2_E(\Omega) := \left\{ u \in H^2(\Omega) \,\Big|\, \frac{\partial u}{\partial \nu} = 0 \right\}\,.$$

Our motivation for the present study is investigating the Cahn-Hilliard regularization (see the previous subsection sub (ii)) for forward-backward equations, whose nonlinearity φ grows at most linearly at infinity (see assumption (A_1)). This is meant as a preliminary step before addressing the singular limit of the problem as $\epsilon \to 0$. Specifically, we prove the existence and uniqueness of global solutions in a suitable function space under the assumption $N \leq 5$ (see Theorem 3.2.2). We also study, using the same approach as in [Z], the asymptotic behaviour as $t \to \infty$ of solutions of the problem

$$(P_\infty) \qquad \begin{cases} u_t = \Delta[\varphi(u) - \epsilon\Delta u] & \text{in } \; Q_\infty := \Omega \times (0,\infty) \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial \Delta u}{\partial \nu} = 0 & \text{on } \; \partial\Omega \times (0,\infty) \\ u = u_0 & \text{in } \; \Omega \times \{0\}\,. \end{cases}$$

(in particular, see Theorem 3.2.6). In doing so, we take advantage of conservation of mass (see Proposition 3.2.3).

Motivated by the asymptotical stability results, for $N = 1$ we study existence and multiplicity of monotonic equilibrium solutions of (P_∞) when $\varphi(u) = \frac{u}{1+u^2}$. By standard energy methods, this problem leads to investigating monotonicity properties of the functions

$$\mathcal{L}(\sigma, b) := \frac{\sqrt{\epsilon}}{2\sqrt{2}} \int_{u_1(\sigma,b)}^{u_2(\sigma,b)} \frac{ds}{\sqrt{\mathcal{W}(s,\sigma) - b}}\,,$$

$$\mathcal{M}(\sigma, b) := \frac{\sqrt{\epsilon}}{2\sqrt{2}} \int_{u_1(\sigma,b)}^{u_2(\sigma,b)} \frac{sds}{\sqrt{\mathcal{W}(s,\sigma) - b}}\,,$$

where $u_1 < u_2$ are solutions of the equation

$$\frac{\log(1+u^2)}{2} - \sigma u - b = 0\,,$$

and

$$\mathcal{W}(u,\sigma) := \frac{\log(1+u^2)}{2} - \sigma u\,;$$

here the parameters (b, σ) take values in a suitable subset of $\mathbb{R}^2$ (the so-called *admissible region*). At variance from the cases of a polynomial φ (see [CGS, NPe, Z]), a complete analytical investigation reveals to be cumbersome, thus recourse to numerical methods has been expedient (see Chapter 3 for a discussion of the numerical results).

Chapter 1

Sobolev regularization of a class of forward-backward parabolic equations

1.1 Introduction

In this paper we consider the initial-boundary value problem

$$(P)\qquad \begin{cases} u_t = \nabla \cdot [\varphi(\nabla u)] & \text{in } Q_T := \Omega \times (0,T) \\ u = 0 & \text{in } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\}\,. \end{cases}$$

Here $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ if $N \geq 2$, $T \in (0,\infty]$ and the dot " $\cdot$ " denotes the scalar product in $\mathbb{R}^N$.

If $N = 1$, on the function $\varphi : \mathbb{R} \to \mathbb{R}$ which appears in problem (P) we shall assume the following:

$$(A_1)\qquad \begin{cases} \text{for any } R > 0 \text{ there exists } L_R > 0 \text{ such that} \\ |\varphi(\xi_1) - \varphi(\xi_2)| \leq L_R\,|\xi_1 - \xi_2| \text{ for any } \xi_1, \xi_2 \in B_R\,, \end{cases}$$

where B_R denotes the ball of radius R in $\mathbb{R}^N$;

$$(A_2)\qquad \begin{cases} \text{there exist } \xi_0 > 0,\ p \in (1,\infty) \text{ and } C_1 > 0 \text{ such that} \\ C_1|\xi|^{p-1} \leq |\varphi(\xi)| \text{ for any } |\xi| > \xi_0\,; \end{cases}$$

$$(A_3)\qquad \varphi(\xi)\xi \geq 0 \text{ for any } \xi \in \mathbb{R}\,.$$

If $N \geq 2$, concerning the map $\varphi : \mathbb{R}^N \to \mathbb{R}^N$, $\varphi \equiv (\varphi_1, \ldots, \varphi_N)$, the following assumptions will be made:

$$(H_1)\qquad \begin{cases} \text{there exists } L > 0 \text{ such that} \\ |\varphi(\xi_1) - \varphi(\xi_2)| \leq L\,|\xi_1 - \xi_2| \text{ for any } \xi_1, \xi_2 \in \mathbb{R}^N\,; \end{cases}$$

$$(H_2) \qquad \begin{cases} \text{there exist } \xi_0 > 0,\ p \in (1,2] \text{ and } C_0 > 0 \text{ such that} \\ |\varphi(\xi)| \le C_0 \left(1 + |\xi|^{p-1}\right) \text{ for any } |\xi| > \xi_0 \,; \end{cases}$$

$$(H_3) \qquad \text{there exists } \Phi \in C^1(\mathbb{R}^N) \text{ such that } \varphi = \nabla\Phi \,.$$

$$(H_4) \qquad \begin{cases} \text{there exist } \xi_0 > 0,\ q \in (1,2] \text{ and } C_1 > 0 \text{ such that} \\ C_1|\xi|^q \le \Phi(\xi) \text{ for any } |\xi| > \xi_0 \,; \end{cases}$$

$$(H_5) \qquad \varphi(\xi)\cdot\xi \ge 0 \text{ for any } \xi \in \mathbb{R}^N \,.$$

Observe that for $N \ge 2$ we assume global Lipschitz continuity of φ, instead of local Lipschitz continuity as in the case $N = 1$; this implies the stronger restriction $p \in (1,2]$ (instead of $p \in (1,\infty)$ as for $N = 1$) on the allowed values of p. Observe also that by (H_2)-(H_4) there holds

$$C_1|\xi|^q \le \Phi(\xi) \le C_3|\xi|^p \text{ for any } |\xi| > \xi_0 \,, \tag{1.1.1}$$

for some constant $C_3 > 0$. This implies the compatibility condition $q \le p$. In the following we always choose $q = p$ (in this connection, see Remark 1.3.1 below). Concerning the initial data function u_0,

- if $N = 1$, we assume $u_0 \in W_0^{1,\infty}(\Omega)$ (by abuse of notation, we set hereafter $W_0^{1,\infty}(\Omega) := C_0(\bar\Omega) \cap W^{1,\infty}(\Omega)$);
- if $N \ge 2$, we assume $u_0 \in W_0^{1,p}(\Omega)$ $(p \in (1,2])$.

Problem (P) was studied in [D, Sl] under assumptions on the growth rate of φ which are more restrictive than (A_2) and (H_2),(H_4); in fact, they correspond to $p \in [2,3)$, $q = 2$ in [Sl], and to the quadratic case $p = q = 2$ in [D].

When the potential Φ is not convex the partial differential equation in problem (P) is a *forward-backward* parabolic equation, thus problem (P) is ill-posed. A well-known case is the *Perona-Malik equation*

$$u_t = \nabla\cdot\left(\frac{\nabla u}{1 + |\nabla u|^2}\right) , \tag{1.1.2}$$

which is parabolic if $|\nabla u| < 1$ and backward parabolic if $|\nabla u| > 1$. Observe that, if $N = 1$, setting $w = u_x$ equation (1.1.2) is formally related to the equation

$$w_t = \left(\frac{w}{1 + w^2}\right)_{xx} . \tag{1.1.3}$$

Equations like (1.1.2) and (1.1.3) arise in a variety of applications, such as edge detection in image processing [PM], aggregation models in population dynamics [Pa], and stratified turbulent shear flow [BBDU]. In fact, our motivation for the present study comes from the need to relax the growth conditions on the potential Φ to address important cases like (1.1.2) (however, observe that the present results do not cover this case, whose potential has logarithmic growth).

A natural approach to treat ill-posed problems is to introduce some regularization, often suggested by physical considerations, which gives rise to well-posed problems. Mainly two classes of regularizing terms have been used in the literature ($\epsilon > 0$ being a small parameter):
(i) $-\epsilon\Delta^2 u$, leading to fourth-order *Cahn-Hilliard type equations;*
(ii) $\epsilon\Delta[\psi(u)]_t$, with $\psi' > 0$, leading to third-order *pseudo-parabolic equations.* If $\psi(u) = u$, this regularization is called *Sobolev regularization.*
Both regularizations have been investigated for the forward-backward parabolic equation

$$u_t = \Delta[f(u)] \tag{1.1.4}$$

with f nonmonotonic, leading respectively to the Cahn-Hilliard equation

$$u_t = \Delta[f(u)] - \epsilon\Delta^2 u \tag{1.1.5}$$

(*e.g.*, see [BFG, BBMN] and references therein), or to the pseudo-parabolic equation

$$u_t = \Delta[f(u)] + \epsilon\Delta[\psi(u)]_t \tag{1.1.6}$$

(see [BBDU, BST, EP, NP, Pl1, Pl2, Pl3, STe, ST1, ST2] and references therein). If $\psi(u) = u$, equation (1.1.6) has been proposed by several authors as a variant of (1.1.5) which includes viscous effects; in fact, both (1.1.5) and (1.1.6) formally are particular cases of the so-called *viscous Cahn-Hilliard equation* (see [BFJ, BS, G, N, NP]).

As $\epsilon \to 0^+$, solutions of either equation (1.1.5), (1.1.6) are expected to converge to some suitably defined solution of (1.1.4). However, in agreement with the nonuniqueness results for forward-backward parabolic equations proven in [Hö, Z], different regularization procedures need not lead to the same solution of (1.1.4), nor to solutions having the same properties (*e.g.*, the asymptotic behaviour for large time). In fact, as pointed out in [NP], for a cubic nonlinearity f even when $\epsilon > 0$ the above regularizations give rise to different dynamics of solutions. Let us also mention that, apart from specific cases, uniqueness is unknown even for solutions of the limiting equation obtained by the same procedure (*e.g.*, by the Sobolev regularization; see [MTT, Pl1]).

The "vanishing viscosity limit" as $\epsilon \to 0^+$ of solutions of the Dirichlet initial-boundary value problem for equation (1.1.6) with $\psi(u) = u$,

$$\begin{cases} u_t = \Delta[f(u)] + \epsilon\Delta u_t & \text{in } Q_T \\ f(u) + \epsilon u_t = 0 & \text{in } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\} \end{cases} \tag{1.1.7}$$

with a cubic f was studied in the seminal paper [Pl1], proving the existence of a *Young measure solution* of the same problem for equation (1.1.4). In fact, let $\{u_\epsilon\}$, $\{v_\epsilon\}$ denote the families of solutions u_ϵ of the regularized problem, respectively of the associated *chemical potential*

$$v_\epsilon := f(u_\epsilon) + \epsilon u_{\epsilon t} \,.$$

It was shown in [Pl1] that, for some vanishing sequence $\{\epsilon_k\}$, the sequence $\{\tau^{\epsilon_k}\}$ of Young measures associated to the sequence $\{u_{\epsilon_k}\}$ converges in the *narrow topology* over $Q \times \mathbb{R}$ to a Young measure τ (see Section 3.2), whose disintegration $\nu_{(x,t)}$ is a superposition of three Dirac masses concentrated on the branches s_0, s_1, s_2 of the graph of f. More precisely, there exist $\lambda_i \in L^\infty(Q)$ $(i = 0, 1, 2)$, $0 \le \lambda_i \le 1$, $\sum_{i=0}^2 \lambda_i = 1$, such that

$$\nu_{(x,t)} = \sum_{i=0}^{2} \lambda_i(x,t)\delta\big(\,\cdot - s_i(v(x,t))\big) \tag{1.1.8}$$

for almost every $(x,t) \in Q$, where $v \in L^\infty(Q)$ is the weak* limit of the sequence $\{v_{\epsilon_k}\}$ in $L^\infty(Q)$.

With respect to the Cahn-Hilliard regularization, a major advantage of the Sobolev regularization is to give rise to a class of inequalities, called *entropy inequalities* by analogy with the case of viscous conservation laws, that are satisfied by a solution of (1.1.7) for any $\epsilon > 0$ [NP]. Also the Young measure solution obtained by the above limiting process as $\epsilon \to 0^+$ satisfies a suitable limiting entropy inequality. Relying on this property, a number of qualitative results has been proven for Young measure solutions obtained by Sobolev regularization, which has no counterpart in the approach based on the Cahn-Hilliard regularization procedure. In particular, when studying solutions that describe the transition between stable phases, as proposed in [EP], *admissibility conditions* follow from the entropy inequality, which can be viewed as prescriptions to select admissible jumps between the stable branches of f (see [Pl1]). Relying on these admissibility conditions, also solutions which exhibit *hysteresis*, or including unstable phases, have been constructed [GT, MTT, Te].

In view of the above results, in this paper we study problem (P) using the Sobolev regularization. Namely, first we address for any $\epsilon > 0$ the initial-boundary value problem

$$(P_\epsilon) \qquad \begin{cases} u_t = \nabla \cdot [\varphi(\nabla u)] + \epsilon \Delta u_t & \text{in } Q_T \\ u = 0 & \text{in } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\}\,. \end{cases}$$

Then we study the limit as $\epsilon \to 0^+$ of the family $\{u_\epsilon\}$ of solutions of the approximating problems (P_ϵ), proving the existence of a Young measure solution of the original problem (P) (see Definition 1.3.1). We also study the asymptotic behaviour as $t \to \infty$ of such solution, using compactness and ω-limit set techniques; in doing so, a major point is the proof of *tightness* of sequences $\{\tau_n\}$ of time translates of the limiting Young measure τ (see Section 3.2 and Lemma 1.6.2). Finally, if $N = 1$ we extend the characterization (1.1.8) to the limiting Young measure τ mentioned in Definition 1.3.1, for any φ satisfying assumptions (A_1)-(A_3). By the change of unknown $w = u_x$ the same result holds for the problem

$$\begin{cases} w_t = [\varphi(w)]_{xx} & \text{in } Q_T \\ w_x = 0 & \text{in } \partial\Omega \times (0,T) \\ w = w_0 := u_0' & \text{in } \Omega \times \{0\}\,. \end{cases}$$

Similar results concerning existence of a Young measure solution and its asymptotic behaviour were proven in [Sl] (yet nothing was proven about its disintegration), both for problem (P) and for the companion problem with Neumann boundary conditions, by studying the limit as $\epsilon \to 0^+$ in a suitable topology of solutions to the corresponding problems for the Cahn-Hilliard type equation

$$u_t = \nabla \cdot [\varphi(\nabla u)] - \epsilon\, \Delta^2 u \qquad (\epsilon > 0)\,. \tag{1.1.9}$$

As already mentioned, in [Sl] the growth assumptions on φ are stronger than ours. Existence of classical solutions to the approximating problems is only discussed by semigroup methods if $N = 1$, supposing that both φ and u_0 are sufficiently smooth, and assumed to hold if $N \geq 2$.

The approach in [Sl] is the same as in the present paper, in that problem (P) is approximated by a family of regularized problems, whose solutions are used to extract a Young measure solution of the original problem. However, in agreement with the above remarks concerning the dependence of the limiting solution on the adopted regularization procedure, the very concept of Young measure solution for a specific problem clearly depends on the procedure of regularization, which necessarily affects the generating sequence of the limiting Young measure. We stress this point in the existence statement of Theorem 1.3.6, where the approximating family used to construct the solution is explicitly mentioned. As stated thereafter, *e.g.* in Theorems 1.3.7 and 1.3.9, all the subsequent qualitative properties we prove refer to this specific solution. There is no reason to expect that the present results concerning existence and asymptotic behaviour of solutions coincide with, or can be derived from those proven in [Sl].

Similar remarks hold true for the study of problem (P) carried out in [D] (see also [KP]). Also the treatment in [D] involves Young measures, but the spirit of the approach, which does not rely on a regularization procedure suggested by physical considerations, is different form the present one. In fact, the approximation procedure used in [D] combines the explicit time discretization method for evolution equations with variational methods used to address the problem obtained at each time step, as outlined in [KP, Theorem 6.1]. At variance from the present paper, in [D] the gradient function φ

is continuous and linear, $u_0 \in H_0^1(\Omega)$ and the first equation in (P) is satisfied in $H^{-1}(Q_T)$. Moreover, it is worth observing that: (i) the Young measure solution exhibited in [D] is a solution of the same problem with the potential Φ replaced by its convexification Φ^{**}; (ii) the support of the disintegration of the relative Young measure is contained in the set where $\Phi = \Phi^{**}$; (iii) the asymptotic behaviour for large time of this solution could be addressed by applying to the equation with the convexified potential standard methods for quasilinear parabolic equations. As a whole, the approach used in [D] seems too limited, with respect to the Sobolev regularization procedure, to describe several important qualitative properties of solutions (see the above discussion concerning equation (1.1.4)).

The paper is organized as follows. In Section 3.2 we describe the mathematical framework, and in Section 1.3 we state our main results, concerning existence and asymptotic behaviour as $t \to \infty$ of Young measure solutions of problem (P) (see Subsections 1.3.1 and 1.3.3), as well as the characterization of the limiting Young measure τ when $N = 1$ (see Subsection 1.3.2). Proofs of the main results are to be found in Sections 1.4, 1.5 and 1.6.

1.2 Mathematical framework

We shall denote by $C_c(\mathbb{R}^N)$ the space of continuous real functions with compact support in $\mathbb{R}^N$, and by $BC(\mathbb{R}^N)$ that of bounded continuous real functions on $\mathbb{R}^N$ endowed with the supremum norm $\|\cdot\|_{C(\mathbb{R}^N)}$. We also set

$$C_0(\mathbb{R}^N) := \{f \in BC(\mathbb{R}^N) \mid f(x) \to 0 \text{ as } |x| \to \infty\}.$$

We shall denote by $\mathcal{M}(\mathbb{R}^N)$ the Banach space of finite Radon measures on $\mathbb{R}^N$, endowed with the norm

$$\|\mu\|_{\mathcal{M}(\mathbb{R}^N)} := |\mu|(\mathbb{R}^N) \quad \text{for any } \mu \in \mathcal{M}(\mathbb{R}^N).$$

By $\mathcal{M}^+(\mathbb{R}^N)$ we denote the cone of positive finite Radon measures, and by $\mathcal{P}(\mathbb{R}^N)$ the convex set of probability measures on $\mathbb{R}^N$:

$$\|\tau\|_{\mathcal{M}(\mathbb{R}^N)} = \tau(\mathbb{R}^N) = 1 \quad \text{for any } \tau \in \mathcal{P}(\mathbb{R}^N).$$

Clearly, $\mathcal{P}(\mathbb{R}^N) \subset \mathcal{M}^+(\mathbb{R}^N) \subset \mathcal{M}(\mathbb{R}^N)$.

The duality map $\langle \cdot, \cdot \rangle_{\mathbb{R}^N}$ between the space $\mathcal{M}(\mathbb{R}^N)$ and $C_c(\mathbb{R}^N)$, namely

$$\langle \mu, \rho \rangle_{\mathbb{R}^N} := \int_{\mathbb{R}^N} \rho \, d\mu$$

can be extended to functions $\rho \in C_0(\mathbb{R}^N)$. By abuse of notation, the above equality will be used to define the quantity $\langle \mu, \rho \rangle_{\mathbb{R}^N}$ for any $\mu \in \mathcal{M}(\mathbb{R}^N)$ and every μ-integrable function ρ.

Similar notations will be used for the function space $BC(Q_T \times \mathbb{R}^N)$ and for the space $\mathcal{M}(Q_T \times \mathbb{R}^N)$ of finite Radon measures on $Q_T \times \mathbb{R}^N$ ($T \in (0,\infty]$). We shall also denote by $\mathcal{L}_{N+1} \equiv \mathcal{L}_{N+1}(Q_T)$ the σ-algebra of Lebesgue measurable subsets $E \subseteq Q_T$, by λ_{N+1} the Lebesgue measure on Q_T and by $\mathcal{B}_N$ the σ-algebra of Borel subsets of $\mathbb{R}^N$; for simplicity we set $|\,Q_T\,| \equiv \lambda_{N+1}(Q_T)$, $|\,\Omega\,| \equiv \lambda_N(\Omega)$. Integration with respect to the Lebesgue measure on Q_T will be denoted by the usual symbol $dxdt \equiv d\lambda_{N+1}$. By $\mathcal{Y}(Q_T;\mathbb{R}^N)$ we denote the set of Young measures on $Q_T \times \mathbb{R}^N$, which are defined as follows (*e.g.*, see [V1]).

Definition 1.2.1. *By a* Young measure on $Q_T \times \mathbb{R}^N$ ($T \in (0,\infty]$) *we mean any positive Radon measure τ on the measurable space* $(Q_T \times \mathbb{R}^N, \mathcal{L}_{N+1} \times \mathcal{B}_N)$ *such that*

$$\tau(E \times \mathbb{R}^N) = \lambda_{N+1}(E) \quad \textit{for any } E \in \mathcal{L}_{N+1} . \tag{1.2.1}$$

If $f : Q_T \to \mathbb{R}^N$ is Lebesgue measurable, the Young measure associated to f *is the measure $\tau \in \mathcal{Y}(Q_T;\mathbb{R}^N)$ such that*

$$\tau(E \times F) = \lambda_{N+1}(E \cap f^{-1}(F)) \quad \textit{for any } E \in \mathcal{L}_{N+1}, F \in \mathcal{B}_N . \tag{1.2.2}$$

Remark 1.2.1. In view of (2.5.2), if τ is the Young measure associated to a Lebesgue measurable function $f : Q_T \to \mathbb{R}^N$, for any τ-integrable function $\psi : Q_T \times \mathbb{R}^N \to \bar{\mathbb{R}}$ there holds

$$\int_{Q_T \times \mathbb{R}^N} \psi \, d\tau = \iint_{Q_T} \psi(x,t,f(x,t))\, dxdt . \tag{1.2.3}$$

Let us recall the following result (*e.g.*, see [GMS]).

Proposition 1.2.1. *Let $\tau \in \mathcal{Y}(Q_T;\mathbb{R}^N)$. Then for almost every $(x,t) \in Q_T$ there exists a measure $\tau_{(x,t)} \in \mathcal{P}(\mathbb{R}^N)$, such that for any $\psi \in BC(Q_T \times \mathbb{R}^N)$:*
(i) the map

$$(x,t) \to \langle \tau_{(x,t)}, \psi(x,t,\cdot) \rangle_{\mathbb{R}^N} = \int_{\mathbb{R}^N} \psi(x,t,\xi)\, d\tau_{(x,t)}(\xi)$$

is Lebesgue measurable;
(ii) there holds

$$\begin{aligned} \langle \tau, \psi \rangle_{Q_T \times \mathbb{R}^N} := \int_{Q_T \times \mathbb{R}^N} \psi \, d\tau &= \iint_{Q_T} \langle \tau_{(x,t)}, \psi(x,t,\cdot) \rangle_{\mathbb{R}^N} \, dxdt \quad (1.2.4) \\ &= \iint_{Q_T} dxdt \int_{\mathbb{R}^N} \psi(x,t,\xi)\, d\tau_{(x,t)}(\xi) . \end{aligned}$$

We shall identify any $\tau \in \mathcal{Y}(Q_T \times \mathbb{R}^N)$ with the associated family $\{\tau_{(x,t)} \,|\, (x,t) \in Q_T\}$, which is called the *disintegration* of τ.

Remark 1.2.2. If τ is the Young measure associated to a Lebesgue measurable function $f : Q_T \to \mathbb{R}^N$, equalities (1.2.3)-(2.5.3) imply

$$\psi(x,t,f(x,t)) = \langle \tau_{(x,t)}, \psi(x,t,\cdot) \rangle_{\mathbb{R}^N} = \int_{\mathbb{R}^N} \psi(x,t,\xi)\, d\tau_{(x,t)}(\xi) \qquad (1.2.5)$$

for almost every $(x,t) \in Q_T$; here $\psi \in BC(Q_T \times \mathbb{R}^N)$ and $\{\tau_{(x,t)}\}$ is the disintegration of τ. In fact, in this case

$$\tau_{(x,t)} = \delta_{f(x,t)} \text{ for almost every } (x,t) \in Q_T\,,$$

where δ_P denotes the Dirac mass concentrated in $P \in \mathbb{R}^N$.

The set $\mathcal{Y}(Q_T;\mathbb{R}^N)$ will be endowed with the following topology. Let $\psi : Q_T \times \mathbb{R}^N \to \mathbb{R}$ be a *Carathéodory integrand*, namely

$$\begin{cases} (i)\ \psi \text{ measurable, } \psi(x,t,\cdot) \in BC(\mathbb{R}^N) \text{ for } a.e.\ (x,t) \in Q_T\,; \\ (ii) \qquad \iint_{Q_T} \|\psi(x,t,\cdot)\|_{C(\mathbb{R}^N)}\, dxdt < \infty\,. \end{cases}$$

Then we have the following definition.

Definition 1.2.2. *Let τ^n, $\tau \in \mathcal{Y}(Q_T;\mathbb{R}^N)$ ($n \in \mathbb{N}$, $T \in (0,\infty]$). We say that $\tau^n \to \tau$* narrowly *in $Q_T \times \mathbb{R}^N$, if*

$$\int_{Q_T \times \mathbb{R}^N} \psi\, d\tau^n \;\to\; \int_{Q_T \times \mathbb{R}^N} \psi\, d\tau \qquad (1.2.6)$$

for any Carathéodory integrand ψ.

Definition 1.2.3. *For any $T \in (0,\infty]$ we denote by $L^\infty(Q_T;\mathcal{M}(\mathbb{R}^N))$ the space of finite Radon measures $\mu \in \mathcal{M}(Q_T \times \mathbb{R}^N)$ which satisfy the following: for almost every $(x,t) \in Q_T$ there exists a measure $\mu_{(x,t)} \in \mathcal{M}(\mathbb{R}^N)$, such that*
(i) for any Carathéodory integrand ψ the map

$$(x,t) \;\to\; \langle \mu_{(x,t)}, \psi(x,t,\cdot) \rangle_{\mathbb{R}^N} = \int_{\mathbb{R}^N} \psi(x,t,\xi)\, d\mu_{(x,t)}(\xi)$$

is Lebesgue measurable, and

$$\begin{aligned} \langle \mu, \psi \rangle_{Q_T \times \mathbb{R}^N} := \int_{Q_T \times \mathbb{R}^N} \psi\, d\mu \;&=\; \iint_{Q_T} \langle \mu_{(x,t)}, \psi(x,t,\cdot) \rangle_{\mathbb{R}^N}\, dxdt \qquad (1.2.7) \\ &=\; \iint_{Q_T} dxdt \int_{\mathbb{R}^N} \psi(x,t,\xi)\, d\mu_{(x,t)}(\xi)\,; \end{aligned}$$

(ii) there holds

$$\text{ess sup}_{(x,t)\in Q_T} \|\mu_{(x,t)}\|_{\mathcal{M}(\mathbb{R}^N)} < \infty\,.$$

As already said for Young measures, we shall identify any $\mu \in \mathcal{M}(Q_T \times \mathbb{R}^N)$ with the associated family $\{\mu_{(x,t)} \mid (x,t) \in Q_T\}$. Observe that $L^\infty(Q_T; \mathcal{M}(\mathbb{R}^N))$ is a Banach space with norm

$$\|\mu\|_\infty := \text{ess sup}_{(x,t)\in Q_T} \|\mu_{(x,t)}\|_{\mathcal{M}(\mathbb{R}^N)} .$$

Similar definitions and remarks hold for the space $L^\infty(\Omega; \mathcal{M}(\mathbb{R}^N))$.

In view of Proposition 2.5.1 and Definition 1.2.3, we have the following result (see [V1, Theorem 2].

Theorem 1.2.2. *The set $\mathcal{Y}(Q_T; \mathbb{R}^N)$ is homeomorphic to a closed subset of $L^\infty(Q_T; \mathcal{P}(\mathbb{R}^N))$. Therefore, every sequence $\{\tau^n\}$ of Young measures contains a subsequence $\{\tau^{n_k}\}$ which converges in the weak* topology of $L^\infty(Q_T; \mathcal{M}(\mathbb{R}^N))$, namely*

$$\int_{Q_T\times\mathbb{R}^N} \psi \, d\tau^{n_k} \;\to\; \int_{Q_T\times\mathbb{R}^N} \psi \, d\tau \tag{1.2.8}$$

for any Carathéodory integrand ψ such that $\psi(x,t,\cdot) \in C_0(\mathbb{R}^N)$ for almost every $(x,t) \in Q_T$.

In Definition 1.2.3 we used the fact that $L^1(Q_T; C_0(\mathbb{R}^N))$ can be identified with the set of Carathéodory integrands ψ such that $\psi(x,t,\cdot) \in C_0(\mathbb{R}^N)$ for almost every $(x,t) \in Q_T$ (see [V1, Lemma A3]).

Definition 1.2.4. *A subset $\mathcal{T} \subseteq \mathcal{Y}(Q_T; \mathbb{R}^N)$ is said to be* tight*, if for any $\sigma > 0$ there exists a compact subset $K_\sigma \subset \mathbb{R}^N$ such that*

$$\tau(Q_T \times (\mathbb{R}^N \setminus K_\sigma)) < \sigma \;\; \textit{for any} \;\; \tau \in \mathcal{T} . \tag{1.2.9}$$

A sequence $\{f_n\}$ of Lebesgue measurable functions from Q_T to $\mathbb{R}^N$ is said to be tight*, if the the sequence of the associated Young measures is tight - namely, if for any $\epsilon > 0$ there exists a compact subset $K_\epsilon \subset \mathbb{R}^N$ such that*

$$\lambda_{N+1}(f_n^{-1}(\{\mathbb{R}^N \setminus K_\epsilon)) < \epsilon \;\; \textit{for any} \;\; n \in \mathbb{N} . \tag{1.2.10}$$

For every $\xi \equiv (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$ $(d \geq 1)$ set $|\xi| := \sqrt{\sum_{i=1}^d \xi_i^2}$.

Definition 1.2.5. *A subset $\mathcal{U} \subseteq L^1(Q_T; \mathbb{R}^d)$ $(d \geq 1)$ is said to be* uniformly integrable *if:*

(i) there exists $M > 0$ such that

$$\|f\|_{L^1(Q_T;\mathbb{R}^d)} := \iint_{Q_T} |f(x,t)| \, dxdt \leq M \;\; \textit{for any} \;\; f \in \mathcal{U} ;$$

(ii) for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $f \in \mathcal{U}$

$$E \in \mathcal{L}_{N+1} , \, \lambda_{N+1}(E) < \delta \quad \mathbb{R}ightarrow \quad \iint_E |f(x,t)| \, dxdt < \epsilon .$$

Let us mention the following result (see [V1, Proposition 10 and Theorem 11]).

Theorem 1.2.3. *(i) A subset $\mathcal{T} \subseteq \mathcal{Y}(Q_T; \mathbb{R}^N)$ is tight if and only if it is relatively compact in the narrow topology of $\mathcal{Y}(Q_T; \mathbb{R}^N)$.*
(ii) The narrow topology on a tight subset $\mathcal{T} \subseteq \mathcal{Y}(Q_T; \mathbb{R}^N)$ coincides with the weak topology mentioned in Theorem 1.2.2.*

By Theorem 1.2.3, every sequence $\{\tau^k\}$ of Young measures contained in a tight subset $\mathcal{T} \subseteq \mathcal{Y}(Q_T; \mathbb{R}^N)$ contains a subsequence $\{\tau^{k_n}\}$ which converges in the narrow topology of $\mathcal{Y}(Q_T; \mathbb{R}^N)$. On the other hand, it is easily seen that the subset of Young measures associated to a bounded subset of $L^1(Q_T; \mathbb{R}^N)$ is tight (see [V1, Proposition 8]). Then we have the following result.

Proposition 1.2.4. *Let $\{f_n\}$ be a bounded sequence in $L^1(Q_T; \mathbb{R}^N)$, and $\{\tau^n\}$ the sequence of associated Young measures. Then there exist subsequences $\{f_k\} \equiv \{f_{n_k}\} \subseteq \{f_n\}$, $\{\tau^k\} \equiv \{\tau^{n_k}\} \subseteq \{\tau^n\}$ and a Young measure τ on $Q_T \times \mathbb{R}^N$ such that $\tau^k \to \tau$ narrowly.*

Observe that the limiting measure $\tau \in \mathcal{Y}(Q_T; \mathbb{R}^N)$ mentioned in the above proposition need not be associated to any measurable function from Q_T to $\mathbb{R}^N$ (see [V1, Theorem 20]). However, under the stronger assumption of uniform integrability we have the following result, which is a consequence of the more general Prokhorov's Theorem (see [V1, Section 4].

Theorem 1.2.5. *Let $\{f_n\}$ be a bounded sequence in $L^1(Q_T; \mathbb{R}^N)$, and $\{\tau^n\}$ the sequence of associated Young measures. Then:*
(i) there exist subsequences $\{f_k\} \equiv \{f_{n_k}\} \subseteq \{f_n\}$, $\{\tau^k\} \equiv \{\tau^{n_k}\} \subseteq \{\tau^n\}$ and a Young measure τ on $Q_T \times \mathbb{R}^N$ such that $\tau^k \to \tau$ narrowly in $Q_T \times \mathbb{R}^N$;
(ii) for any $\rho \in C(\mathbb{R}^N)$ such that the sequence $\{\rho \circ f_n\} \subseteq L^1(Q_T) \equiv L^1(Q_T; \mathbb{R})$ is uniformly integrable, there holds

$$\rho \circ f_k \equiv \rho \circ f_{n_k} \rightharpoonup \rho^* \quad \text{in } L^1(Q_T)\,, \tag{1.2.11}$$

where

$$\rho^*(x,t) := \left\langle \tau_{(x,t)}, \rho \right\rangle_{\mathbb{R}^N} = \int_{\mathbb{R}^N} \rho(\xi)\, d\tau_{(x,t)}(\xi) \qquad \text{for a.e. } (x,t) \in Q_T \tag{1.2.12}$$

and $\{\tau_{(x,t)}\}$ is the disintegration of τ.

Remark 1.2.3. If the sequence $\{f_n\} \subseteq L^1(Q_T; \mathbb{R}^N)$ is uniformly integrable, and $\rho_i(\xi) := \xi_i$ denotes the i-th projection in $\mathbb{R}^N$ $(i = 1, \dots, N)$, from Theorem 2.5.2-(ii) with $\rho = \rho_i$ we have

$$f_k \rightharpoonup f^* \quad \text{in } L^1(Q_T; \mathbb{R}^N)\,, \tag{1.2.13}$$

where $f^* \equiv (f_1^*, \ldots, f_N^*)$,

$$f_i^*(x,t) := \int_{\mathbb{R}^N} \xi_i \, d\tau_{(x,t)}(\xi) \quad \text{for } a.e. \ (x,t) \in Q_T \qquad (i = 1, \ldots, N) \tag{1.2.14}$$

is the *barycenter* of the disintegration $\tau_{(x,t)}$.

1.3 Main results

1.3.1 Existence

In what follows we will prove the existence of Young measure solutions of problem (P) in the sense of the following definition.

Definition 1.3.1. *Let either $N = 1$, $u_0 \in W_0^{1,\infty}(\Omega)$, or $N \geq 2$, $u_0 \in W_0^{1,p}(\Omega)$ $(p \in (1,2])$. By a* Young measure solution *of problem (P) in Q_T we mean a couple (u, τ) such that:*
(i) $u \in L^\infty(\mathbb{R}_+; W_0^{1,\infty}(\Omega)) \cap C([0,\infty); L^2(\Omega))$ if $N = 1$, or
$u \in L^\infty(\mathbb{R}_+; W_0^{1,p}(\Omega)) \cap C([0,\infty); L^p(\Omega))$ $(p \in (1,2])$ if $N \geq 2$;
(ii) $u_t \in L^2(Q_T)$, $\tau \in \mathcal{Y}(Q_T; \mathbb{R}^N)$;
(iii) for almost every $(x,t) \in Q_T$ there holds

$$\nabla u(x,t) = \langle \tau_{(x,t)}, id \rangle_{\mathbb{R}^N} = \int_{\mathbb{R}^N} \xi \, d\tau_{(x,t)}(\xi) \,, \tag{1.3.15}$$

where $id(\xi) := \xi$ $(\xi \in \mathbb{R}^N)$ and $\tau_{(x,t)} \in \mathcal{P}(\mathbb{R}^N)$ denotes the disintegration of τ;
(iv) for any $\zeta \in C^1([0,T]; C_c^1(\Omega))$ and $t \in (0,T)$

$$\int_0^t \int_\Omega \left[u \, \zeta_s - \varphi^* \cdot \nabla \zeta \right](x,s) dx ds = \tag{1.3.16}$$
$$= \int_\Omega u(x,t) \zeta(x,t) \, dx - \int_\Omega u_0(x) \, \zeta(x,0) \, dx \,,$$

where $\varphi^* \equiv (\varphi_1^*, \ldots, \varphi_N^*)$,

$$\varphi_i^*(x,t) := \langle \tau_{(x,t)}, \varphi_i \rangle_{\mathbb{R}^N} = \int_{\mathbb{R}^N} \varphi_i(\xi) \, d\tau_{(x,t)}(\xi) \qquad (i = 1, \ldots, N) \tag{1.3.17}$$

for almost every $(x,t) \in Q_T$.

A Young measure solution of problem (P) in Q_∞ is said to be global.

Remark 1.3.1. In the following, when $N \geq 2$, we often assume that the initial data function u_0 belongs to $W_0^{1,p}(\Omega)$ $(p \in (1,2])$ and assumptions (H_2)-(H_4) are satisfied (in particular, see Proposition 1.3.3, Theorem 1.3.6 and Theorem 1.3.9). Clearly, this means that (H_2)-(H_4) are assumed to hold with some fixed $p = q \in (1,2])$, and that $u_0 \in W_0^{1,p}(\Omega)$ with the same p.

As a particular case of Definition 1.3.1, we have the following

Definition 1.3.2. *By a* Young measure equilibrium solution *of problem* (P) *we mean a couple* $(\bar{u}, \bar{\tau}) \in W_0^{1,p}(\Omega) \times \mathcal{Y}(\Omega; \mathbb{R}^N)$, *such that*

$$\nabla \bar{u}(x) = \int_{\mathbb{R}^N} \xi \, d\bar{\tau}_x(\xi) \tag{1.3.18}$$

for almost every $x \in \Omega$, *and*

$$\int_\Omega \big[\bar{\varphi} \cdot \nabla \rho\big](x) \, dx = 0 \tag{1.3.19}$$

for any $\rho \in C_c^1(\Omega)$. *Here* $\bar{\varphi} \equiv (\bar{\varphi}_1, \dots, \bar{\varphi}_N)$,

$$\bar{\varphi}_i(x) := \langle \bar{\tau}_x, \varphi_i \rangle_{\mathbb{R}^N} = \int_{\mathbb{R}^N} \varphi_i(\xi) \, d\bar{\tau}_x(\xi) \qquad (i = 1, \dots, N) \tag{1.3.20}$$

for almost every $x \in \Omega$, *and* $\bar{\tau}_x \in \mathcal{P}(\mathbb{R}^N)$ *denotes the disintegration of* $\bar{\tau}$.

Consider the problem (P_ϵ) introduced above (see Section 2.1). If $N = 1$, setting

$$v := u_x \,, \quad w := \varphi(v) + \epsilon v_t \,, \tag{1.3.21}$$

problem (P_ϵ) reads

$$\begin{cases} u_t = w_x & \text{in } Q_T \\ u = 0 & \text{in } \partial\Omega \times (0, T) \\ u = u_0 & \text{in } \Omega \times \{0\} \,. \end{cases} \tag{1.3.22}$$

Definition 1.3.3. *Let* $N = 1$, *and let* $u_0 \in W_0^{1,\infty}(\Omega)$. *By a* solution *of problem* (P_ϵ) *in* Q_T *we mean any function* $u_\epsilon \in C^1([0, T); W_0^{1,\infty}(\Omega))$, *with* $w_\epsilon \in C([0, T); W^{2,\infty}(\Omega) \cap C(\bar{\Omega}))$, $w_{\epsilon x} \in C([0, T); W_0^{1,\infty}(\Omega))$, *which satisfies* (P_ϵ) *in classical sense.*

Definition 1.3.4. *Let* $N \geq 2$, *and let* $u_0 \in H_0^1(\Omega)$. *By a* solution *of problem* (P_ϵ) *in* Q_T *we mean any function* $u_\epsilon \in C^1([0, T); H_0^1(\Omega))$ *such that* $u_\epsilon(\cdot, 0) = u_{0\epsilon}$, *and*

$$\int_\Omega u_{\epsilon t}(x, t) \rho(x) \, dx + \int_\Omega \varphi(\nabla u_\epsilon)(x, t) \cdot \nabla \rho(x) \, dx + \epsilon \int_\Omega \nabla u_{\epsilon t}(x, t) \cdot \nabla \rho(x) \, dx = 0 \tag{1.3.23}$$

for any $t \in (0, T)$ *and any* $\rho \in H_0^1(\Omega)$.

Definition 1.3.5. *Let* $N \geq 1$. *A solution of problem* (P_ϵ) *in* Q_∞ *is said to be* global, *if it is a solution in* Q_T *for any* $T \in (0, \infty)$.

Theorem 1.3.1. *Let either* $N = 1$, $u_0 \in W_0^{1,\infty}(\Omega)$ *and assumptions* (A_1)-(A_3) *be satisfied, or let* $N \geq 2$, $u_0 \in H_0^1(\Omega)$ *and* (H_1) *hold. Then for any* $\epsilon > 0$ *there exists a unique global solution* u_ϵ *of problem* (P_ϵ).

The following a priori estimates will play an important role to study the vanishing viscosity limit of the solution u_ϵ as $\epsilon \to 0$. Let us first address the case $N = 1$.

Proposition 1.3.2. *Let $N = 1$, $u_0 \in W_0^{1,\infty}(\Omega)$ and assumptions (A_1)-(A_3) be satisfied. Let u_ϵ be the solution of problem (P_ϵ) given by Theorem 1.3.1. Then there exists $C > 0$ such that for any $\epsilon > 0$*

$$\|u_\epsilon\|_{L^\infty(\mathbb{R}_+;W_0^{1,\infty}(\Omega))} + \|u_{\epsilon t}\|_{L^2(Q_\infty)} + \sqrt{\epsilon}\,\|u_{\epsilon xt}\|_{L^2(Q_\infty)} \leq C\,\|u_0\|_{W_0^{1,\infty}(\Omega)}\,. \tag{1.3.24}$$

To prove analogous estimates when $N \geq 2$, for any fixed $u_0 \in W_0^{1,p}(\Omega)$, with $p \in (1, 2]$, consider a family $\{u_{0\epsilon}\} \subseteq H_0^1(\Omega)$ $(\epsilon > 0)$ such that

$$\|u_{0\epsilon}\|_{W_0^{1,p}(\Omega)} \leq \|u_0\|_{W_0^{1,p}(\Omega)}\,, \quad u_{0\epsilon} \to u_0 \text{ in } W_0^{1,p}(\Omega)\,. \tag{1.3.25}$$

Proposition 1.3.3. *Let $N \geq 2$, $u_0 \in W_0^{1,p}(\Omega)$ $(p \in (1, 2])$ and assumptions (H_1)-(H_4) be satisfied. Let u_ϵ be the solution of problem (P_ϵ) with Cauchy data $u_{0\epsilon} \in H_0^1(\Omega)$ as in (1.3.25), which is given by Theorem 1.3.1. Then there exists $C > 0$ such that for any $\epsilon > 0$*

$$\|u_\epsilon\|_{L^\infty(\mathbb{R}_+;W_0^{1,p}(\Omega))} + \|u_{\epsilon t}\|_{L^2(Q_\infty)} + \sqrt{\epsilon}\,\|\nabla u_{\epsilon t}\|_{L^2(Q_\infty)} \leq C\,\|u_0\|_{W_0^{1,p}(\Omega)}\,. \tag{1.3.26}$$

Remark 1.3.2. Observe that by estimates (1.3.24) and (1.3.26) the family of solutions $\{u_\epsilon\}$ is contained in a bounded subset of $L^\infty(\mathbb{R}_+; W_0^{1,\infty}(\Omega))$ if $N = 1$, respectively of $L^\infty(\mathbb{R}_+; W_0^{1,p}(\Omega))$ if $N \geq 2$. In addition, if $N \geq 2$ and $p \in (1, 2]$, by estimate (1.3.26) for every $T \in (0, \infty)$ there exists a constant $\bar{C}_T > 0$ such that

$$\|u_\epsilon\|_{W^{1,p}(Q_T)} \leq \bar{C}_T\,\|u_0\|_{W_0^{1,p}(\Omega)} \tag{1.3.27}$$

for any $\epsilon > 0$. Similarly, if $N = 1$ by estimate (1.3.24) we have

$$\|u_\epsilon\|_{H^1(Q_T)} \leq \bar{C}_T\,\|u_0\|_{W_0^{1,\infty}(\Omega)} \tag{1.3.28}$$

for any $\epsilon > 0$.

If $N \geq 2$, as a consequence of estimate (1.3.26) we have the following

Proposition 1.3.4. *Let $N \geq 2$, $u_0 \in W_0^{1,p}(\Omega)$ $(p \in (1, 2])$ and assumptions (H_1)-(H_4) be satisfied. Let u_ϵ be the solution of problem (P_ϵ) with Cauchy data $u_{0\epsilon} \in H_0^1(\Omega)$ as in (1.3.25), which is given by Theorem 1.3.1. Then there exist a sequence $\{u_{\epsilon_k}\} \subseteq \{u_\epsilon\}$ and $u \in L^\infty(\mathbb{R}_+; W_0^{1,p}(\Omega)) \cap W^{1,p}(Q_T) \cap C([0, \infty); L^p(\Omega))$, with $u_t \in L^2(Q_\infty)$, such that:*

$$u_{\epsilon_k} \to u \quad \textit{in } C([0, T]; L^p(\Omega)) \tag{1.3.29}$$

for any $T \in (0,\infty)$;

$$u_{\epsilon_k} \rightharpoonup u \quad in \ L^r((0,T);W_0^{1,p}(\Omega)) \tag{1.3.30}$$

for any $T \in (0,\infty)$ *and every* $r \in [1,\infty)$;

$$u_{\epsilon_k} \rightharpoonup u \quad in \ W^{1,p}(Q_T) \tag{1.3.31}$$

for any $T \in (0,\infty)$;

$$u_{\epsilon_k t} \rightharpoonup u_t \quad in \ L^2(Q_\infty)\,; \tag{1.3.32}$$

$$\epsilon_k \,\nabla u_{\epsilon_k t} \to 0 \quad in \ L^2(Q_\infty)\,. \tag{1.3.33}$$

Similarly, if $N = 1$ by estimate (1.3.24) we have the following

Proposition 1.3.5. *Let* $N = 1$, $u_0 \in W_0^{1,\infty}(\Omega)$ *and assumptions* (A_1)-(A_3) *be satisfied. Let* u_ϵ *be the solution of problem* (P_ϵ) *given by Theorem 1.3.1. Then there exist a sequence* $\{u_{\epsilon_k}\} \subseteq \{u_\epsilon\}$ *and* $u \in L^\infty(\mathbb{R}_+;W_0^{1,\infty}(\Omega)) \cap H^1(Q_T) \cap C([0,\infty);L^2(\Omega))$, *with* $u_t \in L^2(Q_\infty)$, *such that:*

$$u_{\epsilon_k} \to u \quad in \ C([0,T];L^2(\Omega)) \tag{1.3.34}$$

for any $T \in (0,\infty)$;

$$u_{\epsilon_k} \rightharpoonup u \quad in \ L^r((0,T);W_0^{1,p}(\Omega)) \tag{1.3.35}$$

for any $T \in (0,\infty)$ *and every* $r \in [1,\infty)$, $p \in (1,\infty)$;

$$u_{\epsilon_k} \rightharpoonup u \quad in \ H^1(Q_T) \tag{1.3.36}$$

for any $T \in (0,\infty)$;

$$u_{\epsilon_k x} \overset{*}{\rightharpoonup} u_x \quad in \ L^\infty(Q_\infty)\,; \tag{1.3.37}$$

$$u_{\epsilon_k t} \rightharpoonup u_t \quad in \ L^2(Q_\infty)\,; \tag{1.3.38}$$

$$\epsilon \, u_{\epsilon_k xt} \to 0 \quad in \ L^2(Q_\infty)\,. \tag{1.3.39}$$

Remark 1.3.3. Let the assumptions of Proposition 1.3.5 hold if $N = 1$, or let $u_0 \in H_0^1(\Omega)$, and let assumptions (H_2)-(H_4) with $p = q = 2$ and assumption (H_5) be satisfied if $N \geq 2$ (thus in particular Proposition 1.3.4 holds with $p = 2$). Let u be the limiting function mentioned in Propositions 1.3.4 and 1.3.5. Let us show for further reference that under these assumptions the map $t \to \|u(\cdot,t)\|^2_{L^2(\Omega)}$ is nonincreasing on $\mathbb{R}_+$.

In fact, since the global solution u_ϵ of problem (P_ϵ) satisfies $u_\epsilon(\cdot,t) \in H_0^1(\Omega)$ for every $t \in \mathbb{R}_+$, from (P_ϵ) we get easily

$$\begin{aligned} \frac{d}{dt}\Big(\|u_\epsilon(\cdot,t)\|^2_{L^2(\Omega)} + \epsilon\,\|\nabla u_\epsilon(\cdot,t)\|^2_{L^2(\Omega)}\Big) \quad &= \\ -2\int_\Omega \left[\varphi(\nabla u_\epsilon)\cdot\nabla u_\epsilon\right](x,t)\,dx \leq 0 & \end{aligned} \tag{1.3.40}$$

by assumption (H_5). This implies that

$$\|u_\epsilon(\cdot,t+\tau)\|^2_{L^2(\Omega)}+\epsilon\,\|\nabla u_\epsilon(\cdot,t+\tau)\|^2_{L^2(\Omega)}\leq\|u_\epsilon(\cdot,t)\|^2_{L^2(\Omega)}+\epsilon\,\|\nabla u_\epsilon(\cdot,t)\|^2_{L^2(\Omega)}$$

for every $t\in\mathbb{R}_+$, $\tau>0$ and $\epsilon>0$. Setting $\epsilon=\epsilon_k$ in the above inequality, where $\{u_{\epsilon_k}\}\subseteq\{u_\epsilon\}$ is the subsequence mentioned in Propositions 1.3.4 and 1.3.5, and letting $\epsilon_k\to 0$ we obtain

$$\|u(\cdot,t+\tau)\|^2_{L^2(\Omega)}\leq\|u(\cdot,t)\|^2_{L^2(\Omega)}$$

for every $t\in\mathbb{R}_+$ and $\tau>0$; here we use the convergence in (1.3.29) and estimate (1.3.26) with $p=2$ if $N\geq 2$, respectively the convergence in (1.3.34) and estimate (1.3.24) if $N=1$. Hence the claim follows.

Consider the sequence $\{u_{\epsilon_k}\}$ of solutions of problems (P_{ϵ_k}) mentioned in Propositions 1.3.4-1.3.5. By inequalities (1.3.24) and (1.3.26) the sequence $\{\nabla u_{\epsilon_k}\}$ is bounded in $L^p(Q_T;\mathbb{R}^N)$ for any $T\in(0,\infty)$, with $p\in(1,2]$ if $N\geq 2$, or for any $p\in(1,\infty)$ if $N=1$. Plainly, this implies that both $\{\nabla u_{\epsilon_k}\}$ and $\{\varphi(\nabla u_{\epsilon_k})\}$ are uniformly integrable. Let $\{\tau^{\epsilon_k}\}\subseteq\mathcal{Y}(Q_T;\mathbb{R}^N)$ be the sequence of Young measures associated to $\{\nabla u_{\epsilon_k}\}$. Then by Theorem 2.5.2-$(i)$ there exist subsequences of $\{\nabla u_{\epsilon_k}\}$ and $\{\tau^{\epsilon_k}\}$, denoted for simplicity by the same notations, and a Young measure τ on $Q_T\times\mathbb{R}^N$ such that $\tau^k\to\tau$ narrowly. Moreover,

$$\varphi(\nabla u_{\epsilon_k})\rightharpoonup\varphi^*\quad\text{in } L^1(Q_T;\mathbb{R}^N)\,, \tag{1.3.41}$$

where φ^* is defined by (1.3.17) (with $\tau_{(x,t)}$ denoting the disintegration of τ). Relying on these facts we shall prove that the couple (u,τ), where u is the limiting function given by Propositions 1.3.4-1.3.5, is a global Young measure solution of problem (P). This gives the following existence result.

Theorem 1.3.6. *Let either $N=1$, $u_0\in W^{1,\infty}_0(\Omega)$ and assumptions (A_1)-(A_3) be satisfied, or let $N\geq 2$, $u_0\in W^{1,p}_0(\Omega)$ $(p\in(1,2])$ and (H_1)-(H_4) hold. Then there exists a global Young measure solution (u,τ) of problem (P), which is the limit of a subsequence $\{u_{\epsilon_k}\}$ of the family $\{u_\epsilon\}$ of solutions to the approximating problems (P_ϵ). Moreover,*
(i) if $N=1$, $u\in L^\infty(\mathbb{R}_+;W^{1,\infty}_0(\Omega))\cap H^1(Q_T)\cap C([0,\infty);L^2(\Omega))$;
(ii) if $N\geq 2$, $u\in L^\infty(\mathbb{R}_+;W^{1,p}_0(\Omega))\cap W^{1,p}(Q_T)\cap C([0,\infty);L^p(\Omega))$ $(p\in(1,2])$;
(iii) $\tau\in\mathcal{Y}(Q_\infty;\mathbb{R}^N)$.

1.3.2 Characterization of the limiting Young measure

Concerning the results stated in this subsection, we assume that the function φ changes monotonicity character a finite number of times, say $2n$ ($n\in\mathbb{N}$;

observe that this number is even by assumptions (A_2)-(A_3)). Hence there holds

$$\mathbb{R} = \left(\bigcup_{l=0}^{n} I_l \right) \bigcup \left(\bigcup_{m=1}^{n} \hat{I}_m \right) ,$$

where

$$I_0 := (-\infty, b_1] , \qquad I_l := (a_l, b_{l+1}] \quad (l = 1, \dots, n-1) , \qquad I_n := (a_n, \infty) ,$$
$$\hat{I}_m := (b_m, a_m] \quad (m = 1, \dots, n)$$

and b_l is a local maximum point, a_l a local minimum point of the graph of φ. Set $J_l := \varphi(I_l)$, $\hat{J}_m := \varphi(\hat{I}_m)$ $(l = 0, \dots, n;\ m = 1, \dots, n)$; since φ is increasing on each interval I_l and decreasing on each interval $\hat{I}_m$, there holds

$$J_0 := (-\infty, \varphi(b_1)] , \qquad J_l := (\varphi(a_l), \varphi(b_{l+1})] \quad (l = 1, \dots, n-1) ,$$
$$J_n := (\varphi(a_n), \infty) , \qquad \hat{J}_m := (\varphi(a_m), \varphi(b_m)] \quad (m = 1, \dots, n) .$$

Now define $n+1$ increasing functions $s_l : J_l \to I_l$ setting

$$(s_l \circ \varphi)(\xi) = \xi \ \text{ for any } \ \xi \in I_l ,$$
$$(\varphi \circ s_l)(\lambda) = \lambda \ \text{ for any } \ \lambda \in J_l \qquad (l = 0, \dots, n) , \tag{1.3.42}$$

and n decreasing functions $t_m : \hat{J}_m \to \hat{I}_m$ such that

$$(t_m \circ \varphi)(\xi) = \xi \ \text{ for any } \ \xi \in \hat{I}_m ,$$
$$(\varphi \circ t_m)(\lambda) = \lambda \ \text{ for any } \ \lambda \in \hat{J}_m \qquad (m = 1, \dots, n) , \tag{1.3.43}$$

Following [Pl1], we shall make use of the following assumption:

$$(C) \qquad \begin{cases} \text{The functions } s'_0, \dots, s'_n,\ t'_1, \dots, t'_n \text{ are} \\ \text{linearly independent on any open subset of } \mathbb{R} . \end{cases}$$

Let $N = 1$, $u_0 \in W_0^{1,\infty}(\Omega)$ and assumptions (A_1)-(A_3) be satisfied. Let u_ϵ be the solution of problem (P_ϵ) given by Theorem 1.3.1; set

$$v_\epsilon := u_{\epsilon x} : , \quad w_\epsilon := \varphi(v_\epsilon) + \epsilon v_{\epsilon t} \tag{1.3.44}$$

as in (1.3.21). Since $u_{\epsilon t} = w_{\epsilon x}$, estimate (1.3.24) also reads

$$\|v_\epsilon\|_{L^\infty(Q_\infty)} + \|w_{\epsilon x}\|_{L^2(Q_\infty)} + \sqrt{\epsilon}\, \|v_{\epsilon t}\|_{L^2(Q_\infty)} \le C \, \|u_0\|_{W_0^{1,\infty}(\Omega)} . \tag{1.3.45}$$

Plainly, this implies that there exists $C > 0$ (only depending on $\|u_0\|_{W_0^{1,\infty}(\Omega)}$) such that for any $\epsilon > 0$ small enough

$$\|\varphi(v_\epsilon)\|_{L^\infty(Q_\infty)} \le C , \tag{1.3.46}$$

$$\|w_\epsilon\|_{L^2(\mathbb{R}_+;H^1(\Omega))} \le C\,. \tag{1.3.47}$$

In fact, by assumption (A_1) and inequality (1.3.45) there exists $C_1 > 0$ (only depending on $\|u_0\|_{W_0^{1,\infty}(\Omega)}$) such that for any $\epsilon > 0$

$$\|\varphi(v_\epsilon)\|_{L^\infty(Q_\infty)} \le C_1\,.$$

Then by the above inequality, the definition of w_ϵ (see (1.3.44)) and inequality (1.3.45) there holds

$$\|w_\epsilon\|_{L^2(Q_\infty)} \le C_1 + \sqrt{\epsilon}\,C\,\|u_0\|_{W_0^{1,\infty}(\Omega)}\,.$$

From the above inequality and inequality (1.3.45) we obtain (1.3.46)-(1.3.47) for some constant $C \ge C_1$.

In view of inequality (1.3.47), there exist a sequence $\{w_{\epsilon_k}\} \subseteq \{w_\epsilon\}$ and $w \in L^2(\mathbb{R}_+; H^1(\Omega))$ such that

$$w_{\epsilon_k} \rightharpoonup w \quad \text{in} \quad L^2(\mathbb{R}_+; H^1(\Omega))\,. \tag{1.3.48}$$

Without loss of generality, we can assume that the convergence results of Proposition 1.3.5 hold with the same sequence of indices $\{\epsilon_k\}$. In particular, there holds (see (1.3.37))

$$v_{\epsilon_k} \overset{*}{\rightharpoonup} v \quad \text{in} \quad L^\infty(Q_\infty) \tag{1.3.49}$$

with $v := u_x$, $u \in L^\infty(\mathbb{R}_+; W_0^{1,\infty}(\Omega))$ being the limiting function mentioned in Proposition 1.3.5.

Now we can state the following result.

Theorem 1.3.7. *Let $N = 1$, $u_0 \in W_0^{1,\infty}(\Omega)$ and assumptions (A_1)-(A_3) be satisfied, and let (u, τ) be any global Young measure solution of problem (P) given by Theorem 1.3.6. Then for almost every $(x, t) \in Q_\infty$ the disintegration $\tau_{(x,t)}$ of the limiting Young measure τ satisfies the equality*

$$\tau_{(x,t)} = \sum_{l=0}^{n} c_l(x,t)\,\delta\big(\,\cdot - s_l\big(w(x,t)\big)\big) + \sum_{m=1}^{n} d_m(x,t)\,\delta\big(\,\cdot - t_m\big(w(x,t)\big)\big)\,, \tag{1.3.50}$$

where $w \in L^2(\mathbb{R}_+; H^1(\Omega)) \cap L^\infty(Q_T)$ $(T \in (0,\infty))$ is the limiting function in (1.3.48).
The coefficients c_l, d_m in (1.3.50) *are functions in $L^\infty(Q_\infty)$, such that for almost every $(x,t) \in Q_\infty$:*
(i) $0 \le c_l(x,t) \le 1$, $0 \le d_m(x,t) \le 1$ $(l = 0, \dots, n;\ m = 1, \dots, n)$;
(ii) $\sum_{l=0}^{n} c_l(x,t) + \sum_{m=1}^{n} d_m(x,t) = 1$;
(iii) $c_0(x,t) = 1$ if $w(x,t) \le A := \min\{\varphi(a_1), \dots, \varphi(a_n)\}$, $c_n(x,t) = 1$ if $w(x,t) \ge B := \max\{\varphi(b_1), \dots, \varphi(b_n)\}$.

Remark 1.3.4. Under the assumptions of Theorem 1.3.7, from equalities (1.3.15) and (1.3.50) we obtain

$$u_x(x,t) = \sum_{l=0}^{n} c_l(x,t)\, s_l\big(w(x,t)\big) + \sum_{m=1}^{n} d_m(x,t)\, t_m\big(w(x,t)\big)$$

for almost every $(x,t) \in Q_\infty$.

To prove Theorem 1.3.7 we need a similar result, concerning the limiting points of the family $\{\varphi(v_\epsilon)\}$. Since this family is uniformly bounded in $L^\infty(Q_\infty)$ (see (1.3.46)), we can consider the associated family $\{\theta^\epsilon\}$ of Young measures. By Proposition 1.2.4 there exist a sequence $\{\theta^{\epsilon_k}\}$ and a Young measure θ over $Q_\infty \times \mathbb{R}$, such that for any $T \in (0,\infty)$

$$\theta^{\epsilon_k} \to \theta \quad \text{narrowly in } Q_T \times \mathbb{R}\,. \tag{1.3.51}$$

As before, there is no loss of generality if we assume that the convergence results of Proposition 1.3.5 hold with the same sequence of indices $\{\epsilon_k\}$.

Let $\theta_{(x,t)}$ denote the disintegration of the Young measure θ. Then we have the following result.

Theorem 1.3.8. *Let the assumptions of Theorem 1.3.7 hold. Let $\theta_{(x,t)}$ be the disintegration of the limiting Young measure θ over $Q_\infty \times \mathbb{R}$ which appears in* (1.3.51)*, and let $w \in L^2(\mathbb{R}_+; H^1(\Omega)) \cap L^\infty(Q_T)$ be the limiting function in* (1.3.48)*. Then for almost every $(x,t) \in Q_\infty$*

$$\theta_{(x,t)} = \delta(\cdot - w(x,t))\,. \tag{1.3.52}$$

1.3.3 Asymptotic behaviour

Since the Young measure solution (u,τ) of problem (P) given by Theorem 1.3.6 is global, it is natural to investigate its asymptotic behaviour as $t \to \infty$. Set for almost every $(x,t) \in Q_T$ and every diverging sequence $\{t_n\} \subset \mathbb{R}_+$:

$$u_n(x,t) := u(x,t+t_n)\,, \quad \tau^n_{(x,t)} := \tau_{(x,t+t_n)}\,, \tag{1.3.53}$$

where $\tau_{(x,t)}$ denotes the disintegration of τ. Denote also $X \equiv W_0^{1,p}(\Omega)$, where $p \in (1,\infty)$ if $N = 1$, or $p \in (1,2]$ if $N \geq 2$. Then we can state the following definition.

Definition 1.3.6. *Let (u,τ) be a global Young measure solution of problem (P). A couple $(\tilde{u},\tilde{\tau})$, with $\tilde{u} \in L^\infty(\mathbb{R}_+; X) \cap C([0,\infty); L^p(\Omega))$ (where $p = 2$ if $N = 1$ and $p \in (1,2]$ if $N \geq 2$), and $\tilde{\tau} \in \mathcal{Y}(Q_\infty; \mathbb{R}^N)$ is called an ω-*limit point *of the solution (u,τ), if*

(*i*) *there exists a diverging sequence $\{t_n\} \subset \mathbb{R}_+$ such that* *(a)* $u_n \rightharpoonup \tilde{u}$ *in* $L^r((0,T);X)$ *for any* $T \in (0,\infty)$ *and every* $r \in [1,\infty)$,

(b) $\tau^n \to \tilde{\tau}$ *narrowly in* $Q_T \times \mathbb{R}^N$ *for any* $T \in (0,\infty)$;

(ii) $(\tilde{u}, \tilde{\tau})$ *is a global Young measure solution of problem* (P) *with initial data function* $\tilde{u}(\cdot, 0)$.

The set of the ω*-limit points of* (u,τ) *is called the* ω-limit set *of* (u,τ).

The following result will be proven.

Theorem 1.3.9. *Let either* $N = 1$, $u_0 \in W_0^{1,\infty}(\Omega)$ *and assumptions* (A_1)-(A_3) *be satisfied, or let* $N \geq 2$, $u_0 \in W_0^{1,p}(\Omega)$ $(p \in (1,2])$ *and* (H_1)-(H_5) *hold. Let* $(u,\tau) \in L^\infty(\mathbb{R}_+;X) \times \mathcal{Y}(Q_\infty;\mathbb{R}^N)$ *be any global Young measure solution of problem* (P), *whose existence is ensured by Theorem 1.3.6. Then:*

(i) the ω*-limit set of* (u,τ) *is nonempty;*

(ii) if $u_0 \in H_0^1(\Omega)$, *for every* ω*-limit point* $(\tilde{u},\tilde{\tau})$ *there holds*

$$\text{supp } \tilde{\tau}_{(x,t)} \subseteq \mathcal{S} := \left\{\xi \in \mathbb{R}^N \,\middle|\, \varphi(\xi)\cdot\xi = 0\right\} \text{ for almost every } (x,t) \in Q_\infty\,, \tag{1.3.54}$$

where $\tilde{\tau}_{(x,t)} \in \mathcal{P}(\mathbb{R}^N)$ *denotes the disintegrations of* $\tilde{\tau}$.

Set

$$\mathcal{S}_0 := \left\{\xi \in \mathbb{R}^N \,\middle|\, \varphi_i(\xi) = 0\,, i = 1,\dots,N\right\}. \tag{1.3.55}$$

Clearly, there holds $\mathcal{S}_0 \subseteq \mathcal{S}$ (see (1.3.54)). If the reverse inclusion holds, we have the following result.

Theorem 1.3.10. *Let the assumptions of Theorem 1.3.9 be satisfied. Suppose that* $\mathcal{S} = \mathcal{S}_0$, *the sets* $\mathcal{S}$, $\mathcal{S}_0$ *being defined in* (1.3.54) *and* (1.3.55). *Then every* ω*-limit point* $(\tilde{u},\tilde{\tau})$ *of* (u,τ) *is a Young measure equilibrium solution of problem* (P).

Remark 1.3.5. Let $u_0 \in H_0^1(\Omega)$ and the assumptions of Theorem 1.3.10 be satisfied. As in [Sl], let us mention the following facts.
(i) If $\mathcal{S}_0 \subseteq K := [a_1, c_1] \times \cdots \times [a_N, c_N]$, then by (1.3.18) and (1.3.54)

$$\nabla \tilde{u}(x) = \int_K \xi \, d\tilde{\tau}_x(\xi)\,,$$

whence

$$a_i \leq \frac{\partial \tilde{u}}{\partial x_i} \leq c_i \text{ a.e. in } \Omega \qquad (i = 1,\dots,N)\,.$$

(ii) If $\mathcal{S}_0 \subseteq \mathbb{R}_+^N$, then by *(i)* above

$$\frac{\partial \tilde{u}}{\partial x_i} \geq 0 \text{ a.e. in } \Omega \qquad (i = 1,\dots,N)\,.$$

Similarly, $\mathcal{S}_0 \subseteq \mathbb{R}^N_- := \big((-\infty, 0]\big)^N$, there holds

$$\frac{\partial \tilde{u}}{\partial x_i} \leq 0 \ \text{ a.e. in } \ \Omega \qquad (i = 1, \dots, N)\,.$$

By a standard argument, since $\tilde{u} \in H^1_0(\Omega)$, it follows in both cases that $\nabla \tilde{u} = 0$, thus $\tilde{u} = 0$ *a.e.* in Ω. Therefore in this cases the ω-limit set of (u, τ) is $\{(0, \delta_0)\}$, δ_0 denoting the Dirac measure concentrated at the origin.

1.4 Existence

Let us first prove Theorem 1.3.1 when $N = 1$. This requires some preliminary steps, the first one being the following local existence and uniqueness result.

Lemma 1.4.1. *Let $N = 1$, $u_0 \in W^{1,\infty}_0(\Omega)$ and assumption (A_1) be satisfied. Then for any $\epsilon > 0$ there exists $T_\epsilon > 0$ such that problem* (1.3.22) *has a unique solution u_ϵ in Q_{T_ϵ}.*

Proof. Consider the problem

$$\begin{cases} v_t = w_{xx} & \text{in } Q_T \\ w_x = 0 & \text{in } \partial\Omega \times (0, T) \\ v = v_0 := u_0' & \text{in } \Omega \times \{0\}\,, \end{cases} \tag{1.4.1}$$

where $w := \varphi(v) + \epsilon v_t$ (see (1.3.21)). Then for any $\epsilon > 0$ there exist $T_\epsilon > 0$ and a unique function $v_\epsilon \in C^1([0, T); L^\infty(\Omega))$, with $w_\epsilon \in C([0, T); W^{2,\infty}(\Omega) \cap C(\bar{\Omega}))$, $w_{\epsilon xx} \in C([0, T); L^\infty(\Omega))$, which satisfies (1.4.1) in Q_{T_ϵ} in classical sense (see [NP, Theorem 2.1]). Suppose $\Omega \equiv (a, b)$ for simplicity; defining

$$u_\epsilon(x, t) := \int_a^x v_\epsilon(y, t)\, dy \qquad ((x, t) \in \Omega \times [0, T_\epsilon))$$

the conclusion easily follows. □

To proceed we need a priori estimates of the local solution u_ϵ of (1.3.22) given by Lemma 1.4.1. Following [NP], set for any $g \in C^1(\mathbb{R})$, $g' \geq 0$:

$$G(v) := \int_0^v g(\varphi(s))\, ds + k \qquad (k \in \mathbb{R})\,. \tag{1.4.2}$$

In Q_{T_ϵ} we have:

$$[G(u_{\epsilon x})]_t = g(\varphi(u_{\epsilon x}))\, u_{\epsilon xt} = g(w_\epsilon)\, w_{\epsilon xx} + [g(\varphi(u_{\epsilon x})) - g(w_\epsilon)] w_{\epsilon xx} =$$

$$= [g(w_\epsilon) w_{\epsilon x}]_x - g'(w_\epsilon)|w_{\epsilon x}|^2 + \underbrace{[g(\varphi(u_{\epsilon x})) - g(w_\epsilon)]\, \frac{w_\epsilon - \varphi(u_{\epsilon x})}{\epsilon}}_{\leq 0}\,. \tag{1.4.3}$$

Then integrating in Ω we obtain:

$$\frac{d}{dt}\int_\Omega G(u_{\epsilon x}(x,t))\,dx \le 0 \quad \text{in } (0,T_\epsilon)\,. \tag{1.4.4}$$

The above inequality is crucial to prove the existence of positively invariant regions for problem (1.3.22). This is the content of the following proposition (see [NP, Proposition 2.7]).

Proposition 1.4.2. *Let*

$$\varphi(v_1) \le \varphi(v) \le \varphi(v_2) \qquad \textit{for any } v \in [v_1, v_2]\,, \tag{1.4.5}$$

and let $u_0'(x) \in [v_1,v_2]$ *for any* $x \in \Omega$. *Then* $u_{\epsilon x}(x,t) \in [v_1,v_2]$ *for any* $(x,t) \in Q_{T_\epsilon}$.

Let us now prove Theorem 1.3.1 in the case $N = 1$.

Proposition 1.4.3. *Let* $N = 1$, $u_0 \in W_0^{1,\infty}(\Omega)$ *and assumptions* (A_1)-(A_3) *be satisfied. Then for any* $\epsilon > 0$ *there exists a unique global solution* u_ϵ *of problem* (P_ϵ).

Proof. By assumptions (A_2)-(A_3) there holds $\varphi(v) \to \pm\infty$ as $v \to \pm\infty$, hence there exists $C > ||u_0'||_\infty$ such that inequality (1.4.5) holds with $[v_1,v_2] = [-C,C]$. Then by Proposition 1.4.2 there holds

$$\|u_\epsilon(\cdot,t)\|_{W_0^{1,\infty}(\Omega)} \le C \quad \text{for any } t \in (0,T_\epsilon)\,. \tag{1.4.6}$$

Hence by Lemma 1.4.1 and standard prolongation arguments the conclusion follows. □

Proof of Proposition 1.3.2. Choosing $g(z) = z$ in (1.4.3) gives:

$$[G(u_{\epsilon x})]_t = w_\epsilon w_{\epsilon xx} - \frac{|w_\epsilon - \varphi(u_{\epsilon x})|^2}{\epsilon} = w_\epsilon w_{\epsilon xx} - \epsilon|u_{\epsilon xt}|^2\,,$$

(see (1.3.21)), whence plainly:

$$\iint_{Q_\infty}\{|w_{\epsilon x}|^2 + \epsilon|u_{\epsilon xt}|^2\}dxdt \le \int_\Omega G(u_0)\,dx\,. \tag{1.4.7}$$

On the other hand, since the solution is global, from (1.4.6) we obtain the a priori estimate

$$\|u_\epsilon\|_{L^\infty(\mathbb{R}_+;W_0^{1,\infty}(\Omega))} \le C\,. \tag{1.4.8}$$

Then from (1.4.7)-(1.4.8) the result follows. □

Let us now address the proof of Theorem 1.3.1 for $N \ge 2$. Denote by $(I-\epsilon\Delta)^{-1}$ $(\epsilon > 0)$ the operator

$$(I-\epsilon\Delta)^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)\,, \quad (I-\epsilon\Delta)^{-1}z := v \text{ for any } z \in H^{-1}(\Omega)\,,$$

where $v \in H^1_0(\Omega)$ is the unique solution of the elliptic problem

$$\begin{cases} -\epsilon\Delta v + v = z & \text{in } \Omega \\ v = 0 & \text{in } \partial\Omega \end{cases} \tag{1.4.9}$$

for any $z \in H^{-1}(\Omega)$.

Proposition 1.4.4. *Let assumption (H_1) hold. Then the operator*

$$\mathcal{L} : H^1_0(\Omega) \to H^1_0(\Omega)\,, \quad \mathcal{L}(u) := (I - \epsilon\Delta)^{-1}\,\nabla \cdot [\varphi(\nabla u)] \qquad (u \in H^1_0(\Omega)) \tag{1.4.10}$$

is Lipschitz continuous.

Proof. Since by assumption $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ is Lipschitz continuous, there holds

$$|\varphi(\nabla u)| \leq L\,|\nabla u| + |\varphi(0)| \quad \text{in } \Omega$$

for any $u \in H^1_0(\Omega)$, where $L > 0$ is the Lipschitz constant in (H_1), thus $\varphi(\nabla u) \in [L^2(\Omega)]^N$. Hence $\nabla \cdot [\varphi(\nabla w)] \in H^{-1}(\Omega)$, thus the elliptic problem

$$\begin{cases} -\epsilon\Delta v + v = \nabla \cdot [\varphi(\nabla u)] & \text{in } \Omega \\ v = 0 & \text{in } \partial\Omega \end{cases}$$

admits a unique solution $v \in H^1_0(\Omega)$, and there holds $v = \mathcal{L}(u)$.

Now set $v_1 := \mathcal{L}(u_1)$, $v_2 := \mathcal{L}(u_2)$ for any $u_1, u_2 \in H^1_0(\Omega)$. Then the difference $v_1 - v_2$ satisfies

$$\begin{cases} -\epsilon\Delta(v_1 - v_2) + (v_1 - v_2) = \nabla \cdot [\varphi(\nabla u_1) - \varphi(\nabla u_2)] & \text{in } \Omega \\ v_1 - v_2 = 0 & \text{in } \partial\Omega \end{cases} \tag{1.4.11}$$

in $H^{-1}(\Omega)$. Using $v_1 - v_2 \in H^1_0(\Omega)$ as test function, from (1.4.11) we obtain

$$\begin{aligned} &\epsilon \int_\Omega |\nabla(v_1 - v_2)|^2 dx + \int_\Omega (v_1 - v_2)^2 dx = \\ = \;& \int_\Omega [\varphi(\nabla u_1) - \varphi(\nabla u_2)] \cdot \nabla(v_1 - v_2) \leq \\ \leq \;& L \int_\Omega |\nabla(u_1 - u_2)|\,|\nabla(v_1 - v_2)|\,dx \leq \\ \leq \;& L \left(\int_\Omega |\nabla(u_1 - u_2)|^2 dx\right)^{1/2} \left(\int_\Omega |\nabla(v_1 - v_2)|^2 dx\right)^{1/2}, \end{aligned}$$

whence

$$\|v_1 - v_2\|_{H^1_0(\Omega)} \leq \epsilon^{-1} L\,\|u_1 - u_2\|_{H^1_0(\Omega)}. \tag{1.4.12}$$

This completes the proof. □

Now we can complete the proof of Theorem 1.3.1.

Proof of Theorem 1.3.1. If $N = 1$, the result follows by Proposition 1.4.3. If $N \geq 2$, by Proposition 1.4.4, for any $u_0 \in H_0^1(\Omega)$ and $\epsilon > 0$ there exists a unique solution $u_\epsilon \in C^1([0,\infty); H_0^1(\Omega))$ of the abstract Cauchy problem

$$\begin{cases} u_t = \mathcal{L}(u) & \text{in } \mathbb{R}_+ \,, \\ u(0) = u_0 \,. \end{cases} \tag{1.4.13}$$

Therefore, u_ϵ satisfies problem (P_ϵ) in Q_T (for any $T \in (0,\infty)$) in the sense of $C^1([0,T]; H^{-1}(\Omega))$, whence equality (1.3.23) follows for any $t \in (0,T)$ and any $\rho \in H_0^1(\Omega)$. Since $T \in (0,\infty)$ is arbitrary, u_ϵ is a global solution. Hence the conclusion follows. □

Proof of Proposition 1.3.3. If $\|u_0\|_{W_0^{1,p}(\Omega)} = 0$, there holds $u_0 = 0$ in Ω, thus the unique solution of problem (P_ϵ) is the trivial one. Therefore, in this case inequality (1.3.26) is trivially satisfied. Let us address the case $\|u_0\|_{W_0^{1,p}(\Omega)} > 0$.

Since $u_{\epsilon t}(\cdot,t) \in H_0^1(\Omega)$ for any $t \in \mathbb{R}_+$, we can choose $\rho = u_{\epsilon t}(\cdot,t)$ in (1.3.23). Then, integrating over $(0,t)$, by assumption (H_3) we obtain

$$\begin{aligned} &\int_0^t\!\!\int_\Omega u_{\epsilon t}^2(x,s)\,dxds = \\ = \; & -\int_0^t \left(\frac{d}{ds}\int_\Omega \left[\Phi(\nabla u_\epsilon)\right](x,s)\,dx\right) ds \; - \; \epsilon \int_0^t\!\!\int_\Omega |\nabla u_{\epsilon t}|^2(x,s)\,dxds \,, \end{aligned}$$

whence for any $t > 0$,

$$\begin{aligned} \int_\Omega \left[\Phi(\nabla u_\epsilon)\right](x,t)\,dx \;\; + \;\; & \int_0^t\!\!\int_\Omega \left\{u_{\epsilon t}^2 + \epsilon|\nabla u_{\epsilon t}|^2\right\}(x,s)\,dxds = \\ = \;\; & \int_\Omega \left[\Phi(\nabla u_{0\epsilon})\right](x)\,dx \,. \end{aligned} \tag{1.4.14}$$

On the other hand, by the first inequality in (1.1.1) and the following remark there holds

$$\begin{aligned} & C_1 \int_\Omega |\nabla u_\epsilon|^p(x,t)\,dx \; = \\ = \;\; & C_1 \int_{\{|\nabla u_\epsilon| > \xi_0\}} |\nabla u_\epsilon|^p(x,t)\,dx + C_1 \int_{\{|\nabla u_\epsilon| \leq \xi_0\}} |\nabla u_\epsilon|^p(x,t)\,dx \; \leq \\ \leq \;\; & \int_{\{|\nabla u_\epsilon| > \xi_0\}} \left[\Phi(\nabla u_\epsilon)\right](x,t)\,dx + C_1 \xi_0^p\,|\Omega| \; = \\ = \;\; & \int_\Omega \left[\Phi(\nabla u_\epsilon)\right](x,t)\,dx - \int_{\{|\nabla u_\epsilon| \leq \xi_0\}} \left[\Phi(\nabla u_\epsilon)\right](x,t)\,dx + C_1 \xi_0^p\,|\Omega| \,. \end{aligned} \tag{1.4.15}$$

Then by equality (1.4.14) we have

$$
\begin{aligned}
&C_1 \int_\Omega |\nabla u_\epsilon|^p(x,t)\,dx + \int_0^t\!\int_\Omega \left\{u_{\epsilon t}^2 + \epsilon|\nabla u_{\epsilon t}|^2\right\}(x,s)\,dxds \,\le \qquad (1.4.16)\\
\le\;& \int_\Omega \left[\Phi(\nabla u_{0\epsilon})\right](x)\,dx - \int_{\{|\nabla u_\epsilon|\le \xi_0\}} \left[\Phi(\nabla u_\epsilon)\right](x,t)\,dx + C_1\xi_0^p\,|\Omega| \,\le\\
\le\;& \int_\Omega \left[\Phi(\nabla u_{0\epsilon})\right](x)\,dx \,+\, K_1
\end{aligned}
$$

with some constant $K_1 > 0$, only depending on ξ_0 and $|\Omega|$ (here use of the continuity of Φ has been made).

Arguing as for (1.4.15), by the second inequality in (1.1.1) and the inequality in (1.3.25) we get

$$
\begin{aligned}
&\int_\Omega \left[\Phi(\nabla u_{0\epsilon})\right](x)\,dx \,= \qquad (1.4.17)\\
=\;& \int_{\{|\nabla u_{0\epsilon}|>\xi_0\}} \left[\Phi(\nabla u_{0\epsilon})\right](x)\,dx + \int_{\{|\nabla u_{0\epsilon}|\le\xi_0\}} \left[\Phi(\nabla u_{0\epsilon})\right](x)\,dx \,\le\\
\le\;& C_3 \int_{\{|\nabla u_{0\epsilon}|>\xi_0\}} |\nabla u_{0\epsilon}|^p(x)\,dx + \int_{\{|\nabla u_{0\epsilon}|\le\xi_0\}} \left[\Phi(\nabla u_{0\epsilon})\right](x)\,dx \,\le\\
\le\;& C_3 \int_\Omega |\nabla u_0|^p(x)\,dx \,+\, K_2
\end{aligned}
$$

with some constant $K_2 > 0$, only depending on ξ_0. Since $u \to \||\nabla u|\|_{L^p(\Omega)}$ is an equivalent norm on $W_0^{1,p}(\Omega)$, from (1.4.16)-(1.4.17) we obtain for some constant $K_3 > 0$

$$
\begin{aligned}
&\|u_\epsilon\|^p_{L^\infty(\mathbb{R}_+;W_0^{1,p}(\Omega))} + \|u_{\epsilon t}\|^2_{L^2(Q_\infty)} + \epsilon\,\|\nabla u_{\epsilon t}\|^2_{L^2(Q_\infty)} \,\le \qquad (1.4.18)\\
\le\;& \frac{1}{\min\{1, C_1K_3\}}\left(C_3\|u_0\|^p_{W_0^{1,p}(\Omega)} \,+\, K_1 \,+\, K_2\right) =: C_4\|u_0\|^p_{W_0^{1,p}(\Omega)} \,+\, C_5\,,
\end{aligned}
$$

whence

$$
\|u_\epsilon\|_{L^\infty(\mathbb{R}_+;W_0^{1,p}(\Omega))} + \|u_{\epsilon t}\|_{L^2(Q_\infty)} + \epsilon\,\|\nabla u_{\epsilon t}\|_{L^2(Q_\infty)} \le C_4\|u_0\|^p_{W_0^{1,p}(\Omega)} \,+\, C_5 \,+\, 3\,.
$$

Then defining

$$
C := C_4\|u_0\|^{p-1}_{W_0^{1,p}(\Omega)} \,+\, \frac{C_5+3}{\|u_0\|_{W_0^{1,p}(\Omega)}}
$$

we obtain inequality (1.3.26). This completes the proof. □

Proof of Proposition 1.3.4. Concerning (1.3.29), observe that by inequality (1.3.27) the family $\{u_\epsilon\}$ is bounded in $W^{1,p}(Q_T)$, hence there exists a sequence $\{u_{\epsilon_k}\} \subseteq \{u_\epsilon\}$ (possibly depending on $T \in (0,\infty)$) which strongly converges in $L^p(Q_T)$. Let us show that $\{u_{\epsilon_k}\}$ is a Cauchy sequence in $C([0,T];L^p(\Omega))$ for any $T\in(0,\infty)$.

In fact, for any $t\in[0,T]$ there holds

$$\begin{aligned}
&\int_\Omega |u_{\epsilon_k}-u_{\epsilon_m}|^p(x,t)\,dx-\int_\Omega |u_{0\epsilon_k}-u_{0\epsilon_m}|^p(x)\,dx\le \\
&\le p\int_0^t\!\!\int_\Omega \left[|u_{\epsilon_k}-u_{\epsilon_m}|^{p-1}\,|u_{\epsilon_k t}-u_{\epsilon_m t}|\right](x,s)\,dxds\le \\
&\le p\left(\int_0^t\!\!\int_\Omega |u_{\epsilon_k}-u_{\epsilon_m}|^p dxds\right)^{1-\frac{1}{p}}\left(\int_0^t\!\!\int_\Omega |u_{\epsilon_k t}-u_{\epsilon_m t}|^p dxds\right)^{\frac{1}{p}}\le \\
&\le p\,|Q_T|^{\frac{2-p}{2p}}\,\|u_{\epsilon_k}-u_{\epsilon_m}\|^{p-1}_{L^p(Q_T)}\|u_{\epsilon_k t}-u_{\epsilon_m t}\|_{L^2(Q_T)}\,,
\end{aligned} \tag{1.4.19}$$

since $p\in(1,2]$. Then by inequality (1.3.26) there holds

$$\begin{aligned}
\|u_{\epsilon_k}-u_{\epsilon_m}\|_{C([0,T];L^p(\Omega))} &\le K\|u_{\epsilon_k}-u_{\epsilon_m}\|^{1-\frac{1}{p}}_{L^p(Q_T)}+ \\
&+ \|u_{0\epsilon_k}-u_{0\epsilon_m}\|_{L^p(\Omega)}\,,
\end{aligned} \tag{1.4.20}$$

where

$$K:=2Cp\,|Q_T|^{\frac{2-p}{2p}}\,\|u_0\|_{W_0^{1,p}(\Omega)}\,.$$

Then by (1.3.25) and (1.4.20) the claim follows. Hence by a diagonal argument there exist a sequence $\{u_{\epsilon_k}\}\subseteq\{u_\epsilon\}$ and $u\in C([0,\infty);L^p(\Omega))$ such that (1.3.29) holds for any $T\in(0,\infty)$.

Concerning the convergence in (1.3.30), observe that by (1.3.26) there holds

$$\|u_{\epsilon_k}\|_{L^r((0,T);W_0^{1,p}(\Omega))}\le C\,T^{\frac{1}{r}}\|u_0\|_{W_0^{1,p}(\Omega)} \tag{1.4.21}$$

for any $T\in(0,\infty)$ and every $r\in[1,\infty)$. Hence by a diagonal argument there exist a subsequence of $\{u_{\epsilon_k}\}$, denoted again by $\{u_{\epsilon_k}\}$ for simplicity, and a function $u:\mathbb{R}_+\to W_0^{1,p}(\Omega)$ such that (1.3.30) holds for any $T\in(0,\infty)$ and every $r\in[1,\infty)$. To prove that $u\in L^\infty(\mathbb{R}_+;W_0^{1,p}(\Omega))$, observe that by (1.3.26)

$$\|u_{\epsilon_k}(\cdot,t)\|_{W_0^{1,p}(\Omega)}\le C\,\|u_0\|_{W_0^{1,p}(\Omega)}\ \text{ for any }\ t\in\mathbb{R}_+\,. \tag{1.4.22}$$

Then there exist a subsequence of $\{u_{\epsilon_k}(\cdot,t)\}$ (possibly depending on t), denoted again by $\{u_{\epsilon_k}(\cdot,t)\}$ for simplicity, and a function $f_t\in W_0^{1,p}(\Omega)$ such that

$$u_{\epsilon_k}(\cdot,t)\rightharpoonup f_t\quad\text{in }\ W_0^{1,p}(\Omega)\,, \tag{1.4.23}$$

thus by inequality (1.4.22) and the lower semicontinuity of the norm

$$\|f_t|_{W_0^{1,p}(\Omega)}\le C\,\|u_0\|_{W_0^{1,p}(\Omega)}\ \text{ for any }\ t\in\mathbb{R}_+\,. \tag{1.4.24}$$

On the other hand, by (1.3.29) and (1.4.23) there holds $f_t=u(\cdot,t)$. Therefore by (1.4.24) we obtain that $u\in L^\infty(\mathbb{R}_+;W_0^{1,p}(\Omega))$, as asserted.

The convergences in (1.3.32)-(1.3.33) follow immediately from (1.3.26), since both sequences $\{u_{\epsilon_k t}\}$ and $\{\sqrt{\epsilon_k}\,\nabla u_{\epsilon_k t}\}$ belong to a bounded subset of $L^2(Q_\infty)$. This completes the proof. □

Proof of Proposition 1.3.5. The proof of (1.3.34) is the same of (1.3.29), using inequality (1.3.28) instead of (1.3.27). Concerning (1.3.35), by (1.3.24) there holds

$$\|u_{\epsilon_k}\|_{L^r((0,T);W_0^{1,p}(\Omega))} \leq C\,T^{\frac{1}{r}}\,|\,\Omega\,|^{\frac{1}{p}}\,\|u_0\|_{W_0^{1,\infty}(\Omega)} \tag{1.4.25}$$

for any $T \in (0,\infty)$ and every $r \in [1,\infty)$, $p \in (1,\infty)$. Hence by a diagonal argument there exist a subsequence of $\{u_{\epsilon_k}\}$, denoted again by $\{u_{\epsilon_k}\}$, and a function $u : \mathbb{R}_+ \to W_0^{1,p}(\Omega)$ such that (1.3.35) holds. To prove that $u \in L^\infty(\mathbb{R}_+; W_0^{1,\infty}(\Omega)) \cap C([0,\infty); L^2(\Omega))$ we argue as in the proof of Proposition 1.3.4. As for (1.3.37), suffice it to observe that inequality (1.3.24) implies

$$\|u_{\epsilon x}\|_{L^\infty(Q_\infty)} \leq C\|u_0\|_{W_0^{1,\infty}(\Omega)}\,. \tag{1.4.26}$$

The proof of (1.3.38)-(1.3.39) is the same of (1.3.32)-(1.3.33), thus the result follows. □

Proof of Theorem 1.3.6. Let us first assume $N \geq 2$. Consider the sequence $\{u_{\epsilon_k}\}$ and the function u mentioned in Proposition 1.3.4, and the weak formulation of problem (P_{ϵ_k}), namely

$$\int_0^t\!\!\int_\Omega \left[u_{\epsilon_k}\,\zeta_s - \varphi(\nabla u_{\epsilon_k})\cdot\nabla\zeta - \epsilon_k \nabla u_{\epsilon_k s}\cdot\nabla\zeta\right](x,s)\,dxds = \tag{1.4.27}$$
$$= \int_\Omega u_{\epsilon_k}(x,t)\zeta(x,t)\,dx - \int_\Omega u_{0\epsilon_k}(x)\,\zeta(x,0)\,dx$$

for any $t > 0$ and $\zeta \in C^1([0,\infty); C_c^1(\Omega))$. By (1.3.29), as $\epsilon_k \to 0$ we get

$$\int_0^t\!\!\int_\Omega u_{\epsilon_k}(x,s)\zeta_s(x,s)\,dxds \to \int_0^t\!\!\int_\Omega u(x,s)\zeta_s(x,s)\,dxds\,, \tag{1.4.28}$$

$$\int_\Omega u_{\epsilon_k}(x,t)\zeta(x,t)\,dx \to \int_\Omega u(x,t)\zeta(x,t)\,dx\,, \tag{1.4.29}$$

whereas by (1.3.33) and (1.3.25) we have

$$\epsilon_k \int_0^t\!\!\int_\Omega \nabla u_{\epsilon_k s}\cdot\nabla\zeta\,dxds \to 0\,, \tag{1.4.30}$$

respectively

$$\int_\Omega u_{0\epsilon_k}(x)\zeta(x,0)\,dx \to \int_\Omega u_0(x)\zeta(x,0)\,dx\,. \tag{1.4.31}$$

Let us show that the sequence $\{\nabla u_{\epsilon_k}\}$ is uniformly integrable; then the same holds for the sequence $\{\varphi(\nabla u_{\epsilon_k})\}$, since by assumption (H_1)

$$|\varphi(\nabla u_{\epsilon_k})| \leq L\,|\nabla u_{\epsilon_k}| + |\varphi(0)|\,.$$

In fact, by inequality (1.3.26) the sequence $\{|\nabla u_{\epsilon_k}|\}$ is uniformly bounded in the Lebesgue space $L^p(Q_T)$ with $p \in (1,2]$, thus in $L^1(Q_T)$ for any $T \in (0,\infty)$. Moreover, by (1.3.26) for any Borel set $E \subseteq Q_T$ we have

$$\begin{aligned} \iint_E |\nabla u_{\epsilon_k}|(x,t)\,dxdt &\leq \left(\iint_E |\nabla u_{\epsilon_k}|^p(x,t)\,dxdt\right)^{\frac{1}{p}} |E|^{1-\frac{1}{p}} \\ &\leq C\,\|u_0\|_{W_0^{1,p}(\Omega)} T^{\frac{1}{p}} |E|^{1-\frac{1}{p}}\,, \end{aligned} \tag{1.4.32}$$

whence the claim immediately follows.

Then by Theorem 2.5.2-(i) and a standard diagonal argument there exist a subsequence of $\{\nabla u_{\epsilon_k}\}$, denoted again by $\{\nabla u_{\epsilon_k}\}$ for simplicity, with associated Young measures $\{\tau^{\epsilon_k}\}$, and a measure $\tau \in \mathcal{M}(Q_\infty \times \mathbb{R}^N)$, with $\tau \in \mathcal{Y}(Q_T;\mathbb{R}^N)$ for any $T \in (0,\infty)$, such that $\tau^{\epsilon_k} \to \tau$ narrowly in $Q_T \times \mathbb{R}^N$. Moreover, we can make use of Theorem 2.5.2-(ii) in $Q_T \times \mathbb{R}^N$ for any $T \in (0,\infty)$, choosing both $\rho = \varphi$ and $\rho = \rho_i$, where $\rho_i(\xi) := \xi_i$ ($i = 1,\dots,N$; see Remark 1.2.3). The first choice gives (1.3.41) with φ^* defined by (1.3.17), $\tau_{(x,t)}$ denoting the disintegration of τ. Then letting $\epsilon_k \to 0$ in (1.4.27), by (1.4.28)-(1.4.31) and (1.3.41) we obtain equality (1.3.16). By the second choice $\rho = \rho_i$ we have that

$$\left(\nabla u_{\epsilon_k}\right)_i \rightharpoonup \left\langle \tau_{(x,t)}, \rho_i \right\rangle_{\mathbb{R}^N} = \int_{\mathbb{R}^N} \xi_i \, d\tau_{(x,t)}(\xi) \quad \text{in } L^1(Q_T) \qquad (i = 1,\dots,N)\,,$$

which together with the convergence in (1.3.35) implies equality (1.3.15).

Therefore, the couple (u,τ) is a Young measure solution of problem (P) in Q_T. It follows from Proposition 1.3.4 that $u \in L^\infty(\mathbb{R}_+; W_0^{1,p}(\Omega)) \cap C([0,\infty); L^p(\Omega))$. To prove that $\tau \in \mathcal{Y}(Q_\infty;\mathbb{R}^N)$ observe that, since $\tau \in \mathcal{Y}(Q_T;\mathbb{R}^N)$ for any $T \in (0,\infty)$, by elementary properties of measures we have

$$\begin{aligned} \lambda_{N+1}(E) &= \lim_{k\to\infty} \lambda_{N+1}(E \cap Q_k) = \\ &= \lim_{k\to\infty} \tau((E \cap Q_k) \times \mathbb{R}^N) = \tau(E \times \mathbb{R}^N) \end{aligned}$$

for any Lebesgue measurable set $E \subseteq Q_\infty$, whence the claim follows.

Since (u,τ) is a Young measure solution of problem (P) in Q_T for any $T \in (0,\infty)$, it is a global solution; hence the result follows in the case $N \geq 2$. The proof when $N = 1$ is the same, using inequality (1.3.24) and Proposition 1.3.5 instead of (1.3.26) and Proposition 1.3.4, respectively. This completes the proof. □

Remark 1.4.1. The argument used in the proof of Theorem 1.3.6 allows us to show that, if $N \geq 2$ and assumption (H_2) is satisfied, every function φ_i^* defined by (1.3.17) ($i = 1,\dots,N$) belongs to $L^r(Q_T)$ for any $T \in (0,\infty)$ and $r = \frac{p}{p-1}$, with $p \in (1,2]$ (clearly if $N = 1$ we have $\varphi^* \in L^\infty(Q_\infty)$ by the uniform estimate (1.3.24)).

In fact, let $T \in (0,\infty)$ be fixed. By (H_2) there exists $M>0$ such that

$$|\varphi_i^*|(x,t) \le M \int_{\mathbb{R}^N} \left(1+|\xi|^{p-1}\right)\, d\tau_{(x,t)}(\xi) \qquad (i=1,\dots,N) \tag{1.4.33}$$

for almost every $(x,t) \in Q_T$. Then by (1.4.33) and Jensen inequality there holds

$$|\varphi_i^*|^r \;\le \bar{M} \int_{\mathbb{R}^N} (1+|\xi|^p)\; d\tau_{(x,t)}(\xi) \qquad (i=1,\dots,N)\,, \tag{1.4.34}$$

with some constant $\bar{M}>0$.

On the other hand, let $\{u_{\epsilon_m}\}$ be the sequence mentioned in Proposition 1.3.4. For every $j \in \mathbb{N}$, let $\eta_j \in C_c^\infty(\mathbb{R})$ be any function such that $0 \le \eta_j \le 1$ in $\mathbb{R}$, $\eta_j(s)=0$ if $|s| \ge j$.

Arguing as in the proof of Theorem 1.3.6 (see (1.4.32)), it is easily seen that for every $j \in \mathbb{N}$ the the sequence $\{\eta_j(|\nabla u_{\epsilon_m}|)\,|\nabla u_{\epsilon_m}|^p\}$ is uniformly bounded in Q_T, hence uniformly integrable. Then by Theorem 2.5.2-(ii) there exists a subsequence $\{\eta_j(|\nabla u_{\epsilon_k}|)\,|\nabla u_{\epsilon_k}|^p\}$ such that

$$\eta_j(|\nabla u_{\epsilon_k}|)|\nabla u_{\epsilon_k}|^p \rightharpoonup f_j^* \quad \text{in } L^1(Q_T)\,, \tag{1.4.35}$$

where

$$f_j^*(x,t) := \int_{\mathbb{R}^N} \eta_j(|\xi|)\,|\xi|^p\, d\tau_{(x,t)}(\xi) \quad \text{for } a.e.\ (x,t) \in Q_T\,. \tag{1.4.36}$$

In particular, we have

$$\begin{aligned} &\iint_{Q_T} dxdt \int_{\mathbb{R}^N} (1+\eta_j(|\xi|)\,|\xi|^p)\; d\tau_{(x,t)}(\xi) \;= \\ &= \lim_{k\to\infty} \iint_{Q_T} (1+\eta_j(|\nabla u_{\epsilon_k}|)\,|\nabla u_{\epsilon_k}|^p)\; dxdt \le \\ &\le \lim_{k\to\infty} \iint_{Q_T} (1+|\nabla u_{\epsilon_k}|^p)\; dxdt \le C, \end{aligned} \tag{1.4.37}$$

where the constant $C>0$ is independent of j (see (1.3.26)). Since in the limit as $j \to \infty$ there holds

$$\eta_j(|\xi|)\,|\xi|^p \to |\xi|^p$$

for every $\xi \in \mathbb{R}^N$, from (1.4.34), (1.4.37) and the Fatou Lemma the claim follows.

1.5 Characterization of the limiting Young measure

Let χ_{I_l} and $\chi_{\hat{I}_m}$ be the characteristic functions of the intervals I_l, respectively $\hat{I}_m$ ($l = 0, \dots, n$; $m = 1, \dots, n$) introduced in Subsection 1.3.2; observe that

$$\sum_{l=0}^{n} \chi_{I_l} + \sum_{m=1}^{n} \chi_{\hat{I}_m} = 1 \quad \text{on } \mathbb{R}\,. \tag{1.5.38}$$

For any $f \in C(\mathbb{R})$ and almost every $(x,t) \in Q_T$ ($T \in (0,\infty)$) set

$$\left\langle \sigma^{(l)}_{(x,t)}, f \right\rangle_{\mathbb{R}} = \int_{\mathbb{R}} f(\lambda)\, d\sigma^{(l)}_{(x,t)}(\lambda) :=$$

$$:= \int_{\mathbb{R}} (f \circ \varphi)(\xi) \chi_{I_l}(\xi)\, d\tau_{(x,t)}(\xi) \quad (l = 0, \dots, n)\,; \tag{1.5.39}$$

$$\left\langle \hat{\sigma}^{(m)}_{(x,t)}, f \right\rangle_{\mathbb{R}} = \int_{\mathbb{R}} f(\lambda)\, d\hat{\sigma}^{(m)}_{(x,t)}(\lambda) :=$$

$$:= \int_{\mathbb{R}} (f \circ \varphi)(\xi) \chi_{\hat{I}_m}(\xi)\, d\tau_{(x,t)}(\xi) \quad (m = 1, \dots, n)\,; \tag{1.5.40}$$

$$\sigma_{(x,t)} := \sum_{l=0}^{n} \sigma^{(l)}_{(x,t)} + \sum_{m=1}^{n} \hat{\sigma}^{(m)}_{(x,t)}\,. \tag{1.5.41}$$

Since $\tau_{(x,t)} \in \mathcal{P}(\mathbb{R})$, from (1.5.38) and (1.5.39)-(1.5.41) with $f \equiv 1$ we obtain that $\sigma_{(x,t)} \in \mathcal{P}(\mathbb{R})$, too. Clearly, for almost every $(x,t) \in Q_T$ all $\sigma^{(l)}_{(x,t)}$ and $\hat{\sigma}^{(m)}_{(x,t)}$, thus $\sigma_{(x,t)}$ are positive Radon measures on $\mathbb{R}$, with compact support since *supp* $\tau_{(x,t)}$ is compact. It is also easily seen that the definitions (1.5.39)-(1.5.40) imply

$$\text{supp } \sigma^{(l)}_{(x,t)} \subseteq J_l\,, \quad \text{supp } \hat{\sigma}^{(m)}_{(x,t)} \subseteq \hat{J}_m \qquad (l = 0, \dots, n;\ m = 1, \dots, n) \tag{1.5.42}$$

for almost every $(x,t) \in Q_T$. In fact, for every $f, g \in C(\mathbb{R})$ such that $\operatorname{supp} f \cap J_l = \emptyset$ and $\operatorname{supp} g \cap \hat{J}_m = \emptyset$, from (1.5.39) we obtain $\left\langle \sigma^{(l)}_{(x,t)}, f \right\rangle = 0$ and $\left\langle \hat{\sigma}^{(m)}_{(x,t)}, g \right\rangle = 0$, whence the claim follows.

Recalling (1.3.42)-(1.3.43), from (1.5.38) and (1.5.39)-(1.5.40) we also have

$$\left\langle \tau_{(x,t)}, f \right\rangle_{\mathbb{R}} = \int_{\mathbb{R}} f(\xi)\, d\tau_{(x,t)}(\xi) = \tag{1.5.43}$$
$$= \sum_{l=0}^{n} \int_{\mathbb{R}} (f \circ s_l \circ \varphi)(\xi) \chi_{I_l}(\xi)\, d\tau_{(x,t)}(\xi) +$$
$$+ \sum_{m=1}^{n} \int_{\mathbb{R}} (f \circ t_m \circ \varphi)(\xi) \chi_{\hat{I}_m}(\xi)\, d\tau_{(x,t)}(\xi) =$$
$$= \sum_{l=0}^{n} \int_{\mathbb{R}} (f \circ s_l)(\lambda)\, d\sigma^{(l)}_{(x,t)}(\lambda) + \sum_{m=1}^{n} \int_{\mathbb{R}} (f \circ t_m)(\lambda)\, d\hat{\sigma}^{(m)}_{(x,t)}(\lambda) =$$
$$= \sum_{l=0}^{n} \left\langle \sigma^{(l)}_{(x,t)}, f \circ s_l \right\rangle_{\mathbb{R}} + \sum_{m=1}^{n} \left\langle \hat{\sigma}^{(m)}_{(x,t)}, f \circ t_m \right\rangle_{\mathbb{R}}.$$

Remark 1.5.1. It is easily seen that the probability measure $\sigma_{(x,t)}$ defined in (1.5.41) coincides with the disintegration $\theta_{(x,t)}$ of the limiting Young measure θ in (1.3.51). In fact, for any $h \in C(\mathbb{R})$ the family $\{(h \circ \varphi)(u_{\epsilon x})\}$ is uniformly bounded in $L^\infty(Q_\infty)$, hence uniformly integrable in $L^1(Q_T)$ for every $T \in (0, \infty)$. Then by Theorem 2.5.2-(ii)

$$(h \circ \varphi)(u_{\epsilon_k x}) \rightharpoonup \int_{\mathbb{R}} h(\lambda)\, d\theta_{(x,t)}(\lambda) \quad \text{in } L^1(Q_T) \tag{1.5.44}$$

along some subsequence $\{u_{\epsilon_k x}\}$. On the other hand, by the same token we also have

$$(h \circ \varphi)(u_{\epsilon_k x}) \rightharpoonup \int_{\mathbb{R}} (h \circ \varphi)(\xi)\, d\tau_{(x,t)}(\xi) \quad \text{in } L^1(Q_T)\,; \tag{1.5.45}$$

moreover, by (1.3.42), (1.3.43), (1.5.41) and (1.5.43) there holds

$$\int_{\mathbb{R}} (h \circ \varphi)(\xi)\, d\tau_{(x,t)}(\xi) = \tag{1.5.46}$$
$$= \sum_{l=0}^{n} \int_{\mathbb{R}} (h \circ \varphi \circ s_l \circ \varphi)(\xi) \chi_{I_l}(\xi)\, d\tau_{(x,t)}(\xi) +$$
$$+ \sum_{m=1}^{n} \int_{\mathbb{R}} (h \circ \varphi \circ t_m \circ \varphi)(\xi) \chi_{\hat{I}_m}(\xi)\, d\tau_{(x,t)}(\xi) =$$
$$= \sum_{l=0}^{n} \int_{\mathbb{R}} h(\lambda)\, d\sigma^{(l)}_{(x,t)}(\lambda) + \sum_{m=1}^{n} \int_{\mathbb{R}} h(\lambda)\, d\hat{\sigma}^{(m)}_{(x,t)}(\lambda) =$$
$$= \int_{\mathbb{R}} h(\lambda)\, d\sigma_{(x,t)}(\lambda)\,.$$

By (1.5.44), (1.5.46) and the arbitrariness of h the claim follows.

As a particular case of (1.3.17), set

$$\varphi_*(x,t) := \int_{\mathbb{R}} \varphi(\xi)\, d\tau_{(x,t)} \qquad ((x,t) \in Q_\infty)\,. \tag{1.5.47}$$

Then we have the following

Proposition 1.5.1. *Let the assumptions of Theorem 1.3.7 be satisfied. Then for almost every $(x,t) \in Q_\infty$ the measure $\sigma_{(x,t)}$ defined in* (1.5.41) *is the Dirac measure concentrated at some point $\lambda_0(x,t) \in \mathbb{R}$.*

Using the above proposition we can prove Theorem 1.3.8.

Proof of Theorem 1.3.8. By Proposition 1.5.1 there holds

$$\lambda_0(x,t) = \int_{\mathbb{R}} \lambda\, d\sigma_{(x,t)}(\lambda) = \int_{\mathbb{R}} \varphi(\xi)\, d\tau_{(x,t)}(\xi) = \varphi_*(x,t)\,.$$

Then by Remark 1.5.1 we have

$$\theta_{(x,t)} = \delta(\cdot - \varphi_*(x,t))$$

for almost every $(x,t) \in Q_\infty$, with φ_* defined in (1.5.47). Therefore equality (1.3.52) will follow, if we prove that $\varphi_* = w$ almost everywhere in Q_∞.

To this purpose, observe that by inequality (1.3.45) for any $T \in (0,\infty)$ there holds

$$\|\varphi(u_{\epsilon_k x}) - w_{\epsilon_k}\|_{L^2(Q_T)} = \epsilon_k\, \|u_{\epsilon_k xt}\|_{L^2(Q_T)} \to 0 \;\text{ as } k \to \infty\,. \tag{1.5.48}$$

By (1.3.48) and (1.5.48) we obtain $\varphi_* = w$ almost everywhere in Q_T for any $T \in (0,\infty)$. This proves the result. □

Let us now prove Theorem 1.3.7.

Proof of Theorem 1.3.7. Let $\{\epsilon_k\}$ be a sequence of indices such that the convergence results in Proposition 1.3.5 and (1.3.48) hold true. Set for any $k \in \mathbb{N}$

$$Q_l^{(k)} := \{(x,t) \in Q_\infty \mid u_{\epsilon_k x}(x,t) \in I_l\} \qquad (l = 0,\dots,n)\,, \tag{1.5.49}$$
$$Q_m^{(k)} := \left\{(x,t) \in Q_\infty \mid u_{\epsilon_k x}(x,t) \in \hat{I}_m\right\} \qquad (m = 1,\dots,n)\,,$$

and denote by $\chi_{Q_l^{(k)}}$, $\chi_{Q_m^{(k)}}$ the characteristic functions of these sets. Then for any $f \in C(\mathbb{R})$ the equality

$$f\left(u_{\epsilon_k x}\right) = \sum_{l=0}^{n} \chi_{Q_l^{(k)}} f\left(s_l\left(\varphi(u_{\epsilon_k x})\right)\right) + \sum_{m=1}^{n} \chi_{Q_m^{(k)}} f\left(t_m\left(\varphi(u_{\epsilon_k x})\right)\right) \tag{1.5.50}$$

holds almost everywhere in Q_∞.

For any fixed l and m the sequences $\left\{\chi_{Q_l^{(k)}}\right\}$, $\left\{\chi_{Q_m^{(k)}}\right\}$ are uniformly bounded in $L^\infty(Q_\infty)$. Hence there exist two subsequences (denoted again $\left\{\chi_{Q_l^{(k)}}\right\}$, $\left\{\chi_{Q_m^{(k)}}\right\}$ for simplicity) and two functions $c_l \in L^\infty(Q_\infty)$, $d_m \in L^\infty(Q_\infty)$ such that

$$\chi_{Q_l^{(k)}} \overset{*}{\rightharpoonup} c_l\,, \qquad \chi_{Q_m^{(k)}} \overset{*}{\rightharpoonup} d_m \quad \text{in } L^\infty(Q_T) \tag{1.5.51}$$

for any $T \in (0,\infty)$. It is easily seen that the functions c_l, d_m ($l = 0,\dots,n$; $m = 1,\dots,n$) have the properties asserted in the statement.

We shall prove that:

$$\lim_{k\to\infty} \chi_{Q_l^{(k)}} f\left(s_l\left(\varphi(u_{\epsilon_k x})\right)\right) = c_l\, f\left(s_l(w)\right) \qquad (l = 0,\dots,n)\,, \tag{1.5.52}$$
$$\lim_{k\to\infty} \chi_{Q_m^{(k)}} f\left(t_m\left(\varphi(u_{\epsilon_k x})\right)\right) = d_m\, f\left(t_m(w)\right) \qquad (m = 1,\dots,n)$$

weakly* in $L^\infty(Q_\infty)$. On the other hand, since the family $\{u_{\epsilon x}\}$ is uniformly integrable, by Theorem 2.5.2-(ii) we have

$$\lim_{k\to\infty} f\left(u_{\epsilon_k x}\right) = f^* \ \text{ in } L^1(Q_T)$$

for any $T \in (0,\infty)$, where

$$f^*(x,t) = \int_{\mathbb{R}} f(\xi)\, d\tau_{(x,t)}(\xi) \qquad \text{for } a.e.\ (x,t) \in Q_T$$

for almost every $(x,t) \in Q_T$. Therefore, letting $k \to \infty$ in (1.5.50) and using (1.5.52) we obtain the equality

$$\int_{\mathbb{R}} f(\xi)\, d\tau_{(x,t)}(\xi) = \sum_{l=0}^{n} c_l\, f\left(s_l(w)\right) + \sum_{m=1}^{n} d_m\, f\left(t_m(w)\right)$$

almost everywhere in Q_T, whence the characterization (1.3.50) follows by the arbitrariness of f.

Now let us prove the first equality in (1.5.52). Observe that for any $T \in (0,\infty)$ and any $\zeta \in L^1(Q_T)$

$$\iint_{Q_T} \left[\chi_{Q_l^{(k)}} f\left(s_l\left(\varphi(u_{\epsilon_k x})\right)\right) - c_l\, f\left(s_l(w)\right)\right]\zeta\, dxdt \;= \tag{1.5.53}$$
$$= \iint_{Q_T} \left[\chi_{Q_l^{(k)}} - c_l\right] f\left(s_l\left(w\right)\right)\zeta\, dxdt \;+$$
$$+ \iint_{Q_T} \chi_{Q_l^{(k)}}\left[f\left(s_l\left(\varphi(u_{\epsilon_k x})\right)\right) - f\left(s_l\left(w\right)\right)\right]\zeta\, dxdt\,.$$

Since $w \in L^\infty(Q_T)$ for any $T \in (0,\infty)$ and $f \circ s_l$ is continuous, the first integral in the right-hand side of the above equality goes to zero as $k \to \infty$ by (1.5.51).

Concerning the second observe that, since by Theorem 1.3.8 the disintegration $\theta_{(x,t)}$ is the Dirac mass concentrated at the point $\{w(x,t)\}$, possibly extracting a subsequence (denoted again $\{u_{\epsilon_k x}\}$ for simplicity) there holds

$$\varphi\left(u_{\epsilon_k x}\right) \to w \quad \text{almost everywhere in } Q\,.$$

(see [V1, Theorem 9]). By the Dominated Convergence Theorem, this implies that also the second integral vanishes as $k \to \infty$, thus the first equality in (1.5.52) follows. The proof of the other is similar, thus it is omitted. This completes the proof. □

It remains to prove Proposition 1.5.1. To this purpose, we generalize to the present situation the proof given in [Pl1] for a cubic φ. Consider the orthonormal basis of $L^2(\Omega)$ given by the eigenfunctions $\eta_i \in H^1(\Omega)$ of the operator $-\Delta$ with homogeneous Neumann conditions. Let $\{\mu_i\}$ be the corresponding sequence of eigenvalues. For any $\epsilon > 0$ let $P_\epsilon, Q_\epsilon : L^2(\Omega) \to H^1(\Omega)$, $P_\epsilon + Q_\epsilon = I$, be the projection operators defined as follows:

$$P_\epsilon f := \sum_{\epsilon\mu_i \leq 1} f_i \eta_i\,, \quad Q_\epsilon f := \sum_{\epsilon\mu_i > 1} f_i \eta_i\,, \quad f_i := \int_\Omega f\, \eta_i\, dx \qquad (f \in L^2(\Omega))\,. \tag{1.5.54}$$

The proof of the following result is the same of [Pl1, Lemma 2.1], thus we omit it.

Lemma 1.5.2. *There exists $C > 0$ such that for any $\epsilon > 0$*

$$\|P_\epsilon\, \varphi(v_\epsilon)\|_{L^2((0,T);H^1_0(\Omega))} + \epsilon^{-1/2} \|Q_\epsilon \varphi(v_\epsilon)\|_{L^2(Q_T)} \leq C\,. \tag{1.5.55}$$

Let $a_0 \in I_0$ be the unique point such that $\varphi(a_0) = A$, where $A := \min\{\varphi(a_1), \ldots, \varphi(a_n)\}$ (see Subsection 1.3.2). Set

$$G(z) := \int_{a_0}^{z} (g \circ \varphi)(r) dr \qquad (z \in \mathbb{R})\,. \tag{1.5.56}$$

Then we have the following result, whose proof is the same of [ST2, Proposition 3.3]); we give it for convenience of the reader.

Proposition 1.5.3. *Let $\tau_{(x,t)}$ be the disintegration of the limiting Young measure τ, and let G be the function* (1.5.56) *with $g \in C^1_c(\mathbb{R})$. Let $f \in C^1(\mathbb{R})$ such that $\|f\|_{L^\infty(\mathbb{R})} + \|f'\|_{L^\infty(\mathbb{R})} \leq C$ for some constant $C > 0$. Then*

$$\left\langle \tau_{(x,t)}, (f \circ \varphi)\, G \right\rangle_{\mathbb{R}} = \left\langle \tau_{(x,t)}, f \circ \varphi \right\rangle_{\mathbb{R}} \left\langle \tau_{(x,t)}, G \right\rangle_{\mathbb{R}}\,. \tag{1.5.57}$$

Proof. Set $F := f \circ \varphi$. Let $T \in (0, \infty)$ be fixed, and let $\{v_{\epsilon_k}\}$ be the converging sequence in (1.3.49). Under the present assumptions the sequences $\{F(v_{\epsilon_k})\}$ and $\{G(v_{\epsilon_k})\}$, thus $\{F(v_{\epsilon_k})G(v_{\epsilon_k})\}$ are uniformly bounded in $L^\infty(Q_T)$ (see (1.3.45)). Therefore, by Theorem 2.5.2-(*ii*)

$$F(v_{\epsilon_k})G(v_{\epsilon_k}) \overset{*}{\rightharpoonup} \left\langle \tau_{(x,t)}, F\, G \right\rangle_{\mathbb{R}} \quad \text{in } L^\infty(Q_T)$$

(see (2.5.5)-(2.5.6)). Then the conclusion follows, if we prove that

$$F(v_{\epsilon_k})G(v_{\epsilon_k}) \stackrel{*}{\rightharpoonup} F_* G_* \quad \text{in } L^\infty(Q_T)\,, \tag{1.5.58}$$

where

$$F_*(x,t) := \left\langle \tau_{(x,t)}, F \right\rangle_{\mathbb{R}}\,, \quad G_*(x,t) := \left\langle \tau_{(x,t)}, G \right\rangle_{\mathbb{R}}$$

are the weak* limits in $L^\infty(Q_T)$ of the sequences $\{F(v_{\epsilon_k})\}$ and $\{G(v_{\epsilon_k})\}$, respectively.

To this purpose, set $F^{\epsilon_k} := f(P_{\epsilon_k}\varphi(v_{\epsilon_k}))$, where P_{ϵ_k} is the projection operator defined in (1.5.54) with $\epsilon = \epsilon_k$. Recalling that by assumption $\|f'\|_{L^\infty(\mathbb{R})} < \infty$, we have

$$\begin{aligned} \|F^{\epsilon_k} - F(v_{\epsilon_k})\|_{L^2(Q_T)} &= \|f(P_{\epsilon_k}\varphi(v_{\epsilon_k})) - f(\varphi(v_{\epsilon_k}))\|_{L^2(Q_T)} \\ &\le \|f'\|_{L^\infty(\mathbb{R})}\,\|Q_{\epsilon_k}\varphi(v_{\epsilon_k})\|_{L^2(Q_T)}\,. \end{aligned} \tag{1.5.59}$$

By (1.5.55) the right-hand side of the above inequality goes to zero as $k \to \infty$. Then we obtain

$$\|\,[F^{\epsilon_k} - F(v_{\epsilon_k})]\,G(v_{\epsilon_k})\|_{L^2(Q_T)} \to 0 \quad \text{as } k \to \infty\,.$$

Hence to prove (1.5.58) it suffices to show that for any $\zeta \in C^1_c(Q_T)$

$$\iint_{Q_T} F^{\epsilon_k} G(v_{\epsilon_k})\,\zeta\,dxdt \to \iint_{Q_T} F_* G_* \zeta\,dxdt\,. \tag{1.5.60}$$

Set $\Omega \equiv (a,b)$, and

$$\Gamma^{\epsilon_k}(x,t) := \int_a^x G(v_{\epsilon_k})(z,t)dz \quad \text{for almost every } (x,t) \in Q_T\,. \tag{1.5.61}$$

Then we have

$$\iint_{Q_T} F^{\epsilon_k} G(v_{\epsilon_k})\,\zeta\,dxdt = -\iint_{Q_T} (F^{\epsilon_k}\zeta)_x \Gamma^{\epsilon_k} dxdt\,. \tag{1.5.62}$$

Since $\|f'\|_{L^\infty(\mathbb{R})} < \infty$, by inequality (1.5.55) there exists $\bar{C} > 0$ such that

$$\|F^{\epsilon_k}_x\|_{L^2(Q_T)} \le \|f'\|_{L^\infty(\mathbb{R})} \|[P_{\epsilon_k}[\varphi(v_{\epsilon_k})]]_x\|_{L^2(Q_T)} \le \bar{C}\,.$$

By estimate (1.3.45) the sequence $\{F^{\epsilon_k}\}$ is uniformly bounded in $L^\infty(Q_T)$, thus in $L^2(Q_T)$; hence by the above inequality it is also uniformly bounded in $L^2((0,T);H^1(\Omega))$. Then there exists $\bar{F} \in L^2((0,T);H^1(\Omega))$ such that

$$F^{\epsilon_k} \rightharpoonup \bar{F} \quad \text{in } L^2((0,T);H^1(\Omega))\,.$$

By (1.5.59) this implies that

$$F(v_{\epsilon_k}) \rightharpoonup \bar{F} \quad \text{in } L^2(Q_T)\,,$$

whence $F_* = \bar{F}$ in Q_T. Therefore $F_* \in L^2((0,T);H^1(\Omega))$ and

$$(F^{\epsilon_k}\,\zeta)_x \rightharpoonup (F_*\,\zeta)_x \quad \text{in} \;\; L^2(Q_T)\,. \tag{1.5.63}$$

On the other hand, since $G(v_{\epsilon_k}) \overset{*}{\rightharpoonup} G_*$ in $L^\infty(Q_T)$, for almost every $(x,t) \in Q_T$ we have

$$\Gamma^{\epsilon_k}(x,t) \to \Gamma_*(x,t) := \int_a^x G_*(z,t)\,dz\,. \tag{1.5.64}$$

Since the family $\{\Gamma^{\epsilon_k}\}$ is uniformly bounded in $L^\infty(Q_T)$, the above convergence implies that

$$\Gamma^{\epsilon_k} \to \Gamma_* \quad \text{in} \;\; L^2(Q_T). \tag{1.5.65}$$

By (1.5.63) and (1.5.65), the right-hand side of equality (1.5.62) converges to

$$-\iint_{Q_T} (F_*\zeta)_x \Gamma_* dxdt = \iint_{Q_T} F_* G_* \zeta\, dxdt$$

(see (1.5.64)). This proves (1.5.60), thus the conclusion follows. □

Denote by $e_1 \leq \ldots e_{2n}$ the set of the local extrema $\varphi(a_l)$, $\varphi(b_l)$ $(l = 1,\ldots,n)$ of the graph of φ; hence $e_1 = A$, $e_{2n} = B$ and

$$\mathbb{R} = (-\infty, A] \bigcup \left(\bigcup_{k=1}^{2n-1} [e_k, e_{k+1}] \right) \bigcup [B, \infty)\,.$$

Let $\mathcal{E} \equiv \mathcal{E}_k$ denote any interval $[e_k, e_{k+1}]$ $(k = 1,\ldots,2n-1)$. Observe that for every $l = 0,\ldots,n$ there holds either $\mathcal{E} \bigcap J_l = \emptyset$, or $\mathcal{E} \bigcap J_l = \mathcal{E}$, and similarly for $\mathcal{E} \cap \hat{J}_m$ $(m = 1,\ldots,n)$. Therefore, there exist $p+1$ intervals J_{l_i} and p intervals $\hat{J}_{m_j}$ $(p = 1,\ldots,n;\ l_i \in \{0,\ldots,p\};\ m_j \in \{1,\ldots,p\})$ such that

$$\mathcal{E} \subseteq \left(\bigcap_{i=1}^{p+1} J_{l_i} \right) \bigcap \left(\bigcap_{j=1}^{p} \hat{J}_{m_j} \right)\,. \tag{1.5.66}$$

We shall suppose without loss of generality that $l_{i-1} \leq l_i$ for any $i = 1,\ldots,p+1$, and $m_{j-1} \leq m_j$ for any $j = 1,\ldots,p$.

Relying on Proposition 1.5.3 we can prove the following lemma, where we set $\sigma \equiv \sigma_{(x,t)}$, $\sigma^{(l)} \equiv \sigma^{(l)}_{(x,t)}$, $\hat{\sigma}^{(m)} \equiv \hat{\sigma}^{(m)}_{(x,t)}$ for simplicity.

Lemma 1.5.4. *(i) Let $\mathcal{E} \equiv \mathcal{E}_k$ denote any interval $[e_k, e_{k+1}]$ $(k = 1,\ldots,2n-1)$, and let $K \subseteq \mathcal{E}$ be any compact subset such that $\sigma(K) > 0$. Then for* *almost every* $\lambda \in \mathcal{E}$

$$\sum_{i=1}^{p+1} D_i(\lambda) \left(\frac{\sigma^{(l_i)}(K)}{\sigma(K)} - \sigma^{(l_i)}(\mathcal{E}) \right) +$$
$$+ \sum_{j=1}^{p} D_{j+1}(\lambda) \left(\frac{\hat{\sigma}^{(m_j)}(K)}{\sigma(K)} - \hat{\sigma}^{(m_j)}(\mathcal{E}) \right) =$$
$$= \sum_{i=1}^{p+1} s'_{l_i}(\lambda) \left(\rho^{(l_i)}(\lambda) - \frac{\rho_K^{(l_i)}(\lambda)}{\sigma(K)} \right) +$$
$$+ \sum_{j=1}^{p} t'_{m_j}(\lambda) \left(\hat{\rho}^{(m_j)}(\lambda) - \frac{\hat{\rho}_K^{(m_j)}(\lambda)}{\sigma(K)} \right) \qquad (1.5.67)$$

where

$$D_1(\lambda) := 0\,, \quad D_i(\lambda) := \sum_{k=1}^{i-1} \left(s'_{l_k} - t'_{m_k} \right)(\lambda) \qquad (i = 2, \ldots, p+1)\,, \quad (1.5.68)$$

$$\rho^{(l_i)}(\lambda) := \sigma^{(l_i)}([\lambda, \varphi(b_{l_i+1})])\,, \quad \hat{\rho}^{(m_j)}(\lambda) := \hat{\sigma}^{(m_j)}([\lambda, \varphi(b_{m_j})])\,, \quad (1.5.69)$$

$$\rho_K^{(l_i)}(\lambda) := \sigma^{(l_i)}([\lambda, \varphi(b_{l_i+1})] \cap K)\,, \quad \hat{\rho}_K^{(m_j)}(\lambda) := \hat{\sigma}^{(m_j)}([\lambda, \varphi(b_{m_j})] \cap K)\,. \quad (1.5.70)$$

(ii) Let $K \subseteq (-\infty, A]$ *(where* $A := \min\{\varphi(a_1), \ldots, \varphi(a_n)\}$*) be any compact subset such that* $\sigma(K) > 0$*. Then for almost every* $\lambda \in (-\infty, A]$

$$\sigma^{(0)}((-\infty, \lambda]) = \frac{\sigma^{(0)}((-\infty, \lambda] \cap K)}{\sigma(K)}\,. \qquad (1.5.71)$$

Similarly, let $K \subseteq [B, \infty)$ *(where* $B := \max\{\varphi(b_1), \ldots, \varphi(b_n)\}$*) be any compact subset such that* $\sigma(K) > 0$*. Then for almost every* $\lambda \in [B, \infty)$

$$\sigma^{(n)}([\lambda, \infty)) = \frac{\sigma^{(n)}([\lambda, \infty) \cap K)}{\sigma(K)}\,. \qquad (1.5.72)$$

Proof. *(i)* Since $K \subseteq \mathcal{E}$ is compact, there exists a sequence $\{f_h\} \subset C_c(\mathcal{E})$, $f_h \geq 0$, $f_h = 1$ on K, such that as $h \to \infty$

$$f_h(\lambda) \to \chi_K(\lambda) \quad \text{for any } \lambda \in \mathcal{E}\,,$$

where χ_K denotes the characteristic function of K. Set $F_h := f_h \circ \varphi$, and consider the function G defined in (1.5.56) with $g \in C_c^1(\mathcal{E})$. By equalities (1.5.43) and (1.5.57) we obtain easily

$$\sum_{l=0}^{n} \left\langle \sigma^{(l)}, f_h(G \circ s_l) \right\rangle_{\mathbb{R}} + \sum_{m=1}^{n} \left\langle \hat{\sigma}^{(m)}, f_h(G \circ t_m) \right\rangle_{\mathbb{R}} =$$
$$= \langle \sigma, f_h \rangle_{\mathbb{R}} \left\{ \sum_{l=0}^{n} \left\langle \sigma^{(l)}, (G \circ s_l) \right\rangle_{\mathbb{R}} + \sum_{m=1}^{n} \left\langle \hat{\sigma}^{(m)}, (G \circ t_m) \right\rangle_{\mathbb{R}} \right\},$$

whence, letting $h \to \infty$:

$$\sum_{l=0}^{n} \int_{\mathbb{R}} (G \circ s_l)(\lambda)\, d\sigma^{(l)}(\lambda) + \sum_{m=1}^{n} \int_{\mathbb{R}} (G \circ t_m)(\lambda)\, d\hat{\sigma}^{(m)}(\lambda) = \tag{1.5.73}$$

$$= \frac{1}{\sigma(K)} \left\{ \sum_{l=0}^{n} \int_{K} (G \circ s_l)(\lambda)\, d\sigma^{(l)}(\lambda) + \sum_{m=1}^{n} \int_{K} (G \circ t_m)(\lambda)\, d\hat{\sigma}^{(m)}(\lambda) \right\}.$$

Choosing $g \in C_c^1(\mathcal{E})$, by (1.5.41), (1.5.42) and the definition of G it is apparent that only the intervals J_{l_i} and $\hat{J}_{m_j}$ ($i = 1, \ldots, p+1$; $j = 1, \ldots, p$; $p = 1, \ldots, n$) in the right-hand side of (1.5.66) contribute to the sums in equality (1.5.73), which thus reads

$$\sum_{i=1}^{p+1} \int_{\mathcal{E}} (G \circ s_{l_i})(\lambda)\, d\sigma^{(l_i)}(\lambda) + \sum_{j=1}^{p} \int_{\mathcal{E}} (G \circ t_{m_j})(\lambda)\, d\hat{\sigma}^{(m_j)}(\lambda) = \tag{1.5.74}$$

$$= \frac{1}{\sigma(K)} \left\{ \sum_{i=1}^{p+1} \int_{K} (G \circ s_{l_i})(\lambda)\, d\sigma^{(l_i)}(\lambda) + \sum_{j=1}^{p} \int_{K} (G \circ t_{m_j})(\lambda)\, d\hat{\sigma}^{(m_j)}(\lambda) \right\}$$

(here (1.5.66) has been used). Now observe that for every $\lambda \in \mathcal{E}$

$$(G \circ s_{l_1})(\lambda) = \int_{a_{l_1}}^{s_{l_1}(\lambda)} g(\varphi(r))\, dr = \int_{\varphi(a_{l_1})}^{\lambda} g(\zeta) s'_{l_1}(\zeta)\, d\zeta\,, \tag{1.5.75}$$

whereas for every $i = 2, \ldots, p+1$ and $\lambda \in \mathcal{E}$

$$\begin{aligned} &(G \circ s_{l_i})(\lambda) = \qquad (1.5.76)\\ = &\sum_{k=1}^{i-1} \left\{ \int_{I_{l_k}} g(\varphi(r))\, dr + \int_{\hat{I}_{m_k}} g(\varphi(r))\, dr \right\} + \int_{a_{l_i}}^{s_{l_i}(\lambda)} g(\varphi(r))\, dr = \\ = &\sum_{k=1}^{i-1} \left\{ \int_{J_{l_k}} g(\zeta) s'_{l_k}(\zeta)\, d\zeta - \int_{\hat{J}_{m_k}} g(\zeta) t'_{m_k}(\zeta)\, d\zeta \right\} + \int_{\varphi(a_{l_i})}^{\lambda} g(\zeta) s'_{l_i}(\zeta)\, d\zeta = \\ = &\int_{\mathcal{E}} g(\zeta) D_i(\zeta)\, d\zeta + \int_{\varphi(a_{l_i})}^{\lambda} g(\zeta) s'_{l_i}(\zeta)\, d\zeta\,, \end{aligned}$$

with D_i defined by (1.5.68). Since $D_1(\zeta) := 0$ by definition, equalities (1.5.75), (1.5.76) take the form

$$(G \circ s_{l_i})(\lambda) = \int_{\mathcal{E}} g(\zeta) D_i(\zeta)\, d\zeta + \int_{\varphi(a_{l_i})}^{\lambda} g(\zeta) s'_{l_i}(\zeta)\, d\zeta \quad \text{for every } i = 1, \ldots, p+1\,. \tag{1.5.77}$$

Similarly, for every $j = 1, \dots, p$ and $\lambda \in \mathcal{E}$

$$
\begin{aligned}
& (G \circ t_{m_j})(\lambda) = \qquad\qquad (1.5.78) \\
= \; & \sum_{k=1}^{j} \left\{ \int_{I_{l_k}} g(\varphi(r))\, dr + \int_{\hat{I}_{m_k}} g(\varphi(r))\, dr \right\} - \int_{t_{m_j}(\lambda)}^{a_{m_j}} g(\varphi(r))\, dr \; = \\
= \; & \sum_{k=1}^{j} \left\{ \int_{J_{l_k}} g(\zeta) s'_{l_k}(\zeta)\, d\zeta - \int_{\hat{J}_{m_k}} g(\zeta) t'_{m_k}(\zeta)\, d\zeta \right\} + \int_{\varphi(a_{m_j})}^{\lambda} g(\zeta) t'_{m_j}(\zeta)\, d\zeta \; = \\
= \; & \int_{\mathcal{E}} g(\zeta) D_{j+1}(\zeta)\, d\zeta + \int_{\varphi(a_{m_j})}^{\lambda} g(\zeta) t'_{m_j}(\zeta)\, d\zeta \, .
\end{aligned}
$$

By (1.5.78), (1.5.77) the sum in the left-hand side of equation (1.5.74) plainly becomes

$$
\begin{aligned}
& \sum_{i=1}^{p+1} \int_{\mathcal{E}} (G \circ s_{l_i})(\lambda)\, d\sigma^{(l_i)}(\lambda) + \sum_{j=1}^{p} \int_{\mathcal{E}} (G \circ t_{m_j})(\lambda)\, d\hat{\sigma}^{(m_j)}(\lambda) \\
= \; & \sum_{i=1}^{p+1} \int_{\mathcal{E}} d\sigma^{(l_i)}(\lambda) \left\{ \int_{\mathcal{E}} g(\zeta) D_i(\zeta)\, d\zeta + \int_{\varphi(a_{l_i})}^{\lambda} g(\zeta) s'_{l_i}(\zeta)\, d\zeta \right\} + \\
+ \; & \sum_{j=1}^{p} \int_{\mathcal{E}} d\hat{\sigma}^{(m_j)}(\lambda) \left\{ \int_{\mathcal{E}} g(\zeta) D_{j+1}(\zeta)\, d\zeta + \int_{\varphi(a_{m_j})}^{\lambda} g(\zeta) t'_{m_j}(\zeta)\, d\zeta \right\} = \\
= \; & \int_{\mathcal{E}} d\zeta\, g(\zeta) \left\{ \sum_{i=1}^{p+1} \Big[D_i(\zeta) \sigma^{(l_i)}(\mathcal{E}) + s'_{l_i}(\zeta)\, \rho^{(l_i)}(\zeta) \Big] + \right. \\
+ \; & \left. \sum_{j=1}^{p} \Big[D_{j+1}(\zeta) \hat{\sigma}^{(m_j)}(\mathcal{E}) + t'_{m_j}(\zeta)\, \hat{\rho}^{(m_j)}(\zeta) \Big] \right\}, \qquad (1.5.79)
\end{aligned}
$$

with $\rho^{(l_i)}$, $\hat{\rho}^{(m_j)}$ defined by (1.5.69). Similarly, the sum in the right-hand side of equation (1.5.74) reads

$$
\begin{aligned}
& \sum_{i=1}^{p+1} \int_{K} (G \circ s_{l_i})(\lambda)\, d\sigma^{(l_i)}(\lambda) + \sum_{j=1}^{p} \int_{K} (G \circ t_{m_j})(\lambda)\, d\hat{\sigma}^{(m_j)}(\lambda) \; = \\
= \; & \int_{\mathcal{E}} d\zeta\, g(\zeta) \left\{ \sum_{i=1}^{p+1} \Big[D_i(\zeta) \sigma^{(l_i)}(K) + s'_{l_i}(\zeta)\, \rho_K^{(l_i)}(\zeta) \Big] + \right. \\
+ \; & \left. \sum_{j=1}^{p} \Big[D_{j+1}(\zeta) \hat{\sigma}^{(m_j)}(K) + t'_{m_j}(\zeta)\, \hat{\rho}_K^{(m_j)}(\zeta) \Big] \right\}, \qquad (1.5.80)
\end{aligned}
$$

with $\rho_K^{(l_i)}$, $\hat{\rho}_K^{(m_j)}$ defined by (1.5.70). Since g is arbitrary, from (1.5.74), (1.5.79) and (1.5.80) we obtain (1.5.67). This proves claim (i).

Concerning (ii), observe that when $K \subseteq (-\infty, A]$ and $g \in C_c^1((-\infty, A])$ equality (1.5.73) reads simply

$$\int_{-\infty}^{A} d\sigma^{(0)}(\lambda) \int_{A}^{\lambda} g(\zeta) s_0'(\zeta)\, d\zeta \;=\; \frac{1}{\sigma(K)} \int_K d\sigma^{(0)}(\lambda) \int_A^{\lambda} g(\zeta) s_0'(\zeta)\, d\zeta\,.$$

Similarly, when $K \subseteq [B, \infty)$ and $g \in C_c^1([B, \infty))$ equality (1.5.73) becomes

$$\int_{B}^{\infty} d\sigma^{(n)}(\lambda) \int_{B}^{\lambda} g(\zeta) s_0'(\zeta)\, d\zeta \;=\; \frac{1}{\sigma(K)} \int_K d\sigma^{(n)}(\lambda) \int_B^{\lambda} g(\zeta) s_0'(\zeta)\, d\zeta\,.$$

Interchanging the order of integration as in the proof of (1.5.79) and (1.5.80) we obtain equalities (1.5.71)-(1.5.72). This completes the proof. □

Now we can prove Proposition 1.5.1.

Proof of Proposition 1.5.1. Let $\mathcal{E} \equiv \mathcal{E}_k$ be any interval $[e_k, e_{k+1}] \subseteq [A, B]$ $(k = 1, \dots, 2n-1)$, and suppose that $\operatorname{supp} \sigma|_{\mathcal{E}} \neq \emptyset$. Set

$$\lambda_0 := \min \{\lambda \in \operatorname{supp} \sigma|_{\mathcal{E}}\}$$

(as before, for simplicity we write $\sigma \equiv \sigma_{(x,t)}$, $\lambda_0 \equiv \lambda_0(x,t)$ and so on). Let us prove that that $\operatorname{supp} \sigma|_{\mathcal{E}} = \{\lambda_0\}$.

If $\lambda_0 = e_{k+1}$, the conclusion follows. Otherwise, consider the compact $K = [\lambda_0, \lambda_0 + \delta]$ with $\delta > 0$ so small that $K \subseteq \mathcal{E}$. Then $\sigma(K) > 0$ and

$$\rho_K^{(l_i)}(\lambda) = \hat{\rho}_K^{(m_j)}(\lambda) = 0 \quad \text{for any } \lambda \in (\lambda_0 + \delta, e_{k+1}]$$

$(i = 1, \dots, p+1,\ j = 1, \dots, p$; see (1.5.70)). Hence

$$\sum_{i=1}^{p+1} s_{l_i}'(\lambda)\, \rho_K^{(l_i)}(\lambda) \;+ \sum_{j=1}^{p} t_{m_j}'(\lambda)\, \hat{\rho}_K^{(m_j)}(\lambda) \;=\; 0 \tag{1.5.81}$$

for any $\lambda \in (\lambda_0 + \delta, e_{k+1}]$, and equality (1.5.67) reads

$$\begin{aligned}\sum_{i=2}^{p+1} D_i(\lambda) \left(\frac{\sigma^{(l_i)}(K)}{\sigma(K)} - \sigma^{(l_i)}(\mathcal{E}) \right) + \sum_{j=1}^{p} D_{j+1}(\lambda) \left(\frac{\hat{\sigma}^{(m_j)}(K)}{\sigma(K)} - \hat{\sigma}^{(m_j)}(\mathcal{E}) \right) = \\ = \sum_{i=1}^{p+1} s_{l_i}'(\lambda)\, \rho^{(l_i)}(\lambda) \;+ \sum_{j=1}^{p} t_{m_j}'(\lambda)\, \hat{\rho}^{(m_j)}(\lambda)\end{aligned} \tag{1.5.82}$$

with $K = [\lambda_0, \lambda_0 + \delta]$, for almost every $\lambda \in (\lambda_0 + \delta, e_{k+1}]$. By the arbitrariness of δ, there exists a sequence $\{\delta_k\}$ such that $\delta_k \to 0$ as $k \to \infty$ and

$$\lim_{k \to \infty} \frac{\sigma^{(l_i)}([\lambda_0, \lambda_0 + \delta_k])}{\sigma([\lambda_0, \lambda_0 + \delta_k])} =: L_{l_i}$$

$$\lim_{k\to\infty} \frac{\hat{\sigma}^{(m_j)}([\lambda_0, \lambda_0+\delta_k])}{\hat{\sigma}([\lambda_0, \lambda_0+\delta_k])} =: \hat{L}_{m_j}$$

for some L_{l_i}, $\hat{L}_{m_j} \leq 1$. Then from (1.5.82) we get

$$\begin{aligned} \sum_{i=2}^{p+1} D_i(\lambda)\, N_i \;+\; \sum_{j=1}^{p} D_{j+1}(\lambda)\, \hat{N}_j \;&=\; \\ =\; \sum_{i=1}^{p+1} s'_{l_i}(\lambda)\, \rho^{(l_i)}(\lambda) \;&+\; \sum_{j=1}^{p} t'_{m_j}(\lambda)\, \hat{\rho}^{(m_j)}(\lambda)\,, \end{aligned} \tag{1.5.83}$$

for almost every $\lambda \in (\lambda_0, e_{k+1}]$, where

$$N_i := L_{l_i} - \sigma^{(l_i)}(\mathcal{E})\,, \quad \hat{N}_j := \hat{L}_{m_j} - \hat{\sigma}^{(m_j)}(\mathcal{E})\,. \tag{1.5.84}$$

Now consider any compact $K \subseteq [\lambda_0, e_{k+1})$. Then there exists an interval (λ^*, e_{k+1}) such that $K \cap (\lambda^*, e_{k+1}) = \emptyset$. For any $\lambda \in (\lambda^*, e_{k+1})$ equality (1.5.81) holds with this choice of K. Therefore, using (1.5.67) again, we obtain that equality (1.5.82) holds for almost every $\lambda \in (\lambda^*, e_{k+1})$. Since $(\lambda^*, e_{k+1}) \subseteq (\lambda_0, e_{k+1}]$, from equalities (1.5.82) and (1.5.83) we obtain

$$\sum_{i=2}^{p+1} \alpha_i\, D_i(\lambda) \;+\; \sum_{j=1}^{p} \beta_j\, D_{j+1}(\lambda) \;=\; 0 \tag{1.5.85}$$

for almost every $\lambda \in (\lambda^*, e_{k+1})$, where

$$\alpha_i := \frac{\sigma^{(l_i)}(K)}{\sigma(K)} - L_{l_i}\,,\,, \quad \beta_j := \frac{\hat{\sigma}^{(m_j)}(K)}{\sigma(K)} - \hat{L}_{m_j}\,. \tag{1.5.86}$$

Recalling definition (1.5.68), equality (1.5.85) plainly reads

$$\begin{aligned} \sum_{i=2}^{p+1} \alpha_i \sum_{k=1}^{i-1} \left(s'_{l_k} - t'_{m_k}\right) \;+\; \sum_{j=1}^{p} \beta_j \sum_{k=1}^{j} \left(s'_{l_k} - t'_{m_k}\right) \;&=\; \\ =\; \sum_{k=1}^{p} \left[\sum_{i=k+1}^{p+1} \alpha_i \;+\; \sum_{j=k}^{p} \beta_j \right] \left(s'_{l_k} - t'_{m_k}\right) \;&=\; 0 \end{aligned} \tag{1.5.87}$$

almost everywhere in $\lambda \in (\lambda^*, e_{k+1})$. Then by the continuity of s'_{l_k}, t'_{m_k} and condition (C) we obtain

$$\sum_{i=k+1}^{p+1} \alpha_i \;+\; \sum_{j=k}^{p} \beta_j \;=\; 0 \quad \text{for every } \; k = 1, \ldots, p\,,$$

namely

$$\alpha_2 + \beta_1 = \cdots = \alpha_{p+1} + \beta_p = 0\,, \tag{1.5.88}$$

for any compact $K \subseteq [\lambda_0, e_{k+1})$. Replacing the above equalities in the left-hand side of (1.5.83), by equalities (1.5.84) and (1.5.86) we get

$$\sum_{i=2}^{p+1} D_i(\lambda)\, N_i \,+\, \sum_{j=1}^{p} D_{j+1}(\lambda)\, \hat{N}_j \,=\, \sum_{i=2}^{p+1} D_i(\lambda)\, \left(N_i \,+\, \hat{N}_{i-1}\right) \,=$$

$$= \sum_{i=2}^{p+1} D_i(\lambda) \left(\frac{\sigma^{(l_i)}(K)}{\sigma(K)} - \sigma^{(l_i)}(\mathcal{E}) + \frac{\hat{\sigma}^{(m_{i-1})}(K)}{\sigma(K)} - \hat{\sigma}^{(m_{i-1})}(\mathcal{E}) - \alpha_i - \beta_{i-1} \right) =$$

$$= \sum_{i=2}^{p+1} D_i(\lambda) \left(\frac{\sigma^{(l_i)}(K)}{\sigma(K)} - \sigma^{(l_i)}(\mathcal{E}) + \frac{\hat{\sigma}^{(m_{i-1})}(K)}{\sigma(K)} - \hat{\sigma}^{(m_{i-1})}(\mathcal{E}) \right),$$

namely the left-hand side of (1.5.82). Therefore, equality (1.5.82) holds for almost every $\lambda \in (\lambda_0, e_{k+1}]$ and any compact $K \subseteq [\lambda_0, e_{k+1})$. In turn, by equality (1.5.67) this implies that equality (1.5.81) holds for the same λ and K.

Now consider any closed interval $K = [\gamma, \delta] \subseteq (\lambda_0, e_{k+1})$. If $\lambda \in (\lambda_0, \gamma)$, there holds $[\lambda, \varphi(b_{l_i+1})] \cap K = [\lambda, e_{k+1}] \cap K = K$ and $[\lambda, \varphi(b_{m_j})] \cap K = [\lambda, e_{k+1}] \cap K = K$. Therefore,

$$\rho^{(l_i)}(\lambda) := \sigma^{(l_i)}(K)\,, \quad \hat{\rho}^{(m_j)}(\lambda) := \hat{\sigma}^{(m_j)}(K) \qquad (i = 2, \ldots, p{+}1;\; j = 1, \ldots, p)\,,$$

whence by equality (1.5.81)

$$\sum_{i=1}^{p+1} s'_{l_i}(\lambda)\, \sigma^{(l_i)}(K) \,+ \sum_{j=1}^{p} t'_{m_j}(\lambda)\, \hat{\sigma}^{(m_j)}(K) \,=\, 0 \tag{1.5.89}$$

for any $\lambda \in (\lambda_0, \gamma)$. By condition (C), the above equality implies $\sigma^{(l_i)}(K) = \hat{\sigma}^{(m_j)}(K) = 0$ for every $i = 2, \ldots, p+1$, $j = 1, \ldots, p$. Since $\lambda_0 < \gamma < \delta < e_{k+1}$ and γ, δ are arbitrary, the measures $\sigma^{(l_i)}$ and $\hat{\sigma}^{(m_j)}$, thus $\sigma|_{\mathcal{E}}$ are concentrated on the set $\{e_{k+1}\} \cup \{\lambda_0\}$.

To conclude the argument, let us prove that $\operatorname{supp} \sigma|_{\mathcal{E}} = \{\lambda_0\}$. Suppose by contradiction that $\{e_{k+1}\} \in \operatorname{supp} \sigma|_{\mathcal{E}}$. Choosing $K = \{e_{k+1}\}$ in equality (1.5.67) gives for almost every $\lambda \in (\lambda_0, e_{k+1})$

$$\rho_K^{(l_i)}(\lambda) = \sigma^{(l_i)}([\lambda, e_{k+1}] \cap \{e_{k+1}\}) = \sigma^{(l_i)}(\{e_{k+1}\}) = \sigma^{(l_i)}(K)\,,$$

$$\rho^{(l_i)}(\lambda) := \sigma^{(l_i)}([\lambda, e_{k+1}]) = \sigma^{(l_i)}([\lambda, e_{k+1}) \cup \{e_{k+1}\}) = \sigma^{(l_i)}(\{e_{k+1}\}) = \sigma^{(l_i)}(K)\,,$$

since

$$\sigma^{(l_i)}([\lambda, e_{k+1})) = \lim_{n \to \infty} \sigma^{(l_i)} \left(\left[\lambda, e_{k+1} - \frac{1}{n} \right] \right) \,=\, 0\,.$$

Similarly,

$$\hat{\rho}^{(m_j)}(\lambda) = \hat{\rho}_K^{(m_j)}(\lambda) = \hat{\sigma}^{(m_j)}(K) \qquad (\lambda \in (\lambda_0, e_{k+1}))\,.$$

Then from equality (1.5.67) we obtain for almost every $\lambda \in (\lambda_0, e_{k+1})$

$$\sum_{i=1}^{p+1} D_i(\lambda) \left(\frac{\sigma^{(l_i)}(K)}{\sigma(K)} - \sigma^{(l_i)}(\mathcal{E}) \right) + \sum_{j=1}^{p} D_{j+1}(\lambda) \left(\frac{\hat{\sigma}^{(m_j)}(K)}{\sigma(K)} - \hat{\sigma}^{(m_j)}(\mathcal{E}) \right) =$$

$$= \sum_{i=1}^{p+1} s'_{l_i}(\lambda) \left(\sigma^{(l_i)}(K) - \frac{\sigma^{(l_i)}(K)}{\sigma(K)} \right) + \sum_{j=1}^{p} t'_{m_j}(\lambda) \left(\hat{\sigma}^{(m_j)}(K) - \frac{\hat{\sigma}^{(m_j)}(K)}{\sigma(K)} \right).$$

Recalling definition (1.5.68), from the above equality we get for almost every $\lambda \in \mathcal{E}$

$$\sum_{k=1}^{p} s'_{l_k}(\lambda) \left[\sum_{i=k+1}^{p+1} \left(\frac{\sigma^{(l_i)}(K)}{\sigma(K)} - \sigma^{(l_i)}(\mathcal{E}) \right) + \sum_{j=k}^{p} \left(\frac{\hat{\sigma}^{(m_j)}(K)}{\sigma(K)} - \hat{\sigma}^{(m_j)}(\mathcal{E}) \right) + \right.$$

$$\left. + \left(\frac{\sigma^{(l_k)}(K)}{\sigma(K)} - \sigma^{(l_k)}(K) \right) \right] + s'_{l_{p+1}}(\lambda) \left(\frac{\sigma^{(l_{p+1})}(K)}{\sigma(K)} - \sigma^{(l_{p+1})}(K) \right) =$$

$$= \sum_{k=1}^{p} t'_{m_k}(\lambda) \left[\sum_{i=k+1}^{p+1} \left(\frac{\sigma^{(l_i)}(K)}{\sigma(K)} - \sigma^{(l_i)}(\mathcal{E}) \right) + \sum_{j=k}^{p} \left(\frac{\hat{\sigma}^{(m_j)}(K)}{\sigma(K)} - \hat{\sigma}^{(m_j)}(\mathcal{E}) \right) + \right.$$

$$\left. + \left(\hat{\sigma}^{(m_k)}(K) - \frac{\hat{\sigma}^{(m_k)}(K)}{\sigma(K)} \right) \right].$$

By condition (C), from the above equality we obtain

$$\sum_{i=k+1}^{p+1} \left(\frac{\sigma^{(l_i)}(K)}{\sigma(K)} - \sigma^{(l_i)}(\mathcal{E}) \right) + \sum_{j=k}^{p} \left(\frac{\hat{\sigma}^{(m_j)}(K)}{\sigma(K)} - \hat{\sigma}^{(m_j)}(\mathcal{E}) \right) + \left(\frac{1}{\sigma(K)} - 1 \right) \sigma^{(l_k)}(K) = 0, \quad (1.5.90)$$

$$\sum_{i=k+1}^{p+1} \left(\frac{\sigma^{(l_i)}(K)}{\sigma(K)} - \sigma^{(l_i)}(\mathcal{E}) \right) + \sum_{j=k}^{p} \left(\frac{\hat{\sigma}^{(m_j)}(K)}{\sigma(K)} - \hat{\sigma}^{(m_j)}(\mathcal{E}) \right) + \left(1 - \frac{1}{\sigma(K)} \right) \hat{\sigma}^{(m_k)}(K) = 0 \quad (1.5.91)$$

for any $k = 1, \ldots, p$, and

$$\left(\frac{1}{\sigma(K)} - 1 \right) \sigma^{(l_{p+1})}(K) = 0. \quad (1.5.92)$$

From equalities (1.5.90)-(1.5.91) we obtain

$$\left(\frac{1}{\sigma(K)} - 1 \right) \left[\sigma^{(l_k)}(K) + \hat{\sigma}^{(m_k)}(K) \right] = 0 \quad \text{for any } k = 1, \ldots, p. \quad (1.5.93)$$

Adding equalities (1.5.93) over $k = 1, \ldots, p$ and (1.5.92), and recalling that

$$\sigma(K) = \sum_{k=1}^{p+1} \sigma^{(l_k)}(K) + \sum_{k=1}^{p} \hat{\sigma}^{(m_k)}(K)\,,$$

we obtain that

$$\left(\frac{1}{\sigma(K)} - 1\right) \sigma(K) = 1 - \sigma(K) = 0 \quad \Rightarrow \quad \sigma(K) = \sigma(\{e_{k+1}\}) = 1\,.$$

However, this contradicts the fact that $\sigma \in \mathcal{P}(\mathbb{R})$, since it implies

$$\sigma(\{e_{k+1}\} \cup \{\lambda_0\}) = \sigma(\{e_{k+1}\}) + \sigma(\{\lambda_0\}) = 1 + \sigma(\{\lambda_0\}) > 1$$

(recall that $\{\lambda_0\} \in \operatorname{supp} \sigma|_{\mathcal{E}}$ by definition of λ_0). The contradiction proves that $\sigma|_{\mathcal{E}}$ is concentrated on $\{\lambda_0\}$.

The above arguments can be readily adapted to the much simpler situation concerning the restrictions $\sigma|_{(-\infty,A]}$ and $\sigma|_{[B,\infty)}$. In fact, suppose that $\operatorname{supp} \sigma|_{(-\infty,A]} \neq \emptyset$; set

$$\lambda_1 := \max\left\{\lambda \in \operatorname{supp} \sigma|_{(-\infty,A]}\right\}\,.$$

Consider for any $\delta > 0$ the compact $K = [\lambda_1 - \delta, \lambda_1]$. Then $\sigma(K) > 0$ and

$$\sigma^{(0)}((-\infty, \lambda] \cap K) = 0 \quad \text{for any } \lambda \in (-\infty, \lambda_1 - \delta]\,.$$

By equality (1.5.71) and the arbitrariness of δ we obtain that $\sigma^{(0)}((-\infty, \lambda_1)$, thus $\operatorname{supp} \sigma|_{(-\infty,A]} = \{\lambda_1\}$. It is similarly seen that $\operatorname{supp} \sigma|_{[B,\infty)}$ consists at most of one point.

The above considerations show that the support of the measure σ consists of at most $2n + 1$ points $\{q_k\}$ $(k = 0, \ldots, 2n)$, such that $q_0 \in (-\infty, A]$, $q_k \in [e_k, e_{k+1}]$ $(k = 1, \ldots, 2n-1)$ and $q_n \in [B, \infty)$. Arguing for each interval $[q_k, q_{k+1}]$ $(k = 0, \ldots, 2n-1)$ as we did before for the intervals $[e_k, e_{k+1}]$, we can prove that $\sigma(\{q_k\}) = 1$ for each $k = 0, \ldots, 2n$. However, since $\sigma \in \mathcal{P}(\mathbb{R})$, this implies that $\operatorname{supp} \sigma$ consists of one point. This completes the proof. □

1.6 Asymptotic behaviour

To prove Theorem 1.3.9 we need some preliminary results. Recall that by X we denote the space $W_0^{1,p}(\Omega)$ with $p \in (1, \infty)$ if $N = 1$, or $p \in (1, 2]$ if $N \geq 2$.

Lemma 1.6.1. *Let the assumptions of Theorem 1.3.9 be satisfied, and let $(u,\tau) \in L^\infty(\mathbb{R}_+;X) \times \mathcal{Y}(Q_\infty;\mathbb{R}^N)$ be any global Young measure solution of problem (P) given by Theorem 1.3.6. Let $u_n(\cdot,t) := u(\cdot,t+t_n)$ be the sequence defined in* (1.3.53) *for almost every $t \in \mathbb{R}_+$ and every diverging sequence $\{t_n\} \subset \mathbb{R}_+$.*
Then there exist a subsequence of $\{u_n\}$, denoted again by $\{u_n\}$ for simplicity, and $\tilde{u} \in L^\infty(\mathbb{R}_+;X) \cap W^{1,p}(Q_T) \cap C([0,\infty);L^p(\Omega))$ $(p \in (1,2])$ with $\tilde{u}_t \in L^2(Q_\infty)$, such that:

$$u_n \to \tilde{u} \quad \text{in } C([0,T];L^p(\Omega)) \qquad (p \in (1,2]) \tag{1.6.1}$$

for any $T \in (0,\infty)$;

$$u_n \rightharpoonup \tilde{u} \quad \text{in } H^1(Q_T) \text{ if } N=1\,, \quad u_n \rightharpoonup \tilde{u} \quad \text{in } W^{1,p}(Q_T) \text{ if } N \geq 2 \tag{1.6.2}$$

for any $T \in (0,\infty)$;

$$u_n \rightharpoonup \tilde{u} \quad \text{in } L^r((0,T);X) \tag{1.6.3}$$

for any $T \in (0,\infty)$ and every $r \in [1,\infty)$;

$$u_n \overset{*}{\rightharpoonup} \tilde{u} \quad \text{in } L^\infty((0,T);X) \tag{1.6.4}$$

for any $T \in (0,\infty)$;

$$\nabla u_n \overset{*}{\rightharpoonup} \nabla\tilde{u} \quad \text{in } L^\infty((0,T);L^p(\Omega)) \qquad (p \in (1,2]) \tag{1.6.5}$$

for any $T \in (0,\infty)$;

$$u_{nt} \rightharpoonup \tilde{u}_t \quad \text{in } L^2(Q_\infty)\,. \tag{1.6.6}$$

Proof. Recall that $u \in L^\infty(\mathbb{R}_+;X)$ and $u_t \in L^2(Q_\infty)$; moreover, for any $T \in (0,\infty)$ $u \in W^{1,p}(Q_T) \cap C([0,\infty);L^p(\Omega))$ ($p \in (1,2]$; see Propositions 1.3.4, 1.3.5 and Theorem 1.3.6).

Clearly, for any $n \in \mathbb{N}$ there holds

$$\|u_n\|_{L^\infty(\mathbb{R}_+;X)} + \|u_{nt}\|_{L^2(Q_\infty)} \leq \|u\|_{L^\infty(\mathbb{R}_+;X)} + \|u_t\|_{L^2(Q_\infty)}\,, \tag{1.6.7}$$

$$\|u_n\|_{H^1(Q_T)} \leq \|u\|_{H^1(Q_T)} \text{ if } N=1\,, \tag{1.6.8}$$

$$\|u_n\|_{W^{1,p}(Q_T)} \leq \|u\|_{W^{1,p}(Q_T)} \text{ if } N \geq 2 \qquad (p \in (1,2]) \tag{1.6.9}$$

for any $T \in (0,\infty)$. Arguing as in the proof of Proposition 1.3.4, the convergence in (1.6.1) follows from inequalities (1.6.9) and (1.6.8), which also imply (1.6.2), whereas the convergences in (1.6.3)-(1.6.6) follow from inequality (1.6.7). This proves the result. □

Lemma 1.6.2. *Let the assumptions of Theorem 1.3.9 be satisfied, and let $(u,\tau) \in L^\infty(\mathbb{R}_+;X) \times \mathcal{Y}(Q_\infty;\mathbb{R}^N)$ be any global Young measure solution of problem (P) given by Theorem 1.3.6. Let $\{\tau_n\} \subseteq \mathcal{Y}(Q_\infty;\mathbb{R}^N)$ be the sequence of Young measures, whose disintegration is defined in* (1.3.53) *for almost every $t \in \mathbb{R}_+$ and every diverging sequence $\{t_n\} \subset \mathbb{R}_+$. Then $\{\tau_n\}$ is tight in $\mathcal{Y}(Q_T;\mathbb{R}^N)$ for any $T \in (0,\infty)$.*

Proof. Let $T \in (0,\infty)$ be fixed, and let $\{u_{\epsilon_m}\}$ be the sequence mentioned in Propositions 1.3.4 and 1.3.5. Then, by estimates (1.3.24), (1.3.26) and Theorem 2.5.2-(ii), for every $q \in (1,p)$ (where $p \in (1,\infty)$ if $N = 1$, or $p \in (1,2]$ if $N \geq 2$) there exists $C > 0$ such that

$$\iint_{Q_T} dxdt \int_{\mathbb{R}^N} |\xi|^q \, d\tau_{(x,t+t_n)}(\xi) \, = \iint_{Q_{n,T}} dxdt \int_{\mathbb{R}^N} |\xi|^q \, d\tau_{(x,t)}(\xi) \, =$$
$$= \lim_{k\to\infty} \iint_{Q_{n,T}} |\nabla u_{\epsilon_k}|^q \, dxdt \, \leq \, C \tag{1.6.10}$$

for every $n \in \mathbb{N}$, where $Q_{n,T} := \Omega \times (t_n, t_n + T)$.

Now set for every $j \in \mathbb{N}$

$$K_j := \left\{\xi \in \mathbb{R}^N \,\middle|\, |\xi| \leq j\right\} .$$

Let us consider any function $f_j \in C(\mathbb{R}_+)$, $0 \leq f_j \leq 1$, such that

$$f_j(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq j-1\,, \\ 1 & \text{if } s \geq j\,. \end{cases}$$

Then we have

$$j^q \, \tau_n(Q_T \times (\mathbb{R}^N \setminus K_j)) \, \leq \, \iint_{Q_{n,T}} dxdt \int_{\mathbb{R}^N} f_j(|\xi|) \, |\xi|^q \, d\tau_{(x,t)}(\xi) \, \leq \, C\,,$$

whence

$$\tau_n(Q_T \times (\mathbb{R}^N \setminus K_j)) \, \leq \, C j^{-q} \quad \text{for every } j,n \in \mathbb{N}\,;$$

here the constant C is independent of n. For every $\sigma > 0$ we can choose $j > \left(\frac{C}{\sigma}\right)^{\frac{1}{q}}$, thus the conclusion follows. □

Lemma 1.6.3. *Let the assumptions of Theorem 1.3.9 be satisfied, and let* $(u,\tau) \in L^\infty(\mathbb{R}_+; X) \times \mathcal{Y}(Q_\infty; \mathbb{R}^N)$ *be any global Young measure solution of problem* (P) *given by Theorem 1.3.6. Let* $\{\tau_n\} \subseteq \mathcal{Y}(Q_\infty; \mathbb{R}^N)$ *be the sequence of Young measures whose disintegration is defined in* (1.3.53) *for almost every* $t \in \mathbb{R}_+$ *and every diverging sequence* $\{t_n\} \subset \mathbb{R}_+$.

Then there exist a subsequence of $\{\tau_n\}$*, denoted again by* $\{\tau_n\}$ *for simplicity, and* $\tilde{\tau} \in \mathcal{Y}(Q_\infty; \mathbb{R}^N)$ *such that* $\tau_n \to \tilde{\tau}$ *narrowly in* $Q_T \times \mathbb{R}^N$ *for any* $T \in (0,\infty)$.

Proof. Follows from Lemma 1.6.2 and Theorem 1.2.3 by a diagonal argument. □

To prove Theorem 1.3.9 we need another technical lemma, which corresponds to [Sl, Lemma 5.5]. A more direct proof is given here for convenience of the reader.

Lemma 1.6.4. *Let the assumptions of Theorem 1.3.9 be satisfied, and let* $(u,\tau) \in L^\infty(\mathbb{R}_+;X) \times \mathcal{Y}(Q_\infty;\mathbb{R}^N)$ *be any global Young measure solution of problem* (P) *given by Theorem 1.3.6. Let* $\{\tau_n\} \subseteq \mathcal{Y}(Q_\infty;\mathbb{R}^N)$ *be the sequence of Young measures whose disintegration is defined in* (1.3.53) *for almost every* $t \in \mathbb{R}_+$ *and every diverging sequence* $\{t_n\} \subset \mathbb{R}_+$.
Let $\{f_j\} \subseteq C(\mathbb{R}_+)$ *any sequence of functions such that*

$$\begin{cases} f_j \equiv 1 & \text{in } [0,j)\,, \\ 0 \le f_j \le 1 & \text{in } [j,j+1)\,, \\ f_j \equiv 0 & \text{in } [j+1,\infty)\,, \end{cases} \tag{1.6.11}$$

and let $\{u_{\epsilon_k}\}$ *denote the converging sequence mentioned in Propositions 1.3.4 and 1.3.5. Let* $\rho \in C(\mathbb{R}^N)$ *satisfy*

$$|\rho(\xi)| \le M\,(1+|\xi|^\gamma) \quad \text{for any } |\xi| > \xi_0\,, \tag{1.6.12}$$

for some $M>0$ *and any* $\gamma>0$ *if* $N=1$*, or any* $\gamma \in (0,p)$ *if* $N \ge 2$.
Then for every $T \in (0,\infty)$ *the map* $(x,t) \to \int_{\mathbb{R}^N} \rho(\xi)\,d\tau_{(x,t)}(\xi)$ *belongs to* $L^1(Q_T)$*, and for any* $\zeta \in L^\infty(Q_T)$ *there holds*

$$\lim_{j\to\infty} \iint_{Q_T} dxdt\,\zeta(x,t) \int_{\mathbb{R}^N} f_j(|\xi|)\rho(\xi)\,d\tau^n_{(x,t)}(\xi) = \tag{1.6.13}$$
$$= \iint_{Q_T} dxdt\,\zeta(x,t) \int_{\mathbb{R}^N} \rho(\xi)\,d\tau^n_{(x,t)}(\xi) \qquad (n \in \mathbb{N})$$

uniformly with respect to n.

Proof. Fix any $T \in (0,\infty)$. The fact that the map $(x,t) \to \int_{\mathbb{R}^N} \rho(\xi)\,d\tau_{(x,t)}(\xi)$ belongs to $L^1(Q_T)$ follows from assumption (1.6.12) arguing as in Remark 1.4.1. Set $g_j := 1 - f_j$, thus $0 \le g_j \le 1$ in $\mathbb{R}_+$ and $g_j \equiv 0$ in $[0,j)$ for every $j \in \mathbb{N}$. For any $\zeta \in L^\infty(Q_T)$ and $j,n \in \mathbb{N}$ there holds

$$\left| \iint_{Q_T} dxdt\,\zeta(x,t) \int_{\mathbb{R}^N} g_j(|\xi|)\rho(\xi)\,d\tau^n_{(x,t)}(\xi) \right| \le \tag{1.6.14}$$
$$\le \|\zeta\|_{L^\infty(Q_T)} \iint_{Q_{n,T}} dxdt \int_{\mathbb{R}^N} g_j(|\xi|)\,|\rho(\xi)|\,d\tau_{(x,t)}(\xi)\,.$$

By inequality (1.6.12), arguing as in the proof of Lemma 1.6.2 we see that for every $j \in \mathbb{N}$ the sequence $\{g_j(|\nabla u_{\epsilon_k}|)|\rho(\nabla u_{\epsilon_k})|\}$ is uniformly integrable in $L^1(Q_{n,T})$. Then by Theorem 2.5.2-(ii) there exists a subsequence (possibly depending on n), denoted again $\{g_j(|\nabla u_{\epsilon_k}|)|\rho(\nabla u_{\epsilon_k})|\}$ for simplicity, such that

$$\iint_{Q_{n,T}} dxdt \int_{\mathbb{R}^N} g_j(|\xi|)\,|\rho(\xi)|\,d\tau_{(x,t)}(\xi) = \tag{1.6.15}$$
$$= \lim_{k\to\infty} \iint_{Q_{n,T}} [g_j(|\nabla u_{\epsilon_k}|)|\rho(\nabla u_{\epsilon_k})|]\,(x,t)\,dxdt \qquad (j \in \mathbb{N})\,.$$

From (1.6.14)-(1.6.15) we get for any $\zeta \in L^\infty(Q_T)$ and $j, n \in \mathbb{N}$

$$\left| \iint_{Q_T} dxdt\, \zeta(x,t) \int_{\mathbb{R}^N} g_j(|\xi|)\rho(\xi)\, d\tau^n_{(x,t)}(\xi) \right| \le \tag{1.6.16}$$
$$\le \quad \|\zeta\|_{L^\infty(Q_T)} \sup_{k\in\mathbb{N}} \iint_{Q_{n,T}} \left[g_j(|\nabla u_{\epsilon_k}|)|\rho(\nabla u_{\epsilon_k})| \right](x,t)\, dxdt\,.$$

Using inequality (1.6.12), for any fixed $j, k, n \in \mathbb{N}$ and since $\gamma < p$ we get

$$\iint_{Q_{n,T}} \left[g_j(|\nabla u_{\epsilon_k}|)|\rho(\nabla u_{\epsilon_k})| \right](x,t)\, dxdt \;= \tag{1.6.17}$$
$$= \quad \iint_{E_{j,k,n}} \left[g_j(|\nabla u_{\epsilon_k}|)|\rho(\nabla u_{\epsilon_k})| \right](x,t)\, dxdt \;\le$$
$$\le \quad M_0 \iint_{E_{j,k,n}} |\nabla u_{\epsilon_k}|^\gamma(x,t)\, dxdt \;\le$$
$$\le \quad M_0 \left(\iint_{E_{j,k,n}} |\nabla u_{\epsilon_k}|^p(x,t)\, dxdt \right)^{\frac{\gamma}{p}} |E_{j,k,n}|^{1-\frac{\gamma}{p}}$$

for some $M_0 > 0$, where

$$E_{j,k,n} := \left\{ (x,t) \in Q_{n,T} \,\middle|\, |\nabla u_{\epsilon_k}|(x,t) \ge j \right\}\,.$$

On the other hand, by Tchebychev inequality we have

$$|E_{j,k,n}| \le \left(\iint_{E_{j,k,n}} |\nabla u_{\epsilon_k}|^p(x,t)\, dxdt \right) j^{-p}\,,$$

which together with inequality (1.6.17) gives

$$\iint_{Q_{n,T}} \left[g_j(|\nabla u_{\epsilon_k}|)|\rho(\nabla u_{\epsilon_k})| \right](x,t)\, dxdt \;\le \tag{1.6.18}$$
$$\le \quad M_0 \left(\iint_{E_{j,k,n}} |\nabla u_{\epsilon_k}|^p(x,t)\, dxdt \right) j^{-(p-\gamma)} \;\le$$
$$\le \quad M_0 T \sup_{t\in\mathbb{R}_+} \left(\int_\Omega |\nabla u_{\epsilon_k}|^p(x,t)\, dx \right) j^{-(p-\gamma)} \;\le K M_0 T\, j^{-(p-\gamma)}\,,$$

for some $K > 0$ depending on the initial data u_0 (see inequalities (1.4.6) if $N = 1$, and (1.4.18) if $N \ge 2$). Then from inequalities (1.6.14) and (1.6.16)-(1.6.18) the conclusion follows. □

Now we can prove Theorem 1.3.9.

Proof of Theorem 1.3.9. Let $(\tilde{u}, \tilde{\tau})$ be the couple whose components are mentioned in Lemmata 1.6.1 and 1.6.3. It follows by these lemmata that requirement (i) of Definition 1.3.6 is satisfied by $(\tilde{u}, \tilde{\tau})$. Then the first statement of the theorem will follow, if we prove that $(\tilde{u}, \tilde{\tau})$ also satisfies equalities (1.3.15) and (1.3.16) (with Cauchy data $\tilde{u}(\cdot, 0)$) of Definition 1.3.1.

Let us first show that $(\tilde{u}, \tilde{\tau})$ satisfies equality (1.3.15), namely that

$$\nabla \tilde{u}(x,t) = \int_{\mathbb{R}^N} \xi \, d\tilde{\tau}_{(x,t)}(\xi) \quad \text{for almost every } (x,t) \in Q_T \,, \tag{1.6.19}$$

where $\tilde{\tau}_{(x,t)} \in \mathcal{P}(\mathbb{R}^N)$ denotes the disintegration of $\tilde{\tau}$. Recalling definitions (1.3.53), from equality (1.3.15) we get

$$\nabla u_n(x,t) = \int_{\mathbb{R}^N} \xi \, d\tau^n_{(x,t)}(\xi) \tag{1.6.20}$$

for almost every $(x,t) \in Q_T$ and any $n \in \mathbb{N}$. By the convergence in (1.6.3), for any $\zeta \in L^\infty(Q_T)$ there holds

$$\begin{aligned} &\lim_{n\to\infty} \iint_{Q_T} \nabla u_n(x,t)\zeta(x,t) \, dxdt = \\ &= \iint_{Q_T} \nabla \tilde{u}(x,t)\zeta(x,t) \, dxdt \,. \end{aligned} \tag{1.6.21}$$

On the other hand, let $\{f_j\} \subseteq C(\mathbb{R}_+)$ $(j \in \mathbb{N})$ satisfy (1.6.11). By Lemma 1.6.4 the limit

$$\begin{aligned} &\lim_{j\to\infty} \iint_{Q_T} dxdt \, \zeta(x,t) \int_{\mathbb{R}^N} f_j(|\xi|)\xi \, d\tau^n_{(x,t)}(\xi) = \\ &= \iint_{Q_T} dxdt \, \zeta(x,t) \int_{\mathbb{R}^N} \xi \, d\tau^n_{(x,t)}(\xi) \qquad (n \in \mathbb{N}) \end{aligned} \tag{1.6.22}$$

exists uniformly with respect to $n \in \mathbb{N}$ (observe that choosing $\rho(\xi) = \xi$ satisfies the growth rate condition (1.6.12)). Hence, using equality (2.2.55), Lemma 1.6.2 and Theorem 1.2.3, we obtain for any $\zeta \in L^\infty(Q_T)$

$$\begin{aligned} &\lim_{n\to\infty} \iint_{Q_T} dxdt \, \zeta(x,t) \int_{\mathbb{R}^N} \xi \, d\tau^n_{(x,t)}(\xi) = \\ &= \lim_{n\to\infty} \lim_{j\to\infty} \iint_{Q_T} dxdt \, \zeta(x,t) \int_{\mathbb{R}^N} f_j(|\xi|) \, \xi \, d\tau^n_{(x,t)}(\xi) = \\ &= \lim_{j\to\infty} \lim_{n\to\infty} \iint_{Q_T} dxdt \, \zeta(x,t) \int_{\mathbb{R}^N} f_j(|\xi|) \, \xi \, d\tau^n_{(x,t)}(\xi) = \\ &= \lim_{j\to\infty} \iint_{Q_T} dxdt \, \zeta(x,t) \int_{\mathbb{R}^N} f_j(|\xi|) \, \xi \, d\tilde{\tau}_{(x,t)}(\xi) = \\ &= \iint_{Q_T} dxdt \, \zeta(x,t) \int_{\mathbb{R}^N} \xi \, d\tilde{\tau}_{(x,t)}(\xi) \end{aligned} \tag{1.6.23}$$

along some subsequence of $\{\tau_n\}$, still denoted by $\{\tau_n\}$ for simplicity; the last-mentioned equality follows by the Dominated Convergence Theorem since

$\tilde{\tau}_{(x,t)} \in \mathcal{P}(\mathbb{R}^N)$. By the arbitrariness of ζ, from equalities (1.6.21)-(1.6.23) we obtain (1.6.19).

It is similarly seen that $(\tilde{u}, \tilde{\tau})$ satisfies equality (1.3.16) with initial data $\tilde{u}(\cdot, 0)$, namely that for any $\zeta \in C^1([0,T); C_c^1(\Omega))$ and $t \in (0,T)$, where $T \in (0,\infty)$ is arbitrary, there holds

$$\int_0^t \int_\Omega \big[\tilde{u}(x,s)\,\zeta_s(x,s) - \sum_{i=1}^N \int_{\mathbb{R}^N} \varphi_i(\xi)\, d\tilde{\tau}_{(x,s)}(\xi)\, \zeta_{x_i}(x,s)\big]\, dxds =$$
$$= \int_\Omega \tilde{u}(x,t)\zeta(x,t)\, dx - \int_\Omega \tilde{u}(x,0)\,\zeta(x,0)\, dx\,. \tag{1.6.24}$$

In fact, let $\zeta \in C^1([0,T); C_c^1(\Omega))$. By abuse of notation, denote by ζ also any extension $\zeta \in C^1((-\infty,T); C_c^1(\Omega))$, thus for any sequence $\{t_n\} \subseteq \mathbb{R}_+$ and $(x,t) \in Q_T$ the translate $\zeta_n(x,t) := \zeta(x, t-t_n)$ is well defined and there holds $\zeta_n \in C^1([0,T); C_c^1(\Omega))$. Then from (1.3.16) we get

$$\begin{aligned} &\int_{t_n}^{t+t_n} \int_\Omega \big[u(x,s)\,\zeta_s(x,s-t_n) - \\ - &\sum_{i=1}^N \left(\int_{\mathbb{R}^N} \varphi_i(\xi)\, d\tau_{(x,s)}(\xi) \right) \zeta_{x_i}(x,s-t_n)\big]\, dxds = \\ = &\int_\Omega u(x,t+t_n)\zeta(x,t)\, dx - \int_\Omega u(x,t_n)\,\zeta(x,0)\, dx \end{aligned}$$

for every sequence $\{t_n\} \subseteq \mathbb{R}_+$, $t \in (0,T)$ and $T \in (0,\infty)$ such that $0 < t_n < t + t_n < T$. By the change of variables $s \to s + t_n$, recalling definition (1.3.53) the above equality reads

$$\int_0^t \int_\Omega \big[u_n(x,s)\,\zeta_s(x,s) - \sum_{i=1}^N \left(\int_{\mathbb{R}^N} \varphi_i(\xi)\, d\tau^n_{(x,s)}(\xi) \right) \zeta_{x_i}(x,s)\big]\, dxds =$$
$$= \int_\Omega u_n(x,t)\zeta(x,t)\, dx - \int_\Omega u_n(x,0)\,\zeta(x,0)\, dx\,. \tag{1.6.25}$$

By the convergence in (1.6.1) and (1.6.3) we have

$$\lim_{n\to\infty} \int_0^t \int_\Omega u_n(x,s)\,\zeta_s(x,s)\, dxds = \int_0^t \int_\Omega \tilde{u}(x,s)\,\zeta_s(x,s)\, dxds\,, \tag{1.6.26}$$

$$\lim_{n\to\infty} \int_\Omega u_n(x,t)\zeta(x,t)\, dx = \int_\Omega \tilde{u}(x,t)\zeta(x,t)\, dx\,, \tag{1.6.27}$$

$$\lim_{n\to\infty} \int_\Omega u_n(x,0)\,\zeta(x,0)\, dx = \int_\Omega \tilde{u}(x,0)\,\zeta(x,0)\, dx\,. \tag{1.6.28}$$

Therefore, letting $n \to \infty$ in equality (1.6.25) gives (1.6.24), if we prove that for every $i = 1, \dots, N$

$$\lim_{n\to\infty} \int_0^t \int_\Omega dxds\, \zeta_{x_i}(x,s) \left(\int_{\mathbb{R}^N} \varphi_i(\xi)\, d\tau^n_{(x,s)}(\xi) \right) = \tag{1.6.29}$$
$$= \int_0^t \int_\Omega dxds\, \zeta_{x_i}(x,s) \int_{\mathbb{R}^N} \varphi_i(\xi)\, d\tilde{\tau}_{(x,s)}(\xi)\,.$$

As before, let $\{f_j\} \subseteq C(\mathbb{R}_+)$ satisfy (1.6.11). By Lemma 1.6.4 the limit

$$\lim_{j\to\infty} \int_0^t \int_\Omega dxds\, \zeta_{x_i}(x,s) \int_{\mathbb{R}^N} f_j(|\xi|) \varphi_i(\xi)\, d\tau^n_{(x,s)}(\xi) = \tag{1.6.30}$$
$$= \int_0^t \int_\Omega dxds\, \zeta_{x_i}(x,s) \left(\int_{\mathbb{R}^N} \varphi_i(\xi)\, d\tau^n_{(x,s)}(\xi) \right) \qquad (n \in \mathbb{N})$$

exists uniformly with respect to $n \in \mathbb{N}$ (observe that the choice $\rho = \varphi_i$ is admissible by Remark 1.4.1). Therefore, using the Dominated Convergence Theorem we get

$$\lim_{n\to\infty} \int_0^t \int_\Omega dxds\, \zeta_{x_i}(x,s) \left(\int_{\mathbb{R}^N} \varphi_i(\xi)\, d\tau^n_{(x,s)}(\xi) \right) = \tag{1.6.31}$$
$$= \lim_{n\to\infty} \lim_{j\to\infty} \int_0^t \int_\Omega dxds\, \zeta_{x_i}(x,s) \int_{\mathbb{R}^N} f_j(|\xi|)\, \varphi_i(\xi)\, d\tau^n_{(x,s)}(\xi) =$$
$$= \lim_{j\to\infty} \lim_{n\to\infty} \int_0^t \int_\Omega dxds\, \zeta_{x_i}(x,s) \int_{\mathbb{R}^N} f_j(|\xi|)\, \varphi_i(\xi)\, d\tau^n_{(x,s)}(\xi) =$$
$$= \lim_{j\to\infty} \int_0^t \int_\Omega dxds\, \zeta_{x_i}(x,s) \int_{\mathbb{R}^N} f_j(|\xi|)\, \varphi_i(\xi)\, d\tilde{\tau}_{(x,s)}(\xi) =$$
$$= \int_0^t \int_\Omega dxds\, \zeta_{x_i}(x,s) \int_{\mathbb{R}^N} \varphi_i(\xi)\, d\tilde{\tau}_{(x,s)}(\xi)$$

(along some subsequence of $\{\tau_n\}$, still denoted by $\{\tau_n\}$) for every $i = 1, \dots, N$, $t \in (0,T)$ and any $T \in (0,\infty)$. This proves equality (1.6.29), which together with (1.6.26)-(1.6.28) gives equality (1.6.24). This completes the proof of statement (i).

To prove (ii), fix any function $f \in C_0(\mathbb{R}_+)$ such that $f(s) = 1$ if $|s| \le 1$, $0 < f(s) < 1$ for every $s \in \mathbb{R}_+$, and $f(s) \to 0$ fast enough to ensure that the function $h : \mathbb{R}^N \to \mathbb{R}_+$, $h(\xi) := f(|\xi|)\, \varphi(\xi) \cdot \xi$ belongs to $C_0(\mathbb{R}^N)$. Clearly,

$$\mathcal{S} = \left\{ \xi \in \mathbb{R}^N \,\middle|\, h(\xi) = 0 \right\};$$

hence the conclusion will follow, if we prove that

$$\iint_{Q_T} dxdt \int_{\mathbb{R}^N} h(\xi)\, d\tilde{\tau}_{(x,t)}(\xi) = 0 \tag{1.6.32}$$

(here use of assumption (H_5) has been made).

To this purpose, observe that by Theorem 1.2.2

$$\iint_{Q_T} dxdt \int_{\mathbb{R}^N} h(\xi)\, d\tilde{\tau}_{(x,t)}(\xi) = \lim_{n\to\infty} \iint_{Q_T} dxdt \int_{\mathbb{R}^N} h(\xi)\, d\tau^n_{(x,t)}(\xi)\,. \tag{1.6.33}$$

for some subsequence of $\{\tau_n\}$, still denoted by $\{\tau_n\}$, since $h \in C_0(\mathbb{R}^N)$. Moreover, since h is bounded in $\mathbb{R}^N$, the sequence $\{h(\nabla u_{\epsilon_k})\}$ is uniformly integrable in Q_T for every $T \in (0,\infty)$, thus by Theorem 2.5.2-(ii)

$$\iint_{Q_T} dxdt \int_{\mathbb{R}^N} h(\xi)\, d\tau^n_{(x,t)}(\xi) = \lim_{k\to\infty} \iint_{Q_{n,T}} h(\nabla u_{\epsilon_k})(x,t)\, dxdt \tag{1.6.34}$$

for every $n \in \mathbb{N}$. Then by (1.6.33)-(1.6.34)

$$\iint_{Q_T} dxdt \int_{\mathbb{R}^N} h(\xi)\, d\tilde{\tau}_{(x,t)}(\xi) = \lim_{n\to\infty}\lim_{k\to\infty} \iint_{Q_{n,T}} h(\nabla u_{\epsilon_k})(x,t)\, dxdt\,. \tag{1.6.35}$$

From inequality (1.3.40) written with $\epsilon = \epsilon_k$ we have

$$\begin{aligned} 0 \le \iint_{Q_{n,T}} & h(\nabla u_{\epsilon_k})(x,t)\, dxdt \le \iint_{Q_{n,T}} \left[\varphi(\nabla u_{\epsilon_k})\cdot \nabla u_{\epsilon_k}\right](x,t)\, dxdt = \\ = \; & -\frac{1}{2}\Big[\big(\|u_{\epsilon_k}(\cdot,T+t_n)\|^2_{L^2(\Omega)} - \|u_{\epsilon_k}(\cdot,t_n)\|^2_{L^2(\Omega)}\big) + \\ + \; & \epsilon_k\big(\|\nabla u_{\epsilon_k}(\cdot,T+t_n)\|^2_{L^2(\Omega)} - \|\nabla u_{\epsilon_k}(\cdot,t_n)\|^2_{L^2(\Omega)}\big)\Big]\,. \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using the convergence in (1.3.29) and estimate (1.3.26) with $p = 2$ if $N \ge 2$, respectively the convergence in (1.3.34) and estimate (1.3.24) if $N = 1$ (see Remark 1.3.3), by equality (1.6.34) we obtain

$$0 \le \iint_{Q_T} dxdt \int_{\mathbb{R}^N} h(\xi)\, d\tau^n_{(x,t)}(\xi) \le -\frac{1}{2}\left(\|u(\cdot,T+t_n)\|^2_{L^2(\Omega)} - \|u(\cdot,t_n)\|^2_{L^2(\Omega)}\right)$$

for every $n \in \mathbb{N}$. As $n \to \infty$ in the above inequality, by Remark 1.3.3 and (1.6.33) we obtain (1.6.32). This completes the proof. □

Finally let us prove Theorem 1.3.10.

Proof of Theorem 1.3.10. By Theorem 1.3.9-(ii) and the assumption $\mathcal{S} \subseteq \mathcal{S}_0$ there holds supp $\tilde{\tau}_{(x,t)} \subseteq \mathcal{S}_0$, thus

$$\int_{\mathbb{R}^N} \varphi_i(\xi)\, d\tau_{(x,t)}(\xi) = 0 \qquad (i = 1,\dots,N)$$

for almost every $(x,t) \in Q_T$. Hence for any $\rho \in C^1_c(\Omega)$ equality (1.3.19) holds, and from equality (1.6.24) we get

$$\int_\Omega \tilde{u}(x,t)\rho(x)dx = \int_\Omega \tilde{u}(x,0)\rho(x)dx$$

for every $t \in \mathbb{R}_+$. Therefore $\tilde{u}$ does not depend on t, and by equality (1.3.15) the same holds for $\tilde{\tau}$. Then the conclusion follows. □

Chapter 2

Passage to the limit over small parameters in the viscous Cahn-Hilliard equation

2.1 Introduction

Forward-backward parabolic equations arise in a variety of applications, such as edge detection in image processing [PM], aggregation models in population dynamics [Pa], and stratified turbulent shear flow [BBDPU]. A well-known equation of this type is the *Perona-Malik equation*,

$$w_t = \operatorname{div}\left(\frac{\nabla w}{1+|\nabla w|^2}\right), \tag{2.1.1}$$

which is parabolic if $|\nabla w| < 1$ and backward parabolic if $|\nabla w| > 1$. Similarly, the equation

$$u_t = \Delta\left(\frac{u}{1+u^2}\right) \tag{2.1.2}$$

is parabolic if $|u| < 1$ and backward parabolic if $|u| > 1$. Observe that in one space dimension the above equations are formally related setting $u = w_x$.

Clearly, forward-backward parabolic equations lead to ill-posed problems. Often a higher order term is added to the right-hand side to regularize the equation. Two main classes of additional terms are encountered in the mathematical literature, which, *e.g.* in case of equation (2.1.2), reduce to:

(i) $\epsilon\Delta[\psi(u)]_t$, with $\psi' > 0$, leading to third order *pseudo-parabolic equations* ($\epsilon > 0$ being a small parameter; for example, see [BBDU, EP, MTT, NP, Pl1, Pl2, S, ST2, ST3]);

(ii) $-\epsilon\Delta^2 u$, leading to fourth-order *Cahn-Hilliard type equations* (for example, see [BBMN, BFG, Pl3, Sl] and references therein).

Remarkably, when $\psi(u) = u$ either of the above regularizations can be regarded as a particular case of the *viscous Cahn-Hilliard equation*,

$$\nu u_t = \Delta[\varphi(u) - \alpha\Delta u + \beta u_t] \qquad (\alpha, \beta, \nu > 0)\,, \tag{2.1.3}$$

choosing either $\alpha = \epsilon$ or $\beta = \epsilon$; here $\varphi(u) = \frac{u}{1+u^2}$ for equation (2.1.2), whereas in general it denotes a *non-monotonic* function.

Equation (2.1.3) has been derived by several authors using different physical considerations (in particular, see [G, JF, N]). *It is worth mentioning the wide literature concerning both the relationship between the viscous Cahn-Hilliard equation and phase field models, and generalized versions of the equation suggested in [G] (e.g., see [R] and references therein).*

Formally, when $\nu = 1$, $\beta = 0$ it gives the *Cahn-Hilliard equation*,

$$u_t = \Delta[\varphi(u) - \alpha\Delta u]\,, \tag{2.1.4}$$

whereas for $\nu = 0$, $\beta = 1$ it reduces to the *Allen-Cahn equation*,

$$u_t = \alpha\Delta u - \varphi(u)\,. \tag{2.1.5}$$

This suggests the choice $\nu = 1 - \beta$ ($\beta \in (0,1)$), as we do hereafter.

It is natural to ask whether the above formal arguments can be given a sound analytical meaning, proving that the *singular limit* of solutions of the equation

$$(1-\beta)u_t = \Delta[\varphi(u) - \alpha\Delta u + \beta u_t]$$

(complemented with suitable initial and boundary conditions), as either $\beta \to 0^+$ or $\beta \to 1^-$, obtains a solution (of the corresponding problem) for equation (2.1.4), respectively (2.1.5). If so, this motivates the use of the above regularizations (i)-(ii), which can be regarded as limiting cases of a more complete physical model.

A related, widely investigated problem is to let the regularizing parameter ϵ in (i)-(ii) to zero, seeking a proper definition of the original ill-posed problem by a "vanishing viscosity" method. Although quite a few results have been obtained in this direction for the pseudoparabolic regularization (i) (in particular, see [Pl1, Pl2, S, ST3]), less is known for the Cahn-Hilliard regularization (ii) [Pl4].

In the light of the above considerations, in this paper we investigate the singular limits as $\beta \to 0^+$ or $\beta \to 1^-$ (for fixed $\alpha > 0$) of solutions to the initial-boundary value problem

$$(P) \qquad \begin{cases} (1-\beta)u_t = \Delta[\varphi(u) - \alpha\Delta u + \beta u_t] & \text{in } \Omega \times (0,T) =: Q \\ u = \Delta u = 0 & \text{on } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\}\,, \end{cases}$$

where $\alpha \in (0,\infty)$, $\beta \in (0,1)$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ if $N \geq 2$, and $T > 0$. As expected, we prove convergence in a suitable sense to solutions of the problem for the Cahn-Hilliard equation:

$$(CH) \qquad \begin{cases} u_t = \Delta[\varphi(u) - \alpha\Delta u] & \text{in } Q \\ u = \Delta u = 0 & \text{on } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\}\,, \end{cases}$$

as $\beta \to 0^+$, or respectively of the problem for the Allen-Cahn equation:

$$(AC) \qquad \begin{cases} u_t = \alpha\Delta u - \varphi(u) & \text{in } Q \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\}\,, \end{cases}$$

as $\beta \to 1^-$ (see Theorems 2.2.4-2.2.5). Concerning the function $\varphi : \mathbb{R} \to \mathbb{R}$, the following assumptions are used:

$$(H_0) \qquad \varphi \in W^{1,\infty}_{loc}(\mathbb{R})\,, \quad \varphi(u)u \geq 0 \text{ for any } u \in \mathbb{R}\,;$$

(H_1) there exists $K > 0$ such that

$$|\varphi'(u)| \leq K(1 + |u|^{q-1}) \tag{2.1.6}$$

for some $q \in (1,\infty)$ if $N = 1,2$, or $q \in \left(1, \frac{N+2}{N-2}\right]$ if $N \geq 3$.
Observe that by assumption (H_0) the function φ is locally Lipschitz continuous and there holds $\varphi(0) = 0$.

Further, we study the limit of solutions of problem (P) as $\alpha \to 0^+$ (for fixed $\beta \in (0,1)$), proving convergence to solutions of the problem for the *Sobolev equation:*

$$(S) \qquad \begin{cases} (1-\beta)u_t = \Delta[\varphi(u) + \beta u_t] & \text{in } Q \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\} \end{cases}$$

under additional assumptions on the function φ (see assumptions (H_2)-(H_3) and Theorem 2.2.6).

Finally, we study the limit of solutions of problem (CH) as $\alpha \to 0^+$, proving the existence of a triple (u, v, μ) - where u, v are functions and μ is a *finite Radon measure* on Q - which satisfies a weak limiting equality (see Theorem 2.2.7, in particular equality (2.2.60)). We cannot maintain that this triple is in some sense a solution of the limiting problem

$$\begin{cases} u_t = \Delta\varphi(u) & \text{in } Q \\ u = \Delta u = 0 & \text{on } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\}\,, \end{cases}$$

since the relation between v and the function $\varphi(u)$, even in the sense of *Young measures*, is unclear. This point was addressed in [Pl4], taking advantage of the cubic-like growth of φ at infinity, which gives rise to better estimates of the family $\{u_\alpha\}$ of solutions of (CH); at the same time, this growth prevented the appearance of a Radon measure in the solution. Instead in the present case, if φ grows linearly at infinity (see assumption (H_4)), we only have L^1-estimates of the family $\{u_\alpha\}$, which are compatible with the need of a Radon measure to describe solutions of the problem (in this connection, see [BBDU, PST, S, ST3]). Similar and more enhanced phenomena can be expected, if φ either has a sublinear growth, or vanishes at infinity, pointing out this behaviour as a major feature for the understanding of the problem.

Our approach is based on a detailed analysis of solutions of problem (P), which relies on an *approximation method* already used in similar cases [BBDU, BST1, BST2, PST, S, ST2, ST3]. Beside lending in a natural way the estimates needed to study the singular limits, it allows to improve in several ways on the available existence results for the viscous Cahn-Hilliard equation ([CD, ES]; see Theorem 2.2.2).

Our main results are presented in Section 2.2, and their proofs are given in Sections 2.3-2.5. Similar results for the case of homogeneous Neumann boundary conditions are discussed in Section 2.6. Let us point out two of their novel features with respect to the situation outlined for the Dirichlet case:
(i) as it can be expected, when $\beta \to 1^-$ we prove convergence to solutions of a *nonlocal* Allen-Cahn equation investigated in [RS];
(ii) taking advantage of the *conservation of mass*, weaker assumptions on the behaviour at infinity of the function φ can be made (see assumption (H_5)).

2.2 Mathematical framework and results

2.2.1 Well-posedness and a priori estimates.

Let us state the following definition.

Definition 2.2.1. *Let $\alpha \in (0,\infty)$, $\beta \in (0,1)$, and let $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$. By a strict solution of problem (P) we mean any function $u \in C([0,T]; H^2(\Omega) \cap H^1_0(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ such that $\varphi(u) \in C([0,T]; L^2(\Omega))$, and*

$$\begin{cases} u_t = \Delta v & \text{in } Q \\ u = u_0 & \text{in } \Omega \times \{0\} \end{cases} \tag{2.2.1}$$

in strong sense. Here $v \in C([0,T]; H^2(\Omega) \cap H^1_0(\Omega))$ and for every $t \in [0,T]$

the function $v(\cdot,t)$ is the unique solution of the elliptic problem

$$\begin{cases} -\beta\Delta v(\cdot,t)+(1-\beta)v(\cdot,t)=\varphi(u)(\cdot,t)-\alpha\Delta u(\cdot,t) & \textit{in } \Omega \\ v(\cdot,t)=0 & \textit{on } \partial\Omega\,. \end{cases} \tag{2.2.2}$$

The function v is called chemical potential.

A first well-posedness result for problem (P), if assumptions (H_0)-(H_1) are replaced by the stronger condition

$$(H_2) \qquad\qquad \varphi\in Lip\,(\mathbb{R})\,,\quad \varphi(u)u\geq 0 \text{ for any } u\in\mathbb{R}\,,$$

is the content of the following theorem.

Theorem 2.2.1. *Let $\alpha\in(0,\infty)$, $\beta\in(0,1)$, and let φ satisfy assumption (H_2). Then for every $u_0\in H^2(\Omega)\cap H^1_0(\Omega)$ there exists a unique strict solution of problem (P).*

Let us now address well-posedness when φ satisfies assumptions (H_0)-(H_1), and the Cauchy data function u_0 belongs to $H^1_0(\Omega)$. In this case solutions of problem (P) are meant in the following sense.

Definition 2.2.2. *Let $\alpha\in(0,\infty)$, $\beta\in(0,1)$, and let $u_0\in H^1_0(\Omega)$. By a solution of problem (P) we mean any function $u\in C([0,T];H^1_0(\Omega))$ such that:*
(i) $u_t\in L^2(Q)$, $\varphi(u)\in L^2(Q)$, $\Delta u\in L^2(Q)$;
(ii) problems (2.2.1) *and* (2.2.2) *are satisfied in strong sense, with $v\in L^\infty((0,T);H^1_0(\Omega))\cap L^2((0,T);H^2(\Omega))$.*

For every $u_0\in H^1_0(\Omega)$, let $\{u_{0n}\}\subseteq C^\infty_c(\Omega)$ be any sequence such that

$$\|u_{0n}\|_{H^1_0(\Omega)}\leq\|u_0\|_{H^1_0(\Omega)}\,, \tag{2.2.3}$$

$$u_{0n}\to u_0 \ \text{ in } H^1_0(\Omega)\,. \tag{2.2.4}$$

For any $n\in\mathbb{N}$ set

$$\varphi_n(u):=\begin{cases}\varphi(u) & \text{if } |u|\leq n\,,\\ \varphi(n)+K(u-n) & \text{if } u>n\,,\\ \varphi(-n)+K(u+n) & \text{if } u<-n\,,\end{cases} \tag{2.2.5}$$

where $K>0$ is the constant in assumption (H_1). It is immediately seen that $\varphi_n(u)u\geq 0$ for any $u\in\mathbb{R}$, and

$$\varphi_n(u)\to\varphi(u) \ \text{ for any } u\in\mathbb{R} \tag{2.2.6}$$

as $n\to\infty$. Moreover, for every $n\in\mathbb{N}$ there holds $\varphi_n\in Lip\,(\mathbb{R})$, since

$$|\varphi_n'(u)|\leq K\left\{\left(1+|u|^{q-1}\right)\chi_{\{|u|\leq n\}}(u)+\chi_{\{|u|>n\}}(u)\right\}\,, \tag{2.2.7}$$

thus in particular

$$|\varphi_n'(u)| \le K\left(1+n^{q-1}\right) \quad \text{for every } n\in\mathbb{N} \qquad (u\in\mathbb{R})\,. \tag{2.2.8}$$

Observe that inequality (2.2.7) also implies the estimate:

$$|\varphi_n'(u)| \le K\left(1+|u|^{q-1}\right) \quad \text{for every } u\in\mathbb{R} \qquad (n\in\mathbb{N})\,. \tag{2.2.9}$$

Remark 2.2.1. *By inequality* (2.2.9) *there exists $K_1>0$ such that*

$$|\varphi_n(u)| \le K_1\left(1+|u|^{q}\right) \quad \textit{for every } u\in\mathbb{R} \qquad (n\in\mathbb{N})\,, \tag{2.2.10}$$

$$0\le \Phi_n(u) \le K_1\left(1+|u|^{q+1}\right) \quad \textit{for every } u\in\mathbb{R}\,, \tag{2.2.11}$$

where

$$\Phi_n(u) := \int_0^u \varphi_n(z)\,dz \qquad (u\in\mathbb{R},\, n\in\mathbb{N}) \tag{2.2.12}$$

(observe that $\Phi_n(u)\ge 0$ for any $u\in\mathbb{R}$, since $\varphi_n(u)u\ge 0$). Clearly, analogous inequalities hold for φ and for its antiderivative

$$\Phi(u) := \int_0^u \varphi(z)\,dz \qquad (u\in\mathbb{R})\,. \tag{2.2.13}$$

Moreover, as $n\to\infty$ there holds

$$\Phi_n(u)\to\Phi(u) \quad \textit{for any } u\in\mathbb{R}\,. \tag{2.2.14}$$

Similar considerations hold true for the following functions:

$$\psi(u) := \int_0^u |z|^{q-1} dz\,, \tag{2.2.15}$$

$$\Psi(u) := \int_0^u \psi(z)\,dz \qquad (u\in\mathbb{R})\,. \tag{2.2.16}$$

Observe that by inequality (2.2.9) *there exists $\tilde{K}>0$ such that*

$$|\varphi_n(u)| \le \tilde{K}\,|\psi(u)| \quad \textit{for every } u\in\mathbb{R} \qquad (n\in\mathbb{N})\,. \tag{2.2.17}$$

Consider the family of approximating problems

$$(P_n) \qquad \begin{cases} (1-\beta)u_{nt} = \Delta[\varphi_n(u_n) - \alpha\Delta u_n + \beta u_{nt}] & \text{in } Q \\ u_n = \Delta u_n = 0 & \text{on } \partial\Omega\times(0,T) \\ u_n = u_{0n} & \text{in } \Omega\times\{0\}\,, \end{cases}$$

with u_{0n} and φ_n defined as above. Observe that problem (P_n) can be recast in the form

$$\begin{cases} u_{nt} = \Delta v_n & \text{in } Q \\ u_n = u_{0n} & \text{in } \Omega\times\{0\} \end{cases} \tag{2.2.18}$$

where for every $t \in [0,T]$ the function $v_n(\cdot,t)$ solves the elliptic problem

$$\begin{cases} -\beta\Delta v_n(\cdot,t) + (1-\beta)v_n(\cdot,t) = \varphi_n(u_n)(\cdot,t) - \alpha\Delta u_n(\cdot,t) & \text{in } \Omega \\ v_n(\cdot,t) = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.2.19}$$

By the above remarks (in particular, see inequality (2.2.8)), every function φ_n satisfies assumption (H_2), whereas u_{0n} belongs to $H^2(\Omega)\cap H^1_0(\Omega)$. Then by Theorem 2.2.1 for every $n \in \mathbb{N}$ there exists a unique strict solution u_n of problem (P_n). By studying the limiting points of the sequence $\{u_n\}$ we shall prove the following result.

Theorem 2.2.2. *Let $\alpha \in (0,\infty)$, $\beta \in (0,1)$, and let φ satisfy assumptions (H_0)-(H_1). Then for every $u_0 \in H^1_0(\Omega)$ there exists a solution of problem (P). If φ satisfies assumption (H_2), the solution is unique.*
Moreover, for every $\bar\alpha > 0$ there exists $M > 0$ (only depending on the norm $\|u_0\|_{H^1_0(\Omega)}$) such that for any $\alpha \in (0,\bar\alpha)$ and $\beta \in (0,1)$

$$\|\Phi(u)\|_{L^\infty((0,T);L^1(\Omega))} \leq M\,, \tag{2.2.20}$$

where the function Φ is defined in (2.2.13);

$$\sqrt{\alpha}\,\|u\|_{L^\infty((0,T);H^1_0(\Omega))} \leq M\,; \tag{2.2.21}$$

$$\sqrt{\beta}\,\|u_t\|_{L^2(Q)} \leq M\,; \tag{2.2.22}$$

$$\sqrt{\alpha}\,\|\varphi(u)\|_{L^2(Q)} \leq M\,; \tag{2.2.23}$$

$$\alpha^{\frac{3}{2}}\,\|\Delta u\|_{L^2(Q)} \leq M\,; \tag{2.2.24}$$

$$\sqrt{1-\beta}\,\|v\|_{L^2((0,T);H^1_0(\Omega))} \leq M\,; \tag{2.2.25}$$

$$\sqrt{\alpha}\,\beta\,\|v\|_{L^\infty((0,T);H^1_0(\Omega))} \leq M\,; \tag{2.2.26}$$

$$\sqrt{\beta(1-\beta)}\,\|v\|_{L^2((0,T);H^2(\Omega))} \leq M\,. \tag{2.2.27}$$

Further estimates of the solution given by Theorem 2.2.2 are the content of the following theorem.

Theorem 2.2.3. *Let $\alpha \in (0,\infty)$, $\beta \in (0,1)$ and $u_0 \in H^1_0(\Omega)$. Let φ satisfy either assumption (H_2), or assumptions (H_0), (H_1) and the following one:*

(H_3) *there exists $u_0 > 0$ such that $\varphi'(u) > 0$ if $|u| \geq u_0$.*

Let u be the solution of problem (P) given by Theorem 2.2.2. Then for every $\alpha \in (0,\infty)$ and $\beta \in (0,1)$

$$\|u\|_{L^\infty((0,T);L^2(\Omega))} \leq \|u_0\|_{H^1_0(\Omega)}\sqrt{\frac{1+e^{\frac{2LT}{\beta}}}{1-\beta}}\,; \tag{2.2.28}$$

$$\|u\|_{L^\infty((0,T);H_0^1(\Omega))} \le \|u_0\|_{H_0^1(\Omega)} \sqrt{\frac{2\left(1+e^{\frac{2LT}{\beta}}\right)}{\beta}}\,; \tag{2.2.29}$$

$$\|\Delta u\|_{L^2(Q)} \le \|u_0\|_{H_0^1(\Omega)} \sqrt{\frac{1+e^{\frac{2LT}{\beta}}}{\alpha}}\,. \tag{2.2.30}$$

Moreover, for every $\bar{\alpha} > 0$ and $\beta \in (0,1)$ there exists $\bar{M} > 0$ (only depending on the norm $\|u_0\|_{H_0^1(\Omega)}$ and on β, and diverging as $\beta \to 0^+$, $\beta \to 1^-$) such that for any $\alpha \in (0,\bar{\alpha})$ and $n \in \mathbb{N}$

$$\|\varphi(u)\|_{L^2(Q)} \le \bar{M}\,. \tag{2.2.31}$$

Remark 2.2.2. *In connection with the above theorem, observe that:*
(i) if (H_2) is satisfied, there exists $L > 0$ such that $|\varphi_n'(u)| \le L$ for any $n \in \mathbb{N}$ and $u \in \mathbb{R}$;
(ii) if (H_3) is satisfied, there holds $\varphi_n'(u) > 0$ for any $n \in \mathbb{N}, n > u_0$ and $u \in \mathbb{R}, |u| \ge u_0$.

2.2.2 Asymptotical limits

The limit $\beta \to 0^+$ (for fixed $\alpha > 0$).

As $\beta \to 0^+$, inequalities (2.2.22) and (2.2.26)-(2.2.31) get lost. Accordingly, solutions of problem (CH) for the Cahn-Hilliard equation are meant in the following sense.

Definition 2.2.3. *Let $\alpha \in (0,\infty)$ and $u_0 \in H_0^1(\Omega)$. By a solution of problem (CH) we mean any function $u \in L^\infty((0,T);H_0^1(\Omega))$ such that:*
(i) $\varphi(u) \in L^2(Q)$, $\Delta u \in L^2(Q)$, and $v := \varphi(u) - \alpha\Delta u \in L^2((0,T);H_0^1(\Omega))$;
(ii) there holds

$$\iint_Q u\,\zeta_t\,dxdt + \iint_Q [\varphi(u) - \alpha\Delta u]\,\Delta\zeta\,dxdt = -\int_\Omega u_0(x)\zeta(x,0)\,dx \tag{2.2.32}$$

for every $\zeta \in C^1([0,T];C_c^2(\Omega))$ such that $\zeta(\cdot,T) = 0$ in Ω.

Theorem 2.2.4. *Let $u_0 \in H_0^1(\Omega)$, and let φ satisfy assumptions (H_0)-(H_1). Let $u_{\alpha,\beta}$ be the solution of problem (P) given by Theorem 2.2.2 ($\alpha \in (0,\infty)$, $\beta \in (0,1)$). Then for every $\alpha \in (0,\infty)$ there exist $u_\alpha \in L^\infty((0,T);H_0^1(\Omega))$, $v_\alpha \in L^2((0,T);H_0^1(\Omega))$ and two subsequences $\{u_{\alpha,\beta_k}\} \subseteq \{u_{\alpha,\beta}\}$, $\{v_{\alpha,\beta_k}\} \subseteq \{v_{\alpha,\beta}\}$ such that*
(i) $\varphi(u_\alpha) \in L^2(Q)$, $\Delta u_\alpha \in L^2(Q)$;
(ii) as $\beta_k \to 0^+$ there holds

$$u_{\alpha,\beta_k} \overset{*}{\rightharpoonup} u_\alpha \quad \text{in } L^\infty((0,T);H_0^1(\Omega))\,, \tag{2.2.33}$$

$$u_{\alpha,\beta_k} \to u_\alpha \quad \textit{almost everywhere in } Q\,, \tag{2.2.34}$$

$$\varphi(u_{\alpha,\beta_k}) \rightharpoonup \varphi(u_\alpha) \quad \textit{in } L^2(Q)\,, \tag{2.2.35}$$

$$\Delta u_{\alpha,\beta_k} \rightharpoonup \Delta u_\alpha \quad \textit{in } L^2(Q)\,; \tag{2.2.36}$$

$$v_{\alpha,\beta_k} \rightharpoonup v_\alpha \quad \textit{in } L^2((0,T);H^1_0(\Omega))\,; \tag{2.2.37}$$

(iii) the function u_α is a solution of problem (CH);

(iv) the function u_α satisfies inequalities (2.2.20)-(2.2.21) *and* (2.2.23)-(2.2.24), *whereas v_α satisfies the a priori estimate*

$$\|v_\alpha\|_{L^2((0,T);H^1_0(\Omega))} \leq M \tag{2.2.38}$$

with some constant $M > 0$ only depending on the norm $\|u_0\|_{H^1_0(\Omega)}$.

The limit $\beta \to 1^-$ (for fixed $\alpha > 0$).

As $\beta \to 1^-$, inequalities (2.2.25), (2.2.27)-(2.2.28) and (2.2.31) are lost. Solutions of the initial-boundary value problem (AC) are defined as follows.

Definition 2.2.4. *Let $\alpha \in (0,\infty)$, and let $u_0 \in H^1_0(\Omega)$. By a solution of problem (AC) we mean any function $u \in L^\infty((0,T);H^1_0(\Omega)) \cap C([0,T];L^2(\Omega))$ such that:*

(i) $u_t \in L^2(Q)$, $\varphi(u) \in L^2(Q)$, $\Delta u \in L^2(Q)$;

(ii) problem (AC) is satisfied in strong sense.

Theorem 2.2.5. *Let $u_0 \in H^1_0(\Omega)$, and let φ satisfy assumptions (H_0)-(H_1). Let $u_{\alpha,\beta}$ be the solution of problem (P) given by Theorem 2.2.2 ($\alpha \in (0,\infty)$, $\beta \in (0,1)$). Then for every $\alpha \in (0,\infty)$ there exist a function $u_\alpha \in L^\infty((0,T);H^1_0(\Omega)) \cap C([0,T];L^2(\Omega))$ and a subsequence $\{u_{\alpha,\beta_k}\} \subseteq \{u_{\alpha,\beta}\}$ such that*

(i) $u_{\alpha t} \in L^2(Q)$, $\varphi(u_\alpha) \in L^2(Q)$, $\Delta u_\alpha \in L^2(Q)$;

(ii) as $\beta_k \to 1^-$ there holds

$$u_{\alpha,\beta_k} \overset{*}{\rightharpoonup} u_\alpha \quad \textit{in } L^\infty((0,T);H^1_0(\Omega))\,, \tag{2.2.39}$$

$$u_{\alpha,\beta_k} \to u_\alpha \quad \textit{almost everywhere in } Q\,, \tag{2.2.40}$$

$$(u_{\alpha,\beta_k})_t \rightharpoonup u_{\alpha t} \quad \textit{in } L^2(Q)\,, \tag{2.2.41}$$

$$\varphi(u_{\alpha,\beta_k}) \rightharpoonup \varphi(u_\alpha) \quad \textit{in } L^2(Q)\,, \tag{2.2.42}$$

$$\Delta u_{\alpha,\beta_k} \rightharpoonup \Delta u_\alpha \quad \textit{in } L^2(Q)\,; \tag{2.2.43}$$

(iii) the function u_α is a solution of problem (AC);

(iv) the function u_α satisfies inequalities (2.2.20)-(2.2.21), (2.2.23)-(2.2.24), *and the a priori estimate*

$$\|u_{\alpha t}\|_{L^2(Q)} \leq M \tag{2.2.44}$$

with some constant $M > 0$ only depending on the norm $\|u_0\|_{H_0^1(\Omega)}$.
Moreover, if φ satisfies either assumption (H_2), or assumptions (H_0), (H_1) and (H_3), then u_α satisfies the a priori estimates

$$\|u_\alpha\|_{L^\infty((0,T);H_0^1(\Omega))} \leq \sqrt{2}\,\|u_0\|_{H_0^1(\Omega)}\left(1+e^{2LT}\right)^{\frac{1}{2}}\ ; \tag{2.2.45}$$

$$\sqrt{\alpha}\,\|\Delta u_\alpha\|_{L^2(Q)} \leq \|u_0\|_{H_0^1(\Omega)}\left(1+e^{2LT}\right)^{\frac{1}{2}} \qquad (\alpha\in(0,\infty))\,. \tag{2.2.46}$$

The limit $\alpha \to 0^+$ (for fixed $\beta \in (0,1)$).

In this case inequalities (2.2.21), (2.2.23)-(2.2.24), (2.2.26) and (2.2.30) are lost. Solutions of problem (S) are meant in the following sense.

Definition 2.2.5. *Let $\beta \in (0,1)$, and let $u_0 \in H_0^1(\Omega)$. By a solution of problem (S) we mean any function $u \in L^\infty((0,T);H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$ such that:*
(i) $u_t \in L^2(Q)$, $\varphi(u) \in L^2(Q)$;
(ii) problem (2.2.1) is satisfied in strong sense with $v := \frac{1}{1-\beta}\left[\varphi(u)+\beta u_t\right] \in L^2((0,T);H_0^1(\Omega)\cap H^2(\Omega))$.

Theorem 2.2.6. *Let $u_0 \in H_0^1(\Omega)$, and let φ satisfy either assumption (H_2), or assumptions (H_0), (H_1) and (H_3). Let $u_{\alpha,\beta}$ be the solution of problem (P) given by Theorem 2.2.2 ($\alpha \in (0,\infty)$, $\beta \in (0,1)$). Then for every $\beta \in (0,1)$ there exist functions $u_\beta \in L^\infty((0,T);H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$, $v_\beta \in L^2((0,T);H_0^1(\Omega)\cap H^2(\Omega))$ and a subsequence $\{u_{\alpha_k,\beta}\} \subseteq \{u_{\alpha,\beta}\}$ such that*
(i) $u_{\beta t} \in L^2(Q)$, $\varphi(u_\beta) \in L^2(Q)$;
(ii) as $\alpha_k \to 0^+$ there holds

$$u_{\alpha_k,\beta} \overset{*}{\rightharpoonup} u_\beta \quad \textit{in } L^\infty((0,T);H_0^1(\Omega))\,, \tag{2.2.47}$$

$$u_{\alpha_k,\beta} \to u_\beta \quad \textit{almost everywhere in } Q\,, \tag{2.2.48}$$

$$(u_{\alpha_k,\beta})_t \rightharpoonup u_{\beta t} \quad \textit{in } L^2(Q)\,, \tag{2.2.49}$$

$$\varphi(u_{\alpha_k,\beta}) \rightharpoonup \varphi(u_\beta) \quad \textit{in } L^2(Q)\,, \tag{2.2.50}$$

$$v_{\alpha_k,\beta} \rightharpoonup v_\beta \quad \textit{in } L^2((0,T);H_0^1(\Omega))\,, \tag{2.2.51}$$

$$\Delta v_{\alpha_k,\beta} \rightharpoonup \Delta v_\beta \quad \textit{in } L^2(Q)\,; \tag{2.2.52}$$

(iii) the function u_β is a solution of problem (S) with $v_\beta = \frac{1}{1-\beta}\left[\varphi(u_\beta)+\beta u_{\beta t}\right]$;

(iv) the function u_β satisfies inequalities (2.2.20), (2.2.22), (2.2.27)-(2.2.29) and (2.2.31), whereas v_β satisfies inequality (2.2.25).

2.2.3 Letting $\alpha \to 0^+$ in problem (CH).

In this case the solution u_α of problem (CH) given by Theorem 2.2.4 satisfies inequalities (2.2.20)-(2.2.21) and (2.2.23)-(2.2.24), whereas $v_\alpha := \varphi(u_\alpha) - \alpha\Delta u_\alpha$ satisfies inequality (2.2.38) (see Theorem 2.2.4-(iv)). As $\alpha \to 0^+$, inequalities (2.2.21), (2.2.23) and (2.2.24) get lost, thus the only a priori estimate of the family $\{u_\alpha\}$ (uniform with respect to α) is given by inequality (2.2.20). As a consequence, different situations will expectedly arise in the limit as $\alpha \to 0^+$, depending on the behaviour at infinity of the function Φ. This makes the following assumption expedient to study the above limit:

(H_4) there exists $k > 0$ such that

$$k\,|u|^r \leq \Phi(u) \tag{2.2.53}$$

for some $r \in [1, \infty)$ if $N = 1, 2$, or $r \in \left[1, \frac{2N}{N-2}\right]$ if $N \geq 3$.

Observe that the above conditions on the exponent r follow from assumption (H_1) and the compatibilty condition $r \leq q + 1$ (see (H_1), (H_4)).

Let us recall for further purposes some results concerning Radon measures on the set Q. By $\mathcal{M}(\Omega)$ we denote the space of finite Radon measures on Ω, and by $\mathcal{M}^+(\Omega)$ the cone of positive (finite) Radon measures on Ω. We denote by $\langle \cdot, \cdot \rangle_\Omega$ the duality map between $\mathcal{M}(\Omega)$ and the space $C_c(\Omega)$ of continuous functions with compact support. For $\mu \in \mathcal{M}(\Omega)$ and $\rho \in L^1(\Omega, \mu)$ we set, by abuse of notation,

$$\langle \mu, \rho \rangle_\Omega := \int_\Omega \rho(x)\, d\mu(x) \quad \text{and} \quad \|\mu\|_{\mathcal{M}(\Omega)} := |\mu|(\overline{\Omega})\,. \tag{2.2.54}$$

Similar notations will be used for the space of finite Radon measures on Q.

We denote by $L^\infty((0,T); \mathcal{M}(\Omega))$ the set of finite Radon measures $u \in \mathcal{M}(Q)$ which satisfy the following: for almost every $t \in (0,T)$ there exists a measure $u(\cdot, t) \in \mathcal{M}(\Omega)$, such that
(i) for every $\zeta \in C(\overline{Q})$ the map $t \to \langle u(\cdot,t), \zeta(\cdot,t)\rangle_\Omega$ is Lebesgue measurable, and

$$\langle u, \zeta \rangle_Q = \int_0^T \langle u(\cdot,t), \zeta(\cdot,t)\rangle_\Omega \, dt\,; \tag{2.2.55}$$

(ii) there holds

$$\text{ess sup}_{t\in(0,T)} \|u(\cdot,t)\|_{\mathcal{M}(\Omega)} < \infty\,.$$

The definition of the positive cone $L^\infty((0,T); \mathcal{M}^+(\Omega))$ should now be obvious.

Let us also recall the following definition.

Definition 2.2.6. *A subset $\mathcal{U} \subseteq L^1(Q)$ is said to be uniformly integrable if:*

(i) *there exists* $M > 0$ *such that*

$$\|f\|_{L^1(Q)} := \iint_Q |f(x,t)|\,dxdt \le M \quad \text{for any } f \in \mathcal{U}\,;$$

(ii) *for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any* $f \in \mathcal{U}$ *and any Lebesgue measurable set* $E \subseteq Q$

$$|E| < \delta \quad \Rightarrow \quad \iint_E |f(x,t)|\,dxdt < \varepsilon\,.$$

Then we can state the following result.

Theorem 2.2.7. *Let* $u_0 \in H_0^1(\Omega)$, *and let* φ *satisfy assumptions* (H_0), (H_1) *and* (H_4). *Let* u_α *be the solution of problem* (CH) *given by Theorem 2.2.4 (*$\alpha \in (0,\infty)$*). Then there exist* $u \in L^\infty((0,T);L^r(\Omega))$, $\mu \in L^\infty((0,T);\mathcal{M}(\Omega))$ *and*
$v \in L^2((0,T);H_0^1(\Omega))$ *with the following properties:*
(i) *there exist two subsequences* $\{u_{\alpha_k}\} \subseteq \{u_\alpha\}$, $\{v_{\alpha_k}\} \subseteq \{v_\alpha\}$ *and a decreasing sequence of measurable sets* $E_k \subseteq Q$ *of Lebesgue measure* $|E_k| \to 0$, *such that the sequence* $\left\{u_{\alpha_k}\chi_{Q\setminus E_k}\right\}$ *is uniformly integrable, and as* $\alpha_k \to 0^+$ *there holds*

$$u_{\alpha_k}\chi_{Q\setminus E_k} \rightharpoonup u \text{ in } L^r(Q)\,, \tag{2.2.56}$$

$$u_{\alpha_k}\chi_{E_k} \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(Q)\,, \tag{2.2.57}$$

$$\varphi(u_{\alpha_k}) \rightharpoonup v \quad \text{in } \mathcal{D}(Q)\,, \tag{2.2.58}$$

$$v_{\alpha_k} \rightharpoonup v \text{ in } L^2((0,T);H_0^1(\Omega))\,; \tag{2.2.59}$$

(ii) *there holds*

$$\iint_Q u\zeta_t\,dxdt + \int_0^T < \mu(\cdot,t),\zeta_t(\cdot,t) >_\Omega dt = \iint_Q \nabla v\cdot\nabla\zeta\,dxdt - \int_\Omega u_0(x)\zeta(x,0)\,dx \tag{2.2.60}$$

for every $\zeta \in C^1([0,T];C_c^1(\Omega))$ *such that* $\zeta(\cdot,T) = 0$ *in* Ω.
Moreover, the measure μ *is equal to 0 if assumption* (H_4) *is satisfied with* $r > 1$. *In this case* $E_k = \emptyset$ *for every* $k \in \mathbb{N}$, *the convergence in* (2.2.56) *reads*

$$u_{\alpha_k} \rightharpoonup u \text{ in } L^r(Q)\,, \tag{2.2.61}$$

and (2.2.57) *is trivially satisfied.*

2.3 Well-posedness : Proofs

Theorem 2.2.1 is easily proven by standard methods of the theory of abstract evolution equations, if problem (P) is rephrased as a Cauchy problem in the Banach space $X = L^2(\Omega)$ - an approach already used in [CD, ES]. To this purpose, denote by $[I - \epsilon\Delta]^{-1}$ $(\epsilon > 0)$ the operator

$$[I - \epsilon\Delta]^{-1} : L^2(\Omega) \to H^2(\Omega) \cap H^1_0(\Omega)\,, \quad [I - \epsilon\Delta]^{-1} z := w \quad (z \in L^2(\Omega))\,,$$

where $w \in H^2(\Omega) \cap H^1_0(\Omega)$ is the unique solution of the elliptic problem

$$\begin{cases} -\epsilon\Delta w + w = z & \text{in } \Omega \\ w = 0 & \text{in } \partial\Omega \end{cases} \tag{2.3.1}$$

for any $z \in L^2(\Omega)$. Observe that the operatorial identity

$$\Delta[(1-\beta)I - \beta\Delta]^{-1} = \frac{1}{\beta}\big\{(1-\beta)[(1-\beta)I - \beta\Delta]^{-1} - I\big\}\,, \tag{2.3.2}$$

where

$$[(1-\beta)I - \beta\Delta]^{-1} := \frac{1}{1-\beta}\left[I - \frac{\beta}{1-\beta}\,\Delta\right]^{-1}\,,$$

holds in the strong sense in $L^2(\Omega)$. Then consider the operator $\mathcal{A} \equiv \mathcal{A}_{\alpha\beta} : D(\mathcal{A}) \subset L^2(\Omega) \to L^2(\Omega)$ defined as follows:

$$\begin{cases} D(\mathcal{A}) := H^2(\Omega) \cap H^1_0(\Omega) \\ \mathcal{A}u := -\alpha\Delta[(1-\beta)I - \beta\Delta]^{-1}\Delta u \qquad (u \in D(\mathcal{A}))\,. \end{cases} \tag{2.3.3}$$

Also observe that, if φ satisfies assumption (H_2), there holds

$$\|\varphi(u)\|_{L^2(\Omega)} \le \|\varphi'\|_{L^\infty(\mathbb{R})}\|u\|_{L^2(\Omega)}$$

for every $u \in L^2(\Omega)$. Hence the nonlinear operator $\mathcal{F} \equiv \mathcal{F}_\beta : L^2(\Omega) \to L^2(\Omega)$,

$$\mathcal{F}(u) := \Delta[(1-\beta)I - \beta\Delta]^{-1}\varphi(u) \qquad (u \in L^2(\Omega)) \tag{2.3.4}$$

is well defined.

From the first equation of problem (P) we plainly obtain

$$(1-\beta)u_t - \beta\Delta u_t = \frac{1}{\beta}\big\{\beta\Delta(\varphi(u) - \alpha\Delta u) - (1-\beta)(\varphi(u) - \alpha\Delta u)\big\} + \frac{(1-\beta)}{\beta}\big[\varphi(u) - \alpha\Delta u\big]\,,$$

whence by (2.3.2)

$$\begin{aligned} u_t &= -\frac{1}{\beta}\big[\varphi(u) - \alpha\Delta u\big] + \frac{(1-\beta)}{\beta}\big[(1-\beta)I - \beta\Delta\big]^{-1}(\varphi(u) - \alpha\Delta u) = \\ &= \Delta[(1-\beta)I - \beta\Delta]^{-1}(\varphi(u) - \alpha\Delta u)\,. \end{aligned} \tag{2.3.5}$$

Then problem (P) reads

$$\begin{cases} u_t = \mathcal{A}u + \mathcal{F}(u) & (t > 0) \\ u(0) = u_0 \in X \,. \end{cases} \tag{2.3.6}$$

To prove existence and uniqueness of solutions to (2.3.6), we need some properties of the operators $\mathcal{A}$ and $\mathcal{F}$.

Proposition 2.3.1. *For every $\alpha \in (0,\infty)$, $\beta \in (0,1)$ the linear operator $\mathcal{A} \equiv \mathcal{A}_{\alpha\beta}$ defined in* (2.3.3) *is self-adjoint.*

Proof. (i) Firstly, let us show that $\mathcal{A}$ is symmetric. For every $u \in H^2(\Omega) \cap H^1_0(\Omega)$ we have

$$\mathcal{A}u = -\alpha \Delta g_u \,, \tag{2.3.7}$$

where $g_u \in H^2(\Omega) \cap H^1_0(\Omega)$ is the unique solution of the elliptic problem

$$\begin{cases} -\beta \Delta g_u + (1-\beta) g_u = \Delta u & \text{in } \Omega \\ g_u = 0 & \text{on } \partial\Omega \,. \end{cases} \tag{2.3.8}$$

For every u, $v \in D(\mathcal{A})$ there holds

$$(\mathcal{A}u, v)_{L^2(\Omega)} - (u, \mathcal{A}v)_{L^2(\Omega)} = -\frac{\alpha(1-\beta)}{\beta} \int_\Omega (g_u v - g_v u)\, dx \,.$$

Hence the claim will follow, if we show that

$$\int_\Omega (g_u v - g_v u)\, dx = 0 \,. \tag{2.3.9}$$

To this purpose, let h_u, $h_v \in H^4(\Omega) \cap H^1_0(\Omega)$ be the unique solutions of the problems

$$\begin{cases} -\beta \Delta h_u + (1-\beta) h_u = u & \text{in } \Omega \\ h_u = 0 & \text{on } \partial\Omega \,, \end{cases} \tag{2.3.10}$$

and

$$\begin{cases} -\beta \Delta h_v + (1-\beta) h_v = v & \text{in } \Omega, \\ h_v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3.11}$$

respectively; here $u \in H^2(\Omega) \cap H^1_0(\Omega)$. In particular, there holds Δh_u, $\Delta h_v \in H^1_0(\Omega)$, hence from (2.3.8)-(2.3.11) by uniqueness we have $\Delta h_u = g_u$ and $\Delta h_v = g_v$. Then we obtain

$$g_u v = -\beta \Delta h_v\, g_u + (1-\beta) h_v g_u = -\beta g_v g_u + (1-\beta) h_v \Delta h_u \,,$$
$$g_v u = -\beta \Delta h_u\, g_v + (1-\beta) h_u g_v = -\beta g_u g_v + (1-\beta) h_u \Delta h_v \,,$$

whence

$$\int_\Omega (g_u v - g_v u)\, dx = (1-\beta) \int_\Omega [h_v \Delta h_u - h_u \Delta h_v]\, dx = 0 \,.$$

Then the claim follows.
(ii) It is easily seen that the operator $\mathcal{A}$ is m-dissipative. In fact, for every $u \in \mathcal{D}(\mathcal{A})$, we have

$$(\mathcal{A}u, u)_{L^2(\Omega)} = -\alpha \int_\Omega u \Delta g_u \, dx \,, \tag{2.3.12}$$

with $g_u \in H^2(\Omega) \cap H^1_0(\Omega)$ as in (2.3.8). Since there holds

$$\begin{aligned}
(\mathcal{A}u, u)_{L^2(\Omega)} &= -\alpha \int_\Omega u \Delta g_u \, dx = -\alpha \int_\Omega g_u \Delta u \, dx = \\
&= \alpha\beta \int_\Omega g_u \Delta g_u \, dx - \alpha(1-\beta) \int_\Omega g_u^2 \, dx = \\
&= -\alpha\beta \int_\Omega |\nabla g_u|^2 \, dx - \alpha(1-\beta) \int_\Omega g_u^2 \, dx \le 0 \,,
\end{aligned}$$

the claim follows.
(iii) Next, let us prove that for every $f \in L^2(\Omega)$ there exists a unique $u \in \mathcal{D}(\mathcal{A})$ such that

$$u - \mathcal{A}u = f \,. \tag{2.3.13}$$

In this connection, observe that

$$u - \mathcal{A}u = u + \alpha\big[[-\beta\Delta + (1-\beta)I]^{-1}(\Delta^2 u) \big],$$

thus equation (2.3.13) can be written as

$$\alpha\Delta^2 u = -\beta\Delta(f-u) + (1-\beta)[f-u].$$

Therefore, it suffices to prove that for every $f \in L^2(\Omega)$ there exists a unique solution $u \in \mathcal{D}(\mathcal{A})$ of the problem

$$\alpha\Delta^2 u - \beta\Delta u + (1-\beta)u = -\beta\Delta f + (1-\beta)f \,, \tag{2.3.14}$$

where by solution of (2.3.14) we mean any $u \in \mathcal{D}(\mathcal{A})$ such that for every $\xi \in \mathcal{D}(\mathcal{A})$

$$\alpha \int_\Omega \Delta u \Delta \xi \, dx + \beta \int_\Omega \nabla u \nabla \xi \, dx + (1-\beta) \int_\Omega u\xi \, dx = -\beta \int_\Omega f \Delta \xi \, dx + (1-\beta) \int_\Omega f\xi \, dx \,.$$

To this purpose, consider the bilinear form $a : \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}) \to \mathbb{R}$ defined by setting

$$a(u,v) := \alpha \int_\Omega \Delta u \Delta v \, dx + \beta \int_\Omega \nabla u \cdot \nabla v \, dx + (1-\beta) \int_\Omega uv \, dx$$

for every u, $v \in \mathcal{D}(\mathcal{A})$. Then there exist $C > 0$ such that

$$\begin{aligned}
|a(u,v)| \;&\le\; \alpha\|\Delta u\|_{L^2(\Omega)}\|\Delta v\|_{L^2(\Omega)} + \beta\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \\
&+\; (1-\beta)\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \le C\|u\|_{H^2(\Omega)}\|v\|_{H^2(\Omega)} \,,
\end{aligned}$$

$$
\begin{aligned}
a(u,u) &= \alpha \int_\Omega |\Delta u|^2\, dx + \beta \int_\Omega |\nabla u|^2 dx + \\
&\quad + (1-\beta) \int_\Omega u^2\, dx \geq C\|u\|^2_{H^2(\Omega)}
\end{aligned}
$$

for every $u,\, v \in \mathcal{D}(\mathcal{A})$. By the above inequalities and the Lax-Milgram Theorem well-posedness of (2.3.14) easily follows. This completes the proof. □

Proposition 2.3.2. *Let φ satisfy assumption (H_2). Then for every $\beta \in (0,1)$ the nonlinear operator $\mathcal{F} \equiv \mathcal{F}_\beta$ defined in* (2.3.4) *is Lipschitz continuous.*

Proof. By equality (2.3.4) and the very definition of the operator $\mathcal{F}$

$$
\|\mathcal{F}(u) - \mathcal{F}(v)\|_{L^2(\Omega)} \leq C\|\varphi(u) - \varphi(v)\|_{L^2(\Omega)} \leq C\|\varphi'\|_{L^\infty(\mathbb{R})}\|u - v\|_{L^2(\Omega)} \tag{2.3.15}
$$

for every $u,\, v \in L^2(\Omega)$. Hence the claim follows. □

Proof of Theorem 2.2.1. The result follows from Propositions 2.3.1-2.3.2 by standard results of semigroup theory (*e.g.*, see [LLMP, Proposition 6.1.2]). □

To prove Theorem 2.2.2 we need some a priori estimates of solutions of the approximating problems (P_n). To this purpose the following lemma is expedient.

Lemma 2.3.3. *Let $\alpha \in (0,\infty)$, $\beta \in (0,1)$, $u_0 \in H^1_0(\Omega)$, and let φ satisfy assumptions (H_0)-(H_1). Let $\{u_n\}$ be the sequence of solutions to problems (P_n) given by Theorem 2.2.1, with $\{u_{0n}\}$ satisfying* (2.2.3)-(2.2.4). *Then for every $t \in (0,T]$ there holds*

$$
\begin{aligned}
&\int_\Omega \Phi_n(u_n)(x,t)\, dx + \frac{\alpha}{2}\int_\Omega |\nabla u_n|^2(x,t)\, dx\ + \\
&+ \beta \int_0^t\!\!\int_\Omega u_{nt}^2\, dxds + (1-\beta)\int_0^t\!\!\int_\Omega |\nabla v_n|^2\, dxds\ = \\
&= \int_\Omega \Phi_n(u_{0n})(x)\, dx + \frac{\alpha}{2}\int_\Omega |\nabla u_{0n}|^2\, dx\,,
\end{aligned}
\tag{2.3.16}
$$

where Φ_n is the function defined in (2.2.12).

Proof. Multiplying the first equation of (2.2.18) by

$$
v_n = \frac{1}{1-\beta}\left[\varphi(u_n) - \alpha\Delta u_n + \beta u_{nt}\right]
$$

and integrating over $\Omega \times (0,t)$ we obtain

$$\begin{aligned}&\int_\Omega \Phi_n(u_n)(x,t)\,dx - \alpha\int_0^t\!\!\int_\Omega \Delta u_n\, u_{nt}\, dxds\, + \\ &+\beta\int_0^t\!\!\int_\Omega u_{nt}^2 dxds + (1-\beta)\int_0^t\!\!\int_\Omega |\nabla v_n|^2 dxds\, = \\ &= \int_\Omega \Phi_n(u_{0n})\,dx\,.\end{aligned} \tag{2.3.17}$$

Since $u_n \in C([0,T]; H^2(\Omega)\cap H_0^1(\Omega))$ and $u_{nt} \in C([0,T]; L^2(\Omega))$, by standard approximation arguments there holds

$$\int_0^t\!\!\int_\Omega \Delta u_n\, u_{nt}\, dxds = \frac{1}{2}\int_\Omega |\nabla u_{0n}|^2\,dx - \frac{1}{2}\int_\Omega |\nabla u_n|^2(x,t)\,dx. \tag{2.3.18}$$

From (2.3.17)-(2.3.18) equality (2.3.16) follows. □

Further a priori estimates of the sequence $\{u_n\}$ are given by the following lemma.

Lemma 2.3.4. *Let* $\alpha \in (0,\infty)$*,* $\beta \in (0,1)$ *and* $u_0 \in H_0^1(\Omega)$*. Let* φ *satisfy either assumption* (H_2)*, or assumptions* (H_0)*,* (H_1) *and* (H_3)*. Let* $\{u_n\}$ *be the sequence of solutions to problems* (P_n) *given by Theorem 2.2.1, with* $\{u_{0n}\}$ *satisfying* (2.2.3)-(2.2.4)*. Then there exists* $L > 0$ *such that for every* $t \in (0,T]$

$$\begin{aligned}&(1-\beta)\int_\Omega u_n^2(x,t)\,dx\, + \frac{\beta}{2}\int_\Omega |\nabla u_n|^2(x,t)\,dx\, + \\ &+\alpha\int_0^t\!\!\int_\Omega (\Delta u_n)^2\,dxds\, \le\, \|u_0\|^2_{H_0^1(\Omega)}\left(1+e^{\frac{2LT}{\beta}}\right)\,.\end{aligned} \tag{2.3.19}$$

Proof. Multiplying the first equation of (P_n) by u_n, integrating over $\Omega\times(0,t)$ and using (2.3.18) we obtain

$$\begin{aligned}&(1-\beta)\int_\Omega u_n^2(x,t)\,dx + \frac{\beta}{2}\int_\Omega |\nabla u_n|^2(x,t)\,dx + \alpha\int_0^t\!\!\int_\Omega (\Delta u_n)^2\,dxds \\ &= -\int_0^t\!\!\int_\Omega \varphi_n'(u_n)|\nabla u_n|^2\,dxds + (1-\beta)\int_\Omega u_{0n}^2\,dx + \frac{\beta}{2}\int_\Omega |\nabla u_{0n}|^2\,dx\,.\end{aligned} \tag{2.3.20}$$

If assumption (H_2) is satisfied, there exists $L > 0$ such that

$$\left|\int_0^t\!\!\int_\Omega \varphi_n'(u_n)|\nabla u_n|^2\,dxds\right| \le L\int_0^t\!\!\int_\Omega |\nabla u_n|^2\,dxds$$

for any $n \in \mathbb{N}$ (see Remark 2.2.2-(i)). On the other hand, if assumption (H_3) is satisfied, for any $n \in \mathbb{N}$, $n > u_0$ there holds

$$\begin{aligned} -\int_0^t\!\!\int_\Omega \varphi_n'(u_n)|\nabla u_n|^2\,dxds &\leq -\int_0^t\!\!\int_{\{|u_n|\leq u_0\}} \varphi'(u_n)|\nabla u_n|^2\,dxds \leq \\ &\leq L\int_0^t\!\!\int_\Omega |\nabla u_n|^2\,dxds \end{aligned}$$

with some constant $L > 0$, since $\varphi \in W^{1,\infty}_{loc}(\mathbb{R})$ by assumption (H_0) (see Remark 2.2.2-(ii)). In either case from equality (2.3.20) we get

$$\begin{aligned} &(1-\beta)\int_\Omega u_n^2(x,t)\,dx + \frac{\beta}{2}\int_\Omega |\nabla u_n|^2(x,t)\,dx + \alpha\int_0^t\!\!\int_\Omega (\Delta u_n)^2\,dxds \leq \\ \leq\;\; &\|u_0\|^2_{H^1_0(\Omega)} + L\int_0^t\!\!\int_\Omega |\nabla u_n|^2\,dxds\,; \end{aligned} \tag{2.3.21}$$

here use of inequality (2.2.3) has been made. Then by the Gronwall Lemma from (2.3.21) we obtain

$$\frac{\beta}{2}\int_\Omega |\nabla u_n|^2(x,t)\,dx \leq \|u_0\|^2_{H^1_0(\Omega)} e^{\frac{2Lt}{\beta}}$$

for every $t \in (0,T]$. Integrating the above inequality on $(0,T]$ we plainly obtain

$$L\int_0^t\!\!\int_\Omega |\nabla u_n|^2\,dxds \leq \|u_0\|^2_{H^1_0(\Omega)} e^{\frac{2LT}{\beta}}\,,$$

which upon substitution in (2.3.21) gives inequality (2.3.19). Then the result follows. □

Proposition 2.3.5. *Let the assumptions of Lemma 2.3.3 be satisfied. Then for every $\bar\alpha > 0$ there exists $M > 0$ (only depending on the norm $\|u_0\|_{H^1_0(\Omega)}$) such that for any $\alpha \in (0,\bar\alpha)$, $\beta \in (0,1)$ and $n \in \mathbb{N}$*

$$\|\Phi_n(u_n)\|_{L^\infty((0,T);L^1(\Omega))} \leq M\,; \tag{2.3.22}$$

$$\sqrt{\alpha}\,\|u_n\|_{L^\infty((0,T);H^1_0(\Omega))} \leq M\,; \tag{2.3.23}$$

$$\sqrt{\beta}\,\|u_{nt}\|_{L^2(Q)} \leq M\,; \tag{2.3.24}$$

$$\sqrt{1-\beta}\,\|v_n\|_{L^2((0,T);H^1_0(\Omega))} \leq M\,; \tag{2.3.25}$$

$$\sqrt{\beta(1-\beta)}\,\|v_n\|_{L^2((0,T);H^2(\Omega))} \leq M\,. \tag{2.3.26}$$

Proof. By inequality (2.2.11) and assumption (H_1) we have

$$0 \leq \int_\Omega \Phi_n(u_{0n})\,dx \leq K_1\left(2|\Omega| + \int_\Omega |u_{0n}|^{q+1}dx\right) \leq$$
$$\leq \begin{cases} K_1\left(2|\Omega| + \int_\Omega |u_{0n}|^{\frac{2N}{N-2}}dx\right) & \text{if } \ N \geq 3\,, \\ K_1|\Omega|\left(1 + \|u_{0n}\|^{q+1}_{L^\infty(\Omega)}\right) & \text{if } \ N = 1, 2\,. \end{cases}$$

By the above estimate, inequality (2.2.3) and Sobolev embedding results there exists $C > 0$ (only depending on the norm $\|u_0\|_{H^1_0(\Omega)}$) such that for every $n \in \mathbb{N}$

$$\int_\Omega \Phi_n(u_{0n})(x)\,dx + \frac{\alpha}{2}\int_\Omega |\nabla u_{0n}|^2\,dx \leq C + \frac{\alpha}{2}\|u_0\|^2_{H^1_0(\Omega)}\,.$$

Then, recalling that $\Phi_n(u_n) \geq 0$, from equality (2.3.16) we obtain estimates (2.3.22)-(2.3.25). Inequality (2.3.26) follows from (2.3.24)-(2.3.25), since $u_{nt} = \Delta v_n$ (see (2.2.18)). □

Proposition 2.3.6. *Let the assumptions of Lemma 2.3.3 be satisfied. Then for every $\bar{\alpha} > 0$ there exists $M > 0$ (only depending on the norm $\|u_0\|_{H^1_0(\Omega)}$) such that for any $\alpha \in (0, \bar{\alpha})$, $\beta \in (0,1)$ and $n \in \mathbb{N}$*

$$\sqrt{\alpha}\,\|\varphi_n(u_n)\|_{L^2(Q)} \leq M\,; \tag{2.3.27}$$

$$\alpha^{\frac{3}{2}}\,\|\Delta u_n\|_{L^2(Q)} \leq M\,; \tag{2.3.28}$$

$$\sqrt{\alpha}\,\beta\,\|v_n\|_{L^\infty((0,T);H^1_0(\Omega))} \leq M\,. \tag{2.3.29}$$

Proof. Observe that by (2.2.18)-(2.2.19) there holds

$$\alpha\Delta u_n = \varphi_n(u_n) + \beta u_{nt} - (1-\beta)v_n \ \text{ in } \ Q\,, \tag{2.3.30}$$

whence by inequalities (2.3.24)-(2.3.25)

$$\alpha\|\Delta u_n\|_{L^2(Q)} \leq \|\varphi_n(u_n)\|_{L^2(Q)} + \beta\|u_{nt}\|_{L^2(Q)} + (1-\beta)\|v_n\|_{L^2(Q)} \leq$$
$$\leq M + \|\varphi_n(u_n)\|_{L^2(Q)}\,. \tag{2.3.31}$$

Therefore inequality (2.3.27), together with (2.3.24)-(2.3.25), implies (2.3.28). To prove (2.3.27), let us distinguish two cases:

(i) either $N = 1, 2$ and $q \in (1, \infty)$, or $N \geq 3$ and $q \in \left(1, \frac{N}{N-2}\right]$;

(ii) $N \geq 3$ and $q \in \left(\frac{N}{N-2}, \frac{N+2}{N-2}\right]$.

(i) By inequality (2.2.10), for every $t \in (0, T]$ we have

$$\|\varphi_n(u_n)(\cdot,t)\|_{L^2(\Omega)} = \left(\int_\Omega |\varphi_n(u_n)|^2(x,t)\,dx\right)^{\frac{1}{2}} \le$$

$$\le \ 2K_1\left(2|\Omega| + \int_\Omega |u_n|^{2q}(x,t)dx\right) \le$$

$$\le \ \begin{cases} 2K_1\left(2|\Omega| + \int_\Omega |u_n|^{\frac{2N}{N-2}}(x,t)\,dx\right) & \text{if } \ N \ge 3 \text{ and } \ q \in \left(1, \frac{N}{N-2}\right], \\ 2K_1|\Omega|\left(1 + \|u_n(\cdot,t)\|^{2q}_{L^\infty(\Omega)}\right) & \text{if } \ N = 1,2 \text{ and } \ q \in (1,\infty). \end{cases}$$

By the above estimate, inequality (2.3.23) and Sobolev embedding results we obtain (2.3.27) in the present case.

(ii) In this case by Sobolev embedding there holds $u_n(\cdot,t) \in L^{\frac{2N}{N-4}}(\Omega)$ and $|\nabla u_n(\cdot,t)| \in L^{\frac{2N}{N-2}}(\Omega)$ for every $t \in (0,T]$, since $u_n \in C([0,T]; H^1_0(\Omega) \cap H^2(\Omega))$. Then by Remark 2.2.1 (in particular, see (2.2.10) and (2.2.15)) there holds

$$\|\psi(u_n)(\cdot,t)\|_{L^2(\Omega)} = \left(\int_\Omega |\psi(u_n)|^2(x,t)\,dx\right)^{\frac{1}{2}}$$

$$\le 2K_1\left(2|\Omega| + \int_\Omega |u_n|^{2q}(x,t)dx\right) \le$$

$$\le 2K_1\left(2|\Omega| + \int_\Omega |u_n|^{\frac{2(N+2)}{N-2}}(x,t)\,dx\right) < \infty,$$

since $L^{\frac{2N}{N-4}}(\Omega) \hookrightarrow L^{\frac{2(N+2)}{N-2}}(\Omega)$. Similarly,

$$\|\left[\psi'(u_n)|\nabla u_n|^2\right](\cdot,t)\|_{L^1(\Omega)} = \int_\Omega \left[\psi'(u_n)|\nabla u_n|^2\right](x,t)\,dx \le$$

$$\le \left(\int_\Omega |\psi'(u_n)|^{\frac{N}{2}}(x,t)\,dx\right)^{\frac{2}{N}} \left(\int_\Omega |\nabla u_n|^{\frac{2N}{N-2}}(x,t)\,dx\right)^{\frac{N-2}{N}} =$$

$$= \left(\int_\Omega |u_n|^{\frac{N}{2}(q-1)}(x,t)\,dx\right)^{\frac{2}{N}} \left(\int_\Omega |\nabla u_n|^{\frac{2N}{N-2}}(x,t)\,dx\right)^{\frac{N-2}{N}} \le$$

$$\le C\left(\int_\Omega |u_n|^{\frac{2N}{N-2}}(x,t)\,dx\right)^{\frac{2}{N}} \left(\int_\Omega |\nabla u_n|^{\frac{2N}{N-2}}(x,t)\,dx\right)^{\frac{N-2}{N}} < \infty$$

for some $C > 0$.

By the above remarks, multiplying equality (2.3.30) by $\psi(u_n)$ and integrating over Q plainly gives

$$\iint_Q |\varphi_n(u_n)\psi(u_n)|\,dxdt + \alpha \iint_Q \left[\psi'(u_n)|\nabla u_n|^2\right](x,t)\,dx =$$

$$\le \ \ \beta \int_\Omega \Psi(u_{0n})\,dx + (1-\beta)\iint_Q v_n\psi(u_n)\,dxdt\,, \tag{2.3.32}$$

where Ψ is the function defined in (2.2.16) and the inequality

$$\varphi_n(u)\psi(u) = \frac{[\varphi_n(u)u][\psi(u)u]}{u^2} \geq 0 \qquad (u \neq 0)$$

has been used.

Concerning the right-hand side of (2.3.32), by Hölder inequality we have

$$\iint_Q |v_n\psi(u_n)|\, dxdt \leq \|v_n\|_{L^{\frac{2N}{N-2}}(Q)} \|\psi(u_n)\|_{L^{\frac{2N}{N+2}}(Q)} . \tag{2.3.33}$$

Let us show that for some constant $M > 0$ there holds

$$\sqrt{\alpha}\, \|\psi(u_n)\|_{L^\infty((0,T);L^{\frac{2N}{N+2}}(\Omega))} \leq M . \tag{2.3.34}$$

In fact, by Remark 2.2.1 there exists $K_2 > 0$ such that for every $t \in (0,T]$

$$\|\psi(u_n)(\cdot,t)\|_{L^{\frac{2N}{N+2}}(\Omega)} \leq K_2 \left(2|\Omega| + \int_\Omega |u_n|^{\frac{2qN}{N+2}}(x,t)dx \right)^{\frac{N+2}{2N}} \leq$$

$$\leq \quad K_2 \left(2|\Omega| + \int_\Omega |u_n|^{\frac{2N}{N-2}}(x,t)dx \right)^{\frac{N+2}{2N}} ,$$

since by assumption $q \leq \frac{N+2}{N-2}$. Then by the above estimate, inequality (2.3.23) and Sobolev embedding we obtain (2.3.34). Further, from inequalities (2.3.25) and (2.3.33)- (2.3.34) by Sobolev embedding we obtain

$$\sqrt{\alpha(1-\beta)} \iint_Q |v_n\psi(u_n)|\, dxdt \leq M \tag{2.3.35}$$

with some constant $M > 0$.

On the other hand, since $|\Psi(u_n)| \leq K_1(1+|u_n|^{\frac{2N}{N-2}})$, by inequality (2.2.3) and Sobolev embedding results there exists $M > 0$ (only depending on the norm $\|u_0\|_{H^1_0(\Omega)}$) such that for every $n \in \mathbb{N}$

$$\int_\Omega \Psi(u_{0n})(x)\, dx \leq M . \tag{2.3.36}$$

Then from (2.3.32), (2.3.35) and (2.3.36) we easily obtain

$$\alpha \iint_Q |\varphi_n(u_n)\psi(u_n)|\, dxdt \leq M$$

for some constant $M > 0$, whence by inequality (2.2.17) the estimate (2.3.27) follows also in this case. This completes the proof of (2.3.27).

To prove inequality (2.3.29), for any $t \in (0,T]$ let us multiply the elliptic equation (2.2.19) by $v_n(\cdot,t)$ and integrate over Ω. Using Hölder inequality as in (2.3.33), by Sobolev embedding we have

$$\begin{aligned}
&\beta \int_\Omega |\nabla v_n(x,t)|^2\,dx + (1-\beta)\int_\Omega v_n^2(x,t)\,dx = \qquad (2.3.37)\\
&= \int_\Omega [v_n\varphi_n(u_n)](x,t)\,dx + \alpha\int_\Omega \big[\nabla v_n \cdot \nabla u_n\big](x,t)\,dx \le\\
&\le \left(\int_\Omega |\varphi_n(u_n)(x,t)|^{\frac{2N}{N+2}}dx\right)^{\frac{N+2}{2N}} \left(\int_\Omega |v_n(x,t)|^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{2N}} +\\
&+\alpha\left(\int_\Omega |\nabla u_n(x,t)|^2 dx\right)^{\frac12}\left(\int_\Omega |\nabla v_n(x,t)|^2 dx\right)^{\frac12}.
\end{aligned}$$

Using inequality (2.2.10) and arguing as in the proof of (2.3.34), it is proven that for some constant $M > 0$

$$\sqrt{\alpha}\,\|\varphi_n(u_n)\|_{L^\infty((0,T);L^{\frac{2N}{N+2}}(\Omega))} \le M\,. \qquad (2.3.38)$$

Then by Sobolev embedding results from inequalities (2.3.23) and (2.3.37)-(2.3.38) we obtain for every $t \in (0,T]$

$$\beta\int_\Omega |\nabla v_n(x,t)|^2\,dx + (1-\beta)\int_\Omega v_n^2(x,t)\,dx \le \frac{M}{\sqrt{\alpha}}\left(\int_\Omega |\nabla v_n(x,t)|^2 dx\right)^{\frac12}$$

for some constant $M > 0$, whence inequality (2.3.29) immediately follows. This completes the proof. □

Since the assumptions of Lemma 2.3.4 imply those of Lemma 2.3.3, we have the following result.

Proposition 2.3.7. *Let the assumptions of Lemma 2.3.4 be satisfied. Then for every $\beta \in (0,1)$ and $n \in \mathbb{N}$*

$$\|u_n\|_{L^\infty((0,T);L^2(\Omega))} \le \|u_0\|_{H_0^1(\Omega)}\sqrt{\frac{1+e^{\frac{2LT}{\beta}}}{1-\beta}}\,; \qquad (2.3.39)$$

$$\|u_n\|_{L^\infty((0,T);H_0^1(\Omega))} \le \|u_0\|_{H_0^1(\Omega)}\sqrt{\frac{2\left(1+e^{\frac{2LT}{\beta}}\right)}{\beta}}\,; \qquad (2.3.40)$$

$$\|\Delta u_n\|_{L^2(Q)} \le \|u_0\|_{H_0^1(\Omega)}\sqrt{\frac{1+e^{\frac{2LT}{\beta}}}{\alpha}} \qquad (\alpha \in (0,\infty))\,. \qquad (2.3.41)$$

Moreover, for every $\bar\alpha > 0$ and $\beta \in (0,1)$ there exists $\bar M > 0$ (only depending on the norm $\|u_0\|_{H_0^1(\Omega)}$ and on β, and diverging as $\beta \to 0^+$, $\beta \to 1^-$) such that for any $\alpha \in (0,\bar\alpha)$ and $n \in \mathbb{N}$

$$\|\varphi_n(u_n)\|_{L^2(Q)} \le \bar M\,. \qquad (2.3.42)$$

Proof. Estimates (2.3.39), (2.3.40) and (2.3.41) follow directly from inequality (2.3.19). Concerning (2.3.42), from the first equality in (2.2.19) and inequalities (2.3.26), (2.3.41) we plainly obtain

$$\begin{aligned}\|\varphi_n(u_n)\|_{L^2(Q)} &\leq \alpha\,\|\Delta u_n\|_{L^2(Q)} + \beta\,\|\Delta v_n\|_{L^2(Q)} + (1-\beta)\,\|v_n\|_{L^2(Q)} \leq \\ &\leq \|u_0\|_{H_0^1(\Omega)}\sqrt{\bar{\alpha}\left(1+e^{\frac{2LT}{\beta}}\right)} + M\left(\sqrt{\frac{\beta}{1-\beta}} + \sqrt{1-\beta}\right).\end{aligned}$$

Then the claim follows. □

Proposition 2.3.8. *Let the assumptions of Lemma 2.3.3 be satisfied. Then there exist* $u \in L^\infty((0,T);H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$, $v \in L^\infty((0,T);H_0^1(\Omega)) \cap L^2((0,T);H^2(\Omega))$ *and subsequences* $\{u_{n_k}\}$, $\{v_{n_k}\}$ *such that*
(i) $u_t \in L^2(Q)$, $\varphi(u) \in L^2(Q)$, $\Delta u \in L^2(Q)$;
(ii) there holds

$$u_{n_k} \overset{*}{\rightharpoonup} u \quad \textit{in } L^\infty((0,T);H_0^1(\Omega))\,, \tag{2.3.43}$$

$$u_{n_k} \to u \quad \textit{almost everywhere in } Q\,, \tag{2.3.44}$$

$$u_{n_k t} \rightharpoonup u_t \quad \textit{in } L^2(Q)\,, \tag{2.3.45}$$

$$\varphi_{n_k}(u_{n_k}) \to \varphi(u) \qquad \textit{almost everywhere in } Q\,, \tag{2.3.46}$$

$$\varphi_{n_k}(u_{n_k}) \rightharpoonup \varphi(u) \quad \textit{in } L^2(Q)\,, \tag{2.3.47}$$

$$\Phi_{n_k}(u_{n_k}) \to \Phi(u) \qquad \textit{almost everywhere in } Q\,, \tag{2.3.48}$$

$$\Delta u_{n_k} \rightharpoonup \Delta u \quad \textit{in } L^2(Q)\,, \tag{2.3.49}$$

$$v_{n_k} \overset{*}{\rightharpoonup} v \quad \textit{in } L^\infty((0,T);H_0^1(\Omega))\,, \tag{2.3.50}$$

$$v_{n_k} \rightharpoonup v \quad \textit{in } L^2((0,T);H^2(\Omega))\,. \tag{2.3.51}$$

Proof. The convergence claims in (2.3.43)-(2.3.44) and in (2.3.50)-(2.3.51) follow from the a priori estimates (2.3.23), respectively (2.3.29) and (2.3.26). Concerning (2.3.45), by estimate (2.3.24) there exists $w \in L^2(Q)$ such that (possibly extracting a subsequence, denoted again $\{u_{n_k t}\}$ for simplicity)

$$u_{n_k t} \rightharpoonup w \quad \text{in } L^2(Q)\,,$$

thus in particular

$$\iint_Q u_{n_k t}\zeta\,dxdt = -\iint_Q u_{n_k}\zeta_t\,dxdt \to \iint_Q w\,\zeta\,dxdt$$

for every $\zeta \in C_c^1(Q)$. Since

$$\iint_Q u_{n_k}\zeta_t\,dxdt \to \iint_Q u\,\zeta_t\,dxdt$$

by (2.3.43), there holds $w = u_t \in L^2(Q)$, thus the first claim in (i) and (2.3.45) follow. The third claim in (i) and (2.3.49) are similarly proven using estimate (2.3.28).

It is easily seen that the convergence in (2.3.46) (respectively in (2.3.48)) follows from that in (2.3.44) by using (2.2.6) and (2.2.9) (respectively (2.2.14) and (2.2.10)). Let us now address (2.3.47). By inequality (2.3.27) there exists $z \in L^2(Q)$ such that (possibly extracting a subsequence, denoted again $\{\varphi_{n_k}(u_{n_k})\}$ for simplicity)

$$\varphi_{n_k}(u_{n_k}) \rightharpoonup z \quad \text{in } L^2(Q)\,. \tag{2.3.52}$$

Inequality (2.3.27) also implies that the sequence $\{\varphi_n(u_n)\}$ is uniformly integrable in $L^1(Q)$. In fact,

$$\|\varphi_n(u_n)\|_{L^1(Q)} \leq \sqrt{|Q|}\,\|\varphi_n(u_n)\|_{L^2(Q)} \leq M\sqrt{|Q|} \ \text{ for any } \ n \in \mathbb{N}\,,$$

and for any measurable subset $E \subseteq Q$ with Lebesgue measure $|E| < \delta$

$$\iint_E |\varphi_n(u_n)|\,dxdt \leq M\sqrt{|\delta|} \ \text{ for any } \ n \in \mathbb{N}\,.$$

Then by the Prokhorov's Theorem and the convergence in (2.3.46) (*e.g.*, see [V2, Proposition 1]) we obtain that (possibly extracting a subsequence, denoted again $\{\varphi_{n_k}(u_{n_k})\}$ for simplicity)

$$\varphi_{n_k}(u_{n_k}) \rightharpoonup \varphi(u) \quad \text{in } L^1(Q)\,. \tag{2.3.53}$$

Comparing (2.3.52) with (2.3.53) shows that $z = \varphi(u) \in L^2(Q)$, thus the second claim in (i) and (2.3.47) follow.

Finally, let us show that $u \in C([0,T];L^2(\Omega))$. By estimate (2.3.24), for each t_1, $t_2 \in [0,T]$ and $n \in \mathbb{N}$ there holds

$$\int_\Omega |u_n(x,t_2) - u_n(x,t_1)|^2 dx = \int_\Omega \left| \int_{t_1}^{t_2} u_{nt}(x,t)\,dt \right|^2 dx \leq$$
$$\leq |t_1 - t_2| \int_{t_1}^{t_2}\!\!\int_\Omega u_{nt}^2(x,t)\,dxdt \leq \frac{M}{\beta}\,|t_1 - t_2|\,.$$

By inequality (2.3.23) and the above estimate the sequence $\{u_n\}$ is equibounded and equicontinuous in $C([0,T];L^2(\Omega))$. Then by the Ascoli-Arzelà Theorem there exists a subsequence $\{u_{n_k}\}$ such that

$$u_{n_k} \to u \quad \text{in } C([0,T];L^2(\Omega))\,, \tag{2.3.54}$$

whence the claim follows. This completes the proof. □

The following result follows from Proposition 2.3.8 (in particular, see (2.3.43), (2.3.46)) by a standard localization argument; we leave the proof to the reader.

Proposition 2.3.9. *Let the assumptions of Proposition 2.3.8 be satisfied. Let u, v be the functions and $\{u_{n_k}\}$, $\{v_{n_k}\}$ the subsequences given by Proposition 2.3.8. Then for almost every $t \in (0,T)$*

$$u_{n_k}(\cdot,t) \rightharpoonup u(\cdot,t) \text{ in } H_0^1(\Omega)\,, \tag{2.3.55}$$

$$\varphi_{n_k}(u_{n_k})(\cdot,t) \rightharpoonup \varphi(u)(\cdot,t) \text{ in } L^{\frac{2N}{N+2}}(\Omega)\,, \tag{2.3.56}$$

$$v_{n_k}(\cdot,t) \rightharpoonup v(\cdot,t) \text{ in } H_0^1(\Omega)\,. \tag{2.3.57}$$

Now we can prove Theorem 2.2.2.

Proof of Theorem 2.2.2. As already proven, the functions u, v considered in Proposition 2.3.8 have the regularity properties stated in Definition 2.2.2. Letting $n_k \to \infty$ in the weak formulation of problems (2.2.18), (2.2.19) (written with $n = n_k$) and using the convergence results of Propositions 2.3.8-2.3.9 shows that problems (2.2.1) and (2.2.2) are satisfied (in $L^2(Q)$, respectively in $L^2(\Omega)$ for almost every $t \in (0,T)$).

Let us prove that $u \in C([0,T]; H_0^1(\Omega))$. Since $u \in C([0,T]; L^2(\Omega)) \cap L^\infty((0,T); H_0^1(\Omega))$ and u_t, $\varphi(u)$, $\Delta u \in L^2(Q)$, multiplying the first equation in (2.2.1) by

$$v = \frac{1}{1-\beta}[\varphi(u) - \alpha\Delta u + \beta u_t]$$

and arguing as in the proof of Lemma 2.3.3 we get

$$\begin{aligned}
&(1-\beta)\int_\Omega \Phi(u)(x,t_2)\,dx + \frac{\alpha(1-\beta)}{2}\int_\Omega |\nabla u|^2(x,t_2)\,dx + \\
&+ \beta(1-\beta)\int_{t_1}^{t_2}\!\!\int_\Omega u_t^2\,dxds + \int_{t_1}^{t_2}\!\!\int_\Omega |\nabla v|^2\,dxds \\
&= (1-\beta)\int_\Omega \Phi(u)(x,t_1)\,dx + \frac{\alpha(1-\beta)}{2}\int_\Omega |\nabla u|^2(x,t_1)\,dx
\end{aligned} \tag{2.3.58}$$

for every $0 \le t_1 < t_2 \le T$ (since $u \in C([0,T]; L^2(\Omega)) \cap L^\infty((0,T); H_0^1(\Omega))$ it is not restrictive to assume that $u(\cdot,t) \in H_0^1(\Omega)$ for every $t \in [0,T]$).

Next, choosing in the above equality (2.3.58) $t_2 = t_n$, where $t_n \to t_1^+$ (the case $t_1 = t_n$, with $t_n \to t_2^-$ being analogous), there holds

$$\begin{aligned}
&\frac{\alpha(1-\beta)}{2}\lim_{n\to\infty}\int_\Omega |\nabla u_n|^2(x,t_n)\,dx = \\
&= -\lim_{n\to\infty}\left\{\beta(1-\beta)\int_{t_1}^{t_n}\!\!\int_\Omega (u_t^2 + |\nabla v|^2)dxdt\right\} + \\
&+ (1-\beta)\lim_{n\to\infty}\left\{\int_\Omega \Phi(u)(x,t_1)dx - \int_\Omega \Phi(u)(x,t_n)\,dx\right\} + \\
&+ \frac{\alpha(1-\beta)}{2}\int_\Omega |\nabla u|^2(x,t_1)\,dx.
\end{aligned}$$

Here we have

$$u(\cdot,t_n) \rightharpoonup u(\cdot,t_1) \quad \text{in } H_0^1(\Omega)\,, \tag{2.3.59}$$

$$\lim_{n\to\infty}\left\{\beta(1-\beta)\int_{t_1}^{t_n}\int_\Omega (u_t^2+|\nabla v|^2)dxdt\right\}=0. \tag{2.3.60}$$

$$\begin{aligned}&\lim_{n\to\infty}\left|\int_\Omega \Phi(u)(x,t_1)dx-\int_\Omega \Phi(u)(x,t_n)\,dx\right|= \qquad (2.3.61)\\ &=\lim_{n\to\infty}\left|\int_{t_1}^{t_n}\int_\Omega \varphi(u)u_t\,dxdt\right|=0\end{aligned}$$

since $\varphi(u)u_t\in L^1(Q)$. By (2.3.60)–(2.3.61) there holds

$$\lim_{n\to\infty}\int_\Omega |\nabla u|^2(x,t_n)dx=\int_\Omega |\nabla u|^2(x,t_1)dx,$$

whence $u(\cdot,t_n)\to u(\cdot,t_1)$ in $H_0^1(\Omega)$ (see also (2.3.59)).

Therefore, the function u is a solution of problem (P), and the a priori estimates (2.2.21)-(2.2.27) follow from the analogous inequalities (2.3.23)-(2.3.29) by the lower semicontinuity of the norm. Concerning (2.2.20), by estimate (2.3.22), the convergence in (2.3.48) and the Fatou Lemma there holds

$$0\le\int_\Omega \Phi(u)(x,t)\,dx\le\liminf_{n_k\to\infty}\int_\Omega \Phi_{n_k}(u_{n_k})(x,t)\,dx\le M$$

for almost every $t\in(0,T)$. Then inequality (2.2.20) follows.

It remains to prove uniqueness. To this purpose, let (u_1,v_1) and (u_2,v_2) be two solutions of problem (P). Then the differences u_1-u_2, v_1-v_2 satisfy the problems

$$\begin{cases}(u_1-u_2)_t=\Delta(v_1-v_2) & \text{in } Q\\ u_1-u_2=0 & \text{in } \Omega\times\{0\}\,,\end{cases} \tag{2.3.62}$$

$$\begin{cases}-\beta[\Delta(v_1-v_2)](\cdot,t)+(1-\beta)(v_1-v_2)(\cdot,t)\;= & \\ =(\varphi(u_1)-\varphi(u_2))(\cdot,t)-\alpha[\Delta(u_1-u_2)](\cdot,t) & \text{in } \Omega\\ (v_1-v_2)(\cdot,t)=0 & \text{on } \partial\Omega\,.\end{cases} \tag{2.3.63}$$

Multiplying the first equation of (2.3.62) by the function

$$\psi(x,t):=-\int_t^\tau (v_1-v_2)(x,s)\,ds\,,$$

for any fixed $\tau\in(0,T)$, and integrating over $Q_\tau:=\Omega\times(0,\tau)$ we obtain

$$\begin{aligned}&\iint_{Q_\tau}(u_1-u_2)(v_1-v_2)\,dxdt=\\ =\;&-\iint_{Q_\tau}[\nabla(v_2-v_2)](x,t)\left(\int_t^\tau[\nabla(v_2-v_2)](x,s)\,ds\right)dxdt=\\ =\;&-\frac12\int_\Omega\left|\int_0^\tau[\nabla(v_2-v_2)](x,s)\,ds\right|^2dx\le 0,\end{aligned}$$

whence

$$\iint_{Q_\tau} (u_1 - u_2)(v_1 - v_2)\, dxdt \le 0\,. \tag{2.3.64}$$

On the other hand, multiplying the first equation of (2.3.63) by the difference $(u_1 - u_2)(\cdot, t)$ and integrating over Q_τ gives

$$(1-\beta) \iint_{Q_\tau} (u_1 - u_2)(v_1 - v_2)\, dxdt = \iint_{Q_\tau} [\varphi(u_1) - \varphi(u_2)]\,[u_1 - u_2]\, dxdt +$$
$$+\alpha \iint_{Q_\tau} |\nabla(u_1 - u_2)|^2\, dxdt + \frac{\beta}{2}\int_\Omega (u_1 - u_2)^2(x,\tau)\, dx\,,$$

whence by inequality (2.3.64) and assumption (H_2)

$$\frac{\beta}{2}\int_\Omega (u_1 - u_2)^2(x,\tau)\, dx \le C \iint_{Q_\tau} (u_1 - u_2)^2\, dxdt$$

with some constant $C > 0$. By the above estimate and the Gronwall Lemma uniqueness follows. This completes the proof. □

Proof of Theorem 2.2.3. Estimates (2.2.28)-(2.2.31) follow from the analogous inequalities (2.3.39)-(2.3.42) by the lower semicontinuity of the norm, in view of the convergence in (2.3.43) and (2.3.47)-(2.3.49). □

2.4 Asymptotic limits : Proofs

Let us first prove Theorem 2.2.4.

Proof of Theorem 2.2.4. The a priori estimates (2.2.21) and (2.2.25) ensure the existence of limiting functions $u_\alpha \in L^\infty((0,T); H^1_0(\Omega))$, $v_\alpha \in L^2((0,T); H^1_0(\Omega))$ such that both the convergences in (2.2.33) and (2.2.37) holds. Moreover, by (2.2.21), (2.2.25) and the first equation in (2.2.1), it can be easily seen that for every $\rho \in H^1_0(\Omega)$ the sequence

$$F^\rho_k(t) = \int_\Omega u_{\alpha,\beta_k}(x,t)\rho(x)\, dx$$

is uniformly bounded – hence weakly relatively compact – in the Sobolev space $H^1((0,T))$. By such consideration and (2.2.33), it follows that for almost every $t \in (0,T)$ there holds

$$F^\rho_k(t) \to \int_\Omega u_\alpha(x,t)\rho(x)\, dx. \tag{2.4.1}$$

Thus, by (2.2.33) and (2.4.1) we obtain

$$u_{\alpha,\beta_k}(\cdot,t) \rightharpoonup u_\alpha(\cdot,t) \quad \text{in } H^1_0(\Omega) \tag{2.4.2}$$

for almost every $t \in (0,T)$. Finally, by (2.4.2), (2.2.33) and the Dominated Convergence Theorem, (2.2.34) follows.

The convergences in (2.2.35)-(2.2.37) and claim (i) follow from (2.2.23)-(2.2.25) arguing as in the proof of Proposition 2.3.6.

To prove claim (iii) let us consider the weak formulation of problems (2.2.1)-(2.2.2) written with $u = u_{\alpha,\beta}$ and $v = v_{\alpha,\beta}$, namely

$$\iint_Q u_{\alpha,\beta}\zeta_t \, dxdt + \iint_Q v_{\alpha,\beta}\Delta\zeta \, dxdt = -\int_\Omega u_0(x)\zeta(x,0)\, dx\,, \tag{2.4.3}$$

$$\beta \iint_Q \nabla v_{\alpha,\beta}\cdot\nabla\zeta \, dxdt + (1-\beta)\iint_Q v_{\alpha,\beta}\zeta \, dxdt = \iint_Q [\varphi(u_{\alpha,\beta}) - \alpha\Delta u_{\alpha,\beta}]\,\zeta \, dxdt \tag{2.4.4}$$

for every $\zeta \in C^1([0,T]; C^2_c(\Omega))$ such that $\zeta(.,T) = 0$ in Ω. Then letting $\beta_k \to 0^+$ in equalities (2.4.3)-(2.4.4) written with $\beta = \beta_k$ and using (2.2.35)-(2.2.37) we obtain

$$\iint_Q u_\alpha\zeta_t \, dxdt + \iint_Q v_\alpha\Delta\zeta \, dxdt = -\int_\Omega u_0(x)\zeta(x,0)\, dx\,,$$

$$\iint_Q v_\alpha\zeta \, dxdt = \iint_Q [\varphi(u_\alpha) - \alpha\Delta u_\alpha]\,\zeta \, dxdt$$

for every ζ as above, whence equality (2.2.32) and claim (iii) follow. Finally, the statements in claim (iv) concerning inequalities (2.2.21), (2.2.23)-(2.2.24) and (2.2.38) follow from the analogous estimates (2.2.21), (2.2.23)-(2.2.25) and (2.2.38) for $u = u_{\alpha,\beta}$ and $v = v_{\alpha,\beta}$, by the convergence in (2.2.33), (2.2.35)-(2.2.37) and the lower semicontinuity of the norm. On the other hand, the statement concerning inequality (2.2.20) follows from the same inequality for $u = u_{\alpha,\beta}$ by the convergence in (2.2.34) and the Fatou Lemma, as in the proof of Theorem 2.2.2. Then the conclusion follows. □

Proof of Theorem 2.2.5. We only prove claim (iii), the proof of the others following by the same arguments used in the proof of Theorem 2.2.4. Consider the weak formulation of problems (2.2.1)-(2.2.2) written with $u = u_{\alpha,\beta}$ and $v = v_{\alpha,\beta}$, namely

$$\iint_Q (u_{\alpha,\beta})_t\, \zeta \, dxdt = -\iint_Q \nabla v_{\alpha,\beta}\cdot\nabla\zeta \, dxdt \tag{2.4.5}$$

$$\beta \iint_Q \nabla v_{\alpha,\beta}\cdot\nabla\zeta \, dxdt + (1-\beta)\iint_Q v_{\alpha,\beta}\zeta \, dxdt = \iint_Q [\varphi(u_{\alpha,\beta}) - \alpha\Delta u_{\alpha,\beta}]\,\zeta \, dxdt \tag{2.4.6}$$

for every $\zeta \in C([0,T]; C^2(\bar\Omega))$ such that $\zeta(.,t) = 0$ on $\partial\Omega$ for every $t \in [0,T]$. By inequality (2.2.26) there exist a function $v_\alpha \in L^\infty((0,T); H^1_0(\Omega))$ and a subsequence $\{v_{\alpha,\beta_k}\} \subseteq \{v_{\alpha,\beta}\}$ such that

$$v_{\alpha,\beta_k} \overset{*}{\rightharpoonup} v_\alpha \quad \text{in } L^\infty((0,T); H^1_0(\Omega))$$

as $\beta_k \to 1^-$. Then letting $\beta_k \to 1^-$ in equalities (2.4.5)-(2.4.6) written with $\beta = \beta_k$ we obtain

$$\begin{aligned}\iint_Q u_{\alpha t}\,\zeta\,dxdt &= -\iint_Q \nabla v_\alpha \cdot \nabla\zeta\,dxdt\,, = \\ &= -\iint_Q [\varphi(u_\alpha) - \alpha\Delta u_\alpha]\,\zeta\,dxdt\end{aligned}$$

for every ζ as above. Using inequality (2.2.44) and arguing as in the proof of Proposition 2.3.8 , it is easily seen that $u_\alpha \in C([0,T]; L^2(\Omega))$. Then the result follows. □

Proof of Theorem 2.2.6. The convergence claims in (2.2.47)-(2.2.48) follow from the a priori estimates (2.2.22) and (2.2.29), those in (2.2.49)-(2.2.50) follow from (2.2.22) and (2.2.31), respectively, and those in (2.2.51)-(2.2.52) follow from (2.2.25) and (2.2.27).

To prove claim (*iii*), consider the following weak formulation of problems (2.2.1)-(2.2.2):

$$\iint_Q (u_{\alpha,\beta})_t\,\zeta\,dxdt = \iint_Q \Delta v_{\alpha,\beta}\zeta\,dxdt\,, \tag{2.4.7}$$

$$\begin{aligned}-\beta\iint_Q \Delta v_{\alpha,\beta}\,\zeta\,dxdt + (1-\beta)\iint_Q v_{\alpha,\beta}\,\zeta\,dxdt &= \\ = \iint_Q \varphi(u_{\alpha,\beta})\,\zeta\,dxdt - \alpha\iint_Q u_{\alpha,\beta}\,\Delta\zeta\,dxdt\end{aligned} \tag{2.4.8}$$

for every $\zeta \in C([0,T]; C_c^2(\Omega))$. Letting $\alpha_k \to 0^+$ in (2.4.8) written with $\alpha = \alpha_k$ and using (2.4.7), by the arbitrariness of ζ we obtain the equalities

$$\iint_Q (u_\beta)_t\,\zeta\,dxdt = \iint_Q \Delta v_\beta\zeta\,dxdt\,, \tag{2.4.9}$$

$$-\beta\iint_Q \Delta v_\beta\,\zeta\,dxdt + (1-\beta)\iint_Q v_\beta\,\zeta\,dxdt = \iint_Q \varphi(u_\beta)\,\zeta\,dxdt\,. \tag{2.4.10}$$

By the arbitrariness of ζ, from (2.4.10) we obtain

$$v_\beta = \frac{1}{1-\beta}\left[\varphi(u_\beta) + \beta u_{\beta t}\right]\,,$$

which upon substitution in (2.4.9) shows that problem (S) is satisfied in strong sense. The proof of claim (*iv*) is analogous to those given for Theorems 2.2.4-2.2.5, hence the result follows. □

2.5 Letting $\alpha \to 0^+$ in problem (CH): Proofs

To prove Theorem 2.2.7 we need some definitions and results concerning Young measures on $Q \times \mathbb{R}$ (*e.g.*, see [GMS, V2] and references therein).

Definition 2.5.1. *By a Young measure on $Q \times \mathbb{R}$ we mean any positive Radon measure τ such that*

$$\tau(E \times \mathbb{R}) = |E| \tag{2.5.1}$$

for any Lebesgue measurable set $E \subseteq Q$. The set of Young measures on $Q \times \mathbb{R}$ will be denoted by $\mathcal{Y}(Q;\mathbb{R})$.
If $f : Q \to \mathbb{R}$ is Lebesgue measurable, the Young measure associated to f is the measure $\tau \in \mathcal{Y}(Q;\mathbb{R})$ such that

$$\tau(E \times F) = |E \cap f^{-1}(F)| \tag{2.5.2}$$

for any Lebesgue measurable set $E \subseteq Q$ and any Borel set $F \subseteq \mathbb{R}$.

Proposition 2.5.1. *Let $\tau \in \mathcal{Y}(Q;\mathbb{R})$. Then for almost every $(x,t) \in Q$ there exists a measure $\tau_{(x,t)} \in \mathcal{P}\infty(\mathbb{R})$, such that for any function $\psi : Q \times \mathbb{R} \to \mathbb{R}$ bounded and continuous:*
(i) the map

$$(x,t) \to \left\langle \tau_{(x,t)}, \psi(x,t,\cdot) \right\rangle_{\mathbb{R}} = \int_{\mathbb{R}} \psi(x,t,\xi)\, d\tau_{(x,t)}(\xi)$$

is Lebesgue measurable;
(ii) there holds

$$\begin{aligned} \langle \tau, \psi \rangle_{Q\times\mathbb{R}} := \int_{Q\times\mathbb{R}} \psi \, d\tau &= \iint_Q \left\langle \tau_{(x,t)}, \psi(x,t,\cdot) \right\rangle_{\mathbb{R}} dxdt \\ &= \iint_Q dxdt \int_{\mathbb{R}} \psi(x,t,\xi)\, d\tau_{(x,t)}(\xi)\,. \end{aligned} \tag{2.5.3}$$

Therefore, every $\tau \in \mathcal{Y}(Q \times \mathbb{R})$ can be identified with the associated family $\{\tau_{(x,t)} \,|\, (x,t) \in Q\}$, which is called the disintegration of τ.

Definition 2.5.2. *Let $\{\tau^n\} \subseteq \mathcal{Y}(Q;\mathbb{R})$, $\tau \in \mathcal{Y}(Q;\mathbb{R})$ $(n \in \mathbb{N})$. We say that $\tau^n \to \tau$ narrowly in $Q \times \mathbb{R}$, if*

$$\int_{Q\times\mathbb{R}} \psi \, d\tau^n \;\to\; \int_{Q\times\mathbb{R}} \psi \, d\tau \tag{2.5.4}$$

for any function $\psi : Q \times \mathbb{R} \to \mathbb{R}$ bounded and measurable, such that $\psi(x,t,\cdot)$ is continuous for almost every $(x,t) \in Q$.

The following result, concerning bounded sequences of functions in $L^1(Q)$, will be used.

Theorem 2.5.2. *Let $\{f_n\}$ be a bounded sequence in $L^1(Q)$, and $\{\tau^n\}$ the sequence of associated Young measures. Then:*
(i) there exist subsequences $\{f_k\} \equiv \{f_{n_k}\} \subseteq \{f_n\}$, $\{\tau^k\} \equiv \{\tau^{n_k}\} \subseteq \{\tau^n\}$ and a Young measure τ on $Q \times \mathbb{R}$ such that $\tau^k \to \tau$ narrowly in $Q \times \mathbb{R}$;
(ii) for any $\rho \in C(\mathbb{R})$ such that the sequence $\{\rho \circ f_n\} \subseteq L^1(Q)$ is uniformly integrable, there holds

$$\rho \circ f_k \equiv \rho \circ f_{n_k} \rightharpoonup \rho^* \quad \text{in } L^1(Q)\,, \tag{2.5.5}$$

where

$$\rho^*(x,t) := \left\langle \tau_{(x,t)}, \rho \right\rangle_{\mathbb{R}} = \int_{\mathbb{R}} \rho(\xi)\, d\tau_{(x,t)}(\xi) \qquad \text{for a.e. } (x,t) \in Q \tag{2.5.6}$$

and $\{\tau_{(x,t)}\}$ is the disintegration of τ.

If a sequence $\{f_n\}$ is bounded in $L^1(Q)$ but not uniformly integrable, we can extract from it a uniformly integrable subsequence "by removing sets of small measure". This is the content of the following theorem (*e.g.*, see [GMS]).

Theorem 2.5.3. (Biting Lemma) *Let $\{f_n\}$ be a bounded sequence in $L^1(Q)$. Then there exist a subsequence $\{f_k\} \equiv \{f_{n_k}\} \subseteq \{f_n\}$ and a decreasing sequence of measurable sets $E_k \subseteq Q$ of Lebesgue measure $|E_k| \to 0$, such that the sequence $\left\{f_k \chi_{Q\setminus E_k}\right\}$ is uniformly integrable.*

Some consequences of the Biting Lemma are discussed in the following.

Remark 2.5.1. *Let $\{f_k\}$ be the subsequence considered in Theorem 2.5.3, and let $\{\tau^k\}$ be the associated sequence of Young measures. Let τ denote the narrow limit of the sequence $\{\tau^k\}$, which exists by Theorem 2.5.2-(i) (possibly extracting a subsequence, still denoted $\{\tau^k\}$ for simplicity), and let $\{E_k\}$ be the sequence of measurable sets considered in Theorem 2.5.3. Since the sequence $\left\{f_k \chi_{Q\setminus E_k}\right\}$ is uniformly integrable, by Theorem 2.5.2-(ii) there holds*

$$f_k \chi_{Q\setminus E_k} \rightharpoonup u := \int_{[0,\infty)} \xi \, d\tau(\xi) \quad \text{in } \; L^1(Q)\,. \tag{2.5.7}$$

The function $u \in L^1(Q)$ in (2.2.56) is called the barycenter of the disintegration $\{\tau_{(x,t)}\}$ of τ. Besides, since the sequence $\{f_k \chi_{E_k}\}$ is bounded in $L^1(Q)$, there exists a measure $\mu \in \mathcal{M}(Q)$ such that

$$f_k \chi_{E_k} \overset{*}{\rightharpoonup} (1-\beta) \;\text{ in } \; \mathcal{M}(Q)\,. \tag{2.5.8}$$

We can now proceed to prove Theorem 2.2.7. Let us first mention the following result.

Proposition 2.5.4. *Let $u_0 \in H_0^1(\Omega)$, and let φ satisfy assumptions (H_0), (H_1) and (H_4). Let u_α be the solution of problem (CH) given by Theorem 2.2.4 ($\alpha \in (0,\infty)$). Then there exists $M > 0$ (only depending on the norm $\|u_0\|_{H_0^1(\Omega)}$) such that for any $\alpha \in (0,\bar{\alpha})$ and r as in (H_4)*

$$\|u_\alpha\|_{L^\infty((0,T);L^r(\Omega))} \leq M\,. \tag{2.5.9}$$

Proof. Follows immediately from inequality (2.2.20) and assumption (H_4). □

We can now prove Theorem 2.2.7.

Proof of Theorem 2.2.7. The claims concerning the function $v \in L^2((0,T);H_0^1(\Omega))$, the subsequence $\{v_{\alpha_k}\} \subseteq \{v_\alpha\}$ and the convergence in (2.2.59) follow from inequality (2.2.44).

On the other hand, by inequality (2.5.9) the family $\{u_\alpha\}$ is bounded in $L^1(Q)$, thus the Biting Lemma and its consequences described in Remark 2.5.1 can be used. In particular, there exist a subsequence $\{u_{\alpha_k}\} \subseteq \{u_\alpha\}$, and a decreasing sequence of measurable sets $E_k \subseteq Q$ of Lebesgue measure $|E_k| \to 0$, such that the convergence in (2.5.7)-(2.5.8) holds with $f_k = u_{\alpha_k}$. Moreover, arguing as in [ST2], it is easily seen that $u \in L^\infty((0,T);L^1(\Omega))$ and $\mu \in L^\infty((0,T);\mathcal{M}(\Omega))$.

To prove claim (ii) and the convergence in (2.2.58), recalling that $v_\alpha := \varphi(u_\alpha) - \alpha\Delta u_\alpha$ belongs to $L^2((0,T);H_0^1(\Omega))$, by standard approximation arguments from equality (2.2.32) we get

$$\iint_Q u_\alpha \zeta_t\,dxdt = \iint_Q \nabla v_\alpha \cdot \nabla\zeta\,dxdt - \int_\Omega u_0(x)\zeta(x,0)\,dx \tag{2.5.10}$$

for every $\zeta \in C^1([0,T];C_c^1(\Omega))$ such that $\zeta(\cdot,T) = 0$ in Ω, and

$$\iint_Q v_\alpha \zeta\,dxdt = \iint_Q \varphi(u_\alpha)\zeta\,dxdt - \alpha\iint_Q u_\alpha\,\Delta\zeta\,dxdt \tag{2.5.11}$$

for every $\zeta \in C^1([0,T];C_c^2(\Omega)$ such that $\zeta(.,T) = 0$ in Ω. Observe that

$$\iint_Q u_{\alpha_k}\zeta_t\,dxdt = \iint_Q u_{\alpha_k}\chi_{E_k}\zeta_t\,dxdt + \iint_Q u_{\alpha_k}\chi_{Q\setminus E_k}\zeta_t\,dxdt\,.$$

Then letting $\alpha_k \to 0^+$ in equality (2.5.10) written with $\alpha = \alpha_k$ and using the convergence in (2.5.7)-(2.5.8) we obtain equality (2.2.60). On the other hand, letting $\alpha_k \to 0^+$ in equality (2.5.11) written with $\alpha = \alpha_k$, by the convergence in (2.2.59) we obtain that in (2.2.58).

Finally, if assumption (H_4) holds with $r > 1$, by inequality (2.5.9) the family $\{u_\alpha\}$ is uniformly integrable in $L^1(Q)$. Then by Theorem 2.5.2-(ii) the convergence in (2.2.61) and the remaining claims follow. This completes the proof. □

2.6 Neumann boundary conditions

Consider the companion problem of (P) with homogeneous Neumann boundary conditions:

$$(NP) \qquad \begin{cases} (1-\beta)u_t = \Delta[\varphi(u) - \alpha\Delta u + \beta u_t] & \text{in } Q \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \Delta u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\}\,. \end{cases}$$

Set

$$H^2_E(\Omega) := \left\{ u \in H^2(\Omega) \mid \frac{\partial u}{\partial n} = 0 \right\}$$

By abuse of notation, hereafter we denote by $[I - \epsilon\Delta]^{-1}$ $(\epsilon > 0)$ the operator

$$[I - \epsilon\Delta]^{-1} : L^2(\Omega) \to H^2_E(\Omega)\,, \quad [I - \epsilon\Delta]^{-1} z := w \quad (z \in L^2(\Omega))\,,$$

where $w \in H^2_E(\Omega)$ is the unique solution of the elliptic problem

$$\begin{cases} -\epsilon\Delta w + w = z & \text{in } \Omega \\ \dfrac{\partial w}{\partial n} = 0 & \text{in } \partial\Omega \end{cases} \tag{2.6.1}$$

for any $z \in L^2(\Omega)$. Accordingly, by $\mathcal{A} \equiv \mathcal{A}_{\alpha\beta} : D(\mathcal{A}) \subset L^2(\Omega) \to L^2(\Omega)$ we denote the following operator:

$$\begin{cases} D(\mathcal{A}) := H^2_E(\Omega) \\ \mathcal{A}u := -\alpha\Delta[(1-\beta)I - \beta\Delta]^{-1}\Delta u & (u \in D(\mathcal{A}))\,. \end{cases} \tag{2.6.2}$$

By the methods used in Section 2.3 we have a first existence result concerning solutions of problem (NP), which is the analogue of Theorem 2.2.1.

Theorem 2.6.1. *Let $\alpha \in (0,\infty)$, $\beta \in (0,1)$, and let φ satisfy assumption (H_2). Then for every $u_0 \in H^2_E(\Omega)$ there exists a unique function u such that:*

(i) $u \in C([0,T]; H^2_E(\Omega)) \cap C^1([0,T]; L^2(\Omega))$, and $\varphi(u) \in C([0,T]; L^2(\Omega))$;

(ii) u satisfies in strong sense problem (2.2.1), where $v \in C([0,T]; H^2_E(\Omega))$ and for every $t \in [0,T]$ the function $v(\cdot,t)$ is the unique solution of the elliptic problem

$$\begin{cases} -\beta\Delta v(\cdot,t) + (1-\beta)v(\cdot,t) = \varphi(u)(\cdot,t) - \alpha\Delta u(\cdot,t) & \text{in } \Omega \\ \dfrac{\partial v(\cdot,t)}{\partial n} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.6.3}$$

A more general well-posedness result for problem (NP), analogous to Theorem 2.2.2, is obtained as before considering the family of approximating problems

$$(NP_n) \qquad \begin{cases} (1-\beta)u_{nt} = \Delta[\varphi_n(u_n) - \alpha\Delta u_n + \beta u_{nt}] & \text{in } Q \\ \dfrac{\partial u_n}{\partial n} = \dfrac{\partial \Delta u_n}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T) \\ u_n = u_{0n} & \text{in } \Omega \times \{0\}\,. \end{cases}$$

Here φ_n is defined as above (see (2.2.5)), and for every $u_0 \in H^1(\Omega)$ $\{u_{0n}\} \subseteq H^2_E(\Omega)$ is any sequence such that

$$\|u_{0n}\|_{H^1(\Omega)} \leq \|u_0\|_{H^1(\Omega)}\,, \tag{2.6.4}$$

$$u_{0n} \to u_0 \quad \text{in } H^1(\Omega)\,. \tag{2.6.5}$$

Observe that by mass conservation, for every $t \in (0,T]$ and $n \in \mathbb{N}$

$$\int_\Omega u_n(x,t)\,dx = \int_\Omega u_{0n}(x)\,dx\,. \tag{2.6.6}$$

Concerning solutions of the approximating problems (NP_n) we have the following estimates, which are the counterpart of those of Proposition 2.3.6.

Proposition 2.6.2. *Let $\alpha \in (0,\infty)$, $\beta \in (0,1)$, $u_0 \in H^1(\Omega)$, and let φ satisfy assumptions (H_0)-(H_1). Let $\{u_n\}$ be the sequence of solutions to problems (NP_n) given by Theorem 2.6.1, with $\{u_{0n}\}$ satisfying* (2.6.4)-(2.6.5). *Then for every $\bar\alpha > 0$ there exists $M > 0$ (only depending on the norm $\|u_0\|_{H^1(\Omega)}$) such that for any $\alpha \in (0,\bar\alpha)$, $\beta \in (0,1)$ and $n \in \mathbb{N}$*

$$\|\Phi_n(u_n)\|_{L^\infty((0,T);L^1(\Omega))} \leq M\,; \tag{2.6.7}$$

$$\sqrt{\alpha}\,\|u_n\|_{L^\infty((0,T);L^2(\Omega))} \leq M\,; \tag{2.6.8}$$

$$\sqrt{\alpha}\,\||\nabla u_n|\|_{L^\infty((0,T);L^2(\Omega))} \leq M\,; \tag{2.6.9}$$

$$\sqrt{\beta}\,\|u_{nt}\|_{L^2(Q)} \leq M\,; \tag{2.6.10}$$

$$\sqrt{\alpha}\,\|\varphi_n(u_n)\|_{L^2(Q)} \leq M\,; \tag{2.6.11}$$

$$\sqrt{\alpha^3\beta}\,\|\Delta u_n\|_{L^2(Q)} \leq M\,; \tag{2.6.12}$$

$$\sqrt{\alpha\beta}\,(1-\beta)\,\|v_n\|_{L^2(Q)} \leq M\,; \tag{2.6.13}$$

$$\sqrt{1-\beta}\,\||\nabla v_n|\|_{L^2(Q)} \leq M\,; \tag{2.6.14}$$

$$\sqrt{\beta}\,\|\Delta v_n\|_{L^2(Q)} \leq M\,. \tag{2.6.15}$$

Proof. It is easily checked that equality (2.3.16) holds under the present assumption, hence inequalities (2.6.7), (2.6.9)-(2.6.10) and (2.6.14) follow. The proof of inequality (2.6.11) is the same of (2.3.27), using (2.6.9) and (2.6.14) instead of (2.3.23) and (2.3.25), respectively. Inequality (2.6.15) follows from (2.6.10) by the equality $u_{nt} = \Delta v_n$.

Concerning (2.6.8), observe that by the Poincaré inequality and mass conservation (see (2.6.6)) there exists $C > 0$ such that

$$\begin{aligned} \|u_n(\cdot,t)\|^2_{L^2(\Omega)} &\leq C\left[\||\nabla u_n|(\cdot,t)\|^2_{L^2(\Omega)} + \left(\int_\Omega u_n(x,t)\,dx\right)^2\right] = \\ &= C\left[\||\nabla u_n|(\cdot,t)\|^2_{L^2(\Omega)} + \left(\int_\Omega u_{0n}(x)\,dx\right)^2\right] \end{aligned}$$

for every $t \in (0,T)$. Then by inequalities (2.6.4) and (2.6.9) we have

$$\|u_n(\cdot,t)\|^2_{L^2(\Omega)} \leq C\left(\frac{M}{\alpha} + |\Omega|\,\|u_0\|^2_{H^1(\Omega)}\right)$$

for every $t \in (0,T)$, whence (2.6.8) follows.

From the first equality in (2.3.37) (which follows from equality (2.2.2) as for Dirichlet boundary conditions) by Sobolev embedding and inequalities (2.6.9), (2.6.11) we have

$$\begin{aligned} &\beta\,\||\nabla v_n|(\cdot,t)\|^2_{L^2(\Omega)} + (1-\beta)\,\|v_n(\cdot,t)\|^2_{L^2(\Omega)} \leq \\ \leq\ &\|v_n(\cdot,t)\|_{L^2(\Omega)}\,\|\varphi_n(u_n)\|_{L^2(\Omega)} + \alpha\,\||\nabla v_n|(\cdot,t)\|_{L^2(\Omega)}\,\||\nabla u_n|(\cdot,t)\|_{L^2(\Omega)} \leq \\ \leq\ &\frac{1-\beta}{2}\,\|v_n(\cdot,t)\|^2_{L^2(\Omega)} + \frac{1}{2(1-\beta)}\,\|\varphi_n(u_n)\|^2_{L^2(\Omega)} + \\ +\ &\frac{\beta}{2}\,\||\nabla v_n|(\cdot,t)\|^2_{L^2(\Omega)} + \frac{\alpha^2}{2\beta}\,\||\nabla u_n|(\cdot,t)\|^2_{L^2(\Omega)}, \end{aligned}$$

namely

$$\begin{aligned} &\beta\,\||\nabla v_n|(\cdot,t)\|^2_{L^2(\Omega)} + (1-\beta)\,\|v_n(\cdot,t)\|^2_{L^2(\Omega)} \leq \\ \leq\ &\frac{1}{1-\beta}\,\|\varphi_n(u_n)\|^2_{L^2(\Omega)} + \frac{\alpha^2}{\beta}\,\||\nabla u_n|(\cdot,t)\|^2_{L^2(\Omega)}\,. \end{aligned}$$

Integrating the above inequality on $(0,T)$ and using inequalities (2.6.9), (2.6.11) we obtain

$$\beta\,\||\nabla v_n|\|^2_{L^2(Q)} + (1-\beta)\,\|v_n\|^2_{L^2(Q)} \leq M^2\left(\frac{1}{\alpha(1-\beta)} + \frac{\alpha}{\beta}\right),$$

whence inequality (2.6.13) follows. Finally, from equality (2.3.30), using (2.6.10), (2.6.11) and (2.6.13), we obtain (2.6.12). This completes the proof. □

Arguing as in Section 2.3, from the above estimates we obtain the following analogue of Theorem 2.2.2.

Theorem 2.6.3. *Let $\alpha \in (0,\infty)$, $\beta \in (0,1)$, and let φ satisfy assumptions (H_0)-(H_1). Then for every $u_0 \in H^1(\Omega)$ there exists a solution of problem (NP), which is meant in the following sense:*
(i) $u \in L^2((0,T);H^2_E(\Omega)) \cap C([0,T];H^1(\Omega))$, $u_t \in L^2(Q)$, $\varphi(u) \in L^2(Q)$;
(ii) problems (2.2.1) *and* (2.6.3) *are satisfied in strong sense, with $v \in L^2((0,T);H^2_E(\Omega))$.*
For every $t \in (0,T]$ there holds

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx\,, \tag{2.6.16}$$

and, if φ satisfies assumption (H_2), the solution is unique.
Moreover, for every $\bar{\alpha} > 0$ there exists $M > 0$ (only depending on the norm $\|u_0\|_{H^1(\Omega)}$) such that for any $\alpha \in (0,\bar{\alpha})$ and $\beta \in (0,1)$

$$\|\Phi(u)\|_{L^\infty((0,T);L^1(\Omega))} \le M\,, \tag{2.6.17}$$

the function Φ being defined in (2.2.13)*;*

$$\sqrt{\alpha}\,\|u\|_{L^\infty((0,T);H^1(\Omega))} \le M\,; \tag{2.6.18}$$

$$\sqrt{\beta}\,\|u_t\|_{L^2(Q)} \le M\,; \tag{2.6.19}$$

$$\sqrt{\alpha}\,\|\varphi(u)\|_{L^2(Q)} \le M\,; \tag{2.6.20}$$

$$\sqrt{\alpha^3\beta}\,\|\Delta u\|_{L^2(Q)} \le M\,; \tag{2.6.21}$$

$$\sqrt{\alpha\beta}\,(1-\beta)\,\|v\|_{L^2(Q))} \le M\,; \tag{2.6.22}$$

$$\sqrt{1-\beta}\,\||\nabla v|\|_{L^2(Q)} \le M\,; \tag{2.6.23}$$

$$\sqrt{\beta}\,\|\Delta v\|_{L^2(Q)} \le M\,. \tag{2.6.24}$$

Remark 2.6.1. *If φ satisfies either assumption (H_2), or assumptions (H_0), (H_1) and (H_3), the counterparts of Lemma 2.3.4, Proposition 2.3.7 and Theorem 2.2.3 are easily proven; we leave their formulation to the reader. Similarly, we shall not discuss in the present case results analogous to Theorem 2.2.6, concerning the limit $\alpha \to 0^+$.*

The following theorem shows that as $\alpha \to 0^+$ the solution of problem (NP) obtained above gives a solution of the Neumann initial-boundary value problem for the Cahn-Hilliard equation:

$$(NCH) \qquad \begin{cases} u_t = \Delta[\varphi(u) - \alpha\Delta u] & \text{in } Q \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \Delta u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\}\,. \end{cases}$$

The proof (which makes use of estimates (2.6.18), (2.6.20), (2.6.22) and (2.6.23)) is similar to that of Theorem 2.2.4, thus is omitted.

Theorem 2.6.4. *Let $u_0 \in H^1(\Omega)$, and let φ satisfy assumptions (H_0)-(H_1). Let $u_{\alpha,\beta}$ be the solution of problem (NP) given by Theorem 2.6.3 ($\alpha \in (0,\infty)$, $\beta \in (0,1)$). Then for every $\alpha \in (0,\infty)$ there exist $u_\alpha \in L^\infty((0,T);H^1(\Omega))$ with $\varphi(u_\alpha) \in L^2(Q)$, $v_\alpha \in L^2((0,T);H^1(\Omega))$ and two subsequences $\{u_{\alpha,\beta_k}\} \subseteq \{u_{\alpha,\beta}\}$, $\{v_{\alpha,\beta_k}\} \subseteq \{v_{\alpha,\beta}\}$ such that*
(i) as $\beta_k \to 0^+$ there holds

$$u_{\alpha,\beta_k} \overset{*}{\rightharpoonup} u_\alpha \quad \text{in } L^\infty((0,T);H^1(\Omega))\,, \tag{2.6.25}$$

$$u_{\alpha,\beta_k} \to u_\alpha \quad \text{almost everywhere in } Q\,, \tag{2.6.26}$$

$$\varphi(u_{\alpha,\beta_k}) \rightharpoonup \varphi(u_\alpha) \quad \text{in } L^2(Q)\,, \tag{2.6.27}$$

$$v_{\alpha,\beta_k} \rightharpoonup v_\alpha \quad \text{in } L^2((0,T);H^1(\Omega))\,; \tag{2.6.28}$$

(ii) the function u_α is a solution of problem (NCH), in the sense that

$$\iint_Q u_\alpha\, \zeta_t\, dxdt - \iint_Q \nabla v_\alpha \cdot \nabla\zeta\, dxdt = -\int_\Omega u_0(x)\zeta(x,0)\, dx\,,$$

$$\iint_Q v_\alpha \zeta\, dxdt = \iint_Q \varphi(u_\alpha)\,\zeta\, dxdt + \alpha \iint_Q \nabla u_\alpha \cdot \nabla\zeta\, dxdt$$

for every $\zeta \in C^1(\bar{Q})$ such that $\zeta(.,T) = 0$ in Ω;
(iii) the function u_α satisfies equality (2.6.16) for almost every $t \in (0,T)$ and inequalities (2.6.17), (2.6.18), (2.6.20), whereas v_α satisfies the a priori estimate

$$\||\nabla v_\alpha|\|_{L^2(Q)} \le M \tag{2.6.29}$$

with some constant $M > 0$ only depending on the norm $\|u_0\|_{H^1(\Omega)}$.

When $\beta \to 1^-$, in the present case we obtain solutions of the problem

$$(NRD) \qquad \begin{cases} u_t = \alpha\Delta u - \varphi(u) + \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \varphi(u)\, dx & \text{in } Q_T \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\} \end{cases}$$

(the presence of the nonlocal term in the right-hand side of the first equation stemming from Neumann boundary condition; see (AC)). This is the content of the following theorem.

Theorem 2.6.5. *Let $u_0 \in H^1(\Omega)$, and let φ satisfy assumptions (H_0)-(H_1). Let $u_{\alpha,\beta}$ be the solution of problem (NP) given by Theorem 2.6.3 ($\alpha \in (0,\infty)$, $\beta \in (0,1)$). Then for every $\alpha \in (0,\infty)$ there exist a function $u_\alpha \in L^\infty((0,T);H^1(\Omega)) \cap C([0,T];L^2(\Omega))$ and a subsequence $\{u_{\alpha,\beta_k}\} \subseteq \{u_{\alpha,\beta}\}$ such that*

(i) $u_{\alpha t} \in L^2(Q)$, $\varphi(u_\alpha) \in L^2(Q)$, $\Delta u_\alpha \in L^2(Q)$;

(ii) as $\beta_k \to 1^-$ there holds

$$u_{\alpha,\beta_k} \overset{*}{\rightharpoonup} u_\alpha \quad \text{in } L^\infty((0,T);H^1(\Omega)), \tag{2.6.30}$$

$$u_{\alpha,\beta_k} \to u_\alpha \quad \text{almost everywhere in } Q, \tag{2.6.31}$$

$$(u_{\alpha,\beta_k})_t \rightharpoonup u_{\alpha t} \quad \text{in } L^2(Q), \tag{2.6.32}$$

$$\varphi(u_{\alpha,\beta_k}) \rightharpoonup \varphi(u_\alpha) \quad \text{in } L^2(Q), \tag{2.6.33}$$

$$\Delta u_{\alpha,\beta_k} \rightharpoonup \Delta u_\alpha \quad \text{in } L^2(Q), \tag{2.6.34}$$

(iii) the function u_α is a solution of problem (NRD), in the sense that

$$\int_Q \left(\int_\Omega \varphi(u_\alpha)(x',t)\,dx' \right) \zeta(x,t)\,dxdt + \int_\Omega u_0(x)\zeta(x,0)\,dx =$$
$$= \iint_Q \varphi(u_\alpha)\,\zeta\,dxdt + \alpha \iint_Q \nabla u_\alpha \cdot \nabla\zeta\,dxdt + \iint_Q u_\alpha\,\zeta_t\,dxdt$$

for every $\zeta \in C^1(\bar{Q})$ such that $\zeta(.,T) = 0$ in Ω;

(iv) the function u_α satisfies equality (2.6.16), *inequalities* (2.6.17), (2.6.18), (2.6.20) *and the a priori estimates*

$$\|u_{\alpha t}\|_{L^2(Q)} \le M, \quad \sqrt{\alpha^3}\,\|\Delta u\|_{L^2(Q)} \le M \tag{2.6.35}$$

with some constant $M > 0$ only depending on the norm $\|u_0\|_{H^1(\Omega)}$.

Proof. We only prove claim (iii). Setting $z_{\alpha,\beta} := (1-\beta)v_{\alpha,\beta}$, by inequalities (2.6.22)-(2.6.23) we have

$$\sqrt{\alpha\beta}\,\|z_{\alpha,\beta}\|_{L^2(Q))} \le M, \quad \||\nabla z_{\alpha,\beta}|\|_{L^2(Q)} \le M\sqrt{1-\beta}.$$

Then for every $\alpha \in (0,\infty)$ there exist a function $z_\alpha \in L^2((0,T);H^1(\Omega))$ and a subsequence $\{z_{\alpha,\beta_k}\} \subseteq \{z_{\alpha,\beta}\}$ such that

$$z_{\alpha,\beta_k} \rightharpoonup z_\alpha \quad \text{in } L^2(Q), \quad \nabla z_{\alpha,\beta_k} \to 0 \quad \text{almost everywhere in } Q. \tag{2.6.36}$$

Therefore z_α is a function of t alone, and $z_\alpha \in L^2(0,T)$.

Now observe that the first equation in (2.6.3) reads

$$z(t) = \varphi(u)(\cdot,t) - \alpha\Delta u(\cdot,t) + \beta u_t(\cdot,t) \text{ in } \Omega \qquad (t \in (0,T)),$$

whence plainly

$$
\begin{aligned}
\iint_Q z_{\alpha,\beta}\,\zeta\,dxdt &= \iint_Q \varphi(u_{\alpha,\beta})\,\zeta\,dxdt\ + \\
&+\ \alpha\iint_Q \nabla u_{\alpha,\beta}\cdot\nabla\zeta\,dxdt + \beta\iint_Q (u_{\alpha,\beta})_t\,\zeta\,dxdt
\end{aligned}
$$

for every $\zeta\in C^1(\bar{Q})$. Writing the above equality with $\beta=\beta_k$ and letting $\beta_k\to 1^-$, by (2.6.30), (2.6.32), (2.6.33) and (2.6.36) we obtain

$$
\begin{aligned}
\int_0^T z_\alpha(t)\int_\Omega \zeta\,dxdt &= \iint_Q \varphi(u_\alpha)\,\zeta\,dxdt\ + \\
&+\ \alpha\iint_Q \nabla u_\alpha\cdot\nabla\zeta\,dxdt + \iint_Q u_{\alpha t}\,\zeta\,dxdt
\end{aligned}
\tag{2.6.37}
$$

for every $\zeta\in C^1(\bar{Q})$. Choosing $\zeta=h\in C_c((0,T))$ in (2.6.37), by conservation of mass (see (2.6.16)) and standard approximation arguments we get

$$
|\Omega|\,z_\alpha(t)=\int_\Omega \varphi(u_\alpha)(x,t)\,dx \tag{2.6.38}
$$

for every $t\in(0,T)$. From equalities (2.6.37)-(2.6.38) the conclusion follows. □

Finally, let us discuss the limit $\alpha\to 0^+$ of problem (NCH). Up to obvious changes, the analogue of Theorem 2.2.7 holds true; we leave its formulation to the reader. Remarkably, thanks to the conservation of mass (see equality (2.6.16) and Theorem 2.6.4-(iii)), the same holds under the weaker assumption

(H_5) there exists $k>0$ such that

$$
k\,u^{\pm}\le\Phi(u)\ \text{for any}\ u\in\mathbb{R}\,, \tag{2.6.39}
$$

where $r^{\pm}:=\max\{\pm r,0\}$ ($r\in\mathbb{R}$), and either sign holds in the above inequality.. In fact, the following holds.

Theorem 2.6.6. *Let $u_0\in H^1(\Omega)$, and let φ satisfy assumptions (H_0), (H_1) and (H_5). Let u_α be the solution of problem (NCH) given by Theorem 2.6.4 ($\alpha\in(0,\infty)$). Then there exist $u\in L^\infty((0,T);L^1(\Omega))$, $\mu\in L^\infty((0,T);\mathcal{M}^+(\Omega))$ and*
$v\in L^2((0,T);H^1(\Omega))$ with the following properties:

(i) there exist two subsequences $\{u_{\alpha_k}\}\subseteq\{u_\alpha\}$, $\{v_{\alpha_k}\}\subseteq\{v_\alpha\}$ and a decreasing sequence of measurable sets $E_k\subseteq Q$ of Lebesgue measure $|E_k|\to 0$, such that the sequence $\{u_{\alpha_k}\chi_{Q\setminus E_k}\}$ is uniformly integrable, and as $\alpha_k\to 0^+$ the convergence in (2.2.56)-(2.2.58) *holds true;*

(ii) as $\alpha_k \to 0^+$ *there holds*

$$v_{\alpha_k} \rightharpoonup v \text{ in } L^2((0,T); H^1(\Omega))\,; \tag{2.6.40}$$

(iii) equality (2.2.60) *is satisfied.*

Proof. By the considerations in Section 2.5, it suffices to prove that the family $\{u_\alpha\}$ is bounded in $L^1(Q)$. To this purpose, observe that for almost every $t \in (0,T)$

$$\begin{aligned}\int_\Omega |u_\alpha(x,t)|\,dx &= \pm\int_\Omega u_\alpha(x,t)\,dx + 2\int_\Omega u_\alpha^\mp(x,t)\,dx \;= \\ &= \pm\int_\Omega u_0(x)\,dx + 2\int_\Omega u_\alpha^\mp(x,t)\,dx \;\leq \\ &\leq \sqrt{|\Omega|}\|u_0\|_{H^1(\Omega)} + 2\int_\Omega \Phi(u_\alpha)(x,t)\,dx \;\leq\; M\,;\end{aligned}$$

here use of equality (2.6.16) and inequality (2.6.17) has been made. Then the conclusion follows. □

Chapter 3

On the Cahn-Hilliard regularization of forward-backward parabolic equations

3.1 Introduction

In this paper we study the problem

$$(P_T) \qquad \begin{cases} u_t = \Delta[\varphi(u) - \epsilon\Delta u] & \text{in } Q_T := \Omega \times (0,T) \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial \Delta u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\}, \end{cases}$$

where $T > 0$, $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, and $\frac{\partial}{\partial \nu}$ denotes the outer normal derivative at $\partial\Omega$. We are interested in nonlinearities φ of the following type:

$$\varphi(u) = \frac{u}{1+u^2}\,, \quad \varphi(u) = u\exp(-u)\,. \tag{3.1.1}$$

Precise assumptions concerning the function φ (and the initial data function u_0) are made below (see Section 3.2).

Our motivation comes from the Perona-Malik equation [PM] in one space dimension

$$z_t = [\varphi(z_x)]_x\,, \tag{3.1.2}$$

where φ is as in (3.1.1), which also appears in a mathematical model for the formation of layers of constant temperature or salinity in the ocean (see [BBDU]). The relationship with problem (P_T), with φ as in (3.1.1), is easily seen: differentiating equation (3.1.2) formally with respect to x and setting $u := z_x$ gives the equation

$$u_t = [\varphi(u)]_{xx}\,. \tag{3.1.3}$$

The first equation in problem (P_T) is a specific regularisation, called of *Cahn-Hilliard type*, of equation (3.1.3). In fact, the Perona-Malik equation is a well-known example of *forward-backward* parabolic equation, which is ill-posed forward in time.

Beside in image processing [PM] and in modelling of stratified turbulent shear flow [BBDU], forward-backward equations arise in many applications, *e.g.*, in aggregation models of population dynamics [Pa]. To regularize these equations, first a higher order term depending on a small parameter $\epsilon > 0$ is added to the right-hand side (on the strength of different physical and biological considerations, *e.g.*, see [BFJ, BS, G]), then the "vanishing viscosity limit" as $\epsilon \to 0$ is investigated. In carrying out this program, mainly two classes of additional terms have been used in the literature:
(*i*) $\epsilon\Delta[\psi(u)]_t$, with $\psi' > 0$, leading to third order *pseudo-parabolic equations* [BBDU, BST1, BST2, EP, MTT, NP, Pl1, Pl2, Pl3, S, STe, ST1, ST2, ST3]. If $\psi(u) = u$, this regularization is called *Sobolev regularization*;
(*ii*) $-\epsilon\Delta^2 u$, leading to fourth-order *Cahn-Hilliard type equations* (see [BFG, Pl4, Sl] and references therein). This kind of regularization has been less addressed, possibly since studying its singular limit as $\epsilon \to 0$ appears to be more difficult than for the regularization mentioned in (*i*).

Cahn-Hilliard type equations have been widely investigated in the context of the theory of phase transitions (in particular, see [EZ, Z]). In this case the non-linearity φ suggested by modelling is cubic, *i.e.*, $\varphi(u) = u^3 - u$. Existence of suitably defined global solutions was proven in [EZ], and their asymptotic behaviour for large time was studied in [Z], under the assumption $N \le 3$ on the space dimension. The singular limit as $\epsilon \to 0$ was studied in [Pl4], taking advantage of the fourth order growth at infinity of the associated free energy Φ. Unfortunately, the approach in [Pl4] does not apply when φ is as in (3.1.1), due to the very slow (only logarithmic) growth at infinity of the associated potential.

In the light of the above remarks, our motivation for the present study is investigating the regularization mentioned in (*ii*) for forward-backward equations, whose nonlinearity φ grows at most linearly at infinity (see assumption (A_1). This is meant as a preliminary step before addressing the singular limit of the problem as $\epsilon \to 0$. Specifically, we prove existence and uniqueness of global solutions in a suitable function space under the assumption $N \le 5$ (see Theorem 3.2.2). We also study, using the same approach as in [Z], the asymptotic behaviour as $t \to \infty$ of solutions of the problem

$$(P_\infty) \qquad \begin{cases} u_t = \Delta[\varphi(u) - \epsilon\Delta u] & \text{in } Q_\infty := \Omega \times (0,\infty) \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial \Delta u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u = u_0 & \text{in } \Omega \times \{0\}\,. \end{cases}$$

(in particular, see Theorem 3.2.6). In doing so, we take advantage of conservation of mass: for any solution u of problem (P_∞)

$$\frac{1}{|\Omega|}\int_\Omega u(x,t)dx = M\,,$$

where

$$M := \frac{1}{|\Omega|}\int_\Omega u_0 dx \qquad (3.1.4)$$

(see Proposition 3.2.3). Finally, in the case $N = 1$ we address existence and multiplicity of equilibrium solutions of (P_∞) when $\varphi(u) = \frac{u}{1+u^2}$. At variance from the cases of a polynomial φ (see [CGS, NPe, Z]), a complete analytical investigation reveals to be cumbersome, thus recourse to numerical methods has been expedient.

The paper is organized as follows. In Section 3.2 we describe the mathematical framework and state our main results. Proofs are to be found in Sections 3.3, 3.4. Equilibrium solutions of (P_∞) in one space dimension are studied in Section 3.5.

3.2 Mathematical framework

3.2.1 Preliminaries

The following function spaces will be used in the sequel:

$$H^2_E(\Omega) := \left\{ u \in H^2(\Omega) \,\Big|\, \frac{\partial u}{\partial \nu} = 0 \right\}\,,$$

$$H^2_{E^*}(\Omega) := \left\{ u \in H^2_E(\Omega) \,\Big|\, \int_\Omega u dx = 0 \right\}\,,$$

$$H^4_E(\Omega) := \left\{ u \in H^4(\Omega) \,\Big|\, \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \right\}\,,$$

$$H^4_{E^*}(\Omega) := \left\{ u \in H^4_E(\Omega) \,\Big|\, \int_\Omega u dx = 0 \right\}\,,$$

$$H^{4,1}(Q_T) := \left\{ u \in L^2(0,T;H^4(\Omega)) \,|\, u_t \in L^2(Q_T) \right\}\,.$$

We always suppose that $\varphi \in C^3(\mathbb{R})$, $\varphi(0) = 0$; moreover, the following assumptions concerning φ will be used:

(A_1) $$\varphi' \in L^\infty(\mathbb{R})\,;$$

(A_2) $$\varphi'' \in L^\infty(\mathbb{R})\,;$$

(A_3) $$\varphi''' \in L^\infty(\mathbb{R})\,;$$

(A_4) $$s\varphi(s) \geq 0 \quad \text{for any} \;\; s \in \mathbb{R}\,.$$

The following proposition (*e.g.*, see [Ze, Proposition 23.23]) will be used to prove existence results.

Proposition 3.2.1. *Let V be a separable reflexive Banach space with dual space V', and let H be a separable Hilbert space such that:*
(i) $V \subset H \subset V'$;
(ii) V is continuously embedded into H and dense in H.
Then for any $p \in (1, \infty)$ the space

$$Z := \left\{ u \,|\, u \in L^p((0,T);V),\, u_t \in L^q((0,T);V') \right\}\,,$$

where $\frac{1}{p} + \frac{1}{q} = 1$, is continuously embedded into $C([0,T];H)$.

3.2.2 Existence

Let us state the following definition.

Definition 3.2.1. *Let $u_0 \in H^2_E(\Omega)$. By a solution of problem (P_T) we mean any function $u = u(x,t)$, $u \in C\left([0,T]; H^2_E(\Omega)\right) \cap H^{4,1}(Q_T)$ such that $\varphi(u) \in C\left([0,T]; H^2_E(\Omega)\right)$, $u(\cdot, 0) = u_0$, and*

$$\iint_{Q_T} u_t \eta \, dxdt = -\iint_{Q_T} \nabla \left[\varphi(u) - \epsilon \Delta u\right] \cdot \nabla \eta \, dxdt \qquad (3.2.5)$$

for any $\eta \in L^2((0,T), H^1(\Omega))$ (the central dot "$\cdot$" denoting the scalar product in $\mathbb{R}^N$).

Theorem 3.2.2. *Let assumptions (A_1)-(A_2) be satisfied. Let $u_0 \in H^2_E(\Omega)$, and let $N \leq 5$. Then for every $T > 0$ there exists a unique solution of problem (P_T).*

Choosing $\eta \equiv 1$ in equality (3.2.5) immediately gives the following result.

Proposition 3.2.3. *Let the assumptions of Theorem 3.2.2 be satisfied, and let u be the solution of problem (P_T) given by the same theorem. Then for every $t \in (0,T)$*

$$\int_\Omega u(x,t)dx = \int_\Omega u_0 dx\,. \qquad (3.2.6)$$

By Theorem 3.2.2 and Proposition 3.2.3 we have the following result.

Corollary 3.2.4. *Let assumptions (A_1)-(A_2) be satisfied. Let $u_0 \in H^2_{E^*}(\Omega)$, and let $N \leq 5$. Then for every $T > 0$ there exists a unique solution of problem (P_T), which belongs to $C\left([0,T]; H^2_{E^*}(\Omega)\right)$.*

3.2.3 Asymptotic behaviour

Let us first state the following definitions.

Definition 3.2.2. *Let $u_0 \in H^2_E(\Omega)$. By a global solution of problem (P_∞) we mean any function $u \in C\left([0,\infty); H^2_E(\Omega)\right) \cap H^{4,1}(Q_\infty)$, with $\varphi(u) \in C\left([0,\infty); H^2_E(\Omega)\right)$, which is a solution of problem (P_T) for every $T > 0$.*

Definition 3.2.3. *Let $u_0 \in H^2_E(\Omega)$. Let u be the global solution of problem (P_∞) given by Theorem 3.2.2. By the $\omega-$limit set of the solution u we mean the set*

$$\omega(u_0) := \{\bar{u} \mid \exists \{t_n\} \subseteq (0,\infty) \text{ such that } u(x,t_n) \to \bar{u} \text{ in } H^2_E(\Omega)\}. \qquad (3.2.7)$$

Definition 3.2.4. *By an equilibrium solution of problem (P_∞) we mean any function $w \in H^4_E(\Omega)$, with $\varphi(w) \in H^2_E(\Omega)$, which satisfies in the strong sense the equality*

$$\Delta[\varphi(w) - \epsilon \Delta w] = 0 \quad \text{in } \Omega\,. \qquad (3.2.8)$$

Remark 3.2.1. *It is immediately seen that there is one-to-one correspondence between equilibrium solutions of problem (P_∞) and functions $w \in H^2_E(\Omega)$, with $\varphi(w) \in H^2_E(\Omega)$, which satisfy in the strong sense the equality*

$$\epsilon \Delta w = \varphi(w) + \sigma \quad \text{in } \Omega \qquad (3.2.9)$$

with some constant $\sigma \in \mathbb{R}$.

By Theorem 3.2.2 and a standard prolongation argument, for every $u_0 \in H^2_E(\Omega)$ there exists a unique global solution of problem (P_∞). Let us study the asymptotic behaviour of this solution as $t \to \infty$.

To this purpose, Proposition 3.2.6 suggests the change of unknown $z := u - M$, where M denotes the mass defined in (3.1.4). Then $u = z + M$ and $z(\cdot,t) \in H^2_{E^*}(\Omega)$ for every $t \in (0,\infty)$. Therefore, it is not restrictive to study problem (P_∞) with initial data $u_0 \in H^2_{E^*}(\Omega)$ (clearly, this implies $u(\cdot,t) \in H^2_{E^*}(\Omega)$ for every $t \in (0,\infty)$). In doing so, the advantage is that we can obtain uniform estimates of the solution on the whole half-line $(0,\infty)$. In fact, the following proposition will be proven.

Proposition 3.2.5. *Let $u_0 \in H^4_{E^*}(\Omega)$, let assumptions (A_1)-(A_4) be satisfied, and let $N \leq 5$. Let u be the unique global solution of problem (P_∞) given by Theorem 3.2.2. Then for every $t \in (0, \infty)$*

$$||u(\cdot, t)||_{H^3(\Omega)} \leq C_2^* \,. \tag{3.2.10}$$

Then we have the following result.

Theorem 3.2.6. *Let $u_0 \in H^4_{E^*}(\Omega)$, let assumptions (A_1)-(A_4) be satisfied, and let $N \leq 5$. Then:*
(i) the ω-limit set $\omega(u_0)$ is nonempty;
(ii) the function $t \to F(u)(t)$, where

$$F(u)(t) := \int_\Omega \left\{ \Phi(u)(x,t) + \frac{\epsilon}{2} |\nabla u(x,t)|^2 \right\} dx \,, \quad \Phi(u) := \int_0^u \varphi(s) ds \tag{3.2.11}$$

for any $u \in C\left([0, \infty); H^1(\Omega)\right)$, is nonicreasing;
(iii) every point of $\omega(u_0)$ is an equilibrium solution of problem (P_∞).

Claim (i) of the above theorem, whose proof is omitted, is an obvious consequence of Proposition 3.2.5, whereas claims (i)-(ii) follow by standard arguments (*e.g.*, see [H]). We leave the details to the reader.

Finally, observe that problem (P_∞) and its solution depends parametrically on ϵ. The following proposition shows that for ϵ sufficiently large the solution decays to zero as $t \to \infty$.

Proposition 3.2.7. *Let $u_0 \in H^4_{E^*}(\Omega)$, let assumptions (A_1)-(A_4) be satisfied, and let $N \leq 5$. Let u be the unique global solution of problem (P_∞) given by Theorem 3.2.2. Then there exists $\epsilon_0 > 0$ such that for any $\epsilon > \epsilon_0$*

$$\|u(\cdot, t)\|_{H^1(\Omega)} \to 0 \qquad as \ t \to \infty \,.$$

In agreement with Proposition 3.2.7, it can be proven that every nontrivial equilibrium solution of problem (P_∞) is trivial if ϵ is large enough. On the other hand, by similar methods it can be proven that problem (P_∞) admits nontrivial equilibrium solutions, if ϵ is sufficiently small and the mass M sufficiently large. The proof of these statements is analogous to those of [Z, Lemma 3.2 and Theorem 3.3]. The latter statement is in agreement with the considerations of Section 3.4 (based on numerical evidence), if $N = 1$ and $\varphi(s) = \frac{s}{1+s^2}$.

3.3 Proof of existence results

This section is devoted to the proof of Theorem 3.2.2. Set

$$v := \varphi(u) - \epsilon\Delta u\,. \tag{3.3.12}$$

Then problem (P_T) can be rewritten in the equivalent form

$$\begin{cases} u_t = \Delta v & \text{in } Q_T \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\}\,. \end{cases} \tag{3.3.13}$$

According to Definition 3.2.2, we seek a couple of functions $u \in C\left([0,T]; H^2_E(\Omega)\right) \cap H^{4,1}(Q_T)$, $v \in C\left([0,T]; H^2_E(\Omega)\right)$ satisfying problem (3.3.13), in the sense that $u(\cdot,0) = u_0$ and

$$\iint_{Q_T} u_t \eta \, dxdt = -\iint_{Q_T} \nabla v \cdot \nabla \eta \, dxdt \tag{3.3.14}$$

for any $\eta \in L^2((0,T), H^1(\Omega))$. This will be achieved using the Faedo-Galerkin method.

Let ψ_k $(k \in \mathbf{N})$ denote the eigenfunctions of the Laplace operator with Neumann boundary conditions

$$\begin{cases} -\Delta\psi_k = \lambda_k \psi_k & \text{in } \Omega \\ \dfrac{\partial \psi_k}{\partial \nu} = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{3.3.15}$$

which combined with the constant function $\Phi_0 \equiv 1$ form an orthogonal basis of $H^2_E(\Omega)$. Since by assumption the boundary $\partial\Omega$ is smooth, the functions ψ_k are smooth and there holds

$$\frac{\partial \Delta\psi_k}{\partial \nu} = 0 \quad \text{on } \partial\Omega \qquad (k \in \mathbf{N} \cup \{0\})\,. \tag{3.3.16}$$

Thus they are a suitable choice for the Faedo-Galerkin method.

In view of the above remarks, we consider approximate solutions of (3.3.13) of the form

$$u_m := \sum_{j=0}^{m} w_{jm}\psi_j\,, \quad v_m := \varphi(u_m) - \epsilon\Delta u_m \qquad (m \in \mathbf{N} \cup \{0\})\,, \tag{3.3.17}$$

with coefficients $w_{jm} = w_{jm}(t)$ $(t \in (0,T)$ to be determined. Since

$$\frac{\partial u_m}{\partial \nu} = \sum_{j=0}^{m} w_{jm} \frac{\partial \psi_j}{\partial \nu}$$

and

$$\frac{\partial v_m}{\partial \nu} = \sum_{j=0}^{m} w_{jm} \left[\varphi'(u_m) \frac{\partial \psi_j}{\partial \nu} - \epsilon \frac{\partial \Delta \psi_j}{\partial \nu} \right] ,$$

by (3.3.15)-(3.3.16) there holds

$$\frac{\partial u_m(\cdot,t)}{\partial \nu} = \frac{\partial v_m(\cdot,t)}{\partial \nu} = 0 \quad \text{on } \partial\Omega \tag{3.3.18}$$

for every $m \in \mathbf{N} \cup \{0\}$. Clearly,

$$u_{mt} = \sum_{j=0}^{m} w'_{jm}(t)\psi_j \, , \quad \Delta u_m = -\sum_{j=0}^{m} \lambda_j w_{jm} \psi_j \, , \quad \Delta^2 u_m = \sum_{j=0}^{m} \lambda_j^2 w_{jm} \psi_j \, . \tag{3.3.19}$$

Denoting by $(\cdot,\cdot)_{L^2(\Omega)}$ the scalar product in $L^2(\Omega)$,

$$(f,g)_{L^2(\Omega)} := \int_\Omega fg dx \qquad \text{for any } f,g \in L^2(\Omega) \, ,$$

by (3.3.19) we have

$$(u_{mt}, \psi_k)_{L^2(\Omega)} = w'_{km}(t) \, ,$$

$$\begin{aligned} (\Delta v_m, \psi_k)_{L^2(\Omega)} &= (v_m, \Delta\psi_k)_{L^2(\Omega)} = -\lambda_k \left(v_m, \psi_k\right)_{L^2(\Omega)} \\ &= -\lambda_k \left\{ (\varphi(u_m), \psi_k)_{L^2(\Omega)} - \epsilon \left(\Delta u_m, \psi_k\right)_{L^2(\Omega)} \right\} \\ &= -\epsilon \lambda_k^2 w_{km}(t) - \lambda_k \left(\varphi(u_m), \psi_k\right)_{L^2(\Omega)} \qquad (k = 0,1,...,m) \, . \end{aligned}$$

According to the Faedo-Galerkin method, we require that the equalities

$$(u_{mt}, \psi_k)_{L^2(\Omega)} = (\Delta v_m, \psi_k)_{L^2(\Omega)}$$

be satisfied for each $m \in \mathbf{N} \cup \{0\}$ and $k = 0,1,...,m$. This gives the system of ordinary differential equations

$$\begin{cases} w'_{km} = -\epsilon \lambda_k^2 w_{km} - \lambda_k \left(\varphi(u_m), \psi_k\right)_{L^2(\Omega)} & \text{in } (0,T) \\ w_{km}(0) = \alpha_{km} \end{cases} \tag{3.3.20}$$

for the coefficients $w_{0m}, w_{1m}, ... w_{mm}$. Here

$$\alpha_{km} := (u_{0m}, \psi_k)_{L^2(\Omega)} \; ,$$

with

$$\begin{cases} u_{0m} := \sum\limits_{j=0}^{m} \alpha_{jm} \psi_j, \quad u_{0m} \to u_0 \text{ in } H^2_E(\Omega) \, , \\ \|u_{0m}\|_{H^2(\Omega)} \le \|u_0\|_{H^2(\Omega)} \, . \end{cases} \tag{3.3.21}$$

For every $m \in \mathbf{N} \cup \{0\}$ and $k = 0$, since $\lambda_0 = 0$ the unique solution of system (3.3.21) is $w_{0m}(t) = w_{0m}(0) = \alpha_{0m}$ (observe that the nonlinear term $(\varphi(u_m), \psi_k)_{L^2(\Omega)}$ is locally Lipschitz continuous with respect to w_{km} by assumption (A_1)). On the other hand, for for every $m \in \mathbf{N}$ and $k = 1, ..., m$ there exists $T_m > 0$ such that system (3.3.20) has a unique solution in the maximal interval $(0, T_m)$. In view of the a priori estimates below, this solution is global - namely, it exists in $(0, T)$ for every $m \in \mathbf{N}$.

Lemma 3.3.1. *Let $u_0 \in H^2_E(\Omega)$, and let assumption (A_1) be satisfied. Let u_m be defined by* (3.3.17) *with coefficients w_{km} satisfying system* (3.3.20)-(3.3.21) *in the maximal interval $(0, T_m)$. Then there exists $C_1 > 0$ (only depending on ϵ, T and the norm $\|u_0\|_{H^2(\Omega)}$) such that for every $t \in (0, T_m)$*

$$||u_m(\cdot, t)||^2_{L^2(\Omega)} + \int_0^t \|\Delta u_m(\cdot, s)\|^2_{L^2(\Omega)} ds \leq C_1 \,, \tag{3.3.22}$$

$$|||\nabla u_m(\cdot, t)|||^2_{L^2(\Omega)} + \int_0^t |||\nabla \Delta u_m(\cdot, s)|||^2_{L^2(\Omega)} ds \leq C_1 \,. \tag{3.3.23}$$

Proof. By (3.3.17) and (3.3.19) we have

$$\|u_m(\cdot, t)\|^2_{L^2(\Omega)} = \sum_{j=0}^m |w_{jm}|^2 \,, \quad \|\Delta u_m(\cdot, t)\|^2_{L^2(\Omega)} = \sum_{j=0}^m \lambda_j^2 |w_{jm}|^2 \,. \tag{3.3.24}$$

Multiplying the first equation of (3.3.20) by w_{km} and summing over $k = 0, ..., m$ plainly we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_m(\cdot, t)\|^2_{L^2(\Omega)} &= -\epsilon \|\Delta u_m(\cdot, t)\|^2_{L^2(\Omega)} + (\varphi(u_m), \Delta u_m)_{L^2(\Omega)} \\ &= -\epsilon \|\Delta u_m(\cdot, t)\|^2_{L^2(\Omega)} - \int_\Omega \varphi'(u_m)|\nabla u_m(x, t)|^2 dx \\ &\leq -\epsilon \|\Delta u_m(\cdot, t)\|^2_{L^2(\Omega)} + C \int_\Omega |\nabla u_m(x, t)|^2 dx \\ &= -\epsilon \|\Delta u_m(\cdot, t)\|^2_{L^2(\Omega)} - C \int_\Omega u_m(x, t) \Delta u_m(x, t) dx \\ &\leq -\frac{\epsilon}{2}\|\Delta u_m(\cdot, t)\|^2_{L^2(\Omega)} + \frac{C^2}{2\epsilon}||u_m(\cdot, t)||^2_{L^2(\Omega)} \end{aligned} \tag{3.3.25}$$

with some constant $C > 0$; here use of assumption (A_1) and equality (3.3.18) has been made. By Gronwall's inequality and (3.3.21), from (3.3.25) we get

$$||u_m(\cdot, t)||^2_{L^2(\Omega)} \leq ||u_0||^2_{H^2(\Omega)} e^{C^2 T/\epsilon} \qquad (t \in (0, T_m)) \,. \tag{3.3.26}$$

Integrating inequality (3.3.25) on $(0, T_m)$ and using (3.3.26) gives inequality (3.3.22).

To prove (3.3.23) observe preliminarily that

$$\||\nabla u_m(\cdot,t)|\|^2_{L^2(\Omega)} = -\left(\Delta u_m, u_m\right)_{L^2(\Omega)} = \sum_{j=0}^{m} \lambda_j |w_{jm}|^2,$$

$$\||\nabla \Delta u_m(\cdot,s)|\|^2_{L^2(\Omega)} = -\left(\Delta u_m, \Delta^2 u_m\right)_{L^2(\Omega)} = \sum_{j=0}^{m} \lambda_j^3 |w_{jm}|^2$$

(see (3.3.16), (3.3.18) and (3.3.19)). Then multiplying the first equation of (3.3.20) by $\lambda_k w_{km}$ and summing over $k = 0, ..., m$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\||\nabla u_m(\cdot,t)|\|^2_{L^2(\Omega)} = -\epsilon\,\||\nabla \Delta u_m(\cdot,t)|\|^2_{L^2(\Omega)} - \left(\varphi(u_m), \Delta^2 u_m\right)_{L^2(\Omega)} \\
= \;& -\epsilon\,\||\nabla \Delta u_m(\cdot,t)|\|^2_{L^2(\Omega)} + \int_\Omega \varphi'(u_m) \nabla u_m(x,t) \cdot \nabla \Delta u_m(x,t)\,dx \\
\le \;& -\epsilon\||\nabla \Delta u_m(\cdot,t)|\|^2_{L^2(\Omega)} + C \int_\Omega |\nabla u_m(x,t)||\nabla \Delta u_m(x,t)|\,dx \\
\le \;& -\frac{\epsilon}{2}\||\nabla \Delta u_m(\cdot,t)|\|^2_{L^2(\Omega)} + \frac{C^2}{2\epsilon} \int_\Omega |\nabla u_m(x,t)|^2\,dx
\end{aligned}$$

with some constant $C > 0$; here use of assumption (A_1) and equality (3.3.16) has been made. Using Gronwall's inequality and and arguing as for (3.3.25), from the above inequality we obtain (3.3.23). This completes the proof. □

Under the stronger assumptions of Theorem 3.2.2 we can improve on the estimates of the above lemma. To this purpose, following [EZ] we shall make use of the Nirenberg inequality:

$$||D^j v||_{L^p(\Omega)} \le K_1 ||D^m v||^a_{L^r(\Omega)} ||v||^{1-a}_{L^q(\Omega)} + K_2 ||v||_{L^q(\Omega)} \tag{3.3.27}$$

where $D \equiv \frac{\partial}{\partial x_k}$ $(k = 1, \dots, N)$, $K_1, K_2 > 0$ and

$$\frac{1}{p} = \frac{j}{N} + a\left(\frac{1}{r} - \frac{m}{N}\right) + \frac{1-a}{q}\,, \tag{3.3.28}$$

with $j \in \mathbf{N} \cup \{0\}$, $m \in \mathbf{N}$, $j \le m$, $a \in [\frac{j}{m}, 1]$ and $p, q, r \in (1, \infty)$ (*e.g.*, see [A]).

Lemma 3.3.2. *Let* $u_0 \in H^2_E(\Omega)$, *let assumptions* (A_1)-(A_2) *be satisfied, and let* $N \le 5$. *Let* u_m *be defined by* (3.3.17) *with coefficients* w_{km} *satisfying system* (3.3.20)-(3.3.21) *in the maximal interval* $(0, T_m)$. *Then there exists* $C_2 > 0$ *(only depending on* ϵ, T *and the norm* $\|u_0\|_{H^2(\Omega)}$*) such that for every* $t \in (0, T_m)$

$$||\Delta u_m(\cdot,t)||^2_{L^2(\Omega)} + \epsilon \int_0^t \|\Delta^2 u_m(\cdot,s)\|^2_{L^2(\Omega)}\,ds \le C_2\,. \tag{3.3.29}$$

Proof. Multiplying the first equation of (3.3.20) by $\lambda_k^2 w_{km}$ and summing over $k = 0, ..., m$, we get

$$\frac{1}{2}\frac{d}{dt}\|\Delta u_m(\cdot,t)\|^2_{L^2(\Omega)} = -\epsilon\,\|\Delta^2 u_m(\cdot,t)\|^2_{L^2(\Omega)} + \big(\Delta\varphi(u_m)(\cdot,t), \Delta^2 u_m(\cdot,t)\big)_{L^2(\Omega)} \tag{3.3.30}$$

(see (3.3.19)). Set $\Delta[\varphi(u_m)] \equiv \Delta\varphi(u_m)(\cdot,t)$, $\Delta^2 u_m \equiv \Delta^2 u_m(\cdot,t)$ for simplicity. Since

$$\Delta[\varphi(u_m)] = \varphi'(u_m)\Delta u_m + \varphi''(u_m)|\nabla u_m|^2\,,$$

using assumptions (A_1)-(A_2) plainly we have

$$\begin{aligned} \left|\big(\Delta\varphi(u_m), \Delta^2 u_m\big)_{L^2(\Omega)}\right| &\le \|\Delta\varphi(u_m)\|_{L^2(\Omega)}\left\|\Delta^2 u_m\right\|_{L^2(\Omega)} \qquad (3.3.31)\\ &\le C\left\|\Delta^2 u_m\right\|_{L^2(\Omega)}\left\{\||\nabla u_m|\|^2_{L^4(\Omega)} + \|\Delta u_m\|_{L^2(\Omega)}\right\}. \end{aligned}$$

To estimate the term $\||\nabla u_m|\|^2_{L^4(\Omega)}$ in the right-hand side of (3.3.31), we use the Nirenberg inequality (3.3.27) with $v = |\nabla u_m|$, $j = 0, m = 3, a = N/12, p = 4, q = r = 2$. Then we obtain

$$\||\nabla u_m|\|_{L^4(\Omega)} \le \tilde{K_1}||\Delta^2 u_m||^a_{L^2(\Omega)}||\,|\nabla u_m|\,||^{1-a}_{L^2(\Omega)} + K_2||\,|\nabla u_m|\,||_{L^2(\Omega)} \tag{3.3.32}$$

for some $\tilde{K_1} > 0$. Since by assumption $2a = N/6 \le 5/6$, by inequalities (3.3.23) and (3.3.32) there exist $M_1 > 0, M_2 > 0$ such that for every $t \in (0, T_m)$

$$\||\nabla u_m|(\cdot,t)\|^2_{L^4(\Omega)} \le M_1||\Delta^2 u_m(\cdot,t)||^{\frac{5}{6}}_{L^2(\Omega)} + M_2\,. \tag{3.3.33}$$

Similarly, the term $\|\Delta u_m\|_{L^2(\Omega)}$ in the right-hand side of (3.3.31) can be estimated using the Nirenberg inequality with $v = |\nabla u_m|$, $j = 1, m = 3, a = 1/3, p = q = r = 2$. This gives

$$\|\Delta u_m\|_{L^2(\Omega)} \le \bar{K_1}||\Delta^2 u_m||^{\frac{1}{3}}_{L^2(\Omega)}||\,|\nabla u_m|\,||^{\frac{2}{3}}_{L^2(\Omega)} + K_2||\,|\nabla u_m|\,||_{L^2(\Omega)} \tag{3.3.34}$$

for some $\bar{K_1} > 0$. Hence by inequalities (3.3.23) and (3.3.33) there exist $N_1 > 0, N_2 > 0$ such that for every $t \in (0, T_m)$

$$\|\Delta u_m(\cdot,t)\|_{L^2(\Omega)} \le N_1||\Delta^2 u_m(\cdot,t)||^{\frac{1}{3}}_{L^2(\Omega)} + N_2\,. \tag{3.3.35}$$

By equality (3.3.30) and inequalities (3.3.31), (3.3.33) and (3.3.35), it is easily seen that there exists $M > 0$ (depending on ϵ) such that

$$\frac{1}{2}\frac{d}{dt}\|\Delta u_m(\cdot,t)\|^2_{L^2(\Omega)} + \frac{\epsilon}{2}\,\|\Delta^2 u_m(\cdot,t)\|^2_{L^2(\Omega)} \le M$$

for every $t \in (0, T_m)$, whence inequality (3.3.29) immediately follows. This completes the proof. □

Now we can prove Theorem 3.2.2.

Proof of Theorem 3.2.2. Let $\{u_m\}$, $\{v_m\}$ be the sequences defined in (3.3.17), with coefficients w_{jm} satisfying system (3.3.20). Observe preliminarily that by estimate (3.3.22) there holds $T_m = T$ for every $m \in \mathbf{N}$, thus estimates (3.3.22), (3.3.23) and (3.3.29) hold for every $t \in (0, T)$ and $m \in \mathbf{N}$. As a consequence, the sequence $\{u_m\}$ belongs to a bounded subset of $L^2((0, T); H^4(\Omega))$. Then there exist a subsequence $\{u_k\} \equiv \{u_{m_k}\} \subseteq \{u_m\}$ and a function $u \in L^2((0, T); H^4(\Omega))$ such that

$$u_k \rightharpoonup u \text{ in } L^2((0, T); H^4(\Omega)) \,. \tag{3.3.36}$$

Moreover, by estimates (3.3.22)-(3.3.23), it is not restrictive to assume that

$$u_k \to u \text{ almost everywhere in } Q_T \,. \tag{3.3.37}$$

By the assumed properties of φ, there holds $\varphi(u) \in L^2\left((0, T); H^2(\Omega)\right)$. Let us prove that u is a solution of problem (P_T).

For every $k \in \mathbf{N}$ and any $\eta \in L^2(0, T; H^1(\Omega))$ there holds

$$\begin{aligned} &\iint_{Q_T} u_{kt}\eta\, dxdt = -\iint_{Q_T} \nabla\left[\varphi(u_k)\right] \cdot \nabla\eta\, dxdt - \epsilon\iint_{Q_T} \Delta^2 u_k \eta\, dxdt \\ = \;& -\iint_{Q_T} \varphi'(u_k)\nabla u_k \cdot \nabla\eta\, dxdt - \epsilon\iint_{Q_T} \Delta^2 u_k \eta\, dxdt \,. \end{aligned} \tag{3.3.38}$$

As $k \to \infty$, using the convergence in (3.3.36)-(3.3.37) and assumption (A_1), by the Dominated Convergence Theorem we easily get

$$\iint_{Q_T} \varphi'(u_k)\nabla u_k \cdot \nabla\eta\, dxdt \quad \to \quad \iint_{Q_T} \varphi'(u)\nabla u \cdot \nabla\eta\, dxdt \,,$$

whereas by (3.3.36)

$$\iint_{Q_T} \Delta^2 u_k \eta\, dxdt \quad \to \quad \iint_{Q_T} \Delta^2 u\, \eta\, dxdt$$

for any η as above. By the above convergence results, letting $k \to \infty$ in equality (3.3.38) for any $\eta \in C_0^\infty(Q_T)$ we have

$$\begin{aligned} \iint_{Q_T} u_{kt}\eta\, dxdt \;&=\; -\iint_{Q_T} u_k \eta_t\, dxdt \quad \to \quad -\iint_{Q_T} u\eta_t\, dxdt \\ &=\; -\iint_{Q_T} \left\{\Delta\left[\varphi(u)\right] - \epsilon\Delta^2 u\right\} \eta\, dxdt \,. \end{aligned}$$

By the regularity of u, it follows that the distributional derivative u_t belongs to $L^2(Q_T)$, thus $u \in H^{4,1}(Q_T)$. Moreover, equality (3.3.14) holds for any $\eta \in L^2(0, T; H^1(\Omega))$.

Further observe that, in view of estimate (3.3.23), it is not restrictive to assume that

$$u_k \to u \text{ in } L^2((0,T); H^2(\Omega))\,. \tag{3.3.39}$$

Since $\{u_k\} \subseteq L^2((0,T); H^2_E(\Omega))$, from (3.3.39) we obtain that $u \in L^2((0,T); H^2_E(\Omega))$. Observe that $H^4(\Omega) \cap H^2_E(\Omega)$ endowed with the norm $\|\cdot\|_{H^4(\Omega)}$ is a closed subspace of $H^4(\Omega)$ hence is a reflexive Banach space. Therefore, applying Proposition 3.2.1 with $p = 2$, $V = H^4(\Omega) \cap H^2_E(\Omega)$ and $H = H^2_E(\Omega)$ we obtain that $u \in C\left([0,T]; H^2_E(\Omega)\right)$, thus $\varphi(u) \in C\left([0,T]; H^2_E(\Omega)\right)$.

Finally, observe that by (3.3.38) and subsequent remarks there holds

$$u_{kt} \to u_t \text{ in } L^2(Q_T)\,, \tag{3.3.40}$$

thus by Sobolev embedding

$$u_k \to u \text{ in } C\left([0,T]; L^2(\Omega)\right)\,.$$

In particular,

$$u_k(\cdot,0) = u_{0k} \to u(\cdot,0) \text{ in } L^2(\Omega)\,,$$

whence $u(\cdot,0) = u_0$ by (3.3.21).

It remains to prove uniqueness. By a standard argument, let u, v be solutions of problem (P_T). Then we have

$$(u-v)_t + \epsilon\,\Delta^2(u-v) = \Delta[\varphi(u) - \varphi(v)]\,.$$

Multiplying by $u - v$ and integrating over Ω yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(u-v)(\cdot,t)\|^2_{L^2(\Omega)} + \epsilon\,\|\Delta(u-v)(\cdot,t)\|^2_{L^2(\Omega)} \\
=\;& \int_\Omega \left[(\varphi(u) - \varphi(v))\,\Delta(u-v)\right](x,t)dx \\
\le\;& C\int_\Omega \left[|u-v|\,|\Delta(u-v)|\right](x,t)dx \\
\le\;& \frac{\epsilon}{2}\|\Delta(u-v)(\cdot,t)\|^2_{L^2(\Omega)} + \frac{C^2}{2\epsilon}\|(u-v)(\cdot,t)\|^2_{L^2(\Omega)}
\end{aligned}$$

(here use of assumption (A_1) has been made).Then we have

$$\frac{d}{dt}\|(u-v)(\cdot,t)\|^2_{L^2(\Omega)} + \epsilon\|\Delta(u-v)(\cdot,t)\|^2_{L^2(\Omega)} \le \frac{C^2}{\epsilon}\|(u-v)(\cdot,t)\|^2_{L^2(\Omega)}\,,$$

whence by Gronwall's inequality the equality $u = v$ immediately follows. This completes the proof. □

Remark 3.3.1. *Let $\langle\cdot,\cdot\rangle$ denote the duality map between the spaces $(H^1(\Omega))'$ and $H^1(\Omega)$. It is worth observing that, under the weaker assumptions of*

Lemma 3.3.1, for any $u_0 \in H^2_E(\Omega)$ *a solution of problem* (P_T) *exists in the following weaker sense:*
(i) $u \in C\left([0,T]; H^2_E(\Omega)\right) \cap L^2((0,T); H^3(\Omega))$, $u(\cdot,0) = u_0$, $\varphi(u) \in C\left([0,T]; H^2_E(\Omega)\right)$ *and* $u_t \in L^2((0,T); (H^1(\Omega))')$;
(ii) *there holds*

$$\int_0^T \langle u_t, \eta \rangle dt = -\iint_{Q_T} \nabla \left[\varphi(u) - \epsilon \Delta u\right] \cdot \nabla \eta \, dxdt \tag{3.3.41}$$

for every $\eta \in L^2((0,T); H^1(\Omega))$

In fact, by estimates (3.3.22)-(3.3.23) *there exist a subsequence* $\{u_k\} \equiv \{u_{m_k}\} \subseteq \{u_m\}$ *and a function* $u \in L^2((0,T); H^3(\Omega))$ *such that the convergence in* (3.3.37) *and* (3.3.39) *holds, and moreover*

$$u_k \rightharpoonup u \text{ in } L^2\left((0,T); H^3(\Omega)\right) . \tag{3.3.42}$$

Then applying Proposition 3.2.1 with $p = 2$, $V = H^3(\Omega) \cap H^2_E(\Omega)$ *and* $H = H^2_E(\Omega)$ *we obtain that* $u \in C\left([0,T]; H^2_E(\Omega)\right)$, *thus* $\varphi(u) \in C\left([0,T]; H^2_E(\Omega)\right)$.

Further observe that for any $\xi \in H^1(\Omega)$ *and* $t \in [0,T]$ *there holds*

$$\begin{aligned}\langle u_{mt}(\cdot,t), \xi \rangle &= (u_{mt}(\cdot,t), \xi)_{L^2(\Omega)} \\ &= (\Delta v_m, \xi)_{L^2(\Omega)} = -\left(\nabla v_m, \nabla \xi\right)_{L^2(\Omega)} \\ &= -\left([\varphi'(u_m)\nabla u_m - \epsilon \nabla \Delta u_m](\cdot,t), \nabla \xi\right)_{L^2(\Omega)} ,\end{aligned}$$

whence

$$\|u_{mt}(\cdot,t)\|_{(H^1(\Omega))'} \leq \left\|[\varphi'(u_m)\nabla u_m - \epsilon \nabla \Delta u_m](\cdot,t)\right\|_{L^2(\Omega)}$$

for any $t \in [0,T]$. *By estimate* (3.3.23) *and assumption* (A_1) *this plainly gives*

$$\int_0^T \|u_{mt}(\cdot,t)\|^2_{(H^1(\Omega))'} \leq 2C_1\left(C^2 + \epsilon^2\right)$$

for every $m \in \mathbf{N}$, *proving that the sequence* $\{u_{mt}\}$ *belongs to a bounded subset of* $L^2((0,T); (H^1(\Omega))')$. *Then it is not restrictive to assume that*

$$u_{kt} \rightharpoonup u_t \text{ in } L^2((0,T); (H^1(\Omega))') . \tag{3.3.43}$$

By the above remarks, letting $k \to \infty$ *in the equality*

$$\iint_{Q_T} u_{kt} \eta \, dxdt = -\iint_{Q_T} \nabla \left[\varphi(u_k) - \epsilon \Delta u_k\right] \cdot \nabla \eta \, dxdt$$

(which holds for any $k \in \mathbf{N}$ *and* $\eta \in L^2((0,T); H^1(\Omega))$*) we obtain* (3.3.41). *Arguing as in the proof of Theorem 3.2.2, the equality* $u(x,0) = u_0(x)$ *is easily proven. Hence the claim follows.*

We conclude this section by proving for further reference the following analogue of Lemma 3.3.2.

Lemma 3.3.3. *Let $u_0 \in H^4_E(\Omega)$, let assumptions (A_1)-(A_3) be satisfied, and let $N \leq 5$. Let u_m be defined by (3.3.17) with coefficients w_{km} satisfying system (3.3.20) in the maximal interval $(0, T_m)$, with initial data α_{km} such that*

$$\begin{cases} u_{0m} := \sum_{j=0}^{m} \alpha_{jm}\psi_j, \quad u_{0m} \to u_0 \text{ in } H^4_E(\Omega)\,, \\ \|u_{0m}\|_{H^4(\Omega)} \leq \|u_0\|_{H^4(\Omega)}\,. \end{cases} \tag{3.3.44}$$

Then there exists $C_3 > 0$ (only depending on ϵ, T and the norm $\|u_0\|_{H^4(\Omega)}$) such that for every $t \in (0, T_m)$

$$|| \, |\nabla \Delta u_m(\cdot, t)| \, ||^2_{L^2(\Omega)} + \epsilon \int_0^t \| \, |\nabla \Delta^2 u_m(\cdot, s)| \, \|^2_{L^2(\Omega)} ds \leq C_3\,. \tag{3.3.45}$$

Proof. Multiplying the first equation of (3.3.20) by $\lambda_k^3 w_{km}$ and summing over $k = 0, ..., m$, we get

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\| \, |\nabla \Delta u_m(\cdot, t)| \, \|^2_{L^2(\Omega)} \qquad (3.3.46) \\ = \; & -\epsilon \| \, |\nabla \Delta^2 u_m(\cdot, t)| \, \|^2_{L^2(\Omega)} - \big(\nabla[\Delta\varphi(u_m)](\cdot, t), \nabla[\Delta^2 u_m](\cdot, t)\big)_{L^2(\Omega)} \\ \leq \; & -\frac{\epsilon}{2} \| \, |\nabla \Delta^2 u_m(\cdot, t)| \, \|^2_{L^2(\Omega)} + \frac{1}{2\epsilon}\| \, |\nabla[\Delta\varphi(u_m)](\cdot, t)| \, \|^2_{L^2(\Omega)}\,; \end{aligned}$$

here use of the equalities

$$\sum_{j=0}^{m} \lambda_k^3 \, |w_{km}|^2 = || \, |\nabla \Delta u_m(\cdot, t)| \, ||^2_{L^2(\Omega)}\,,$$

$$\sum_{k=0}^{m} \lambda_k^5 \, |w_{km}|^2 = \| \, |\nabla \Delta^2 u_m(\cdot, t)| \, \|^2_{L^2(\Omega)}\,,$$

$$\sum_{k=0}^{m} \lambda_k^4 w_{km} \, (\varphi(u_m), \psi_k)_{L^2(\Omega)} = \big(\nabla[\Delta\varphi(u_m)](\cdot, t), \nabla[\Delta^2 u_m](\cdot, t)\big)_{L^2(\Omega)}$$

has been made (see (3.3.19)). Set $\nabla[\Delta\varphi(u_m)] \equiv \nabla[\Delta\varphi(u_m)](\cdot, t)$ for simplicity. Since

$$\begin{aligned} \nabla[\Delta\varphi(u_m)] &= \nabla[\varphi'(u_m)\Delta u_m + \varphi''(u_m)|\nabla u_m|^2] \\ &= \varphi'(u_m)\nabla\Delta u_m + \varphi'''(u_m)\nabla u_m |\nabla u_m|^2 \\ &\quad + \varphi''(u_m)\left(\nabla u_m + \sum_{j=1}^{N} \frac{\partial u_m}{\partial x_j}\frac{\partial \nabla u_m}{\partial x_j}\right)\,, \end{aligned}$$

using assumptions (A_1)-(A_3) and estimates (3.3.22), (3.3.23) and (3.3.29) plainly we have

$$\| |\nabla[\Delta\varphi(u_m)]| \|_{L^2(\Omega)} \leq \bar{C}_1 \left\{ \| |\nabla\Delta u_m| \|_{L^2(\Omega)} + \| |\nabla u_m| \|^3_{L^6(\Omega)} \right\} + \bar{C}_2 \tag{3.3.47}$$

with some $\bar{C}_1$, $\bar{C}_2 > 0$. To estimate the term $\||\nabla u_m|\|^3_{L^6(\Omega)}$ in the right-hand side of (3.3.47), we use the Nirenberg inequality (3.3.27) with $v = u_m$, $j = 1, m = 5, p = 6, = 2$. As for q, by estimate (3.3.29) and Sobolev embedding we can choose any $q \in (1, \infty)$ if $N \leq 3$, respectively $q \in (1, \frac{2N}{N-4})$ if $N = 4, 5$. If $N \leq 3$ equality (3.3.28) is satisfied with

$$a = a(q, N) := \frac{1}{3} \frac{(6-N)q + 6N}{(10-N)q + 2N} < \frac{1}{3},$$

respectively with $a = \frac{N-3}{9} \leq \frac{2}{9}$ if $N = 4, 5$. Then we obtain

$$\||\nabla u_m|\|_{L^6(\Omega)} \leq \tilde{K_1} \| |\nabla\Delta^2 u_m| \|^a_{L^2(\Omega)} \| |\nabla u_m| \|^{1-a}_{L^q(\Omega)} + K_2 \| |\nabla u_m| \|_{L^q(\Omega)} \tag{3.3.48}$$

for some $\tilde{K_1} > 0$. Then by estimate (3.3.29) there exist $P_1 > 0, P_2 > 0$ such that for every $t \in (0, T_m)$

$$\||\nabla u_m|(\cdot, t)\|^3_{L^6(\Omega)} \leq P_1 \| |\nabla\Delta^2 u_m| \|^{3a}_{L^2(\Omega)} + P_2 , \tag{3.3.49}$$

with $3a < 1$.

Similarly, the term $\| |\nabla\Delta u_m| \|_{L^2(\Omega)}$ in the right-hand side of (3.3.47) can be estimated using the Nirenberg inequality with $v = \Delta u_m$, $j = 1, m = 3, a = 1/3, p = q = r = 2$. This gives

$$\| |\nabla\Delta u_m| \|_{L^2(\Omega)} \leq \tilde{K_1} \| |\nabla\Delta^2 u_m| \|^{\frac{1}{3}}_{L^2(\Omega)} \|\Delta u_m\|^{\frac{2}{3}}_{L^2(\Omega)} + K_2 \|\Delta u_m\|_{L^2(\Omega)} \tag{3.3.50}$$

for some $\tilde{K_1} > 0$. Hence by estimate (3.3.29) there exist $Q_1 > 0, Q_2 > 0$ such that for every $t \in (0, T_m)$

$$\| |\nabla\Delta u_m(\cdot, t)| \|_{L^2(\Omega)} \leq Q_1 \| |\nabla\Delta^2 u_m(\cdot, t)| \|^{\frac{1}{3}}_{L^2(\Omega)} + Q_2 . \tag{3.3.51}$$

By the last inequality in (3.3.46) and inequalities (3.3.47), (3.3.49) and (3.3.51), it is easily seen that there exists $\tilde{M} > 0$ (depending on ϵ) such that

$$\frac{1}{2} \frac{d}{dt} \| |\nabla\Delta u_m(\cdot, t)| \|^2_{L^2(\Omega)} + \frac{\epsilon}{2} \| |\nabla\Delta^2 u_m(\cdot, t)| \|^2_{L^2(\Omega)} \leq \tilde{M}$$

for every $t \in (0, T_m)$, whence inequality (3.3.45) follows. This completes the proof. □

3.4 Asymptotic behaviour: Proofs

Let $u_0 \in H^2_{E^*}(\Omega)$. In this case the approximating sequence in (3.3.21) becomes

$$\begin{cases} u_{0m} := \sum_{j=1}^{m} \alpha_{jm}\psi_j, \quad u_{0m} \to u_0 \text{ in } H^2_E(\Omega)\,, \\ \|u_{0m}\|_{H^2(\Omega)} \le \|u_0\|_{H^2(\Omega)} \qquad (m \in \mathbf{N})\,, \end{cases} \tag{3.4.52}$$

thus

$$\int_\Omega u_{0m}\,dx = 0 \quad \text{for any } m \in \mathbf{N}\,.$$

Accordingly, in the proof of Theorem 3.2.2 now we have

$$u_m := \sum_{k=1}^{m} w_{km}\psi_k \tag{3.4.53}$$

with coefficients w_{km} $(k = 1, ..., m)$ determined by system (3.3.20). Hence there holds

$$\int_\Omega u_m(x,t)\,dx = \sum_{k=1}^{m} w_{km}(t) \int_\Omega \psi_k\,dx = 0 \tag{3.4.54}$$

for any $m \in \mathbf{N}$ and $t \in (0, T_m)$.

Since $u_m(\cdot, t) \in H^2_{E^*}(\Omega)$, it is easily seen that estimates analogous to those of Lemmata 3.3.1, 3.3.2 and 3.3.3 hold with constants independent of T. In fact, the following holds.

Lemma 3.4.1. *Let $u_0 \in H^2_{E^*}(\Omega)$, and let assumptions (A_1) and (A_4) be satisfied. Let u_m be defined by (3.4.53) with coefficients w_{km} satisfying system (3.3.20), (3.4.52) in the maximal interval $(0, T_m)$. Then there exists $C_1^* > 0$ (only depending on ϵ and the norm $\|u_0\|_{H^2(\Omega)}$) such that for every $t \in (0, T_m)$*

$$||u_m(\cdot, t)||_{H^1(\Omega)} \le C_1^*\,. \tag{3.4.55}$$

Lemma 3.4.2. *Let $u_0 \in H^4_{E^*}(\Omega)$, let assumptions (A_1)-(A_4) be satisfied, and let $N \le 5$. Let u_m be defined by (3.4.53) with coefficients w_{km} satisfying system (3.3.20), (3.4.52) in the maximal interval $(0, T_m)$. Then there exists $C_2^* > 0$ (only depending on ϵ and the norm $\|u_0\|_{H^4(\Omega)}$) such that for every $t \in (0, T_m)$*

$$||u_m(\cdot, t)||_{H^3(\Omega)} \le C_2^*\,. \tag{3.4.56}$$

The proof of Lemma 3.4.2 is similar to that of Lemmata 3.3.2 and 3.3.3, using estimate (3.4.55) instead of (3.3.22)-(3.3.23). We leave the details to the reader.

Proof of Lemma 3.4.1. Let F be the functional defined in (3.2.11). Using (3.3.20) plainly gives

$$\frac{d}{dt}\left[F(u_m)\right](t) = \int_\Omega [\varphi(u_m)(x,t) - \epsilon\Delta u_m(x,t)]u_{mt}(x,t)\,dx$$
$$= \int_\Omega v_m(x,t)u_{mt}(x,t)\,dx = -\sum_{k=1}^{m}\lambda_k|(v_m(\cdot,t),\psi_k)_{L^2(\Omega)}|^2 \le 0\,.$$

This yields $F(u_m)(t) \le F(u_m)(0)$, namely

$$\int_\Omega \left\{\Phi(u_m)(x,t) + \frac{\epsilon}{2}|\nabla u_m(x,t)|^2\right\}dx \le \int_\Omega \left\{\Phi(u_{0m}) + \frac{\epsilon}{2}|\nabla u_{0m}|^2\right\}dx$$
$$\le (M+\epsilon)\|u_0\|_{H^2(\Omega)}$$

with some constant $M > 0$, for any $t \in (0, T_m)$; here use of assumption (A_1) has been made. Since by assumption $(A5)$ there holds $\Phi(u) \ge 0$ for any $u \in \mathbb{R}$, from the above equality we get

$$||\,|\nabla u_m(\cdot,t)|\,||_{L^2(\Omega)} \le \frac{2(M+\epsilon)}{\epsilon}\|u_0\|_{H^2(\Omega)}\,,$$

whence (3.4.55) follows by Poincaré's inequality. This proves the result. □

Now we can prove Proposition 3.2.5.

Proof of Proposition 3.2.5. Let $\{u_k\}$ be the subsequence considered in the proof of Theorem 3.2.2. By estimate (3.3.45) and a diagonal argument, there exists a subsequence $\{u_l\} \equiv \{u_{k_l}\} \subseteq \{u_k\}$ such that

$$u_l \to u \text{ in } L^2((0,T); H^3(\Omega))\,.$$

Plainly, this gives

$$\int_0^T \left|\, ||u_l(\cdot,t)||_{H^3(\Omega)} - ||u(\cdot,t)||_{H^3(\Omega)}\right|\,dt$$
$$\le \int_0^T ||u_l(\cdot,t) - u(\cdot,t)||_{H^3(\Omega)}\,dt$$
$$\le \sqrt{T}\left(\int_0^T ||u_l(\cdot,t) - u(\cdot,t)||^2_{H^3(\Omega)}\,dt\right)^{\frac{1}{2}} \to 0$$

as $l \to \infty$. Hence there exists a subsequence, denoted again by $\{u_l\}$ for simplicity, such that

$$||u_l(\cdot,t)||_{H^3(\Omega)} \to ||u(\cdot,t)||_{H^3(\Omega)} \quad \text{for almost every } t \in (0,T)\,.$$

By inequality (3.4.56), this yields inequality (3.2.10) for any $t \in (0,T)$. Then by the arbitrariness of $T \in (0,\infty)$ the conclusion follows. □

Proof of Theorem 3.2.7. Multiplying the first equation of (P_∞) by u and integrating over Ω we get

$$\frac{1}{2}\frac{d}{dt}\| \, |\nabla u(\cdot,t)| \, \|^2_{L^2(\Omega)} + \epsilon \, \|\nabla \Delta u(\cdot,t)\|^2_{L^2(\Omega)} = -\int_\Omega \varphi'(u)\nabla u(x,t)\nabla\Delta u(x,t)dx$$

$$\leq \frac{\epsilon}{2}\|\nabla\Delta u(\cdot,t)\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon}\int_\Omega (\varphi'(u))^2|\nabla u(x,t)|^2 dx\,, \qquad (3.4.57)$$

whence by assumption (A_1)

$$\frac{d}{dt}\| \, |\nabla u(\cdot,t)| \, \|^2_{L^2(\Omega)} + \epsilon \, \|\nabla \Delta u(\cdot,t)\|^2_{L^2(\Omega)} \leq \frac{C^2}{\epsilon}\| \, |\nabla u(\cdot,t)| \, \|^2_{L^2(\Omega)}$$

for some $C > 0$. Since $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, by Poincaré's inequality there holds

$$\begin{aligned}
\| \, |\nabla u(\cdot,t)| \, \|^2_{L^2(\Omega)} &\leq C_0 \|\Delta u(\cdot,t)\|^2_{L^2(\Omega)} \\
&= -C_0 \int_\Omega \nabla u(x,t)\cdot\nabla\Delta u(x,t)dx \\
&\leq \frac{1}{2}\| \, |\nabla u(\cdot,t)| \, \|^2_{L^2(\Omega)} + \frac{C_0^2}{2}\| \, |\nabla \Delta u(\cdot,t)| \, \|^2_{L^2(\Omega)}\,,
\end{aligned}$$

namely

$$\| \, |\nabla u(\cdot,t)| \, \|^2_{L^2(\Omega)} \leq C_0^2 \| \, |\nabla \Delta u(\cdot,t)| \, \|^2_{L^2(\Omega)} \qquad (3.4.58)$$

for some $C_0 > 0$. By inequalities (3.4.57)-(3.4.58) we have

$$\frac{d}{dt}\| \, |\nabla u(\cdot,t)| \, \|^2_{L^2(\Omega)} + \left(\frac{\epsilon}{C_0^2} - \frac{C^2}{\epsilon}\right)\| \, |\nabla u(\cdot,t)| \, \|^2_{L^2(\Omega)} \leq 0\,,$$

whence

$$\lim_{t\to\infty} \| \, |\nabla u(\cdot,t)| \, \|_{L^2(\Omega)} = 0$$

if $\epsilon > \epsilon_0 := C_0 C$. Since $u(\cdot,t) \in H^2_{E^*}(\Omega)$ for any $t \in (0,\infty)$, by Poincaré's inequality the above equality implies

$$\lim_{t\to\infty} \|u(\cdot,t)\|_{L^2(\Omega)} = 0\,.$$

Then the conclusion follows. □

3.5 Stationary problem in one space dimension

In this section we address existence and multiplicity of solutions of the problem

$$\begin{cases} [\varphi(u) - \epsilon u'']'' = 0 \text{ in } (-L, L) \\ u'(-L) = u'(L) = u'''(-L) = u'''(L) = 0 \\ \dfrac{1}{2M}\displaystyle\int_\Omega u(x)dx = L\,, \end{cases} \qquad (3.5.59)$$

where the primes denote differentiation and M is the mass defined by (3.1.4). Integrating twice the first equation of (3.5.59) we obtain (in agreement with Remark 3.2.1) the equivalent second order problem

$$(E)\qquad \begin{cases} \epsilon u'' = \varphi(u) - \sigma \text{ in } (-L, L) \\ u'(-L) = u'(L) = 0 \\ \dfrac{1}{2M} \displaystyle\int_\Omega u(x)dx = L\,, \end{cases}$$

where $\sigma \in \mathbb{R}$ is a constant to be chosen. The above problem with $\varphi(u) = u^3 - u$ (and $\epsilon = 1$) was investigated in [Z, NPe]. As motivated in the Introduction, we are interested in obtaining similar results choosing

$$\varphi(u) = \frac{u}{1+u^2}\,. \tag{3.5.60}$$

Specifically, we address the existence of *simple* solutions, *i.e.*, solutions which are strictly monotone and bounded.

Let $u = u(x)$ be a solution of problem (E). Multiplying the first equation of problem (E) by u' and integrating we obtain

$$\frac{\epsilon}{2}\,[u'(x)]^2 = \mathcal{W}(u(x), \sigma) - b\,, \tag{3.5.61}$$

where

$$\mathcal{W}(u, \sigma) := \frac{\log(1+u^2)}{2} - \sigma u$$

and b is another constant of integration.

Due to the boundary conditions, we must have

$$\mathcal{W}(u(\pm L), \sigma) - b = 0\,. \tag{3.5.62}$$

If u is a simple solution, there holds $u(-L) \neq u(L)$. Therefore, the equation $\mathcal{W}(u, \sigma) - b = 0$ must have at least two roots $u_1 < u_2$ such that $\mathcal{W}(u, \sigma) - b > 0$ for $u_1 < u < u_2$ (see (3.5.61)). If φ is of the form (3.5.60) and $\sigma > 0$, it is easily seen that this implies

$$\mathcal{W}(\alpha, \sigma) < b < \mathcal{W}(\beta, \sigma)\,,$$

where α and β are local extremum points of $\mathcal{W}$ - namely, solve the equation

$$\mathcal{W}_u(u, \sigma) = u - \sigma(1+u^2) = 0\,.$$

Clearly, the above situation gives the condition $\sigma < \frac{1}{2}$; if this is the case, there holds

$$\alpha \equiv \alpha(\sigma) = \frac{1-\sqrt{1-4\sigma^2}}{2\sigma}\,, \quad \beta \equiv \beta(\sigma) = \frac{1+\sqrt{1-4\sigma^2}}{2\sigma}\,.$$

Similar considerations hold for $\sigma < 0$, yielding the condition $\sigma > -\frac{1}{2}$. We conclude that the above strategy to determine simple solutions of problem (E) requires that the parameters (σ, b) belong to the following *admissible region* (see Figure 3.1)

$$\Sigma := \left\{ (\sigma, b) \,\middle|\, \sigma \in \left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right), \; \mathcal{W}(\alpha, \sigma) < b < \mathcal{W}(\beta, \sigma) \right\} . \quad (3.5.63)$$

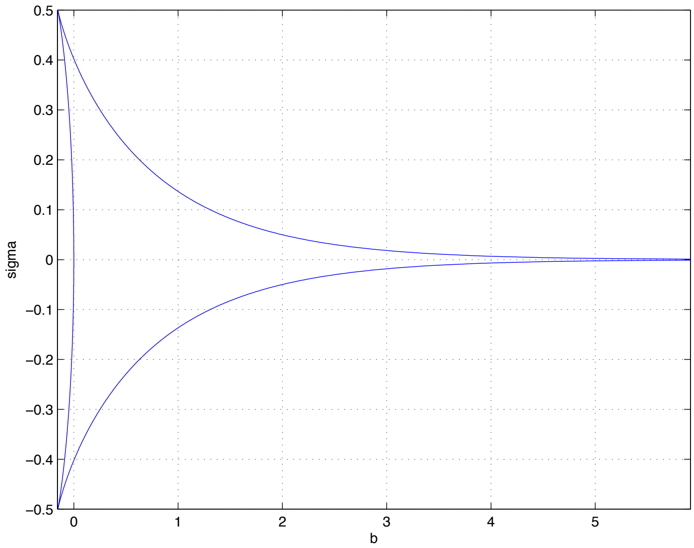

Figure 3.1: Admissible region.

Now let $(\sigma, b) \in \Sigma$, and let u be a simple solution of problem (E); suppose $u' > 0$ without loss of generality. Then equality (3.5.61) gives

$$u'(x) = \sqrt{\frac{2}{\epsilon}} \sqrt{\mathcal{W}(u(x), \sigma) - b} \,,$$

whence by integration

$$2L = \int_{-L}^{L} dx = \sqrt{\frac{\epsilon}{2}} \int_{u_1(\sigma,b)}^{u_2(\sigma,b)} \frac{ds}{\sqrt{\mathcal{W}(s, \sigma) - b}} =: 2\mathcal{L}(\sigma, b) \,. \quad (3.5.64)$$

(see (3.5.62)). By the same token, the third equation of problem (E) gives

$$2LM = \int_{-L}^{L} u(x) dx = \sqrt{\frac{\epsilon}{2}} \int_{u_1(\sigma,b)}^{u_2(\sigma,b)} \frac{s ds}{\sqrt{\mathcal{W}(s, \sigma) - b}} =: 2\mathcal{M}(\sigma, b). \quad (3.5.65)$$

Therefore, proving the existence of a simple solution of problem (E) amounts to finding a pair $(\sigma, b) \in \Sigma$ such that

$$\begin{cases} \mathcal{L}(\sigma, b) = L \\ \mathcal{M}(\sigma, b) = LM \,. \end{cases} \quad (3.5.66)$$

Investigating existence and multiplicity of solutions of system (3.5.66) requires information about the monotonicity properties of the functions $\mathcal{L}$ and $\mathcal{M}$. Unfortunately, at variance from the cases of φ of polynomial type dealt with in [CGS, NPe, Z], for the present choice of φ a complete analytical investigation of this point could not be carried out (similar difficulties are encountered if $\varphi(u) = u \exp(-u)$; see (3.1.1)). Therefore, the investigation has been pursued by a numerical method, whose main steps are described as follows in the case $\sigma \in (0, \frac{1}{2})$ (similarly it is possible to obtain the case with $\sigma \in (-\frac{1}{2}, 0)$):

Step 1 We fix the step size $\Delta\sigma = 0.001$ and $\Delta b = 0.005$ and we construct the vector $\sigma \in (0, \frac{1}{2})$ with grid step $\Delta\sigma$. In this way we have 499 nodes for σ.

Step 2 For each σ we calculate $\mathcal{W}(\alpha, \sigma)$ and $\mathcal{W}(\beta, \sigma)$ in order to construct the matrix of the values for b.

Step 3 For each value of σ and each value of b inside the admissible region visible in Fig. 3.1, we calculate the two roots $u_1(\sigma, b)$ and $u_2(\sigma, b)$ by using the bisection method.

Step 4 By a recursive adaptive Simpson quadrature method, we numerically evaluate the integral defined between the two roots calculated in the previous step in order to calculate the functions $\mathcal{L}(\sigma, b)$ and $\mathcal{M}(\sigma, b)$ defined in (3.5.64) and (3.5.65), respectively.

Step 5 We calculate the partial derivative of the functions $\mathcal{L}(\sigma, b)$ and $\mathcal{M}(\sigma, b)$ with respect to σ and b by replacing the derivatives with their incremental ratios, that is by using the centered finite differences method.

We can arrive to our conclusion by using Figures 3.2, 3.3 and 3.4. In order to obtain a more clear vision, just some level curves are represented. The computations were done on a computer Mac OS X version 10.6.8 with processor 2.66 GHz Intel Core 2 Duo, RAM 4 GB.

On the strength of numerical evidence, the following conclusions can be drawn.

Statement 1 (Monotonicity properties of $\mathcal{L}$ and $\mathcal{M}$). *For any $(\sigma, b) \in \Sigma$ there holds*

$$\frac{\partial \mathcal{L}}{\partial b}, \frac{\partial \mathcal{L}}{\partial \sigma}, \frac{\partial \mathcal{M}}{\partial b}, \frac{\partial \mathcal{M}}{\partial \sigma} \leq 0.$$

Consider the level curves

$$C_{\mathcal{L}} := \{(\sigma, b) \in \Sigma | \mathcal{L}(\sigma, b) = L\}, \quad D_{\mathcal{M}} := \{(\sigma, b) \in \Sigma | \mathcal{M}(\sigma, b) = LM\}.$$

Statement 2 (Monotonicity properties of the curves $C_{\mathcal{L}}$, $D_{\mathcal{M}}$). *For any $L > 0$, both curves $C_{\mathcal{L}}$ and $D_{\mathcal{M}}$ are decreasing in the admissible region Σ.*

Statement 3 (Intersections of the curves $C_{\mathcal{L}}$, $D_{\mathcal{M}}$) *For any $L > 0$ large enough, there exist some $M > 0$ such that problem (E) has a simple solution.*

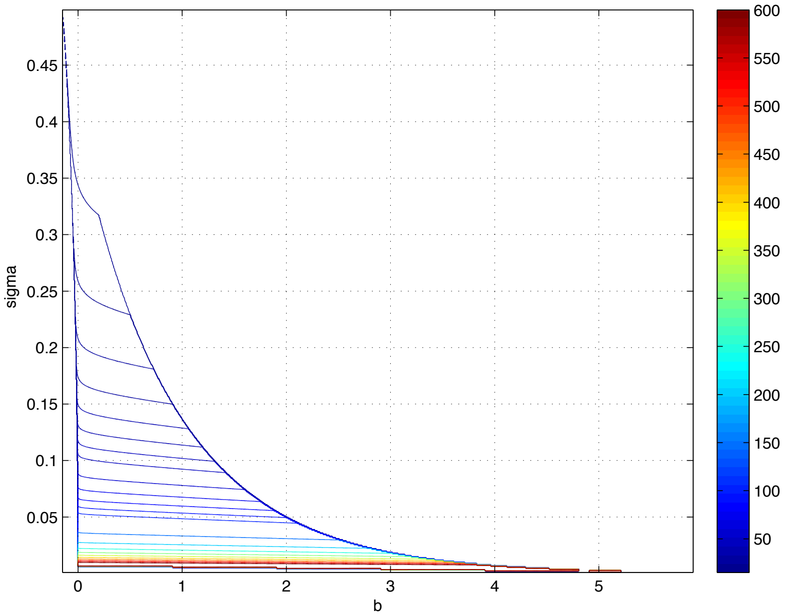

Figure 3.2: Level curves of $\mathcal{L}(\sigma, b)$.

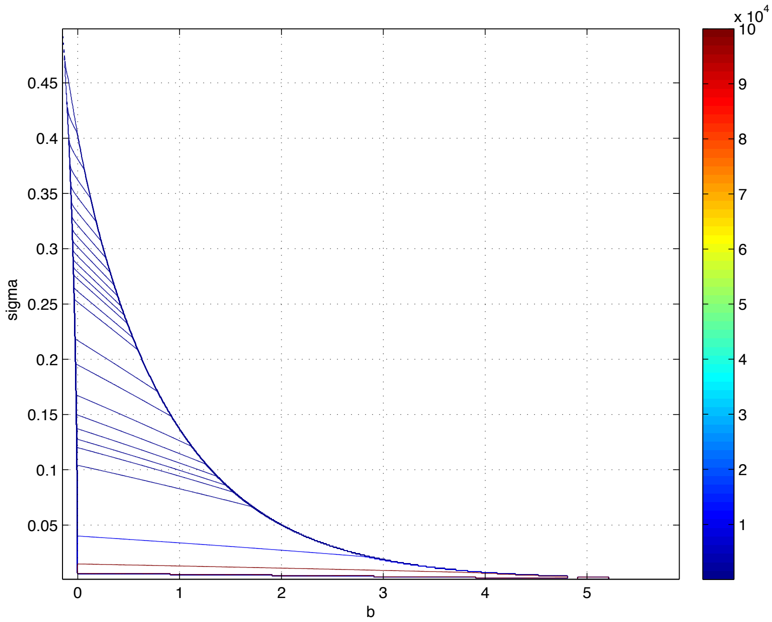

Figure 3.3: Level curves of $\mathcal{M}(\sigma, b)$.

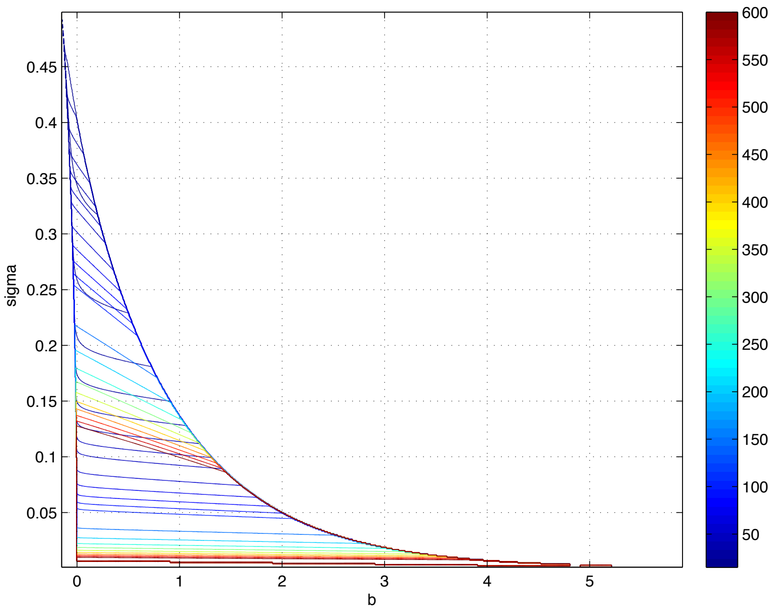

Figure 3.4: Level curves of $\mathcal{L}(\sigma, b)$ and $\mathcal{M}(\sigma, b)$.

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