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Dipartimento di Matematica “G. Castelnuovo”

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Structure and Invertibility in Cellular Automata

Silvio Capobianco

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Chapter 1

Introduction

Cellular automata were conceived by John von Neumann in the 1950s as models for self-reproduction.
The basilar idea was the following: A regular mesh (the "support space") hosts an array of identical devices, which evolve at discrete time steps by changing their internal state (by a "local evolution function") according to those of a finite number of other ones (the "neighborhood") whose relative displacement is the same for all points. This induces a transformation (the "global evolution function") of the set of the "snapshots" of the whole mesh, taken at different times (the "configurations"). Interest was focused on finite portions of configurations (the "patterns"): considerations were made about the possibility that a pattern, initially present in a given region, could later spawn again, possibly in another region. With time, other uses for cellular automata emerged: from simulation of population dynamics, to implementation of general-purpose computing machines: the latter followed from the observation of a "natural" way to simulate in real-time an arbitrary Turing machine (which, according to *Church's Thesis*, is capable of universal computation) with a one-dimensional cellular automaton.
The fundamental concept underlying cellular automata dynamics is *locality*: that is, the future state of a single point of the space can be exactly computed from the current states of all points of the space within a certain distance from the first one. Since this is the observed behavior of most physical phenomena, the use of cellular automata as models of physical systems was immediate; indeed, the first attempts to simulate the evolution of a set of gas particles by means of a device similar to a cellular automaton (named "lattice gas") dates at least from mid-1970s. However, an emerging prob-

lem appeared: physical laws are not only local, but also *invertible* (at least on the microscopic scale); and any representation of a dynamical system by means of a local law, necessarily conceals some of its *global* properties: *this is exactly what happens to invertibility itself,* as well as to other properties that depend on it, like conservation laws. In other words: invertibility is *not* immediately evident from the standard "shape" of a cellular automaton.

Actually, the study of invertibility in cellular automata can be tracked down to the 1972 papers by Richardson [24] and Amoroso and Patt [1]. The first one expanded previous papers by Moore [21] and Myhill [22] that related the existence of *Garden-of-Eden* configurations (that can only be initial, but cannot be reached) to that of *mutually erasable* patterns (finite subconfigurations that "are forgotten" in the evolution): Richardson gave a wider portrait of the argument by proving that any *injective* cellular automaton is invertible, and that the inverse of a cellular automaton is a cellular automaton. On the other hand, Amoroso and Patt developed an algorithm to decide invertibility for one-dimensional cellular automata; later, Patt applied the technique to determine which one-dimensional cellular automata with "small" range were both invertible and nontrivial, indeed discovering that they were rare.

Since then, studies on invertible cellular automata flourished. However, these objects were considered somewhat exotic as well as probably uncapable of universal computation, until Toffoli [30] proved that invertible cellular automata are both computation- and construction-universal. On the same line, Fredkin developed a universal computation model (the *Billiard Ball Model*) that is immediately realizable as an invertible cellular automaton.

With time, new ideas emerged in cellular automata theory. Studying invertibility of large systems as a consequence of invertibility of their components, Fredkin and Toffoli [10] devised two classes of universal invertible logical ports; and actually, the search for complete sets of invertible primitives is an important field of research (see [16], [17], or also [31]). On the lattice gas front, aiming to overcome the limitations of a prevoius model, Frisch, Hasslacher and Pomeau [11] introduced a lattice gas model based on an hexagonal grid rather than a square one: this was perhaps the first major extension of the usual definition, which only considered hypercubic grids. Other specialized kinds of cellular automata appeared, such as second-order ones (see [2] for an example). The question about decidability of cellular automata remained open until 1990, when Kari [18] proved its undecidability in dimension 2, which implies undecidability in greater dimensions. Kari's result showed that the existence of a general-purpose computer program capable to decide

if an arbitrary cellular automaton is invertible, *only looking at its standard-form description,* is strictly linked to the geometry of the support space. The fundamental work of Toffoli and Margolus [32] is a compendium of everything that had been said about invertibility and cellular automata, and asks new questions about that, thus representing a true milestone in the field.
At the same time, people started to consider broader contexts where locality could still be defined, namely, Cayley graphs of finitely generated groups: this class includes both hypercubic and hexagonal meshes. Also, in analogy with symbolic dynamics (see [19] for a complete treaty on this subject), works like [14] appeared, studying the effects of one-dimensional cellular automata over one-dimensional *shift subspaces*, that is, closed, translation-invariant sets of configurations over $\mathbb{Z}$. The work of Fiorenzi [8] suggested that a larger class of dynamical systems, whose configuration space was not "full", but whose evolution function still changed the value in a point only according to the values of the points in a finite neighborhood, were still worth to be called cellular automata: her thesis work deals with extensions to these broader families of known results about the links between injectivity and invertibility [24], and between the existence of Garden-of-Eden configurations and mutually erasable patterns [21, 22]. Also very interesting are the works of Formenti [9] about quotient topologies and Kolmogorov complexity for one-dimensional cellular automata.

The present work started as an attempt to study the invertibility problem for cellular automata in contexts different (and possibly broader) than the most used one, both by considerations on local structure and with the tools described by [8]. When the search for an extension of Amoroso and Patt's theorem led to an isomorphism result for configuration spaces, more focus was put on the conditions that a dynamical system must satisfy to have a presentation as a cellular automaton, and on the possibility of employing the modern formalism also to describe specialized objects such as lattice gas and second-order cellular automata.

This work is divided as follows. In Chapter 2 we introduce the tools we are going to use: dynamical systems, finitely generated groups, configuration spaces, and uniformly locally definable functions. In Chapter 3 we discuss cellular automata according to recent formalizations, examine its advantages and disadvantages with respect to the original one, show a characterization result for dynamical systems that can be presented as cellular automata, and explain the invertibility problem. In Chapter 4 we deal with the special subfamily of lattice gases, showing that, with the modern formulation, they are presentations of the same dynamical systems as cellular automata. In Chapter 5 we deal with the problem of transferring part of the complexity of a cellular automaton from its group to its alphabet: this is done when the group is a semi-direct product of a finitely generated group and a finite one, extending the subject of [3] to a more general case; the application of this technique yields a nontrivial example of a class of cellular automata where the invertibility problem is decidable. In Chapter 6 we introduce second-order dynamics and show an interesting similarity between lattice gases and second-order cellular automata, which occurs at *structural* (rather than functional) level.

Chapter 2

Fundamental notions

This section is meant to prepare the ground for our constructions. It can either be read before entering the actual thesis, or be skipped now and referred later.

2.1 Semigroups and actions

A *semigroup* is a set with an associative binary operation; a semigroup having an identity element is called a *monoid*. A group is a monoid where for all s there exists the inverse s^{-1}.
The binary operation of a semigroup can be extended to its subsets *in the Frobenius sense*: if $H, K \subseteq S$, then:

$$HK = \{s \in S : \exists\, h \in H, k \in K : s = hk\}$$

If $H = \{h\}$ is a singleton, then hK is a shortcut for $\{h\}K$: similarly if $K = \{k\}$.
If S is a group, another operation on subsets is given by:

$$U^{-1} = \{s \in S : \exists u \in U : s = u^{-1}\}$$

If $U = U^{-1}$ we say that U is *symmetric*.

Definition 2.1.1 *Let X be a set. Let S be a semigroup. A* right action of S over X *is a map $\varphi : X \times S \to X$ such that:*

1. $\varphi(\varphi(x, s_1), s_2) = \varphi(x, s_1 s_2)$ *for all $x \in X$, $s_1, s_2 \in S$;*

2. *if S is a monoid with identity 1_S, then $\varphi(x, 1_S) = x$ for all $x \in X$.*

When φ is clear from the context, the element $\varphi(x, s)$ is indicated by with x^s.
A right action $\varphi : X \times S \to X$ induces a family of transformations $\{\varphi_s\}_{s\in S} \subseteq X^X$ defined by $\varphi_s(x) = \varphi(x, s)$: this family has the property that $\varphi_{s_1 s_2} = \varphi_{s_2} \circ \varphi_{s_1}$, that is, the semigroup operation *between maps* is analogous to the semigroup operation *between indices* (we are using the convention that the product fg is the composition $g \circ f$); moreover, if S is a monoid, then $\varphi_{1_S} = \mathrm{id}_X$. On the other hand, if $\{T_s\}_{s\in S} \subseteq X^X$ satisfies $T_{s_1 s_2} = T_{s_2} \circ T_{s_1}$ for all $s_1, s_2 \in S$ (and $T_{1_S} = \mathrm{id}_X$ if S is a monoid), then $\varphi : X \times S \to X$ defined by $\varphi(x, s) = T_s(x)$ is a right action of S over X.
Left actions can be defined in a similar way as maps $\psi : S \times X \to X$ such that $\psi(s_1, \psi(s_2, x)) = \psi(s_1 s_2, x)$ for all $x \in X$, $s_1, s_2 \in S$. If the semigroup is commutative, then there is a one-to-one correspondance between left and right actions obtained by putting $\psi(s, x) = \varphi(x, s)$; if the semigroup is a group, then there is a one-to-one correspondance between left and right actions obtained by putting $\psi(s, x) = \varphi(x, s^{-1})$.
Since a right action is a collection of transformations, the definitions given for transformations can be extended to actions by saying that the action has property P if and only if all the induced maps have property P. We do this in some important cases.

Definition 2.1.2 *Let X be a set. Let S be a semigroup. Let φ be a right action of S over X. We say that $Y \subseteq X$ is* invariant under φ *if $\varphi_s(Y) \subseteq Y$ for all $s \in S$.*

Definition 2.1.3 *Let X be a set. Let S be a semigroup. Let φ be a right action of S over X. A map $F : X \to X$* commutes with φ *if for every $s \in S$, $x \in X$ we have $\varphi(F(x), s) = F(\varphi(x, s))$.*

Definition 2.1.4 *Let X be a topological space. Let S be a semigroup. A right action φ of S over X is* continuous *if φ_s is continuous for all $s \in S$.*

2.2 Dynamical systems

A dynamics can be seen as the action of a semigroup on a space. Some dynamics show enough regularity to be presented as follows.

Definition 2.2.1 *A* discrete dynamical system *is a pair* (X, F) *where* X *is a compact metrizable space and* $F : X \to X$ *is a continuous function.*
The set X *is called the* phase space *of the dynamical system. The map* F *is called the* evolution *or* transition function *of the dynamical system.*

The word "discrete" in Definition 2.2.1 refers to *time*, whose flow is represented by repeated applications of the evolution function; and actually, a dynamical system with phase space X can be seen as a continuous right action over X of the monoid $\mathbb{N}$, defined by $\varphi(x, n) = F^n(x)$. On the other hand, if $\varphi : X \times \mathbb{N} \to X$ is a right action, then by putting $F = \varphi(\cdot, 1)$ one obtains $\varphi(\cdot, n)$ as F^n.
As an example, put $X = [0, 1]$ with the Euclidean topology and $F(x) = x^2$: then (X, F) is a dynamical system.
In general, for every metrizable space X, there are many distances inducing the topology of X: we fix one of these distances d_X and always refer to it.

Definition 2.2.2 *Let* (X, F) *be a dynamical system. A* dynamical subsystem *of* (X, F) *is a pair* (Y, F) *where* $Y \subseteq X$ *is topologically closed and satisfies* $F(Y) \subseteq Y$*.*

Since X is both metrizable and compact, a subset of X is compact iff it is closed, so in Definition 2.2.2 (Y, F) is itself a dynamical system.

Definition 2.2.3 *Let* (X, F) *and* (X', F') *be dynamical systems. A* morphism *from* (X, F) *to* (X', F') *is a continuous map* $\vartheta : X \to X'$ *such that* $\vartheta \circ F = F' \circ \vartheta$*.*
An injective morphism is called an embedding*. A surjective morphism is called a* factor map*. A bijective morphism is called a* conjugacy*. Two dynamical systems are* conjugate *if there exist a conjugacy between them.*

As an example, let X be obtained from the interval $[0, 1]$ by identifying 0 and 1, and let $F(x) = 2x \bmod 1$; let X' be the unit circle in the complex plane and let $F'(\alpha) = \alpha^2$. Then $\vartheta : X \to X'$ given by $\vartheta(x) = e^{2\pi i x}$ is a conjugacy from (X, F) to (X', F').
A result in General Topology says that, if X and Y are topological spaces with X compact and Y metrizable, then $\varphi(K)$ is closed in Y for every closed $K \subseteq X$ and continuous $\varphi : X \to Y$; in particular, if φ is continuous and bijective, then φ^{-1} is continuous too, and φ is a *homeomorphism*. This allows to define invertible dynamical systems.

Definition 2.2.4 *A dynamical system* (X, F) *is* invertible *if* F *is bijective. The dynamical system* (X, F^{-1}) *is called the* inverse *of* (X, F).

An invertible dynamical system with phase space X can be seen as a continuous right action on X of the group $\mathbb{Z}$, defined by $\varphi(x, k) = F^k(x)$.

2.3 Configuration spaces

Definition 2.3.1 *An* alphabet *is a finite set with at least two elements.*

Alphabets are always seen as discrete topological spaces.

Definition 2.3.2 *Let* G *be a group. Let* $S \subseteq G$. *The* subgroup of G generated by S *is the set:*

$$\langle S \rangle = \{g \in G : \exists n \in \mathbb{N} : \exists \{s_1, \ldots, s_n\} \subseteq S \cup S^{-1} : g = s_1 \ldots s_n\}$$

A set of generators for G *is a subset* S *of* G *such that* $\langle S \rangle = G$. *The group* G *is* finitely generated *(briefly f.g.) if it has a finite set of generators.*

The group $G = \mathbb{Z}^2$ is finitely generated, $S = \{\mathbf{e}_1.\mathbf{e}_2\}$ being a finite set of generators for G. On the other hand, the group $(\mathbb{Q}, +)$ is not finitely generated, because if $a_1, b_1, \ldots, a_n, b_n \in \mathbb{Z} \setminus \{0\}$, $GCD(a_i, b_i) = 1$ for all $i \in \{1, \ldots, n\}$, and $p > |b_1 \ldots b_n|$ is prime, then $\frac{1}{p} \notin \left\langle \frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n} \right\rangle$.
Every f.g. group G has a finite symmetric set of generators that does not contain 1_G: from now on, unless stated otherwise, only sets of generators of this kind will be considered.

Definition 2.3.3 *Let* G *be a f.g. group. Let* S *be a finite set of generators for* G. *The* Cayley graph of G w.r.t. S *is the graph* $\mathrm{Cay}(G, S)$ *whose nodes are the elements of* G *and whose set of arcs is* $\mathcal{E} = \{(g, gs), g \in G, s \in S\}$.

As an example, if $G = \mathbb{Z}^2$ and $S = \{\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_1, -\mathbf{e}_2\}$, then $\mathrm{Cay}(G, S)$ is the square grid on the plane.

Definition 2.3.4 *Let* G *be a f.g. group. Let* S *be a finite set of generators for* G. *The* length of $g \in G$ with respect to S *is the smallest number* n *such that* g *is the product of* n *elements of* S, *and is indicated by* $\|g\|_S^G$.

It is not difficult to prove that $\|\cdot\|_S^G$ is a metric, whose value is strongly dependent from the set of generators S.
As an example, put $G = \mathbb{Z}^2$, $g = (-3, 5)$: if:

$$S = \{\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_1, -\mathbf{e}_2\}$$

and:

$$S' = S \cup \{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_1 - \mathbf{e}_2, -\mathbf{e}_1 + \mathbf{e}_2\}$$

then $\|g\|_S^G = 8$ and $\|g\|_{S'}^G = 5$.
Since $\|\cdot\|_S^G$ is a metric, the quantity:

$$d_S^G(g, h) = \|g^{-1}h\|_S^G \tag{2.1}$$

defines a distance, called the *distance between g and h w.r.t. S*.
Observe that $\|g^{-1}h\|_S^G$ is the length of a shortest path from g to h in $\mathrm{Cay}(G, S)$; the *disk of center g and radius r w.r.t. S* is the set:

$$D_{r,S}^G(g) = \{h \in G : d_S^G(g, h) \leq r\}$$

We write $D_{r,S}^G$ for $D_{r,S}^G(1_G)$. Observe that $D_{n,S}^G(g) = gD_{n,S}^G$, and that d_S^G induces the discrete topology over G.
Other quantities are the *diameter of U w.r.t. S*:

$$\mathrm{diam}\, U = \sup\{d_S^G(u_1, u_2), u_1, u_2 \in U\}$$

and the *range of U w.r.t. S*:

$$r_S^G(U) = \sup\{\|g\|_S^G, g \in U\} = \inf\{n \in \mathbb{N} : U \subseteq D_{n,S}^G\}$$

If X is a set and A is an alphabet, a *configuration of A over X* is a function $c : X \to A$, and c_x indicates the value of $c \in A^X$ at the point $x \in X$.

Theorem 2.3.5 *Let A be an alphabet. Let G be a f.g. group. If G is infinite, then A^G is a Cantor space.*

Proof:
G is infinite but finitely generated, hence it is countable: since A has at least two distinct elements, A^G has the power of continuum. But A^G is a topological product of compact and totally disconnected spaces, hence is itself compact and totally disconnected.

Let S be a finite set of generators for G. Define the *distance of* c_1 *and* c_2 *w.r.t.* S as:

$$d_S(c_1, c_2) = 2^{-\inf\left\{n \geq 0 \colon \exists g \in D^G_{n,S} \colon (c_1)_g \neq (c_2)_g\right\}} \tag{2.2}$$

with the conventions $\inf \emptyset = \infty$, $2^{-\infty} = 0$. It is easy to prove that d_S is a distance. To prove that d_S induces the product topology, observe first that the projections are continuous w.r.t. d_S: actually, if $\|g\|^G_S = n$ and $d_G(c_1, c_2) < 2^{-n}$, then $(c_1)_g = (c_2)_g$. Furthermore, any topology where the projections are continuous must contain all the sets of A^G obtained by fixing $c \in A^G$ and a finite $E \subseteq G$ and taking all the configurations that coincide with c over E: in particular, the open disks of d_S. Therefore the topology induced by d_S is the coarser between those where the projections are continuous, that is, it is the product topology: hence A^G is metrizable.
To prove that A^G is a Cantor space one only needs to prove that every configuration is an accumulation point. Fix $c \in A^G$; for every $n \in \mathbb{N}$, let c_n be a configuration that coincides with c in every point, except one single point g_n such that $\|g_n\|^G_S = n$. Then $d_S(c, c_n) = 2^{-n}$, $\lim_{n\to\infty} c_n = c$, and $\{c_n\}$ is not ultimately constant. □

Definition 2.3.6 *Let A be an alphabet. Let G be a finitely generated group. The* natural right action of G over A^G *is the action* $\sigma_G : A^G \times G \to A^G$ *defined by:*

$$(\sigma_G(c, g))_i = c_{gi} \quad \forall g, i \in G \; \forall c \in A^G \tag{2.3}$$

For $g \in G$, the induced map $\sigma_g : X \to X$ defined by $\sigma_g(c) = \sigma_G(c, g)$ is called the shift map in the direction g.

If $A = \{0, 1\}$ and $G = \mathbb{Z}$, then σ_1 is simply called the *shift map.*
The map defined by (2.3) is actually a right action: for every $g, h, i \in G$, $c \in A^G$:

$$((c^g)^h)_i = (c^g)_{hi} = c_{ghi} = (c^{gh})_i$$

Proposition 2.3.7 *Let A be an alphabet. Let G be a finitely generated group. The natural right action of G over A^G is continuous.*

Proof:
Let $\varepsilon > 0$. Let $n \in \mathbb{N}$ be such that $2^{-n} < \varepsilon$. Fix a finite set of generators S for G. Fix $g \in G$. Choose $c_1, c_2 \in A^G$ such that $d_S(c_1, c_2) < 2^{-(n+\|g\|^G_S)}$. If $i \in D^G_{n,S}$, then surely $gi \in D^G_{n+\|g\|^G_S, S}$, therefore:

$$((c_1)^g)_i = (c_1)_{gi} = (c_2)_{gi} = ((c_2)^g)_i$$

hence $d_S((c_1)^g, ((c_2)^g)) < 2^{-n} < \varepsilon$. $\square$

2.4 Shift subspaces and patterns

Symbolic dynamics studies the properties of the subsystems of the *shift dynamical system* $(A^{\mathbb{Z}}, \sigma_1)$: the phase spaces of these subsystems are called *shift subspaces*. The definition extends to more general configuration spaces, as is done in [8].

Definition 2.4.1 *Let A be an alphabet. Let G be a f.g. group. A* shift subspace *of A^G is a closed subspace $X \subseteq A^G$ invariant under the natural right action of G over A^G.*

If $X \subseteq A^G$ is a shift subspace, then the restriction to X of the natural action of G over A^G is a (continuous) right action of G over X: we keep on calling it the *natural* right action.
An example of a shift subspace is the set $X \subseteq A^G$ of those configurations that assume the value a at most at one point: in fact, X is invariant under the natural action of G, and is topologically closed because, if $c_i = c_j = 1$ with $i \neq j$, given a set of generators S for X and $n \in \mathbb{N}$ such that $i, j \in D^G_{n,S}$, no configuration that coincides with c on $D^G_{n,S}$ can be inside X.
Observe that not all subsets are shift subspaces, nor any of the two conditions implies the other one. To see this, fix $a \in A$: if G is nontrivial, then the set $X' = \{c \in A^G : c_{1_G} = a\}$ is closed but not translation invariant; and if G is infinite, then the set X'' of those $c \in A^G$ such that $c_g \neq a$ for at most finitely many $g \in G$ is translation invariant but not closed.

Definition 2.4.2 *Let A be an alphabet. Let G be a f.g. group. Let $E \subseteq G$ be finite. A* pattern over A^G with support E *is a function $p : E \to A$.*
A pattern p with support E occurs in $c \in A^G$ *if there exists $g \in G$ such that $(c^g)_i = p_i$ for all $i \in E$. A pattern p is* forbidden for $X \subseteq A^G$ *if it does not occur in any of the elements of X.*

The support of a pattern p can be indicated by $\operatorname{supp} p$.
Let $\mathcal{F}$ be a set of patterns over A^G: the set of all the configurations $c \in A^G$ such that no $p \in \mathcal{F}$ occurs in c is indicated by $\mathsf{X}_{\mathcal{F}}$.

Proposition 2.4.3 *Let A be an alphabet. Let G be a f.g. group. Let $X \subseteq A^G$. The following are equivalent:*

1. *X is a shift subspace;*

2. *(X, σ_g) is a dynamical subsystem of (A^G, σ_g) for every $g \in G$;*

3. *(X, σ_s) is a dynamical subsystem of (A^G, σ_s) for every $s \in S \cup S^{-1}$, where S is a finite set of generators for G;*

4. *there exists a set $\mathcal{F}$ of patterns over A^G such that $X = \mathsf{X}_{\mathcal{F}}$.*

Proof:
($1 \Leftrightarrow 2 \Leftrightarrow 3$) Since every σ_g is a continuous function, point 1 is equivalent to point 2. Moreover, if $g = s_1 \dots s_n$ then $\sigma_g = \sigma_{s_n} \circ \dots \circ \sigma_{s_1}$, so that point 2 is actually equivalent to point 3.
($4 \Rightarrow 1$) A set of the form $\mathsf{X}_{\mathcal{F}}$ is invariant under the natural action of G over A^G. Moreover, if $\{c_n\}$ converges to c and $p \in \mathcal{F}$ occurs in c, take $g \in G$ such that c^g coincides with p over E: since $\{(c_n)^g\}$ converges to c^g, for all n large enough $(c_n)^g$ must coincide with p over E, so that $\{c_n\}$ cannot be entirely contained in $\mathsf{X}_{\mathcal{F}}$.
($1 \Rightarrow 4$) Let X be a shift subspace, and let:

$$D_{n,S}(c) = \{c' \in A^G : d_S(c, c') < 2^{-n}\}$$

Since X is closed, for all $c \notin X$ there exists $n_c \in \mathbb{N}$ such that $D_{n_c,S}(c) \cap X = \emptyset$. Put:

$$\mathcal{F} = \left\{ c_{|D^G_{n_c,S}},\ c \notin X \right\}$$

If $c \notin X$, then surely $c \notin \mathsf{X}_{\mathcal{F}}$. On the other hand, if $c \notin \mathsf{X}_{\mathcal{F}}$, then there exist $c' \notin X$ and $g \in G$ such that $(c^g)_{|D^G_{n_{c'},S}} = c'_{|D^G_{n_{c'},S}}$: but then $d_S(c^g, c') < 2^{-n_{c'}}$, which by definition of $n_{c'}$ implies $c^g \notin X$. Since X is a shift subspace, $c \notin X$.
□

Let X be a shift subspace: if $X = \mathsf{X}_{\mathcal{F}}$, then $\mathcal{F}$ is a *presentation* of X, because all of the structure of X is encoded in $\mathcal{F}$. Sometimes, shift subspaces are "simple" enough to have *finite* presentations.

Definition 2.4.4 *Let A be an alphabet. Let G be a finitely generated group. Let S be a finite set of generators for G. A shift subspace $X \subseteq A^G$ is* of finite type *if there exists a finite set $\mathcal{F}$ of patterns over A^G such that $X = \mathsf{X}_{\mathcal{F}}$. A shift subspace $X \subseteq A^G$* has finite memory w.r.t. *S if there exist $M \geq 0$ and $\mathcal{F} \subseteq A^{D^G_{M,S}}$ such that $X = \mathsf{X}_{\mathcal{F}}$. The minimum of the set of possible M's is called the* memory *of X w.r.t. S.*

Proposition 2.4.5 *Let A be an alphabet. Let G be a finitely generated group. Let $X \subseteq A^G$ be a shift subspace. The following are equivalent:*

1. *X is of finite type;*
2. *there exists a finite set of generators S of G such that X has finite memory w.r.t. S;*
3. *for all finite sets of generators S of G, X has finite memory w.r.t. S.*

Proof:
$(2 \Rightarrow 1)$ Follows from the definitions.
$(3 \Rightarrow 2)$ Point 2 is a special case of Point 3.
$(1 \Rightarrow 3)$ Let X be a shift of finite type: then $X = \mathsf{X}_{\mathcal{F}}$ for some finite set of patterns $\mathcal{F}$. Let S be a finite set of generators for G: then there exists $n \geq 0$ such that $D^G_{n,S}$ contains $\operatorname{supp} p$ for all $p \in \mathcal{F}$. Let:

$$\mathcal{F}' = \left\{ p' \in A^{D^G_{n,S}} : \exists p \in \mathcal{F} : p'_{|\operatorname{supp} p} = p \right\}$$

Then $\mathsf{X}_{\mathcal{F}'} = \mathsf{X}_{\mathcal{F}} = X$, so that X has memory $M \leq n$ w.r.t. S. $\square$

2.5 Uniformly locally definable functions

A function $F : A^G \to A^G$ is locally definable at a point $g \in G$, if the value of $F(c)$ in g depends only on the values of c in some points "near" g, i.e. the points of a finite set $\mathcal{N} = \mathcal{N}(g)$; if the $\mathcal{N}(g)$'s also "have the same shape", in the sense that there exists $\mathcal{N} \subseteq G$ such that $\mathcal{N}(g) = g\mathcal{N}$, then we speak of uniform locality.

Definition 2.5.1 *Let G be a group. A* neighborhood index *is a finite subset $\mathcal{N} \subseteq G$. A point $h \in G$ is a* neighbor of g w.r.t. $\mathcal{N}$ *if $h \in g\mathcal{N}$.*

The relation "being a neighbor w.r.t. $\mathcal{N}$" is reflexive iff $1_G \in \mathcal{N}$, and symmetric iff $\mathcal{N}$ is symmetric; moreover, if h is a neighbor of g w.r.t. $\mathcal{N}$ and u is a neighbor of h w.r.t. $\mathcal{N}'$, then u is a neighbor of g w.r.t. $\mathcal{N}\mathcal{N}'$.
Neighborhood indices allow to introduce the following notation. Let A be an alphabet, G a f.g. group, $\mathcal{N} \subseteq G$ a neighborhood index: for $c \in A^G$ put:

$$\langle c_h \rangle_{h \in g\mathcal{N}} = (c^g)_{|\mathcal{N}} \tag{2.4}$$

In fact, since $\mathcal{N}$ is finite, one can define an ordering $\mathcal{N} = \{v_1, \ldots, v_N\}$ and induce a corresponding ordering $g\mathcal{N} = \{gv_1, \ldots, gv_N\}$ for any $g \in G$: then, $f\left(\langle c_h \rangle_{h \in g\mathcal{N}}\right)$ is a writing for $f\left(c_{gv_1}, \ldots, c_{gv_N}\right)$.
This allows to define uniform local definability as follows:

Definition 2.5.2 *Let A be an alphabet. Let G be a finitely generated group. A function $F : A^G \to A^G$ is* uniformly locally definable*, or briefly* UL-definable*, if there exist a finite set $\mathcal{N} \subseteq G$ and a map $f : A^{\mathcal{N}} \to A$ such that for every $c \in A^G$, $g \in G$:*

$$(F(c))_g = f\left(\langle c_h \rangle_{h \in g\mathcal{N}}\right) \tag{2.5}$$

The set $\mathcal{N}$ is called the neighborhood index *of F.*

From Definition 2.5.2 and the compactness of A^G follows a classical result, stated for the first time in [13] in the case $G = \mathbb{Z}$ and $F = \sigma_1$: its proof in our context deserves a look.

Theorem 2.5.3 (Hedlund's Theorem) *A function $F : A^G \to A^G$ is uniformly locally definable if and only if it is continuous and commutes with the natural action of G over A^G.*

Proof:
($\Rightarrow$) Suppose F satisfies Equation 2.5 for some finite set $\mathcal{N}$ and function $f : A^{\mathcal{N}} \to A$. Let $\varepsilon > 0$. Let n be such that $2^{-n} < \varepsilon$. Fix a finite set of generators S, and let $r \geq 0$ be such that $\mathcal{N} \subseteq D^G_{r,S}$.
Suppose $d_S(c_1, c_2) < 2^{-(n+r)}$: then c_1 and c_2 are equal over $D^G_{n+r,S}$. Let $g \in D^G_{n,S}$: then $g\mathcal{N} \subseteq D^G_{r,S}(g) \subseteq D^G_{n+r,S}$, therefore:

$$\begin{aligned}(F(c_1))_g &= f\left(\langle (c_1)_h \rangle_{h \in g\mathcal{N}}\right) \\ &= f\left(\langle (c_2)_h \rangle_{h \in g\mathcal{N}}\right) \\ &= (F(c_2))_g\end{aligned}$$

From the arbitrariness of g follows that $F(c_1)$ and $F(c_2)$ are equal over $D^G_{n,S}$: hence $d_S(F(c_1), F(c_2)) < 2^{-n} < \varepsilon$. From the arbitrariness of ε follows the continuity of F.

Moreover, for every fixed $g \in G$ and for every $h \in G$:

$$\begin{aligned}(F(c^g))_h &= f\left(\langle (c^g)_i \rangle_{i \in h\mathcal{N}}\right) \\ &= f\left(\langle c_{gi} \rangle_{i \in h\mathcal{N}}\right) \\ &= f\left(\langle c_j \rangle_{j \in gh\mathcal{N}}\right) \\ &= (F(c))_{gh} \\ &= ((F(c))^g)_h\end{aligned}$$

therefore F commutes with the natural action of G over A^G.
($\Leftarrow$) Suppose that F is continuous and commutes with the natural action. Since A^G is compact and metrizable, F is uniformly continuous: in particular, for every finite set of generators S for G, there exists $r \in \mathbb{N}$ such that, if c_1 and c_2 are equal over $D^G_{r,S}$, then $(F(c_1))_{1_G} = (F(c_2))_{1_G}$. Choose S, fix r accordingly, and put $\mathcal{N} = D^G_{r,S}$.
For $\alpha \in A^{\mathcal{N}}$, let $f(\alpha) = (F(c_\alpha))_{1_G}$, where c_α is any configuration whose restriction to $\mathcal{N}$ is identical to α: then $f : A^{\mathcal{N}} \to A$ is well defined because of the choice of r. To complete the proof, one must show that F and f satisfy Equation 2.5: and indeed, since F commutes with the natural action, for every $c \in A^G$ and $g \in G$:

$$\begin{aligned}(F(c))_g &= (F(c^g))_{1_G} \\ &= f\left(\langle (c^g)_j \rangle_{j \in \mathcal{N}}\right) \\ &= f\left(\langle c_{gj} \rangle_{j \in \mathcal{N}}\right) \\ &= f\left(\langle c_i \rangle_{i \in g\mathcal{N}}\right)\end{aligned}$$

□

Hedlund's Theorem has the following important consequence.

Theorem 2.5.4 *Let A be an alphabet. Let G be a finitely generated group. The class of the transformations of A^G that are UL-definable is a monoid with respect to the composition rule.*

Proof:
Suppose $F_1, F_2 : A^G \to A^G$ are both UL-definable: then they are both continuous, so the composition $F_2 \circ F_1$ is continuous; moreover, for any $c \in A^G$, $g \in G$:

$$(F_2 \circ F_1)(c^g) = F_2(F_1(c^g)) = F_2((F_1(c))^g) = ((F_2(F_1(c)))^g) = ((F_2 \circ F_1)(c))^g$$

that is, $F_2 \circ F_1$ commutes with the natural right action of G over A^G. Then $F_2 \circ F_1$ is UL-definable by Hedlund's Theorem.
The proof is concluded by observing that the identity map of A^G is UL-definable. $\square$
As an example, consider the shift map σ_1 of $\{0,1\}^{\mathbb{Z}}$ into itself: since σ_1 satisfies the hypotheses of Hedlund's Theorem, it must be UL-definable. And indeed, if $\mathcal{N} = -1, 0, +1$ and $f(c_{-1}, c_0, c_{+1}) = c_{+1}$, then σ_1 is defined by f in the sense of (2.5).
As a counterexample on the same configuration space, the function F that switches the value of c in 0 and leaves it unchanged elsewhere, cannot be UL-definable: in fact, if $c_i = 0$ for all $i \in \mathbb{Z}$, then $((F(c))^{+1})_{-1} = 1$ but $(F(c^{+1}))_{-1} = 0$.
Observe that if the hypotheses of Hedlund's Theorem are satisfied on a shift subspace X, then the map $F : X \to A^G$ can be seen as the restriction to X of a uniformly locally definable transformation of A^G. More formally:

Proposition 2.5.5 *Let A be an alphabet. Let G be a f.g. group. Let $X \subseteq A^G$ be a shift subspace. If $F : X \to A^G$ is continuous and commutes with the natural right action of G over X, then F is the restriction to X of a UL-definable transformation of A^G.*

Proof:
By definition, X is compact and inherits the metric from A^G. Fix a finite set of generators S for G: as in proof of Hedlund's Theorem, there must exist $r \geq 0$ such that, if $c_1, c_2 \in X$ are equal over $D^G_{r,S}$,, then $F(c_1)$ and $F(c_2)$ assume the same value on 1_G. Let:

$$P = \{\alpha \in A^{D^G_{r,S}} : \exists c \in X : c_{|D^G_{r,S}} = \alpha\}$$

Define $f : A^{D^G_{r,S}} \to A$ by defining $f(\alpha)$ as the common value $a = (F(c))_{1_G}$ of the $c \in X$ such that $c_{|D^G_{r,S}} = \alpha$ if $\alpha \in P$, and arbitrarily otherwise: let F_f be the UL-definable transformation of A^G induced by f by means of (2.5). Then for every $c \in X$ and $g \in G$:

$$(F_f(c))_g = (F_f(c^g))_{1_G} = f\left((c^g)_{|D^G_{r,S}}\right) = (F(c^g))_{1_G} = (F(c))_g$$

so that $(F_f)_{|X} = F$. $\square$

Chapter 3

Cellular automata

Cellular automata are a class of finitary descriptions of dynamical systems that has encountered a very large appraisal from many fields — and as a side effect, has undergone many redefinitions.
The reason for this success lies both in the simplicity of the idea and the similarity with "real world" phenomena: in fact, many complex structures can be seen as aggregates of simpler ones, interacting at finite range in finite time.
The drawback of a description so simple for *the part*, is difficulty of reconstructing the properties of *the whole*: this is especially true for invertibility.
In this chapter, cellular automata are presented both with the old-fashioned formulation and the more recent point of view based on finitely generated (not necessarily Abelian) groups and possibly incomplete (but closed and translation-invariant) spaces. The chapter continues by showing a characterization of the dynamical systemd that can be presented in terms of these "new-styled" cellular automata. Finally, the invertibility problem is formulated and the known main results are stated.

3.1 Definitions and properties

Cellular automata are described in terms of set of states, dimension, finite neighborhood, and local evolution function. Instead of starting with formulas and theorems, we give some examples.

The first one is a very famous two-dimensional conceptual experiment: *Con-*

way's game of life.
Consider "living" entities that occupy the nodes of a square lattice. These beings are born, survive, and die according to the following rules:

- if exactly three alive entities are near an empty node, then a new entity is born in the node;
- if an alive entity has either two or three alive neighbors, it stays alive;
- all other beings that are alive, die and nothing else is born

where a "neighbor" occupies the nearest cell in one of the eight directions: north, south, east, west, northeast, northwest, southeast, southwest.
The dynamics above can be described by means of four items:

- a set of states, $\{0 = \text{dead}, 1 = \text{alive}\}$;
- an underlying space, $\mathbb{Z} \times \mathbb{Z}$;
- a set of "possible displacement of neighbors", $\{(0,0),(0,1),(0,-1),(1,0),(-1,0),(1,1),(-1,1),(-1,1),(-1,-1)\}$;
- a function that transforms 9-tuples of states into states, according to the rules given before.

This is a *finite* way of describing an infinite system, since the function can be given by means of its (finite) *look-up table*, that is, the table that associates each possible input to its output.
The next example is on the line. Again, the states are 0 and 1; consider the map that assigns at every node of the cell the sum modulo 2 of the values of its leftmost and rightmost neighbor.
Again, one has:

- a set of states, $\{0, 1\}$;
- an underlying space, $\mathbb{Z}$;
- a set of "possible displacement of neighbors" $\{+1, -1\}$;
- a functions that transforms couples of states into states.

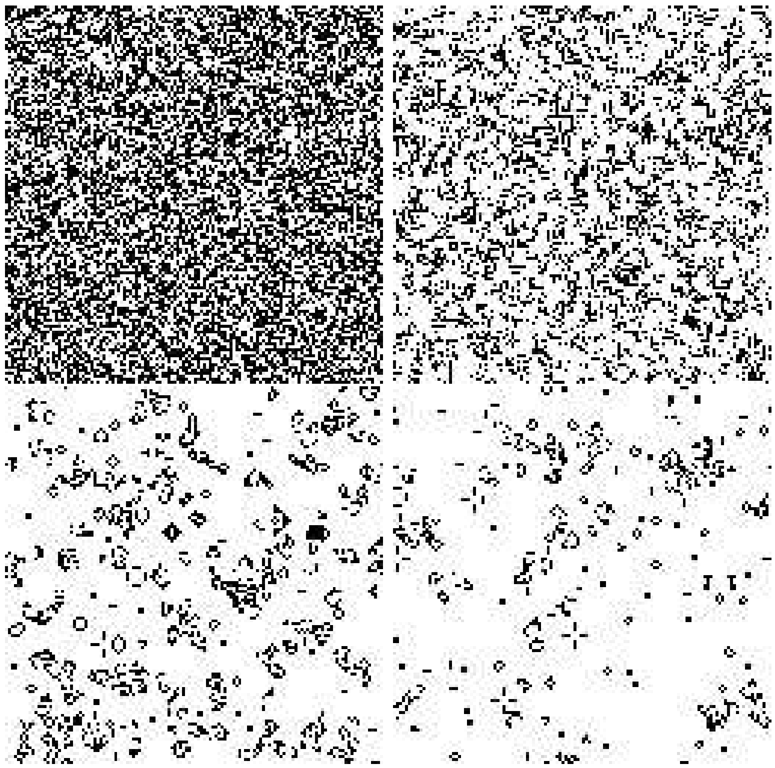

Figure 3.1: The evolution of a configuration of the Game of Life cellular automaton. From left to right, from top to bottom: initial random situation, first iteration, 50th iteration, 250th iteration.

This is called *Wolfram's rule 90*. Again, this is a finite description of an infinite system.
The interesting thing that one can observe from this discussion, is the possibility to obtain finite descriptions of infinite dynamical systems: actually, one needs a special description of the state space and a *local expression* for the evolution function. It is actually possible to give such a formal description for a class of systems that includes the two previous examples.
Suggestions for extending the concept of cellular automaton to non-Euclidean context can already be found in the work of Machì and Mignosi [20] and of Ceccherini-Silberstein, Machì and Scarabotti [4] on Moore-Myhill property (see [21] and [22]) for cellular automata whose support space is the set of configurations on an amenable group. Later, extending some results in sym-

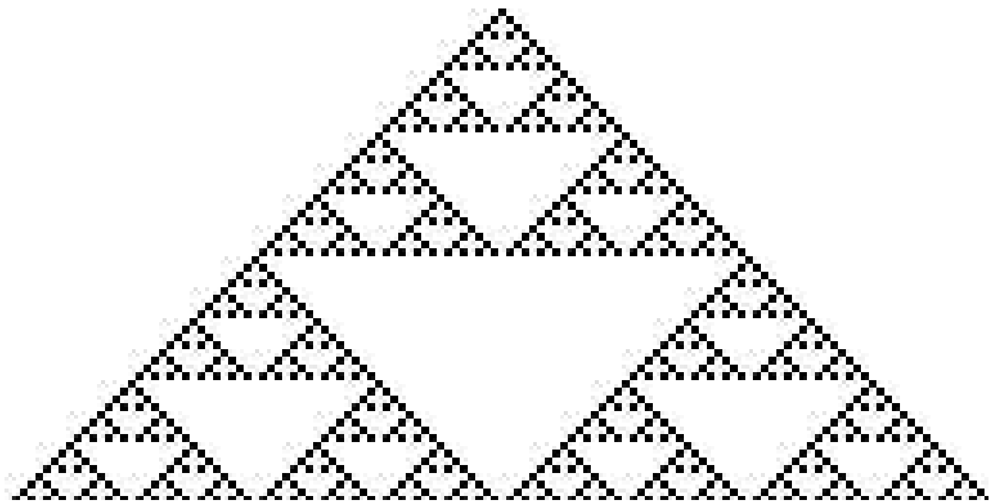

Figure 3.2: The evolution of Wolfram's Rule 90 from an initial configuration with a single cell having value 1. The "shape" is that of Pascal Triangle modulo 2.

bolic dynamics about shift dynamical systems over the integers, Fiorenzi [8] included general finitely generated groups and non-complete sets of configurations. It is now possible to give a definition in this direction, with an additional remark on the role of the neighborhood, which can have a definite importance in some contexts (see e.g. [28]).

Definition 3.1.1 *Let G be a finitely generated group. Let A be an alphabet. A* cellular automaton with alphabet A and tessellation group G *is a triple $\langle X, \mathcal{N}, f \rangle$ where:*

1. *X is a shift subspace of A^G;*

2. *$\mathcal{N}$ is a finite subset of G;*

3. *f is a function from $A^{\mathcal{N}}$ to A such that the function F defined by (2.4) satisfies $F(X) \subseteq X$.*

The shift subspace X is called the support *of the cellular automaton. The set $\mathcal{N}$ is called the* neighborhood index *of the cellular automaton. The function f is called the* local evolution function *of the cellular automaton.*

We introduce some alternative notations to be used in special cases. If $\mathcal{N} = D^G_{r,S}$ for some integer $r \geq 0$ and finite set of generators S for G, we write $\langle X, S, r, f\rangle$ instead of $\left\langle X, D^G_{r,S}, f\right\rangle$. If $X = A^G$ with $G = \mathbb{Z}^d$, we write $\langle A, d, \mathcal{N}, f\rangle$ instead of $\left\langle A^{\mathbb{Z}^d}, \mathcal{N}, f\right\rangle$.

Properties of cellular automata can be based on properties of their components. For example, a cellular automaton with tessellation group $\mathbb{Z}^d$ will be called *d-dimensional* or *free Abelian*; if its support space is a full shift, *full*; if its alphabet is binary, *binary*; and so on.

As an example, let $G = \mathbb{Z}$, $A = \{0, 1\}$; put $X = A^G$, $S = \{+1, -1\}$, $r = 1$, and define $f : A^{\{-1,0,+1\}} \to A$ as $f(a_{-1}, a_0, a_1) = a_1$: then $\langle X, S, r, f\rangle$ is a cellular automaton.

For a less trivial example, let A, G, X, S and r as before, and define $f : A^{\{-1,0,+1\}} \to A$ as $f(a_{-1}, a_0, a_1) = (a_{-1} + a_1) \bmod 2$: this is Wolfram's Rule 90.

As Toffoli [29] points out, cellular automata are not dynamical systems *per se*: rather, they are *presentations* of dynamical systems, that is, finite "encodings" of their behavior.

Definition 3.1.2 *Let $\langle X, \mathcal{N}, f\rangle$ be a cellular automaton. The* associate dynamical system *is the pair (X, F) where F is defined by (2.5). The function F is called the* global evolution function *of the cellular automaton.*

Observe that, by Hedlund's Theorem, the global evolution function of a cellular automaton is continuous.

For example, if $f(a_{-1}, a_0, a_1) = a_1$, then the dynamical system associate to $\left\langle \{0, 1\}^{\mathbb{Z}}, \{-1, 0, +1\}, f\right\rangle$ is the shift dynamical system.

Actually it is clear that, if $\mathcal{N} \subseteq \mathcal{N}'$ and $f'\left(\langle c_h\rangle_{h\in g\mathcal{N}'}\right) = f\left(\langle c_h\rangle_{h\in g\mathcal{N}}\right)$, then $\langle X, \mathcal{N}, f\rangle$ and $\langle X, \mathcal{N}', f'\rangle$ have the same associate dynamical system: thus we will often use the form $\langle X, S, r, f\rangle$, with r and S such that $\mathcal{N} \subseteq D^G_{r,S}$. The *range* of f w.r.t. S is the range of $\mathcal{N}$ w.r.t. S.

Observe that the passage to a subsystem is preserved in a natural way by the structure of cellular automata.

Proposition 3.1.3 *The dynamical system associate to a cellular automaton $\langle X, \mathcal{N}, f\rangle$ with alphabet A and tessellation group G is a dynamical subsystem of the dynamical system associate to $\left\langle A^G, \mathcal{N}, f\right\rangle$.*

One can observe that from Hedlund's Theorem follows that the global evolution function of a cellular automaton commutes with the natural right action

of its tessellation group on the phase space of the associate dynamical system.
It is also interesting to show that different elements of the phase space can always be told from one another by "pointing a microscope in the proper place", as can be seen in the following:

Proposition 3.1.4 *Let $\langle X, \mathcal{N}, f \rangle$ be a cellular automaton with alphabet A and tessellation group G. Then there exists a continuous function $\pi : X \to A$ such that:*

1. *for every pair of configurations $c_1, c_2 \in X$ such that $c_1 \neq c_2$, there exists $c \in G$ such that $\pi((c_1)^g) \neq \pi((c_2)^g)$;*
2. *the function $\tau : A^G \to A^G$ defined by $(\tau(c))_g = \pi(c^g)$ for all $g \in G$ is continuous.*

Proof:
Define π by $\pi(c) = c_{1_G}$. Then π is continuous by the definition of product topology (π is the projection on the coordinate 1_G).
(1.) If $c_1 \neq c_2$, then there must exist $g \in G$ such that $(c_1)_g \neq (c_2)_g$. Therefore:

$$\pi((c_1)^g) = ((c_1)^g)_{1_G} = (c_1)_g \neq (c_2)_g = ((c_2)^g)_{1_G} = \pi(c_2)$$

(2.) Let $c \in A^G$, $g \in G$: then $(\tau(c))_g = \pi(c^g) = (c^g)_{1_G} = c_g$, so τ is actually the identity. □
Two more properties of the class of cellular automata are proved in the next two statements.

Theorem 3.1.5 *Let A be an alphabet. Let G be a f.g. group. Let $X \subseteq A^G$ be a shift subspace. The class of cellular automata with support X is a monoid with respect to the composition rule.*

Proof:
It known from Theorem 2.5.4 that the family of UL-definable maps over A^G is a monoid. This is also true for the family of those UL-definable functions F satisfying $F(X) \subseteq X$. □

Proposition 3.1.6 *Let A be an alphabet. Let G be a finitely generated group. Let $X \subseteq A^G$ be a shift subspace. If F is the global evolution function of a cellular automaton with alphabet A and tessellation group G, then $F(X) \subseteq X$ is a shift subspace of A^G.*

Proof:
Since A^G is both compact and metrizable, $X \subseteq A^G$ is closed if and only if it is compact: and it is known from General Topology that the image of a compact set by means of a continuous function is compact. Therefore, $F(X)$ is closed.
Let $c \in F(X)$: then $c = F(c')$ for some $c' \in X$. Let $g \in G$: since X is a shift subspace, $(c')^g \in X$: since F commutes with the natural action of G over A^G, $c^g = (F(c'))^g = F((c')^g) \in F(X)$. Hence $F(X)$ is invariant under the natural action of G over A^G. □

3.2 The characterization theorem

Cellular automata provide short descriptions of dynamical systems, as every cellular automaton describes its associate dynamical system. It is interesting to find exactly the dynamical systems that can be described by cellular automata.

Definition 3.2.1 *A dynamical system* (X, F) has a presentation as a cellular automaton *if there exists a cellular automaton* $\langle Y, \mathcal{N}, f \rangle$ *whose associate dynamical system is conjugate to* (X, F)*. The cellular automaton* $\langle Y, \mathcal{N}, f \rangle$ *is called a* presentation of (X, F) as a cellular automaton.

Point 1 of Proposition 3.1.4 tells that, if (X, F) is associate to a cellular automaton with alphabet A and tessellation group G, then there is a way to tell the points of the phase space from one another, just by using a finite alphabet in the right way. This suggests the following definition:

Definition 3.2.2 *Let X be a set. Let A be an alphabet. Let G be a group. Let φ be a right action of G over X.*
X is discernible on A by φ *if there exists a continuous function* $\pi : X \to A$ *such that, for every* $x_1, x_2 \in X$ *such that* $x_1 \neq x_2$*, there exists* $g \in G$ *such that* $\pi(\varphi(x_1, g)) \neq \pi(\varphi(x_2, g))$.

Discernibility on an alphabet by a group action, is precisely a way to tell points of the phase space from one another, exactly as it can be done with supports of cellular automata: this property, together with commutation of the transition function with a group action, is shared by all dynamical systems that have a presentation as cellular automata.

Proposition 3.2.3 *Let (X, F) be a dynamical system. If (X, F) has a presentation as a cellular automaton, then F commutes with a continuous right action of the tessellation group of the cellular automaton.*

Proof:
Let $\langle X', \mathcal{N}, f' \rangle$ be a presentation of (X, F) as a cellular automaton; let A be its alphabet and G ist tessellation group. Let ϑ be a conjugacy from (X, F) to (X', F'). Define a right action $\varphi : X \times G \to X$ by putting for all $x \in X$, $g \in G$:

$$\varphi(x, g) = \vartheta^{-1}(\sigma_G(\vartheta(x), g)) \tag{3.1}$$

From now on, $\varphi(x, g)$ will be written as x^g and $\sigma_G(c, g)$ as c^g.
Observe that φ is actually a right action of G on X. Indeed, for all $x \in X$: $x^{1_G} = \vartheta^{-1}((\vartheta(x))^{1_G}) = \vartheta^{-1}(\vartheta(x)) = x$ (remember that the action of G over X' is the restriction to X' of the natural action of G over A^G); moreover, for any $g_1, g_2 \in G$, $x \in X$:

$$\begin{aligned} (x^{g_1})^{g_2} &= (\vartheta^{-1}((\vartheta(x))^{g_1}))^{g_2} \\ &= \vartheta^{-1}((\vartheta(\vartheta^{-1}((\vartheta(x))^{g_1})))^{g_2}) \\ &= \vartheta^{-1}(((\vartheta(x))^{g_1})^{g_2}) \\ &= \vartheta^{-1}((\vartheta(x))^{g_1 g_2}) \\ &= x^{g_1 g_2} \end{aligned}$$

Moreover, for every $g \in G$, the map φ_g is a composition of continuous functions, and is therefore continuous. From the definition also follows that $\vartheta(x^g) = (\vartheta(x))^g$. Putting $c = \vartheta(x) \in X'$ and applying ϑ^{-1} to the last equation, gives $(\vartheta^{-1}(c))^g = x^g = \vartheta^{-1}(c^g)$ also.
Now, let $x \in X$, $g \in G$: then:

$$\begin{aligned} (F(x))^g &= ((\vartheta^{-1} \circ F' \circ \vartheta)(x))^g \\ &= \vartheta^{-1}(((F' \circ \vartheta)(x))^g) \\ &= (\vartheta^{-1} \circ F')((\vartheta(x))^g) \\ &= (\vartheta^{-1} \circ F' \circ \vartheta)(x^g) \\ &= F(x^g) \end{aligned}$$

□

Proposition 3.2.4 *Let (X, F) be a dynamical system. If (X, F) has a presentation as a cellular automaton, then X is discernible on the alphabet of the cellular automaton by a continuous right action of the tessellation group of the cellular automaton.*

Proof:
Let $\langle X', \mathcal{N}, f' \rangle$, A, G, ϑ and φ as in proof of Proposition 3.2.3; again, x^g stands for $\varphi(x, g)$ and c^g for $\sigma_G(c, g)$.
Define $\pi : X \to A$ by putting $\pi(x) = (\vartheta(x))_{1_G}$: this function is continuous as composition of continuous functions. Suppose that $x_1 \neq x_2$: then $\vartheta(x_1) \neq \vartheta(x_2)$ too, because ϑ is a conjugacy; hence, there exists $g \in G$ such that $(\vartheta(x_1))_g \neq (\vartheta(x_2))_g$, that is, $((\vartheta(x_1))^g)_{1_G} \neq ((\vartheta(x_2))^g)_{1_G}$. But $(\vartheta(x))^g = \vartheta(x^g)$ for all $x \in X$, $g \in G$: so $\pi((x_1)^g) \neq \pi((x_2)^g)$. □

At the aim of understanding what happens if (X, F) commutes with a right action of a group and is discernible on some alphabet by the same action, Proposition 2.5.5 is restated in a more convenient form.

Theorem 3.2.5 *Let (X, F) be a dynamical system such that X is a shift subspace of A^G for some alphabet A and finitely generated group G. If F commutes with the natural right action of G over A^G, then (X, F) is the dynamical system associate to a cellular automaton with alphabet A, tessellation group G and support X.*

Theorem 3.2.5 has a somewhat surprising consequence: a f.g. group is Abelian if and only if all the maps induced by its natural action on any configuration space A^G are global evolution functions of cellular automata.

Corollary 3.2.6 *Let A be an alphabet. Let G be a finitely generated group. Let $g \in G$. Then (A^G, σ_g) has a presentation as a cellular automaton if and only if g commutes with every element of G.*

Proof:
By Proposition 2.3.7 and Theorem 3.2.5, (A^G, σ_g) has a presentation as a cellular automaton if and only if $c^{gh} = c^{hg}$ for all $h \in G$ and all $c \in A^G$: this is equivalent to $gh = hg$ for all $h \in G$. □
Now, fix $g \in G$ and a finite set of generators $S \subseteq G$, and let $r = \|g\|_S^G$: by Hedlund's Theorem, the function $F_g : A^G \to A^G$ induced by the map $f_g : A^{D_{r,S}^G} \to A$ defined by $f_g(\alpha) = \alpha_g$, is continuous and commutes with the

natural right action. The apparent contradiction vanishes when one observes that, in general, the UL-definable map F_g induced by f_g does *not* coincide with σ_g: indeed, computation shows that for every $c \in A^G$ and $i \in G$:

$$\begin{aligned}(F_g(c))_i &= f_g\left(\langle c_j\rangle_{j\in D^G_{n,S}(i)}\right)\\ &= f_g\left(\left\langle (c^i)_j\right\rangle_{j\in D^G_{n,S}}\right)\\ &= (c^i)_g\\ &= c_{ig}\end{aligned}$$

while $(\sigma_g(c))_i = c_{gi}$, which in general is different from c_{ig}.

From Propositions 3.2.3 and 3.2.4 follows that, if a dynamical system (X, F) has a presentation as a cellular automaton, then it necessarily has the following property: there exist an alphabet A, a finitely generated group G and a *continuous* right action φ of G over X such that F commutes with φ and X is discernible on A by φ. Actually, this property is also sufficient for a dynamical system to have a presentation as a cellular automaton.

Proposition 3.2.7 *Let (X, F) be a dynamical system. Let A be an alphabet, G a f.g. group, φ a continuous right action of G over X such that F commutes with φ and X is discernible over A by φ. Then (X, F) has a presentation as a cellular automaton with alphabet A and tessellation group G.*

Proof:
For every $x \in X$ define $c_x \in A^G$ by putting $(c_x)_g = \pi(\varphi(x, g))$.
The application τ that maps x into c_x is injective. In fact, suppose $x_1 \neq x_2$: since X is discernible on A by φ, there exists $g \in G$ such that $\pi((x_1)^g) \neq \pi((x_2)^g)$. But $\pi(\varphi(x_1, g)) = (c_{x_1})_g$ and $\pi(\varphi(x_2, g)) = (c_{x_2})_g$, so $c_{x_1} \neq c_{x_2}$. Put $Y = \tau(X)$: then τ is bijective *as a map from X to Y*. Observe that for all $g, h \in G$ and all $x \in X$:

$$\begin{aligned}(c_{\varphi(x,g)})_h &= \pi(\varphi(\varphi(x, g), h))\\ &= \pi(\varphi(x, gh))\\ &= (c_x)_{gh}\\ &= ((c_x)^g)_h\end{aligned}$$

that is, $c_{x^g} = (c_x)^g$ for all $x \in X$, $g \in G$. Now, given $c \in Y$, it is $c = c_x$ for some $x \in X$: but then, $c^g = (c_x)^g = c_{x^g}$ is still in Y, that is, Y is invariant under the natural right action of G.
Now, by Definition 3.2.2, π is continuous: since A is discrete, there must exist $\eta > 0$ such that $d_X(x_1, x_2) < \eta$ implies $\pi(x_1) = \pi(x_2)$.
Let $\varepsilon > 0$. Consider $n \in \mathbb{N}$ such that $2^{-n} < \varepsilon$.
Fix a finite set of generators S for G. Since $D^G_{n,S}$ is finite and the action is continuous, there must exist $\delta > 0$ such that, if $d_X(x_1, x_2) < \delta$, then $d_X(\varphi(x_1, g), \varphi(x_2, g)) < \eta$ for all $g \in D^G_{n,S}$. By joining the requirements one finds that, if $d_X(x_1, x_2) < \delta$, then $\pi(\varphi(x_1, g)) = \pi(\varphi(x_2, g))$ for all $g \in D^G_{n,S}$; but then:

$$(c_{x_1})_g = \pi(\varphi(x_1, g)) = \pi(\varphi(x_2, g)) = (c_{x_2})_g$$

for all $g \in D^G_{n,S}$. So, if $d_X(x_1, x_2) < \delta$, then $d_S(c_{x_1}, c_{x_2}) < 2^{-n} < \varepsilon$.
From the arbitrariness of ε follows that τ is continuous.
Since X is compact and τ is continuous, Y is also compact; since A^G is metrizable, Y is closed in A^G. Thus Y is a shift subspace of A^G.
Since X and Y are compact metrizable spaces and τ is continuous and invertible, $\tau^{-1} : Y \to X$ is continuous too, so τ is a homeomorphism between X and Y.
Define $F' : Y \to Y$ by $F' = \tau \circ F \circ \tau^{-1}$: then (Y, F') is a dynamical system and τ is a conjugacy between (X, F) and (Y, F').
Let $c \in Y$: then $c = c_x$ for one and only one $x \in X$. Then:

$$\begin{aligned} (F'(c))^g &= ((\tau \circ F \circ \tau^{-1})(c_x))^g \\ &= ((\tau \circ F)(x))^g \\ &= (c_{F(x)})^g \\ &= c_{(F(x))^g} \\ &= c_{F(x^g)} \\ &= \tau(F(x^g)) \\ &= (\tau \circ F)(\tau^{-1}(c_{x^g})) \\ &= (\tau \circ F)(\tau^{-1}((c_x)^g)) \\ &= F'(c^g) \end{aligned}$$

so F' commutes with the natural right action of G over A^G. By Theorem 3.2.5, (Y, F') is the dynamical system associate to a cellular automaton with alphabet A, tessellation group G, and support Y: since (X, F) is con-

jugate to (Y, F'), (X, F) has a presentation as a cellular automaton. □
From Propositions 3.2.3, 3.2.4 and 3.2.7 follows then:

Theorem 3.2.8 (The Characterization Theorem) *Let (X, F) be a dynamical system. The following are equivalent:*

1. *(X, F) has a presentation as a cellular automaton;*
2. *there exist an alphabet A, a finitely generated group G and a continuous right action φ of G over X such that F commutes with φ and X is discernible on A by φ.*

In this case, (X, F) has a presentation as a cellular automaton with alphabet A and tessellation group G.

Theorem 3.2.8 could be restated for a more restrictive definition of cellular automata: for instance, one could consider only cellular automata with Abelian f.g. tessellation group, and get the corresponding version of the theorem.
On the other hand, the hypothesis that the support of a cellular automaton is a shift subspace rather than the whole configuration space, *cannot* be abandoned: in fact, if (X, F) is discernible over A by φ and $Y \subseteq X$ is a closed subset that is invariant with respect to both F and φ, then (Y, F) is discernible over A by φ too; and discernibility is exactly what allows to define an embedding of the original dynamical system into the associate dynamical system of one of its presentations as a cellular automaton.
Regarding the choice of generators for the tesselletion group, different sets of generators can produce different Cayley graphs, however the associate dynamical system remains the same.

Lemma 3.2.9 *Let $\langle X, S, r, f \rangle$ be a cellular automaton with alphabet A and tessellation group G and let (X, F) its associate dynamical system. For every set S' of generators of G, there exist a number $r' \geq 0$ and a function $f' : A^{D^G_{r',S'}} \to A$ such that $\langle X, S', r', f' \rangle$ is a presentation of (X, F).*

Proof:
Put $r' = \max\{\|g\|^G_{S'}, g \in D^G_{r,S}\}$: then $D^G_{r,S} \subseteq D^G_{r',S'}$, and $f' : A^{D^G_{r',S'}}$ defined by:

$$f'\left(\langle a_g \rangle_{g \in D^G_{r',S'}}\right) = f\left(\langle a_g \rangle_{g \in D^G_{r,S}}\right)$$

satisfies for all $g \in G$:

$$(F(c))^g = f\left(\langle a_h \rangle_{h \in D^G_{r,S}(g)}\right) = f'\left(\langle a_h \rangle_{h \in D^G_{r',S'}(g)}\right)$$

□

3.3 Discussion

The fact of considering non-Abelian tessellation groups and non-full configuration spaces, gives a definition that is potentially broader. This suggests the question: does a dynamical system exist, having a presentation as a cellular automaton, but not as a free Abelian full cellular automaton? and if this happens, what is really making the difference?
A simple argument answers positively the first part of the question.

Proposition 3.3.1 *Let G be a finitely generated group. If G has more than two elements, then there exist cellular automata with tessellation group G that are not conjugate to any cellular automaton over a full shift.*

Proof:
Let $a, b \in A$ be distinct.
If G is finite, let $X \subseteq A^G$ be the set of those configurations such that:

1. $c_g \in \{a, b\}$ for all $g \in G$;

2. there exist $g, h \in G$ such that $c_g = a$ and $c_h = b$.

Then X is a shift subspace, and $|X| = 2^{|G|} - 2 = 2(2^{|G|-1} - 1)$, that is never a perfect power if $|G| > 2$.
If G is infinite, then it is countable. Let X be the set of those configurations c such that $c_g = a$ except for at most a single $g \in G$, where $c_g = b$: then X is countable, invariant under the natural action of G over A^G, and also topologically closed, because if $\{c_n\} \subseteq X$ converges to $c \in A^G$, then there is at most one $g \in G$ such that $c_g \neq a$, and one must have $c_g = b$: hence X is a shift subspace.
In either case, there cannot exist a bijection from X to a full shift, and no cellular automaton with support X can be conjugate to a cellular automaton over a full shift: this is especially true for $\langle X, \{1_G\}, \mathrm{id}_A \rangle$. □

Regarding the second question, observe that the new definition makes easier to think about *subsystems*. In fact, suppose that (X, F) satisfies point 2 of Theorem 3.2.8: if $Y \subseteq X$ is closed and invariant both by F and φ, then not only Y has a presentation as a cellular automaton, but this presentation can be chosen so that its associate dynamical system is a subsystem of the one associate to a presentation of (X, F) as a cellular automaton, whatever it is. In other words: with the new formulation, the operations of "finding a subsystem" and "present as a cellular automaton" can *commute*, at least under reasonable hypotheses.

However, there is always the other side of the coin. Adding shift subspaces is not painless, because if X is a shift subspace, and c is a configuration, then in general it is *not* possible to state *in a finite time* if $c \in X$ or not: the problem would be decidable if both X and $A^G \setminus X$ were shifts of finite type, but in general, the complement of a shift subspace is not a shift subspace. In other words, this is like having some kind of "hidden device", that does "bad things" if the initial configuration is not "correct", and that can possibly be out of control from the inside.

Another thing that can be lost with the new formulation, is *finite* presentability. In fact, with the canonical formulation, a cellular automaton can always be described by means of a finite sequence of symbols: but this can be impossible, for example, if the shift subspace is not full. On the other hand, the neighborhood and local evolution function part of a cellular automaton are always finitely presentable: this means that the "weak point" is actually given by the support. Hence, one has to look at classes of shift subspaces that have a finite presentation.

Of course, shifts of finite type are a good candidate: one only needs a finite list of forbidden patterns. But it is possible to do better than this: in fact, Proposition 3.1.6 allows to define a new class of objects.

Definition 3.3.2 *A shift subspace $X \subseteq A^G$ is* sofic *if there exist a uniformly locally definable map F and a shift of finite type $Y \subseteq A^G$ such that $X = F(Y)$.*

Sofic shifts (from a Hebrew word meaning "finite") are important in one-dimensional discrete symbolic dynamics, because a shift subspace of $A^{\mathbb{Z}}$ is sofic if and only if it has a presentation as *finite state automaton*. But a sofic shift on an arbitrary f.g. group still has a finite presentation: *as a cellular automaton whose support is a shift of finite type.*

3.4 The invertibility problem

When dealing with cellular automata, one has two laws to take into account: the local one, that maps states of neighborhoods into states of points; and the global one, that maps states of the space into states of the space. Talking about invertibility of the local law is usually pointless, because it maps finite sets into smaller (except for trivial cases) finite sets; instead, the beautiful thing would be to reconstruct the previous global state from the current one. In other words, invertibility for a cellular automaton should be equivalent to invertibility of the associate dynamical system.
This leads to the following statement:

Definition 3.4.1 *A cellular automaton $\langle X, \mathcal{N}, f \rangle$ is* invertible *if its local evolution function F is invertible over X.*

The main difficulty in dealing with invertibility of dynamical system presented as cellular automata, lies in the fact that a presentation in local terms is not the best tool to derive global properties: this is true, for example, for conservation laws, where results such as [23] or [12] are the exception rather than the norm. This is also true for invertibility.
The following result, originally stated in a slightly different context (locally definable *relations* on $\mathbb{Z}^d$ whose inverse is a function), states that the class of cellular automata is closed with respect to inversion.

Proposition 3.4.2 (Richardson's Lemma) *Let $\langle X, \mathcal{N}, f \rangle$ be an invertible cellular automaton. Let F be its global evolution function. Then (X, F^{-1}) has a presentation as a cellular automaton.*

In other words: the inverse of an invertible cellular automaton is itself a cellular automaton.
Proof:
First of all, $F : X \to X$ is continuous and invertible with X being compact and metrizable: therefore F^{-1} is continuous.
Let G be the tessellation group of $\langle X, \mathcal{N}, f \rangle$: for every $c \in X$, $g \in G$, putting $c' = F^{-1}(c)$, one has for every $i \in G$:

$$\begin{aligned} (F^{-1}(c^g))_i &= (F^{-1}((F(c'))^g))_i \\ &= (F^{-1}(F((c')^g)))_i \\ &= ((c')^g)_i \end{aligned}$$

$$= ((F^{-1}(F(c')))^g)_i$$
$$= ((F^{-1}(c))^g)_i$$

that is, $F^{-1}(c^g) = (F^{-1}(c))^g$. Since $g \in G$ and $c \in X$ arbitrary, F^{-1} commutes with the natural action of G over X: then the thesis follows by Theorem 3.2.5. □

As a consequence of Richardson's Lemma together with Theorem 3.1.5 one finds:

Corollary 3.4.3 *Let A be an alphabet. Let G be a finitely generated group. Let $X \subseteq A^G$ be a shift subspace. The class of invertible cellular automata with support X is a group with respect to the composition rule.*

The following question was asked by Richardson:

> Is it possible to deduce the invertibility of a dynamical system only from a presentation as a cellular automaton?

or better:

> Does a presentation of a dynamical system as a cellular automaton contain enough information to reduce the control of the invertibility of the system to a purely mechanical process?

In the language of Computability Theory, this problem is formulated as follows:

Definition 3.4.4 *Let $\mathcal{C}$ be a class of cellular automata. We say that* the invertibility problem is decidable for $\mathcal{C}$ *if there exists an algorithm that, given an arbitrary element $\mathcal{A}$ of $\mathcal{C}$ as input, returns within a finite time* True *if $\mathcal{A}$ is invertible, and* False *otherwise.*

The problem was stated by Richardson and studied by many people including Richardson himself, Amoroso, Patt, and Kari. We can summarize the main results with the following statements (we omit the proofs).

Theorem 3.4.5 (Richardson, 1972) *Let $\mathcal{C}$ be the class of cellular automata with tessellation group $\mathbb{Z}^d$ and full configuration space. If the global evolution function of $\mathcal{A} \in \mathcal{C}$ is injective, then $\mathcal{A}$ is invertible.*

Theorem 3.4.6 (Amoroso and Patt, 1972) *Let $\mathcal{C}$ be the class of full cellular automata with tessellation group $\mathbb{Z}$. Then the invertibility problem for $\mathcal{C}$ is decidable.*

Theorem 3.4.7 (Toffoli, 1977) *Every d-dimensional full cellular automaton can be embedded into a $d+1$-dimensional invertible full cellular automaton.*

An extension of Theorem 3.4.6 can be found in Section 1.6 of [8]: it states that the invertibility problem is still decidable for the class of cellular automata with tessellation group $\mathbb{Z}$ and support of finite type.
Amoroso and Patt suggested that the technique they used to prove decidability in one dimension, "should be extended to higher dimensions, even with some difficulty". This prediction was proved wrong eighteen years later.

Theorem 3.4.8 (Kari, 1990) *Let $\mathcal{C}$ be the class of full cellular automata with tessellation group $\mathbb{Z}^2$. Then the invertibility problem for $\mathcal{C}$ is undecidable.*

The techniques adopted by Kari to prove his theorem, was also used by Clementi [6] to show:

Theorem 3.4.9 (Clementi, Mentrasti, Pierini 1995) *There exists no recursive function that, given an arbitrary d-dimensional invertible full cellular automaton, returns an upper bound for the range of the inverse cellular automaton.*

Of course, what does not hold for a large class, can still possibly hold for smaller classes: so it is meaningful to focus on peculiar characteristics and see if they yield more information. An interesting result in this direction is:

Theorem 3.4.10 (Sato, 1993) *Let $\mathcal{C}$ be the class of cellular automata with tessellation group $\mathbb{Z}^d$ with $d \geq 1$ and full configuration space, such that $A = R$ is a finite ring and f only depends on a linear combination of its input values. Then the invertibility problem for $\mathcal{C}$ is decidable.*

Those who want to study invertibility problems, however, must keep in mind that simulation is not a good tool. In fact, Clementi [5] showed that:

Theorem 3.4.11 (Clementi, 1994) *There exist an alphabet A, a finite set $\mathcal{N} \subseteq A^{\mathbb{Z}^2}$ and a map $f : A^{\mathcal{N}} \to A$ such that, calling $X_n \subseteq A^{\mathbb{Z}^2}$ the set of configurations that satisfy $c_{(i+hn,j+kn)} = c_{(i,j)}$ for all $i, j, h, k \in \mathbb{Z}$:*

1. *$\langle X_n, \mathcal{N}, f \rangle$ is invertible for all $n \geq 1$;*
2. *$\langle A, 2, \mathcal{N}, f \rangle$ is not invertible.*

Chapter 4

Lattice gases

Lattice gases are a special class of cellular automata developed with the main purpose to model perfect gases. In a lattice gas, particles move along the edges of a regular graph and interact with each other at its nodes.
The very first lattice gas, named HPP (Hardy, de Pazzis, and Pomeau [15]), has a very simple, bidimensional structure: on a square grid, particles ("gas molecules") move from node to node, where they can collide according to a rule that preserves momentum.
HPP is capable to display gaslike behavior, e.g. formation of shock waves. However, further research (see e.g. [33]) showed that HPP lacks other important properties of perfect gases: in particular, there is conservation of spurious quantities (for instance, the horizontal component of total momentum is a constant on each horizontal line) that has no correspondance with real physics, and the macroscopic limit does not yield the *Navier-Stokes equations*, that are the continuous model for bidimensional perfect gas.
In response to this, FHP (Frisch, Hasslacher, and Pomeau [11]) was developed: this is perhaps the first example of a cellular automaton whose underlying structure is not a square grid: in fact, FHP particles move on an *hexagonal* grid, that is the Cayley graph of $\mathbb{Z}^2$ with respect to the set of generators:

$$\{\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_1, -\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2, -\mathbf{e}_1 - \mathbf{e}_2\}$$

(It must also be observed that the collisions in FHP are not completely deterministic, which is the other factor that allows it to recover Navier-Stokes equations in the macroscopic limit.)
An important difference between cellular automata and lattice gases lies in the field of preferred use: in fact, cellular automata are a natural way to

model *dissipative* systems, where invertibility is not a fundamental issue; on the other hand, lattice gases are the main tool for modelization of *conservative* systems, because invertibility on the space of all possible configurations is very easy to check. However, similarities between the two tools do exist, and the question if they actually model the same class of dynamics, is still open.
The aim is now to view lattice gases in the conceptual scheme used for cellular automata. This formalization is of interest for models more general than the simulation of perfect gas dynamics. This will lead to the noteworthy result that, if their supports are allowed to be non-full, then cellular automata and lattice gases *do* schematize the same dynamics.

4.1 Definitions and properties

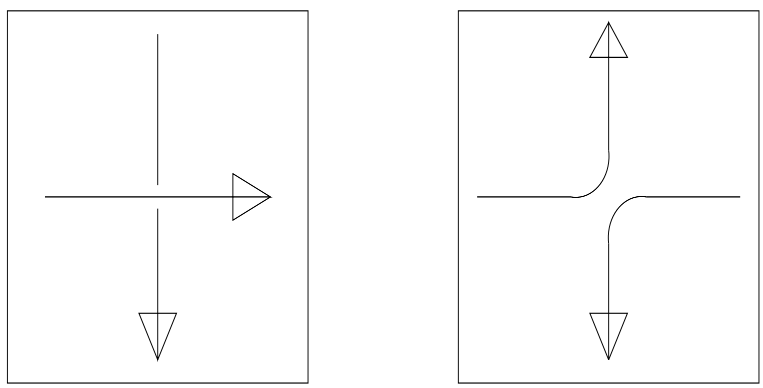

Figure 4.1: The collisions of two particles in the HPP lattice gas.

Our example of choice is HPP. The model lies on a square grid; particles move along the wires of the grid, collide in its nodes, and collisions preserve momentum.
It turns out that the only nontrivial collision function satisfying these constraints is defined as follows:

- if in a node exactly two particles are present, coming from opposite directions, then they bounce off in opposite directions on the *orthogonal* axis;

- otherwise, each particle keeps going its way.

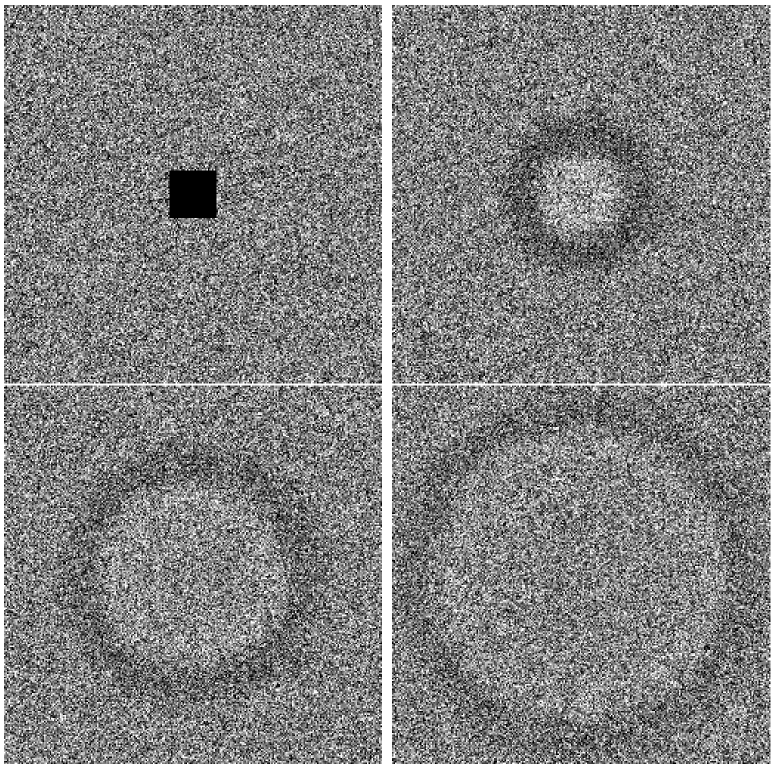

Figure 4.2: The evolution of the HPP lattice gas from a "stone in the pond" configuration. From left to right, from top to bottom: initial random situation, 50th iteration, 100th iteration, 150th iteration. Observe the propagation of shock waves.

Observe that the number of possible directions from any point is fixed; that a particle that can go in each of the four directions can exist in each node; and this can be represented by a 4-bit array, or equivalently by a neighborhood with four elements. Therefore, collisions take 4-bit arrays into 4-bit arrays, and also the sum of their values is the same before and after this operation, because the number of particles in a node is not changed by a collision. Finally, during the propagation phase, each particle going in direction i moves from node x to node $x + i$.

The aim now is to adapt the cellular automata formalism to lattice gases,

taking into account the propagation-collision discipline.
The first thing to observe is the one-to-one correspondance between the *number* of neighbors and the *quantity* of state: this means that, if the neighborhood index is $\mathcal{N}$, then the alphabet should be of the form $A^{\mathcal{N}}$ for some alphabet A.
The next remarkable thing, is that instead of speaking about propagation and collision of particles, one can equivalently talk about transmission and interaction: actually, particles can take the role of signal values, transmitted by propagation, and interacting by collisions. This change of viewpoint actually allows to become free from a strictly phisical point of view, and focus on algebraic properties.

Definition 4.1.1 *Let A be an alphabet. Let G be a finitely generated group. Let $\mathcal{N}$ be a finite,* symmetric *subset of G. Let $f : A^{\mathcal{N}} \to A^{\mathcal{N}}$. The* interaction function induced by f *is the map $\mathcal{I}_f : (A^{\mathcal{N}})^G \to (A^{\mathcal{N}})^G$ defined by $(\mathcal{I}_f(c))_g = f(c_g)$ for all $c \in (A^{\mathcal{N}})^G$, $g \in G$.*

Observe that $\mathcal{I}_f$ is the global evolution function of a cellular automaton with alphabet $A^{\mathcal{N}}$, tessellation group G, and range 0.

Definition 4.1.2 *Let A be an alphabet. Let G be a finitely generated group. Let $\mathcal{N}$ be a finite symmetric subset of G. The* transmission function induced by $\mathcal{N}$ over $(A^{\mathcal{N}})^G$ *is the function $\mathcal{T}_{\mathcal{N}} : (A^{\mathcal{N}})^G \to (A^{\mathcal{N}})^G$ defined by:*

$$((\mathcal{T}_{\mathcal{N}}(c))_g)_i = (c_{gi})_{i^{-1}} \tag{4.1}$$

for all $c \in (A^{\mathcal{N}})^G$, $g \in G$, $i \in \mathcal{N}$.

Definition 4.1.2 is consistent with the fact that a particle arriving from direction i, is moving in direction i^{-1}. For example, in the HPP model: before the collision phase, the particle moving north in a given point was the particle moving north in the southmost point immediately before.

Proposition 4.1.3 *Let A be an alphabet. Let G be a finitely generated group. Let $\mathcal{N}$ be a finite symmetric subset of G. Then:*

1. *$\mathcal{T}_{\mathcal{N}}$ is continuous;*
2. *$\mathcal{T}_{\mathcal{N}}$ commutes with the natural action of G over $(A^{\mathcal{N}})^G$;*
3. *$\mathcal{T}_{\mathcal{N}}$ is an involution*

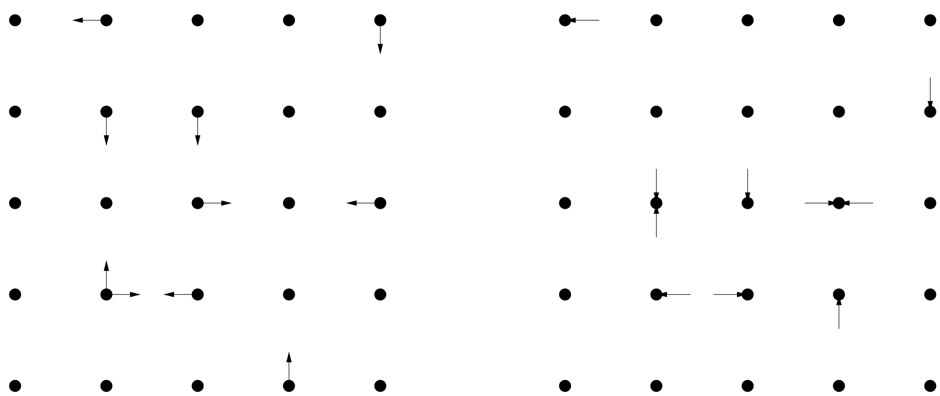

Figure 4.3: Propagation in the HPP lattice gas. Observe that particles coming from north are going south, and so on.

that is, $\mathcal{T}_\mathcal{N}$ is the local function of an invertible cellular automaton with alphabet $A^\mathcal{N}$ and tessellation group G.

Proof:
(1.) Let $c_1, c_2 \in (A^\mathcal{N})^G$. Let r be the range of $\mathcal{N}$. It follows from the definition that, if c_1 and c_2 agree on $D^G_{n+r,S}$, then $\mathcal{T}_\mathcal{N}(c_1)$ and $\mathcal{T}_\mathcal{N}(c_2)$ agree on $D^G_{n,S}$.
(2.) For every $g, h \in G$, $i \in \mathcal{N}$, $c \in (A^\mathcal{N})^G$:

$$\begin{aligned}((\mathcal{T}_\mathcal{N}(c^g))_h)_i &= ((c^g)_{hi})_{i^{-1}} \\ &= (c_{ghi})_{i^{-1}} \\ &= ((\mathcal{T}_\mathcal{N}(c))_{gh})_i \\ &= (((\mathcal{T}_\mathcal{N}(c))^g)_h)_i\end{aligned}$$

so $\mathcal{T}_\mathcal{N}$ commutes with the natural action of G over $(A^\mathcal{N})^G$.
(3.) For every $c \in (A^\mathcal{N})^G$, $g \in G$, $i \in \mathcal{N}$ we have:

$$\begin{aligned}((\mathcal{T}_\mathcal{N}(\mathcal{T}_\mathcal{N}(c)))_g)_i &= ((\mathcal{T}_\mathcal{N}(c))_{gi})_{i^{-1}} \\ &= (c_{gii^{-1}})_{(i^{-1})^{-1}} \\ &= (c_g)_i\end{aligned}$$

hence $\mathcal{T}_\mathcal{N}$ is an involution. □
It is now possible to give a formal definition of lattice gases and their associate dynamical systems.

Definition 4.1.4 *Let A be an alphabet. Let G be a finitely generated group. A* lattice gas with alphabet A and tessellation group G *is a triple $\langle \mathcal{N}, X, f \rangle$ where $\mathcal{N}$ is a finite symmetric subset of G, X is a shift subspace of $(A^{\mathcal{N}})^G$, $f : A^{\mathcal{N}} \to A^{\mathcal{N}}$ is such that $\mathcal{I}_f(\mathcal{T}_{\mathcal{N}}(X)) \subseteq X$.*
The shift subspace X is called the support *of the lattice gas. The map $f : A^{\mathcal{N}} \to A^{\mathcal{N}}$ is called the* local interaction function *of the lattice gas.*

Definition 4.1.5 *Let $\langle \mathcal{N}, X, f \rangle$ be a lattice gas. The* associate dynamical system *is the pair $(X, \mathcal{I}_f \circ \mathcal{T}_{\mathcal{N}})$. The function $\mathcal{I}_f \circ \mathcal{T}_{\mathcal{N}}$ is called the* global evolution function *of the lattice gas.*

Some authors, e.g. [33], define the global evolution function of $\langle \mathcal{N}, X, f \rangle$ as $\mathcal{T}_{\mathcal{N}} \circ \mathcal{I}_f$ instead of $\mathcal{I}_f \circ \mathcal{T}_{\mathcal{N}}$. Of course, $\mathcal{T}_{\mathcal{N}}$ being UL-definable and invertible, the two formulations are equivalent, because the associate dynamical systems will be conjugate ($\mathcal{T}_{\mathcal{N}}$ being a conjugacy).
In both cases, lattice gases are cellular automata of a special kind.

Theorem 4.1.6 *Every global evolution function of a lattice gas with alphabet A and tessellation group G is the composition of the global evolution functions of two cellular automata with alphabet $A^{\mathcal{N}}$ and tessellation group G, one having range 0 and the other one being an involution.*

Proof:
Follows from Definition 4.1.1 and Proposition 4.1.3 □
Again, lattice gas can be used as presentations for dynamical systems.

Definition 4.1.7 *A dynamical system (X, F)* has a presentation as a lattice gas *if there exists a lattice gas $\langle \mathcal{N}, Y, f \rangle$ such that (X, F) is conjugate to $(Y, \mathcal{I}_f \circ \mathcal{T}_{\mathcal{N}})$. The lattice gas $\langle \mathcal{N}, Y, f \rangle$ is called a* presentation of (X, F) as a lattice gas.

From Theorems 3.1.5 and 4.1.6 follows:

Corollary 4.1.8 *If a dynamical system (X, F) has a presentation as a lattice gas with alphabet A and tessellation group G, then it also has a presentation as a cellular automaton with alphabet $A^{\mathcal{N}}$ and tessellation group G, for some neighborhood index $\mathcal{N} \subseteq G$.*

The definition of invertibility for lattice gases is immediate.

Definition 4.1.9 *A lattice gas* $\langle \mathcal{N}, X, f \rangle$ *is* invertible *if the map* $\mathcal{I}_f \circ \mathcal{T}_\mathcal{N} : X \to X$ *is invertible.*

In other words, a lattice gas is said to be invertible if and only if its associate dynamical system is invertible, or equivalently, if and only if the lattice gas is invertible *as a cellular automaton.*
Observe that, while the inverse of a cellular automaton is still a cellular automaton by Richardson's Lemma, the inverse of a lattice gas, in general, is not a lattice gas: more precisely, the global evolution function of an invertible cellular automaton is still the global evolution function of a cellular automaton, while the inverse of the global evolution function of an invertible lattice gas is not, *a priori*, the global evolution function of a lattice gas. However, since a scheme of the form "first interaction, then transmission" is conjugate to a scheme of the form "first transmission, then interaction", it should be possible to prove that the inverse of the dynamical system associate to an invertible lattice gas has a presentation as a lattice gas. This will be done in the next section.
A result that marks an important conceptual difference between full lattice gases and full cellular automata, is the following:

Theorem 4.1.10 *Let* $\left\langle \mathcal{N}, (A^\mathcal{N})^G, f \right\rangle$ *be a full lattice gas. The following are equivalent:*

1. $\left\langle \mathcal{N}, (A^\mathcal{N})^G, f \right\rangle$ *is invertible;*
2. $\mathcal{I}_f$ *is invertible;*
3. f *is a permutation.*

Proof:
($1 \Leftrightarrow 2$) This follows from $\mathcal{T}_\mathcal{N}$ being invertible.
($3 \Rightarrow 1$) Suppose that f is a permutation. Let $c \in (A^\mathcal{N})^G$: define $c' \in (A^\mathcal{N})^G$ by $(c')_g = f^{-1}(c_g)$ for all $g \in G$. Then $\mathcal{I}_f(c') = c$. Thus $\mathcal{I}_f$ is surjective. Moreover, if $c_1 \neq c_2$, then there exists $g \in G$ such that $(c_1)_g \neq (c_2)_g$: since f is a permutation, $(\mathcal{I}_f(c_1))_g = f((c_1)_g) \neq f((c_2)_g) = (\mathcal{I}_f(c_2))_g$, hence $\mathcal{I}_f(c_1) \neq \mathcal{I}_f(c_2)$: thus $\mathcal{I}_f$ is injective.
($1 \Rightarrow 3$) If f is not a permutation, then it is not injective, hence there exist $\alpha_1, \alpha_2 \in A^\mathcal{N}$ such that $\alpha_1 \neq \alpha_2$ but $f(\alpha_1) = f(\alpha_2)$. Let $(c_1)_g = \alpha_1$ for all $g \in G$, and let $(c_2)_{1_G} = \alpha_2$, $(c_2)_g = \alpha_1$ if $g \neq 1_G$: then $c_1 \neq c_2$ but

$\mathcal{I}_f(c_1) = \mathcal{I}_f(c_2)$, therefore $\mathcal{I}_f$ is not injective, and in particular it is not invertible. □

Theorem 4.1.10 can be restated as such: a lattice gas with full support is *functionally* invertible, if and only if it is *structurally* invertible, that is, invertibility of *dynamics* is equivalent to invertibility of *point process*.

Observe that, in general, $\langle \mathcal{N}, X, f \rangle$ is invertible if and only if the map $\mathcal{I}_f$ is invertible *as a map from* $\mathcal{T}_\mathcal{N}(X)$ *to* X. Moreover, in the proof of Theorem 4.1.10, the hypothesis that the support space of a lattice gas is full plays a crucial role both in finding a preimage of c when proving that point 3 implies point 2, and in finding two different configurations with the same image when proving hat point 2 implies point 3: hence, the hypothesis $X = (A^\mathcal{N})^G$ cannot be easily discarded.

Corollary 4.1.11 *For full lattice gases, the invertibility problem is decidable.*

Proof:
By Theorem 4.1.10, a full lattice gas is invertible if and only if its local interaction function f is invertible. Since the domain of f is finite, this is a decidable problem. □

4.2 Equivalence with cellular automata

Theorem 4.1.6 says that every lattice gas can be seen as a cellular automaton. One focuses now on the opposite case: can a cellular automaton always be seen as a lattice gas?

Before giving the answer, remember that if a dynamical system has a presentation as a cellular automaton, then it also has one whose neighborhood is symmetric: in fact, one can always replace a neighborhood with a larger one, and in particular, $\mathcal{N}$ with $\mathcal{N} \cup \mathcal{N}^{-1}$.

Proposition 4.2.1 *Let* $\langle X, \mathcal{N}, f \rangle$ *be a cellular automaton with symmetric neighborhood. Let* (X, F) *be its associate dynamical system. Then:*

1. *there exists a lattice gas* $\langle \mathcal{N}, X', f' \rangle$ *with alphabet* A *and tessellation group* G*, that is a presentation of* (X, F) *as a lattice gas;*

2. *if* X *is a shift of a finite type, then* X' *is a shift of finite type;*

3. *if X is sofic, then X' is sofic.*

Proof:
(1.) Suppose initially $X = A^G$. Let:

$$Z = \{c \in (A^{\mathcal{N}})^G : (c_g)_i = (c_g)_j \ \forall i, j \in \mathcal{N} \ \forall g \in G\}$$

Then Z is a shift subspace of $(A^{\mathcal{N}})^G$. Consider the function $f' : A^{\mathcal{N}} \to A^{\mathcal{N}}$ defined by:

$$(f'(\alpha))_i = f(\alpha) \ \ \forall \alpha \in A^{\mathcal{N}} \ \forall i \in \mathcal{N}$$

Consider the lattice gas $\langle \mathcal{N}, Y, f' \rangle$: then $(\mathcal{I}_{f'} \circ \mathcal{T}_{\mathcal{N}})(Z) \subseteq Z$.
Let $\varphi : A^G \to Z$ be defined by $((\varphi(c))_g)_i = c_g$ for all $i \in \mathcal{N}$, $g \in G$: then φ is both continuous and invertible, hence it is a homeomorphism.
Let $c \in A^G$, $g \in G$. Then for every $i \in \mathcal{N}$:

$$\begin{aligned}
(((\varphi \circ F)(c))_g)_i &= (F(c))_g \\
&= f(\langle c_j \rangle_{j \in g\mathcal{N}}) \\
&= f(\langle c_{gj} \rangle_{j \in \mathcal{N}}) \\
&= f(\langle ((\varphi(c))_{gj})_{j^{-1}} \rangle_{j \in \mathcal{N}}) \\
&= f(\langle (((\mathcal{T}_{\mathcal{N}} \circ \varphi)(c))_g)_j \rangle_{j \in \mathcal{N}}) \\
&= (f'(((\mathcal{T}_{\mathcal{N}} \circ \varphi)(c))_g))_i \\
&= (((\mathcal{I}_{f'} \circ \mathcal{T}_{\mathcal{N}} \circ \varphi)(c))_g)_i
\end{aligned}$$

From the arbitrariness of $g \in G$ and $c \in (A^{\mathcal{N}})^G$ follows that φ is a conjugacy between (A^G, F) and $(Z, \mathcal{I}_{f'} \circ \mathcal{T}_{\mathcal{N}})$.
Now, let X be an arbitrary shift subspace of A^G: put $X' = \varphi(X)$. Then X' is a subset of Z and is closed in $(A^{\mathcal{N}})^G$ as a homeomorphic image of a closed set of A^G. Let $c' \in X'$: there exist exactly one $c \in X$ such that $c' = \varphi(c)$. Then for every $g \in G$, $i \in \mathcal{N}$:

$$\begin{aligned}
((\varphi(c^g))_h)_i &= (c^g)_h \\
&= c_{gh} \\
&= ((c')_{gh})_i \\
&= (((c')^g)_h)_i
\end{aligned}$$

so that $(c')^g = \varphi(c^g)$. Since X is a shift subspace, $c^g \in X$ and $(c')^g \in X'$: from the arbitrariness of c and g follows that X' is a shift subspace.

Since $\varphi^{-1}(X') = X$, one gets $F(\varphi^{-1}(X')) \subseteq X$, and finally $\varphi(F(\varphi^{-1}(X'))) \subseteq X'$. But $\varphi \circ F \circ \varphi^{-1} = \mathcal{I}_{f'} \circ \mathcal{T}_{\mathcal{N}}$, hence $(X', \mathcal{I}_{f'} \circ \mathcal{T}_{\mathcal{N}})$ is a subsystem of $(Y, \mathcal{I}_{f'} \circ \mathcal{T}_{\mathcal{N}})$ and φ is a conjugacy from (X, F) to $(X', \mathcal{I}_{f'} \circ \mathcal{T}_{\mathcal{N}})$.
(2.) Suppose X is a shift of finite type. Let S be a finite set of generators for G: by Proposition 2.4.5, there exist $M \geq 0$ and $\mathcal{F} \subseteq A^{D^G_{M,S}}$ such that $X = \mathsf{X}_{\mathcal{F}}$. Let $\mathcal{F}' \subseteq (A^{\mathcal{N}})^{D^G_{M,S}}$ be the set of those patterns p' such that:

- either $(p'_h)_i \neq (p'_h)_j$ for some $h \in D^G_{M,S}$, $i, j \in \mathcal{N}$;
- or there exists $p \in \mathcal{F}$ such that $(p'_h)_i = p_h$ for all $h \in D^G_{M,S}$, $i \in \mathcal{N}$.

It is then clear that $\varphi(X) = X' = \mathsf{X}_{\mathcal{F}'}$: since $\mathcal{F}'$ is finite by construction, X' is a shift of finite type.
(3.) Suppose $X = \psi(Y)$ for a shift of finite type $Y \subseteq A^G$ and a UL-definable function $\psi : A^G \to A^G$. Let $Y' = \varphi(Y)$: we know from the proof of point 2 that Y' is a shift of finite type. Put $\psi' = \varphi \circ \psi \circ \varphi^{-1}$: then by construction ψ' is continuous and $\psi'(Y') = X'$. Moreover for all $c \in Y'$ $g, h \in G$, $i \in \mathcal{N}$:

$$((\psi'(c^g))_h)_i = ((\varphi \circ \psi \circ \varphi^{-1})(c^g))_h)_i = (\psi(\varphi^{-1}(c^g)))_h$$

But for all $c \in Z$, $g, h \in G$, $i \in \mathcal{N}$:

$$\begin{aligned} (\varphi^{-1}(c^g))_h &= ((c^g)_h)_i \\ &= (c_{gh})_i \\ &= (\varphi(c))_{gh} \\ &= ((\varphi(c))^g)_h \end{aligned}$$

that is, $\varphi^{-1}(c^g) = (\varphi^{-1}(c))^g$ for all $c \in Z$, $g \in G$: from this and the fact that ψ is UL-definable follows:

$$\begin{aligned} (\psi(\varphi^{-1}(c^g)))_h &= (\psi((\varphi^{-1}(c))^g))_h \\ &= (\psi(\varphi^{-1}(c)))_{gh} \\ &= ((\varphi((\psi \circ \varphi^{-1})(c)))_{gh})_i \\ &= ((\psi'(c))_{gh})_i \end{aligned}$$

Therefore $\psi'(c^g) = (\psi'(c))^g$ for all $g \in G$, that is, ψ' commutes with the natural action of G over Y'. By Proposition 2.5.5, ψ' is the restriction to Y' of a UL-definable function $\Psi' : (A^{\mathcal{N}})^G \to (A^{\mathcal{N}})^G$: thus $X' = \Psi'(Y')$ is a sofic shift. □
A definition of conjugacy for presentations of dynamical systems is immediate:

Definition 4.2.2 *Two cellular automata (or two lattice gases, or a cellular automaton and a lattice gas) are* conjugate *if their associate dynamical systems are conjugate.*

Putting everything together one finds:

- for any lattice gas $\langle \mathcal{N}, X, f \rangle$ with alphabet A and tessellation group G, there exists a conjugate cellular automaton $\langle X', \mathcal{N}, f' \rangle$ with alphabet $A^{\mathcal{N}}$ and tessellation group G;

- for any cellular automaton $\langle X, \mathcal{N}, f \rangle$ with alphabet A and tessellation group G, there exists a conjugate lattice gas $\langle \mathcal{N}, X', f' \rangle$ with alphabet A and tessellation group G.

From this, and from the fact that conjugacy is an equivalence relation, follows:

Theorem 4.2.3 (The Equivalence Theorem) *Let (X, F) be a dynamical system. Then (X, F) has a presentation as a cellular automaton if and only if it has a presentation as a lattice gas.*

Corollary 4.2.4 *If $\langle \mathcal{N}, X, f \rangle$ is an invertible lattice gas, then $(X, (\mathcal{I}_f \circ \mathcal{T}_{\mathcal{N}})^{-1})$ has a presentation as a lattice gas.*

Proof:
If $\langle \mathcal{N}, X, f \rangle$ is an invertible lattice gas, then $(X, \mathcal{I}_f \circ \mathcal{T}_{\mathcal{N}})$ is the dynamical system associate to an invertible cellular automaton: by Richardson's Lemma, $(X, (\mathcal{I}_f \circ \mathcal{T}_{\mathcal{N}})^{-1})$ has a presentation as a cellular automaton, therefore by Theorem 4.2.3 it also has a presentation as a lattice gas. $\square$
Observe that Theorem 4.2.3 can be achieved precisely because the configuration space is allowed not to be a full shift: the problem if full cellular automata and full lattice gases are presentations of the same class of dynamical systems, is still open. The reason is very simple:

Theorem 4.2.5 *There is no algorithm that, given an arbitrary full cellular automaton, returns in finite time a full lattice gas that is a presentation of the dynamical system associate to the cellular automaton.*

Proof:
If such an algorithm existed, then invertibility for full cellular automata would be reducible in finite time to invertibility of full lattice gases: it would

be sufficient to construct a conjugate full lattice gas, then solve the invertibility problem for it. But by Theorem 4.1.10, invertibility for full lattice gases is a decidable problem that does not depend on the tessellation group, therefore invertibility for bidimensional full cellular automata would be decidable, contradicting Theorem 3.4.8. □

As a consequence, either there is a full cellular automata that is not conjugate to a lattice gas (but there is no known example yet), or a proof of the equivalence between full cellular automata and full lattice gases is very difficult to obtain. A step in this direction has been obtained by Durand-Lose, who in [7] proves that every invertible full cellular automaton can be simulated by an invertible *block cellular automaton* (another class of cellular automata having a decidable invertibility problem): but the construction in his paper requires to know both the range of the cellular automaton and its inverse, so by Theorem 3.4.9 it cannot be recursive.

Chapter 5

Groups with finite parts

Theorems 3.4.6 and 3.4.8 solve the invertibility problem in the free Abelian case: the problem is decidable in dimension 1, and undecidable in higher dimension. However, the new formulation opens new paths to explore.

First of all, the Abelian case is *a priori* larger than the free Abelian case, so one can try to extend Amoroso and Patt's result to the case of finitely generated Abelian groups having rank 1. Moreover, even if this was not possible, it is still interesting to verify if the Abelian case is a real extension of the free Abelian case: that is, if every cellular automaton with Abelian tessellation group is conjugate to a cellular automaton with free Abelian tessellation group.

This bears some similarities to what happens with many-dimensional models in Theoretical Phisics: current theories work on an 11-dimensional space-time, where 7 of the 10 spatial dimensions are "curled up" and are perceived as *state* rather than *direction*. If the Abelian and free Abelian case for cellular automata were essentially the same in the aforementioned sense, that would mean that a similar argument would be valid for them.

Indeed, what is shown in this chapter is exactly that, for a rather wide class of tessellation groups, not only the finite components can always be seen as parts of *the alphabet*, but, under suitable hypotheses, the "change of viewpoint" is computable. This means, first of all, that the transformation of dimension into state happens in cellular automata exactly as in Physics, thus confirming the usefulness of such a presentation; then, that algorithms working on "simpler" spaces can be adapted to work on "more complicated" ones. Of course, one of the first consequences of the second observation will be an extension of Amoroso and Patt's theorem to a much larger case: this will lead to a nontrivial (i.e., neither finite nor free Abelian) example of a group where the invertibility problem is decidable.

5.1 Semi-direct products of groups

Semi-direct products are a generalization of direct products, where the multiplication is still performed *componentwise*, but not *independently*: that is, in a semi-direct product of two groups, one of the factor groupss influences the other one's multiplication.

Definition 5.1.1 *Let H and K be groups. Let τ be a homomorphism of H into the group* Aut(K) *of automorphisms of K. The* semi-direct product of H by K with respect to τ *is the group $H \ltimes_\tau K$ of the ordered pairs (h,k), $h \in H$, $k \in K$, with the operation $(h_1,k_1)(h_2,k_2) = (h_1h_2, \tau_{h_2}(k_1)k_2)$.*

Observe that the direct product $H \times K$ is just the semi-direct product of H by K with respect to the trivial morphism from H into $\mathrm{Aut}(K)$.
Definition 5.1.1 is similar to the one in Rose's textbook [25], while many authors prefer that in Rotman's book [26], where the role of H and K are exchanged. The reason why the first one was chosen, is that the product of two applications α and β is defined as the composition $\beta \circ \alpha$, while Rotman [26] adopts the opposite order; of course, *mutatis mutandis*, the properties remain the same.
It is not difficult to prove that $H \ltimes_\tau K$ is a group, just remember that, due to the definition of the product of morphisms, $\tau_{h_1h_2} = \tau_{h_2} \circ \tau_{h_1}$. Definition 5.1.1 easily extends to products with a finite number of factors.

Lemma 5.1.2 *For every $n \in \mathbb{N}$, $h_1 \dots h_n \in H$, $k_1 \dots k_n \in K$:*

$$(h_1,k_1)\dots(h_n,k_n) = (h_1h_2\dots h_n, \tau_{h_2\dots h_n}(k_1)\tau_{h_3\dots h_n}(k_2)\dots\tau_{h_n}(k_{n-1})k_n) \tag{5.1}$$

in $H \ltimes_\tau K$.

Proof:
By induction. The thesis is true for $n = 1, 2$. Suppose it is true for n. Then:

$$\begin{aligned}(h_1,k_1)\dots(h_{n+1},k_{n+1}) &= ((h_1,k_1)\dots(h_n,k_n))(h_{n+1},k_{n+1}) \\ &= (h_1h_2\dots h_n, \tau_{h_2\dots h_n}(k_1)\tau_{h_3\dots h_n}(k_2)\dots\tau_{h_n}(k_{n-1})k_n) \\ &\quad (h_{n+1},k_{n+1})\end{aligned}$$

$$\begin{aligned}
&= (h_1 h_2 \dots h_n h_{n+1}, \\
&\quad \tau_{h_{n+1}}(\tau_{h_2 \dots h_n}(k_1)\tau_{h_3 \dots h_n}(k_2) \dots \tau_{h_n}(k_{n-1})k_n)k_{n+1}) \\
&= (h_1 h_2 \dots h_n h_{n+1}, \\
&\quad \tau_{h_{n+1}}(\tau_{h_2 \dots h_n}(k_1))\tau_{h_{n+1}}(\tau_{h_3 \dots h_n}(k_2)) \\
&\quad \dots \tau_{h_{n+1}}(\tau_{h_n}(k_{n-1}))\tau_{h_{n+1}}(k_n)k_{n+1}) \\
&= (h_1 h_2 \dots h_n h_{n+1}, \\
&\quad \tau_{h_2 \dots h_n h_{n+1}}(k_1)\tau_{h_3 \dots h_n h_{n+1}}(k_2) \\
&\quad \dots \tau_{h_{n+1}}(\tau_{h_n}(k_{n-1}))\tau_{h_{n+1}}(k_n)k_{n+1})
\end{aligned}$$

because $\tau_h \circ \tau_i = \tau_{ih}$. □

Lemma 5.1.3 *Let H, K be groups. Let $\tau : H \to \mathrm{Aut}(K)$ be a homomorphism. Then in $H \ltimes_\tau K$, for every $h, i \in H$, $k, j \in K$:*

1. $(h, 1_K)(i, j) = (hi, j) = (h, k)(i, \tau_i(k^{-1})j)$
2. $(h, k)(i, j) = (h, 1_K)(i, \tau_i(k)j)$
3. $(h, k)(1_H, j) = (h, kj)$
4. $(1_H, \tau_{i^{-1}}(k))(i, j) = (i, kj)$

Proof:
Follows from the definition and the fact that τ is a homomorphism, so in particular $\tau_{1_H} = \mathrm{id}_K$ and $\tau_h(1_K) = \mathrm{id}_K$ for all $h \in H$. □

Corollary 5.1.4 *In the hypotheses of Lemma 5.1.3, if $h = s_1 s_2 \dots s_n$ and $k = t_1 t_2 \dots t_m$, then:*

$$(h, k) = (s_1, 1_K)(s_2, 1_K) \dots (s_n, 1_K)(1_H, t_1)(1_H, t_2) \dots (1_H, t_m)$$

In particular, if $H = \langle S \rangle$ and $K = \langle T \rangle$, then:

$$H \ltimes_\tau K = \langle (S \times \{1_K\}) \cup (\{1_H\} \times T) \rangle$$

Proof:
Apply points 1 and 3 of Lemma 5.1.3. □
From now on, given a set S of generators for H and a set T of generators for

K, the semi-direct product $H \ltimes_\tau K$ will always be thought of as generated by $U = (S \times \{1_K\}) \cup (\{1_H\} \times T)$. Observe that U is symmetrical if S and T are.
The next results will be useful later.

Lemma 5.1.5 *Suppose K is f.g., T is a finite set of generators for K, H is finite, $\tau : H \to \mathrm{Aut}(K)$ is a homomorphism. Then there exists $M \geq 1$ such that $\|\tau_h(k)\|_T^K \leq M\|k\|_T^K$ for every $k \in K$, $h \in H$.*

Proof:
Put $M = \max_{h\in H} \max_{t\in T} \|\tau_h(t)\|_T^K$: then $M \geq 1$, because each τ_h is an automorphism of K.
Let $k \in K$, and let $k = t_1 t_2 \dots t_n$ be a writing of minimal length. Then for every $h \in H$:

$$\begin{aligned}
\|\tau_h(k)\|_T^K &= \|\tau_h(t_1 t_2 \dots t_n)\|_T^K \\
&= \|\tau_h(t_1)\tau_h(t_2)\dots\tau_h(t_n)\|_T^K \\
&\leq \|\tau_h(t_1)\|_T^K + \|\tau_h(t_2)\|_T^K = \dots + \|\tau_h(t_n)\|_T^K \\
&\leq nM \\
&= M\|k\|_T^K
\end{aligned}$$

From the arbitrariness of $k \in K$ the thesis follows. □

Proposition 5.1.6 *Let H and K be f.g. groups with finite sets of generators S and T respectively. Let $\tau : H \to \mathrm{Aut}(K)$ be a homomorphism. Let $G = H \ltimes_\tau K$. Then:*

1. *if $h \in D_r^H$ and $k \in D_{r'}^K$, then $g = (h, k) \in D_{r+r'}^G$;*
2. *if $g = (h, k) \in D_r^G$, then $h \in D_r^H$;*
3. *if H is finite, then there exists $M \geq 1$ such that, if $g = (h, k) \in D_r^G$, then $\tau_i(k) \in D_{Mr}^K$ for every $i \in H$.*

Proof:
(1.) Let $h = s_1 s_2 \dots s_n$ and $k = t_1 t_2 \dots t_{n'}$ be writings of minimal length. Then:

$$(h, k) = (h_1, 1_K)(h_2, 1_K)\dots(h_n, 1_K)(1_H, k_1)(1_H, k_2)\dots(1_H, k_{n'})$$

is a writing of length $n + n' \leq r + r'$.
(2.) Let $g = (h, k) = (s_1, t_1)(s_2, t_2) \dots (s_n, t_n)$ be a writing of minimal length. Then $h = s_1 s_2 \dots s_n$ is a writing of length $n \leq r$ as a product of elements of $S \cup \{1_H\}$. Therefore $h \in D_r^H$.
(3.) Suppose now that H is finite. Put M as in proof of Lemma 5.1.5. Then by Lemma 5.1.2 for every $i \in H$:

$$\begin{aligned} \|\tau_i(k)\|_T^K &= \|\tau_i(\tau_{h_2 \dots h_n}(t_1)\tau_{h_3 \dots h_n}(t_2) \dots \tau_{h_n}(t_{n-1})t_n)\|_T^K \\ &\leq \|\tau_{h_2 \dots h_n i}(t_1)\|_T^K + \|\tau_{h_3 \dots h_n i}(t_2)\|_T^K + \dots \\ &\quad + \|\tau_{h_n i}(t_{n-1})\|_T^K + \|\tau_i(t_n)\|_T^K \\ &\leq Mn \\ &\leq Mr \end{aligned}$$

so that $\tau_i(k) \in D_{Mr}^K$. $\square$

Corollary 5.1.7 *Under the hypotheses of Proposition 5.1.6, if $(i, j) \in D_r^G((h, k))$, then:*

1. *$i \in D_r^H(h)$;*
2. *if H is finite, then there exists $M \geq 1$ such that $\tau_{i^{-1}h}(j) \in D_{M^2 r}^K(k)$.*

Proof:
(1.) Since $(h, k)^{-1} = (h^{-1}, \tau_{h^{-1}}(k^{-1}))$, $(i, j) \in D_r^G((h, k))$ iff $(h^{-1}i, \tau_{h^{-1}i}(k^{-1})j) \in D_r^G$: in this case, by point 2 of Proposition 5.1.6 $h^{-1}i \in D_r^H$, or equivalently $i \in D_r^H(h)$.
(2.) If $(h, k)^{-1}(i, j) = (s_1, t_1)(s_2, t_2) \dots (s_n, t_n)$ is a writing of minimal length, then for M as in proof of Lemma 5.1.5:

$$\begin{aligned} \|\tau_{h^{-1}i}(k^{-1})j\|_T^K &= \|\tau_{h_2 \dots h_n}(t_1)\tau_{h_3 \dots h_n}(t_2) \dots \tau_{h_n}(t_{n-1})t_n\|_T^K \\ &\leq \|\tau_{h_2 \dots h_n}(t_1)\|_T^K + \|\tau_{h_3 \dots h_n}(t_2)\|_T^K + \dots \\ &\quad + \|\tau_{h_n}(t_{n-1})\|_T^K + \|t_n\|_T^K \\ &\leq Mn \\ &\leq Mr \end{aligned}$$

hence by point 3 of Proposition 5.1.6:

$$\|k^{-1}\tau_{i^{-1}h}(j)\|_T^K = \|\tau_{i^{-1}h}(\tau_{h^{-1}i}(k)j)\|_T^K \leq M\|\tau_{h^{-1}i}(k)j\|_T^K \leq M^2 r$$

which is equivalent to $\tau_{i^{-1}h}(j) \in D_{M^2 r}^K(k)$. $\square$

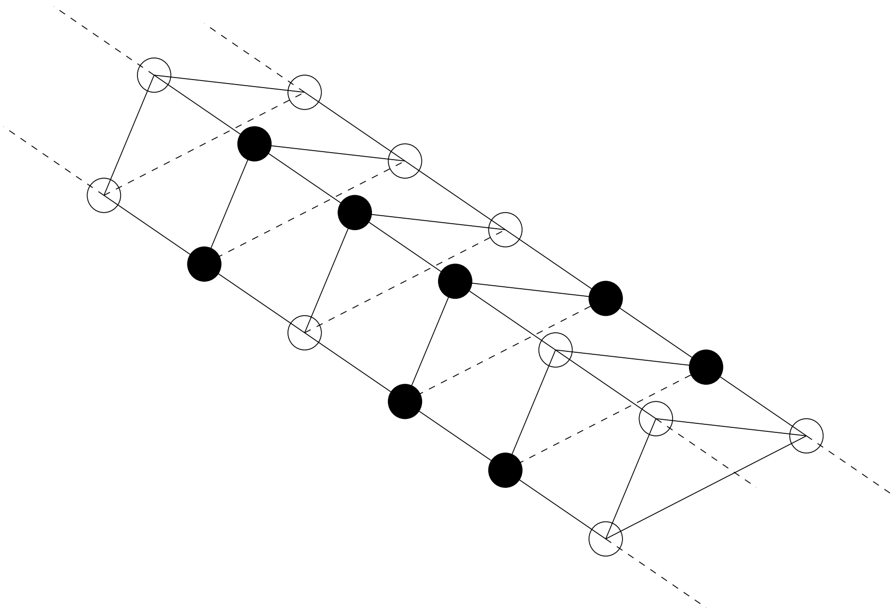

Figure 5.1: A configuration of a cellular automaton with tessellation group $\mathbb{Z} \times \mathbb{Z}_3$ and alphabet $\{0, 1\}$. White represents 0 and black 1.

Let H and K be finitely generated groups, at least one of which is finite, and let $\tau : H \to \mathrm{Aut}(K)$ be a homomorphism: let $\langle X, S, r, f \rangle$ be a cellular automaton with tessellation group $H \ltimes_\tau K$: one asks whether it is possible to transfer some of the complexity of the structure from the tessellation group to the alphabet. This is actually possible, but requires different treatment according to the finite component being H or K, and will be the subject of the next two sections.

5.2 The case with K finite

The easier case is when the finite group is K. The idea is to study a cellular automaton with alphabet A and tessellation group $H \ltimes_\tau K$ as it was a cellular automaton with alphabet A^K and tessellation group H. In this context, the "change of viewpoint" necessary to determine a homeomorphism between the two configuration spaces, and consequently, a conjugacy between cellular automata, is rather simple.

For the rest of this section, H will be a f.g. group, K a finite group, $\tau : H \to \mathrm{Aut}(K)$ a homomorphism, and A an alphabet; put $G = H \ltimes_\tau K$, $\mathcal{C} = A^G$, $\mathcal{C}' = (A^K)^H$.
Consider the function $\varphi : \mathcal{C} \to \mathcal{C}'$ given by:

$$((\varphi(c))_h)_k = c_{(h,k)} \quad \forall h \in H \; \forall k \in K \tag{5.2}$$

This is clearly an invertible function, whose inverse ψ is given by:

$$(\psi(c'))_{(h,k)} = ((c')_h)_k \quad \forall k \in K \; \forall h \in H \tag{5.3}$$

Observe an intuitive, but interesting, property of φ:

Proposition 5.2.1 *Let $c_1, c_2 \in \mathcal{C}$, $c'_1, c'_2 \in \mathcal{C}'$. Then:*

1. *if c_1 and c_2 agree on $D^{H \ltimes_\tau K}_{r+|K|}$, then $\varphi(c_1)$ and $\varphi(c_2)$ agree on D^H_r;*
2. *if c'_1 and c'_2 agree on D^K_r, then $\psi(c'_1)$ and $\psi(c'_2)$ agree on $D^{H \ltimes_\tau K}_r$.*

In particular: φ is a homeomorphism.

Proof:
(1.) Suppose that c_1 and c_2 agree on $D^{H \ltimes_\tau K}_{r+|K|}$. Let $h \in D^H_r$, $k \in K = D^K_{|K|}$: then $(h,k) \in D^G_{r+|K|}$ by point 1 of Proposition 5.1.6 and:

$$((\varphi(c_1))_h)_k = (c_1)_{(h,k)} = (c_2)_{(h,k)} = ((\varphi(c_2))_h)_k$$

and so $(\varphi(c_1))_h = (\varphi(c_2))_h$.
(2.) Suppose that c'_1 and c'_2 agree over D^H_r. Let $(h,k) \in D^G_r$. Then $h \in D^H_r$ by point 2 of Proposition 5.1.6 and so:

$$(\psi(c'_1))_{(h,k)} = ((c'_1)_h)_k = ((c'_2)_h)_k = (\psi(c'_2))_{(h,k)}$$

□
The homeomorphism φ has two more important properties: to prove them, a preliminary result is needed.

Lemma 5.2.2 *Let $h \in H$.*

1. *For every $c \in \mathcal{C}$, $(\varphi(c))^h = \varphi(c^{(h,1_K)})$.*
2. *For every $c' \in \mathcal{C}'$, $(\psi(c'))^{(h,1_K)} = \psi((c')^h)$.*

Proof:
Let $i \in H$, $j \in K$. Then by part 1 of Lemma 5.1.3:

$$\begin{aligned}(((\varphi(c))^h)_i)_j &= ((\varphi(c))_{hi})_j \\ &= c_{(hi,j)} \\ &= c_{(h,1_K)(i,j)} \\ &= (c^{(h,1_K)})_{(i,j)} \\ &= ((\varphi(c^{(h,1_K)}))_i)_j\end{aligned}$$

and:

$$\begin{aligned}((\psi(c'))^{(h,1_K)})_{(i,j)} &= (\psi(c'))_{(h,1_K)(i,j)} \\ &= (\psi(c'))_{(hi,j)} \\ &= ((c')_{hi})_j \\ &= (((c')^h)_i)_j \\ &= (\psi((c')^h))_{(i,j)}\end{aligned}$$

From the arbitrariness of i and j the thesis follows. □

Proposition 5.2.3 *Let $F : \mathcal{C} \to \mathcal{C}$ and let $F' : \mathcal{C}' \to \mathcal{C}'$ be defined by $F' = \varphi \circ F \circ \psi$. If F is UL-definable then F' is UL-definable.*

Proof:
If F is UL-definable, then it is continuous, so F' is composition of continuous functions. Let $c' \in \mathcal{C}'$, $h \in H$: by Lemma 5.2.2 follows that:

$$\begin{aligned}F'(c^h) &= \varphi(F(\psi(c^h))) \\ &= \varphi(F((\psi(c))^{(h,1_K)})) \\ &= \varphi((F(\psi(c)))^{(h,1_K)}) \\ &= (\varphi(F(\psi(c))))^h \\ &= (F'(c))^h\end{aligned}$$

From the arbitrariness of h and c' follows that F' commutes with the shift action of H over $\mathcal{C}'$: by Hedlund's Theorem follows that F' is UL-definable. □

Proposition 5.2.4 *Let $X \subseteq \mathcal{C}$ be a shift subspace. Then:*

1. *$\varphi(X) \subseteq \mathcal{C}'$ is a shift subspace;*

2. *if X is a shift of finite type, then $\varphi(X)$ is a shift of finite type;*

3. *if X is sofic, then $\varphi(X)$ is sofic.*

Proof:
(1.) Since φ is a homeomorphism, $Y = \varphi(X)$ is closed in $\mathcal{C}'$.
Let $c' \in Y$: then $c' = \varphi(c)$ for one and only one $c \in X$. Let $h \in H$: by Lemma 5.2.2, $(c')^h = (\varphi(c))^h = \varphi(c^{(h,1_K)})$. But $c^{(h,1_K)} \in X$ because $c \in X$ and X is a subshift: hence $(c')^h \in Y$. From the arbitrariness of $c' \in Y$, $h \in H$ follows that $Y \subseteq \mathcal{C}'$ is a shift subspace.
(2.) Suppose $X = \mathsf{X}_{\mathcal{F}}$ for a finite set $\mathcal{F}$: one can assume $\mathcal{F} \subseteq A^{D_r^G}$ for some $r \geq 0$. Put:

$$\begin{aligned} \mathcal{F}' &= \Big\{ p' \in (A^K)^{D_r^H} : \exists p \in \mathcal{F} : \exists k \in K : \\ &\quad \forall i \in H : \forall j \in K : (i,j) \in D_r^G \Rightarrow (p'_i)_{\tau_i(k)j} = p_{(i,j)} \Big\} \end{aligned}$$

$\mathcal{F}' \subseteq (A^K)^{D_r^H}$ is clearly finite; the idea is to show that $\varphi(X) = \mathsf{X}_{\mathcal{F}'}$.
Suppose $c' \notin \varphi(X)$. Then $\psi(c') \notin X$, so there are $g = (h,k) \in G$, $p \in \mathcal{F}$ such that $((\psi(c'))^g)_{|D_r^G} = p$. Put $(p'_i)_j = ((\psi(c'))^{(h,1_K)})_{(i,j)}$ for $i \in D_r^H$, $j \in K$: from point 2 of Lemma 5.1.3 follows that if $(i,j) \in D_r^G$ then:

$$(p'_i)_{\tau_i(k)j} = ((\psi(c'))^{(h,1_K)})_{(i,\tau_i(k)j)} = ((\psi(c'))^{(h,k)})_{(i,j)} = p_{(i,j)}$$

so that $p' \in \mathcal{F}'$; moreover, for all $i \in D_r^H$, $j \in K$:

$$(((c')^h)_i)_j = ((\psi(c'))^{(h,1_K)})_{(i,j)} = (p'_i)_j$$

so the pattern $p' \in \mathcal{F}'$ occurs in c'.
Suppose that a pattern $p' \in \mathcal{F}'$ occurs in c'. Then there is $h \in H$ such that $(((c')^h)_i)_j = (p'_i)_j$ for every $i \in D_r^H$, $j \in K$; in particular, given the structure of $\mathcal{F}'$, there are $p \in \mathcal{F}$, $k \in K$ such that, for every $i \in H$, $j \in K$ such that $(i,j) \in D_r^G$:

$$((\psi(c'))^{(h,1_K)})_{(i,\tau_i(k)j)} = p_{(i,j)}$$

But then by point 2 of Lemma 5.1.3:

$$((\psi(c'))^{(h,k)})_{(i,j)} = p_{(i,j)}$$

for every $i \in H$, $j \in K$ such that $(i, j) \in D_r^G$: thus $\psi(c') \notin \mathsf{X}_{\mathcal{F}} = X$ and so $c' \notin \varphi(X)$.
(3.) If X is sofic, then there exists a shift of finite type $Y \subseteq \mathcal{C}$ and a UL-definable function $F : \mathcal{C} \to \mathcal{C}$ such that $X = F(Y)$. By point 2, $Y' = \varphi(Y)$ is a shift of finite type; by Lemma 5.2.2, $F' = \varphi \circ F \circ \psi$ is UL-definable. But $\varphi(X) = \varphi(F(Y)) = \varphi(F(\psi(Y'))) = F'(Y')$, so $\varphi(X)$ is sofic. □
Observe that Proposition 5.2.4 cannot be reversed, because $\psi(Y)$ can possibly not be a shift subspace of $\mathcal{C}$, even if $Y \subseteq \mathcal{C}'$ is a shift of finite type and the product is direct. This is not surprising, because a less complicated tessellation group means less restrictive conditions for commutation with the group action.
To prove this, take $H = \mathbb{Z}$, $K = \mathbb{Z}_2$, $A = \{a, b\}$. Let $f_{xy} : \mathbb{Z}_2 \to A$ be the function such that $f_{xy}(0) = x$, $f_{xy}(1) = y$: then $A^{\mathbb{Z}_2} = \{f_{aa}, f_{ab}, f_{ba}, f_{bb}\}$. Let $Y = \{c' \in (A^{\mathbb{Z}_2})^{\mathbb{Z}} : ((c')_h)_1 = b \ \forall h \in \mathbb{Z}\}$: then $Y = \mathsf{X}_{\{0 \mapsto f_{aa}, 0 \mapsto f_{ba}\}}$ is a shift of finite type. But $\psi(Y) = \{c \in A^{\mathbb{Z} \times \mathbb{Z}_2} : c_{(h,1)} = b \ \forall h \in \mathbb{Z}\}$ is not a shift subspace, because if $\bar{c} \in A^{\mathbb{Z} \times \mathbb{Z}_2}$ is such that $\bar{c}_{(h,k)} = a$ if $k = 0$ and $\bar{c}_{(h,k)} = b$ if $k = 1$, then $\bar{c} \in \psi(Y)$ but $\bar{c}^{(0,1)} \notin \psi(Y)$.

The way to the main result of this section is now paved.

Theorem 5.2.5 *Let G be a f.g. group. If $G \cong H \ltimes_\tau K$ with H finitely generated and K finite, then every cellular automaton with tessellation group G is conjugate to a cellular automaton with tessellation group H. The conjugacy is such that the new cellular automaton is of finite type, or sofic, if the old one is.*

Proof:
Let $\langle X, U, r, f \rangle$ be a cellular automaton with tessellation group $H \ltimes_\tau K$. Let $F : \mathcal{C} \to \mathcal{C}$ be its global evolution function.
By Proposition 5.2.4, $\varphi(X)$ is a shift subspace of $\mathcal{C}'$ and is of finite type if X is of finite type, sofic if X is sofic. Define $F' : \mathcal{C}' \to \mathcal{C}'$ by $F' = \varphi \circ F \circ \psi$: by Proposition 5.2.3, F' is UL-definable. Thus $(\varphi(X), F')$ has a presentation as a cellular automaton with alphabet A^K and tessellation group H: let $\langle \varphi(X), S, r', f' \rangle$ be this presentation. Then $\langle X, U, r, f \rangle$ and $\langle \varphi(X), S, r', f' \rangle$ are conjugate, $\varphi_{|X}$ being a conjugacy between the two cellular automata. □

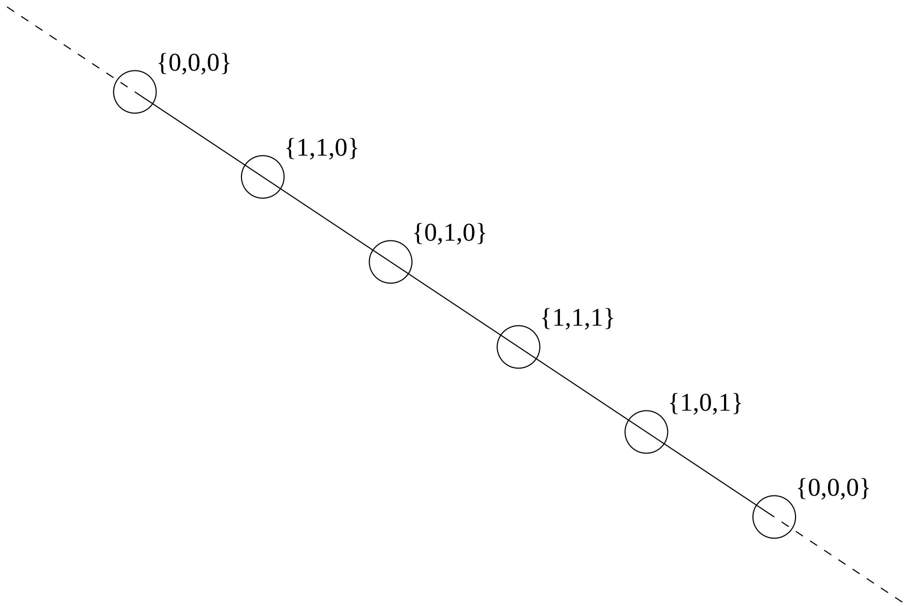

Figure 5.2: The cellular automaton with tessellation group $\mathbb{Z}$ and alphabet $\{0,1\}^{\mathbb{Z}_3}$ conjugate to the cellular automaton of Figure 5.1 constructed with the technique of Theorem 5.2.5. The configuration corresponds to that of Figure 5.1 too.

5.3 The case with H finite

The case with H finite is more complicated, even if similar. Indeed, one might guess that an intuitive function from $A^{H \ltimes_\tau K}$ to $(A^H)^K$ such as:

$$((\varphi'(c))_k)_h = c_{(h,k)} \;\; \forall h \in H \;\; \forall k \in K$$

whose inverse is given by:

$$(\psi'(c'))_{(h,k)} = ((c')_k)_h \;\; \forall h \in H \;\; \forall k \in K$$

would match one's need. This is not the case, because φ' does not have the property that $\varphi'(X)$ is a subshift if X is, not even if K is finite too.
As a counterexample, take $A = \{0,1\}$, $H = \mathbb{Z}_2$, $K = \mathbb{Z}_4$, $\tau_0 = \mathrm{id}_K$, $\tau_1(k) = -k$ $\forall k \in K$: then $H \ltimes_\tau K$ is isomorphic to the dihedral group D_4. Let $xyzt$ be the function from $D_1^{H \ltimes_\tau K}$ to A such that $xyzt((0,0)) = x$, $xyzt((0,1)) = y$, $xyzt((0,-1)) = z$, $xyzt((1,0)) = t$. Put $X = \mathsf{X}_{\{1111\}}$: it

is a shift subspace by construction. Consider the configuration $c \in A^{H \ltimes_\tau K}$ such that $c_{(0,0)} = c_{(0,1)} = c_{(0,2)} = c_{(1,1)} = 1$, $c_{(h,k)} = 0$ otherwise: then $c \in X$ because $D_1^{H \ltimes_\tau K}((0,1)) = \{(0,1),(0,2),(0,0),(1,-1)\}$. If $\varphi'(X)$ was a subshift, then $c' = (\varphi'(c))^{(-1)}$, the configuration obtained by translating $\varphi'(c)$ by $-1 \in \mathbb{Z}_4$, would still belong to $\varphi'(X)$: but this is equivalent to $\psi'(c') \in X$. A quick computation shows that $(\psi'(c'))_{(0,0)} = (\psi'(c'))_{(0,1)} = (\psi'(c'))_{(0,-1)} = (\psi'(c'))_{(1,0)} = 1$, $(\psi'(c'))_{(h,k)} = 0$ otherwise: then 1111 occurs in $\psi'(c') \in X = \mathsf{X}_{\{1111\}}$. This is a contradiction.
Observe that the counterexample still holds in the case $K = \mathbb{Z}$.
The reason for this lack is that φ' wastes all the information regarding the action of H over K determined by τ: this was admissible in the former case, when the group that undertakes the action enters the alphabet, but is not in the new one, when it is the group performing the action that "disappears" into the alphabet. This suggests to define the transformation of $A^{H \ltimes_\tau K}$ into $(A^H)^K$ so that it takes into account the action induced by τ; and indeed, this allows to obtain results similar to those of the former section.
For the rest of this section, H will be a finite group, K a finitely generated group, $\tau : H \to \mathrm{Aut}(K)$ a homomorphism, and A an alphabet; put $G = H \ltimes_\tau K$, $\mathcal{C} = A^G$, $\mathcal{C}' = (A^H)^K$.

Consider the following application from $\mathcal{C}$ to $\mathcal{C}'$:

$$((\varphi_\tau(c))_k)_h = c_{(h,\tau_h(k))} \tag{5.4}$$

and the following application from $\mathcal{C}'$ to $\mathcal{C}$:

$$(\psi_\tau(c'))_{(h,k)} = ((c')_{\tau_{h^{-1}}(k)})_h \tag{5.5}$$

Proposition 5.3.1 *Let $c_1, c_2 \in \mathcal{C}$, $c_1', c_2' \in \mathcal{C}'$. Let M be as in proof of Lemma 5.1.5. Then:*

1. *if c_1 and c_2 agree on $D_{Mr+|H|}^{H \ltimes_\tau K}$, then $\varphi_\tau(c_1)$ and $\varphi_\tau(c_2)$ agree on D_r^K;*

2. *if c_1' and c_2' agree on D_{Mr}^K, then $\psi(c_1')$ and $\psi(c_2')$ agree on $D_r^{H \ltimes_\tau K}$;*

3. *φ_τ and ψ_τ are each other's inverse.*

In particular: φ_τ is a homeomorphism.

Proof:
Let M as in proof of Lemma 5.1.5.
(1.) Suppose that c_1 and c_2 agree on $D^G_{Mr+|H|}$. Let $k \in D^K_r$: then $\tau_h(k) \in D^K_{Mr}$ by Lemma 5.1.5. Let $h \in H = D^H_{|H|}$: by point 1 of Proposition 5.1.6, $(h, \tau_h(k)) \in D^G_{Mr+|H|}$. Hence:

$$((\varphi_\tau(c_1))_k)_h = (c_1)_{(h,\tau_h(k))} = (c_2)_{(h,\tau_h(k))} = ((\varphi_\tau(c_2))_k)_h$$

and from the arbitrariness of h follows $(\varphi_\tau(c_1))_k = (\varphi_\tau(c_2))_k$.
(2.) Suppose that c'_1 and c'_2 agree on D^K_{Mr}. Let $(h, k) \in D^G_r$: then $\tau_{h^{-1}}(k) \in D^K_{Mr}$ by point 3 of Proposition 5.1.6, and for all $h \in H$:

$$(\psi_\tau(c'_1))_{(h,k)} = ((c'_1)_{\tau_{h^{-1}}(k)})_h = ((c'_2)_{\tau_{h^{-1}}(k)})_h = (\psi(c'_2))_{(h,k)}$$

(3.) For every $h \in H$, $k \in K$, $c \in \mathcal{C}$, $c' \in \mathcal{C}'$:

$$\begin{aligned}
(\psi_\tau(\varphi_\tau(c)))_{(h,k)} &= ((\varphi_\tau(c))_{\tau_{h^{-1}}(k)})_h \\
&= c_{(h,\tau_h(\tau_{h^{-1}}(k)))} \\
&= c_{(h,\tau_{h^{-1}h}(k))} \\
&= c_{(h,\tau_{1_H}(k))} \\
&= c_{(h,\mathrm{id}_K(k))} \\
&= c_{(h,k)}
\end{aligned}$$

and:

$$\begin{aligned}
((\varphi_\tau(\psi_\tau(c')))_k)_h &= (\psi_\tau(c'))_{h,\tau_h(k)} \\
&= ((c')_{\tau_{h^{-1}}(\tau_h(k))})_h \\
&= ((c')_{\tau_{hh^{-1}}(k)})_h \\
&= ((c')_{\tau_{1_H}(k)})_h \\
&= ((c')_{\mathrm{id}_K(k)})_h \\
&= ((c')_k)_h
\end{aligned}$$

From the arbitrariness of $h \in H$, $k \in K$ follows that $\psi_\tau(\varphi_\tau(c)) = c$ for all $c \in \mathcal{C}$ and $\varphi_\tau(\psi_\tau(c')) = c'$ for all $c' \in \mathcal{C}'$: hence ψ_τ is the inverse of φ_τ. $\square$

Lemma 5.3.2 *Let $k \in K$.*

1. *For all* $c \in \mathcal{C}$, $(\varphi_\tau(c))^k = \varphi_\tau(c^{(1_H,k)})$.
2. *For all* $c' \in \mathcal{C}'$, $(\psi_\tau(c'))^{(1_H,k)} = \psi_\tau((c')^k)$.

Proof:
For every $i \in H$, $j \in K$:

$$\begin{aligned}
(((\varphi_\tau(c))^k)_j)_i &= ((\varphi_\tau(c))_{kj})_i \\
&= c_{(i,\tau_i(kj))} \\
&= c_{(i,\tau_i(k)\tau_i(j))} \\
&= c_{(1_H,k)(i,\tau_i(j))} \\
&= (c^{(1_H,k)})_{(i,\tau_i(j))} \\
&= ((\varphi_\tau(c^{(1_H,k)}))_j)_i
\end{aligned}$$

and:

$$\begin{aligned}
((\psi_\tau(c'))^{(1_H,k)})_{(i,j)} &= (\psi_\tau(c'))_{(1_H,k)(i,j)} \\
&= (\psi_\tau(c'))_{(i,\tau_i(k)j)} \\
&= ((c')_{\tau_{i^{-1}}(\tau_i(k)j)})_i \\
&= ((c')_{\tau_{i^{-1}i}(k)\tau_{i^{-1}}(j)}))_i \\
&= ((c')_{k\tau_{i^{-1}}(j)})_i \\
&= (((c')^k)_{\tau_{i^{-1}}(j)})_i \\
&= (\psi_\tau((c')^k))_{(i,j)}
\end{aligned}$$

From the arbitrariness of i and j the thesis follows. □
Lemma 5.3.2 has two important consequences.

Proposition 5.3.3 *Let* $F : \mathcal{C} \to \mathcal{C}$ *and let* $F' : \mathcal{C}' \to \mathcal{C}'$ *be defined by* $F' = \varphi_\tau \circ F \circ \psi_\tau$. *If* F *is UL-definable then* F' *is UL-definable.*

Proof:
If F is UL-definable, then it is continuous, so F' is a composition of continuous functions. Let $c' \in \mathcal{C}'$, $k \in K$: by Lemma 5.3.2:

$$\begin{aligned}
F'((c')^k) &= \varphi_\tau(F(\psi_\tau((c')^k))) \\
&= \varphi_\tau(F((\psi_\tau(c'))^{(1_H,k)})) \\
&= \varphi_\tau((F(\psi_\tau(c')))^{(1_H,k)}) \\
&= (\varphi_\tau(F(\psi_\tau(c'))))^k \\
&= (F'(c'))^k
\end{aligned}$$

so F' commutes with the action of K over $\mathcal{C}'$: by Hedlund's Theorem follows that F' is UL-definable. $\square$

Proposition 5.3.4 *Let $X \subseteq \mathcal{C}$ be a shift subspace. Then:*

1. *$\varphi_\tau(X) \subseteq \mathcal{C}'$ is a shift subspace;*
2. *if X is a shift of finite type, then $\varphi_\tau(X)$ is a shift of finite type;*
3. *if X is a sofic shift, then $\varphi_\tau(X)$ is a sofic shift.*

Proof:
(1.) Let $c' \in \varphi_\tau(X)$. There is exactly one $c \in X$ such that $c' = \varphi_\tau(c)$ Fix $k \in K$. By Lemma 5.3.2:

$$(c')^k = (\varphi_\tau(c))^k = \varphi_\tau(c^{(1_H,k)})$$

But $c^{(1_H,k)} \in X$ because $c \in X$ and X is a subshift: hence $(\varphi_\tau(c))^k \in \varphi_\tau(X)$. From the arbitrariness of $k \in K$, $c' \in X$ follows that $\varphi_\tau(X)$ is a shift subspace.
(2.) Suppose that X is a shift of finite type: then there exist $r > 0$, $\mathcal{F} \subseteq D_r^G$ such that $X = \mathsf{X}_\mathcal{F}$. Put:

$$\begin{aligned} \mathcal{F}' &= \Big\{ p' \in (A^H)^{D^K_{Mr}} : \exists p \in \mathcal{F} : \exists h \in H : \\ &\quad \forall i \in H : \forall j \in K : (i,j) \in D_r^G \Rightarrow ((p')_{\tau_{i^{-1}}(j)})_{hi} = p_{(i.j)} \Big\} \end{aligned}$$

where $M \geq 1$ is as in Lemma 5.1.5. The idea is to show that $\varphi_\tau(X) = \mathsf{X}_{\mathcal{F}'}$.
Suppose that $c' \notin \varphi_\tau(X)$. Then $c = \psi_\tau(c') \notin X$, so there must exist $p \in \mathcal{F}$, $g = (h,k) \in G$ such that $(c^g)_{|D_r^G} = p$. Define $p' \in D^K_{Mr}$ by $((p')_j)_i = (c^{(1_H,\tau_{h^{-1}}(k)})_{(i,\tau_i(j))}$: then for all $i \in H$, $j \in K$ such that $(i,j) \in D_r^G$:

$$\begin{aligned} p_{(i,j)} &= \left(c^{(h,k)}\right)_{(i,j)} \\ &= c_{(h,k)(i,j)} \\ &= c_{(1_H,\tau_{h^{-1}}(k))(h,1_K)(i,j)} \\ &= c_{(1_H,\tau_{h^{-1}}(k))(hi,j)} \\ &= \left(c^{(1_H,\tau_{h^{-1}}(k))}\right)_{(hi,j)} \\ &= \left(c^{(1_H,\tau_{h^{-1}}(k))}\right)_{(hi,\tau_i(\tau_{i^{-1}}(j)))} \\ &= ((p')_{\tau_{i^{-1}}(j)})_{hi} \end{aligned}$$

so that $p' \in \mathcal{F}'$, and for $j \in D^K_{Mr}$, $i \in H$:

$$\begin{aligned} ((p')_j)_i &= (c^{(1_H, \tau_{h^{-1}}(k))})_{(i,\tau_i(j))} \\ &= ((\psi_\tau(c'))^{(1_H, \tau_{h^{-1}}(k))})_{(i,\tau_i(j))} \\ &= (\psi_\tau((c')^{\tau_{h^{-1}}(k)}))_{(i,\tau_i(j))} \\ &= (((c')^{\tau_{h^{-1}}(k)})_{\tau_{i^{-1}}(\tau_i(j))})_i \\ &= (((c')^{\tau_{h^{-1}}(k)})_j)_i \end{aligned}$$

so that c' has a pattern in $\mathcal{F}'$.
Suppose that c' has a pattern in $\mathcal{F}'$. Then there are $k \in K$, $p' \in \mathcal{F}'$ such that $((c')^k)_{|D^K_{Mr}} = p'$. In particular, given the structure of $\mathcal{F}'$, there exist $p \in \mathcal{F}$, $h \in H$ such that, for every $i \in H$, $j \in K$ such that $(i,j) \in D^G_r$:

$$(((c')^k)_{\tau_{i^{-1}}(j)})_{hi} = ((p')_{\tau_{i^{-1}}(j)})_{hi} = p_{(i,j)}$$

Let $c = \psi_\tau(c')$: then for all $i \in H$, $j \in K$ such that $(i,j) \in D^G_r$:

$$\begin{aligned} p_{(i,j)} &= (((c')^k)_{\tau_{i^{-1}}(j)})_{hi} \\ &= ((c')_{k\tau_{i^{-1}}(j)})_{hi} \\ &= ((c')_{\tau_{i^{-1}}(\tau_i(k)j)})_{hi} \\ &= (\psi_\tau(c'))_{(hi,\tau_i(k)j)} \\ &= c_{(h,k)(i,j)} \\ &= (c^{(h,k)})_{(i,j)} \end{aligned}$$

so that p occurs in c: therefore $c \notin X$ and $c' \notin \varphi_\tau(X)$.
(3.) If X is sofic, then there exists a shift of finite type $Y \subseteq \mathcal{C}$ and a UL-definable function $F : \mathcal{C} \to \mathcal{C}$ such that $X = F(Y)$. By point 2, $Y' = \varphi_\tau(Y)$ is a shift of finite type; by Lemma 5.3.2, $F' = \varphi_\tau \circ F \circ \psi_\tau$ is UL-definable. But $\varphi_\tau(X) = \varphi_\tau(F(Y)) = \varphi_\tau(F(\psi_\tau(Y'))) = F'(Y')$, so $\varphi_\tau(X)$ is sofic. □
The way to the main theorem of this section is now paved.

Theorem 5.3.5 *Let G be a f.g. group. If $G \cong H \ltimes_\tau K$ with H finite and K finitely generated, then every cellular automaton with tessellation group G is conjugate to a cellular automaton with tessellation group K. The conjugacy is such that the new cellular automaton is of finite type, or sofic, if the old one is.*

Proof:
By Proposition 5.3.4, $\varphi_\tau(X)$ is a shift subspace of $\mathcal{C}'$, and is of finite type or sofic if X is. Let F be the global evolution function of $\langle X, U, r, f \rangle$: then $F' = \varphi_\tau \circ F \circ \psi_\tau$ is UL-definable. Thus, $(\varphi_\tau(X), F')$ has a presentation as a cellular automaton with alphabet A^H and tessellation group K: let $\langle \varphi_\tau(X), T, r', f' \rangle$ be this presentation. Then $\langle X, U, r, f \rangle$ and $\langle \varphi_\tau(X), T, r', f' \rangle$ are conjugate, $\varphi_{\tau|X}$ being a conjugacy between the two cellular automata. □

5.4 An application to the invertibility problem

Theorems 5.2.5 and 5.3.5 suggest an extension of Amoroso and Patt's theorem on the decidability of invertibility problem. After all, under the light shed by these results, a semi-direct product of $\mathbb{Z}$ and a finite group is not very different from $\mathbb{Z}$ itself for what regards the construction of cellular automata: therefore, if a problem is decidable on $\mathbb{Z}$, it should still be decidable on the semi-direct product. And actually, under very reasonable hypotheses, this is exactly the case for the invertibility problem.
Before achieving this result, a definition must be given.

Definition 5.4.1 *Let G be a f.g. group with finite set of generators S. We say that* the word problem is decidable for G *if the set of all the words over S that are equal to 1_G is recursive.*

Equivalently, a f.g. group G has decidable word problem if there exists an algorithm that, given a word w over $S \cup S^{-1}$ and a finite set $U \subseteq G$, determines if w represents some element of U. The class of groups with decidable word problem is not too small, because it includes finite groups, free groups, and finite semi-direct products of groups with decidable word problem. In this last case, one can say more:

Lemma 5.4.2 *Let H and K be two f.g. groups and let $\tau : H \to \mathrm{Aut}(K)$ be a group homomorphism. If the word problem is decidable for both H and K, then τ is computable.*

Proof:
One must construct an algorithm to compute $\tau_h(k)$ whatever $h \in H$ and

$k \in K$ are.
Let S be a finite set of generators for H and let T be a finite set of generators for K: the set $\{\tau_s(t), s \in S, t \in T\}$ is finite. Consider the following algorithm:

INPUT: a pair $(h, k) \in H \times K$
OUTPUT: the element $\tau_h(k) \in K$

put $x = k$
construct a writing $h = s_1 \dots s_N \in S^*$
for i from 1 to N:
    construct a writing $x = t_1 \dots t_M \in T^*$
    replace x with $\tau_{s_i}(t_1) \dots \tau_{s_i}(t_M)$
end for
return x

One must prove that this algorithm computes $\tau_h(k)$.
First, observe that the construction of the writing $h = s_1 \dots s_N$ is computable: indeed, one only needs to compare h with 1_H, then with each element of S, then with each element of H that can be written as a product of two elements of S, and so on: each comparison terminates in a finite time, and eventually one of them returns true.
Next, observe that the same is true for every construction inside the "for" cycle.
Finally, consider the new value of x after every cycle over i: for $i = 1$ it is $\tau_{s_1}(k)$, for $i = 2$ it is $\tau_{s_2}(\tau_{s_1}(k)) = \tau_{s_1 s_2}(k)$, and so on, up to $\tau_{s_N}(\tau_{s_1 \dots s_{N-1}}(k)) = \tau_{s_1 \dots s_N}(k) = \tau_h(k)$ when $i = N$. Observe that if $N = 0$, the cycle over i is not entered, but in this case $h = 1_H$, $\tau_{1_H} = \mathrm{id}_K$, and the returned value is $\mathrm{id}_K(k) = k$. $\Box$
As before, the simpler case is examined first.

Theorem 5.4.3 *If the word problem is decidable over H, then the construction of f' from f in Theorem 5.2.5 is computable.*

Proof:
By Lemma 5.4.2, the hypothesis on H and the fact that K is finite imply computability of τ and decidability of the word problem over G.
Represent $(F'(c'))_h$ by the sequence of its values $((F'(c'))_h)_k$, for k in K. Consider the following procedure:

INPUT: the list $\langle (c')_i \rangle_{i \in D_r^H(h)}$
OUTPUT: the value $(F'(c'))_h$

```
X = an empty list
for k in K:
    s = a list of |D_r^{H ⋉_τ K}| elements of A
    for i in D_r^H(h):
        for j in K:
            if (i, j) in D_r^{H ⋉_τ K}((h, k)):
                replace with ((c')_i)_j the element of s
                in the position corresponding to (h, k)^{-1}(i, j)
                in the defined ordering of D_r^{H ⋉_τ K}
            end if
        end for
    end for
    append f(s) to X
end for
return X
```

First of all, observe that, since τ is computable, the multiplications are all computable, and because the word problem is decidable over H and $H \ltimes_\tau K$, the fact that an element appears in a finite subset of one of these groups is decidable; therefore this procedure is actually an algorithm. One must now show that it correctly computes $(F'(c'))_h$.
Observe that:

$$(F(c))_{(h,k)} = f\left(\left\langle c_{(i,j)} \right\rangle_{(i,j) \in D_r^{H \ltimes_\tau K}((h,k))}\right)$$

thus:

$$\begin{aligned} (F(\psi(c')))_{(h,k)} &= f\left(\left\langle (\psi(c'))_{(i,j)} \right\rangle_{(i,j) \in D_r^{H \ltimes_\tau K}((h,k))}\right) \\ &= f\left(\left\langle (c'_i)_j \right\rangle_{(i,j) \in D_r^{H \ltimes_\tau K}((h,k))}\right) \end{aligned}$$

and so:

$$((F'(c'))_h)_k = f\left(\left\langle (c'_i)_j \right\rangle_{(i,j) \in D_r^{H \ltimes_\tau K}((h,k))}\right)$$

But for each k in K, the cycle over i transforms the sequence s in the sequence $\left\langle (c'_i)_j \right\rangle_{(i,j) \in D_r^{H \ltimes_\tau K}((h,k))}$.

Indeed, if $(i,j) \in D_r^{H \ltimes_\tau K}((h,k))$ then surely $j \in K$ and $i \in D_r^H(h)$ by Corollary 5.1.7,
This means that the double iteration over i and j surely catches all the elements in $D_r^{H \ltimes_\tau K}((h,k))$: hence the next instruction appends to the list X precisely the value $f\left(\langle (c_i')_j \rangle_{(i,j) \in D_r^{H \ltimes_\tau K}((h,k))}\right) = ((F'(c'))_h)_k$. In the end, the returned list X is precisely the sequence $((F'(c'))_h)_{k_1} \ldots ((F'(c'))_h)_{k_{|K|}}$. $\square$
From Theorems 5.2.5 and 5.4.3 follows:

Theorem 5.4.4 *Let $G \cong H \ltimes_\tau K$, with H finitely generated and K finite. Suppose that the word problem is decidable over H. Then the following are true:*

1. *If invertibility of full cellular automata with tessellation group H is decidable, then invertibility of full cellular automata with tessellation group G is decidable too.*
2. *If invertibility of cellular automata of finite type with tessellation group H is decidable, then invertibility of cellular automata of finite type with tessellation group G is decidable too.*
3. *If invertibility of sofic cellular automata with tessellation group H is decidable, then invertibility of sofic cellular automata with tessellation group G is decidable too.*

It is now time to deal with the harder case.

Theorem 5.4.5 *If the word problem is decidable over K, then the construction of f' from f in Theorem 5.3.5 is computable.*

Proof:
By Lemma 5.4.2, the hypothesis on K and the fact that H is finite imply computability of τ and decidability of the word problem over G.
Represent $(F'(c'))_k$ by the sequence of its values $((F'(c'))_k)_h$, for h in H. Let M be as in Lemma 5.1.5. Consider the following procedure:

INPUT: the list $\langle (c')_j \rangle_{j \in D^K_{M^3 r}(k)}$
OUTPUT: the value $(F'(c'))_k$

$X =$ an empty list
for h in H:

```
        s = a list of |D_r^{H⋉_τ K}| elements of A
        for i in H:
            for j in D_{M^3 r}^K(k)):
                if (i, τ_i(j)) in D_r^{H⋉_τ K}((h, τ_h(k))):
                    replace with ((c')_j)_i the element of s
                    in the position corresponding to (h, τ_h(k))^{-1}(i, τ_i(j))
                    in the defined ordering of D_r^{H⋉_τ K}
                end if
            end for
        end for
        append f(s) to X
    end for
    return X
```

First of all, observe that, since τ is computable, the multiplications are all computable, and because the word problem is decidable over K and over $H \ltimes_\tau K$, the fact that an element appears in a finite subset of one of these groups is decidable; therefore this procedure is actually an algorithm. One must now show that it correctly computes $(F'(c'))_h$.
Observe that:

$$(F(c))_{(h,k)} = f\left(\left\langle c_{(i,j)}\right\rangle_{(i,j)\in D_r^{H\ltimes_\tau K}((h,k))}\right)$$

thus:

$$\begin{aligned}(F(\psi_\tau(c')))_{(h,k)} &= f\left(\left\langle(\psi_\tau(c'))_{(i,j)}\right\rangle_{(i,j)\in D_r^{H\ltimes_\tau K}((h,k))}\right)\\ &= f\left(\left\langle((c')_{\tau_{i^{-1}}(j)})_i\right\rangle_{(i,j)\in D_r^{H\ltimes_\tau K}((h,k))}\right)\end{aligned}$$

and so:

$$\begin{aligned}((F'(c'))_k)_h &= ((\varphi_\tau(F(\psi_\tau(c'))))_k)_h\\ &= (F(\psi_\tau(c')))_{(h,\tau_h(k))}\\ &= f\left(\left\langle((c')_{\tau_{i^{-1}}(j)})_i\right\rangle_{(i,j)\in D_r^{H\ltimes_\tau K}((h,\tau_h(k)))}\right)\\ &= f\left(\langle((c')_j)_i\rangle_{(i,\tau_i(j))\in D_r^{H\ltimes_\tau K}((h,\tau_h(k)))}\right)\end{aligned}$$

But for each h in H, the cycle over j transforms the sequence s in the sequence $\left\langle(c'_j)_i\right\rangle_{(i,\tau_i(j))\in D_r^{H\ltimes_\tau K}((h,\tau_h(k)))}$.

Indeed, if $(i, \tau_i(j)) \in D_r^{H \ltimes_\tau K}((h, \tau_h(k)))$, then, by Corollary 5.1.7, $i \in D_r^H(h)$ and $\tau_h(j) = \tau_{i^{-1}h}(\tau_i(j)) \in D_{M^2r}^K(\tau_h(k))$, that is, $\tau_h(k^{-1}j) \in D_{M^2r}^K$ which implies by Lemma 5.1.5 $k^{-1}j \in D_{M^3r}^K$, that is, $j \in D_{M^3r}^K(k)$.
This means that the double cycle over i and j captures all the pairs (i, j) such that $(i, \tau_i(j)) \in D_r^{H \ltimes_\tau K}((h, \tau_h(k)))$. In the end, the returned list X is precisely the sequence $((F'(c'))_k)_{h_1} \ldots ((F'(c'))_k)_{h_{|H|}}$. □
From Theorems 5.3.5 and 5.4.5 follows:

Theorem 5.4.6 *Let $G \cong H \ltimes_\tau K$, with H finite and K finitely generated. Suppose that the word problem is decidable over K. Then the following are true:*

1. *If invertibility of full cellular automata with tessellation group K is decidable, then invertibility of full cellular automata with tessellation group G is decidable too.*

2. *If invertibility of cellular automata of finite type with tessellation group K is decidable, then invertibility of cellular automata of finite type with tessellation group G is decidable too.*

3. *If invertibility of sofic cellular automata with tessellation group K is decidable, then invertibility of sofic cellular automata with tessellation group G is decidable too.*

Consider now the case when the tessellation group is Abelian. A very well known structure theorem states that every finitely generated Abelian group is isomorphic to a finite direct product of cyclic groups; that is, for every f.g. Abelian group G there exist $N, n_1, \ldots, n_k$ such that $G \cong \mathbb{Z}^N \times \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_k}$. The number N is called the *rank* of the f.g. Abelian group G; isomorphic finitely generated Abelian groups have the same rank. Theorems 5.2.5 and 5.4.3 give us a technique to decide the invertibility of a full cellular automaton over a f.g. Abelian group having rank 1, because in this case the conditions over $H = \mathbb{Z}$ and K are satisfied.
Consider a full cellular automaton whose tessellation group is $\mathbb{Z} \times \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_k}$: by applying Theorem 5.2.5 with $H = \mathbb{Z}$ and $K = \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_k}$ one finds a conjugate full cellular automaton with tessellation group $\mathbb{Z}$, and the construction is computable because of Theorem 5.4.3. But because of Theorem 3.4.6, invertibility for the new cellular automaton is decidable, and because of conjugacy, it is equivalent to invertibility of the original cellular

automaton.
This, together with Theorem 3.4.8, proves:

Theorem 5.4.7 *Let G be a finitely generated Abelian group. Then invertibility for the class of full cellular automata over G is decidable if and only if G has at most rank 1.*

Remark that the procedure is applicable even if the tessellation group is not Abelian: for example, invertibility of cellular automata with tessellation group $\mathbb{Z} \times S_3$, where S_3 is the group of permutation of three distinct objects, is still decidable. The last theorem of this section states exactly this fact, thus extending the results of Section 1.6 of [8].

Theorem 5.4.8 *Let G be a finite group. Let A be an alphabet. Suppose one of the following is verified:*

1. *$\tau : \mathbb{Z} \to \mathrm{Aut}(G)$ is a homomorphism and $X \subseteq A^{\mathbb{Z} \ltimes_\tau G}$ is a shift of finite type;*

2. *$\tau : G \to \mathrm{Aut}(\mathbb{Z})$ is a homomorphism and $X \subseteq A^{G \ltimes_\tau \mathbb{Z}}$ is a shift of finite type.*

Then the invertibility problem for the class of cellular automata with support X is decidable.

Proof:
Follows from Theorems 5.4.4 and 5.4.6, together with the fact that the invertibility problem for shifts of finite type of $\mathbb{Z}$ is decidable. □

Corollary 5.4.9 *Let $G = D_\infty$ be the infinite dihedral group. Let A be an alphabet. Let $X \subseteq A^G$ be a shift of finite type. Then the invertibility problem for the class of cellular automata with support X is decidable.*

Proof:
The group D_∞ is isomorphic to the semi-direct product $\mathbb{Z}_2 \ltimes_\tau \mathbb{Z}$, where $\tau_0 = \mathrm{id}_\mathbb{Z}$ and $\tau_1(n) = -n$ for all $n \in \mathbb{Z}$: this is an instance of case 2 of Theorem 5.4.8, and the thesis follows. □

5.5 Remarks

Theorems 5.2.5 and 5.3.5 say that the "finite part" of the tessellation group is unessential to the dynamics, because it can be seen as a component of the alphabet instead of the group. This implies that the Abelian case essentially reduces to the free Abelian case, where the tessellation group is finite or is $\mathbb{Z}^d$ for some $d > 0$: hence, study of "non-classical" cellular automata dynamics should be oriented to the case of non-Abelian tessellation group.
On the other hand, Theorems 5.4.4 says that the question on the decidability of the invertibility problem has a known answer for cellular automata over Abelian groups: therefore, further study of the question should then consider either special subcases of the classical case or cellular automata over non-Abelian groups. In the last case, the most interesting groups are perhaps the free groups with two or more (but still finitely many) generators.

Chapter 6

Second-order systems

All the dynamics examined in the previous chapters are *first-order*: the next state is completely determined by the current one. This is analogous to a system of first-order differential equations, because if time is discrete, then knowing a velocity from a position, is the same as knowing a position from another position.
However, many real-world phenomena are *second-order*: first and foremost, the second principle of dynamics. It is then interesting to do with dynamical systems the same thing that is done with systems of differential equations, that is, considering orders greater than the first.
The aim of this section is to present second-order dynamical systems and second-order cellular automata inside the same formalism.

6.1 Second-order dynamical systems

It is natural to consider second-order evolution functions as maps transforming pairs of phases into phases. Second-order dynamical systems are now defined, their main properties shown, and links with "canonical" dynamical systems (called *first-order* from now on) found.

Definition 6.1.1 *A* second-order dynamical system *is a pair* (X, F) *where* X *is a compact metrizable space and* $F : X \times X \to X$ *is a continuous function.*
The set X *is called the* phase space *of the second-order dynamical system.*
The map F *is called the* transition function *of the second-order dynamical system.*

The definition of invertibility is somewhat peculiar. Indeed, defining a second-order dynamical system (X, F) to be invertible if and only if F is invertible, would imply that no second-order dynamical system with $1 < |X| < \infty$ can be invertible: this would be an undesirable feature, so it is necessary to change definitions a little.
A first-order dynamical system (X, F) is a structure that allows to define sequences $\{x^t\}$ by means of a recurrence of the form $x^{t+1} = F(x^t)$: since "invertibility" means "being able to go backwards in time", saying that (X, F) is invertible, means that it is always possible to reconstruct x^t from x^{t+1} by means of another recurrence relation of the form $x^t = G(x^{t+1})$.
A second-order dynamical system does a similar thing, but using two data instead of one: that is, the recurrence has the form $x^{t+1} = F(x^t, x^{t-1})$. Again, since "invertibility" means "being able to go backwards in time", saying that a second-order dynamical system is invertible, means that it must be possible to reconstruct x^{t-1} from x^t and x^{t+1}: that is, a recurrence relation of the form $x^{t-1} = G(x^t, x^{t+1})$ must exist. But switching from the "forward" dynamics to the "backward" one, only swaps the roles *of* x^{t-1} *and* x^{t+1}: that is, while an invertible first-order dynamical system swaps *the future and the present*, a second-order invertible dynamical system swaps *the future and the past*, and the role of the present in this swapping is only *to influence the way the swapping occurs*.
Therefore, the right definition of invertibility for a second-order dynamical system must be the following one:

Definition 6.1.2 *A second-order dynamical system* (X, F) *is* invertible *if, for every* $\bar{x} \in X$*, the map* $x \mapsto F(\bar{x}, x)$ *is invertible.*

In other words, a second-order dynamical system is invertible if its transition function is a permutation of its second argument, parameterized by its first argument.
Actually, one wants to be able to consider second-order dynamical systems as special cases of first-order dynamical systems, exactly as a system of second-order differential equations can be transformed in a system of first-order differential equations. This is performed as follows:

Definition 6.1.3 *Let* (X, F) *be a second-order dynamical system. The* first-order transform*, or briefly* 1-transform*, of* (X, F) *is the first-order dynamical system* $(X \times X, F^\star)$*, where* $F^\star(x_1, x_2) = (F(x_1, x_2), x_1)$*.*

The 1-transform is the tool that allows to consider second-order dynamical systems as if they were first-order: this is similar to what happens with differential equations, where a system of N second-order equations can always be thought of as a system of $2N$ first-order equations.

Proposition 6.1.4 *Let (X, F) be a second-order dynamical system. Then (X, F) is invertible if and only if its 1-transform is invertible.*

Proof:
Suppose that (X, F) is invertible.
Let $(x_1, x_2) \in X \times X$: there exists x_3 such that $x_1 = F(x_2, x_3)$. But then, $(x_1, x_2) = F^\star(x_2, x_3)$: hence $F^\star$ is surjective.
Let $F^\star(x_1, x_2) = F^\star(x_1', x_2')$: then $x_1 = x_1'$ and $F(x_1, x_2) = F(x_1', x_2')$. Since $x_1 = x_1'$ and $x \mapsto F(x_1, x)$ is invertible, $x_2 = x_2'$. Hence $F^\star$ is injective.
Suppose now that $(X \times X, F^\star)$ is invertible.
Fix $\bar{x} \in X$.
Let $x \in X$: there exists $x_1, x_2 \in X$ such that $F^\star(x_1, x_2) = (x, \bar{x})$, but this implies $x_1 = \bar{x}$ and $F(\bar{x}, x_2) = F(x_1, x_2) = x$: therefore the map $x \mapsto F(\bar{x}, x)$ is surjective.
Let $F(\bar{x}, x_1) = F(\bar{x}, x_2)$: then $(F(\bar{x}, x_1), \bar{x}) = (F(\bar{x}, x_2), \bar{x})$, that is, $F^\star(\bar{x}, x_1) = F^\star(\bar{x}, x_2)$: this implies $x_1 = x_2$. Hence $x \mapsto F(\bar{x}, x)$ is injective.
From the arbitrariness of $\bar{x}$ follows that (X, F) is invertible. $\square$
The aim is now to find a way to see when two second order dynamical systems essentially describe the same dynamics. When dealing with first-order dynamical systems, this was performed by employing conjugacies: now, together with a second-order dynamical system, its 1-transform is also present, and it can be useful to consider the *two* possible meanings of the phrase "describing the same dynamics".
Recall that, if $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, the *tensor product* $f_1 \otimes f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is defined by $(f_1 \otimes f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$.

Definition 6.1.5 *Two second-order dynamical systems (X, F), (X', F') are* (strongly) conjugate *if there exists a homeomorphism $\vartheta : X \to X'$ such that $\vartheta \circ F = F' \circ (\vartheta \otimes \vartheta)$. The map ϑ is called a* (strong) conjugacy *from (X, F) to (X', F').*

Definition 6.1.6 *Two second-order dynamical systems (X, F), (X', F') are* weakly conjugate *if their 1-transforms are conjugate.*

The reason for talking of "weak" and "strong" conjugacies is given by:

Proposition 6.1.7 *The 1-transforms of two conjugate second-order dynamical systems are conjugate.*

In other words, two conjugate second-order dynamical systems are weakly conjugate.
Proof:
The trick is to prove that, if ϑ be a conjugacy from (X, F) to (X', F'), then $\vartheta \otimes \vartheta$ is a conjugacy from $(X \times X, F^\star)$ to $(X' \times X', (F')^\star)$.
Now, $\vartheta \otimes \vartheta$ is continuous by construction, and $(\vartheta^{-1}) \otimes (\vartheta^{-1})$ is its inverse; since $X \times X$ is both compact and metrizable, $\vartheta \otimes \vartheta$ is a homeomorphism.
Let now $(x_1, x_2) \in X \times X$: then:

$$\begin{aligned}((\vartheta \otimes \vartheta) \circ F^\star)(x_1, x_2) &= (\vartheta \otimes \vartheta)(F(x_1, x_2), x_1) \\ &= ((\vartheta \circ F)(x_1, x_2), \vartheta(x_1)) \\ &= ((F' \circ (\vartheta \otimes \vartheta))(x_1, x_2), \vartheta(x_1)) \\ &= (F'(\vartheta(x_1), \vartheta(x_2)), \vartheta(x_1)) \\ &= (F')^\star(\vartheta(x_1), \vartheta(x_2)) \\ &= ((F')^\star \circ (\vartheta \otimes \vartheta))(x_1, x_2)\end{aligned}$$

From the arbitrariness of x_1 and x_2 follows that $\vartheta \otimes \vartheta$ is a conjugacy. □
The problem if two weakly conjugate dynamical systems are also strongly conjugate, is much harder. This happens because, if $|X| > 1$, there are maps $\tau : X \times X \to X \times X$ that do not have the form $\tau = \vartheta \otimes \vartheta$ for some $\vartheta : X \to X$: just think to $\tau(x_1, x_2) = (x_2, x_1)$. Anyway, invertibility is not affected by this problem.

Proposition 6.1.8 *Two weakly conjugate second-order dynamical systems are either both invertible or both noninvertible.*

Proof:
A second-order dynamical system is invertible if and only if its 1-transform is invertible. But any two conjugate first-order dynamical systems are either both invertible or both noninvertible. □
One can now make the following reasoning: A first-order dynamical system can be thought of as a second-order dynamical system whose transition function does not depend on its second component. This "faking" of a first-order dynamical system into a first-order one is tempting; but it is also pointless, just like talking of a second-order differential equation that only depends on

first-order terms, or a second-degree polynomial of the form $p(x) = bx + c$: it is nothing more than a fake. Indeed, one cannot hope that properties of a first-order dynamical system can be deduced by those of its "faking" as a second-order dynamical system: this is especially true for invertibility.

Proposition 6.1.9 *Let (X, F) be a second-order dynamical system such that $F(x_1, x_2) = G(x_1)$ for some $G : X \to X$. If (X, F) is invertible then $|X| = 1$.*

Proof:
If X has two distinct elements x_1, x_2, then the map $x \mapsto F(x_1, x)$ takes the same value $G(x_1)$ on both x_1 and x_2, and cannot be invertible. □
Since 1-transform allows to see second-order dynamics as they were first-order, it also allows to define conjugacies between second-order and first-order dynamical systems.

Definition 6.1.10 *A second-order dynamical system (X, F) and a first-order dynamical system (X', F') are* conjugate *if the 1-transform of (X, F) is conjugate to (X', F').*

6.2 Second-order cellular automata

At the aim of defining second-order cellular automata, the idea is: the elements of the phase space should be presented as configurations over a finitely generated group, in a way such that what happens in a point of the group, must only depend on its surroundings. However, more information is now taken into account: precisely, the *next* state of a point must depend both on the *current* state of its *surroundings*, and on the *previous* state of *the point itself.*

Definition 6.2.1 *Let A be an alphabet. Let G be a finitely generated group. A* second-order cellular automaton *with alphabet A* and tessellation group *G is a triple $\langle X, \mathcal{N}, f \rangle$ where:*

1. *X is a shift subspace of A^G;*

2. *$\mathcal{N}$ is a finite subset of G;*

3. *$f : A^{\mathcal{N}} \times A \to A$ is such that the function $F : A^G \times A^G \to A^G$ defined by:*

$$(F(c_1, c_2))_g = f\left(\langle (c_1)_h \rangle_{h \in g\mathcal{N}}, (c_2)_g\right) \tag{6.1}$$

satisfies $F(X \times X) \subseteq X$.

The shift subspace X is called the support *of the second-order cellular automaton. The function f is called the* local transition function *of the second-order cellular automaton. The function F defined by (6.1) is called the* global transition function *of the second-order cellular automaton.*

As for first-order cellular automata, if $\mathcal{N} = D^G_{r,S}$ for some $r \geq 0$ and finite set of generators S for G, the writing $\langle X, S, r, f \rangle$ can be used instead of $\left\langle X, D^G_{r,S}, f \right\rangle$. The maximum length of an element of $\mathcal{N}$ is called the *range* of the second-order cellular automaton.

Definition 6.2.2 *Let $\langle X, \mathcal{N}, f \rangle$ be a second-order cellular automaton. The* associate second-order dynamical system *is the pair (X, F) where F is defined by (6.1).*

As for second-order dynamical systems, the 1-transform of a second-order cellular automaton must be defined. However, a problem arises: the phase space of the 1-transform of a dynamical system is a product of two phase spaces, while the configuration space of a first-order cellular automaton must be a shift subspace; but in general, $A^G \times A^G$ is *not* a configuration space, so one cannot just take $X \times X$ as a support space.
Observe however that, after all, $A^G \times A^G$ is not very different from $(A \times A)^G$: the former is made of pairs of maps from G to A, while the latter is made of maps from G to pairs of elements to A. So perhaps there is a way to overcome this little problem.

Proposition 6.2.3 *Let A be an alphabet. Let G be a finitely generated group. The map $\varphi_{A,G} : (A^G \times A^G) \to (A \times A)^G$ defined by:*

$$(\varphi_{A,G}(c_1, c_2))_g = ((c_1)_g, (c_2)_g) \;\; \forall g \in G \;\; \forall c_1, c_2 \in X \tag{6.2}$$

is a homeomorphism.

Proof:
The map $\varphi_{A,G}$ is invertible, its inverse $\varphi^{-1}_{A,G}$ being given by the equations:

$$\begin{aligned} ((\pi_1 \circ \varphi^{-1}_{A,G})(c))_g &= (c_g)_1 \\ ((\pi_2 \circ \varphi^{-1}_{A,G})(c))_g &= (c_g)_2 \end{aligned}$$

for all $g \in G$, $c \in Y$. The map $\varphi_{A,G}$ is also continuous, because if c_1 and c_1' agree on $D^G_{r,S}$ and c_2 and c_2' do the same, then $\varphi_{A,G}(c_1, c_2)$ and $\varphi_{A,G}(c_1', c_2')$ agree over $D^G_{r,S}$. Since $A^G \times A^G$ is compact and $(A \times A)^G$ is metrizable, $\varphi^{-1}_{A,G}$ is continuous too. □

The function defined by (6.2) will be called the *natural homeomorphism*. Observe that it has an important "commutation" property.

Lemma 6.2.4 *Let A, be an alphabet, G a f.g. group, $g \in G$. Then:*

1. *for every $c_1, c_2 \in A^G$:*

$$\varphi_{A,G}(c_1^g, c_2^g) = (\varphi_{A,G}(c_1, c_2))^g$$

2. *for every $c \in (A \times A)^G$ and $i = 1, 2$:*

$$(\pi_i \circ \varphi^{-1}_{A,G})(c^g) = ((\pi_i \circ \varphi^{-1}_{A,G})(c))^g$$

Proof:
Let $h \in G$: then:

$$\begin{aligned}
(\varphi_{A,G}(c_1^g, c_2^g))_h &= ((c_1^g)_h, (c_2^g)_h) \\
&= ((c_1)_{gh}, (c_2)_{gh}) \\
&= (\varphi_{A,G}(c_1, c_2))_{gh} \\
&= ((\varphi_{A,G}(c_1, c_2))^g)_h
\end{aligned}$$

and:

$$\begin{aligned}
((\pi_i \circ \varphi^{-1}_{A,G})(c^g))_h &= ((c^g)_h)_i \\
&= (c_{gh})_i \\
&= ((\pi_i \circ \varphi^{-1}_{A,G})(c))_{gh} \\
&= (((\pi_i \circ \varphi^{-1}_{A,G})(c))^g)_h
\end{aligned}$$

From the arbitrariness of h the thesis follows. □

Coming back to the original problem, the change of viewpoint defined by $\varphi_{A,G}$ behaves very well with respect to shift subspaces.

Proposition 6.2.5 *Let A be an alphabet. Let G be a finitely generated group. Let $\varphi_{A,G}$ be defined by (6.2). Let $X \subseteq A^G$ and let $Y = \varphi_{A,G}(X \times X)$*

1. *If X is a shift subspace, then Y is a shift subspace.*
2. *If X is a shift of finite type, then Y is a shift of finite type.*
3. *If X is a sofic shift, then Y is a sofic shift.*

Proof:
(1.) Let X be a shift subspace: then $X \times X$ is closed in $A^G \times A^G$, so $Y = \varphi_{A,G}(X \times X)$ is closed in $(A \times A)^G$. Every element of Y has the form $c = \varphi_{A,G}(c_1, c_2)$ with $c_1, c_2 \in X$: by Lemma 6.2.4, for every $g \in G$ the element $c^g = (\varphi_{A,G}(c_1, c_2))^g = \varphi_{A,G}(c_1^g, c_2^g)$ is still in Y, because X is a shift subspace.
(2.) Suppose $X = \mathsf{X}_{\mathcal{F}}$ for a set $\mathcal{F} \subseteq A^{D^G_{M,S}}$. Then $c' = (c_1, c_2) \in (A \times A)^G$ belongs to Y if and only if no pattern of $\mathcal{F}$ occurs in either c_1 or c_2: this is the same as saying that no pattern in:

$$\mathcal{F}' = \varphi_{A,G}((\mathcal{F} \times A^{D^G_{M,S}}) \cup (A^{D^G_{M,S}} \times \mathcal{F}))$$

occurs in c'. Therefore $Y = \mathsf{X}_{\mathcal{F}'}$ is a shift of finite type.
(3.) Suppose that there exist a shift of finite type $Y \subseteq A^G$ and a UL-definable map $F : A^G \to A^G$ such that $F(Y) = X$: by point 2, $Y' = \varphi_{A,G}(Y \times Y) \subseteq (A \times A)^G$ is a shift of finite type. Define $F' = (A \times A)^G \to (A \times A)^G$ as:

$$F' = \varphi_{A,G} \circ (F \otimes F) \circ \varphi_{A,G}^{-1}$$

It is then clear that:

$$F'(Y') = \varphi_{A,G}(F(Y) \times F(Y)) = \varphi_{A,G}(X \times X)$$

Moreover F' is clearly continuous, and from Lemma 6.2.4 and the fact that F is UL-definable follows that F' commutes with the natural action of G over $(A \times A)^G$: hence F' is UL-definable and $X' = F'(Y')$ is sofic. □
The definition of 1-transform for second-order dynamical system can now be stated.

Definition 6.2.6 *Let A be an alphabet. Let G be a finitely generated group. Let $\langle X, \mathcal{N}, f \rangle$ be a second-order cellular automaton with alphabet A and tessellation group G. The* first-order transform*, or simply* 1-transform*, of $\langle X, \mathcal{N}, f \rangle$ is the first-order cellular automaton $\langle \varphi_{A,G}(X \times X), \mathcal{N}, f^\star \rangle$, where $\varphi_{A,G}$ is the natural homeomorphism from $A^G \times A^G$ to $(A \times A)^G$ and $f^\star : (A \times A)^{\mathcal{N}} \to (A \times A)$ is defined by:*

$$f^\star\left(\langle (a_1, a_2)_i \rangle_{i \in \mathcal{N}}\right) = \left(f\left(\langle (a_1)_i \rangle_{i \in \mathcal{N}}, (a_2)_{1_G}\right), (a_1)_{1_G}\right) \tag{6.3}$$

The definition is a "natural" one. Indeed, not only the tessellation group is the same and the alphabet "is what it is expected to be", but two important properties (finiteness of type and soficity) are preserved in the passage to the 1-transform. But there is more: the operations of "presenting as a cellular automaton" and "seeing as it was first-order" *commute* (up to a conjugacy).

Lemma 6.2.7 *Let A be an alphabet. Let G be a finitely generated group. Let $\langle X, \mathcal{N}, f\rangle$ be a second-order cellular automaton with alphabet A and tessellation group G. The 1-transform of the dynamical system associate to $\langle X, \mathcal{N}, f\rangle$ is conjugate to the dynamical system associate to its 1-transform via the natural homeomorphism from $A^G \times A^G$ to $(A \times A)^G$.*

Proof:
Let $\langle X, \mathcal{N}, f\rangle$ be a second-order cellular automaton; let (X, F) be its associate dynamical system, and let $\langle \varphi_{A,G}(X \times X), \mathcal{N}, f^\star\rangle$ be its 1-transform. The 1-transform of (X, F) is $(X \times X, F^\star)$ with:

$$\begin{aligned} ((\pi_1 \circ F^\star)(c_1, c_2))_g &= (F(c_1, c_2))_g \\ &= f\left(\langle (c_1)_h\rangle_{h \in g\mathcal{N}}, (c_2)_g\right) \end{aligned}$$

and:

$$((\pi_2 \circ F^\star)(c_1, c_2))_g = (c_1)_g$$

On the other hand, the associate dynamical system of $\langle \varphi_{A,G}(X \times X), \mathcal{N}, f^\star\rangle$ is $(\varphi_{A,G}(X \times X), \Phi^\star)$, where:

$$\begin{aligned} (\Phi^\star(c))_g &= f^\star\left(\langle c_h\rangle_{h \in g\mathcal{N}}\right) \\ &= \left(f\left(\langle (\pi_1(c))_h\rangle_{h \in g\mathcal{N}}, (\pi_2(c))_g\right), (\pi_1(c))_g\right) \end{aligned}$$

Then, for every $g \in G$:

$$\begin{aligned} ((\Phi^\star \circ \varphi_{A,G})(c_1, c_2))_g &= (\Phi^\star(\varphi_{A,G}(c_1, c_2)))_g \\ &= f^\star\left(\langle (\varphi_{A,G}(c_1, c_2))_h\rangle_{h \in g\mathcal{N}}\right) \\ &= \left(f\left(\langle (c_1)_h\rangle_{h \in g\mathcal{N}}, (c_2)_g\right), (c_1))_g\right) \\ &= \varphi_{A,G}\left(((\pi_1 \circ F^\star)(c_1, c_2))_g, ((\pi_2 \circ F^\star)(c_1, c_2))_g\right) \\ &= ((\varphi_{A,G} \circ F^\star)(c_1, c_2))_g \end{aligned}$$

Therefore $\varphi_{A,G}$ is a conjugacy from $(X \times X, F^\star)$ to $(\varphi_{A,G}(X \times X), \Phi^\star)$. □
To employ second-order cellular automata as presentations for second-order dynamical systems, one must choose, between the two notions of conjugacy introduced, the one that better represents the situation "first transform then evolve, is the same as first evolve then transform": since this transformation must be consistent with dynamics, the right type of conjugacy is the strong one.

Definition 6.2.8 *A second-order dynamical system* (X, F) admits a presentation as a second-order cellular automaton *if there exists a second-order cellular automaton* $\langle Y, \mathcal{N}, f\rangle$ *whose associate second-order dynamical system is strongly conjugate to* (X, F)*.*
The second-order cellular automaton $\langle Y, \mathcal{N}, f\rangle$ *is called a* presentation of (X, F) as a second-order cellular automaton*.*

From Definition 6.2.8, Proposition 6.1.7 and Lemma 6.2.7 follows:

Proposition 6.2.9 *If a second-order dynamical system has a presentation as a second-order cellular automaton, then its 1-transform has a presentation as a first-order cellular automaton.*

Proof:
Let (X, F) be a second-order dynamical system and let $\langle Y, \mathcal{N}, f\rangle$ be a presentation of (X, F) as a second-order cellular automaton, whose global evolution function is indicated by F_f. Let ϑ be a strong conjugacy from (X, F) to (Y, F_f): by Proposition 6.1.7, $\vartheta \otimes \vartheta$ is a conjugacy between $(X \times X, F^\star)$ and $(Y \times Y, F_f^\star)$. But by Lemma 6.2.7, $(Y \times Y, F_f^\star)$ is conjugate to the first-order dynamical system associate to the 1-transform of $\langle Y, \mathcal{N}, f\rangle$: this 1-transform is thus a presentation of $(X \times X, F^*)$ as a cellular automaton. □
The definition of invertibility for second-order cellular automata is similar to the one seen in the first-order case.

Definition 6.2.10 *A second-order cellular automaton* $\langle X, \mathcal{N}, f\rangle$ *is* invertible *if its associate second-order dynamical system* (X, F) *is invertible.*

From Lemma 6.2.7 and Proposition 6.1.4 follows:

Corollary 6.2.11 *A second-order cellular automaton is invertible if and only if its 1-transform is invertible.*

But their special structure gives second-order cellular automaton a very important property:

Theorem 6.2.12 *Let $\langle A^G, \mathcal{N}, f\rangle$ be a full second-order cellular automaton. The following are equivalent:*

1. *$\langle A^G, \mathcal{N}, f\rangle$ is invertible;*
2. *for every $\alpha \in A^{\mathcal{N}}$, the map $a \mapsto f(\alpha, a)$ is invertible.*

In other words: a second-order cellular automaton is invertible if and only if its local evolution function is a permutation of the past state of the cell parameterized by the present state of its neighborhood.
Proof:
($2 \Rightarrow 1$) If point 2 is satisfied, fix $\overline{c} \in A^G$ and consider the map $F_{\overline{c}} : A^G \to A^G$ defined by $F_{\overline{c}}(c) = F(\overline{c}, c)$: if $c' = F_{\overline{c}}(c)$, then for every $g \in G$:

$$(c')_g = f\left(\langle(\overline{c})_h\rangle_{h \in g\mathcal{N}}, c_g\right) = f_{\langle(\overline{c})_h\rangle_{h \in g\mathcal{N}}}(c_g)$$

Then, from condition 2 follows:

$$c_g = \left(f_{\langle(\overline{c})_h\rangle_{h \in g\mathcal{N}}}\right)^{-1}((c')_g)$$

that is, c is uniquely determined by c'. From the arbitrariness of $\overline{c}$ follows that $\langle A^G, \mathcal{N}, f\rangle$ is invertible.
($1 \Rightarrow 2$) If point 2 is not satisfied, then there exists $a_1, a_2 \in A$ and $\alpha \in A^{\mathcal{N}}$ such that:

$$f(\alpha, a_1) = f(\alpha, a_2)$$

Let $\overline{c} \in A^G$ be such that $\overline{c}_{|D^G_{r,S}} = \alpha$; let c_1 and c_2 be two configurations such that $(c_1)_{1_G} = a_1$, $(c_2)_{1_G} = a_2$, and $(c_1)_g = (c_2)_g$ for every $g \neq 1_G$. Then $F(\overline{c}, c_1) = F(\overline{c}, c_2)$ with $c_1 \neq c_2$, so the second-order cellular automaton is not invertible. □
This leads to the result of [31]:

Corollary 6.2.13 *For full second-order cellular automata, invertibility is decidable.*

Proof:
From Theorem 6.2.12 follows that, to check the invertibility of a full second-order CA, one only needs to check if a finite number of finitary maps is invertible, which is a decidable problem. □

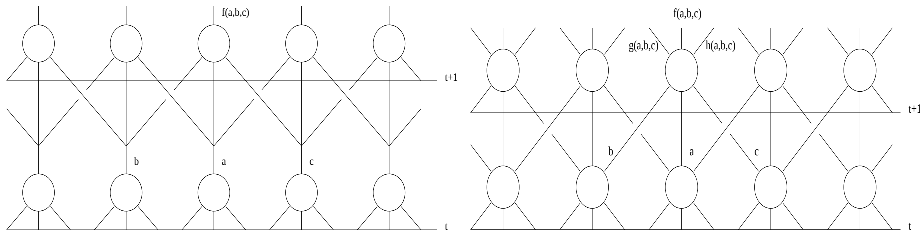

Figure 6.1: A schematic representation of a cellular automaton (left) and a lattice gas (right) as function composition schemes.

6.3 A structural phenomenon in Euclidean space

Theorems 4.1.10 and 6.2.12 are very similar in shape: they both assert, each in its range of validity, that for dynamical systems having a very special structure, invertibility is a structural property. This is a very strong similarity between full second-order cellular automata and full lattice gases, and suggests the existence of a stronger link than that with ordinary cellular automata. Such a link does actually exist, at least in the Euclidean case.

Remark that a full first-order cellular automaton with tessellation group G can be thought of as a special type of *function composition scheme*, having enough regularity to be represented as an infinite "circuits" whose "switches" are on the node of $G \times \mathbb{Z}$ and whose "wires" connect the neighbors of a node at time t to the node itself at time $t+1$, and having the additional constraint that all signals exiting a switch must be equal — or equivalently, that every switch has a single output, that is later replicated.
A similar argument works for lattice gases, which show an additional property: *signal conservation*. In detail: each switch must have as many outputs as inputs, and every signal can be used only once as an input to a switch.

Let $G = \mathbb{Z}^d$ and let $\mathcal{A} = \left\langle A^G, \mathcal{N}, f \right\rangle$ be a second-order cellular automaton. It is not restrictive to suppose that $\mathcal{N}$ is a parallelepiped whose sides are parallel to the main axes of the space: in this case, $\mathcal{N}$ tiles the space, and there is a *subgroup* $H \subseteq \mathbb{Z}^d$ such that $\mathbb{Z}^d = H + \mathcal{N} = \{h+n, h \in H, n \in \mathcal{N}\}$: we observe that it must be $H = \bigoplus_{1 \leq i \leq d} N_i \mathbb{Z} \cong \mathbb{Z}^d$. (Direct sum is used

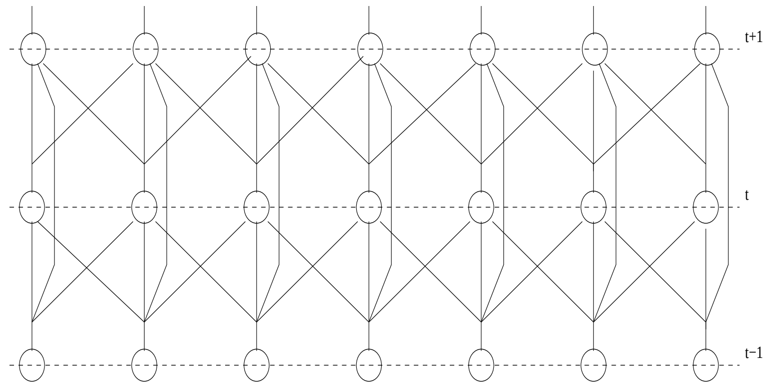

Figure 6.2: A representation of a second-order cellular automaton as a graph. Arcs exiting from the same node are written as having the first part in common, to stress that the value assigned to each of them is the same.

in place of direct product, because in Abelian groups the product is often indicated by a plus symbol; the meaning is the same.)
We can represent $\mathcal{A}$ as follows: consider a *directed* graph whose nodes are the point of $\mathbb{Z}^{d+1}$ and whose arcs obey the following rules:

1. if y is a neighbor of x w.r.t. $\mathcal{N}$, then there is an arc from (y, t) to $(x, t+1)$ for all $t \in \mathbb{Z}$;

2. for all $t \in \mathbb{Z}$, there is an arc from (x, t) to $(x, t+2)$;

3. there is no other arc.

This is a *space-time* representation of $\mathcal{A}$, in the sense that we can reproduce *any* evolution of the dynamical system associate to $\mathcal{A}$, simply by assigning a set of values c_x^t from A to the arcs *exiting* from nodes at times $t = 0$ and $t = 1$ (with arcs exiting from the same node having the same value), and doing the same on each node at time $t > 1$ with the rule that the common value of the arcs exiting from node x at time t, must equal $f\left(\left\langle c_y^{t-1}\right\rangle_{y \in x+\mathcal{N}}, c_x^{t-2}\right)$. After that, it is not important that the t-th update of a cell really occurs at time t: it can happen at *any time not beofore* t *and before* $t+1$, and we still have a representation of $\mathcal{A}$. So we do as such: we enumerate $\mathcal{N} = \{n_0, \ldots, n_{N-1}\}$ (where $N = N_1 N_2 \ldots N_d$) and for all $g \in \mathbb{Z}$, if $g = h + n_i$ with $h \in H$, point g is marked with label i *and evolves at times* $t + i/N$. Shortly, a single time

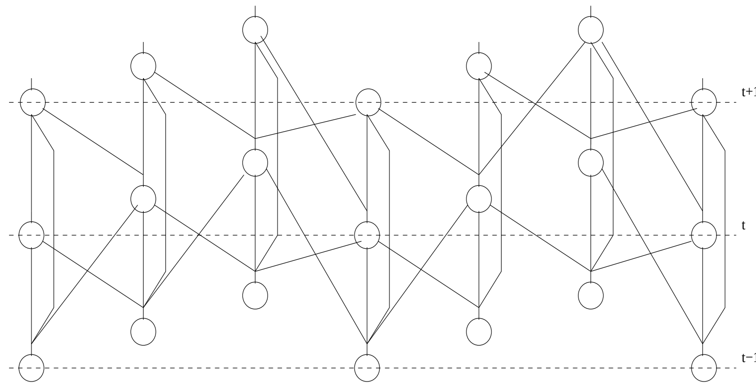

Figure 6.3: Implementation of the time slack allowed by second-order structure. This is again a representation, if we identify the state at time t with the collection of the "output" values from nodes at time $\tau \in [t, t+1)$.

step is divided into N partial time steps.
At this point, there is no need for replicated signals: actually, *one* signal generated by node x at time $t + i/N$, can pass once by each node that have x as a neighbor between times $t + 1$ (included) and $t + 2$ (excluded) (this is possible because of the introduced time slack, which in turn is allowed by second order), before arriving at node x at time $t + 2 + i/N$, where it is processed.
Now, instead of having "small" cells in a "fine" group, the structure has "large" cells in a "coarse" group: but the behavior is the same, because the "state" of a macrocell is the collection of signals that, in each cell, *are spawned* and *have been spawned from the same cell at the previous time*, and there is a one-to-one correspondance both in *state* and *dynamics*. (And actually, this is a representation of the dynamical system associated to the 1-transform of $\mathcal{A}$.) But this is a system where each processing element has as many inputs as outputs, and signal is never replicated: in other words, a lattice gas. We conclude by observing that the configuration space of the new system, is basically a rearrangement of the old one's: in particular, it is a full shift.

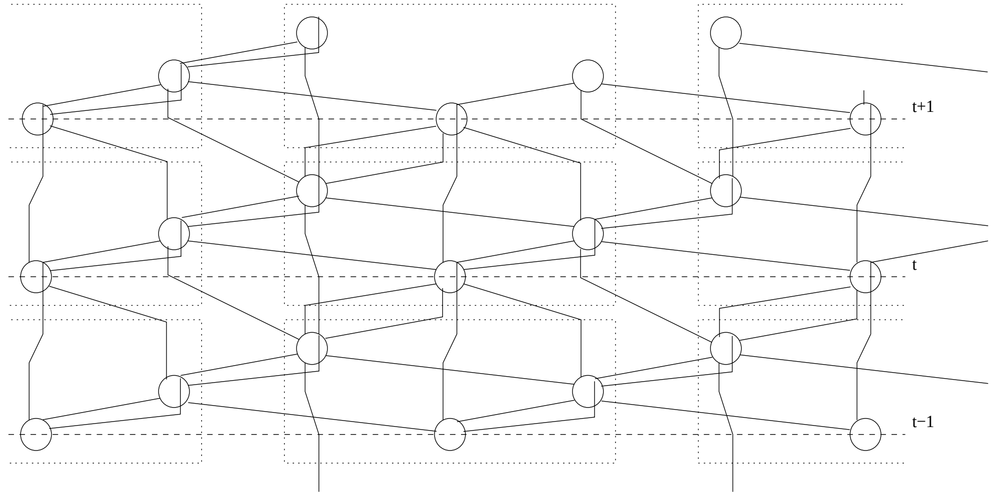

Figure 6.4: The time slack actually allows a single copy of the signal to pass through each node it must. Dotted lines remark macrocells.

Chapter 7

Conclusions

The formalization of cellular automata by means of group theory and symbolic dynamics not only allows to restate old results in a new way, but also solves many problems, both old and new.
First of all, it gives a context where an old conjecture on dynamical systems is true: this sentence appears in a work of Toffoli from the first Eighties, and asserts that commutation and discernibility with respect to an action of $\mathbb{Z}^d$, were sufficient to have a presentation as a cellular automaton — without specifying whether it had to be full or not. Actually, it is the possibility to choose incomplete configuration spaces as support of the cellular automaton, that yields the greater freedom one needs to prove Theorem 3.2.8.
This same fact also allows to see cellular automata and lattice gases as two sides of the same coin: this is the content of Theorem 4.2.3. This time, the greater freedom (and consequently, the more relaxed discipline) in the choice of the support space, actually spoils the effect of the stronger structural requirements a lattice gas must satisfy.
Moreover, at least under some hypotheses, a part of the complexity can be transferred from the group to the alphabet without altering the underlying dynamics, and this can be done in a computable way: the theorems in Chapter 5 allow to extend results for cellular automata with a given tessellation group, to cellular automata with a slightly more complicated tessellation group.
Finally, the formalism is also well suited in the study of the second-order case, which is of great interest because of the strong resemblance with real-world phenomena; Lemma 6.2.7 allows to swap the operations of "interpreting as first-order" and "presenting as a cellular automaton", and the greater dis-

cipline imposed by second order fetches an invertibility result similar to the one for lattice gases.

Of course, many interesting open problems remain:

1. find additional conditions to characterize the dynamical systems that allow a presentation as full cellular automata;

2. prove the equivalence between full cellular automata and full lattice gases, or find a counterexample;

3. find non-Abelian groups that are not semi-direct products with $\mathbb{Z}$ and where the invertibility problem is decidable;

4. study cases when the cardinality of the alphabet can be increased or reduced without altering the dynamics;

5. understand which properties of $\mathbb{Z}^d$ allow the construction of Section 6.3, so that it can be done on every group that satisfies them.

The use of tools from group theory and symbolic dynamics, allowed from the current formalization, will be of great importance in the study of these problems, as it has been in the present work.

Bibliography

[1] S. Amoroso, Y. N. Patt, Decision procedures for surjectivity and injectivity of parallel maps for tessellation structures, *J. Comp. Syst. Sci.* **6** (1972) 448–464

[2] C. Bennett, N. Margolus, T. Toffoli, Bond-energy variables for Ising spin-glass dynamics, *Phys. Rev. B* **37** (1988) 2254

[3] S. Capobianco, Cellular automata over groups with finite parts and decidability of the invertibility problem, Preprint n. 41/2003, Dipartimento di Matematica, Università di Roma "La Sapienza", October 2003

[4] T. G. Ceccherini-Silberstein, A. Machì, F. Scarabotti, Amenable groups and cellular automata, *Ann. Inst. Fourier, Grenoble* **42** (1999) 673–685

[5] A. Clementi, On the complexity of cellular automata, Ph.D. Thesis, Dipartimento di Informatica, Università di Roma "La Sapienza", 1994

[6] A. Clementi, P. Mentrasti, P. Pierini, Some results on invertible cellular automata, *Complex Systems* **9** (1995) 235–250

[7] J. Durand-Lose, Representing reversible cellular automata with reversible block cellular automata, in R. Cori, J. Mazoyer, M. Morvan, and R. Mosery eds., *Discrete models, combinatorics, computation and geometry (DM-CCG'01)*, volume AA, Discrete Mathematics and Theor. Comput. Sci., 2001

[8] F. Fiorenzi, Cellular automata and finitely generated groups, *Tesi di Dottorato, Dipartimento di Matematica "Guido Castelnuovo", Università degli Studi di Roma "La Sapienza", 2000*

[9] E. Formenti, Cellular automata and chaos: from topology to algorithmic complexity, *Ph.D. Thesis, Laboratoire de l'Informatique du Parallélisme, ENS Lyon, 1998*

[10] E. Fredkin, T. Toffoli, Conservative logic, *Int. J. Theor. Phys.* **21** (1982) 219–253

[11] U. Frisch, B. Hasslacher, Y. Pomeau, Lattice-gas automata for the Navier-Stokes equation, *Phys. Rev. Lett.* **56** (1986) 1505–1508

[12] E. Goles, G. Vichniac, Invariants in automata networks, *J.Phys. A: Math* **19** (1986) L961–L965

[13] G. A. Hedlund, Endomorphisms and automorphisms of the shift dynamical system, *Math. Syst. Th.* **3** (1969) 320–375

[14] A. Hof, O. Knill, Cellular automata with almost periodic initial conditions, *Nonlinearity* **8** (1995) 477–491

[15] J. Hardy, O. de Pazzis, Y. Pomeau, Molecular dynamics of a classical lattice gas: Transport properties and time correlation functions, *Phys. Rev.* **A18** (1976) 1949–1960

[16] G. Jacopini, P. Mentrasti, Generation of invertible functions, *Theor. Comput. Sci.* **66** (1989) 289-287

[17] G. Jacopini, P. Mentrasti, G. Sontacchi, Reversible Turing machines and polynomial time reversibly computable functions, *SIAM J. Disc. Math.* **3** (1990) 241–254

[18] J. Kari, Reversibility of 2D cellular automata is undecidable, *Physica D* **45** (1990) 379–385

[19] D. Lind, B. Marcus, An introduction to symbolic dynamics and coding, Cambridge University Press (1995)

[20] A. Machì, F. Mignosi, Garden of Eden configurations for cellular automata on Cayley graphs on groups, *SIAM J. Disc. Math.* **6** (1993) 44–56

[21] E. F. Moore, Machine models of self-reproduction, *Proc. Symp. Appl. Math.* **14** (1962) 17–33

[22] J. Myhill, The converse of Moore's Garden-of-Eden theorem, *Proc. Am. Math. Soc.* **14** (1962) 685–686

[23] Y. Pomeau, Invariant in cellular automata, *J. Phys.* **A17** (1984) L415–L418

[24] D. Richardson, Tessellations with local transformations, *J. Comp. Syst. Sci.* **6** (1972) 373–388

[25] J. S. Rose, A course in Group Theory, Cambridge University Press 1978

[26] J. J. Rotman, An Introduction to the Theory of Groups, *Graduate Texts in Mathematics* **148**, Springer-Verlag, New York 1995

[27] T. Sato, Decidability for some problems of linear cellular automata over finite commutative rings, *Inf. Proc. Lett.* **46** (1993) 151–155

[28] V. Terrier, Two-dimensional cellular automata and their neighborhoods, *Theor. Comput. Sci.* **312** (2004) 203–222

[29] T. Toffoli, Cellular automata mechanics, *Ph.D. Thesis, Technical Report No. 208, University of Michigan, 1977*

[30] T. Toffoli, Computation and construction universality in reversible cellular automata, *J. Comp. Syst. Sci.* **15** (1977) 213–231

[31] T. Toffoli, S. Capobianco, P. Mentrasti, How to turn a second-order cellular automaton into a lattice gas: A new inversion scheme, *Theor. Comput. Sci.* **325** (2004) 329–344

[32] T. Toffoli, N. Margolus, Invertible cellular automata: A review, *Physica D* **45** (1990) 229–253

[33] D. Wolf-Gladrow, Lattice Gas Cellular Automata and Lattice Boltzmann models: An introduction, *Lecture Notes in Mathematics* 1725, Springer 2000