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Spacetime asymptotics for Schrödinger Equations

Tesi di Dottorato

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Introduction

The Schrödinger equation appears in a large number of different physical and mathematical contexts. It was formulated in 1926 by the Austrian physicist Erwin Schrödinger: in [76] he defined the concept of wave function, solution to an appropriate partial differential equation, in order to describe how the state of a physical system changes with time in Quantum Mechanics.

The nonlinear Schrödinger equation (later on NLS) is a nonlinear variation of the Schrödinger equation. It is a classical field equation whose principal applications are to the propagation of light in nonlinear optical fibers and planar waveguides and to Bose-Einstein condensates confined to highly anisotropic cigar-shaped traps, in the mean-field regime. Additionally, the equation appears in the studies of small-amplitude gravity waves on the surface of deep inviscid (zero-viscosity) water; the Langmuir waves in hot plasmas; the propagation of plane-diffracted wave beams in the focusing regions of the ionosphere; the propagation of Davydov's alpha-helix solitons, which are responsible for energy transport along molecular chains; the evolution of vortex filaments; and many others. More generally, the NLS appears as one of universal equations that describe the evolution of slowly varying packets of quasi-monochromatic waves in weakly nonlinear media that have dispersion.

The Schrödinger equation is one of the principal examples of *dispersive equations*: this category is tranversal with respect to the usual classification in hyperbolic, parabolic and elliptic partial differential equations, and shows in its equations some peculiar and common behaviours, linked to *dispersion* properties. Let us consider a plane wave function

$$u(t,x) = Ae^{i\xi\cdot x}e^{i\omega t}, \tag{0.0.1}$$

where A is the amplitude of the solution, ξ is the wave number and ω is the frequency, and a general partial differential equation

$$\partial_t + ih(D)u = 0, \tag{0.0.2}$$

with $h(D) = \mathcal{F}^{-1}(h(\xi)\mathcal{F})$, where $\mathcal{F}$ is the Fourier transform with respect to the spatial variable x and $D = -i\nabla$. We see that a plane wave function u as in (0.0.1) is a solution of (0.0.2) if and only if the following *dispersion relation* hold:

$$\omega = -h(\xi) \tag{0.0.3}$$

that is $u(t,x) = Ae^{i\xi\cdot(x+h(\xi)\xi/|\xi|^2 t)}$: we define its *phase velocity* is $c_p(\xi) := \omega\xi/|\xi|^2$ and its *group velocity* is $c_g(\xi) := -\nabla h(\xi)$. The equation (0.0.2) is *dispersive* if $\nabla_\xi\omega$ is not constant. As said before, the linear Schrödinger equation

$$i\partial_t u - \Delta u = 0 \tag{0.0.4}$$

is a dispersive equation as in (0.0.2), with $h(\xi) = |\xi|^2$: each plane wave solution translates with group velocity $c_g(\xi) := -\nabla h(\xi) = -2\xi$.

For simplicity, we consider here a nonlinear Schrödinger equation with a power type nonlinearity in $\mathbb{R}^n$:

$$\begin{cases} i\partial_t u - \Delta u + |u|^{p-1}u = 0, \\ u(0,\cdot) = u_0. \end{cases} \tag{0.0.5}$$

It is well known (we refer to the book [13] and to Chapters 1 and 2) that for $1 \leqslant p < 1 + 4/(n-2)$ the initial value problem (0.0.5) admits a global solution $u \in C(\mathbb{R}, H^1(\mathbb{R}^n))$ for every $u_0 \in H^1(\mathbb{R}^n)$.

The study of *scattering* for solutions to the nonlinear Schrödinger equation (0.0.5) consists in understanding the asymptotic behaviour of the solutions u to (0.0.5) for time $t \to \pm\infty$. Intuitively, if for big time the solution is "small", then the $|u|^{p-1}u$ term will be even smaller: it will be possible to neglect this term and solutions to the linear Schrödinger equation (0.0.4) will be "close" to solutions to the nonlinear equation (0.0.5) for big time. More precisely, we say that *scattering* holds if for every $u_+ \in H^1(\mathbb{R}^n)$ there exists a unique $u_0 \in H^1(\mathbb{R}^n)$ such that, if u is the unique solution to (0.0.5) with u_0 as initial datum, we have

$$\lim_{t\to+\infty} \left\| u(t,\cdot) - e^{-it\Delta}u_\pm \right\|_{H^1(\mathbb{R}^n)} = 0.$$

In this case we say that u *scatters* to $e^{-it\Delta}u_+$. We assign Ω_+, the *wave operator* for $t \to \infty$: $\Omega_+ : u_+ \in H^1(\mathbb{R}^n) \mapsto u_0 \in H^1(\mathbb{R}^n)$; if the wave operator is surjective, we say that the equation is *asymptotically complete*. Analogous definitions are given for $t \to -\infty$, defining the *wave operator* Ω_- for $t \to -\infty$. We remark that we are defining the scattering in $H^1(\mathbb{R}^n)$, the energy space for the Schrödinger equation, but analogous definitions can be given in other spaces.

In the case that $1 + 4/n < p < 1 + 4/(n-2)$, scattering and asymptotic completeness hold for the equation (0.0.5): in this case, we define the *scattering operator* $S := \Omega_+^{-1} \circ \Omega_- : u_- \in H^1(\mathbb{R}^n) \mapsto u_+ \in H^1(\mathbb{R}^n)$.

In order to prove scattering, it is necessary to have some control on the behaviour of the solutions, namely to prove *a priori* estimates. This can be accomplished thanks to the dispersive nature of the Schrödinger equation: since any function can be thought as superposition of plane waves, we see from the previous arguments that a solution will spread in many waves of different spacial frequency, each propagating with velocity depending on the wave number. This phenomenon implies decay properties on the solutions of the linear and nonlinear Schrödinger equations, described by means of Strichartz estimates and smoothing estimates.

Strichartz estimates for the Schrödinger equation were introduced by R. Strichartz in [81], as a consequence of Fourier restriction theorems. In the fundamental paper [44] by J. Ginibre and G. Velo, using the so called TT^* argument, they proved Strichartz estimates as consequence of decay estimates. In the paper [52], M. Keel and T. Tao completed the program with the proof of the endpoint estimates.

The natural norms which are considered in this family of estimates are of mixed type, namely we deal with $L^p_t L^q_x$–spaces. If u is a solution of (0.0.4), then the following estimates

$$\left\| e^{-it\Delta} f \right\|_{L^p_t L^q_x} \leqslant C \|f\|_{L^2}$$

hold for any couple (p, q) satisfying the Schrödinger admissibility condition

$$\frac{2}{p} = \frac{n}{2} - \frac{n}{q}, \quad p \geqslant 2, \quad (n, p, q) \neq (2, 2, \infty).$$

Strichartz estimates represent a crucial instrument to perform fixed point arguments in the study of nonlinear problems. One of the first examples of nonlinear application of Strichartz estimates was given in [42] for the NLS.

It is frequent for equations with infinite speed of propagation, such as the Schrödinger equation, that the solution is more regular then the initial data. The gain of derivatives, which is in fact related to the algebraic structure of the equations, is a very interesting fact, and is often a crucial improvement for the nonlinear techniques. The smoothing effect was discovered by T. Kato for the Korteweg-de Vries equation; for the Schrödinger equation, Kato and Yajima in [49] proved the well known inequality

$$\left\| \langle x \rangle^{-\frac{1}{2}-} |D|^{\frac{1}{2}} e^{it\Delta} u \right\|_{L^2_t L^2_x} \leqslant C \|u\|_{L^2};$$

a stronger local version of the previous inequality (see the standard references [20], [78] and [86]) is the following

$$\sup_{R>0} \frac{1}{R} \int_{\mathbb{R}_t} \int_{B_R} |||(\|e^{it\Delta} u)|^2 \, dxdt \leqslant C \|u(0)\|_{\dot{H}^{\frac{1}{2}}},$$

where $|D|^{\frac{1}{2}} = \mathcal{F}^{-1}(|\xi|^{\frac{1}{2}} \mathcal{F})$, B_R is the ball of radius R centered in the origin and $\dot{H}^{\frac{1}{2}}$ is the usual homogeneous Sobolev space with the norm

$$\|f\|_{\dot{H}^{\frac{1}{2}}} = \||D|^{\frac{1}{2}} f\|_{L^2}.$$

One of the techniques used for proving smoothing estimates is the Morawetz multiplier method: namely one multiplies the equation for the appropriate quantity and, after some manipulation and integrations by parts, gets the required control on the solution.

This approach can be used for dealing with nonlinearities in the equation, in order to obtain smoothing estimates for the NLS. This was done the first time by C. Morawetz in [63] for the Klein-Gordon equation with a general nonlinearity and were successively used for proving the asymptotic completeness by Lin and Strauss in [54] and Ginibre and Velo in [43]. Recently, there have been introduced new bilinear smoothing estimates, named interaction of quadratic Morawetz inequalities: specificly Morawetz estimates for two solutions (possibly the same solution taken twice) are computed at once. We quote in this direction the papers [16], [15], [14], [17], [83], [71] and finally the survey [41] (see moreover Chapters 1 and 2).

The dispersive nature of the Schrödinger equation prevents solutions to be spatially concentrated for long time, influencing hence the asymptotic behaviour in the space variable. In detail, the zero solution is the unique solution of the linear Schrödinger equation (0.0.4) that in two different times has a gaussian profile, with variances $\alpha > 0$ and $\beta > 0$, i.e.

$$\left\| e^{\frac{|x|^2}{\alpha^2}} u(0, \cdot) \right\|_{L^2(\mathbb{R}^n)} + \left\| e^{\frac{|x|^2}{\beta^2}} u(T, \cdot) \right\|_{L^2(\mathbb{R}^n)} < +\infty,$$

with $\alpha\beta < 4T$. This property is closely related with the Hardy Uncertainty Principle: *if* $f(x) = O\left(e^{-|x|^2/\beta^2}\right)$ *and its Fourier transform* $\hat{f}(\xi) = O\left(e^{-4|\xi|^2/\alpha^2}\right)$, *then*

$$\alpha\beta < 4 \Rightarrow f \equiv 0$$
$$\alpha\beta = 4 \Rightarrow f \textit{ is a constant multiple of } e^{-\frac{|x|^2}{\beta^2}}.$$

The connection between the two phenomena is suggested by the formula for solutions to the free Schrödinger equation, namely u is a solution to (0.0.4) with initial datum f if

$$u(x,t) := e^{it\Delta}f(x) = (2\pi it)^{-\frac{n}{2}} e^{i\frac{|x|^2}{4t}} \mathcal{F}\left(e^{i\frac{|\cdot|^2}{4t}} f\right)\left(\frac{x}{2t}\right).$$

We see than that the Schrödinger propagator is, apart from a multiplication by a phase, the Fourier Transform. It is possible then, in principle, to get results for the Schrödinger propagator from analogous results on the Hardy Uncertainty Principle.

L. Escauriaza, C. Kenig, G. Ponce, and L. Vega in the sequel of papers [33, 31, 32, 34, 35], and with M. Cowling in [21] have developed this approach and have extended these results to the case of a perturbed linear equation

$$\partial_t u = i(\Delta + V)u, \tag{0.0.6}$$

with $V = V(t,x) \in L^\infty(\mathbb{R}\times\mathbb{R}^n) \cap L^1(\mathbb{R}_t, \dot{H}^1(\mathbb{R}^n))$. Fundamental steps in this program were a proof by means of Real Analysis tools of the Hardy Uncertainty Principle, that allowed the presence of (possibly rough) potentials V in (0.0.6), and a deep understanding of logarithmic convexity properties of Schrödinger evolutions, namely the possibility of estimate weighted norms of a solution in a time interval by means of the weighted norms of the solution in the extreme points of it.

Aim of the thesis and plan of the work

The aim of this PhD work is to examine in depth and generalize the classical Theory on the Schrödinger equation considering variable coefficients perturbations: this is done including the presence of electromagnetic and electrostatic potentials and considering perturbations of the second order term for the nonlinear Scrödinger equation. Moreover, in the case of lower spatial dimension, but without the presence of variable coefficients perturbations, we extend the theory of scattering to the system framework.

Each chapter of this thesis is almost completely self-contained, and consists of a different and independent paper: we give here a rapid outline of the results we have proved, referring to the single introductions for greater details.

In Chapter 1, we prove scattering for a system of weakly coupled Schrödinger equations in dimensions $1, 2$ and 3: we develop new techniques for tackling the problem of scattering in the system framework and in the low dimensional case. The reference for the results in this Chapter is [12].

In Chapter 2, we consider the NLS with variable coefficients in dimension $n \geqslant 3$

$$i\partial_t u - Lu + f(u) = 0, \qquad Lv = \nabla^b \cdot (a(x)\nabla^b v) - c(x)v, \qquad \nabla^b = \nabla + ib(x),$$

on $\mathbb{R}^n$ or more generally on an exterior domain with Dirichlet boundary conditions, for a gauge invariant, defocusing nonlinearity of power type $f(u) \simeq |u|^{\gamma-1}u$. We assume that L is a small, long range perturbation of Δ, plus a potential with a large positive part. The first main result of the paper is a bilinear smoothing (interaction Morawetz) estimate for the solution. As an application, under the conditional assumption that Strichartz estimates are valid for the linear flow e^{itL}, we prove global well posedness in the energy space for subcritical powers $\gamma < 1 + \frac{4}{n-2}$, and scattering provided $\gamma > 1 + \frac{4}{n}$. When the domain is $\mathbb{R}^n$, by extending the Strichartz estimates due to Tataru [84], we prove that the conditional assumption is satisfied and deduce well posedness and scattering in the energy space. The reference for the results in this Chapter is [10].

In Chapter 3, we prove a sharp version of the Hardy uncertainty principle for Schrödinger equations with external bounded electromagnetic potentials, based on logarithmic convexity properties of Schrödinger evolutions. We provide, in addition, an example of a real electromagnetic potential which produces the existence of solutions with critical gaussian decay, at two distinct times. The results in this Chapter are proved in [11].

Notations.

In this thesis we will use often the following notation: given any two positive real numbers a, b, we write $a \lesssim b$ to indicate $a \leqslant Cb$, with $C > 0$, we unfold the constant only when needed.

For any $1 \leqslant r \leqslant \infty$ we denote by $1 \leqslant r' \leqslant \infty$ its Hölder conjugate exponent. We indicate by L^r_x the Lebesgue space $L^r(\mathbb{R}^n)$, and respectively by $W^{1,r}_x$ and H^1_x the inhomogeneous Sobolev spaces $W^{1,r}(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ (for more details see [1]). For any $N \in \mathbb{N}$, we also set $\mathcal{L}^r_x = L^r(\mathbb{R}^n)^N$ and define the Sobolev spaces $\mathcal{W}^{1,r}_x = W^{1,r}(\mathbb{R}^n)^N$ and $\mathcal{H}^1_x = H^1(\mathbb{R}^n)^N$. We will make frequent use of the basic properties of Lorentz spaces $L^{p,q}$, in particular precised Hölder, Young and Sobolev inequalities, for which we refer to Section 2.9.

For any differential operator $\mathscr{D}$ we use the symbol $\mathscr{D}_x$ (resp. $\mathscr{D}_y$) to explicit the dependence on the x (resp. y) variable. Finally, in Chapter 3 we will denote by f_t the time derivative $\partial_t f$ of any function f.

Chapter 1

H^1-scattering for systems of N-defocusing weakly coupled NLS equations in low space dimensions

In this chapter we prove scattering for a system of weakly coupled Schrödinger equations: because of the nonlineary considered, a theorem of existence and uniqueness is available only if the problem is set in $\mathbb{R}^n$, with $n \leqslant 3$. We develop hence here new techniques for tackling the problem of scattering in the system framework and in the low dimensional case. The reference for the results in the present chapter is [12].

1.1 Introduction

The main object of this chapter is the study of decay and scattering properties of the solution to the following system of $N \geqslant 2$ defocusing nonlinear Schrödinger equations in dimension $1 \leqslant n \leqslant 3$:

$$\begin{cases} i\partial_t u_\mu + \Delta u_\mu - \sum_{\mu,\nu=1}^{N} \beta_{\mu\nu} |u_\nu|^{p+1} |u_\mu|^{p-1} u_\mu = 0, & \mu = 1, \dots, N, \\ (u_\mu(0,\cdot))_{\mu=1}^N = (u_{\mu,0})_{\mu=1}^N \in H^1(\mathbb{R}^d)^N. \end{cases} \tag{1.1.1}$$

Here, for all $\mu, \nu = 1, \dots, N$, $u_\mu = u_\mu(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, $\beta_{\mu\nu} \geqslant 0$, $\beta_{\mu\mu} \neq 0$ are coupling parameters, and we require that the nonlinearity parameter p satisfies the following conditions:

$$1 \leqslant p < p^*(n), \qquad p^*(n) = \begin{cases} +\infty & \text{if} \quad n = 1, 2, \\ 2 & \text{if} \quad n = 3. \end{cases}, \tag{1.1.2}$$

$$\frac{2}{n} < p. \tag{1.1.3}$$

We recall that the power nonlinearity $p^*(n)$ corresponds to the H^1-critical exponent for the single NLS in $\mathbb{R}^n$, while the lower bound $\max(1, \frac{2}{n})$ arises from limitations associated to the well-posedness in the product space $H^1(\mathbb{R}^n)^N$ for the solutions to (1.1.1), as we see later in the Remarks 1.1.2 and 1.3.4. There is a vast literature regarding the global well-posedness theory as well as the bound state theory for the problem (1.1.1), and moreover the system of Schrödinger equations plays an important role in many models of mathematical physics: it describes the interactions of M–wave packets, the nonlinear waveguides, the optical pulse propagation in birefringent fibers, the propagation of polarized laser beam in Kerr-like photorefractive media and in the Bose-Einstein condensates theory, just to name a few. We refer to [19], [37], [72], [59] and [36] in the case $N = 2$ and to [55] and [67] in the general case $N \geqslant 2$ for a complete set of references both on mathematical and on physical setting and applications.

We study the scattering theory in $H^1(\mathbb{R}^n)^N$ for (1.1.1) in analogy with the case of the single defocusing Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u - |u|^{2p} u = 0 \\ u(0) = u_0 \in H^1(\mathbb{R}^n), \end{cases} \tag{1.1.4}$$

with $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ and $p > 0$.

The basic strategy in this work is to use the interaction Morawetz etimates for exploiting decay properties of L^q-norms of the solutions to (1.1.1) as $t \to \pm\infty$, provided $2 < q < 6$ for $n = 3$ and $2 < q < \infty$ for $n = 1, 2$, as suggested by the classical theory available for (1.1.4). Namely, in a first step we obtain Morawetz identities, interaction Morawetz identities and their corresponding inequalities in the framework of the system (1.1.1), following the spirit of the paper [87] (we follow the same path in Chapter 2, Sections 2.4 and 2.5). Then, by localizing the nonlinear part of Morawetz inequalities above on space-time cubes we are in position, as a second step, to give a contradiction argument which enables us to say that the solutions $(u_\mu)_{\mu=1}^N$ decay (see also Proposition 2.7.3 in Chapter 2). We remark that, in order to close this contradiction argument we use some terms coming from the nonlinear terms of the equations, but in fact this is not necessary: one can indeed get a contradiction with a similar argument using the linear terms, as done in the proof of Proposition 2.7.3 in Chapter 2; we follow this approach in order to simplify the arguments in dimension $n = 1, 2$, and in order to give a more complete exposition of the topic in the present Thesis.

Once proved decay properties for solutions to (1.1.1) thanks to a generalization of the nonlinear theory developed in [13], we obtain existence of the wave operators and asymptotic completeness in the energy space $H^1(\mathbb{R}^n)^N$ for the system (1.1.1). We emphasize that our results rely on an argument which yields the asymptotics in a single stroke and which does not distinguish the number N of coupled equations. In fact, by writing the linear part of the interaction Morawetz in an appropriate form and dealing only with its nonlinear part, it is possible to overcome the mathematical difficulties, and moreover to provide a further simple proof of scattering results appearing in [43], [71] and specially in [65]. In this last paper, the author produces a set of weighted Morawetz estimate and uses the *separation of localized energy method* to achieve that the wave operators and the scattering operators for (1.1.4) when $n = 1, 2$ are well-defined and bijective in H^1, but this is very difficult to extend to a system of coupled nonlinear Schrödinger equations.

We state now the main result of this chapter.

Theorem 1.1.1. *Let* $1 \leqslant n \leqslant 3$, $p \in \mathbb{R}$ *such that* (1.1.2), (1.1.3) *hold, then:*

- (existence of wave operators) *For every* $(u^{\pm}_{\mu,0})_{\mu=1}^N \in H^1(\mathbb{R}^n)^N$ *there exist unique initial data* $(u_{\mu,0})_{\mu=1}^N \in H^1(\mathbb{R}^n)^N$ *such that the global solution to* (1.1.1) $(u_\mu)_{\mu=1}^N \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^n)^N)$ *satisfies*

$$\lim_{t\to\pm\infty} \left\| u_\mu(t,\cdot) - e^{it\Delta} u^{\pm}_{\mu,0}(\cdot) \right\|_{H^1} = 0 \quad \text{for all } \mu = 1, \dots, N. \tag{1.1.5}$$

- (asymptotic completeness) *If* $(u_{\mu,0})_{\mu=1}^N \in H^1(\mathbb{R}^n)^N$, *then there exist* $(u^{\pm}_{\mu,0})_{\mu=1}^N \in H^1(\mathbb{R}^n))^N$ *such that* (1.1.5) *holds.*

Remark 1.1.2 (Case $\beta_{\mu\nu} = 0$, $\mu \neq \nu$)**.** If, for $\mu \neq \nu$, some of the $\beta_{\mu\nu}$ is nonvanishing, we are forced to assume $p \geqslant 1$ in order to treat the coupling nonlinearity considered in (1.1.1): this excludes the analisys of the system in dimension $n \geqslant 4$, since in this case an existence theorem is not available (see Prop. 1.3.1). By the way, in the trivial case $\beta_{\mu\nu} = 0$ for all $\mu \neq \nu$, we are no longer obliged to assume $p \geqslant 1$, and hence, as a byproduct of this theory, we get decay (for $0 < p < 2/(n-2)$) and scattering (for $2/n < p < 2/(n-2)$) results for the solution to the Cauchy problem (1.1.4) in all dimensions $n \geqslant 1$. We remark that such results were already established in [87], however our techniques simplify some arguments present in it. Finally, we underline that this approach eases the well known results [65, 66] for the scattering of (1.1.4) in lower dimension $n = 1, 2$.

Morawetz and interaction Morawetz estimates are not available in the system framework to our knowledge: we recall here some of the known results, other than the already cited [87], [65] and [66] connected with the problem (1.1.4).

In order to shed light on scattering properties for solutions to (1.1.4) it is necessary to get fundamental tools such as the Morawetz multiplier technique and the resulting estimates. These were obtained for the first time in [63] for the Klein-Gordon equation with a general nonlinearity and were successively used for proving the asymptotic completeness in [54] for the cubic NLS in $\mathbb{R}^3$ and in [43] for the Schrödinger equation in $\mathbb{R}^n$ and with a pure power nonlinearity as in (1.1.4) for $2/n < p < 2/(n-2)$ (that is, L^2-supercritical and H^1-subcritical). Recently, a new approach has simplified the proof of scattering, consisting in getting bilinear Morawetz inequalities, also named interaction of quadratic Morawetz inequalities, specificly Morawetz estimates for two solutions (possibly the same solution taken twice) are computed at once. We quote in this direction the papers [16], [15], where cubic and quintic defocusing NLS in $\mathbb{R}^3$ are considered, the [14] in which interaction Morawetz and then asymptotic completeness are proved for the cubic defocusing NLS in $\mathbb{R}^2$, the paper [71] where the interaction Morawetz estimates which do not involve the bilaplacian of the Morawetz multipliers are given for the L^2-supercritical and H^1-subcritical NLS in $\mathbb{R}^n$ with $n \geqslant 1$, providing also application to various nonlinear problems also settled on $3D$ exterior domains and finally the survey [41] where the Authors show quadratic Morawetz estimates and scattering for the NLS and the Hartree equation in the L^2-supercritical and H^1-subcritical cases. We quote also [25], [38], [73], [40] (and references therein), where such a theory is applied considering the presence of electromagnetic potentials and the paper [64], where the interaction Morawetz technique is extended to the partially periodic setting in the scattering analysis of the NLS posed on the product space $\mathbb{R}^n \times \mathbb{T}$, with $n \geqslant 1$.

In Section 1.2 we establish the interaction Morawetz identities and inequalities (in Lemmas 1.2.2) and the corresponding Morawetz estimates (in Propositions 1.2.4 and 1.2.5) for the system of NLS 1.1.1 ancillary for proving the Theorem 1.1.1. The Section 1.3 is divided in two part: in the former we show how the interactive Morawetz inequalities give a relevant advantage in the exploitation of the decay of solutions to 1.1.1, this is contained in the Proposition (1.3.3), which has its own interest; in the latter we look at the existence of scattering states and wave operators by an extension of scattering techniques to the systems frame. Finally in the Appendix 1.4 a generalized Gagliardo-Nirenberg inequality is obtained (see for instance [87]).

1.2 Morawetz and interaction Morawetz identities

We provide in this section the fundamental tools for the proof of our main theorem. We start by obtaining Morawetz-type identities, which are similar to the ones in Chapter 2, which hold for the single NLS: : we will sketch them for the sake of completeness, since some care is needed in handling more functions at once. We introduce the following notations: given a function $f \in H^1(\mathbb{R}^n, \mathbb{C})$, we denote by

$$m_f(x) := |f(x)|^2, \qquad j_f(x) := \Im\left[\overline{f}\nabla f(x)\right] \in \mathbb{C}^n. \tag{1.2.1}$$

We have the following Lemma.

Lemma 1.2.1 (Morawetz)**.** *Let $n \geqslant 1$, and $(u_\mu)_{\mu=1}^N \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^n)^N)$ be a global solution to system (1.1.1), let $\phi = \phi(x) : \mathbb{R}^n \to \mathbb{R}$ be a sufficiently regular and decaying function, and denote by*

$$V(t) := \sum_{\mu=1}^N \int_{\mathbb{R}^n} \phi(x)\, m_{u_\mu}(x)\, dx.$$

The following identities hold:

$$\dot V(t) = \sum_{\mu=1}^N \int_{\mathbb{R}^n} \phi(x)\dot m_{u_\mu}(x)\, dx = 2\sum_{\mu=1}^N \int_{\mathbb{R}^n} j_{u_\mu}(x)\cdot\nabla\phi(x)\, dx \tag{1.2.2}$$

$$\begin{aligned}\ddot V(t) =& \sum_{\mu=1}^N \int_{\mathbb{R}^n} \phi(x)\ddot m_{u_\mu}(x)\, dx \\ =& \sum_{\mu=1}^N \left[-\int_{\mathbb{R}^n} m_{u_\mu}(x)\Delta^2\phi(x)\, dx + 4\int_{\mathbb{R}^n} \nabla u_\mu(x) D^2\phi(x)\cdot\nabla\overline{u}_\mu(x)\, dx\right] \\ &+ \frac{2p}{p+1}\sum_{\mu,\nu=1}^N \beta_{\mu\nu}\int_{\mathbb{R}^n} |u_\mu(x)|^{p+1}|u_\nu(x)|^{p+1}\Delta\phi(x)\, dx,\end{aligned} \tag{1.2.3}$$

where $D^2\phi \in \mathcal{M}_{n\times n}(\mathbb{R}^n)$ is the hessian matrix of ϕ, and $\Delta^2\phi = \Delta(\Delta\phi)$ the bi-laplacian operator.

Proof. We prove the identities for a smooth solution $(u_\mu)_\mu$, letting the general case $(u_\mu)_{\mu=1}^N \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^n)^N)$ to a final standard density argument (see for instance [13, Theorem 7.6.4, Step 2], [41, Appendix 4], or the remarks in the beginning of Section 2.3). The equation (1.2.2) is easy to check. We give some details for obtaining (1.2.3). By means of an integration by parts and thanks to (1.1.1), we have for every fixed μ

$$\begin{aligned} 2\partial_t &\int_{\mathbb{R}^n} j_{u_\mu}(x)\cdot\nabla\phi(x)\,dx \\ &= -2\Im\int_{\mathbb{R}^n}\partial_t u_\mu(x)[\Delta\phi(x)\bar{u}_\mu(x)+2\nabla\phi(x)\cdot\nabla\bar{u}_\mu(x)]\,dx \\ &= 2\Re\int_{\mathbb{R}^n} i\partial_t u_\mu(x)[\Delta\phi(x)\bar{u}_\mu(x)+2\nabla\phi(x)\cdot\nabla\bar{u}_\mu(x)]\,dx \\ &= 2\Re\int_{\mathbb{R}^n}\Big[-\Delta u_\mu(x)+\sum_{\nu=1}^N\beta_{\mu\nu}|u_\nu(x)|^{p+1}|u_\mu(x)|^{p-1}u_\mu(x)\Big] \\ &\qquad\cdot[\Delta\phi(x)\bar{u}_\mu(x)+2\nabla\phi(x)\cdot\nabla\bar{u}_\mu(x)]\,dx. \end{aligned} \tag{1.2.4}$$

We have

$$\begin{aligned} 2\Re\int_{\mathbb{R}^n} &-\Delta u_\mu(x)[\Delta\phi(x)\bar{u}_\mu(x)+2\nabla\phi(x)\cdot\nabla\bar{u}_\mu(x)]\,dx \\ &= -\int_{\mathbb{R}^n}\Delta^2\phi(x)|u_\mu(x)|^2\,dx+4\int_{\mathbb{R}^n}\nabla u_\mu(x)D^2\phi(x)\nabla\bar{u}_\mu(x)\,dx. \end{aligned} \tag{1.2.5}$$

Moreover

$$\begin{aligned} 2\sum_{\nu=1}^N\beta_{\mu\nu}\Re\int_{\mathbb{R}^n}&|u_\nu|^{p+1}|u_\mu|^{p-1}u_\mu(x)\cdot[\Delta\phi(x)\bar{u}_\mu(x)+2\nabla\phi(x)\cdot\nabla\bar{u}_\mu(x)]\,dx \\ &= 2\sum_{\nu=1}^N\beta_{\mu\nu}\Re\int_{\mathbb{R}^n}|u_\mu u_\nu|^{p+1}\Delta\phi(x)+2\nabla\phi(x)\cdot\frac{\nabla|u_\mu|^{p+1}}{p+1}|u_\nu|^{p+1}\,dx, \end{aligned}$$

and, summing over $\nu,\mu=1,\dots,N$,

$$\begin{aligned} 2&\sum_{\nu,\mu=1}^N\beta_{\mu\nu}\Re\int_{\mathbb{R}^n}|u_\mu u_\nu|^{p+1}\Delta\phi(x)+\nabla\phi(x)\cdot\frac{2\nabla|u_\mu|^{p+1}}{p+1}|u_\nu|^{p+1}\,dx \\ =2&\sum_{\nu,\mu=1}^N\beta_{\mu\nu}\Re\int_{\mathbb{R}^n}|u_\mu u_\nu|^{p+1}\Delta\phi(x)+\nabla\phi(x)\cdot\frac{\nabla(|u_\mu|^{p+1}|u_\nu|^{p+1})}{p+1}\,dx \\ =2&\sum_{\nu,\mu=1}^N\beta_{\mu\nu}\left(1-\frac{1}{p+1}\right)\Re\int_{\mathbb{R}^n}|u_\mu u_\nu|^{p+1}\Delta\phi(x)\,dx, \end{aligned} \tag{1.2.6}$$

where in the last equality we have used integration by parts. Taking in account (1.2.4), (1.2.5), summing over $\mu=1,\dots,N$, and considering (1.2.6), we get the thesis. □

By means of the previous Lemma, we can now prove the following interaction Morawetz identities.

Lemma 1.2.2 (Interaction Morawetz)**.** *Let* $(u_\mu)_{\mu=1}^N \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^n)^N)$ *be a global solution to system* (1.1.1)*, let* $\phi = \phi(|x|) : \mathbb{R}^n \to \mathbb{R}$ *be a convex radial function, regular and decaying enough, and denote by* $\psi = \psi(x,y) := \phi(|x-y|) : \mathbb{R}^{2n} \to \mathbb{R}$*,*

$$I(t) := \sum_{\mu,\kappa=1}^N \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi(x,y) m_{u_\mu}(x) m_{u_\kappa}(y)\, dx\, dy.$$

The following holds:

$$\dot{I}(t) = 2 \sum_{\mu,\kappa=1}^N \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} j_{u_\mu}(x) \cdot \nabla_x \psi(x,y)\, m_{u_\kappa}(y)\, dxdy, \tag{1.2.7}$$

$$\ddot{I}(t) \geqslant 2 \sum_{\mu,\kappa=1}^N \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Delta_x \psi(x,y) \nabla_x m_{u_\mu}(t,x) \cdot \nabla_y m_{u_\kappa}(t,y)\, dxdy + N_{(p,\psi)}, \tag{1.2.8}$$

with

$$N_{(p,\psi)} = \frac{4p}{p+1} \sum_{\mu,\nu,\kappa=1}^N \beta_{\mu\nu} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u_\mu(x)|^{p+1} |u_\nu(x)|^{p+1} m_{u_\kappa}(y) \Delta_x \psi(x,y)\, dxdy. \tag{1.2.9}$$

Proof. As for the previous lemma, we prove the identities for a smooth solution $(u_\nu)_{\nu=1}^N$, letting the general case $(u_\mu)_{\mu=1}^N \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^n)^N)$ to a final standard density argument. First one has

$$\dot{I}(t) = \sum_{\mu,\kappa=1}^N \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\dot{m}_{u_\mu}(x) m_{u_\kappa}(y) + m_{u_\mu}(x) \dot{m}_{u_\kappa}(y) \right) \psi(x,y)\, dx\, dy, \tag{1.2.10}$$

then, due to the symmetry of $\psi(x,y) = \phi(|x-y|)$, we obtain that the equality above is equivalent to

$$\dot{I}(t) = 2 \sum_{\mu,\kappa=1}^N \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \dot{m}_{u_\mu}(x) m_{u_\kappa}(y) \psi(x,y)\, dx\, dy.$$

Therefore, (1.2.7) immediately follows by (1.2.2) and the Fubini's Theorem. Analogously, we can differentiate again and get the identity

$$\begin{aligned} \ddot{I}(t) = &\sum_{\mu,\kappa=1}^N \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \ddot{m}_{u_\mu}(x) m_{u_\kappa}(y) \psi(x,y)\, dx\, dy \\ &+ \sum_{\mu,\kappa=1}^N \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} m_{u_\mu}(x) \ddot{m}_{u_\kappa}(y) \psi(x,y)\, dx\, dy \\ &+2 \sum_{\mu,\kappa=1}^N \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \dot{m}_{u_\mu}(x) \dot{m}_{u_\kappa}(y) \psi(x,y)\, dx\, dy. \end{aligned} \tag{1.2.11}$$

We can write $\ddot{I}(t) := A + B$: by (1.2.3), an application of the Fubini's Theorem and using once again the symmetry of $\psi(x,y)$ we are allowed to set

$$
\begin{aligned}
A =& -2\sum_{\mu,\kappa=1}^{N}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m_{u_\mu}(x)m_{u_\kappa}(y)\Delta_x^2\psi(x,y)\,dxdy \\
&+\frac{4p}{p+1}\sum_{\mu,\kappa=1}^{N}\beta_{\mu\mu}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|u_\mu(x)|^{2p+2}m_{u_\kappa}(y)\Delta_x\psi(x,y)\,dxdy \qquad (1.2.12)\\
&+\frac{4p}{p+1}\sum_{\mu,\nu,\kappa=1\mu\neq\nu}^{N}\beta_{\mu\nu}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|u_\mu(x)|^{p+1}|u_\nu(x)|^{p+1}m_{u_\kappa}(y)\Delta_x\psi(x,y)\,dxdy,
\end{aligned}
$$

notice that the second and third line of the (1.2.12) above are sum of terms coming from the nonlinearity in the equation, while the r.h.s. of the first line consists of sums of terms related to the linear part of the equation. We reshape the linear term in the previous identity (1.2.12) as follows

$$
\begin{aligned}
&-2\sum_{\mu,\kappa=1}^{N}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m_{u_\mu}(t,x)m_{u_\kappa}(t,y)\Delta^2\psi(x,y)\,dxdy \\
=&2\sum_{\mu,\kappa=1}^{N}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m_{u_\mu}(t,x)m_{u_\kappa}(t,y)\partial_{x_i}\partial_{y_i}\Delta\psi(x,y)\,dxdy \qquad (1.2.13)\\
=&2\sum_{\mu,\kappa=1}^{N}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_{x_i}m_{u_\mu}(t,x)\partial_{y_i}m_{u_\kappa}(t,y)\Delta\psi(x,y)\,dxdy,
\end{aligned}
$$

applying integration by parts (with no boundary terms) and using the property $\partial_{x_k}\psi = -\partial_{y_k}\psi$. In conclusion, we get

$$
A = 2\sum_{\mu,\kappa=1}^{N}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Delta_x\psi(x,y)\nabla_x m_{u_\mu}(t,x)\cdot\nabla_y m_{u_\kappa}(t,y)\,dxdy + N_{(p,\psi)}. \qquad (1.2.14)
$$

Moreover by (1.2.2), (1.2.3) and the Fubini's Theorem we introduce

$$
\begin{aligned}
B =& 4\sum_{\mu,\kappa=1}^{N}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\nabla u_\mu(x)D_x^2\psi(x,y)\nabla\overline{u}_\mu(x)m_{u_\kappa}(y)\,dxdy \\
&+4\sum_{\mu,\kappa=1}^{N}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m_{u_\mu}(x)\nabla u_\kappa(y)D_y^2\psi(x,y)\nabla\overline{u}_\kappa(y)\,dxdy \\
&+8\sum_{\mu,\kappa=1}^{N}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} j_{u_\mu}(x)D_{xy}^2\psi(x,y)\cdot j_{u_\kappa}(y)\,dxdy,
\end{aligned}
$$

here we used, at least at this level, the symmetry of $D^2\psi$ to eliminate the real part condition in the first two summands of the equality above. Let us focalize on B : it is the sum of two terms, $B_{\mu=\kappa}$, and $B_{\mu\neq\kappa}$. We deal with each of them separately, then we start with the summand with $\mu=\kappa$ that is

$$B_{\mu=\kappa} = \sum_{\mu=1}^{N} B^{\mu\mu}, \tag{1.2.15}$$

where, for each $\mu = 1, ..., N$ the $B^{\mu\mu}$ term is defined by the chain of equalities

$$\begin{aligned} B^{\mu\mu} =& 4\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m_{u_\mu}(x)\nabla_y u_\mu(y) D_y^2\psi(x,y)\nabla_y \overline{u}_\mu(y)\, dxdy \\ &+ 4\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m_{u_\mu}(y)\nabla_x u_\mu(x) D_x^2\psi(x,y)\nabla_x \overline{u}_\mu(x)\, dxdy \\ &+ 8\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} j_{u_\mu}(x) D_{xy}^2\psi(x,y)\cdot j_{u_\mu}(y)\, dxdy \\ =& 4\sum_{j,k=1}^{d}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u_\mu(x)|^2 \partial_{y_j} u_\mu(y)\partial_{y_j y_k}^2\phi(|x-y|)\partial_{y_k}\overline{u}_\mu(y)\, dxdy \\ &+ 4\sum_{j,k=1}^{d}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u_\mu(y)|^2 \partial_{x_j} u_\mu(x)\partial_{x_j x_k}^2\phi(|x-y|)\partial_{x_k}\overline{u}_\mu(x)\, dxdy \\ &+ 8\sum_{j,k=1}^{d}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Im(\overline{u}_\mu(x)\partial_{x_j} u_\mu(x))\partial_{x_j y_k}^2\phi(|x-y|)\Im(\overline{u}_\mu(y)\partial_{y_k} u_\mu(y))\, dxdy. \end{aligned} \tag{1.2.16}$$

Since $\partial_{x_j}\psi = -\partial_{y_j}\psi$, for all $j = 1, \ldots, n$, one can check (after a rearrangement) that the last identity of the (1.2.15) above is equal to

$$\begin{aligned} &-4\sum_{j,k=1}^{d}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_{x_j y_k}^2\phi(|x-y|)|u_\mu(x)|^2\Re(\partial_{y_j} u_\mu(y)\partial_{y_k}\overline{u}_\mu(y))\, dxdy \\ &-4\sum_{j,k=1}^{d}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_{x_j y_k}^2\phi(|x-y|)|u_\mu(y)|^2\Re(\partial_{x_j} u_\mu(x)\partial_{x_k}\overline{u}_\mu(x))\, dxdy \\ &+8\sum_{j,k=1}^{d}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Im(\overline{u}_\mu(x)\partial_{x_j} u_\mu(x))\partial_{x_j y_k}^2\phi(|x-y|)\Im(\overline{u}_\mu(y)\partial_{y_k} u_\mu(y))\, dxdy, \end{aligned} \tag{1.2.17}$$

and finally to

$$\begin{aligned} = -2\Big[&\sum_{j,k=1}^{d}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_{x_j y_k}^2\phi(|x-y|)|u_\mu(x)|^2(\partial_{y_j} u_\mu(y)\partial_{y_k}\overline{u}_\mu(y) + \partial_{y_j}\overline{u}_\mu(y)\partial_{y_k}\overline{u}_\mu(y))\, dxdy \\ +&\sum_{j,k=1}^{d}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_{x_j y_k}^2\phi(|x-y|)|u_\mu(y)|^2(\partial_{x_j} u_\mu(x)\partial_{x_k}\overline{u}_\mu(x) + \partial_{x_j}\overline{u}_\mu(x)\partial_{x_k} u_\mu(x))\, dxdy \\ +&\sum_{j,k=1}^{d}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_{x_j y_k}^2\phi(|x-y|)(\overline{u}_\mu(x)\partial_{x_j} u_\mu(x) - u_\mu(x)\partial_{x_j}\overline{u}_\mu(x)) \\ &\qquad\cdot(\overline{u}_\mu(y)\partial_{y_k} u_\mu(y) - u_\mu(y)\partial_{y_k}\overline{u}_\mu(y))\, dxdy\Big]. \end{aligned}$$

If we set

$$C_j^{\mu\mu} := u_\mu(t,x)\partial_{y_j}\overline{u_\mu(t,y)} + \partial_{x_j}u_\mu(t,x)\overline{u_\mu(t,y)},$$
$$D_j^{\mu\mu} := u_\mu(t,x)\partial_{y_j}u_\mu(t,y) - \partial_{x_j}u_\mu(t,x)u_\mu(t,y),$$

then by gathering (1.2.15) and (1.2.17) we earn

$$B_{\mu=\kappa} = 2\sum_{\mu=1}^{N}\sum_{j,k=1}^{d}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\partial^2_{x_jx_k}\phi(|x-y|)\left[C_j^{\mu\mu}\overline{C_k^{\mu\mu}} + D_j^{\mu\mu}\overline{D_k^{\mu\mu}}\right]dxdy. \tag{1.2.18}$$

Take into account now the summand with $\mu \neq \kappa$ that is

$$B_{\mu\neq\kappa} = \sum_{\substack{\mu,\kappa=1\\ \mu\neq\kappa}}^{N} B^{\mu\kappa}, \tag{1.2.19}$$

with the $B^{\mu\kappa}$ term given by

$$\begin{aligned}
B^{\mu\kappa} =& 4\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m_{u_\mu}(x)\nabla_y u_\kappa(y)D_y^2\psi(x,y)\nabla_y\overline{u}_\kappa(y)\,dxdy\\
&+ 4\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m_{u_\kappa}(y)\nabla_x u_\mu(x)D_x^2\psi(x,y)\nabla_x\overline{u}_\mu(x)\,dxdy\\
&- 8\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} j_{u_\mu}(x)D_x^2\psi(x,y)\cdot j_{u_\kappa}(y)\,dxdy\\
=& 4\sum_{j,k=1}^{d}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|u_\mu(x)|^2\partial_{y_j}u_\kappa(y)\partial^2_{y_jy_k}\phi(|x-y|)\partial_{y_k}\overline{u}_\kappa(y)\,dxdy\\
&+ 4\sum_{j,k=1}^{d}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|u_\kappa(y)|^2\partial_{x_j}u_\mu(x)\partial^2_{x_jx_k}\phi(|x-y|)\partial_{x_k}\overline{u}_\mu(x)\,dxdy\\
&+ 8\sum_{j,k=1}^{d}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Im(\overline{u}_\mu(x)\partial_{x_j}u_\mu(x))\partial^2_{x_jy_k}\phi(|x-y|)\Im(\overline{u}_\kappa(y)\partial_{y_k}u_\kappa(y))\,dxdy,
\end{aligned} \tag{1.2.20}$$

thus arguing as for the proof of (1.2.18), once one set

$$E_j^{\mu\kappa} := u_\mu(t,x)\partial_{y_j}\overline{u_\kappa(t,y)} + \partial_{x_j}u_\mu(t,x)\overline{u_\kappa(t,y)},$$
$$F_j^{\mu\kappa} := u_\mu(t,x)\partial_{y_j}u_\kappa(t,y) - \partial_{x_j}u_\mu(t,x)u_\kappa(t,y),$$

we arrive at the equality

$$B_{\mu\neq\kappa} = 2\sum_{\substack{\mu,\kappa=1\\ \mu\neq\kappa}}^{N}\sum_{j,k=1}^{d}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\partial^2_{x_jx_k}\phi(|x-y|)\left[E_j^{\mu\kappa}\overline{E_k^{\mu\kappa}} + F_j^{\mu\kappa}\overline{F_k^{\mu\kappa}}\right]dxdy. \tag{1.2.21}$$

Therefore the identities (1.2.18), (1.2.21) and the fact that ϕ is a convex function give $B \geqslant 0$. This argument implies, in combination with (1.2.11), (1.2.14), the proof of (1.2.8). □

By using the identity (1.2.13) which appears in the proof of Lemma 1.2.2 we have an equivalent way to the (1.2.8) useful when the quantity $\Delta_x^2\psi(x,y)$ is nonpositive. This is contained in the following Corollary, whose easy proof we omit.

Corollary 1.2.3. *Let be* $(u_\mu)_{\mu=1}^N \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^n)^N)$, $\psi=\psi(x,y)$ *and* $N_{(p,\psi)}$ *as in Lemma 1.2.2, then the following holds*

$$\ddot{I}(t) \geqslant -2\sum_{\mu,\kappa=1}^N \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m_{u_\mu}(x)m_{u_\kappa}(y)\Delta_x^2\psi(x,y)\,dxdy + N_{(p,\psi)}. \tag{1.2.22}$$

We remark that a regular function $\psi\colon \mathbb{R}^n \to \mathbb{R}$ such that $\Delta\psi \geqslant 0$ and $-\Delta(\Delta\psi)\geqslant 0$ can not exist in $\mathbb{R}^n$ if $n=1,2$, hence in fact Corollary 1.2.3 will be useful only when dealing with the 3D case.

As an immediate consequence of Lemma 1.2.2 and Corollary 1.2.3, we prove the following results.

Proposition 1.2.4. *Let* $n=3$, $p\in\mathbb{R}$ *such that* (1.1.2) *holds, and let* $(u_\mu)_{\mu=1}^N \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^3)^N)$ *be a global solution to* (1.1.1). *Then one has*

$$\sum_{\mu=1}^N \int_{\mathbb{R}}\int_{\mathbb{R}^3} |u_\mu(t,x)|^4\,dx\,dt < \infty, \tag{1.2.23}$$

$$\sum_{\mu=1}^N \beta_{\mu\mu}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_\mu(t,x)|^{2p+2}|u_\mu(t,y)|^2}{|x-y|}\,dx\,dy\,dt < \infty. \tag{1.2.24}$$

Proof. Integrating (1.2.22) to time variable one obtains by (1.2.7)

$$2\sum_{\mu,\kappa=1}^N\left[\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} j_{u_\mu}(t,x)\cdot\nabla_x\psi(x,y)m_{u_\kappa}(t,y)\,dxdy\right]_{t=S}^{t=T} \tag{1.2.25}$$
$$\geqslant -2\sum_{\mu,\kappa=1}^N\int_S^T m_{u_\mu}(t,x)m_{u_\kappa}(t,y)\Delta_x^2\psi(x,y)\,dxdydt$$
$$+\frac{4p}{p+1}\sum_{\mu,\kappa=1}^N \beta_{\mu\mu}\int_S^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |u_\mu(t,x)|^{2p+2}m_{u_\kappa}(t,y)\Delta_x\psi(x,y)\,dxdydt$$
$$+\frac{4p}{p+1}\sum_{\substack{\mu,\nu,\kappa=1\\ \mu\neq\nu}}^N \beta_{\mu\nu}\int_S^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |u_\mu(t,x)|^{p+1}|u_\nu(t,x)|^{p+1}m_{u_\kappa}(t,y)\Delta_x\psi(x,y)\,dxdydt$$

Now choose $\psi(x,y)=|x-y|$. For the l.h.s of the (1.2.25) we have the immediate bound

$$2\sum_{\mu,\kappa=1}^N\left[\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} j_{u_\mu}(t,x)\cdot\nabla_x\psi(x,y)m_{u_\kappa}(t,y)\,dxdy\right]_{t=S}^{t=T} \tag{1.2.26}$$
$$\leqslant C_1\left(\sum_{\mu=1}^N \|u_\mu(T)\|_{H_x^1} + \sum_{\mu=1}^N \|u_\mu(S)\|_{H_x^1}\right)$$
$$\leqslant C_2\sum_{\mu=1}^N \|u_{\mu,0}\|_{H_x^1} < \infty,$$

for some $C_1, C_2 > 0$ and any $T, S \in \mathbb{R}$, since the H^1_x-norm is preserved. We have

$$\Delta_x |x-y| = \frac{n-1}{|x-y|}, \qquad \Delta_x^2 |x-y| = -4\pi\delta_{x=y} \leqslant 0,$$

and hence

$$\begin{aligned}
&-2\sum_{\mu,\kappa=1}^{N}\int_S^T m_{u_\mu}(t,x)m_{u_\kappa}(t,y)\Delta_x^2\psi(x,y)\,dxdydt \\
&+\frac{4p}{p+1}\Bigg(\sum_{\mu,\kappa=1}^{N}\beta_{\mu\mu}\int_S^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|u_\mu(t,x)|^{2p+2}m_{u_\kappa}(t,y)\Delta_x\psi(x,y)\,dxdydt \\
&+\sum_{\substack{\mu,\nu,\kappa=1\\ \mu\neq\nu}}^{N}\beta_{\mu\nu}\int_S^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|u_\mu(t,x)|^{p+1}|u_\nu(t,x)|^{p+1}m_{u_\kappa}(t,y)\Delta_x\psi(x,y)\,dxdydt\Bigg) \\
&\geqslant C\sum_{\mu=1}^{N}\left(\int_S^T\int_{\mathbb{R}^3}|u_\mu(t,x)|^4\,dt\,dx+\beta_{\mu\mu}\int_S^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_\mu(t,x)|^{2p+2}|u_\mu(t,y)|^2}{|x-y|}\,dx\,dy\,dt\right),
\end{aligned} \tag{1.2.27}$$

for some $C > 0$, and any $T, S \in \mathbb{R}$. The thesis follows by (1.2.25), (1.2.26), and (1.2.27), letting $T \to \infty, S \to -\infty$. □

Proposition 1.2.5. *Let $n = 1, 2$, $p > 0$ as in (1.1.2), and let $(u_\mu)_{\mu=1}^N \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^n)^N)$ be a global solution to (1.1.1). Then*

- *for $n = 1$ we have*

$$\sum_{\mu=1}^{N}\beta_{\mu\mu}\int_{\mathbb{R}}\int_{\mathbb{R}}|u_\mu(t,x)|^{2p+4}\,dt\,dx < \infty, \tag{1.2.28}$$

- *for $n = 2$ we have*

$$\sum_{\mu=1}^{N}\beta_{\mu\mu}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|u_\mu(t,x)|^{2p+2}|u_\mu(t,y)|^2}{|x-y|}\,dt\,dx\,dy < \infty. \tag{1.2.29}$$

Proof. The cases $n = 1, 2$ can be treated by a direct application of the inequality (1.2.8). Pick up once again $\psi(x, y) = |x - y|$, then we have

$$\Delta_x\psi = \begin{cases} \frac{1}{|x-y|} & \text{if } n = 2, \\ 2\delta_{x=y} & \text{if } n = 1. \end{cases} \tag{1.2.30}$$

Arguing as in the proof of Proposition 1.2.4, we get in the case $n = 1$ that

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\sum_{\mu=1}^{N}\partial_x m_{u_\mu}(t,x)\right|^2 dt\,dx + \sum_{\mu=1}^{N}\beta_{\mu\mu}\int_{\mathbb{R}}\int_{\mathbb{R}}|u_\mu(t,x)|^{2p+4}\,dt\,dx < \infty, \tag{1.2.31}$$

from which we infer the inequality (1.2.28).

In the case $n = 2$, first one needs to recall the property (for more details see [41])

$$\sum_{\mu,\kappa=1}^{N} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Delta_x \psi(x,y) \nabla_x m_{u_\mu}(t,x) \cdot \nabla_y m_{u_\kappa}(t,y)\, dxdydt \tag{1.2.32}$$
$$= \int_{\mathbb{R}} \Big\| \sum_{\mu=1}^{N} (-\Delta)^{\frac{1}{4}} m_{u_\mu}(t,x) \Big\|_{L_x^2}^2 dt.$$

Then we get the following

$$\int_{\mathbb{R}} \Big\| \sum_{\mu=1}^{N} (-\Delta)^{\frac{1}{4}} |u_\mu(t,x)|^2 \Big\|_{L_x^2}^2 dt$$
$$+ \sum_{\mu,\kappa=1}^{N} \beta_{\mu\mu} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|u_\mu(t,x)|^{2p+2} |u_\kappa(t,y)|^2}{|x-y|}\, dt\, dx\, dy < \infty,$$

that yields the inequality (1.2.29). □

Remark 1.2.6. We observe also that, for $n = 2$, an application of the Sobolev embedding theorem implies, in similarity with the case $n = 3$, also the following bound

$$\sum_{\mu=1}^{N} \|u_\mu(t,x)\|_{L^4(\mathbb{R}, L_x^8)}^4 < \infty.$$

In fact this estimate can be used for proving scattering, as in [87, 71], but here we will use the nonlinear term (1.2.29).

Remark 1.2.7. One could prove the interaction inequalities of the Proposition 1.2.5 by following the theory developed for a single NLS in the paper [14] and based on a suitable choice of the function $\psi(x,y)$, built case by case. To be more precise: it is introduced for $n = 1$

$$\psi(x,y) = 2 \int_{-\infty}^{\frac{x-y}{\varepsilon}} e^{-t^2}\, dt \quad \text{with} \quad \varepsilon > 0, \tag{1.2.33}$$

and then integration by parts are performed in combination with the limiting argument $\varepsilon \to 0$; for $n = 2$ it is selected a even function $\Delta_x^2 \psi$ satisfying the property

$$-\Delta_x^2 \psi = \frac{2\pi}{a} - h_a(|x-y|),$$

for some real number $a > 0$ and with

$$h_a(|x-y|) = \begin{cases} \frac{1}{|x-y|^3} & \text{if} \quad |x-y| \geqslant a, \\ 0 & \text{elesewhere,} \end{cases}$$

then it is used a bilinear Morawetz inequality similar to (1.2.8). We elaborate our own method which is easier to technicalities used above and well-suited also to treat the case of system with more than two nonlinear coupled equations.

1.3 Proof of Theorem 1.1.1

We split the proof of the main Theorem 1.1.1 in two steps. In the first one we shall show, by transposing the method of [87], some decay properties of the solution of the system (1.1.1). In the second one we present the proof of the scattering by combining the argument of the first step with the theory estabilished in [13] and [41], here applied to the case of the system of equations. In this section we will denote $w(t,x) = (u_\mu(t,x))_{\mu=1}^N$.

We start this section observing that Theorem 3.3.9 and Remark 3.3.12 in [13], in connection with the defocusing nature of the system, give a well-known result concerning global well-posedness for (1.1.1) (see also [37]):

Proposition 1.3.1. *Let $1 \leqslant n \leqslant 3$ and $p \in \mathbb{R}$ such that (1.1.2) holds. Then for all $(u_{\mu,0})_{\mu=1}^N \in \mathcal{H}_x^1$ there exists a unique $(u_\mu)_{\mu=1}^N \in C(\mathbb{R}, \mathcal{H}_x^1)$ solution to (1.1.1), moreover*

$$\|u_\mu(t)\|_{L_x^2} = \|u_\mu(0)\|_{L_x^2} \quad \text{for all } \mu = 1, \dots, N, \tag{1.3.1}$$

$$E(u_1(t), \dots, u_N(t)) = E(u_1(0), \dots, u_N(0)), \tag{1.3.2}$$

with

$$E(u_1, \dots, u_N) = \int_{\mathbb{R}^n} \sum_{\mu=1}^N |\nabla u_\mu|^2 \, dx + \sum_{\mu,\nu=1}^N \beta_{\mu\nu} \frac{|u_\mu u_\nu|^{p+1}}{p+1} \, dx.$$

Remark 1.3.2. The conservation laws (1.3.1) and (1.3.2) for the solution to (1.1.1) yield also that

$$\sum_{\mu=1}^N \|u_\mu(t)\|_{H_x^1} \leqslant \sum_{\mu=1}^N \|u_\mu(0)\|_{H_x^1} < \infty. \tag{1.3.3}$$

1.3.1 Decay of solutions to (1.1.1)

In this section we show some decay properties of the solution to (1.1.1), fundamental in the proof of scattering. We have the following.

Proposition 1.3.3. *Let $1 \leqslant n \leqslant 3$ and $p \in \mathbb{R}$ such that (1.1.2) holds. If $w \in \mathcal{C}(\mathbb{R}, \mathcal{H}_x^1)$ is a global solution to (1.1.1), then we have*

$$\lim_{t \to \pm\infty} \|w(t)\|_{\mathcal{L}_x^q} = 0, \tag{1.3.4}$$

with $2 < q < 6$, for $n = 3$ and with $2 < q < +\infty$, for $n = 1, 2$. In addition, if $n = 1$ one gets

$$\lim_{t \to \pm\infty} \|w(t)\|_{\mathcal{L}_x^\infty} = 0. \tag{1.3.5}$$

Proof. We treat only the case $t \to \infty$, the case $t \to -\infty$ being analogous; we split the proof in two part: we deal first with $n = 3$, and then $n = 1, 2$.

Case $n = 3$. Following the approach of [43] it is sufficient to prove (1.3.4) for a suitable $2 < q < 6$, since the thesis for the general case can be then obtained by the conservation of mass (1.3.1), the kinetic energy (1.3.3) and interpolation. In order to do this we shall prove that

$$\lim_{t\to\pm\infty} \|w(t)\|_{\mathcal{L}_x^{\frac{10}{3}}} = 0. \tag{1.3.6}$$

For this aim we argue as in [87] and we assume by the absurd that there exists $\{t_k\}$ such that

$$\lim_{k\to\infty} t_k = \infty \quad \text{and} \quad \inf_k \|w(t_k,x)\|_{\mathcal{L}_x^{\frac{10}{3}}} = \epsilon_0 > 0. \tag{1.3.7}$$

Next recall the following localized Gagliardo-Nirenberg inequality given in Section 1.4 (see also [56] and [57]):

$$\|\varphi\|_{\mathcal{L}_x^{\frac{2n+4}{n}}}^{\frac{2n+4}{n}} \leqslant C \left(\sup_{x\in\mathbb{R}^3} \|\varphi\|_{\mathcal{L}^2(Q_x)} \right)^{\frac{4}{n}} \|\varphi\|_{\mathcal{H}_x^1}^2, \tag{1.3.8}$$

where Q_x is the unit cube in $\mathbb{R}^3$ centered in x. By combining (1.3.7), (1.3.8) (where we choose $\varphi = w(t_k,x)$) with the bound $\|w(t_k,x)\|_{\mathcal{H}_x^1} < +\infty$, we deduce that

$$\exists x_k \in \mathbb{R}^n \quad \text{such that} \quad \|w(t_k,x)\|_{\mathcal{L}^2(Q_{x_k})} = \delta_0 > 0. \tag{1.3.9}$$

We claim that

$$\exists \bar{t} > 0 \quad \text{such that} \quad \|w(t,x)\|_{\mathcal{L}^2(\tilde{Q}_{x_k})} \geqslant \delta_0/2, \quad \forall t \in (t_k, t_n + \bar{t}), \tag{1.3.10}$$

where $\tilde{Q}_x$ denotes the cube in $\mathbb{R}^n$ of radius 2 centered in x. In order to prove (1.3.10) we fix a cut–off function $\chi(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\chi(x) = 1$ for $|x| < 1$ and $\chi(x) = 0$ for $|x| > 2$. Then by using (1.2.2) where we choose $\phi(x) = \chi(x - x_n)$ we get

$$\left| \frac{d}{dt} \int_{\mathbb{R}^n} \chi(x - x_k)|w(t,x)|^2 dx \right| < C \sup_t \|w(t,x)\|_{\mathcal{H}_x^1}^2.$$

Hence by (1.3.3) and the fundamental theorem of calculus we deduce

$$\left| \int_{\mathbb{R}^n} \chi(x - x_k)|w(s,x)|^2 dx - \int_{\mathbb{R}^n} \chi(x - x_k)|w(t,x)|^2 dx \right| \leqslant C_0|t - s|, \tag{1.3.11}$$

for some $C_0 > 0$ independent of k. Hence if we choose $t = t_k$ we get the elementary inequality

$$\int_{\mathbb{R}^n} \chi(x - x_k)|w(s,x)|^2 dx \geqslant \int_{\mathbb{R}^n} \chi(x - x_k)|w(t_k,x)|^2 dx - C_0|t_k - s|, \tag{1.3.12}$$

which implies (by the compact support property of the function χ)

$$\int_{\tilde{Q}_{x_k}} |w(s,x)|^2 dx \geqslant \int_{Q_{x_k}} |w(t_k,x)|^2 dx - C_0|t_k - s|. \tag{1.3.13}$$

Hence (1.3.10) follows provided that we choose $\bar{t} > 0$ such that $\delta_0^2 - C_0\bar{t} > \delta_0^2/4$. The estimate (1.3.10) contradicts the Morawetz estimates (1.2.24). In fact, the lower bounds (1.3.10) means that

$$\sum_{\mu=1}^{N} \|u_\mu(t)\|^2_{L^2_x(\tilde{Q}_{x_k})} \geqslant C(n)\delta_0^2 > 0, \tag{1.3.14}$$

for any $t \in (t_k, t_k + \bar{t})$ with $\bar{t}$ as above, where we selected the intervals $t \in (t_k, t_k + \bar{t})$ disjoint, and whence, by Hölder inequality, there exists $\bar{\mu} \in \{1, \dots, N\}$ such that

$$\|u_{\bar{\mu}}(t)\|^{\bar{p}}_{L^{\bar{p}}_x(\tilde{Q}_{x_k})} \geqslant C(n)\frac{\delta_0^2}{N}, \tag{1.3.15}$$

for any $\bar{p} \geqslant 2$ and with $t \in (t_k, t_k + \bar{t})$ and $\bar{t}$ as above. Thus we can write the following

$$\begin{aligned} \min_{\mu=1,\dots,N} \beta_{\mu\mu} \sum_{\mu=1}^{N} \int_{\mathbb{R}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} & \frac{|u_\mu(t,x)|^{2p+2}|u_\mu(t,y)|^2}{|x-y|}\, dx\, dy\, dt \\ & \geqslant C \sum_{\mu=1}^{N} \sum_{n} \int_{t_k}^{t_k+\bar{t}} \int_{\mathbb{R}^n} \int_{\tilde{Q}_{x_k} \times \tilde{Q}_{x_k}} |u_\mu(t,x)|^{2p+2}|u_\mu(t,y)|^2\, dx\, dy\, dt \\ & \geqslant C \sum_{n} \int_{t_k}^{t_k+\bar{t}} \delta_0^4\, dt = \infty, \end{aligned} \tag{1.3.16}$$

where in the last inequality we used (1.3.10) in combination with (1.3.14), (1.3.15) and Fubini's Theorem. This leads to the contradiction with (1.2.24).

Case $n = 1, 2$. We can argue as in the previous case just replacing the inequality (1.3.8) by the following version

$$\|\varphi\|^3_{\mathcal{L}^3_x} \leqslant C \left(\sup_{x \in \mathbb{R}^n} \|\varphi\|_{\mathcal{L}^2(Q_x)} \right) \|\varphi\|^2_{\mathcal{H}^1_x}, \tag{1.3.17}$$

(or alternatively by the (1.4.1)) displayed in Section 1.4, with the function φ defined as above. Then proceeding as in the previous step we achieve, for $n = 2$, exactly the same chain of inequalities as in (1.3.16) which is in contradiction with (1.2.29). For $n = 1$ we instead arrive at

$$\begin{aligned} \min_{\mu=1,\dots,N} \beta_{\mu\mu} \sum_{\mu=1}^{N} \int_{\mathbb{R}} \int_{\mathbb{R}} & |u_\mu(t,x)|^{2p+4}\, dt\, dx \\ & \geqslant C \sum_{\mu=1}^{N} \sum_{n} \int_{t_k}^{t_k+\bar{t}} \int_{\tilde{Q}_{x_k}} |u_\mu(t,x)|^{2p+4}\, dt\, dx = \infty, \end{aligned} \tag{1.3.18}$$

but this contradicts the interaction estimate (1.2.28). □

Remark 1.3.4. As stated in the Introduction, we need to have the more stringent lower bound $\max(1, \frac{2}{n})$ with respect to similar one earned in Theorem 0.1 in [87]. Indeed, if we select $0 < p < 1$ the coupling terms $\sum_{\mu,\nu=1}^{N} \beta_{\mu\nu}|u_\nu|^{p+1}|u_\mu|^{p-1}u_\mu$ with $\mu \neq \nu$ give rise to a kind of nonlinearity which could forbid the local well-posedness result for the associated Cauchy problem (1.1.1) such as in Proposition 1.3.1. If one replaces the nonlinear term in (1.1.1) with another model satisfying the assumptions given in the Remark 3.3.12 in [13], then by repeating the argument of this section it should be possible to eliminate the lower bound conditions given in (1.1.2) and (1.1.3). But as of now we are unaware of such references.

1.3.2 Scattering for the NLS system (1.1.1).

This section is devoted to prove Theorem 1.1.1. The results are quite classic (see [13], [43], Chapter 2 and references therein), anyway we present them in the more general form of system framework. We recall from [52] the following.

Definition 1.3.5. An exponent pair (q, r) is Schrödinger-admissible if $2 \leqslant q, r \leqslant \infty$, $(q, r, n) \neq (2, \infty, 2)$, and

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}. \tag{1.3.19}$$

In order to prove Theorem 1.1.1 we need the following lemma.

Lemma 1.3.6. *Assume p is as in (1.1.2), (1.1.3). Then, for any $w \in \mathcal{C}(\mathbb{R}, \mathcal{H}^1_x)$ global solution to (1.1.1), we have*

$$w \in L^q(\mathbb{R}, \mathcal{W}^{1,r}_x), \tag{1.3.20}$$

for every Schrödinger-admissible pair (q, r).

Proof. The proof is a transposition of the Theorem 7.7.3, in [13] and is similar to the proof of the analogous Proposition 2.7.4 in Chapter 2. Let us consider the integral operator associated to (1.1.1)

$$w(t+T) = e^{it\Delta_x}w_0 + \int_0^t e^{i(t-\tau)\Delta_x} g(u(T+\tau), v(T+\tau), p)d\tau \tag{1.3.21}$$

where $t > T > 0$ and

$$w(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_N(t) \end{pmatrix}, \quad w_0 = \begin{pmatrix} u_{1,0} \\ \vdots \\ u_{N,0} \end{pmatrix},$$

$$g(w,p) = \begin{pmatrix} g_1(u_1, \dots, u_N, p) \\ \vdots \\ g_N(u_1, \dots, u_N, p) \end{pmatrix} = \begin{pmatrix} \sum_{\nu=1}^{N} \beta_{1\nu}|u_1|^{p+1}|u_1|^{p-1}u_1 \\ \vdots \\ \sum_{\nu=1}^{N} \beta_{N\nu}|v_\nu|^{p+1}|u_1|^{p-1}u_1 \end{pmatrix}.$$

The thesis is obtained by making an use of the classical inhomogeneous Strichartz estimates (see once again [52]). We point out the details in handling the nonlinear part in (1.3.21), that is the estimate of the following

$$\sum_{\mu=1}^{N} \|g_\mu(u_1, \dots, u_N, p)\|_{L^{q'}((T,t), W^{1,r'}_x)}, \tag{1.3.22}$$

for an appropriate (q,r) Schrödinger-admissible couple: we select (q,r) such that

$$(q,r) := \left(\frac{4(p+1)}{np}, 2p+2\right). \tag{1.3.23}$$

We consider for μ fixed the term

$$g_\mu(u_1,\ldots,u_N,p) = \sum_{\nu=1}^N \beta_{\mu\nu}|u_\nu|^{p+1}|u_\mu|^{p-1}u_\mu,$$

since the others can be handled in a similar way. The Hölder inequality combined with Leibniz fractional rule gives

$$\begin{aligned}
&\|g_\mu(u_1,\ldots,u_N,p)\|_{L^{q'}((T,t),W_x^{1,r'})}\\
&\leqslant C\Big\|\sum_{\nu=1}^N \beta_{\mu\nu}\|u_\mu\|_{W_x^{1,r}}\||u_\nu|^{p+1}|u_\mu|^{p-1}\|_{L_x^{\frac{2p+2}{2p}}}\Big\|_{L^{q'}((T,t))}\\
&\leqslant C\max_{\mu,\nu=1,\ldots,N}\beta_{\mu\nu}\Big\|\|u_\mu\|_{W_x^{1,r}}\sum_{\nu=1}^N\||u_\nu|^{p+1}|u_\mu|^{p-1}\|_{L_x^{\frac{2p+2}{2p}}}\Big\|_{L^{q'}((T,t))}.
\end{aligned} \tag{1.3.24}$$

From the following pointwise Young inequality (see for instance [46])

$$|u_\nu|^{p+1}|u_\mu|^{p-1} + |u_\mu|^{p+1}|u_\nu|^{p-1} \leqslant C(p)\left(|u_\mu|^{2p} + |u_\nu|^{2p}\right),$$

and setting $\beta = \max_{\mu,\nu=1,\ldots,N}\beta_{\mu\nu}$, we see that the last term of the inequality above is not greater than

$$\begin{aligned}
&\widetilde{C}(p,\beta)\Big\|\|u\|_{W_x^{1,r}}\sum_{\nu=1}^N\|u_\nu\|^{2p}_{L_x^{2p+2}}\Big\|_{L^q((T,t))}\\
&\lesssim \Big\|\sum_{\mu=1}^N\|u_\mu\|_{W_x^{1,r}}\cdot\sum_{\nu=1}^N\|u_\nu\|^{2p}_{L_x^{2p+2}}\Big\|_{L^{q'}((T,t))}\\
&\lesssim \Big\|\sum_{\mu=1}^N\|u_\mu\|_{W_x^{1,r}}\cdot\Big(\sum_{\nu=1}^N\|u_\nu\|^{2p+1-\frac{q}{q'}}_{L_x^{2p+2}}\|u_\nu\|^{\frac{q}{q'}-1}_{L_x^{2p+2}}\Big)\Big\|_{L^{q'}((T,t))}\\
&\lesssim \Big\|\sum_{\mu=1}^N\|u_\mu\|_{W_x^{1,r}}\cdot\Big(\sum_{\nu=1}^N\|u_\nu\|_{L_x^{2p+2}}\Big)^{\frac{q}{q'}-1}\cdot\sum_{\kappa=1}^N\|u_\kappa\|^{2p+1-\frac{q}{q'}}_{L_x^{2p+2}}\Big\|_{L^{q'}((T,t))},
\end{aligned} \tag{1.3.25}$$

with all the constants involved in the inequalities above independent from t,T. Notice that $(2p+1)q'-q>0$ so the last term of the above chain of inequalities can be bounded by

$$\widetilde{C}(p,\beta)\Big\|\Big(\sum_{\kappa=1}^N\|u_\kappa\|_{W_x^{1,r}}\Big)^{\frac{q}{q'}}\Big(\sum_{\nu=1}^N\|u_\nu\|_{L_x^{2p+2}}\Big)^{2p+1-\frac{q}{q'}}\Big\|_{L^{q'}((T,t))}, \tag{1.3.26}$$

here we used without any distinction the dummy indices μ,ν and κ because defined on the same set. Summing in μ the (1.3.26) above we get that the quantity in (1.3.22) is bounded by

$$C\sup_{\tau>T}\Big(\sum_{\nu=1}^N\|u_\nu(\tau)\|_{L_x^{2p+2}}\Big)^{2p+1-\frac{q}{q'}}\Big(\sum_{\kappa=1}^N\|u_\kappa\|_{L^q((T,t)),W_x^{1,r})}\Big)^{q-1}, \tag{1.3.27}$$

with $C > 0$. The premises above, the equation (1.3.21) and the Proposition 1.3.3, in connection with an use of the inhomogenehouse Strichartz estimates bring to

$$\|w\|_{L^q((T,t),\mathcal{W}_x^{1,r})} \leqslant C\|w_0\|_{\mathcal{H}_x^1} + \eta(T)\left(\|w\|_{L^q((T,t),\mathcal{W}_x^{1,r})}\right)^{q-1}, \tag{1.3.28}$$

where $\eta(T) \to 0$ as $T \to \infty$.

Thanks to the Lemma 7.7.4 in [13], for T large enough we have

$$\|w\|_{L^q((T,t),\mathcal{W}_x^{1,r})} \leqslant \bar{C},$$

with the constant $\bar{C}$ independent from t. In that way we get that $w \in L^q((T,\infty),\mathcal{W}_x^{1,r})$, and one can use a similar argument in order to have $w \in L^q((-\infty,-T),\mathcal{W}_x^{1,r})$. From this fact we conclude immediately that $w \in L^q(\mathbb{R},\mathcal{W}_x^{1,r})$. □

Proof of Theorem 1.1.1. The proof of Theorem 1.1.1 is now a straightforward adaptation of Theorem 7.8.1 and Theorem 7.8.4 in [13]: we shortly prove it here for the sake of completeness.
Asymptotic completeness: Let us write $\overline{w}(t) = e^{-it\Delta_x}w(t)$, we get

$$\overline{w}(t) = w_0 + i\int_0^t e^{-is\Delta_x}g(w,p)ds, \tag{1.3.29}$$

moreover one has, for $0 < t < t_1$,

$$\overline{w}(t) - \overline{w}(t_1) = i\int_{t_1}^t e^{-is\Delta_x}g(w,p)ds. \tag{1.3.30}$$

An application of classical Strichartz estimates yields

$$\begin{aligned}\|\overline{w}(t) - \overline{w}(t_1)\|_{\mathcal{H}_x^1} &\lesssim \|e^{it\Delta_x}(\overline{w}(t) - \overline{w}(t_1))\|_{\mathcal{H}_x^1}\\ &\lesssim \|g(w,p)\|_{L^{q'}((t,t_1),\mathcal{W}_x^{1,r})}\end{aligned} \tag{1.3.31}$$

where (q,r) is a Schrödinger-admissible pair as in (1.3.23). Following the proof of Proposition 1.3.6 we achieve

$$\lim_{t,t_1\to\infty}\|\overline{w}(t) - \overline{w}(t_1)\|_{\mathcal{H}_x^1} = 0.$$

Thus we can say that there exist $(u_{1,0}^{\pm},\dots,u_{N,0}^{\pm}) \in H^1(\mathbb{R}^n)^N$ such that exist $(u_1(t),\dots,u_N(t)) \to (u_{1,0}^{\pm},\dots,u_{N,0}^{\pm})$ in $H^1(\mathbb{R}^n)^N$ as $t \to \pm\infty$. Notice also that, by Proposition 1.3.1, we have also the following properties verified

$$\begin{aligned}&\|(u_{1,0}^{\pm},\dots,u_{N,0}^{\pm})\|_{\mathcal{L}_x^2} = \|(u_{1,0},\dots,u_{N,0})\|_{\mathcal{L}_x^2},\\ &\sum_{\mu=1}^N\int_{\mathbb{R}^n}|\nabla u_{\mu,0}^{\pm}|dx = E(u_{1,0},\dots,u_{N,0}).\end{aligned} \tag{1.3.32}$$

Existence of wave operators: Let us select a Schrödinger-admissible pair as in (1.3.23) and introduce $v(t) = e^{it\Delta_x}w_0^+$ (the proof for $w_0^-(t)$ is analogous). Then by the Strichartz estimates and Corollary 2.3.7 in [13] we get that, for $T > 0$,

$$\widetilde{\mathscr{H}}(T) = \|v(t)\|_{L^q([T,\infty],\mathcal{W}_x^{1,r})} + \sup_{t\geqslant T}\|v(t)\|_{\mathcal{L}_x^r}, \tag{1.3.33}$$

is a decreasing function w.r.t. the T variable and such that $\widetilde{\mathscr{H}}(T) \to 0$ as $T \to \infty$. As a consequence we are allowed to introduce the complete metric space $Z \subseteq L^q([T,\infty), \mathcal{W}_x^{1,r})$ defined as

$$Z = \left\{ w(t) : \|w(t)\|_{L^q((T,\infty),\mathcal{W}_x^{1,r})} + \sup_{t \geqslant T} \|w(t)\|_{\mathcal{L}_x^r} \leqslant 2\widetilde{\mathscr{H}}(T) \right\} \tag{1.3.34}$$

and equipped with the topology induced by $\|.\|_{L^q([T,\infty),\mathcal{L}_x^r)}$. Let be

$$\mathcal{I}(w)(t) = -i \int_t^{\pm\infty} e^{i(t-\tau)\Delta_x} g(u(\tau), v(\tau), p) d\tau, \tag{1.3.35}$$

with

$$\mathcal{I}(w) \in C([T,\infty), \mathcal{H}_x^1) \cap L^q((T,\infty), \mathcal{W}_x^{1,r}).$$

(see [52] or the Corollary 2.3.6 in [13], easily generalizable to a system of equations). Furthermore, by the inequality (see Lemma 1.3.6)

$$\|g(w,p)\|_{L^{q'}((T,\infty),\mathcal{W}_x^{1,r})} \leqslant C(\widetilde{\mathscr{H}}(T))^{2p+1}$$

with $w = (u,v) \in Z$, in combination with the behavior of $\widetilde{\mathscr{H}}(T)$ for T large enough and the Sobolev embedding inequality we achieve the following estimate

$$\|w(t)\|_{L^q((T,\infty),\mathcal{W}_x^{1,r})} + \sup_{t \geqslant T} \|w(t)\|_{L^\infty((T,\infty),\mathcal{L}_x^r)} \leqslant \widetilde{\mathscr{H}}(T), \tag{1.3.36}$$

for T large enough. By the estimate (1.3.33) and the (1.3.36) above we conclude that the operator

$$\mathcal{K}(w) = e^{it\Delta_x} w_0^+ + \mathcal{I}(w) \tag{1.3.37}$$

is a contraction on Z with respect the norm $\|.\|_{L^q([T,\infty),\mathcal{W}_x^{1,r})}$. By applying a fixed point argument we get that there exists $w \in Z$ satisfying the equation (1.3.37). In addition $w \in C([T,\infty), \mathcal{H}_x^1)$. By classical arguments one can show also that w is a global solution to the equation (1.1.1) and then $w(0) = w_0 \in \mathcal{H}_x^1$ is well defined. Furthermore the properties (1.1.5) is fulfilled. The proof of the remaining part regarding the uniqueness reads as in Theorem 7.8.4 in [13], so we skip it. □

1.4 A localized Gagliardo-Nirenberg inequality

The principal target of this section is to prove of the localized inequality (1.3.8) used in the proof of Proposition 1.3.3 (see Section 1.3.1). Albeit it already appeared in the literature (see for example [87], [56, 57] or [85] in the context of product space $\mathbb{R}^n \times M$, with M^k any k-dimensional compact manifold), we recall it in a more general form. We have

Proposition 1.4.1. *Let be $n \geqslant 1$ and $\alpha \in \mathbb{N}$, then for all vector-valued functions $\phi = (\phi_\ell)_{\ell=1}^\alpha \in H^1(\mathbb{R}^n)^\alpha$ one gets the following*

$$\|\phi\|_{L^{\frac{2n+4}{n}}(\mathbb{R}^n)^\alpha}^{\frac{2n+4}{n}} \leqslant C \left(\sup_{x \in \mathbb{R}^n} \|\phi\|_{L^2(Q_x)^\alpha} \right)^{\frac{4}{n}} \|\phi\|_{H^1(\mathbb{R}^n)^\alpha}^2. \tag{1.4.1}$$

Proof. Consider an open covering of $\mathbb{R}^n$ given by a family of disjoint cubes $\{Q_s\}_{s\in\mathbb{N}}$.

Let us look at the high dimensional case $n \geqslant 3$. For any component of $\phi = (\phi_1, \dots, \phi_\alpha)$ one has that $\phi_\ell \in L^{\frac{2n}{n-2}}$, $\ell = 1, \dots, \alpha$, then the Sobolev embedding and an application of Hölder inequality bring to the chain of inequalities

$$\begin{aligned} \sum_{\ell=1}^{\alpha} \int_{Q_s} |\phi_\ell|^{\frac{2n+4}{n}} \leqslant & C \sum_{\ell=1}^{\alpha} \left(\int_{Q_s} |\phi_\ell|^2 \right)^{\frac{2}{n}} \left(\int_{Q_s} |\phi_\ell|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \\ \leqslant & C \sum_{\ell=1}^{\alpha} \left(\int_{Q_s} |\phi_\ell|^2 \right)^{\frac{2}{n}} \left(\int_{Q_s} |\phi_\ell|^{\frac{2n}{n-2}} \right)^{\frac{2(n-2)}{2n}} \\ \leqslant & C(\alpha) \left(\sum_{\ell=1}^{\alpha} \|\phi_\ell\|_{L^2(Q_s)} \right)^{\frac{4}{n}} \sum_{\ell=1}^{\alpha} \|\phi_\ell\|^2_{H^1(Q_s)}. \end{aligned} \tag{1.4.2}$$

The estimates above can be rewritten as

$$\|\phi\|^{\frac{2n+4}{n}}_{L^{\frac{2n+4}{n}}(Q_s)^\alpha} \leqslant C \left(\|\phi\|_{L^2(Q_s)^\alpha} \right)^{\frac{4}{n}} \|\phi\|^2_{H^1(Q_s)^\alpha}, \tag{1.4.3}$$

and hence summing over s we arrive at

$$\begin{aligned} \|\phi\|^{\frac{2n+4}{n}}_{L^{\frac{2n+4}{n}}(\mathbb{R}^n)^\alpha} \leqslant & C \left(\sup_{s\in\mathbb{N}} \|\phi\|_{L^2(Q_s)^\alpha} \right)^{\frac{4}{n}} \sum_{s\in\mathbb{N}} \|\phi\|^2_{H^1(Q_s)^\alpha} \\ \leqslant & C \left(\sup_{s\in\mathbb{N}} \|\phi\|_{L^2(Q_s)^\alpha} \right)^{\frac{4}{n}} \|\phi\|^2_{H^1(\mathbb{R}^n)^\alpha}. \end{aligned} \tag{1.4.4}$$

We remark that the estimate above is translation invariant, hence the constants are independent from s. Rhe estimate (1.4.1) follows from the above (1.4.4) in combination with the fact $\{Q_s\}_{s\in\mathbb{N}} \subset \{Q_x\}_{x\in\mathbb{R}^n}$.

The remaining cases, $n = 1, 2$, can be handled in the same way as before with minor changes. For $n = 2$ we need to replace the estimate (1.4.2) by the following

$$\begin{aligned} \sum_{\ell=1}^{\alpha} \int_{Q_s} |\phi_\ell|^4 \leqslant & C \sum_{\ell=1}^{\alpha} \left(\int_{Q_s} |\phi_\ell|^2 \right) \|\phi_\ell\|^2_{H^1(Q_s)} \\ \leqslant & C(\alpha) \left(\sum_{\ell=1}^{\alpha} \|\phi_\ell\|_{L^2(Q_s)} \right)^2 \sum_{\ell=1}^{\alpha} \|\phi_\ell\|^2_{H^1(Q_s)}, \end{aligned} \tag{1.4.5}$$

which can be carried out taking $u = |\phi_l|^2$ in the following Sobolev inequality

$$\|u\|_{L^2(Q_s)} \lesssim \|u\|_{L^1(Q_s)} + \|\nabla u\|_{L^1(Q_s)},$$

and by an use of Leibniz chain rule. Then one argues as in the proof for the higher dimensions case. For the last case, that is $n = 1$, we use instead of (1.4.5) the following

$$\begin{aligned} \sum_{\ell=1}^{\alpha} \int_{Q_s} |\phi_\ell|^6 \leqslant & C \sum_{\ell=1}^{\alpha} \left(\int_{Q_s} |\phi_\ell|^2 \right)^2 \|\phi_\ell\|^2_{H^1(Q_s)} \\ \leqslant & C \left(\sum_{\ell=1}^{\alpha} \|\phi_\ell\|_{L^2(Q_s)} \right)^4 \sum_{\ell=1}^{\alpha} \|\phi_\ell\|^2_{H^1(Q_s)}, \end{aligned} \tag{1.4.6}$$

which can be earned by using once again the inhomogeneous Sobolev embedding $W_x^{1,1} \subset L_x^\infty$ and Leibniz chain rule. The proof is then concluded analogously. □

Remark 1.4.2. Following the paper [87], we can also obtain a variant of the inequality (1.4.1) in the cases $n = 1, 2$. The Sobolev embedding $W_x^{1,1} \subset L_x^2$ enables us to write the easy localized estimate

$$\begin{aligned} \sum_{\ell=1}^{\alpha} \int_{Q_s} |\phi_\ell|^3 &\leqslant C \sum_{\ell=1}^{\alpha} \left(\int_{Q_s} |\phi_\ell|^2 \right)^{\frac{1}{2}} \| |\phi_\ell|^2 \|_{W^{1,1}(Q_s)} \\ &\leqslant C(\alpha) \left(\sum_{\ell=1}^{\alpha} \|\phi_\ell\|_{L^2(Q_s)} \right) \sum_{\ell=1}^{\alpha} \|\phi_\ell\|_{H^1(Q_s)}^2, \end{aligned} \tag{1.4.7}$$

this fact, arguing as in the proof of the Lemma 1.4.1 leads to the estimate

$$\|\varphi\|_{L^3(\mathbb{R}^n)^\alpha}^3 \leqslant C \left(\sup_{x \in \mathbb{R}^n} \|\phi\|_{L^2(Q_x)^\alpha} \right) \|\phi\|_{H^1(\mathbb{R}^n)^\alpha}^2, \tag{1.4.8}$$

that is the inequality (1.3.17).

Chapter 2

Scattering in the energy space for the NLS with variable coefficients

We consider the NLS with variable coefficients in dimension $n \geqslant 3$

$$i\partial_t u - Lu + f(u) = 0, \qquad Lv = \nabla^b \cdot (a(x)\nabla^b v) - c(x)v, \qquad \nabla^b = \nabla + ib(x),$$

on $\mathbb{R}^n$ or more generally on an exterior domain with Dirichlet boundary conditions, for a gauge invariant, defocusing nonlinearity of power type $f(u) \simeq |u|^{\gamma-1}u$. We assume that L is a small, long range perturbation of Δ, plus a potential with a large positive part. The first main result of the chapter is a bilinear smoothing (interaction Morawetz) estimate for the solution.

As an application, under the conditional assumption that Strichartz estimates are valid for the linear flow e^{itL}, we prove global well posedness in the energy space for subcritical powers $\gamma < 1 + \frac{4}{n-2}$, and scattering provided $\gamma > 1 + \frac{4}{n}$. When the domain is $\mathbb{R}^n$, by extending the Strichartz estimates due to Tataru [84], we prove that the conditional assumption is satisfied and deduce well posedness and scattering in the energy space. The reference for the present Chapter is [10].

2.1 Introduction

We study the Cauchy problem in the energy space for the semilinear Schrödinger equation

$$i\partial_t u - Lu + f(u) = 0, \qquad u(0,x) = u_0(x) \tag{2.1.1}$$

on an exterior domain $\Omega = \mathbb{R}^n \setminus \omega$ with C^1 boundary, in dimension $n \geqslant 3$, where ω is compact and possibly empty. Here L is a second order elliptic operator defined on Ω with Dirichlet boundary conditions, of the form

$$Lv = \nabla^b \cdot (a(x)\nabla^b v) - c(x)v, \qquad \nabla^b = \nabla + ib(x), \tag{2.1.2}$$

where $a(x) = [a_{jk}(x)]_{j,k=1}^n$, $b(x) = (b_1(x), \dots, b_n(x))$ and $c(x)$ satisfy

$$a, b, c \text{ are real valued, } a_{jk} = a_{kj} \text{ and } NI \geqslant a(x) \geqslant \nu I \text{ for some } N \geqslant \nu > 0. \tag{2.1.3}$$

The low dimensional cases $n \leqslant 2$ require substantial modifications of our techniques and will be the object of future work.

Our main results can be summarized as follows. Assume that

(i) the principal part of L is a small, long range perturbation of Δ;

(ii) b, c have an almost critical decay, with b and $c_- := \max\{0, -c\}$ small;

(iii) the boundary $\partial\Omega$ is starshaped with respect to the metric induced by $a(x)$;

(iv) the nonlinearity $f(u) \simeq |u|^{\gamma-1}u$ is of power type, gauge invariant, defocusing, with γ in the subcritical range $1 \leqslant \gamma < 1 + \frac{4}{n-2}$.

Then we prove:

1. a virial identity for (2.1.1), from which we deduce a smoothing and a bilinear smoothing (interaction Morawetz) estimate for solutions of (2.1.1).

2. global well posedness and scattering in the energy space for the Cauchy problem (2.1.1), under the black box assumption that Strichartz estimates are valid for the linear flow e^{itL}; scattering requires $\gamma > 1 + \frac{4}{n}$.

3. in the case $\Omega = \mathbb{R}^n$, we extend the Strichartz estimates proved by Tataru [84] to the case of large electric potentials; hence we can drop the black box assumption and we obtain well posedness and scattering in the energy space for (2.1.1).

Note that for exterior domains, Strichartz estimates are known but only locally in time, see e.g. [60], [3] and the references therein. However, research on this topic is advancing rapidly, thus in the general case $\Omega \neq \mathbb{R}^n$ we decided to assume *a priori* the validity of Strichartz estimates. In the case $\Omega = \mathbb{R}^n$ sufficiently strong results are already available and we use them to close the proof of scattering. On a related note we mention the global smoothing estimates on the exterior of polygonal domains proved in [3].

The theory of Strichartz estimates on $\mathbb{R}^n$ is extensive and many results are known. We mention in particular [90], [89], [89], [75] [23] for the case of electric potentials, [24] and [29] for magnetic potentials, and, for operators with fully variable coefficients, [79], [74] and [84] (see also the refences therein). Note that large perturbations in the second order terms require suitable nontrapping assumptions, which are implicit here in the assumption that $|a(x) - I|$ is sufficiently small.

Scattering theory is a important subject and the number of references is huge. For a comprehensive review of the classical theory and an extensive bibliography we refer to [13] (see also [43]). Smoothing estimates are also a classical subject, originated in [50] and [63], [62]. The bilinear version of smoothing estimates, also called *interaction Morawetz* estimates, was introduced as a tool in scattering theory in [17], [83] and recently adapted to Schrödinger equations with an electromagnetic potential in [18]. We mention that here we follow the simpler approach developed in [87], [12].

We conclude the introduction with a detailed exposition of our results. Here and in the rest of the chapter we make frequent use of the basic properties of Lorentz spaces $L^{p,q}$, in particular precised Hölder, Young and Sobolev inequalities, for which we refer to Section 2.9.

In the following we denote by $|a(x)|$ the operator norm of the matrix $a(x)$, and we use the notations

$$|a'| = \textstyle\sum_{|\alpha|=1} |\partial^\alpha a(x)|, \qquad |a''| = \sum_{|\alpha|=2} |\partial^\alpha a(x)|, \qquad |a'''| = \sum_{|\alpha|=3} |\partial^\alpha a(x)|,$$

$$|b'| = \textstyle\sum_{j,k} |\partial_{x_j} b_k|, \qquad |c'| = \sum_j |\partial_{x_j} c|.$$

2.1.1 The operator L and its heat kernel e^{tL}

The results of this section are valid for all dimensions $n \geqslant 3$. Very mild conditions on the coefficients of L are sufficient for selfadjointness: in Proposition 2.6.1 we prove by standard arguments that if

$$b \in L^{n,\infty}, \qquad c \in L^{\frac{n}{2},\infty}, \qquad \|c_-\|_{L^{\frac{n}{2},\infty}} < \epsilon, \tag{2.1.4}$$

with ϵ small enough (and $a(x) \in L^\infty$), then the operator L defined on $C_c^\infty(\Omega)$ extends in the sense of forms to a selfadjoint, nonpositive operator with domain $H_0^1(\Omega) \cap H^2(\Omega)$. Throughout the chapter, this operator will be referred to as the operator L with Dirichlet boundary conditions; note that in all our results the assumptions are stronger than (2.1.4).

Under the additional assumption

$$b^2 + |\nabla \cdot b| \in L^2_{loc}, \qquad c \in L^{\frac{n}{2},1}, \qquad \|c_-\|_{L^{\frac{n}{2},1}} < \epsilon$$

with ϵ small enough, we prove in Proposition 2.6.2 that the heat kernel of L satisfies a gaussian upper estimate of the form

$$|e^{tL}(x,y)| \leqslant C' t^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{Ct}}, \qquad t > 0.$$

In Proposition 2.6.3, assuming further that

$$\|a - I\|_{L^\infty} + \||b| + |a'|\|_{L^{n,\infty}} + \|b'\|_{L^{\frac{n}{2},\infty}} < \epsilon$$

for ϵ small enough, using the previous bound we deduce the equivalence

$$\|(-L)^\sigma v\|_{L^p} \simeq \|(-\Delta)^\sigma v\|_{L^p}, \qquad 1 < p < \frac{n}{2\sigma}, \qquad 0 \leqslant \sigma \leqslant 1. \tag{2.1.5}$$

2.1.2 Morawetz and interaction Morawetz estimates

From now on we restrict to the case when the operator L is a suitable long range perturbation of Δ on Ω; the precise conditions are the following.

Let $n \geqslant 3$ and assume that for some $0 < \delta \leqslant 1$

$$|a'(x)| + |x||a''(x)| + |x|^2|a'''(x)| \leqslant C_a \langle x \rangle^{-1-\delta}, \tag{2.1.6}$$

where $\langle x \rangle := (1 + |x|^2)^{1/2}$. Moreover, b and the matrix $db(x) := [\partial_j b_\ell - \partial_\ell b_j]_{j,\ell=1}^n$ satisfy

$$b \in L^{n,\infty}, \qquad |db(x)| \leqslant \tfrac{C_b}{|x|^{2+\delta} + |x|^{2-\delta}}. \tag{2.1.7}$$

The potential $c(x)$ satisfies

$$-\tfrac{C_-^2}{|x|^2} \leqslant c(x) \leqslant \tfrac{C_+^2}{|x|^2} \tag{2.1.8}$$

(which implies $c \in L^{\frac{n}{2},\infty}$) and is *repulsive* with respect to the metric $a(x)$, meaning that

$$a(x)x \cdot \nabla c(x) \leqslant \tfrac{C_c}{|x|\langle x\rangle^{1+\delta}}. \tag{2.1.9}$$

The nonlinearity $f\colon \mathbb{C} \to \mathbb{C}$ is such that $f(0) = 0$ and, for some $1 \leqslant \gamma < 1 + \frac{4}{n-2}$,

$$|f(z) - f(w)| \leqslant (|z| + |w|)^{\gamma-1}|z - w|, \qquad \text{for all } z, w \in \mathbb{C}. \tag{2.1.10}$$

We remark that in fact we can consider more general nonlinearities f such that $|f(z) - f(w)| \leqslant (1 + |z|^{\gamma-1} + |w|^{\gamma-1})|z - w|$: for the sake of simplicity, we will just assume (2.1.10), since the more general case can be easily obtained adapting our proofs. We also assume that f is *gauge invariant*, that is to say

$$f(\mathbb{R}) \subseteq \mathbb{R} \quad \text{and} \quad f(e^{i\theta}z) = e^{i\theta}f(z) \quad \text{for all } \theta \in \mathbb{R},\ z \in \mathbb{C}. \tag{2.1.11}$$

Moreover, writing

$$F(z) := \textstyle\int_0^{|z|} f(s)\, ds, \tag{2.1.12}$$

we assume that f is *repulsive*, i.e.,

$$f(z)\bar{z} - 2F(z) \geqslant 0 \ \text{ for all } z \in \mathbb{C}. \tag{2.1.13}$$

Finally, concerning the domain Ω, we assume that $\partial\Omega$ is C^1 and $a(x)$–*starshaped*, meaning that at all points $x \in \partial\Omega$ the exterior normal $\vec{\nu}$ to $\partial\Omega$ satisfies

$$a(x)x \cdot \vec{\nu}(x) \leqslant 0. \tag{2.1.14}$$

In the following statement we use the Morrey-Campanato type norms defined by

$$\|v\|^2_{\dot{X}} := \sup_{R>0} \tfrac{1}{R^2} \textstyle\int_{\Omega\cap\{|x|=R\}} |v|^2 dS, \qquad \|v\|^2_{\dot{Y}} := \sup_{R>0} \tfrac{1}{R} \textstyle\int_{\Omega\cap\{|x|\leqslant R\}} |v|^2 dx.$$

Moreover we use the notation $L^2_T = L^2(0,T)$ to denote integration in t on the interval $[0,T]$, while $L^p_T L^q = L^p(0,T; L^q(\Omega))$ and $L^p L^q = L^p(\mathbb{R}; L^q(\Omega))$.

Theorem 2.1.1 (Smoothing)**.** *Let $n \geqslant 4$, L the operator in (2.1.2), (2.1.3) with Dirichlet b.c. on the exterior domain Ω, and assume (2.1.6), (2.1.7), (2.1.9) and (2.1.14). Let $u \in C(\mathbb{R}, H^1_0(\Omega))$ be a solution of Problem (2.1.1). Then, if $N/\nu - 1$ and the constants C_a, C_b, C_-, C_c are sufficiently small, u satisfies for all $T > 0$ the estimate*

$$\|u\|^2_{\dot{X}_x L^2_T} + \|\nabla^b u\|^2_{\dot{Y}_x L^2_T} + \textstyle\int_0^T \int_\Omega \frac{f(u)\overline{u} - 2F(u)}{|x|}\, dxdt \lesssim \|u(0)\|^2_{\dot{H}^{\frac{1}{2}}} + \|u(T)\|^2_{\dot{H}^{\frac{1}{2}}} \tag{2.1.15}$$

with an implicit constant independent of T.

Theorem 2.1.1 actually holds even in the case $n = 3$, but we need a condition on $a(x)$ which essentially forces it to be diagonal, and this is of course too restrictive for our purposes (see (2.4.2) below). Thus in the 3D case we modify our approach and prove an estimate in terms of *nonhomogeneous* Morrey-Campanato norms

$$\|v\|^2_{X} := \sup_{R>0} \tfrac{1}{\langle R\rangle^2} \textstyle\int_{\Omega\cap\{|x|=R\}} |v|^2 dS, \qquad \|v\|^2_{Y} := \sup_{R>1} \tfrac{1}{R} \textstyle\int_{\Omega\cap\{|x|\leqslant R\}} |v|^2 dx.$$

We also need some slightly stronger assumptions on the coefficients: we require

$$|a(x) - I| \leqslant C_I \langle x \rangle^{-\delta}, \qquad C_I < 1, \tag{2.1.16}$$

moreover we assume

$$b \in L^{3,\infty}, \qquad |db(x)| \leqslant \tfrac{C_b}{|x|^{2+\delta}+|x|}. \tag{2.1.17}$$

Then we have:

Theorem 2.1.2 (Smoothing, $n = 3$). *Let L the operator in (2.1.2), (2.1.3) with Dirichlet b.c. on the exterior domain Ω, and assume (2.1.6), (2.1.16) (2.1.17), (2.1.8), (2.1.9), (2.1.11), (2.1.13), and (2.1.14). Let $u \in C(\mathbb{R}, H^1_0(\Omega))$ be a solution of Problem (2.1.1). Then, if $N/\nu - 1$ and the constants C_a, C_I, C_b, C_-, C_c are sufficiently small, the solution u satisfies for all $T > 0$ the estimate*

$$\|u\|^2_{X_x L^2_T} + \|\nabla^b u\|^2_{Y_x L^2_T} + \int_0^T \int_\Omega \tfrac{f(u)\overline{u} - 2F(u)}{\langle x \rangle}\, dxdt \lesssim \|u(0)\|^2_{\dot{H}^{\frac{1}{2}}} + \|u(T)\|^2_{\dot{H}^{\frac{1}{2}}} \tag{2.1.18}$$

with an implicit constant independent of T.

The previous results are *a priori* estimates on a global solution u, for which conservation of energy might not hold; this is why we state estimates (2.1.15),(2.1.18) on a finite time interval $[0, T]$ and we need the norm of u both at $t = 0$ and at $t = T$ at the right hand side. Note that it is possible to give explicit bounds on the smallness assumption on the coefficients, see Remark 2.4.1.

Remark 2.1.3. The proofs of Theorems 2.1.1 and 2.1.2 have a substantial overlap with the proof in [8] of resolvent estimates for the Helmholtz equation

$$Lu + zu = f, \qquad z \in \mathbb{C} \setminus \mathbb{R}.$$

One can indeed deduce estimates for the *linear* Schrödinger equation from the corresponding estimates for Helmholtz, via Kato's theory of smoothing [51], but with a loss in the sharpness of the estimates (see Corollary 1.3 in [8] for details; see also [6] for earlier results in a simpler setting).

Remark 2.1.4. Note that in (2.1.15) and (2.1.18) the space-time norms are reversed in (x, t), due to the method of proof. In the hypoteses of Theorem 2.1.1, thanks to (2.1.15) and (2.2.7), (2.2.9), and in the hypoteses of Theorem 2.1.2, thanks to (2.1.18) and (2.2.9), (2.2.12), we deduce the standard weighted L^2 estimate

$$\|\langle x \rangle^{-3/2-} u\|^2_{L^2_T L^2_x} + \|\langle x \rangle^{-1/2-} \nabla^b u\|^2_{L^2_T L^2_x} + \int_0^T \int_\Omega \tfrac{f(u)\overline{u} - 2F(u)}{|x|}\, dxdt \lesssim \|u(0)\|^2_{\dot{H}^{\frac{1}{2}}} + \|u(T)\|^2_{\dot{H}^{\frac{1}{2}}}. \tag{2.1.19}$$

By (2.2.16) we can replace ∇^b with ∇ at the left hand side, obtaining

$$\|\langle x \rangle^{-3/2-} u\|^2_{L^2_T L^2_x} + \|\langle x \rangle^{-1/2-} \nabla u\|^2_{L^2_T L^2_x} \lesssim \|u(0)\|^2_{\dot{H}^{\frac{1}{2}}} + \|u(T)\|^2_{\dot{H}^{\frac{1}{2}}}. \tag{2.1.20}$$

If the assumptions on b, c are slightly stronger so that the heat kernel e^{tL} satisfies an upper gaussian bound, we can apply the techniques in [7] to obtain a further estimate of weighted L^2 tipe. In the next Corollary we assume $\Omega = \mathbb{R}^n$ to keep the proof simple but this would not be necessary.

Corollary 2.1.5. *Let $n \geqslant 3$, $\Omega = \mathbb{R}^n$, let L be as in Theorem 2.1.1 or as in Theorem 2.1.2, and assume that*

$$b^2 + |\nabla \cdot b| \in L^2_{loc}, \qquad c \in L^{\frac{n}{2},1}, \qquad \|c_-\|_{L^{\frac{n}{2},1}} < \epsilon.$$

Then for ϵ small enough the flow e^{itL} satisfies the estimate

$$\|\langle x\rangle^{-1-} e^{itL} u_0\|_{L^2_t L^2_x} \lesssim \|u_0\|_{L^2}. \tag{2.1.21}$$

The next results are bilinear smoothing (interaction Morawetz) estimates for equation (2.1.1), which are the crucial tool in the proof of scattering. Note that the assumptions are essentially the same as in Theorems 2.1.1, 2.1.2, and the constant $C_{c'}$ may be large.

Theorem 2.1.6 (Bilinear smoothing, $n \geqslant 4$)**.** *Let $n \geqslant 4$ and let Ω, L be as in Theorem 2.1.1. In addition, assume that*

$$|x|^2 |\nabla c| \leqslant C_{c'} \langle x\rangle^{-1-\delta}. \tag{2.1.22}$$

Let $u \in C(\mathbb{R}, H^1_0(\Omega))$ be a solution of (2.1.1). Then, if the constants C_a, C_b, C_-, C_c and $N/\nu - 1$ are small enough, u satisfies the estimate

$$\int_0^T \int_{\Omega\times\Omega} \frac{|u(t,x)|^2 |u(t,y)|^2}{|x-y|^3} dxdydt \lesssim \|u(0)\|^2_{L^2} \left[\|u(0)\|_{\dot H^{\frac12}} + \|u(T)\|_{\dot H^{\frac12}}\right]^2. \tag{2.1.23}$$

Theorem 2.1.7 (Bilinear smoothing, $n = 3$)**.** *Let $n = 3$ and let Ω, L be as in Theorem 2.1.2. In addition, assume (2.1.22). Let $u \in C(\mathbb{R}, H^1_0(\Omega))$ be a solution of (2.1.1). Then, if the constants C_a, C_I, C_b, C_-, C_c and $N/\nu - 1$ are small enough, u satisfies the estimate*

$$\|u\|^4_{L^4(0,T;L^4(\Omega))} \lesssim \|u(0)\|^2_{L^2} \left[\|u(0)\|_{\dot H^{\frac12}} + \|u(T)\|_{\dot H^{\frac12}}\right]^2. \tag{2.1.24}$$

2.1.3 Global existence and scattering

The proof of well posedness and scattering for (2.1.1) in the energy space relies in an essential way on Strichartz estimates for the linear flow e^{itL}. As mentioned above, these are known in the case $\Omega = \mathbb{R}^n$ under various assumptions on the coefficients, while the results for exterior domains are far from complete. For this reason we decided to state our main results by assuming the validity of Strichartz estimates in a black box form, and then specialize them to some situations where Strichartz estimates are already available. Recalling that an *admissible (non endpoint) couple* is a couple of indices (p, q) with $2 < p \leqslant \infty$ and $2/p + n/q = n/2$, our black box assumption has the following form:

Assumption (S). *The Schrödinger flow e^{itL} satisfies the Strichartz estimates*

$$\|e^{itL} u_0\|_{L^{p_1} L^{q_1}} \lesssim \|u_0\|_{L^2}, \qquad \|\textstyle\int_0^t e^{i(t-s)L} F ds\|_{L^{p_1} L^{q_1}} \lesssim \|F\|_{L^{p_2'} L^{q_2'}} \tag{2.1.25}$$

for all admissible couples (p_j, q_j), while the derivative of the flow ∇e^{itL} satisfies

$$\|\nabla e^{itL} u_0\|_{L^{p_1} L^{q_1}} \lesssim \|\nabla u_0\|_{L^2}, \qquad \|\nabla \textstyle\int_0^t e^{i(t-s)L} F ds\|_{L^{p_1} L^{q_1}} \lesssim \|\nabla F\|_{L^{p_2'} L^{q_2'}} \tag{2.1.26}$$

for admissible couples (p_j, q_j) such that $q_1 < n$.

Note that it is not trivial to deduce (2.1.26) from (2.1.25): indeed, for this step one needs the equivalence of norms

$$\|(-L)^{\frac{1}{2}}v\|_{L^q} \simeq \|\nabla v\|_{L^q}$$

with q in the appropriate range. Under fairly general assumptions on L, we are able to prove this equivalence for all $1 < q < n$ (see (2.1.5)), and this is the reason for the restriction on q_1 in (S).

Using Assumption (S) we can prove local well posedness in the energy space, and global well posedness provided the nonlinearity is *defocusing*, i.e.,

$$F(r) = \int_0^r f(s)ds \geqslant 0 \quad \text{for} \quad s \in \mathbb{R} \tag{2.1.27}$$

(this is the content of Proposition 2.7.1 and Theorem 2.7.2):

Theorem 2.1.8. *Let $n \geqslant 3$, let $\Omega = \mathbb{R}^n \setminus \omega$ be an exterior domain with compact and possibly empty C^1 boundary, let L be the selfadjoint operator with Dirichlet b.c. defined by (2.1.2), (2.1.3), (2.1.4), and assume (S) holds.*

(i) (Local existence in H^1). If $f \in C^1(\mathbb{C}, \mathbb{C})$ satisfies $f(0) = 0$ and $|f(z) - f(w)| \lesssim (|z| + |w|)^{\gamma-1}|z - w|$ for some $1 \leqslant \gamma < 1 + \frac{4}{n-2}$, then for all $u_0 \in H_0^1(\Omega)$ there exists $T = T(\|u_0\|_{H^1})$ and a unique solution $u \in C([-T, T]; H_0^1(\Omega))$.

(ii) (Global existence in H^1). Assume in addition that $b^2 + |\nabla \cdot b| \in L^2_{loc}$, $c \in L^{\frac{n}{2},1}$,

$$\|a - I\|_{L^\infty} + \||b| + |a'|\|_{L^{n,\infty}} + \|b'\|_{L^{\frac{n}{2},\infty}} + \|c_-\|_{L^{\frac{n}{2},1}} < \epsilon$$

for ϵ small enough, and that $f(u)$ is gauge invariant (2.1.11) and defocusing (2.1.27). Then for all initial data $u_0 \in H_0^1(\Omega)$ problem (2.7.1) has a unique global solution $u \in C \cap L^\infty(\mathbb{R}; H_0^1(\Omega))$. The solution has constant energy for all $t \in \mathbb{R}$:

$$E(t) = \frac{1}{2}\int_\Omega a(x)\nabla^b u \cdot \overline{\nabla^b u} dx + \frac{1}{2}\int_\Omega c(x)|u|^2 dx + \int_\Omega F(u)dx \equiv E(0).$$

Combining the global existence result with the bilinear smoothing estimate in Theorems 2.1.6 and 2.1.7, we obtain the main results of this chapter. Note that a power nonlinearity $f(u) = |u|^{\gamma-1}u$ with $1 + \frac{4}{n} < \gamma < 1 + \frac{4}{n-2}$ satisfies all conditions of the following Theorems:

Theorem 2.1.9 (Scattering on Ω, under (S))**.** *Let $n \geqslant 3$, $\Omega = \mathbb{R}^n \setminus \omega$ an exterior domain with C^1 compact and possibly empty boundary satisfying (2.1.14), L the operator (2.1.2) with Dirichlet b.c. on Ω. Assume a, b, c satisfy, for some $\epsilon, C > 0$, $\delta \in (0, 1]$*

$$|x|a(x)x \cdot \nabla c < \epsilon\langle x\rangle^{-1-\delta}, \qquad |x||c| + |x|^2|c'| < C\langle x\rangle^{-1-\delta},$$

and in addition

$$\|a - I\|_{L^\infty} + |x|^2 c_- + \|c_-\|_{L^{\frac{n}{2},1}} < \epsilon, \quad |x||b| + |x|^2|b'| < \epsilon|x|^\delta\langle x\rangle^{-2\delta}, \qquad \text{if } n \geqslant 4;$$

$$\langle x\rangle^\delta\|a - I\|_{L^\infty} + \langle x\rangle^2 c_- + \|c_-\|_{L^{\frac{n}{2},1}} < \epsilon, \quad |x||b| + |x|\langle x\rangle^{1+\delta}|b'| < \epsilon, \qquad \text{if } n = 3.$$

Finally $|a'| + |x||a''| + |x|^2|a'''| < \epsilon\langle x\rangle^{-1-\delta}$, *and* $f : \mathbb{C} \to \mathbb{C}$ *is gauge invariant* (2.1.11)*, repulsive* (2.1.13)*, defocusing* (2.1.27) *and satisfies* $f(0) = 0$, $|f(z)-f(w)| \lesssim (|z| + |w|)^{\gamma-1}|z - w|$ *for some* $1 + \frac{4}{n} < \gamma < 1 + \frac{4}{n-2}$*. Then if (S) holds and* ϵ *is small enough we have:*

(i) (Existence of wave operators) For every $u_+ \in H^1_0(\Omega)$ *there exists a unique* $u_0 \in H^1_0(\Omega)$ *such that the global solution* $u(t)$ *to* (2.1.1) *satisfies* $\|e^{-itL}u_+ - u(t)\|_{H^1} \to 0$ *as* $t \to +\infty$*. An analogous result holds for* $t \to -\infty$*.*

(ii) (Asymptotic completeness) For every $u_0 \in H^1_0(\Omega)$ *there exists a unique* $u_+ \in H^1_0(\Omega)$ *such that the global solution* $u(t)$ *to* (2.1.1) *satisfies* $\|e^{-itL}u_+-u(t)\|_{H^1} \to 0$ *as* $t \to +\infty$*. An analogous result holds for* $t \to -\infty$*.*

When $\Omega = \mathbb{R}^n$, Strichartz estimates for e^{itL} were proved by Tataru [84] in the case L is a small, long range perturbations of Δ. In Theorems 2.8.1 - 2.8.2 we adapt the result in [84] to our situation, and in particular, combining it with the smoothing estimate (2.1.15), we extend Strichartz estimates to potentials $c(x)$ with a large positive part. In addition we deduce the necessary estimates also for the derivative of the flow ∇e^{itL} (Corollary 2.8.3). As a consequence, Assumption (S) is satisfied and we obtain the final result of the chapter:

Theorem 2.1.10 (Scattering on $\mathbb{R}^n$)**.** *Let* $n \geqslant 3$*, assume* a, b, c *satisfy* $c \in L^n_{loc}$ *and*

$$|a - I| + \langle x\rangle(|a'| + |b|) + \langle x\rangle^2(|a''| + |b'|) + \langle x\rangle^3|a'''| < \epsilon\langle x\rangle^{-\delta},$$
$$|x|\langle x\rangle a(x)x \cdot \nabla c < \epsilon\langle x\rangle^{-\delta}, \quad \|c_-\|_{L^{\frac{n}{2},1}} < \epsilon, \quad |x||c| + |x|^2|c'| < C\langle x\rangle^{-1-\delta}.$$
$$|x|^2c_- < \epsilon, \quad \text{if } n \geqslant 4, \qquad \langle x\rangle^2c_- < \epsilon, \quad \text{if } n = 3,$$

for some $C > 0$*,* $\delta \in (0, 1]$ *and some* ϵ *small enough, and let* L *be the selfadjoint operator defined by* (2.1.2)-(2.1.3) *on* $\mathbb{R}^n$*. Finally, assume* $f : \mathbb{C} \to \mathbb{C}$ *is gauge invariant* (2.1.11)*, repulsive* (2.1.13)*, defocusing* (2.1.27) *and satisfies* $f(0) = 0$*,* $|f(z) - f(w)| \lesssim (|z| + |w|)^{\gamma-1}|z - w|$ *for some* $1 + \frac{4}{n} < \gamma < 1 + \frac{4}{n-2}$*.*

Then the conclusions (i), (ii) of Theorem 2.1.9 are valid.

2.2 Notations and elementary identities

Using the convention of implicit summation over repeated indices, we define the operators

$$A^bv := \nabla^b \cdot (a(x)\nabla^bv) = \partial^b_j(a_{jk}(x)\partial^b_kv), \qquad Av := \nabla \cdot (a(x)\nabla v) = \partial_j(a_{jk}(x)\partial_kv) \tag{2.2.1}$$

so that $L = A^b - c$. The quadratic form associated with A is given by

$$a(w, z) := a_{jk}(x)w_k\overline{z}_j.$$

We shall use the notations

$$\widehat{x} = \tfrac{x}{|x|} = (\widehat{x}_1, \dots, \widehat{x}_n), \qquad \widehat{x}_j = \tfrac{x_j}{|x|},$$

$$\widehat{a}(x) = a_{\ell m}(x)\widehat{x}_\ell\widehat{x}_m, \qquad \overline{a}(x) = \operatorname{trace} a(x) = a_{mm}(x).$$

Since $a(x)$ is positive definite, we have

$$0 \leqslant \widehat{a} = a\widehat{x} \cdot \widehat{x} \leqslant |a\widehat{x}| \leqslant \overline{a}.$$

Indices after a semicolon refer to partial derivatives:

$$a_{jk;\ell} := \partial_\ell a_{jk}, \qquad a_{jk;\ell m} := \partial_\ell \partial_m a_{jk}, \qquad a_{jk;\ell mp} := \partial_\ell \partial_m \partial_p a_{jk}.$$

Notice the formulas

$$\begin{aligned}
\partial_k(\widehat{x}_\ell) &= |x|^{-1}[\delta_{k\ell} - \widehat{x}_k\widehat{x}_\ell],\\
\partial_k(\widehat{x}_\ell\widehat{x}_m) &= |x|^{-1}[\delta_{k\ell}\widehat{x}_m + \delta_{km}\widehat{x}_\ell - 2\widehat{x}_k\widehat{x}_\ell\widehat{x}_m],\\
\partial_j\partial_k(\widehat{x}_\ell\widehat{x}_m) =& \frac{1}{|x|^2}[\delta_{k\ell}\delta_{jm} + \delta_{km}\delta_{j\ell} + 8\widehat{x}_j\widehat{x}_k\widehat{x}_\ell\widehat{x}_m\\
&- 2\delta_{k\ell}\widehat{x}_j\widehat{x}_m - 2\delta_{km}\widehat{x}_j\widehat{x}_\ell - 2\delta_{jk}\widehat{x}_\ell\widehat{x}_m - 2\delta_{j\ell}\widehat{x}_k\widehat{x}_m - 2\delta_{jm}\widehat{x}_k\widehat{x}_\ell]
\end{aligned}$$

which imply

$$a_{jk}a_{\ell m}\widehat{x}_j\partial_k(\widehat{x}_\ell\widehat{x}_m) = 2|x|^{-1}[|a\widehat{x}|^2 - \widehat{a}^2],$$

and

$$a_{jk}a_{\ell m}\partial_j\partial_k(\widehat{x}_\ell\widehat{x}_m) = \tfrac{2}{|x|^2}[a_{\ell m}a_{\ell m} - 4(|a\widehat{x}|^2 - \widehat{a}^2) - \overline{a}\widehat{a}].$$

Using the previous identities, we see that for any radial function $\psi(x) = \psi(|x|)$ we can write

$$A\psi(x) = \partial_\ell(a_{\ell m}\widehat{x}_m\psi') = \widehat{a}\psi'' + \frac{\overline{a} - \widehat{a}}{|x|}\psi' + a_{\ell m;\ell}\widehat{x}_m\psi' \tag{2.2.2}$$

where ψ' denotes the derivative of $\psi(r)$ with respect to the radial variable.

We now give the definitions of the Morrey-Campanato type norms $\dot{X}, \dot{Y}, X, Y$ and recall some relations between them and usual weighted L^2 norms.

For an open subset $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, we use the notations

$$\Omega_{=R} = \Omega \cap \{x\colon |x| = R\}, \quad \Omega_{\leqslant R} = \Omega \cap \{x\colon |x| \leqslant R\}, \quad \Omega_{\geqslant R} = \Omega \cap \{x\colon |x| \geqslant R\},$$

$$\Omega_{R_1 \leqslant |x| \leqslant R_2} = \Omega \cap \{x\colon R_1 \leqslant |x| \leqslant R_2\}.$$

The homogeneous and inhomogeneous norms $\dot{X}$ and X of a function $v : \Omega \to \mathbb{C}$ are defined as

$$\|v\|_{\dot{X}}^2 := \sup_{R>0} \tfrac{1}{R^2} \int_{\Omega_{=R}} |v|^2 dS, \quad \|v\|_X^2 := \sup_{R>0} \tfrac{1}{\langle R\rangle^2} \int_{\Omega_{=R}} |v|^2 dS,$$

where dS is the surface measure on $\Omega_{=R}$ and $\langle R\rangle = \sqrt{1+R^2}$. We shall also need proper Morrey-Campanato spaces, both in the homogeneous version $\dot{Y}$ and in the non homogenous version Y; their norms are defined as

$$\|v\|_{\dot{Y}}^2 := \sup_{R>0} \tfrac{1}{R} \int_{\Omega_{\leqslant R}} |v|^2 dx, \qquad \|v\|_Y^2 := \sup_{R>0} \tfrac{1}{\langle R\rangle} \int_{\Omega_{\leqslant R}} |v|^2 dx. \tag{2.2.3}$$

The following equivalence is easy to prove:

$$\|v\|_Y^2 \leqslant \sup_{R\geqslant 1} \tfrac{1}{R} \int_{\Omega_{\leqslant R}} |v|^2 \leqslant \sqrt{2}\|v\|_Y^2. \tag{2.2.4}$$

The following Lemmas collect a few estimates to be used in the rest of the chapter, which follow immediately from the definitions (proofs are straightforward, and full details can be found in [8]).

Lemma 2.2.1. *For any $v \in C^\infty(\mathbb{R}^n)$,*

$$\||x|^{-1}v\|_{\dot{Y}} \leqslant \|v\|_{\dot{X}}, \qquad \|\langle x\rangle^{-1}v\|_Y \leqslant \|v\|_X, \tag{2.2.5}$$

$$\sup_{R>0} \int_{\Omega_{\geqslant R}} \frac{R^{n-1}}{|x|^{n+2}}|v|^2dx \leqslant \frac{1}{n-1}\|v\|^2_{\dot{X}}, \qquad \sup_{R>1} \int_{\Omega_{\geqslant R}} \frac{R^{n-1}}{|x|^{n+2}}|v|^2dx \leqslant \frac{2}{n-1}\|v\|^2_{X}. \tag{2.2.6}$$

Lemma 2.2.2. *For any $0 < \delta < 1$ and $v \in C^\infty(\mathbb{R}^n)$,*

$$\int_\Omega \frac{|v|^2}{|x|^2\langle x\rangle^{1+\delta}} \leqslant 2\delta^{-1}\|v\|^2_{\dot{X}}, \tag{2.2.7}$$

$$\int_{\Omega_{\geqslant 1}} \frac{|v|^2}{|x|^3\langle x\rangle^{\delta}} \leqslant \int_{\Omega_{\geqslant 1}} \frac{|v|^2}{|x|^{3+\delta}} \leqslant 2\delta^{-1}\|v\|^2_{X}, \tag{2.2.8}$$

$$\int_\Omega \frac{|v|^2}{\langle x\rangle^{1+\delta}} \leqslant 8\delta^{-1}\|v\|^2_{Y} \leqslant 8\delta^{-1}\|v\|^2_{\dot{Y}}. \tag{2.2.9}$$

Lemma 2.2.3. *For any $R > 0$, $0 < \delta < 1$ and $v, w \in C^\infty(\mathbb{R}^n)$,*

$$\int_{\Omega_{\leqslant 1}} \frac{|vw|}{|x|^{2-\delta}} + \int_{\Omega_{\geqslant 1}} \frac{|vw|}{|x|^{2+\delta}} \leqslant 9\delta^{-1}\|v\|_{\dot{X}}\|w\|_{\dot{Y}}. \tag{2.2.10}$$

In the following Lemma we prove some magnetic Hardy type inequalities, which require $n \geqslant 3$, expressed in terms of the nonhomogeneous X, Y norms (compare (2.2.11) with Theorem A.1 in [38]):

Lemma 2.2.4. *Let $n \geqslant 3$ and assume $b(x) = (b_1(x), \dots, b_n(x))$ is continuous up to the boundary of Ω with values in $\mathbb{R}^n$. For any $0 < \delta < 1, y \in \Omega$ and $v \in C_c^\infty(\Omega)$, we have:*

$$\||x-y|^{-1}v\|_{L^2(\Omega)} \leqslant \frac{2}{n-2}\|\nabla^b v\|_{L^2(\Omega)}, \tag{2.2.11}$$

$$\||x|^{-1}v\|^2_{Y} \leqslant 6\|\nabla^b v\|^2_Y + 3\|v\|^2_X, \tag{2.2.12}$$

$$\int_{\Omega_{\leqslant 1}} \frac{|\nabla^b v||v|}{|x|}dx + \int_{\Omega_{\geqslant 1}} \frac{|\nabla^b v||v|}{|x|^{2+\delta}}dx \leqslant 9\delta^{-1}(\|\nabla^b v\|^2_Y + \|v\|^2_X), \tag{2.2.13}$$

$$\|v\|_X \leqslant 4\sup_{R>1} R^{-2}\int_{\Omega_{=R}} |v|^2dS + 13\|\nabla^b v\|^2_Y. \tag{2.2.14}$$

Proof. We give the complete proof of (2.2.11); the remaining inequalities are proved in [8]. Integrating on Ω the identity

$$\nabla \cdot \left\{\frac{x-y}{|x-y|^2}|v|^2\right\} = \Re\left[2c(x)\overline{\nabla^b f(x)}\frac{x-y}{|x-y|^2}\right] + (n-2)\frac{|c(x)|^2}{|x-y|^2}$$

and noticing that boundary term vanishes, we get

$$\frac{n-2}{2}\int_\Omega \frac{|f(x)|^2}{|x-y|^2}\,dx \leqslant \Re\int_\Omega \frac{(x-y)f(x)}{|x-y|^2}\overline{\nabla^b f(x)}\,dx \leqslant \left(\int_\Omega \frac{|f(x)|^2}{|x-y|^2}\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla^b f(x)|^2\,dx\right)^{\frac{1}{2}}. \ \square$$

By a density argument, it is clear that the previous estimates are valid not only for smooth functions but also for functions in $D(L) = H^1_0(\Omega) \cap H^2(\Omega)$.

We conclude this section with some additional properties of the magnetic norms.

Lemma 2.2.5. *Let $n \geqslant 3$. If $b \in L^{n,\infty}(\Omega)$, the following equivalence holds:*

$$\|\nabla^b v\|_{L^2(\Omega)} \simeq \|\nabla v\|_{L^2(\Omega)}. \tag{2.2.15}$$

Moreover, for $s > 0$ we have

$$\|\langle x\rangle^{-s}\nabla^b v\|_{L^2(\Omega)} + \|\langle x\rangle^{-s-1}v\|_{L^2(\Omega)} \simeq \|\langle x\rangle^{-s}\nabla v\|_{L^2(\Omega)} + \|\langle x\rangle^{-s-1}v\|_{L^2(\Omega)}. \tag{2.2.16}$$

Proof. By Hölder inequality and Sobolev embedding in Lorentz spaces, we can write

$$\|\nabla^b v\|_{L^2} \leqslant \|\nabla v\|_{L^2} + \|bv\|_{L^2} \leqslant \|\nabla v\|_{L^2} + \|b\|_{L^{n,\infty}}\|v\|_{L^{\frac{2n}{n-2},2}} \lesssim (1+\|b\|_{L^{n,\infty}})\|\nabla v\|_{L^2}.$$

Conversely, writing $\nabla = \nabla^b - ib$, we have

$$\|\nabla v\|_{L^2} \leqslant \|\nabla^b v\|_{L^2} + \|bv\|_{L^2} \lesssim \|\nabla^b v\|_{L^2} + \|b\|_{L^{n,\infty}}\|v\|_{L^{\frac{2n}{n-2},2}}.$$

Recall now the pointwise diamagnetic inequality

$$|\nabla|v|| \leqslant |\nabla^b v| \tag{2.2.17}$$

which is true for $b \in L^2_{loc}$. Thus, again by Sobolev-Lorentz embedding,

$$\|v\|_{L^{\frac{2n}{n-2},2}} \lesssim \|\nabla|v|\|_{L^2} \leqslant \|\nabla^b v\|_{L^2}$$

and we obtain (2.2.15). Next we can write

$$\|\langle x\rangle^{-s}\nabla v\|_{L^2} + \|\langle x\rangle^{-s-1}v\|_{L^2} \simeq \|\nabla(\langle x\rangle^{-s}v)\|_{L^2} + \|\langle x\rangle^{-s-1}v\|_{L^2}$$

and

$$\|\langle x\rangle^{-s}\nabla^b v\|_{L^2} + \|\langle x\rangle^{-s-1}v\|_{L^2} \simeq \|\nabla^b(\langle x\rangle^{-s}v)\|_{L^2} + \|\langle x\rangle^{-s-1}v\|_{L^2}$$

which, together with (2.2.15), imply (2.2.16). □

Lemma 2.2.6. *Let $n \geqslant 3$ and consider the operator $L = A^b - c$ with Dirichlet b.c. on Ω, under assumptions (2.1.3), (2.1.6), (2.1.7), (2.1.9) and (2.1.14). If the constant C_- is sufficiently small, the operator L is selfadjoint and nonpositive. If in addition $b \in L^{n,\infty}(\Omega)$ then for all $0 \leqslant s \leqslant 1$ we have the equivalence*

$$\|(-L)^{\frac{s}{2}}v\|_{L^2(\Omega)} \simeq \|v\|_{\dot H^s(\Omega)}. \tag{2.2.18}$$

Proof. Selfadjointness and positivity are standard, and actually hold under less restrictive assumptions on the coefficients (see Proposition 2.6.3 below for a more general result). Next, (2.2.18) is trivial for $s = 0$, while for $s = 1$ we have

$$\|(-L)^{\frac{1}{2}}v\|^2_{L^2} = (-Lv, v)_{L^2(\Omega)} = a(\nabla^b v, \nabla^b v) + \int_\Omega c|v|^2 dx$$

which implies, using (2.2.15),

$$\|(-L)^{\frac{1}{2}}v\|^2_{L^2} \simeq \|\nabla^b v\|^2_{L^2} + \int_\Omega c|v|^2 dx \simeq \|\nabla v\|^2_{L^2} + \int_\Omega c|v|^2 dx.$$

By Hardy's inequality we obtain the claim for $s = 1$, provided C_- is sufficiently small, and by complex interpolation we conclude the proof (recalling the complex interpolation formula $[D(H^{\sigma_0}), D(H^{\sigma_1})]_\theta = D(H^{\sigma_\theta})$ with $\sigma_\theta = (1-\theta)\sigma_0 + \theta\sigma_1$ which is valid for any selfadjoint operator H). □

2.3 Virial identity

In [8] a virial identity for the Helmholtz equation with variable coefficients was obtained by adapting the Morawetz multiplier method. We show here how to modify the technique in order to prove the analogous virial identity for the nonlinear Schrödinger equation (2.1.1). To make sense of the formal manipulations, one needs some additional smoothness (e.g., $u \in H^2(\Omega)$ is enough), which can be obtained by an approximation procedure similar to the proof of the conservation of energy in Theorem 2.7.2 below; we omit the details. The identity is the following:

Proposition 2.3.1 (Virial Identity). *Assume $a, b, c, f(z)$ are as in Theorem 2.1.1, let u be a solution of (2.1.1) and $\psi\colon \mathbb{R}^n \to \mathbb{R}$ an arbitrary weight. Then the following identity holds:*

$$\begin{aligned}\partial_t[\Im(a(\nabla\psi, \nabla^b u)u)] = &-\tfrac{1}{2}A^2\psi|u|^2 + \Re(\alpha_{\ell m}\ \partial_m^b u\ \overline{\partial_\ell^b u}) \\ &- a(\nabla\psi, \nabla c)|u|^2 \\ &+ 2\Im(a_{jk}\partial_k^b u(\partial_j b_\ell - \partial_\ell b_j)a_{\ell m}\partial_m\psi\ \overline{u}) \\ &+ A\psi[f(u)\bar{u} - 2F(u)] \\ &+ \partial_j\{-\Re Q_j + 2F(u)a_{jk}\partial_k\psi + \Im[u_t\bar{u}a_{jk}\partial_k\psi]\},\end{aligned} \tag{2.3.1}$$

where

$$\alpha_{\ell m} = 2a_{jm}\partial_j(a_{\ell k}\partial_k\psi) - a_{jk}\partial_k\psi\partial_j a_{\ell m}, \tag{2.3.2}$$

$$Q_j = a_{jk}\partial_k^b u \cdot [A^b, \psi]\overline{u} - \frac{1}{2}a_{jk}(\partial_k A\psi)|u|^2 - a_{jk}\partial_k\psi\left[c|u|^2 + a(\nabla^b u, \nabla^b u)\right]. \tag{2.3.3}$$

Proof. We multiply both sides of (2.1.1) by the multiplier

$$[A^b, \psi]\bar{u} = (A\psi)\bar{u} + 2a(\nabla\psi, \nabla u)$$

and take real parts. We recall the following identity (which however can be checked directly with some lengthy but elementary computations) from Proposition 2.1 of [8]:

$$\begin{aligned}\Re[(-A^b u + cu)[A^b, \psi]\overline{u}] = &-\tfrac{1}{2}A^2\psi|u|^2 + \Re(\alpha_{\ell m}\ \partial_m^b u\ \overline{\partial_\ell^b u}) \\ &- a(\nabla\psi, \nabla c)|u|^2 \\ &+ 2\Im(a_{jk}\partial_k^b u(\partial_j b_\ell - \partial_\ell b_j)a_{\ell m}\partial_m\psi\ \overline{u}) \\ &- \Re\,\partial_j Q_j,\end{aligned} \tag{2.3.4}$$

where α_{lm} are defined by (2.3.2) and Q_j by (2.3.3). For the terms containing $f(u)$ we can write

$$\Re(f(u)[A^b, \psi]\bar{u}) = A\psi[f(u)\bar{u} - 2F(u)] + \nabla \cdot \{2F(u)a\nabla\psi\}. \tag{2.3.5}$$

Indeed, by the assumptions on f, there exists a function $g\colon [0, +\infty) \to \mathbb{R}$ such that $f(z) = g(|z|^2)z$. As a consequence,

$$\begin{aligned}\nabla F(u) &= \nabla \int_0^{|u|} f(s)\,ds = \nabla \int_0^{|u|} g(s^2)s\,ds = \\ &= \frac{1}{2}\nabla \int_0^{|u|^2} g(t)\,dt = \Re(g(|u|^2)u\nabla\bar{u}) = \Re(f(u)\nabla\bar{u}) = \\ &= \Re(f(u)\overline{\nabla^b u}),\end{aligned}$$

since $\Re(f(u)ib\bar{u}) = 0$. We conclude that

$$\begin{aligned}\Re[f(u)[A\psi\bar{u} + 2a(\nabla\psi, \nabla^b u)]] =& A\psi f(u)\bar{u} + 2\nabla\psi^t a\Re(f(u)\overline{\nabla^b u}) = \\ =& A\psi f(u)\bar{u} + 2\nabla\psi^t a\nabla F(u) = \\ =& A\psi f(u)\bar{u} + 2[a\nabla\psi]\cdot\nabla F(u) = \\ =& A\psi[f(u)\bar{u} - 2F(u)] + \nabla\cdot\{2F(u)a\nabla\psi\},\end{aligned}$$

and (2.3.5) is proved. Finally, for the terms containing iu_t we have the identity

$$\Re(i\partial_t u[A^b, \psi]\bar{u}) = \partial_t[-\Im a(\nabla\psi, \nabla^b u)u] + \nabla\cdot\{\Im(u_t\bar{u}a\nabla\psi)\}. \tag{2.3.6}$$

This can be proved directly as follows:

$$\begin{aligned}&\Re[iu_t[A\psi\bar{u} + 2a(\nabla\psi, \nabla^b u)]] = -\Im[u_t\nabla\cdot(a\nabla\psi)\bar{u} + 2\nabla\psi^t a\overline{\nabla u}u_t - 2i\nabla\psi^t ab\bar{u}u_t] = \\ &\quad = -\Im[-\nabla u_t^t a\nabla\psi\bar{u} - u_t\overline{\nabla u}^t a\nabla\psi + 2\nabla\psi^t a\overline{\nabla u}u_t - 2i\nabla\psi^t ab\bar{u}u_t + \nabla\cdot\{u_t\bar{u}a\nabla\psi\}] = \\ &\quad = -\Im[\overline{\nabla u_t}^t a\nabla\psi u + \overline{\nabla u}^t a\nabla\psi u_t] + 2\Im[i\nabla\psi^t a(b\bar{u})u_t] + \nabla\cdot\{\Im[u_t\bar{u}a\nabla\psi]\} = \\ &\quad = -\Im[\partial_t(\overline{\nabla u}^t a\nabla\psi u)] + \Im[i\partial_t(\nabla\psi^t a\overline{bu}u)] + \nabla\cdot\{\Im[u_t\bar{u}a\nabla\psi]\} = \\ &\quad = -\Im[\partial_t(\overline{\nabla u}^t a\nabla\psi u)] + \Im[i\partial_t(\nabla\psi^t a\overline{bu}u)] + \nabla\cdot\{\Im[u_t\bar{u}a\nabla\psi]\} = \\ &\quad = \partial_t[-\Im a(\nabla\psi, \nabla^b u)u] + \nabla\cdot\{\Im[u_t\bar{u}a\nabla\psi]\}.\end{aligned}$$

Gathering (2.3.4), (2.3.5) and (2.3.6) we obtain (2.3.1). □

2.4 Proof of Theorems 2.1.1, 2.1.2: the smoothing estimate

Since the arguments for Theorems 2.1.1 and 2.1.2 largely overlap, we shall proceed with both proofs in parallel. The proof consists in integrating the virial identity (2.3.1) on Ω and estimating carefully all the terms. Note that some of the following computations are similar to those of Section 4 in [8].

Remark 2.4.1. At several steps, we shall need to assume that the constants $N/\nu - 1$, C_a, C_I, C_c, C_b, C_- are small enough. We can give explicit conditions on the smallness required in Theorem 2.1.1 and in Theorem 2.1.2. In both the Theorems the smallness of C_- is only required in order to make L a selfadjoint, nonpositive operator. In view of the magnetic Hardy inequality (2.2.11), it is sufficient to assume

$$C_- \leqslant \tfrac{2\sqrt{\nu}}{n-2}. \tag{2.4.1}$$

In Theorem 2.1.1 it is sufficient that

$$\tfrac{N}{\nu} \leqslant \sqrt{\tfrac{n^2+2n+15}{6(n+2)}} \quad \text{for } 3 \leqslant n \leqslant 25, \qquad \tfrac{N}{\nu} < \tfrac{7n-1}{3(n+3)} \quad \text{for } n \geqslant 26 \tag{2.4.2}$$

and that the following quantities are positive:

$$\tfrac{K_0}{2}\nu^2 - \tfrac{5N^2C_b + 12nC_a(N+C_a) + C_c}{\delta} > 0,$$
$$\tfrac{n-1}{3n}\nu^2 - \tfrac{5N^2C_b + 24NC_a}{\delta} > 0, \qquad \left(n - \tfrac{N}{\nu}\right) - \tfrac{n}{n-1}\nu C_a > 0,$$

where
$$K_0 := \tfrac{n-1}{6} - \tfrac{n+3}{2}\tfrac{N}{\nu} + n > 0.$$
We remark that $n - N/\nu > 0$ thanks to (2.4.2). On the other hand, the condition $K_0 > 0$ is equivalent to the second equation in (2.4.2) and is implied by the first equation in (2.4.2) in the case $n \leqslant 26$.

In Theorem 2.1.2 it is sufficient that the following quantities are positive:

$$\tfrac{(1-C_I)^2}{78} - 8\delta^{-1}[C_c + 9C_I + 41C_a(N + C_a)] - 9\delta^{-1}N^2C_b > 0,$$
$$\tfrac{(1-C_I)^2}{6} - 13\delta^{-1}[C_c + 38C_a(N + C_a)] - 9\delta^{-1}N^2C_b > 0$$
$$\left(n - \tfrac{N}{\nu}\right) - \tfrac{n}{n-1}\nu C_a > 0.$$

The proof is divided into several steps.

2.4.1 Choice of the weight ψ

Define
$$\psi_1(r) = \int_0^r \psi_1'(s)ds \tag{2.4.3}$$
where
$$\psi_1'(r) = \begin{cases} \frac{n-1}{2n} r, & r \leqslant 1 \\ \frac{1}{2} - \frac{1}{2nr^{n-1}}, & r > 1. \end{cases}$$
Then ψ is the radial function, depending on a scaling parameter $R > 0$,
$$\psi(|x|) \equiv \psi_R(|x|) := R\psi_1\left(\frac{|x|}{R}\right).$$
Here and in the following, with a slight abuse, we shall use the same letter ψ to denote a function $\psi(r)$ defined for $r \in \mathbb{R}^+$ and the radial function $\psi(x) = \psi(|x|)$ defined on $\mathbb{R}^n$. We compute the first radial derivatives $\psi^{(j)}(r) = (\frac{x}{|x|} \cdot \nabla)^j \psi(x)$ for $|x| > 0$:
$$\psi'(x) = \begin{cases} \frac{n-1}{2n} \cdot \frac{|x|}{R}, & |x| \leqslant R \\ \frac{1}{2} - \frac{R^{n-1}}{2n|x|^{n-1}}, & |x| > R \end{cases} \tag{2.4.4}$$
which can be equivalently written as
$$\psi'(x) = \frac{|x|}{2nR}\left[n\frac{R}{R\vee|x|} - (\frac{R}{R\vee|x|})^n\right]$$
and implies in particular
$$0 \leqslant \psi' \leqslant \tfrac{1}{2}. \tag{2.4.5}$$
Then we have
$$\psi''(x) = \frac{n-1}{2n} \cdot \frac{R^{n-1}}{(R\vee|x|)^n} = \frac{n-1}{2n} \cdot \begin{cases} \frac{1}{R} & |x| \leqslant R \\ \frac{R^{n-1}}{|x|^n} & |x| > R, \end{cases} \tag{2.4.6}$$
$$\psi'''(x) = -\frac{n-1}{2}\frac{R^{n-1}}{|x|^{n+1}}\mathbf{1}_{|x|\geqslant R} \tag{2.4.7}$$
$$\psi^{IV}(x) = \frac{n^2-1}{2} \cdot \frac{R^{n-1}}{|x|^{n+2}}\mathbf{1}_{|x|\geqslant R} - \frac{n-1}{2}\frac{1}{R^2}\delta_{|x|=R}. \tag{2.4.8}$$

$$\psi'' - \frac{\psi'}{|x|} = \begin{cases} 0 & |x| \leqslant R \\ -\frac{1}{2|x|}\left(1 - \frac{R^{n-1}}{|x|^{n-1}}\right) & |x| > R. \end{cases} \tag{2.4.9}$$

Moreover the function (see (2.2.2))

$$A\psi = \widehat{a}\psi'' + \tfrac{\overline{a}-\widehat{a}}{|x|}\psi' + a_{\ell m;\ell}\widehat{x}_m\psi'. \tag{2.4.10}$$

is continuous and piecewise Lipschitz.

2.4.2 Estimate of the terms in $|u|^2$

In this section we consider the terms

$$I_{|u|^2} = -\frac{1}{2}A^2\psi|u|^2 - a(\nabla\psi, \nabla c)|u|^2. \tag{2.4.11}$$

We compute the quantity $A^2\psi$: after some long but elementary computations (see [8]) we have

$$A^2\psi(x) = S(x) + R(x) \tag{2.4.12}$$

where

$$\begin{aligned} S(x) =& \widehat{a}^2\psi^{IV} + [2\overline{a}\widehat{a} - 6\widehat{a}^2 + 4|a\widehat{x}|^2]\tfrac{\psi'''}{|x|} + \\ &+ [2a_{\ell m}a_{\ell m} + \overline{a}^2 - 6\overline{a}\widehat{a} + 15\widehat{a}^2 - 12|a\widehat{x}|^2]\left(\tfrac{\psi''}{|x|^2} - \tfrac{\psi'}{|x|^3}\right) \end{aligned} \tag{2.4.13}$$

and

$$\begin{aligned} R(x) =& \widehat{a}a_{\ell m;\ell}\widehat{x}_m\psi''' + (\overline{a} - \widehat{a})a_{jk;j}\widehat{x}_k\left(\tfrac{\psi''}{|x|} - \tfrac{\psi'}{|x|^2}\right) + \\ &+ [\partial_j(a_{jk}a_{\ell m;k}\widehat{x}_\ell\widehat{x}_m) + \partial_j(a_{jk}a_{\ell m})\partial_k(\widehat{x}_\ell\widehat{x}_m)]\left(\psi'' - \tfrac{\psi'}{|x|}\right) + (A\overline{a})\tfrac{\psi'}{|x|} + \\ &+ 2a_{jk}a_{\ell m;k}\widehat{x}_\ell\widehat{x}_m\widehat{x}_j\left(\psi''' - \tfrac{\psi''}{|x|}\right) + 2a(\nabla\overline{a}, \nabla\tfrac{\psi'}{|x|}) + \\ &+ A(a_{\ell m;\ell}\widehat{x}_m\psi'). \end{aligned}$$

The remainder $R(x)$ can be estimated as follows: recalling that, by definition of ψ, we have

$$|\psi'| \leqslant \tfrac{|x|}{2(R\vee|x|)}, \qquad |\psi''| \leqslant \tfrac{n-1}{2n(R\vee|x|)}, \qquad |\psi'''| \leqslant \tfrac{n-1}{2|x|(R\vee|x|)}$$

and using assumption (2.1.6), we find that

$$|R(x)| \leqslant \frac{12nC_a(N + C_a)}{|x|\langle x\rangle^{1+\delta}(R \vee |x|)}. \tag{2.4.14}$$

Proof of Theorem 2.1.1

We prove that, assuming (2.1.9), (2.1.6), (2.1.3), (2.4.2), we have

$$\begin{aligned} \textstyle\int_\Omega\int_0^T I_{|u|^2}\,dtdx \geqslant& \tfrac{n-1}{2}\nu\tfrac{1}{R^2}\textstyle\int_{\Omega_{=R}}\widehat{a}\|u\|_{L^2_T}dS \\ &- \left[\tfrac{n+3}{2}N - n\nu\right](n-1)\textstyle\int_{\Omega_{\geqslant R}}\widehat{a}\tfrac{R^{n-1}}{|x|^{n+2}}\|u\|_{L^2_T}\,dx \\ &- (12nC_a(N + C_a) + C_c)\delta^{-1}\|u\|^2_{\dot{X}_xL^2_T}. \end{aligned} \tag{2.4.15}$$

We focus on the main term $S(x)$. With our choice of the weight ψ we have in the region $|x| \leqslant R$

$$S(x) = -\tfrac{n-1}{2}\widehat{a}^2 \tfrac{1}{R^2}\delta_{|x|=R} \tag{2.4.16}$$

while in the region $|x| > R$

$$\begin{aligned} S(x) =&(n-1)\left[\tfrac{n+3}{2}\widehat{a} - \overline{a}\right]\widehat{a}\tfrac{R^{n-1}}{|x|^{n+2}} - 2(n-1)[|a\widehat{x}|^2 - \widehat{a}^2]\tfrac{R^{n-1}}{|x|^{n+2}} \\ &- [2a_{\ell m}a_{\ell m} + \overline{a}^2 - 6\overline{a}\widehat{a} + 15\widehat{a}^2 - 12|a\widehat{x}|^2]\big(1 - (\tfrac{R}{|x|})^{n-1}\big)\tfrac{1}{2|x|^3}. \end{aligned} \tag{2.4.17}$$

Note that $a_{\ell m}a_{\ell m}$ is the square of the Hilbert-Schmidt norm of the matrix $a(x)$. We deduce from assumption (2.1.3)

$$nN \geqslant \overline{a} \geqslant n\nu, \qquad N \geqslant |a\widehat{x}| \geqslant \widehat{a} \geqslant \nu, \qquad a_{\ell m}a_{\ell m} \geqslant n\nu^2,$$

so that

$$S(x) \leqslant -\tfrac{n-1}{2}\nu\widehat{a}\tfrac{1}{R^2}\delta_{|x|=R} \qquad \text{for } |x| \leqslant R. \tag{2.4.18}$$

On the other hand, we have

$$2|a(x)|^2_{HS} + \overline{a}^2 - 6\overline{a}(x)\widehat{a}(x) + 15\widehat{a}^2(x) - 12|a(x)\widehat{x}|^2 \geqslant (2n+n^2+15)\nu^2 - 6(n+2)N^2 \geqslant 0 \tag{2.4.19}$$

by (2.4.2) (note that the second condition in (2.4.2) implies the first one when $n \geqslant 26$), thus we get

$$S(x) \leqslant (n-1)\left[\tfrac{n+3}{2}N - n\nu\right]\widehat{a}\tfrac{R^{n-1}}{|x|^{n+2}} \qquad \text{for } |x| \geqslant R. \tag{2.4.20}$$

Now we can estimate from below the integral

$$-\int_\Omega \int_0^T A^2\psi|u|^2\,dtdx = -\int_\Omega A^2\psi\|u(x)\|^2_{L^2_T}\,dx = I + II$$

where

$$I = -\int_\Omega S(x)\|u(x)\|^2_{L^2_T}dx, \qquad II = -\int_\Omega R(x)\|u(x)\|^2_{L^2_T}dx.$$

By (2.4.14) and (2.2.7) we have immediately for any $R > 0$

$$II \geqslant -24n\delta^{-1}C_a(N + C_a)\|u\|^2_{\dot{X}_x L^2_T}. \tag{2.4.21}$$

Note that we must first integrate in time over $[0, T]$, then in space over $\Omega_{=R}$ and finally divide by R^2 and take the sup in $R > 0$; this gives a reverse norm $\dot{X}_x L^2_t$ in the previous estimate. Concerning the $S(x)$ term I, we have by (2.4.18), (2.4.20)

$$I \geqslant \tfrac{n-1}{2}\nu\tfrac{1}{R^2}\int_{\Omega_{=R}}\widehat{a}\|u\|^2_{L^2_T}dS - \left[\tfrac{n+3}{2}N - n\nu\right](n-1)\int_{\Omega_{\geqslant R}}\widehat{a}\tfrac{R^{n-1}}{|x|^{n+2}}\|u\|^2_{L^2_T}\,dx \tag{2.4.22}$$

for all $R > 0$.

It remains to consider the second term in (2.4.11); we have

$$-a(\nabla\psi, \nabla c)|u|^2 = -a(\widehat{x}, \nabla c)\psi'|u|^2 \geqslant -\tfrac{C_c}{|x|^2\langle x\rangle^{1+\delta}}\psi'|u|^2 \tag{2.4.23}$$

thanks to assumption (2.1.9). Since $0 < \psi' < 1/2$, by estimate (2.2.7) we obtain

$$-\int_\Omega\int_0^T a(\nabla\psi, \nabla c)|u|^2\,dtdx \geqslant -C_c\delta^{-1}\|u\|^2_{\dot{X}_x L^2_T} \tag{2.4.24}$$

Collecting (2.4.22), (2.4.21), and (2.4.24) we get (2.4.15).

Proof of Theorem 2.1.2

We prove that, assuming (2.1.3), (2.1.6), (2.1.16), (2.1.9), we have for all $R > 1$

$$\begin{aligned} \textstyle\int_\Omega \int_0^T I_{|v|^2}\, dtdx \geqslant (1-C_I)\frac{1}{R^2}\int_{\Omega_{=R}} \|u\|^2_{L^2_T} dS \\ - 8\delta^{-1}[C_c + 9C_I + 41C_a(N+C_a)]\|u\|^2_{XL^2_T} \\ - 13\delta^{-1}[C_c + 36C_a(N+C_a)]\|\nabla^b u\|^2_{YL^2_T}. \end{aligned} \tag{2.4.25}$$

Writing $a(x) = I + q(x)$ i.e. $q_{\ell m} := a_{\ell m} - \delta_{\ell m}$ we have, with the notations $\widehat{q} = q_{\ell m}\widehat{x}_\ell \widehat{x}_m$ and $\overline{q} = q_{\ell\ell}$,

$$a_{\ell m}a_{\ell m} = \delta_{\ell m}\delta_{\ell m} + 2\delta_{\ell m}q_{\ell m} + q_{\ell m}q_{\ell m} = 3 + 2\overline{q} + q_{\ell m}q_{\ell m}$$

and also

$$\widehat{a} = 1 + \widehat{q}, \qquad \overline{a} = 3 + \overline{q}, \qquad |a\widehat{x}|^2 = 1 + 2\widehat{q} + |q\widehat{x}|^2.$$

Note that $|q| = |a(x) - I| \leqslant C_I\langle x\rangle^{-\delta} < 1$ by assumption (2.1.16), which implies

$$|\overline{q}| \leqslant 3C_I\langle x\rangle^{-\delta}, \qquad |\widehat{q}| \leqslant C_I\langle x\rangle^{-\delta}, \qquad |q\widehat{x}| \leqslant C_I\langle x\rangle^{-\delta}$$

so that

$$\begin{aligned} 2a_{\ell m}a_{\ell m} + \overline{a}^2 - 6\overline{a}\widehat{a} + 15\widehat{a}^2 - 12|a\widehat{x}|^2 =& 4\overline{q} - 12\widehat{q} + 2q_{\ell m}q_{\ell m} + \overline{q}^2 - 6\overline{q}\widehat{q} + 15\widehat{q}^2 - 12|q\widehat{x}|^2 \\ \geqslant& 4\overline{q} - 12\widehat{q} - 6\overline{q}\widehat{q} - 12|q\widehat{x}|^2 \geqslant -46C_I\langle x\rangle^{-\delta}. \end{aligned}$$

We have also $1 - C_I \leqslant \widehat{a} \leqslant 1 + C_I$ so that $(n = 3)$

$$-\tfrac{n-1}{2}\widehat{a}^2 \leqslant -(1-C_I)^2, \quad \left(\tfrac{n+3}{2}\widehat{a} - \overline{a}\right)\widehat{a} \leqslant 6C_I(1+C_I) < 12C_I$$

Thus under the assumptions of Theorem 2.1.2 we obtain the estimates

$$S(x) \leqslant -(1-C_I)^2 \tfrac{1}{R^2}\delta_{|x|=R} \quad \text{for } |x| \leqslant R \tag{2.4.26}$$

and

$$S(x) \leqslant 24C_I\left[\tfrac{R^2}{|x|^5} + \tfrac{1}{|x|^3\langle x\rangle^\delta}\right] \quad \text{for } |x| > R. \tag{2.4.27}$$

Now we can estimate from below the integral

$$-\textstyle\int_\Omega \int_0^T A^2\psi|u|^2\, dtdx = -\int_\Omega A^2\psi\|u(x)\|^2_{L^2_T}\, dx = I + II$$

where

$$I = -\textstyle\int_\Omega S(x)\|u(x)\|^2_{L^2_T} dx, \qquad II = -\int_\Omega R(x)\|u(x)\|^2_{L^2_T} dx.$$

Concerning the $S(x)$ term I, using (2.2.6) and (2.2.8) in (2.4.26), (2.4.27), we have for all $R > 1$

$$I \geqslant (1-C_I)^2\frac{1}{R^2}\int_{\Omega_{=R}} \|u\|^2_{L^2_T}\, dS - 72C_I\delta^{-1}\|u\|^2_{X_x L^2_T}. \tag{2.4.28}$$

We estimate the now the II–term: for all $R>1$, thanks to (2.4.14), we have

$$\begin{aligned} II \geqslant & -36C_a(N+C_a)\int_0^T\int_\Omega |x|^{-1}\langle x\rangle^{-1-\delta}(R\vee|x|)^{-1}|u(t,x)|^2\,dxdt \\ \geqslant & -36C_a(N+C_a)\textstyle\int_0^T\left[\int_{\Omega_{\leqslant 1}}+\int_{\Omega_{\geqslant 1}}\right]|x|^{-2}\langle x\rangle^{-1-\delta}|u(t,x)|^2\,dxdt \end{aligned} \tag{2.4.29}$$

We observe that, thanks to (2.2.8), we have

$$\textstyle\int_0^T\int_{\Omega_{\geqslant 1}}\frac{|u|^2}{|x|^2\langle x\rangle^{1+\delta}}\,dxdt=\int_{\Omega_{\geqslant 1}}\frac{\|u(x)\|_{L^2_T}}{|x|^2\langle x\rangle^{1+\delta}}\leqslant 2\delta^{-1}\|u\|_{XL^2_T}. \tag{2.4.30}$$

Moreover, thanks to (2.2.11) and (2.2.4), we estimate

$$\begin{aligned} \textstyle\int_0^T\int_{\Omega_{\leqslant 1}}\frac{|u|^2}{|x|^2\langle x\rangle^{1+\delta}}\,dxdt\leqslant\int_0^T\int_{\Omega_{\leqslant 1}}|x|^{-2}|u|^2\,dxdt\leqslant\ 4\int_0^T\int_{\Omega_{\leqslant 1}}|\nabla^b u|^2\,dxdt \\ =4\|\nabla^b u\|^2_{L^2(\Omega_{\leqslant 1})L^2_T} \end{aligned} \tag{2.4.31}$$

Gathering (2.4.30) and (2.4.31), we have

$$\textstyle\int_0^T\int_\Omega\frac{|u|^2}{|x|^2\langle x\rangle^{1+\delta}}\,dxdt\leqslant 2\delta^{-1}\|u\|^2_{XL^2_T}+4\|\nabla^b u\|^2_{L^2(\Omega_{\leqslant 1})L^2_T}. \tag{2.4.32}$$

We get immediately from (2.4.29) and (2.4.32) that

$$II\geqslant -324\delta^{-1}C_a(N+C_a)\left[\|u\|^2_{XL^2_T}+\|\nabla^b u\|^2_{L^2(\Omega_{\leqslant 1})L^2_T}\right]. \tag{2.4.33}$$

We consider the second term in (2.4.11); thanks to (2.1.9) and (2.4.32) we have

$$\begin{aligned} -\textstyle\int_0^T\int_\Omega a(\nabla\psi,\nabla c)|u|^2\,dxdt\geqslant & -\displaystyle\int_0^T\int_\Omega\frac{C_c}{2}\frac{|u|^2}{|x|^2\langle x\rangle^{1+\delta}}\,dxdt \\ & \geqslant -C_c\delta^{-1}\|u\|^2_{XL^2_T}-2C_c\|\nabla^b u\|^2_{L^2(\Omega_{\leqslant 1})L^2_T}. \end{aligned} \tag{2.4.34}$$

Recalling (2.4.33), (2.4.28) and (2.4.34) we finally get

$$\begin{aligned} \textstyle\int_\Omega\int_0^T I_{|u|^2}\,dtdx\geqslant & (1-C_I)^2\tfrac{1}{R^2}\textstyle\int_{\Omega_{=R}}\|u\|^2_{L^2_T}\,dS \\ & -(72C_I\delta^{-1}+\delta^{-1}C_c)\|u\|^2_{XL^2_T}-2C_c\|\nabla^b u\|^2_{L^2(\Omega_{\leqslant 1})L^2_T} \\ & -324\delta^{-1}C_a(N+C_a)\left[\|u\|^2_{XL^2_T}+\|\nabla^b u\|^2_{L^2(\Omega_{\leqslant 1})L^2_T}\right] \end{aligned}$$

whence, noticing that $\|w\|_{L^2(\Omega_{\leqslant 1})}\leqslant\sqrt{2}\|w\|_Y$, we have (2.4.25) for all $R>1$.

2.4.3 Estimate of the terms in $|\nabla^b u|^2$

For such terms, using assumption (2.1.6), we shall prove for all $R>0$ the estimate

$$\textstyle\int_\Omega \alpha_{lm}\Re(\partial_l^b u\overline{\partial_m^b u})\,dx\geqslant\frac{n-1}{nR}\nu^2\int_{\Omega_{\leqslant R}}\|\nabla^b u(x)\|^2_{L^2_T}\,dx-24NC_a\delta^{-1}\|\nabla^b u\|^2_{Y_xL^2_T}, \tag{2.4.35}$$

where α_{lm} are the quantities defined in (2.3.2). The computations here are very similar to those in Section 4 of [8]. We split the quantities $\alpha_{\ell m}$ as

$$\alpha_{\ell m}(x) = s_{\ell m}(x) + r_{\ell m}(x)$$

where the remainder $r_{\ell m}$ gathers all terms containing some derivative of the a_{jk}. Since the weight ψ is radial, we have

$$s_{\ell m}(x) = 2a_{jm}a_{\ell k}\widehat{x}_j\widehat{x}_k\left(\psi'' - \tfrac{\psi'}{|x|}\right) + 2a_{jm}a_{j\ell}\tfrac{\psi'}{|x|}$$

while

$$r_{\ell m}(x) = [2a_{jm}a_{\ell k;j} - a_{jk}a_{\ell m;j}]\widehat{x}_k\psi'.$$

We estimate directly

$$|r_{\ell m}(x)\Re(\partial_\ell^b u\overline{\partial_m^b u})| \leqslant 3|a(x)||a'(x)||\nabla^b u(x)|^2$$

and by assumption (2.1.6) we obtain

$$|r_{\ell m}(x)\Re(\partial_\ell^b u\overline{\partial_m^b u})| \leqslant 3NC_a\langle x\rangle^{-1-\delta}|\nabla^b u|^2.$$

Integrating in $t \in [0,T]$ first and then in $x \in \Omega$, we get

$$\begin{aligned}\int_\Omega\int_0^T |r_{\ell m}(x)\Re(\partial_\ell^b u\overline{\partial_m^b u})|\,dtdx &\leqslant 3NC_a\int_\Omega\langle x\rangle^{-1-\delta}\int_0^T|\nabla^b u|^2\,dt\,dx\\ &= 3NC_a\int_\Omega\langle x\rangle^{-1-\delta}\|\nabla^b u(x)\|_{L_T^2}^2\,dx.\end{aligned}$$

Thus, using (2.2.9), we obtain the estimate

$$\int_\Omega\int_0^T |r_{\ell m}\Re(\partial_\ell^b u\overline{\partial_m^b u})|\,dtdx \leqslant 24NC_a\delta^{-1}\|\nabla^b u\|_{Y_xL_T^2}^2. \tag{2.4.36}$$

Concerning the terms $s_{\ell m}$, in the region $|x| > R$ we have

$$s_{\ell m}(x) = [a_{jm}a_{j\ell} - a_{jm}a_{\ell k}\widehat{x}_j\widehat{x}_k]\tfrac{1}{|x|} + \tfrac{R^{n-1}}{|x|^n}a_{jm}a_{\ell k}\widehat{x}_j\widehat{x}_k - a_{jm}a_{j\ell}\tfrac{R^{n-1}}{n|x|^n}$$

so that, in the sense of positivity of matrices,

$$s_{\ell m}(x) \geqslant [a_{jm}a_{j\ell} - a_{jm}a_{\ell k}\widehat{x}_j\widehat{x}_k]\tfrac{n-1}{n|x|} \geqslant 0 \quad \text{for } |x| > R$$

(indeed, one has $a_{jm}a_{j\ell} \geqslant a_{jm}a_{\ell k}\widehat{x}_j\widehat{x}_k$ as matrices); on the other hand, in the region $|x| \leqslant R$ we have

$$s_{\ell m}(x) = a_{jm}a_{j\ell}\tfrac{n-1}{nR} \quad \text{for } |x| \leqslant R.$$

Thus, by the assumption $a(x) \geqslant \nu I$, one has for all x

$$s_{\ell m}(x)\Re(\partial_\ell^b u\overline{\partial_m^b u}) \geqslant \tfrac{n-1}{nR}\nu^2\mathbf{1}_{|x|\leqslant R}(x)|\nabla^b u|^2. \tag{2.4.37}$$

Integrating (2.4.37) with respect to $t \in [0,T]$ and $x \in \Omega$, and recalling (2.4.36), we conclude the proof of (2.4.35).

2.4.4 Estimate of the magnetic terms

We now consider the terms

$$I_b := 2\Im[a_{jk}\partial_k^b u(\partial_j b_\ell - \partial_\ell b_j)a_{\ell m}\partial_m \psi\ \overline{u}] = 2\Im\left[(db \cdot a\widehat{x}) \cdot (a\nabla^b u)\overline{u}\psi'\right]$$

where the identity holds for any radial ψ, while db is the matrix

$$db = [\partial_j b_\ell - \partial_\ell b_j]_{j,\ell=1}^n.$$

Proof of Theorem 2.1.1

We shall prove the estimate

$$\int_\Omega \int_0^T |I_b|\, dx \leqslant 5\delta^{-1}N^2 C_b(\|\nabla^b u\|^2_{\dot{Y}_x L^2_T} + \|u\|^2_{\dot{X}_x L^2_T}). \tag{2.4.38}$$

Indeed, since $0 \leqslant \psi' \leqslant 1/2$ and $|a(x)| \leqslant N$, by (2.1.7) we have

$$|I_b(x)| \leqslant 2N^2|db(x)| \cdot |\nabla^b u||u|\psi' \leqslant N^2 \tfrac{|\nabla^b u||u|}{|x|^{2+\delta}+|x|^{2-\delta}}.$$

We integrate in $t \in [0,T]$, then in $x \in \Omega$, and we use the Hölder inequality in time:

$$\int_\Omega \int_0^T |I_b(x)|\, dtdx \leqslant N^2 \int_\Omega \int_0^T \tfrac{|\nabla^b u||u|}{|x|^{2+\delta}+|x|^{2-\delta}}\, dtdx \leqslant N^2 \int_\Omega \tfrac{\|\nabla^b u\|_{L^2_T}\|u\|_{L^2_T}}{|x|^{2+\delta}+|x|^{2-\delta}}\, dx$$

and by estimate (2.2.10) we obtain (2.4.38).

Proof of Theorem 2.1.2

In this case we prove the estimate

$$\int_\Omega \int_0^T |I_b|\, dtdx \leqslant 9\delta^{-1}N^2 C_b(\|\nabla^b u\|^2_{Y_x L^2_T} + \|u\|^2_{X_x L^2_T}). \tag{2.4.39}$$

The proof is completely analogous to the previous one, using (2.1.17) and (2.2.13).

2.4.5 Estimate of the terms containing $f(u)$

We prove here that there exists a $\gamma_0 > 0$ such that

$$A\psi[f(u)\bar{u} - 2F(u)] \geqslant \frac{\gamma_0}{R \vee |x|}[f(u)\bar{u} - 2F(u)]. \tag{2.4.40}$$

Thanks to (2.1.13), it is sufficient to check the pointwise inequality

$$A\psi(x) \geqslant \frac{\gamma_0}{R \vee |x|}.$$

Indeed, for $|x| \leqslant R$,

$$\widehat{a}\psi'' + \tfrac{\overline{a}-\widehat{a}}{|x|}\psi' = \tfrac{n-1}{2n}\left[\tfrac{\widehat{a}}{R} + \tfrac{\overline{a}-\widehat{a}}{R}\right] = \tfrac{n-1}{2n}\tfrac{\overline{a}}{R}$$

while for $|x| > R$

$$\widehat{a}\psi'' + \tfrac{\overline{a}-\widehat{a}}{|x|}\psi' = \tfrac{\widehat{a}}{|x|}\tfrac{n-1}{2n}\tfrac{R^{n-1}}{|x|^{n-1}} + \tfrac{\overline{a}-\widehat{a}}{|x|}\left(\tfrac{1}{2} - \tfrac{1}{2n}\tfrac{R^{n-1}}{|x|^{n-1}}\right) \geqslant \tfrac{\overline{a}-\widehat{a}}{|x|}\tfrac{n-1}{2n}.$$

Moreover, by (2.1.6),

$$a_{lm;l}\widehat{x}_m\psi' \geqslant -\tfrac{C_a}{\langle x\rangle^{1+\delta}}|\psi'| \geqslant -\tfrac{C_a}{|x|}|\psi'|.$$

Summing up we get

$$\begin{aligned} A\psi \geqslant & \begin{cases} \frac{n-1}{2nR}\left(\overline{a}-C_a\right) & |x|\leqslant R \\ \frac{1}{2|x|}\left[\frac{n-1}{n}(\overline{a}-\widehat{a})-C_a\right] & |x|>R, \end{cases} \\ \geqslant & \frac{\gamma_0}{R\vee|x|}, \end{aligned}$$

for any $\gamma_0 > 0$ such that

$$\gamma_0 < \begin{cases} \frac{n-1}{2n}\left(\overline{a}-C_a\right) & |x|\leqslant R \\ \frac{1}{2}\left[\frac{n-1}{n}(\overline{a}-\widehat{a})-C_a\right] & |x|>R, \end{cases} \tag{2.4.41}$$

which is possible provided C_a is so small that $C_a < \frac{n-1}{n}(\overline{a}(x)-\widehat{a}(x))$ (see Remark 2.4.1).

2.4.6 Estimate of the boundary terms

We now prove that

$$\int_\Omega \partial_j\{-\Re Q_j + 2F(u)a_{jk}\partial_k\psi + \Im[u_t\bar{u}a_{jk}\partial_k\psi]\}\,dx \geqslant 0. \tag{2.4.42}$$

Indeed, proceeding exactly as in [8], we see that assumption (2.1.14) implies

$$\textstyle\int_\Omega \partial_j\Re Q_j\,dx \leqslant 0.$$

Moreover, at any fixed $t\in[0,T]$ we have

$$\int_\Omega \nabla\cdot\{2F(u)a\nabla\psi + \Im[u_t\bar{u}a\nabla\psi]\} = 0.$$

To see this, we integrate $\nabla\cdot\{2F(u)a\nabla\psi + \Im[u_t\bar{u}a\nabla\psi]\}$ over the set $\Omega\cap\{|x|\leqslant R\}$ and let $R\to+\infty$: the integral over $|x|=R$ tends to 0 since $a\nabla\psi\in L^\infty(\Omega)$ and thanks to (2.1.10)

$$|F(u)| \leqslant \left|\textstyle\int_0^{|u|} f(s)\,ds\right| \lesssim |u|^{\gamma+1}\in L^1(\Omega), \tag{2.4.43}$$

(recall that $u\in H_0^1(\Omega)$), while the integral over $\partial\Omega$ vanishes by the Diriclet boundary condition since $F(0)=0$.

2.4.7 Estimate of the derivative term

We finally estimate the term at the left hand side of (2.3.1). We need the following Lemma:

Lemma 2.4.2. *Let $v \in H^1_0(\Omega)$ and $\psi\colon \mathbb{R}^n \to \mathbb{R}$ be such that $\nabla\psi$ and $|x|A\psi$ are bounded. Then there exist $C = C(\|a\|_{L^\infty}, \|\nabla\psi\|_{L^\infty}, \||x|A\psi\|_{L^\infty}) > 0$ such that*

$$\left|\int_\Omega a(\nabla\psi, \nabla^b v)v\,dx\right| \leqslant C\|v\|^2_{\dot H^{\frac12}},$$

Proof. Define for $f, g \in C^\infty_c(\Omega)$

$$T(f,g) := \int_\Omega \nabla\psi(x)\cdot a(x)\overline{\nabla^b f(x)}g(x)\,dx = \int_\Omega [a(x)\nabla\psi(x)]\cdot\overline{\nabla^b f(x)}g(x)\,dx.$$

We have trivially

$$|T(f,g)| \leqslant \int_\Omega |[a(x)\nabla\psi(x)]\cdot\overline{\nabla^b f(x)}g(x)|\,dx \leqslant C\|\nabla^b f\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)}$$

with $C = \|a\nabla\psi\|_{L^\infty}$. On the other hand, integration by parts gives

$$\begin{aligned}|T(f,g)| =& \left|\int_{\mathbb{R}^n}[a(x)\nabla\psi(x)]\overline{\nabla^b f(x)}g(x)\,dx\right| =\\ =&\left|\int_{\mathbb{R}^n}[a(x)\nabla\psi(x)]\nabla^b g(x)\overline{f(x)}\,dx + \int_{\mathbb{R}^n}\nabla\cdot[a(x)\nabla\psi(x)]g(x)\overline{f(x)}\,dx +\right.\\ &\left. - \int_{\mathbb{R}^n}\nabla\cdot\{[a(x)\nabla\psi(x)]g(x)\overline{f(x)}\}\,dx\right|.\end{aligned}$$

Discarding the divergence term and using the boundedness of $|x|A\psi$ we have, for some $C = C(\|a\|_{L^\infty}, \|\nabla\psi\|_{L^\infty}, \||x|A\psi\|_{L^\infty}) > 0$,

$$|T(f,g)| \leqslant C\left[\|f\|_{L^2(\Omega)}\|\nabla^b g\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\||x|^{-1}g\|_{L^2(\Omega)}\right]$$

which implies, using the magnetic Hardy inequality (2.2.11),

$$|T(f,g)| \leqslant C\|f\|_{L^2(\Omega)}\|\nabla^b g\|_{L^2(\Omega)}$$

for a different $C = C(\|a\|_{L^\infty}, \|\nabla\psi\|_{L^\infty}, \||x|A\psi\|_{L^\infty}) > 0$. The claim then follows by the equivalence $\|\nabla^b v\|_{L^2} \simeq \|\nabla v\|_{L^2}$ proved in Lemma 2.2.5, by complex interpolation and by density. □

Applying Lemma 2.4.2 we get

$$\Im\int_\Omega a(\nabla\psi, \nabla^b u)u\,dx \leqslant \left|\int_\Omega a(\nabla\psi, \nabla^b u)u\,dx\right| \leqslant \widetilde C\|u\|^2_{\dot H^{\frac12}} \tag{2.4.44}$$

for some $\widetilde C$ depending on $\|a\|_{L^\infty}$, $\|\nabla\psi\|_{L^\infty}$, $\||x|A\psi\|_{L^\infty}$. Note that even if ψ depends on $R > 0$, the constant $\widetilde C$ does not, since by (2.4.5), (2.1.6),

$$\|a\nabla\psi\|_{L^\infty} \leqslant \tfrac12\|a\|_{L^\infty}, \qquad \||x|A\psi\|_{L^\infty} \leqslant C(C_a, \|a\|_{L^\infty}).$$

2.4.8 Conclusion of the proof

From (2.3.1), using (2.4.40), we have

$$\begin{aligned}\partial_t[\Im(a(\nabla\psi,\nabla^b u)u)] \geqslant &-\tfrac{1}{2}A^2\psi|u|^2 - \Re a(\nabla\psi,\nabla c)|u|^2 + \Re(\alpha_{\ell m}\,\partial^b_m u\,\overline{\partial^b_\ell u})\\ &+ 2\Im(a_{jk}\partial^b_k u(\partial_j b_\ell - \partial_\ell b_j)a_{\ell m}\partial_m\psi\,\overline{u})\\ &+ \gamma_0[f(u)\bar{u} - 2F(u)](R\vee|x|)^{-1}\\ &+ \partial_j\{-\Re Q_j + 2F(u)a_{jk}\partial_k\psi + \Im[u_t\bar{u}a_{jk}\partial_k\psi]\}.\end{aligned}$$

Integrating with respect to $t\in[0,T]$ and then $x\in\Omega$ we obtain

$$\int_\Omega\int_0^T \partial_t\Im[a(\nabla\psi,\nabla^b u)u]\,dtdx \geqslant \tag{2.4.45}$$

$$-\int_\Omega\int_0^T\left[\tfrac{1}{2}A^2\psi + \Re a(\nabla\psi,\nabla c)\right]|u|^2\,dtdx \tag{2.4.46}$$

$$+\int_\Omega\int_0^T\Re\left[\alpha_{lm}\partial^b_m u\overline{\partial^b_l u}\right]\,dtdx \tag{2.4.47}$$

$$+2\int_\Omega\int_0^T\Im[a_{jk}\partial^b_k u(\partial_j b_l - \partial_l b_j)a_{lm}\partial_m\psi\bar{u}]\,dtdx \tag{2.4.48}$$

$$+\gamma_0\int_\Omega\int_0^T\frac{f(u)\bar{u}-2F(u)}{R\vee|x|}\,dtdx \tag{2.4.49}$$

$$+\int_\Omega\int_0^T\partial_j\{-\Re Q_j + 2F(u)a_{jk}\partial_k\psi + \Im[u_t\bar{u}a_{jk}\partial_k\psi]\}\,dtdx \tag{2.4.50}$$

We now use the estimates from the previous sections.

For the term (2.4.45), we use (2.4.44):

$$\begin{aligned}&\int_\Omega\int_0^T\partial_t\Im[a(\nabla\psi,\nabla^b u)u]\,dtdx\\ &\quad\leqslant\int_\Omega\Im a(\nabla\psi,\nabla^b u(0))u(0)\,dx + \int_\Omega\Im a(\nabla\psi,\nabla^b u(T))u(T)\,dx\leqslant\\ &\quad\leqslant C\left(\|u(0)\|^2_{\dot{H}^{\frac{1}{2}}} + \|u(T)\|^2_{\dot{H}^{\frac{1}{2}}}\right),\end{aligned}$$

where C depends on $\|a\|_{L^\infty}$, $\|\nabla\psi\|_\infty$, $\||x|A\psi\|_{L^\infty}$, but not on $R>0$.

For (2.4.50) we swap the integrals, then using (2.4.42) we see that this term can be discarded.

Proof of Theorem 2.1.1

We estimate (2.4.47) using (2.4.35) and recalling that $\|\cdot\|_Y\leqslant\|\cdot\|_{\dot{Y}}$, while (2.4.48) is estimated using (2.4.38). Summing up, we have obtained

$$\begin{aligned}&C\big(\|u(0)\|^2_{\dot{H}^{\frac{1}{2}}} + \|u(T)\|^2_{\dot{H}^{\frac{1}{2}}}\big)\geqslant\\ &\tfrac{1}{2}\left(\tfrac{n-1}{2}\nu\tfrac{1}{R^2}\int_{\Omega_{=R}}\widehat{a}\|u\|_{L^2_T}dS - \left[\tfrac{n+3}{2}N - n\nu\right](n-1)\int_{\Omega_{\geqslant R}}\widehat{a}\tfrac{R^{n-1}}{|x|^{n+2}}\|u\|_{L^2_T}\,dx\right)\\ &-(12nC_a(N+C_a)+C_c)\delta^{-1}\|u\|^2_{\dot{X}_xL^2_T}\\ &+\tfrac{n-1}{nR}\nu^2\int_{\Omega_{\leqslant R}}\|\nabla^b u(x)\|^2_{L^2_T}\,dx - 24NC_a\delta^{-1}\|\nabla^b u\|^2_{\dot{Y}_xL^2_T}\\ &-5\delta^{-1}N^2C_b\left(\|\nabla^b u\|^2_{\dot{Y}_xL^2_T} + \|u\|^2_{\dot{X}_xL^2_T}\right)\\ &+\gamma_0\int_\Omega\int_0^T\tfrac{f(u)\bar{u}-2F(u)}{R\vee|x|}\,dtdx.\end{aligned}\tag{2.4.51}$$

We now take the sup over $R > 0$ at the right hand side. Denote with $\Sigma(R)$ all the terms which depend on R:

$$\Sigma(R) :=\tfrac{1}{2}\left(\tfrac{n-1}{2}\nu\tfrac{1}{R^2}\int_{\Omega_{=R}}\widehat{a}\|u\|_{L^2_T}dS\left[\tfrac{n+3}{2}N-n\nu\right](n-1)\int_{\Omega_{\geqslant R}}\widehat{a}\tfrac{R^{n-1}}{|x|^{n+2}}\|u\|_{L^2_T}dx\right) + \tfrac{n-1}{nR}\nu^2\int_{\Omega_{\leqslant R}}\|\nabla^b u(x)\|^2_{L^2_T}dx+\gamma_0\int_\Omega\int_0^T\tfrac{f(u)\bar{u}-2F(u)}{R\vee|x|}dtdx.$$

We shall use the simple remark that if three nonnegative quantities f, g, h depend on R, then

$$\sup_{R>0}[f(R)+g(R)+h(R)] \geqslant \frac{1}{3}\left(\sup_{R>0}f(R)+\sup_{R>0}g(R)+\sup_{R>0}h(R)\right). \tag{2.4.52}$$

We now distinguish two cases.

First case: $\frac{n+3}{2}N \geqslant n\nu$. Then by (2.2.6) we get

$$\Sigma(R) \geqslant Z(R) - \tfrac{1}{2}\left[\tfrac{n+3}{2}N-n\nu\right]\|\widehat{a}^{1/2}u\|^2_{\dot{X}_xL^2_T},$$

where

$$Z(R) := \tfrac{n-1}{4}\nu\tfrac{1}{R^2}\int_{\Omega_{=R}}\widehat{a}\|u\|^2_{L^2_T}dS+\tfrac{n-1}{nR}\nu^2\int_{\Omega_{\leqslant R}}\|\nabla^b u(x)\|^2_{L^2_T}dx+\gamma_0\int_\Omega\int_0^T\tfrac{f(u)\bar{u}-2F(u)}{R\vee|x|}dtdx.$$

Thanks to (2.2.6), (2.4.52), and recalling that $\widehat{a} \geqslant \nu$, we obtain

$$\sup_{R>0}Z(R) \geqslant \tfrac{n-1}{12}\nu^2\|u\|^2_{\dot{X}_xL^2_T}+\tfrac{n-1}{3n}\nu^2\|\nabla^b u\|^2_{\dot{Y}_xL^2_T}+\tfrac{\gamma_0}{3}\int_\Omega\int_0^T\tfrac{f(u(x))\bar{u}(x)-2F(u)}{|x|}dtdx$$

and consequently, again by $\widehat{a} \geqslant \nu$,

$$\sup_{R>0}\Sigma(R) \geqslant \tfrac{K_0}{2}\nu^2\|u\|^2_{\dot{X}_xL^2_T}+\tfrac{n-1}{3n}\nu^2\|\nabla^b u\|^2_{\dot{Y}_xL^2_T}+\tfrac{\gamma_0}{3}\int_\Omega\int_0^T\tfrac{f(u(x))\bar{u}(x)-2F(u)}{|x|}dtdx, \tag{2.4.53}$$

provided we define

$$K_0 := \tfrac{n-1}{6} - \tfrac{n+3}{2}\tfrac{N}{\nu}+n \tag{2.4.54}$$

which is a strictly positive quantity provided we assume N/ν is small enough (like in (2.4.2)).

Second case: $\frac{n+3}{2}N \leqslant n\nu$. Then we have

$$\Sigma(R) \geqslant \tfrac{n-1}{4}\nu\tfrac{1}{R^2}\int_{\Omega_{=R}}\widehat{a}\|u\|^2_{L^2_T}dS+\tfrac{n-1}{nR}\nu^2\int_{\Omega_{\leqslant R}}\|\nabla^b u(x)\|^2_{L^2_T}dx+\gamma\int_\Omega\int_0^T\tfrac{f(u)\bar{u}-2F(u)}{R\vee|x|}dtdx. \tag{2.4.55}$$

Thanks to (2.4.52), recalling that $\widehat{a} \geqslant \nu$, and observing that in this case $K_0 \leqslant \frac{n-1}{6}$, we obtain again (2.4.53).

By (2.4.51), (2.4.53) we conclude that

$$M_1\|u\|^2_{\dot{X}_xL^2_T}+M_2\|\nabla^b u\|^2_{\dot{Y}_xL^2_T}+M_3\int_\Omega\int_0^T\tfrac{f(u(x))\bar{u}(x)-2F(u)}{|x|}dtdx \leqslant C\big(\|u(0)\|^2_{\dot{H}^{\frac{1}{2}}}+\|u(T)\|^2_{\dot{H}^{\frac{1}{2}}}\big) \tag{2.4.56}$$

for some $C > 0$, where γ_0 is defined in (2.4.41) and

$$M_1 := \tfrac{K_0}{2}\nu^2 - \tfrac{5N^2C_b+12nC_a(N+C_a)+C_c}{\delta},$$

$$M_2 := \tfrac{n-1}{3n}\nu^2 - \tfrac{5N^2C_b+24NC_a}{\delta}, \qquad M_3 := \tfrac{\gamma_0}{3}.$$

If the constants C_a, C_b and C_c are sufficiently small, these quantities are positive, and the estimate (2.4.56) is effective.

Proof of Theorem 2.1.2

We estimate (2.4.47) using (2.4.35) and (2.4.48) thanks to (2.4.39). Summing up, we have obtained

$$
\begin{aligned}
C\big(\|u(0)\|^2_{\dot H^{\frac12}} + \|u(T)\|^2_{\dot H^{\frac12}}\big) \geqslant&\\
&(1-C_I)^2\frac{1}{R^2}\int_{\Omega_{=R}}\|u\|^2_{L^2_T}\,dS && (2.4.57)\\
&-8\delta^{-1}[C_c+9C_I+41C_a(N+C_a)]\|u\|^2_{XL^2_T} && (2.4.58)\\
&-13\delta^{-1}[C_c+36C_a(N+C_a)]\|\nabla^b u\|^2_{YL^2_T} && (2.4.59)\\
&+\tfrac{n-1}{nR}\nu^2\textstyle\int_{\Omega_{\leqslant R}}\|\nabla^b u(x)\|^2_{L^2_T}\,dx-24NC_a\delta^{-1}\|\nabla^b u\|^2_{Y_xL^2_T}\\
&-9\delta^{-1}N^2C_b(\|\nabla^b u\|^2_{Y_xL^2_T}+\|u\|^2_{X_xL^2_T})\\
&+\gamma_0\textstyle\int_\Omega\int_0^T\frac{f(u)\bar u-2F(u)}{R\vee|x|}\,dtdx. && (2.4.60)
\end{aligned}
$$

We now take the sup over $R>1$ at the right hand side. We denote with $\Sigma(R)$ all the terms which depend on R:

$$
\begin{aligned}
\Sigma(R) :=&(1-C_I)^2\tfrac{1}{R^2}\textstyle\int_{\Omega_{=R}}\|u\|^2_{L^2_T}\,dS+\tfrac{n-1}{nR}\nu^2\int_{\Omega_{\leqslant R}}\|\nabla^b u(x)\|^2_{L^2_T}\,dx\\
&+\gamma\int_\Omega\int_0^T\frac{f(u)\bar u-2F(u)}{R\vee|x|}\,dtdx
\end{aligned}
$$

Thanks to (2.2.14), we have, for $0<\theta<1$,

$$
\begin{aligned}
(1-C_I)^2\sup_{R>1}\frac{1}{R^2}\int_{\Omega_{=R}}\|u\|^2_{L^2_T}\,dS \geqslant& (1-\theta)(1-C_I)^2\sup_{R>1}\frac{1}{R^2}\int_{\Omega_{=R}}\|u\|^2_{L^2_T}\,dS\\
&+\theta(1-C_I)^2\left(\frac14\|u\|^2_{XL^2_T}-\frac{13}{4}\|\nabla^b u\|^2_{YL^2_T}\right).
\end{aligned}
\tag{2.4.61}
$$

Note also that we can take $\nu=1-C_I$ and $N=1+C_I$ by assumption (2.1.16), while $n=3$. We obtain

$$
\sup_{R>1}\frac{n-1}{nR}\nu^2\int_{\Omega_{\leqslant R}}\|\nabla^b u(x)\|^2_{L^2_T}\,dx \geqslant \frac23(1-C_I)^2\|\nabla^b u\|^2_{YL^2_T} \tag{2.4.62}
$$

Finally

$$
\gamma_0\sup_{R>1}\int_\Omega\int_0^T\frac{f(u)\bar u-2F(u)}{R\vee|x|}\,dtdx \geqslant \gamma_0\int_\Omega\int_0^T\frac{f(u)\bar u-2F(u)}{\langle x\rangle}\,dtdx. \tag{2.4.63}
$$

We take $\theta:=2/13$ (it is enough to choose θ such that $2/3\geqslant(13\theta)/4$). Thanks to (2.4.52), (2.4.61), (2.4.62), (2.4.63), we get

$$
\begin{aligned}
\sup_{R>1}\Sigma(R)\geqslant&\frac{(1-C_I)^2}{78}\|u\|^2_{XL^2_T}+\frac{(1-C_I)^2}{6}\|\nabla^b u\|^2_{YL^2_T}\\
&+\frac{\gamma_0}{3}\int_\Omega\int_0^T\frac{f(u)\bar u-2F(u)}{\langle x\rangle}\,dtdx.
\end{aligned}
\tag{2.4.64}
$$

By (2.4.60), (2.4.64) we conclude that

$$M_1\|u\|^2_{X_x L^2_T} + M_2\|\nabla^b u\|^2_{Y_x L^2_T} + M_3 \int_\Omega \int_0^T \frac{f(u(x))\bar{u}(x) - 2F(u)}{\langle x\rangle}\,dtdx \leqslant C\big(\|u(0)\|^2_{\dot H^{\frac12}} + \|u(T)\|^2_{\dot H^{\frac12}}\big) \tag{2.4.65}$$

for some $C > 0$, where

$$M_1' := \frac{(1-C_I)^2}{78} - 8\delta^{-1}[C_c + 9C_I + 41C_a(N + C_a)] - 9\delta^{-1}N^2C_b,$$
$$M_2' := \frac{(1-C_I)^2}{6} - 13\delta^{-1}[C_c + 38C_a(N + C_a)] - 9\delta^{-1}N^2C_b,$$
$$M_3 = \frac{\gamma_0}{3},$$

and γ_0 is defined in (2.4.41). If the constants C_a, C_b, C_c and C_I are sufficiently small, these quantities are positive and the estimate (2.4.65) is effective.

2.4.9 Proof of Corollary 2.1.5

Since $u = e^{itL}u_0$ satisfies equation (2.1.1) with the choice $f \equiv 0$, we see that u satisfies the smoothing estimate (2.1.20). By complex interpolation, we deduce from (2.1.20) the estimate

$$\|\langle x\rangle^{-1-}(-\Delta)^{\frac14}u\|_{L^2_T L^2} \lesssim \|u_0\|_{\dot H^{\frac12}} + \|u(T)\|_{\dot H^{\frac12}}$$

for all $T > 0$. Proceeding exactly as in the proof of Corollary 1.4 in [7], from the gaussian bound for e^{tL} in Proposition 2.6.2 we deduce the weighted estimate

$$\|w(x)(-L)^{\frac14}v\|_{L^2} \lesssim \|w(x)(-\Delta)^{\frac14}v\|_{L^2}$$

for any A_2 weight w, and in particular for $w(x) = \langle x\rangle^{-s}$ for any $s > 0$. Thus we have

$$\|\langle x\rangle^{-1-}(-L)^{\frac14}u\|_{L^2_T L^2} \lesssim \|\langle x\rangle^{-1-}(-\Delta)^{\frac14}u\|_{L^2_T L^2} \lesssim \|u_0\|_{\dot H^{\frac12}} + \|u(T)\|_{\dot H^{\frac12}}$$

and commuting $(-L)^{\frac14}$ with e^{itL}, and recalling the equivalence (2.6.2), we obtain

$$\|\langle x\rangle^{-1-}u\|_{L^2_T L^2} \lesssim \|u_0\|_{L^2} + \|u(T)\|_{L^2} \simeq \|u_0\|_{L^2}$$

by the conservation of the L^2 norm.

2.5 Proof of Theorems 2.1.6, 2.1.7: the bilinear smoothing estimate

Since the arguments for Theorems 2.1.6 and 2.1.7 largely overlap, we shall again proceed with both proofs in parallel.

Let u be a solution of (2.1.1), and write identity (2.3.1) with a weight of the form $\psi = \psi(x-y)$, for $x, y \in \Omega$. In the following formulas, to make notations lighter, we shall write simply $u(x)$, $u(y)$ instead of $u(t,x)$, $u(t,y)$. We have

$$\begin{aligned}
M(x,y) = & -\tfrac{1}{2}A_x^2\psi(x-y)|u(x)|^2 + \Re(\alpha_{\ell m}(x)\ \partial^{b(x)}_{x_m}u(x)\ \overline{\partial^{b(x)}_{x_\ell}u(x)}) \\
& - \Re a(x)(\nabla_x\psi(x-y), \nabla_x c(x))|u(x)|^2 \\
& + 2\Im(a_{jk}(x)\partial^{b(x)}_{x_k}u(x)(\partial_{x_j}b(x)_\ell - \partial_{x_\ell}b(x)_j)a_{\ell m}(x)\partial_{x_m}\psi(x-y)\ \overline{u(x)}) \\
& + A_x\psi(x-y)[f(u(x))\bar{u}(x) - 2F(u(x))] \\
& + \partial_{x_j}\{-\Re Q_j(x) + 2F(u(x))a_{jk}(x)\partial_{x_k}\psi(x-y) + \Im[u_t(x)\bar{u}(x)a_{jk}(x)\partial_{x_k}\psi(x-y)]\}.
\end{aligned}$$

where $M(x,y)$ is defined by

$$M(x,y) := \partial_t\{\Im(a_x(\nabla_x\psi(x-y), \nabla^{b(x)}_x u(x))u(x))\}.$$

Note that in order to distinguish between the two groups of variables x and y, here and in the following we used the notations

$$a(x)(z,w) = a_{jk}(x)z_j\overline{w_k}, \qquad \partial^{b(x)}_{x_j} = \partial_{x_j} + ib_j(x), \qquad A^{b(x)}_x v = \partial^{b(x)}_{x_j}(a_{jk}(x)\partial^{b(x)}_{x_k}v(x,y))$$

and similarly A_x, $\nabla^{b(x)}_x$; we shall however stick to simpler notations whenever possible. The starting point for the proof is the identity

$$\begin{aligned}
&\partial_t\{\Im[a(x)(\nabla_x\psi(x-y), \nabla^b u(x))u(x)]|u(y)|^2\} = \\
&\qquad = M(x,y)|u(y)|^2 + \Im[a(x)(\nabla\psi(x-y), \nabla^b u(x))u(x)]\partial_t\{|u(y)|^2\}.
\end{aligned} \tag{2.5.1}$$

Since u is a solution of (2.1.1) and c, $f(u)\bar{u}$ are real valued, we have

$$\begin{aligned}
\partial_t|u|^2 = & 2\Re[u_t\bar{u}] = 2\Re[\bar{u}(-iA^b u + icu + if(u))] = \\
= & 2\Re[-iA^b u\bar{u} + ic|u|^2 + if(u)\bar{u}] = 2\Im[A^b u\bar{u}]
\end{aligned}$$

and using the identity

$$A^b u\bar{u} = -a(\nabla^b u, \nabla^b u) + \nabla\cdot\{a\nabla^b u\bar{u}\},$$

by the reality of $a(\nabla^b u, \nabla^b u)$ we have

$$\partial_t|u(y)|^2 = 2\Im[A^{b(y)}u(y)\overline{u(y)}] = 2\nabla_y\cdot\{\Im[a(y)\nabla^{b(y)}_y u(y)\bar{u}(y)]\}.$$

Thus the last term in (2.5.1) can be manipulated as follows:

$$\begin{aligned}
&\Im[a(x)(\nabla\psi(x-y), \nabla^b u(x))u(x)]\partial_t[|u(y)|^2] = \\
&\quad = 2\Im[a(x)(\nabla\psi(x-y), \nabla^b u(x))u(x)]\nabla_y\cdot\{\Im[a(y)\nabla^{b(y)}u(y)\bar{u}(y)]\} = \\
&\quad = 2\Im[(a(x)\overline{\nabla^b_x u(x)})^t u(x)]D^2\psi(x-y)\Im[(a(y)\nabla^b_y u(y))\overline{u(y)}] \\
&\qquad + \nabla_y\cdot\{2\Im[a(x)(\nabla\psi(x-y), \nabla^b u(x))u(x)]\Im[a(y)\nabla^{b(y)}u(y)\bar{u}(y)]\}.
\end{aligned}$$

Moreover, we rewrite the quantities $\alpha_{\ell m}$ in the form

$$\alpha_{\ell m} = 2(a(x)D^2_x\psi(x-y)a(x))_{\ell m} + r_{\ell m}$$

where the first term is the ℓm entry of the matrix $a \cdot D^2\psi \cdot a$ and

$$r_{lm} = \partial_k \psi_y (2a_{jm}a_{lk;j} - a_{jk}a_{lm;j}). \tag{2.5.2}$$

We choose different weights for the proofs of Theorem 2.1.6 and Theorem 2.1.7: in the proof of Theorem 2.1.7 we set

$$\psi(x-y) := \langle x-y\rangle_\sigma, \tag{2.5.3}$$

for $\sigma > 0$, where we use the following notation:

$$\langle x-y\rangle_\sigma := (\sigma^2 + |x-y|^2)^{1/2},$$

while in the proof of Theorem 2.1.6 we take $\sigma = 0$ in (2.5.3), that is to say, we choose

$$\psi(x-y) := |x-y|,$$

Note that in the following, with a small abuse, we shall use the same notation for the radial weight function $\psi(x)$ and for $\psi(r) = \psi(|x|)$. We gather here some identities satisfied by $\psi(r) = \langle r\rangle_\sigma$ for $\sigma \geqslant 0$:

$$\begin{gathered}\psi' = \frac{r}{\langle r\rangle_\sigma}, \quad \psi'' = \frac{\sigma^2}{\langle r\rangle_\sigma^3}, \quad \psi''' = -3\sigma^2\frac{r}{\langle r\rangle_\sigma^5}, \quad \psi^{IV} = 12\frac{\sigma^2}{\langle r\rangle_\sigma^5} - 15\frac{\sigma^4}{\langle r\rangle_\sigma^7}, \\ \frac{1}{r^2}\left(\psi'' - \frac{\psi'}{r}\right) = -\frac{1}{\langle r\rangle_\sigma^3}, \quad \psi''' - \frac{\psi''}{r} = -\sigma^2\left(\frac{r}{\langle r\rangle_\sigma^5} + \frac{1}{\langle r\rangle_\sigma^3 r}\right) \leqslant \frac{4\sigma^2}{\langle r\rangle_\sigma^3 r}.\end{gathered} \tag{2.5.4}$$

Moreover, for $\sigma \geqslant 0$, we introduce the notation

$$\widehat{(x-y)}_m^\sigma := \frac{x_m - y_m}{\langle x-y\rangle_\sigma}.$$

We have (see (2.4.10))

$$\begin{aligned}A_x\langle x-y\rangle_\sigma =& a_{\ell m;\ell}(x)\widehat{(x-y)}_m^\sigma + \\ &+ \frac{\sigma^2}{\langle x-y\rangle_\sigma}a_{\ell m}(x)\widehat{(x-y)}_\ell^\sigma\widehat{(x-y)}_m^\sigma + \frac{\bar a(x) - \widehat{(x-y)}_\ell^\sigma a_{\ell m}(x)\widehat{(x-y)}_m^\sigma}{\langle x-y\rangle_\sigma}\end{aligned}$$

which implies, since the last two terms are non negative,

$$A_x\langle x-y\rangle_\sigma \geqslant a_{\ell m;\ell}(x)\widehat{(x-y)}_m^\sigma \geqslant -|a'(x)| \geqslant -\frac{C_a}{\langle x\rangle^{1+\delta}},$$

and, using assumption (2.1.13),

$$A_x\langle x-y\rangle_\sigma[f(u(x))\overline{u(x)} - 2F(u(x))]|u(y)|^2 \geqslant -\frac{C_a}{\langle x\rangle^{1+\delta}}[f(u(x))\overline{u(x)} - 2F(u(x))]|u(y)|^2.$$

Now we integrate (2.5.1) on $\Omega^2 = \Omega_x \times \Omega_y$. The divergence terms in ∇_x can be neglected exactly as in the proof of Theorems 2.1.1 and 2.1.2, while the divergence terms in ∇_y vanish on $\partial\Omega_y$ and at infinity. Taking into account the previous computations we obtain the inequality

$$\begin{aligned}
&2 \int_{\Omega^2} \Re[(a(x)\overline{\nabla^b u(x)})^t D^2\psi(x-y)(a(x)\nabla^b u(x))]|u(y)|^2 dxdy+ \\
&\quad +2 \int_{\Omega^2} \Im[(a(x)\overline{\nabla^b u(x)})^t u(x)] D^2\psi(x-y)\Im[(a(y)\nabla^b u(y))\overline{u(y)}] dxdy+ \\
&\quad -\tfrac{1}{2} \int_{\Omega^2} A^2\psi(x-y)|u(x)|^2|u(y)|^2 dxdy \leqslant \\
&\leqslant \partial_t \int_{\Omega^2} \Im[a(x)(\nabla\psi(x-y), \nabla^b u(x))u(x)]|u(y)|^2 dxdy + \int_{\Omega^2} R(x,y)|u(y)|^2 dxdy,
\end{aligned} \tag{2.5.5}$$

where

$$\begin{aligned}
R(x,y) = &- r_{lm}(x)\partial^b_m u(x)\overline{\partial^b_l u(x)} \\
&+ \Re[a(x)(\nabla\psi(x-y), \nabla c(x))]|u(x)|^2 \\
&- 2\Im[a_{jk}(x)\partial_k^{b(x)} u(x)(\partial_j b_\ell(x) - \partial_\ell b_j(x))a_{\ell m}(x)\partial_m\psi(x-y)\ \overline{u(x)}] \\
&- C_a\langle x\rangle^{-1-\delta}[f(u(x))\overline{u(x)} - 2F(u(x))]|u(y)|^2.
\end{aligned} \tag{2.5.6}$$

We remark that $R(x,y)$ depends on y only through ψ. In the following sections we integrate (2.5.5) in time on an interval $[0,T]$ and we estimate each term individually.

2.5.1 Positivity of quadratic terms in the derivative

The first two terms in (2.5.5) can be dropped from the inequality since their sum is nonnegative. To check this fact, we rewrite them as the sum

$$\begin{aligned}
&\int_{\Omega^2} (a(x)\overline{\nabla^b u(x)})^t D^2\psi(x-y)(a(x)\nabla^b u(x))|u(y)|^2 dxdy \\
&\quad + \int_{\Omega^2} (a(y)\overline{\nabla^b u(y)})^t D^2\psi(x-y)(a(y)\nabla^b u(y))|u(x)|^2 dxdy \\
&\quad + 2\int_{\Omega^2} \Im[(a(x)\overline{\nabla^b u(x)})^t u(x)] D^2\psi(x-y)\Im[(a(y)\nabla^b u(y))\overline{u(y)}] dxdy
\end{aligned}$$

and then positivity follows from the the following algebraic lemma with the choice $T(x,y) = D^2_x\psi(x-y)$:

Lemma 2.5.1. *Let $T(x,y)$ be a real, symmetric, nonnegative matrix depending on $x, y \in \mathbb{R}^n$. Then the following quantity is nonnegative a.e.:*

$$\begin{aligned}
&(a(x)\overline{\nabla^b u(x)})^t T(x,y)(a(x)\nabla^b u(x))|u(y)|^2 + (a(y)\overline{\nabla^b u(y)})^t T(x,y)(a(y)\nabla^b u(y))|u(x)|^2 \\
&\qquad + 2\Im[(a(x)\overline{\nabla^b u(x)})^t u(x)] T(x,y)\Im[(a(y)\nabla^b u(y))\overline{u(y)}] \geqslant 0.
\end{aligned}$$

Proof. Let $\Sigma(x,y)$ be the quantity in the statement. Assume first $T = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ is diagonal at a point (x,y), with $\lambda_j \geqslant 0$. We have then

$$\begin{aligned}
\Sigma(x,y) &= \textstyle\sum_{j=1}^n \lambda_j \big\{|(a(x)\nabla^b u(x))_j|^2|u(y)|^2 + |(a(y)\nabla^b u(y))_j|^2|u(x)|^2 \\
&\qquad + 2\Im[(a(x)\overline{\nabla^b u(x)})_j u(x)]\Im[(a(y)\nabla^b u(y))_j\overline{u(y)}]\big\} \\
&\geqslant \textstyle\sum_{j=1}^n \lambda_j \big\{|(a(x)\nabla^b u(x))_j|^2|u(y)|^2 + |(a(y)\nabla^b u(y))_j|^2|u(x)|^2 \\
&\qquad - 2|(a(x)\overline{\nabla^b u(x)})_j||u(x)||(a(y)\nabla^b u(y))_j||\overline{u(y)}|\big\} \geqslant 0.
\end{aligned}$$

If $T(x,y)$ is non diagonal, we can find an orthonormal matrix $S = S(x,y)$ with real entries such that $T = S^t \Lambda S$, with $\Lambda \geqslant 0$ real and diagonal. This implies

$$\begin{aligned}\Sigma(x,y) =& (Sa(x)\overline{\nabla^b u(x)})^t \Lambda (Sa(x)\nabla^b u(x))|u(y)|^2 \\ &+ (Sa(y)\overline{\nabla^b u(y)})^t \Lambda (Sa(y)\nabla^b u(y))|u(x)|^2 \\ &+ 2\Im[(Sa(x)\overline{\nabla^b u(x)})^t u(x)]\Lambda \Im[(Sa(y)\nabla^b u(y))\overline{u(y)}],\end{aligned}$$

and the claim follows from the previous step. □

2.5.2 The ∂_t term

We now consider the first term at the right hand side of (2.5.5), which is differentiated in time. We need a Lemma:

Lemma 2.5.2. *Let $\psi(x-y) = \langle x-y\rangle_\sigma$, for $\sigma \geqslant 0$. Then the following estimate holds:*

$$\left| \int_{\Omega^2} a(x)(\nabla\psi(x-y), \nabla^b u(x))u(x)\,\varphi(y)\,dxdy \right| \lesssim \|\varphi\|_{L^1}\|u\|^2_{\dot H^{\frac12}},$$

for an implicit constant independent on σ.

Proof. For $f, g \in C_c^\infty(\Omega)$, set

$$T(f,g) := \textstyle\int_{\Omega^2} a(x)(\nabla\psi(x-y), \nabla^b f(x))g(x)\,\varphi(y)\,dxdy.$$

We have immediately

$$|T(f,g)| \leqslant \|a\|_{L^\infty}\|\varphi\|_{L^1}\|\nabla^b f\|_{L^2}\|g\|_{L^2}. \tag{2.5.7}$$

On the other hand, integrating by parts we get

$$\begin{aligned}|T(f,g)| \leqslant& \left|\textstyle\int_{\Omega^2} a(x)(\nabla\psi(x-y), \nabla^b g(x))f(x)\overline{\varphi(y)}dxdy\right| \\ &+ \left|\textstyle\int_{\Omega^2} \partial_{x_m} a_{\ell m}(x)\partial_{x_\ell}\psi(x-y)\overline{f(x)}g(x)\varphi(y)\,dxdy\right| \\ &+ \left|\textstyle\int_{\Omega^2} a_{\ell m}(x)\partial_{x_\ell x_m}\psi(x-y)\overline{f(x)}g(x)\varphi(y)\,dydx\right|.\end{aligned} \tag{2.5.8}$$

By assumption (2.1.6), we have

$$\begin{aligned}&|\textstyle\int_{\Omega^2} \partial_{x_m} a_{\ell m}(x)\partial_{x_\ell}\psi(x-y)\overline{f(x)}g(x)\varphi(y)\,dxdy| \leqslant \\ &\qquad \leqslant C_a\|\varphi\|_{L^1}\|f\|_{L^2}\left\|\langle x\rangle^{-1-\delta}g\right\|_{L^2} \lesssim \|\varphi\|_{L^1}\|f\|_{L^2}\|\nabla^b g\|_{L^2},\end{aligned} \tag{2.5.9}$$

where in the last step we used (2.2.11). By direct computation

$$\begin{aligned}&|\textstyle\int_{\Omega^2} a_{\ell m}(x)\partial_{x_\ell x_m}\psi(x-y)\overline{f(x)}g(x)\varphi(y)dydx| \\ &\qquad \leqslant \textstyle\int_{\mathbb{R}^{2n}} |\bar a(x) - a(x)(\widehat{(x-y)}^\sigma, \widehat{(x-y)}^\sigma)|\cdot\langle x-y\rangle_\sigma^{-1}\cdot|\overline{f(x)}g(x)\varphi(y)|dxdy \\ &\qquad \leqslant Nn\|\varphi\|_{L^1}\|f\|_{L^2}\sup_y\left(\textstyle\int_{\mathbb{R}^n}\frac{|g(x)|^2}{|x-y|^2}\,dx\right)^{\frac12} \\ &\qquad \lesssim \|\varphi\|_{L^1}\|f\|_{L^2}\|\nabla^b g\|_{L^2},\end{aligned} \tag{2.5.10}$$

again using (2.2.11) in the last inequality. By (2.5.9) and (2.5.10), we deduce from (2.5.8)

$$|T(f,g)| \lesssim \|\varphi\|_{L^1}\|f\|_{L^2}\|\nabla^b g\|_{L^2}.$$

Recalling now the equivalence (2.2.15), by complex interpolation beetwen this estimate and (2.5.7) we obtain

$$|\int_{\mathbb{R}^n} a(x)(\nabla\psi(x-y),\nabla^b f(x))g(x)\,\varphi(y)\,dxdy| \lesssim \|\varphi\|_{L^1}\|f\|_{\dot H^{\frac12}}\|g\|_{\dot H^{\frac12}}$$

and taking $f=g=u$ we conclude the proof. □

If we choose $\varphi=|u|^2$ in the previous Lemma, we obtain

$$\begin{aligned}\left|\int_0^T \partial_t\int_{\Omega^2}\Im[a(x)(\nabla\psi(x-y),\nabla^b u(x))u(x)]|u(y)|^2dxdy\,dt\right|\\ \lesssim \|u(0)\|_{L^2}^2\left[\|u(0)\|_{\dot H^{\frac12}}^2+\|u(T)\|_{\dot H^{\frac12}}^2\right],\end{aligned} \tag{2.5.11}$$

since the L^2-norm of the solution is constant in time.

2.5.3 The $R(x,y)$ term

We now focus on the last term in (2.5.5). Our goal is to prove

$$\left|\int_0^T\int_{\Omega^2}R(x,y)|u(y)|^2dxdy\,dt\right| \lesssim \|u(0)\|_{L^2}^2\left[\|u(0)\|_{\dot H^{\frac12}}^2+\|u(T)\|_{\dot H^{\frac12}}^2\right]. \tag{2.5.12}$$

The quantity $R(x,y)$, defined by (2.5.6), gives rise to four terms.

For the term containing $r_{\ell m}$ (see (2.5.2)) we notice that for all $\sigma\geqslant 0$ we have $|\nabla\psi|\leqslant 1$, hence both in the proof of Theorem 2.1.6 and Theorem 2.1.7 we have

$$|r_{\ell m}(x)|\leqslant 2N|a'(x)|\leqslant 2NC_a\langle x\rangle^{-1-\delta}$$

using (2.1.6). This implies

$$|\int_0^T\int_{\Omega^2}r_{\ell m}(x)\partial_m^b u(x)\overline{\partial_\ell^b u(x)}|u(y)|^2dxdydt| \lesssim \|u(0)\|_{L^2}^2\int_{\mathbb{R}^n}\langle x\rangle^{-1-\delta}\|\nabla^b u(x)\|_{L_t^2}^2dx$$

by the conservation of the L^2 norm. In the proof of Theorem 2.1.6, by estimate (2.2.9) and (2.1.15) we obtain

$$|\int_0^T\int_{\Omega^2}r_{\ell m}(x)\partial_m^b u(x)\overline{\partial_\ell^b u(x)}|u(y)|^2dxdydt| \lesssim \|u(0)\|_{L^2}^2\left[\|u(0)\|_{\dot H^{\frac12}}^2+\|u(T)\|_{\dot H^{\frac12}}^2\right],$$

and in the proof of Theorem 2.1.7 we get the same result thanks to (2.2.9) and (2.1.18).

We estimate differently the term containing c in the two proofs. In the proof of Theorem 2.1.6, recalling assumption (2.1.22), we have

$$\begin{aligned}&|\int_0^T\int_{\Omega^2}a(x)(\nabla\psi(x-y),\nabla c(x))|u(x)|^2|u(y)|^2dxdydt|\\ &\qquad\lesssim \|u(0)\|_{L^2}\int_\Omega\|u(x)\|_{L_t^2}^2|x|^{-2}\langle x\rangle^{-1-\delta}\,dx \lesssim \|u(0)\|_{L_x^2}^2\|u\|_{\dot X L_t^2}^2\end{aligned}$$

using the inequality (2.2.7), and, thanks to (2.1.15),

$$|\int_0^T\int_{\Omega^2} a(x)(\nabla\psi(x-y),\nabla c)|u(x)|^2|u(y)|^2dxdy\,dt| \lesssim \|u(0)\|_{L^2}^2\left[\|u(0)\|^2_{\dot H^{\frac12}}+\|u(T)\|^2_{\dot H^{\frac12}}\right].$$

In the proof of Theorem 2.1.7, recalling assumption (2.1.22) and thanks to (2.4.32), we have

$$\begin{aligned}&|\int_0^T\int_{\Omega^2} a(x)(\nabla\psi(x-y),\nabla c(x))|u(x)|^2|u(y)|^2dxdydt|\\ &\qquad\lesssim \|u(0)\|_{L^2}\int_0^T\int_\Omega |x|^{-2}\langle x\rangle^{-1-\delta}|u(x)|^2\,dx\,dt\\ &\qquad\lesssim \|u(0)\|_{L^2}\left[\|u\|^2_{XL^2_T}+\|\nabla^b u\|^2_{YL^2_T}\right]\\ &\qquad\lesssim \|u(0)\|_{L^2}\left[\|u(0)\|^2_{\dot H^{\frac12}}+\|u(T)\|^2_{\dot H^{\frac12}}\right].\end{aligned}\tag{2.5.13}$$

We turn to the estimate of the term containing $b(x)$. In the proof of Theorem 2.1.6, b satisfies (2.1.7), and we proceed exactly as in Section 2.4.4 above, and then use (2.1.15). In the proof of Theorem 2.1.7, b satisfies (2.1.17) and we proceed exactly as in Section 2.4.4 above, and then use (2.1.18). In both cases we get

$$\int_0^T\int_{\Omega^2}|I_b(x)||u(y)|^2dxdy\,dt \lesssim \|u(0)\|_{L^2}^2\left[\|u(0)\|^2_{\dot H^{\frac12}}+\|u(T)\|^2_{\dot H^{\frac12}}\right].$$

For the term containing $f(u)$ we write

$$\begin{aligned}&C_a\int_0^T|\int_{\Omega^2}\langle x\rangle^{-1-\delta}[f(u(x))\overline{u(x)}-2F(u(x))]|u(y)|^2dxdydt\\ &\qquad\lesssim \|u(0)\|_{L^2}\left[\|u(0)\|^2_{\dot H^{\frac12}}+\|u(T)\|^2_{\dot H^{\frac12}}\right],\end{aligned}$$

by (2.1.15) in the proof of Theorem 2.1.6 and (2.1.18) in the proof of Theorem 2.1.7, and this concludes the proof of (2.5.12).

2.5.4 The main term

Integrating in time the inequality (2.5.5) on $[0,T]$ and collecting estimates (2.5.11), (2.5.12) and the results of Section 2.5.1, we have proved that

$$-\int_0^T\int_{\Omega^2} A_x^2\psi(x-y)|u(x)|^2|u(y)|^2\,dxdydt \lesssim \|u(0)\|_{L^2}^2\left[\|u(0)\|^2_{\dot H^{\frac12}}+\|u(T)\|^2_{\dot H^{\frac12}}\right].\tag{2.5.14}$$

We now compute explicitly the quantity $A_x^2\psi$: we have

$$A_x^2\psi(x-y)=S(x,y)+E(x,y)$$

where, using the notations

$$\widetilde a=\widetilde a(x,y)=a(x)\widehat{(x-y)}\cdot\widehat{(x-y)},\qquad \widehat x=\frac{x}{|x|},$$

$S(x,y)$ and $E(x,y)$ are given by

$$\begin{aligned}S(x,y)=&\widetilde a^2\psi^{IV}(x-y)+[2\overline a(x)\widetilde a-6\widetilde a^2+4|a(x)\widehat{(x-y)}|^2]\tfrac{\psi'''(x-y)}{|x-y|}+\\ &+[2a_{\ell m}(x)a_{\ell m}(x)+\overline a^2(x)-6\overline a(x)\widetilde a+15\widetilde a^2-12|a(x)\widehat{(x-y)}|^2]\times\\ &\times\left(\tfrac{\psi''(x-y)}{|x-y|^2}-\tfrac{\psi'(x-y)}{|x-y|^3}\right)\end{aligned}\tag{2.5.15}$$

and

$$
\begin{aligned}
E(x,y) =& \widetilde{a} a_{\ell m;\ell}(x)\widehat{(x-y)}_m \psi'''(x-y) + (\overline{a}(x)-\widetilde{a}) a_{jk;j}(x)\widehat{(x-y)}_k \left(\tfrac{\psi''(x-y)}{|x-y|} - \tfrac{\psi'(x-y)}{|x-y|^2}\right) + \\
&+ [\partial_j(a_{jk}(x)a_{\ell m;k}(x)\widehat{(x-y)}_\ell \widehat{(x-y)}_m) + \partial_j(a_{jk}(x)a_{\ell m}(x))\partial_k(\widehat{(x-y)}_\ell \widehat{(x-y)}_m)] \times \\
&\quad \times \left(\psi''(x-y) - \tfrac{\psi'(x-y)}{|x-y|}\right) \\
&+ (A_x \overline{a}(x)) \tfrac{\psi'(x-y)}{|x-y|} \\
&+ 2a_{jk}(x)a_{\ell m;k}(x)\widehat{(x-y)}_\ell \widehat{(x-y)}_m \widehat{(x-y)}_j \left(\psi'''(x-y) - \tfrac{\psi''(x-y)}{|x-y|}\right) \\
&+ 2a(x)(\nabla \overline{a}(x), \nabla \tfrac{\psi'(x-y)}{|x-y|}) + A_x(a_{\ell m;\ell}(x)\widehat{(x-y)}_m \psi'(x-y)).
\end{aligned}
$$

With long but elementary computations, for $n \geqslant 3$ and $\sigma \geqslant 0$ we have that

$$
|E(x,y)| \leqslant 5nC_a(N+C_a)\left[\frac{1}{\langle x\rangle^{1+\delta}|x-y|\langle x-y\rangle_\sigma} + \frac{1}{\langle x\rangle^{1+\delta}|x|\langle x-y\rangle_\sigma} + \frac{1}{\langle x\rangle^{1+\delta}|x|^2}\right],
$$

whence

$$
\int_{\Omega^2} E(x,y)|u(x)|^2|u(y)|^2\, dxdy \lesssim C_a[I + II + III]
$$

with an implicit constant depending on N and n, where

$$
I = \int_{\Omega^2} \frac{|u(x)|^2|u(y)|^2}{\langle x\rangle^{1+\delta}|x-y|^2}\, dxdy, \qquad II = \int_{\Omega^2} \frac{|u(x)|^2|u(y)|^2}{\langle x\rangle^{1+\delta}|x||x-y|}\, dxdy
$$

and

$$
III = \int_{\Omega^2} \frac{|u(x)|^2|u(y)|^2}{\langle x\rangle^{1+\delta}|x|^2}\, dxdy.
$$

We now extend $u(t,x)$ as zero outside Ω, i.e. we define the function $U(t,x)$ as

$$
U(t,x) = u(t,x) \text{ for } x \in \Omega, \qquad U(t,x) = 0 \text{ for } x \notin \Omega.
$$

Before proceeding further, we need the following Lemma:

Lemma 2.5.3. *Let $n \geqslant 3$, $\delta \in (0,1]$. There exist $\eta = \eta(n,\delta) > 0$ such that for all $f \in \mathscr{S}$*

$$
\left\| |D|^{\frac{3-n}{2}-1} \frac{f}{\langle \cdot \rangle^{1+\delta}} \right\|_{L^2(\mathbb{R}^n)} \leqslant \eta \| |D|^{\frac{3-n}{2}} f\|_{L^2(\mathbb{R}^n)},
$$

$$
\left\| |D|^{\frac{3-n}{2}-1} \frac{f}{|\cdot|^{\frac12}\langle \cdot \rangle^{\frac12+\delta}} \right\|_{L^2(\mathbb{R}^n)} \leqslant \eta \| |D|^{\frac{3-n}{2}} f\|_{L^2(\mathbb{R}^n)}.
$$

Proof. We prove the first inequality. By duality, it is equivalent to prove that

$$
\left\| |D|^{\frac{n-3}{2}} \frac{f}{\langle x \rangle^{1+\delta}} \right\|_{L^2(\mathbb{R}^n)} \lesssim \| |D|^{\frac{n-3}{2}+1} f\|_{L^2(\mathbb{R}^n)}. \tag{2.5.16}
$$

If $n = 3$, (2.5.16) is a simple consequence of Hardy inequality (2.2.11), in the case $y = 0$, $b \equiv 0$. If $n \geqslant 4$, by the Kato-Ponce inequality (see e.g. [45]) and Sobolev embedding, we have

$$\begin{aligned}
&\left\| |D|^{\frac{n-3}{2}} \frac{f}{\langle x\rangle^{1+\delta}} \right\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \||D|^{\frac{n-3}{2}} f\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \|\langle x\rangle^{-1-\delta}\|_{L^n} + \||D|^{\frac{n-3}{2}} \langle x\rangle^{-1}\|_{L^{\frac{2n}{n-1}}} \|f\|_{L^{2n}} \\
&\lesssim \||D|^{\frac{n-3}{2}+1} f\|_{L^2}
\end{aligned} \tag{2.5.17}$$

where the implicit constants clearly depend only on n and δ. The proof of the second inequality is analogous. □

Now, to estimate I we write

$$\begin{aligned}
I &= \int_{\mathbb{R}^n} \tfrac{|U(x)|^2}{\langle x\rangle^{1+\delta}} \int_{\mathbb{R}^n} \tfrac{|U(y)|^2}{|x-y|^2} dy dx \simeq \int_{\mathbb{R}^n} \tfrac{|U(x)|^2}{\langle x\rangle^{1+\delta}} |D|^{2-n} |U(x)|^2 \, dx \\
&= \int_{\mathbb{R}^n} |D|^{\frac{3-n}{2}-1} (\langle x\rangle^{-1-\delta} |U(x)|^2) |D|^{\frac{3-n}{2}} |U(x)|^2 \, dx \\
&\leqslant \left\| |D|^{\frac{3-n}{2}-1} \frac{|U|^2}{\langle x\rangle^{1+\delta}} \right\|_{L^2} \||D|^{\frac{3-n}{2}} |U|^2\|_{L^2}
\end{aligned}$$

and applying Lemma 2.5.3 we obtain

$$I \leqslant C(n,\delta) \||D|^{\frac{3-n}{2}} |U|^2\|_{L^2}^2.$$

Next we split the integral II

$$II = \int_{\mathbb{R}^{2n}} \frac{|U(x)|^2 |U(y)|^2}{\langle x\rangle^{1+\delta} |x||x-y|} dx dy = \int_A + \int_B$$

in the regions $A = \{(x,y) : 2|x| \geqslant |y|\}$ and $B = \{(x,y) : 2|x| < |y|\}$. On A we have

$$\begin{aligned}
\int_A \frac{|U(x)|^2 |U(y)|^2}{\langle x\rangle^{1+\delta} |x||x-y|} dx dy &\lesssim \int_A \frac{|U(x)|^2 |U(y)|^2}{\langle x\rangle^{\frac{1}{2}+\frac{\delta}{2}} |x|^{\frac{1}{2}} \langle y\rangle^{\frac{1}{2}+\frac{\delta}{2}} |y|^{\frac{1}{2}} |x-y|} dx dy \\
&\leqslant \int_{\mathbb{R}^{2n}} \frac{|U(x)|^2}{\langle x\rangle^{\frac{1}{2}+\frac{\delta}{2}} |x|^{\frac{1}{2}}} \frac{1}{|x-y|} \frac{|U(y)|^2}{\langle y\rangle^{\frac{1}{2}+\frac{\delta}{2}} |y|^{\frac{1}{2}}} dx dy \\
&= \int_{\mathbb{R}^n} \frac{|U(x)|^2}{\langle x\rangle^{\frac{1}{2}+\frac{\delta}{2}} |x|^{\frac{1}{2}}} |D|^{1-n} \frac{|U(x)|^2}{\langle x\rangle^{\frac{1}{2}+\frac{\delta}{2}} |x|^{\frac{1}{2}}} dx \\
&= \left\| |D|^{\frac{1-n}{2}} \frac{|U|^2}{|\cdot|^{\frac{1}{2}} \langle \cdot\rangle^{\frac{1}{2}+\frac{\delta}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 \leqslant C(n,\delta) \||D|^{\frac{3-n}{2}} |U|^2\|_{L^2(\mathbb{R}^n)}^2,
\end{aligned}$$

where in the last step we used Lemma 2.5.3. On the region B we have $|x| \lesssim |x-y|$, hence

$$\int_B \frac{|U(x)|^2 |U(y)|^2}{\langle x\rangle^{1+\delta} |x||x-y|} dx dy \lesssim \int_B \frac{|U(x)|^2 |U(y)|^2}{\langle x\rangle^{1+\delta} |x|^2} dx dy \leqslant III$$

Summing up, we have proved the estimate

$$
\begin{aligned}
-\textstyle\int_{\Omega^2} A_x^2\psi(x-y)|u(x)|^2|u(y)|^2dxdy \\
\gtrsim -\int_{\Omega^2} S(x,y)\,dxdy - III - C(n,N,\delta)C_a \||D|^{\frac{3-n}{2}}|U|^2\|^2_{L^2(\mathbb{R}^n)}
\end{aligned}
\tag{2.5.18}
$$

with an implicit constant depending on N, n only.

Proof of Theorem 2.1.6

In this case, the expression for S simplifies:

$$S(x,y) = -|x-y|^{-3}\left[2a_{lm}(x)a_{lm}(x) + \overline{a}^2(x) - 6\overline{a}(x)\widetilde{a}(x) + 15\widetilde{a}^2 - 12|a(x)\widehat{(x-y)}|^2\right].$$

Now recalling (2.4.19), we see that if $N/\nu - 1$ is small enough we have

$$-S(x,y) \geqslant \epsilon_0 |x-y|^{-3}$$

for some strictly positive constant ϵ_0. This implies

$$
\begin{aligned}
\textstyle\int_{\Omega^2} -S(x,y)|u(x)|^2|u(y)|^2dxdy \geqslant \epsilon_0 \int_{\Omega^2} \frac{|u(x)|^2|u(y)|^2}{|x-y|^3}dxdy \\
= \epsilon_0 \||D|^{\frac{3-n}{2}}|U|^2\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}
\tag{2.5.19}
$$

and, from (2.5.18), we get

$$-\textstyle\int_{\Omega^2} A_x^2\psi(x-y)|u(x)|^2|u(y)|^2dxdy \gtrsim -III + (\epsilon_0 - C(n,N,\delta)C_a)\||D|^{\frac{3-n}{2}}|U|^2\|^2_{L^2(\mathbb{R}^n)}$$

with an implicit constant depending on N, n only. If C_a is sufficiently small (with respect to N, n, ν and δ), we obtain

$$\gtrsim -III + \||D|^{\frac{3-n}{2}}|U|^2\|^2_{L^2(\mathbb{R}^n)}$$

and integrating in time on $[0,T]$ and recalling (2.5.14), we arrive at the estimate

$$\||D|^{\frac{3-n}{2}}|U|^2\|^2_{L^2_T L^2_x} \lesssim \textstyle\int_0^T III dt + \|u(0)\|^2_{L^2}\left[\|u(0)\|^2_{\dot H^{\frac12}} + \|u(T)\|^2_{\dot H^{\frac12}}\right].$$

Note that by (2.2.7) we can write

$$\textstyle\int_0^T III dt \leqslant \|u(0)\|^2_{L^2}\||x|^{-1}\langle x\rangle^{-\frac12-\frac{\delta}{2}}u\|^2_{L^2_x L^2_T} \lesssim \|u(0)\|^2_{L^2}\|u\|^2_{\dot X L^2_T}$$

and recalling (2.1.15) this gives

$$\textstyle\int_0^T III dt \leqslant \|u\|^2_{L^2_x}\|u\|^2_{L^2_T \dot X} \leqslant \|u(0)\|^2_{L^2}\left[\|u(0)\|^2_{\dot H^{\frac12}} + \|u(T)\|^2_{\dot H^{\frac12}}\right].$$

In conclusion we have

$$\||D|^{\frac{3-n}{2}}|U|^2\|^2_{L^2_T L^2_x} \lesssim \|u(0)\|^2_{L^2}\left[\|u(0)\|^2_{\dot H^{\frac12}} + \|u(T)\|^2_{\dot H^{\frac12}}\right].$$

Note that

$$\||D|^{\frac{3-n}{2}}|U|^2\|^2_{L^2_x} = \textstyle\int_{\mathbb{R}^n} |D|^{\frac{3-n}{2}}|U|^2 \cdot |D|^{\frac{3-n}{2}}|U|^2 dx = \int_{\mathbb{R}^n} |U|^2 \cdot |D|^{3-n}|U|^2 dx$$

and this can be written, apart from a constant,

$$= \textstyle\int_{\mathbb{R}^{2n}} \frac{|U(x)|^2|U(y)|^2}{|x-y|^3}dxdy = \int_{\Omega^2} \frac{|u(x)|^2|u(y)|^2}{|x-y|^3}dxdy$$

which concludes the proof of the Theorem.

Proof of Theorem 2.1.7

We recall the following identities for a:

$$a = I + q, \quad a_{lm} = \delta_{lm} + q_{lm},$$
$$\bar{a} = 3 + \bar{q}, \quad a_{lm}a_{lm} = 3 + 2\bar{q} + q_{lm}q_{lm},$$
$$\widetilde{a} = 1 + \widetilde{q}, \quad |a(\widehat{x-y})|^2 = 1 + 2\widetilde{q} + |q(\widehat{x-y})|^2.$$

Starting from (2.5.15) and using formulas (2.5.4) and the previous identities, we obtain

$$\begin{aligned}-S(x,y) \geqslant & 15\frac{\sigma^4}{\langle x-y\rangle_\sigma^7} + 30\widetilde{q}\frac{\sigma^4}{\langle x-y\rangle_\sigma^7} + (2\bar{q} - 6\widetilde{q} + 2\bar{q}\widetilde{q})\frac{3\sigma^2}{\langle x-y\rangle_\sigma^5} \\ & + \left(4\bar{q} - 12\widetilde{q} - 6\bar{q}\widetilde{q} - 3\widetilde{q}^2 - 12|q(\widehat{x-y})|^2\right)\frac{1}{\langle x-y\rangle_\sigma^3}.\end{aligned}$$

Since we have by assumption

$$|\bar{q}| \leqslant 3C_I\langle x\rangle^{-\delta}, \quad |\widetilde{q}| \leqslant C_I\langle x\rangle^{-\delta}, \quad |q(\widehat{x-y})| \leqslant C_I\langle x\rangle^{-\delta}.$$

this implies

$$-S(x,y) \geqslant 15\sigma^4\langle x-y\rangle_\sigma^{-7} - 46C_I\langle x\rangle^{-\delta}\langle x-y\rangle_\sigma^{-3}. \tag{2.5.20}$$

From (2.5.18) and (2.5.20) we have

$$\begin{aligned}&-\textstyle\int_{\Omega^2} A_x^2\psi(x-y)|u(x)|^2|u(y)|^2dxdy \\ &\qquad \gtrsim \int_{\Omega^2}\left(15\frac{\sigma^4}{\langle x-y\rangle_\sigma^7} - \frac{46C_I}{\langle x-y\rangle_\sigma^3}\right)|u(x)|^2|u(y)|^2\,dxdy \\ &\qquad\quad - III - C(n,N,\delta)C_a\|u\|_{L^4}^4\end{aligned}$$

with an implicit constant depending on N, n only. We let $\sigma \to 0$ and integrate in t on $[0,T]$: recalling (2.5.14), we get

$$(15-46C_I-C(n,N,\delta)C_a)\|u\|_{L_T^4L^4}^4 \lesssim \|u(0)\|_{L^2}^2\left[\|u(0)\|_{\dot{H}^{\frac{1}{2}}}^2 + \|u(T)\|_{\dot{H}^{\frac{1}{2}}}^2\right] + \textstyle\int_0^T III\,dt. \tag{2.5.21}$$

Note that by (2.4.32), (2.2.4), and (2.1.18) we have

$$\begin{aligned}\textstyle\int_0^T III dt &\leqslant \|u(0)\|_{L^2}^2\||x|^{-1}\langle x\rangle^{-(1+\delta)/2}u\|_{L_x^2L_T^2}^2 \\ &\leqslant \|u(0)\|_{L^2}^2\delta^{-1}\left[\|u\|_{XL_T^2}^2 + \|\nabla^b u\|_{YL_T^2}^2\right] \\ &\lesssim \|u(0)\|_{L^2}^2\left[\|u(0)\|_{\dot{H}^{\frac{1}{2}}}^2 + \|u(T)\|_{\dot{H}^{\frac{1}{2}}}^2\right].\end{aligned} \tag{2.5.22}$$

If C_I and C_a are small enough, we get immediately the claim from (2.5.21) and (2.5.22).

2.6 Gaussian bounds and applications

Let L be the operator (2.1.2), (2.1.3) defined on an open set $\Omega \subseteq \mathbb{R}^n$. For the results in this section it is not necessary to assume any condition on Ω which may be an arbitrary open set; we shall anyway assume $\partial\Omega \in C^1$ for the sake of simplicity. First of all, we chack that L can be realized as a selfadjoint operator, with Dirichlet b.c., under very weak assumptions on the coefficients:

Proposition 2.6.1. *Let $n \geqslant 3$ and $\Omega \subseteq \mathbb{R}^n$ an open set with C^1 boundary. Consider the operator L defined on $C_c^\infty(\Omega)$ by (2.1.2), (2.1.3), under the assumptions*

$$a \in L^\infty, \qquad b \in L^{n,\infty}, \qquad c \in L^{\frac{n}{2},\infty}, \qquad \|c_-\|_{L^{\frac{n}{2},\infty}} < \epsilon. \tag{2.6.1}$$

Then, if ϵ sufficiently small, $-L$ extends to a selfadjoint nonnegative operator in the sense of forms, and $D(-L) = H_0^1(\Omega) \cap H^2(\Omega)$ is a form core. Moreover we have

$$(-Lv, v)_{L^2} = \|(-L)^{\frac{1}{2}} v\|_{L^2}^2 \simeq \|\nabla v\|_{L^2}^2, \qquad \|(-L)^{\frac{1}{4}} v\|_{L^2} \simeq \|v\|_{\dot{H}^{\frac{1}{2}}}. \tag{2.6.2}$$

Proof. We sketch the proof which is mostly standard, apart from the use of Lorentz spaces. The form

$$q(v) = (-Lv, v)_{L^2} = \int_\Omega a(\nabla^b v, \nabla^b v) dx + \int_\Omega c|v|^2 dx$$

is bounded on $H_0^1(\Omega)$: indeed, by Hölder and Sobolev inequalities in Lorentz spaces,

$$\int_\Omega |c| \cdot |v|^2 dx \lesssim \|c\|_{L^{\frac{n}{2},\infty}} \||v|^2\|_{L^{\frac{n}{n-2},1}} \lesssim \|c\|_{L^{\frac{n}{2},\infty}} \|v\|^2_{L^{\frac{2n}{n-2},2}} \lesssim \|c\|_{L^{\frac{n}{2},\infty}} \|\nabla v\|_{L^2}^2$$

while by (2.2.15) we have $\|\nabla^b v\|_{L^2} \simeq \|\nabla v\|_{L^2}$. Thus if ϵ is sufficiently small we have $q(v) \simeq \|\nabla v\|_{L^2}$; in particular $q(v)$ is a symmetric, closed, nonnegative form on $H_0^1(\Omega)$, and defines a selfadjoint operator with $D(-L) = H^2(\Omega) \cap H_0^1(\Omega)$ which is also a core for q. The last property in (2.6.2) follows by complex interpolation, since $D((-L)^s)$ for $0 \leqslant s \leqslant 1$ is an interpolation family. □

Under slightly stronger assumptions, we can see that the heat flow e^{tL} satisfies an upper gaussian bound; this will be a crucial tool in the following. Compare with [26] and [25] for similar results in the case $a = I$, $\Omega = \mathbb{R}^n$. Note that for $a, b, c \in L^\infty$ with $c \geqslant 0$ the bound is proved in Corollary 6.14 of [70]. The following result is sufficient for our purposes, although the assumptions on the coefficients could be further relaxed.

Proposition 2.6.2. *Let $n \geqslant 3$. Assume the operator L is defined as in (2.1.2), (2.1.3) on the open set $\Omega \subseteq \mathbb{R}^n$ with C^1 boundary, and that a, b, c satisfy*

$$a \in L^\infty, \quad b \in L^4_{loc} \cap L^{n,\infty}, \quad \nabla \cdot b \in L^2_{loc}, \quad c \in L^{\frac{n}{2},1}, \quad \|c_-\|_{L^{\frac{n}{2},1}} < \epsilon. \tag{2.6.3}$$

Then, if ϵ is sufficiently small, the heat kernel e^{tL} satisfies, for some $C, C' > 0$,

$$|e^{tL}(x,y)| \leqslant C' t^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{Ct}}, \qquad t > 0, \; x, y \in \Omega. \tag{2.6.4}$$

Proof. We can apply Proposition 2.6.1 since the assumptions are stronger. When $b = c = 0$, the gaussian bound follows directly from Corollary 6.14 in [69]; note that in this case the kernel of e^{tL} is $\geqslant 0$.

Next, in order to handle the case $b \neq 0$, $c = 0$, we adapt the proof of Lemma 10 in [53]. Let $\phi \in C_c^\infty(\Omega)$ and write $\phi_\delta = \sqrt{|\phi|^2 + \delta^2}$ for $\delta > 0$. It is easy to prove the pointwise inequality (recall notations (2.2.1))

$$A\phi_\delta \geqslant \Re(\frac{\overline{\phi}}{\phi_\delta} A^b \phi)$$

which implies, for all $\lambda > 0$,

$$(-A+\lambda)\phi_\delta \leqslant \Re(\frac{\overline{\phi}}{\phi_\delta}(-A^b+\lambda)\phi) + \lambda(\phi_\delta - \frac{|\phi|^2}{\phi_\delta}).$$

Proceeding as in [53], we obtain

$$|(-A^b+\lambda)^{-1}f| \leqslant (-A+\lambda)^{-1}|f|$$

and iterating we have for all $k \geqslant 0$

$$|(-A^b+\lambda)^{-k}f| \leqslant (-A+\lambda)^{-k}|f| \tag{2.6.5}$$

since $(-A+\lambda)^{-1}$ is positivity preserving (see Remark 1 in [53]). Then we deduce

$$|e^{tA^b}\phi| \leqslant e^{tA}|\phi|$$

via $e^{tA^b} = \lim_{k\to\infty}(I - tA^b/n)^{-n}$, and applying the last formula to a delta sequence $\phi = \phi_j$ made of nonnegative functions, we conclude that the gaussian bound (2.6.4) is valid for e^{tA^b}.

It remains to consider the case $c \neq 0$. To this end we apply the theory of [58]. Let $U(t,s)$ be the propagator defined as $U(t,s)f = e^{(t-s)A^b}f$, for $t \geqslant s \geqslant 0$. By the gaussian bound just proved we have that $U(t,s)$ extends to a uniformly bounded operator $L^1 \to L^1$ and $L^\infty \to L^\infty$, moreover $\|U(t,s)\|_{L^1\to L^\infty} \lesssim |t-s|^{-\frac{n}{2}}$; finally, the adjoint propagator $U_*(t,s) := (U(s,t))^*$ for $s \geqslant t \geqslant 0$ coincides with $U(s,t)$ since A^b is selfadjoint, so that U_* is strongly continuous on L^1 (notice that this last assumption is not actually necessary, as mentioned in the chapter). Then by applying Theorem 3.10 from [58] we conclude that the gaussian bound, with possibly different constants, is satisfied also by the perturbed propagator $U_c = e^{t(A^b - c)}$, provided the potential c is a Miyadera perturbation of both U and U_* such that c_- is Miyadera small with constants (∞, γ), $\gamma < 1$. The verification of this last condition, in the special case considered here, reduces to the following inequality, for all $s \geqslant 0$

$$I := \int_s^{+\infty} \|c(x)e^{(t-s)A^b}f\|_{L^1}dt \leqslant \gamma\|f\|_{L^1} \tag{2.6.6}$$

(we are using formula (2.5) in [58] with the choices $\alpha = \infty$, $J = \mathbb{R}^+$ and $p = 1$) and the same inequality with $\gamma < 1$ for c_-. The gaussian bound already proved for e^{tA^b} implies

$$I \lesssim \int_\Omega \int_\Omega |c(x)||f(y)| \int_0^{+\infty} t^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{Ct}} dt dy dx \lesssim \|f\|_{L^1} \sup_{y\in\Omega} \int_\Omega \frac{|c(x)|}{|x-y|^{n-2}} dx$$

and by the Young inequality in Lorentz spaces we get

$$I \lesssim \|c\|_{L^{\frac{n}{2},1}} \|f\|_{L^1},$$

which concludes the proof (compare with the proof of Lemma 5.1 in [88]). $\square$

Proposition 2.6.3. *Let $n \geqslant 3$. Assume the operator L is defined as in (2.1.2), (2.1.3) on the open set $\Omega \subseteq \mathbb{R}^n$ with C^1 boundary, and that a, b, c satisfy*

$$b^2+|\nabla\cdot b| \in L^2_{loc}, \quad c \in L^{\frac{n}{2},1}, \quad \|a-I\|_{L^\infty}+\||b|+|a'|\|_{L^{n,\infty}}+\|b'\|_{L^{\frac{n}{2},\infty}}+\|c_-\|_{L^{\frac{n}{2},1}} < \epsilon. \tag{2.6.7}$$

If ϵ sufficiently small then for all $0 \leqslant \sigma \leqslant 1$ we have

$$\|(-L)^\sigma v\|_{L^p} \simeq \|(-\Delta)^\sigma v\|_{L^p}, \qquad 1 < p < \frac{n}{2\sigma}. \tag{2.6.8}$$

Proof. The assumptions of the two previous Propositions are satisfied, thus $-L$ is selfadjoint, nonnegative, and the gaussian bound (2.6.4) is valid.

Consider first the case $\sigma = 1$. Write the operator L in the form

$$Lv = \textstyle\sum_{jk} a_{jk}\partial_j\partial_k v + \sum_j \beta_j \partial_j v + \gamma_0 v - c_+ v$$

where

$$\beta_k = \textstyle\sum_j(\partial_j a_{jk} + 2i a_{jk} b_k), \qquad \gamma_0 = \sum_{j,k} i\partial_j(a_{jk}b_k) - a(b,b) + c_-.$$

Then by Hölder and Sobolev inequalities in Lorentz spaces we have for $1 < p < \frac{n}{2}$

$$\|Lv\|_{L^p} \leqslant \|a\|_{L^\infty}\|D^2 v\|_{L^p}+\|\beta\|_{L^{n,\infty}}\|Dv\|_{L^{\frac{np}{n-p},p}}+\|\gamma_0-c_+\|_{L^{\frac{n}{2},\infty}}\|v\|_{L^{\frac{np}{n-2p},p}} \lesssim \|\Delta v\|_{L^p}.$$

To prove the converse inequality, we first represent the operator $(-\Delta + c_+)^{-1}$ in the form

$$(-\Delta + c_+)^{-1} = c(n)\textstyle\int_0^{+\infty} e^{t(\Delta - c_+)}dt$$

and we apply the gaussian bound to obtain

$$|(-\Delta + c_+)^{-1}| \lesssim \textstyle\int_0^{+\infty} e^{-\frac{|x-y|^2}{Ct}} t^{-\frac{n}{2}} dt \lesssim |x-y|^{2-n}.$$

As a consequence, using the Hardy-Sobolev inequality we get

$$\|(-\Delta + c_+)^{-1} v\|_{L^{\frac{np}{n-2p}}} \lesssim \|v\|_{L^p} \quad \text{i.e.} \quad \|v\|_{L^{\frac{np}{n-2p}}} \lesssim \|(-\Delta + c_+)v\|_{L^p}$$

for all

$$1 < p < \tfrac{n}{2}.$$

In particular this gives (since $\|c_+\|_{L^{\frac{n}{2},\infty}} \lesssim \|c_+\|_{L^{\frac{n}{2},1}}$)

$$\|\Delta v\|_{L^p} \leqslant \|(\Delta - c_+)v\|_{L^p} + \|c_+\|_{L^{\frac{n}{2},\infty}}\|v\|_{L^{\frac{np}{n-2p}}} \lesssim \|(\Delta - c_+)v\|_{L^p}, \qquad 1 < p < \tfrac{n}{2}.$$

Adding and subtracting the remaining terms in L in the last term, we obtain

$$\|(\Delta - c_+)v\|_{L^p} \leqslant \|Lv\|_{L^p} + \|\textstyle\sum(a_{jk} - \delta_{jk})\partial_j\partial_k v\|_{L^p} + \|\sum \beta_k \partial_k v\|_{L^p} + \|\gamma_0 v\|_{L^p}$$

and a last application of Hölder and Sobolev inequalities gives

$$\|\Delta v\|_{L^p} \lesssim \|(\Delta - c_+)v\|_{L^p} \lesssim \|Lv\|_{L^p} + \epsilon\|\Delta v\|_{L^p}.$$

If ϵ is sufficiently small we can subtract the last term from the left hand side, and the proof of the case $\sigma = 1$ is concluded. The case $\sigma = 0$ is trivial, and the remaining cases will be handled by Stein-Weiss complex interpolation.

Indeed, consider the family of operators $T_z = (-L)^z(-\Delta)^{-z}$ for $0 \leqslant \Re z \leqslant 1$; our first goal is to prove that $T_z : L^p \to L^p$ is bounded provided $1 < p < n/(2\Re z)$, which implies the inequality $\lesssim$ in (2.6.8). Note that the following arguments work with trivial modifications also for $-1 \leqslant \Re z \leqslant 0$ and give then the converse inequality $\gtrsim$.

T_z is obviously an analytic family of operators, and T^{iy} for real y is bounded on all L^p with $1 < p < \infty$, with a norm growing at most polynomially as $|y| \to \infty$. This property is well known for $(-\Delta)^{iy}$, while for L^{iy} it follows from the theory developed in [28] (see also [7] for the case $\Omega = \mathbb{R}^n$), which requires the sole assumption that L satisfies a gaussian bound like (2.6.4). A standard application of the Stein-Weiss theorem then gives the claim. □

To conclude this section we construct a family of regularizing operators which will be needed later in the proof of H^1 well posedness; what follows is an adaptation of Section 1.5 in [13]. Assume that Ω and L satisfy the assumptions of the previous Proposition. We define for $0 < \epsilon \leqslant 1$ the operators

$$J_\epsilon := (I - \epsilon L)^{-1} \equiv \epsilon^{-1} R(-\epsilon^{-1}) \tag{2.6.9}$$

where $R(z) = (-L-z)^{-1}$ is the resolvent operator of $-L$. Then for every $f \in H^{-1}(\Omega)$ the function $u = J_\epsilon f \in H^1_0(\Omega)$ is well defined as the unique weak solution of the elliptic equation

$$-Lu + \epsilon^{-1}u = \epsilon^{-1}f.$$

Thus $J_\epsilon : H^{-1}(\Omega) \to H^1_0(\Omega)$ is a bounded operator, $L : H^1_0(\Omega) \to H^{-1}(\Omega)$ is bounded, we have the equivalence $\|(I-L)v\|_{H^{-1}(\Omega)} \simeq \|v\|_{H^1_0(\Omega)}$ and the estimates

$$\|J_\epsilon v\|_{H^1_0(\Omega)} \leqslant C\epsilon^{-1}\|v\|_{H^{-1}(\Omega)}, \qquad \|J_\epsilon v\|_{H^2(\Omega)} \leqslant C\epsilon^{-1}\|v\|_{L^2(\Omega)} \tag{2.6.10}$$

by standard elliptic theory, with a C independent of ϵ. Further we have

$$\|J_\epsilon v\|_{H^1_0(\Omega)} \leqslant C\|v\|_{H^1_0(\Omega)}, \quad \|J_\epsilon v\|_{L^2(\Omega)} \leqslant C\|v\|_{L^2(\Omega)}, \quad \|J_\epsilon v\|_{H^{-1}(\Omega)} \leqslant C\|v\|_{H^{-1}(\Omega)} \tag{2.6.11}$$

and by complex interpolation

$$\|J_\epsilon v\|_{H^1_0(\Omega)} \leqslant C\epsilon^{-\frac{1}{2}}\|v\|_{L^2(\Omega)}, \qquad \|J_\epsilon v\|_{L^2(\Omega)} \leqslant C\epsilon^{-\frac{1}{2}}\|v\|_{H^{-1}(\Omega)}.$$

Then, using the identity $J_\epsilon - I = J_\epsilon(I - I + \epsilon L) = \epsilon J_\epsilon L$, we deduce

$$\|(J_\epsilon - I)v\|_{H^{-1}(\Omega)} \leqslant C\epsilon\|Lv\|_{H^{-1}(\Omega)} \leqslant C'\epsilon\|v\|_{H^1_0(\Omega)}. \tag{2.6.12}$$

Note that if $v \in H^{-1}(\Omega)$ only, we can still approximate it with $\phi \in C^\infty_c(\Omega)$ to get

$$\|(J_\epsilon - I)v\|_{H^{-1}(\Omega)} \leqslant C\|v-\phi\|_{H^{-1}(\Omega)} + C\epsilon\|\phi\|_{H^1_0(\Omega)}$$

and this implies

$$\forall v \in H^{-1}(\Omega) \qquad J_\epsilon v \to v \ \text{ in } H^{-1}(\Omega) \text{ as } \epsilon \to 0. \tag{2.6.13}$$

We also obtain

$$\|(J_\epsilon - I)v\|_{L^2(\Omega)} \leqslant C\|(J_\epsilon - I)v\|^{\frac{1}{2}}_{H^1_0(\Omega)}\|(J_\epsilon - I)v\|^{\frac{1}{2}}_{H^{-1}(\Omega)} \leqslant C'\epsilon^{\frac{1}{2}}\|v\|_{H^1_0(\Omega)} \tag{2.6.14}$$

and an argument similar to the previous one gives

$$\forall v \in L^2(\Omega) \qquad J_\epsilon v \to v \ \text{ in } L^2(\Omega) \text{ as } \epsilon \to 0. \tag{2.6.15}$$

Finally, by the equivalence $\|(J_\epsilon - I)v\|_{H_0^1(\Omega)} \simeq \|(J_\epsilon - I)(I-L)v\|_{H^{-1}(\Omega)}$ we get

$$\forall v \in H_0^1(\Omega) \qquad J_\epsilon v \to v \ \text{ in } H_0^1(\Omega) \text{ as } \epsilon \to 0. \tag{2.6.16}$$

Concerning the convergence in $L^p(\Omega)$ we have:

Proposition 2.6.4. *Let $p \in [1,\infty)$ and let Ω and L satisfy the assumptions of Proposition 2.6.3. Then J_ϵ extends to a bounded operator on $L^p(\Omega)$ and the following estimate holds for $0 < \epsilon \leqslant 1$*

$$\|J_\epsilon v\|_{L^p(\Omega)} \leqslant C\|v\|_{L^p(\Omega)} \tag{2.6.17}$$

with a constant depending on p but not of ϵ. Moreover, for $1 < p < \infty$ we have

$$\forall v \in L^p(\Omega) \qquad J_\epsilon v \to v \ \textit{ in } L^p(\Omega) \textit{ as } \epsilon \to 0. \tag{2.6.18}$$

Proof. Let $\phi : (0,\infty) \to [0,\infty)$ be a smooth nondecreasing function with $\phi(s), s\phi'(s)$ bounded. Starting from the identity

$$\begin{aligned}\Re(-Lv \cdot \phi(|v|)\overline{v}) + \nabla \cdot \{\Re(\overline{v}\phi(|v|)a\nabla^b v)\} =& \phi(|v|)a(\nabla^b v, \nabla^b v) + \frac{\phi'(|v|)}{|v|}|\Re(\overline{v} \cdot a\nabla^b v)|^2 \\ &+ c\phi(|v|)|v|^2,\end{aligned}$$

and proceeding exactly as in the proof of Proposition 1.5.1 in [13], we obtain (2.6.17). In order to prove (2.6.18), we can assume $v \in C_c^\infty(\Omega)$ (as above). Then by the interpolation inequality in L^p we can write for all $0 < \theta < 1$

$$\|(J_\epsilon - I)v\|_{L^{\frac{2}{1-\theta}}} \leqslant \|(J_\epsilon - I)v\|_{L^1}^\theta \|(J_\epsilon - I)v\|_{L^2}^{1-\theta} \leqslant C\|v\|_{L^1}^\theta \cdot \|(J_\epsilon - I)v\|_{L^2}^\theta$$

where we used (2.6.17), and by (2.6.15) we conclude that $J_\epsilon v \to v$ in $L^p(\Omega)$ for all $p = \frac{2}{1-\theta} \in (1,2)$. A similar argument gives the result for $p \in (2,\infty)$, and the case $p = 2$ we already know. □

2.7 Global existence and Scattering: proof of Theorem 2.1.9

Throughout this section $\Omega \subseteq \mathbb{R}^n$ is an open set with C^1 boundary, $n \geqslant 3$, while L is the unbounded operator on $L^2(\Omega)$ with Dirichlet boundary conditions under the assumptions of Proposition 2.6.1. As explained in the Introduction, we shall work under the black box Assumption (S) which ensures that the necessary Strichartz estimates are available. Notice that we are restricting the range of admissible indices at the left hand side for the derivative of the flow ∇e^{itL}.

Our goal is to extend the usual local and global H^1 theory to the NLS with variable coefficients

$$iu_t - Lu + f(u) = 0, \qquad u(0,x) = u_0(x). \tag{2.7.1}$$

We shall sketch only the essential results which will be needed in the proof of scattering, and not aim at the greatest possible generality. In the following we use the notations

$$L^p_T L^q = L^p(0,T;L^q(\Omega)), \qquad C_T H^1_0 = C([0,T],H^1_0(\Omega)).$$

Proposition 2.7.1 (Local existence in $H^1_0(\Omega)$). *Let $n \geqslant 3$ and assume (S) holds, while $f \in C^1(\mathbb{C},\mathbb{C})$ satisfies*

$$|f(z)| \lesssim |z|^\gamma, \quad |f(z)-f(w)| \lesssim (1+|z|+|w|)^{\gamma-1}|z-w| \ \ \text{for some } 1 \leqslant \gamma < 1+\tfrac{4}{n-2}. \tag{2.7.2}$$

Then for all $u_0 \in H^1_0(\Omega)$ there exists $T = T(\|u_0\|_{H^1})$ and a unique solution $u \in C([0,T];H^1_0(\Omega))$.

Proof. The proof is standard; we sketch the main steps in order to check that the restriction $q_1 < n$ imposed in (S) is harmless. We apply a fixed point argument to the map $\Phi : v \mapsto u$ defined as the solution of $iu_t - Lu + f(v) = 0$, $u(0,x) = u_0$, working in a suitable bounded subset of the space $X_T = C([0,T];H^1_0(\Omega)) \cap L^p(0,T;W^{1,q}(\Omega))$ for an appropriate choice of (p,q), endowed with the distance $d(u,v) = \|u-v\|_{C_T L^2 \cap L^p_T L^q}$; note that bounded subsets of X_T are complete with this distance.

In order to choose the indices we pick a real number k such that

$$n < 2kn < n+2, \qquad \gamma(n-4)+2 < 2kn < \gamma(n-2)+2. \tag{2.7.3}$$

Note that for all $n \geqslant 3$ and all $1 < \gamma < \frac{n+2}{n-2}$ the two intervals in (2.7.3) have a nonempty intersection. Moreover, the couples (p_j,q_j) defined by

$$p_1 = \tfrac{4\gamma}{2+\gamma(n-2)-2kn}, \quad q_1 = \tfrac{\gamma n}{kn+\gamma-1}, \qquad p_2 = \tfrac{4}{2kn-n}, \quad q_2 = \tfrac{1}{1-k}$$

are admissible and we can use the estimates in (S), provided $q_1 < n$ which will be checked at the end. We choose then $(p,q) = (p_1,q_1)$ in the definition of X_T. Applying Strichartz estimates on a time interval $[0,T]$ with T to be chosen, we have for $u = \Phi(v)$

$$\|\nabla u\|_{L^{p_1}_T L^{q_1}} + \|\nabla u\|_{L^\infty_T L^2} \lesssim \|u_0\|_{\dot H^1} + \|\nabla f(v)\|_{L^{p_2'}_T L^{q_2'}}.$$

By Hölder and Sobolev inequalities, using the assumptions on f, we have

$$\|\nabla f(v)\|_{L^{p_2'}_T L^{q_2'}} \lesssim \left\| \|v\|^{\gamma-1}_{L^{\frac{\gamma n}{kn-1}}} \|\nabla v\|_{L^{q_1}} \right\|_{L^{p_2'}_T} \lesssim \|\nabla v\|^\gamma_{L^{\gamma p_2'}_T L^{q_1}}.$$

Now we note that the condition $\gamma < \frac{n+2}{n-2}$ is equivalent to $\gamma p_2' < p_1$, thus Hölder inequality on $[0,T]$ gives

$$\|\nabla u\|_{L^{p_1}_T L^{q_1}} + \|\nabla u\|_{L^\infty_T L^2} \lesssim \|u_0\|_{\dot H^1} + T^{\frac{1}{p_2'}-\frac{\gamma}{p_1}} \|\nabla v\|^\gamma_{L^{p_1}_T L^{q_1}}$$

with a strictly positive power of T. An analogous computation gives

$$\|u\|_{L^{p_1}_T L^{q_1}} + \|u\|_{L^\infty_T L^2} \lesssim \|u_0\|_{L^2} + T^{\frac{1}{p_2'}-\frac{\gamma}{p_1}} \left[\|\nabla v\|^\gamma_{L^{p_1}_T L^{q_1}} + \|v\|^\gamma_{L^{p_1}_T L^{q_1}} \right]$$

and summing up we have proved

$$\|\Phi(v)\|_{X_T} \lesssim \|u_0\|_{H^1} + T^\sigma \|v\|_{X_T}^\gamma, \qquad \sigma = \tfrac{1}{p_2'} - \tfrac{\gamma}{p_1} > 0.$$

Similar computations give

$$d(\Phi(v_1), \Phi(v_2)) \lesssim T^\sigma (1 + \|v_1\|_{X_T} + \|v_2\|_{X_T})^{\gamma-1} \|v_1 - v_2\|_{L_T^{p_1} L^{q_1}}$$

and by a standard contraction argument on a suitable ball of X_T we obtain the existence of a fixed point i.e. a solution of (2.7.1) provided T is smaller than a quantity $T(\|u_0\|_{H^1})$ which depends only on the H^1 norm of the initial data.

It remains to check the claim $q_1 < n$. Since $2kn > n$ and $\gamma < \frac{n+2}{n-2}$ we have

$$q_1 = \tfrac{2\gamma n}{2kn+2\gamma-2} < \tfrac{2\gamma n}{n+2\gamma-2} < \tfrac{2n(n+2)}{n^2-2n+8}$$

and the last fraction is $\leqslant 3$ for all integers $n \geqslant 5$, while it is equal to $70/33 < 4$ for $n = 4$ and to $30/11 < 3$ when $n = 3$.

To prove uniqueness, if u, v are two solutions in $C_T H^1$ for some $T > 0$, we can write

$$\|u - v\|_{L_T^p L^{\gamma+1}} \lesssim \|f(u) - f(v)\|_{L_T^{p'} L^{(\gamma+1)'}} \lesssim \|u - v\|_{L_T^b L^{\gamma+1}} \||u| + |v|\|_{L_T^{p_0} L^{\gamma+1}}^{\gamma-1}$$

where

$$p = \tfrac{4}{n}\tfrac{\gamma+1}{\gamma-1}, \qquad \tfrac{1}{p_0} = \tfrac{1}{p} - \tfrac{1}{2}, \qquad \tfrac{1}{b} = \tfrac{\gamma}{2} - \tfrac{\gamma}{p} + \tfrac{1}{2}.$$

(note that we are not using Strichartz estimates of ∇u), hence by Sobolev embedding

$$\|u - v\|_{L_T^p L^{\gamma+1}} \lesssim (\|u\|_{L_T^{p_0} H^1} + \|v\|_{L_T^{p_0} H^1})^{\gamma-1} \|u - v\|_{L_T^b L^{\gamma+1}}$$

It is easy to check that $b < p$, thus we get

$$\lesssim T^\epsilon (\|u\|_{L_T^\infty H^1} + \|v\|_{L_T^\infty H^1}) \|u - v\|_{L_T^p L^{\gamma+1}}$$

for some $\epsilon > 0$ and this implies $u - v \equiv 0$ if T is small enough. □

Define the *energy* of a solution $u \in C([0,T]; H_0^1(\Omega))$ as

$$E(t) = \tfrac{1}{2} \int_\Omega a(\nabla^b u, \nabla^b u) dx + \tfrac{1}{2} \int_\Omega c(x)|u|^2 dx + \int_\Omega F(u) dx \tag{2.7.4}$$

Theorem 2.7.2 (Global existence in H^1). *Let $n \geqslant 3$ and assume the coefficients of L satisfy*

$$b^2 + |\nabla \cdot b| \in L^2_{loc}, \quad c \in L^{\frac{n}{2},1}, \quad \|a - I\|_{L^\infty} + \||b| + |a'|\|_{L^{n,\infty}} + \|b'\|_{L^{\frac{n}{2},\infty}} + \|c_-\|_{L^{\frac{n}{2},1}} < \epsilon. \tag{2.7.5}$$

Assume $f(u)$ satisfies the conditions (2.7.2) of the previous result, and in addition it is gauge invariant (2.1.11) with $F(r) = \int_0^r f(s)ds \geqslant 0$ for $s \in \mathbb{R}$. Moreover, assume condition (S) holds.

Then, if ϵ is sufficiently small, for all initial data $u_0 \in H_0^1(\Omega)$ problem (2.7.1) has a unique global solution $u \in C \cap L^\infty(\mathbb{R}; H_0^1(\Omega))$. In addition the solution has constant energy $E(t) \equiv E(0)$ for all $t \in \mathbb{R}$.

Proof. Since the lifespan of the local solution only depends on the H^1 norm of the data, in order to prove the claim it is sufficient to prove that the energy $E(t)$ of the solution is conserved. Indeed, $E(t)$ controls the H^1 norm of u, and then global existence follows from a standard continuation argument.

Let $e(u)$ be the energy density

$$e(u)(t,x) = \tfrac{1}{2}a(x)\nabla^b u \cdot \overline{\nabla^b u} + \tfrac{1}{2}c(x)|u|^2 + F(u)$$

so that $E(t) = \int_\Omega e(u)dx$. By gauge invariance and the definition of F we have $\partial_t F(u) = \partial_t \int_0^{|u|} f(s)ds = \Re\left(f(|u|)\frac{u}{|u|}\bar{u}_t\right) = \Re(f(u)\bar{u}_t)$. If the function u satisfies $u(t) \in H^2(\Omega)$, we can write

$$\partial_t e(u) + \nabla\cdot\{\Re\overline{u_t}a(x)\nabla^b u\} = \Re\overline{u_t}(iu_t - Lu + f(u)) \equiv 0 \tag{2.7.6}$$

and integrating on Ω, since $u_t|_{\partial\Omega} = 0$ by the Dirichlet boundary conditions, we obtain that $E(u)(t) \equiv E(u)(0)$ is constant in time.

Since we know only $u(t) \in H_0^1(\Omega)$, in order to use (2.7.6) we need a regularization procedure; we use the operators J_ϵ constructed at the end of Section 2.6. Thus we define $u_\epsilon = J_\epsilon u$ and note that u_ϵ belongs to $C_T H^2(\Omega)$ and satisfies

$$i\partial_t u_\epsilon - Lu_\epsilon + J_\epsilon f(u) = 0.$$

Using (2.7.6) we obtain, after an integration on $[t_1, t_2]\times\Omega$, with $0 \leqslant t_1 < t_2 \leqslant T$,

$$\int_\Omega e(u_\epsilon)|_{t_1}^{t_2} dx = \Re\int_{t_1}^{t_2}\int_\Omega \partial_t\overline{u_\epsilon}\cdot(f(u_\epsilon) - J_\epsilon f(u))dxdt.$$

Substituting $\partial_t u_\epsilon$ from the equation and using the Cauchy-Schwartz inequality and the assumption $a_{jk} \in L^\infty$ we get

$$\left|\int_\Omega e(u_\epsilon)|_{t_1}^{t_2} dx\right| \lesssim \int_{t_1}^{t_2}[\phi_\epsilon(t) + \psi_\epsilon(t) + \chi_\epsilon(t)]dt \tag{2.7.7}$$

where

$$\phi_\epsilon = \int_\Omega |\nabla^b u_\epsilon|\cdot|\nabla^b(f(u_\epsilon) - J_\epsilon f(u))|dx, \qquad \psi_\epsilon(t) = \int_\Omega |J_\epsilon f(u)|\cdot|f(u_\epsilon) - J_\epsilon f(u)|dx.$$

$$\chi_\epsilon(t) = \int_\Omega |c||u_\epsilon|\cdot|f(u_\epsilon) - J_\epsilon f(u)|dx$$

Since $u_\epsilon \to u$ in H_0^1 and hence by Sobolev embedding in $L^{\gamma+1}$, we see that $E(u_\epsilon) \to E(u)$. Thus to conclude the proof it is sufficient to show that the right hand side of (2.7.7) tends to 0 as $\epsilon \to 0$, possibly through a subsequence; to this end we shall apply dominated convergence on the interval $[0, T]$.

Consider first the case $n \geqslant 4$, so that $\gamma + 1 < n$. We prepare a few additional inequalities:

$$\|\nabla u_\epsilon\|_{L^{\gamma+1}} \simeq \|(-L)^{\frac{1}{2}}J_\epsilon u\|_{L^{\gamma+1}} = \|J_\epsilon(-L)^{\frac{1}{2}}u\|_{L^{\gamma+1}} \lesssim \|(-L)^{\frac{1}{2}}u\|_{L^{\gamma+1}} \simeq \|\nabla u\|_{L^{\gamma+1}}$$

by the L^p boundedness of J_ϵ and (2.6.8) for $\sigma = 1/2$. By Hölder and Sobolev inequalities in Lorentz spaces, using $b \in L^{n,\infty}$, we have also

$$\|bu_\epsilon\|_{L^{\gamma+1}} \lesssim \|u_\epsilon\|_{L^{q,\gamma+1}} \lesssim \|\nabla u_\epsilon\|_{L^{\gamma+1}} \lesssim \|\nabla u\|_{L^{\gamma+1}}, \qquad \frac{1}{\gamma+1} = \frac{1}{n} + \frac{1}{q}$$

and summing the two

$$\|\nabla^b u_\epsilon\|_{L^{\gamma+1}} \lesssim \|\nabla u\|_{L^{\gamma+1}}.$$

Thus we have

$$\phi_\epsilon(t) \lesssim \|\nabla u\|_{L^{\gamma+1}} \|\nabla(f(u_\epsilon) - J_\epsilon f(u)))\|_{L^{\frac{\gamma+1}{\gamma}}} \lesssim \|\nabla u\|^2_{L^{\gamma+1}} \|u\|^{\gamma-1}_{L^{\gamma+1}} =: \phi(t).$$

Note that $\phi \in L^1(0,T)$ since

$$\int_0^T \phi dt \leqslant \|\nabla u\|^2_{L^2_T L^{\gamma+1}} \|u\|^{\gamma-1}_{L^\infty L^{\gamma+1}}$$

and $\nabla u \in L^p_T L^{\gamma+1}$ for some $p > 2$ by Strichartz estimates, while $u \in C_T H^1_0 \hookrightarrow L^\infty_T L^{\gamma+1}$ by Sobolev embedding. For ψ_ϵ we have easily

$$\psi_\epsilon(t) \lesssim \|u\|^{2\gamma}_{L^{2\gamma}} =: \psi(t),$$

and by the interpolation and Sobolev inequalities

$$\|u\|^{2\gamma}_{L^{2\gamma}} \leqslant \|u\|^{2\gamma-\sigma}_{L^{\gamma+1}} \|u\|^\sigma_{L^{\frac{n(\gamma+1)}{n-(\gamma+1)}}} \lesssim \|u\|^{2\gamma-\sigma}_{L^{\gamma+1}} \|\nabla u\|^\sigma_{L^{\gamma+1}}, \qquad \sigma = \frac{\gamma-1}{\gamma+1} n$$

so that

$$\int_0^T \psi dt \lesssim \|u\|^{2\gamma-\sigma}_{L^\infty_T L^{\gamma+1}} \|\nabla u\|^\sigma_{L^\sigma_T L^{\gamma+1}}$$

and again we obtain $\psi \in L^1(0,T)$ since $0 < \sigma < 2$ for $1 < \gamma < \frac{n+2}{n-2}$. As to χ_ϵ, recalling that $|c|^{\frac{1}{2}} \in L^{n,\infty}$, we can write

$$\|cu_\epsilon J_\epsilon f(u)\|_{L^1} \leqslant \||c|^{\frac{1}{2}} u_\epsilon\|_{L^{\gamma+1}} \||c|^{\frac{1}{2}} J_\epsilon f(u)\|_{L^{\frac{\gamma+1}{\gamma}}} \lesssim \|\nabla u\|_{L^{\gamma+1}} \|\nabla J_\epsilon f(u)\|_{L^{\frac{\gamma+1}{\gamma}}} \lesssim \phi(t)$$

proceeding as in the estimate of bu_ϵ; the term $cu_\epsilon f(u_\epsilon)$ is similar. Thus the sequences $\phi_\epsilon, \psi_\epsilon, \chi_\epsilon$ are dominated. Moreover, it is easy to check, using exactly the previous estimates and properties (2.6.11), (2.6.16), (2.6.17) and (2.6.18), that for a.e. $t \in [0,T]$ one has $\phi_\epsilon(t), \psi_\epsilon(t), \chi_\epsilon(t) \to 0$ as $\epsilon \to 0$.

In the case $n = 3$, the quantity $\gamma + 1$ is in the range $2 \leqslant \gamma + 1 < 6$ and can be larger than n. The previous computations work fine for $1 \leqslant \gamma < 2$; when $2 \leqslant \gamma < 5$ it is not difficult to modify the choice of indices so to use only the allowed Strichartz norms. For the estimate of $\phi_\epsilon(t)$ we can write for $\frac{1}{4} < \epsilon < \frac{1}{2}$

$$\phi_\epsilon(t) \lesssim \|\nabla u\|^2_{L^{\frac{3}{1+\epsilon}}} \|u\|^{\gamma-1}_{L^{\frac{3(\gamma-1)}{1-2\epsilon}}} \lesssim \|\nabla u\|^2_{L^{\frac{3}{1+\epsilon}}} \|\nabla u\|^{\gamma-1}_{L^{\frac{3(\gamma-1)}{\gamma-2\epsilon}}} =: \phi(t)$$

by Hölder and Sobolev inequalitites, and hence

$$\int_0^T \phi(t) dt \leqslant \|\nabla u\|^2_{L_T^{\frac{4}{1-2\epsilon}} L^{\frac{3}{1+\epsilon}}} \|\nabla u\|^{\gamma-1}_{L_T^{\frac{2(\gamma-1)}{1+2\epsilon}} L^{\frac{3(\gamma-1)}{\gamma-2\epsilon}}}.$$

Notice that the first factor is an (allowed) Strichartz norm, while the second factor can be estimated by Hölder inequality in time with the Strichartz norm

$$\|\nabla u\|^{\gamma-1}_{L_T^{\frac{4(\gamma-1)}{\gamma-3+4\epsilon}} L^{\frac{3(\gamma-1)}{\gamma-2\epsilon}}},$$

(which is allowed and meaningful for $\frac{1}{4} < \epsilon < \frac{1}{2}$) since the condition $\frac{4(\gamma-1)}{\gamma-3+4\epsilon} > \frac{2(\gamma-1)}{1+2\epsilon}$ is equivalent to $\gamma < 5$. The reamining estimates can be modified in a similar way; we omit the details. □

The next Proposition is the crucial step in the proof of scattering. We follow the simpler approach to scattering developed in [87] and [12]. We prefer this to the more technical method of [83], which could also be used here.

Proposition 2.7.3. *Let $n \geqslant 3$, and consider Problem (2.7.1) under the assumptions of Theorem 2.1.6 if $n \geqslant 4$ or of Theorem 2.1.7 if $n = 3$. Then any solution $u \in C \cap L^\infty(\mathbb{R}; H_0^1(\Omega))$ satisfies*

$$\lim_{t\to\pm\infty} \|u(t,\cdot)\|_{L^r} = 0 \quad \textit{for all} \quad 2 < r < \frac{2n}{n-2}. \tag{2.7.8}$$

Proof. We consider only the case $t \to +\infty$; the proof in the case $t \to -\infty$ is identical. It is enough to prove (2.7.8) for $r = 2 + \frac{4}{n}$, i.e.,

$$\lim_{t\to+\infty} \|u(t,\cdot)\|_{L^{2+\frac{4}{n}}} = 0. \tag{2.7.9}$$

Since the H^1 norm of u is bounded for $t \in \mathbb{R}$, by Sobolev inequality we have

$$\|u(t,\cdot)\|_{L^{\frac{2n}{n-2}}} + \|u(t,\cdot)\|_{L^2} \lesssim \|u(t,\cdot)\|_{\dot H^1} + \|u(t,\cdot)\|_{L^2} \leqslant C \tag{2.7.10}$$

with C independent of t, and interpolating with (2.7.9) we obtain the full claim (2.7.8).

Assume by contradiction that there exist an $\epsilon_0 > 0$ and a sequence of times $t_k \uparrow +\infty$ such that for all k

$$\|u(t_k,\cdot)\|_{L^{2+\frac{4}{n}}} \geqslant \epsilon_0. \tag{2.7.11}$$

Denote with $Q_R(x)$ the intersection with Ω of the cube of side R and center x (with sides parallel to the axes). By interpolation in L^p spaces and Sobolev embedding, we have for all $v \in H_0^1(\Omega)$ and $x \in \Omega$

$$\|v\|^{2+\frac{4}{n}}_{L^{2+\frac{4}{n}}(Q_1(x))} \leqslant \|v\|^2_{L^{\frac{2n}{n-2}}(Q_1(x))} \cdot \|v\|^{\frac{4}{n}}_{L^2(Q_1(x))} \lesssim \|v\|^2_{H^1(Q_1(x))} \cdot \|v\|^{\frac{4}{n}}_{L^2(Q_1(x))}$$

which implies, for all $x \in \Omega$,

$$\|v\|^{2+\frac{4}{n}}_{L^{2+\frac{4}{n}}(Q_1(x))} \lesssim \|v\|^2_{H^1(Q_1(x))} \cdot \sup_{y\in\Omega} \|v\|^{\frac{4}{n}}_{L^2(Q_1(y))}.$$

Choosing a sequence of centers $x \in \Omega$ such that the cubes $Q_1(x)$ cover Ω and are almost disjoint, and summing over all cubes, we obtain the inequality (see also Proposition 1.4.1 in Chapter 1)

$$\|v\|^{2+\frac{4}{n}}_{L^{2+\frac{4}{n}}(\Omega)} \lesssim \|v\|^2_{H^1(\Omega)} \cdot \sup_{x\in\Omega} \|v\|^{\frac{4}{n}}_{L^2(Q_1(x))}. \tag{2.7.12}$$

Combining (2.7.12) with the energy bound (2.7.10) and recalling (2.7.11), we obtain that there exists a sequence of points $x_k \in \Omega$ such that

$$\|u(t_k,\cdot)\|_{L^2(Q_1(x_k))} \geqslant \epsilon_1 > 0.$$

We claim that we can find $\bar{t} > 0$ such that

$$\|u(t,\cdot)\|_{L^2(Q_2(x_k))} \geqslant \epsilon_1/2 \quad \text{for all } t \in (t_k, t_k + \bar{t}). \tag{2.7.13}$$

Indeed, consider a cut-off function $\chi \in C_c^\infty(\mathbb{R}^n)$ such that $\chi(x) = 1$ on the cube of side 1 with center x_k, and $\chi(x) = 0$ outside the cube of side 2 with center x_k. We integrate the elementary identity

$$\tfrac{d}{dt}\left[\chi(x)|u(t,x)|^2\right] = 2\chi(x)\nabla \cdot \{\Im[a(x)\nabla^b u(t,x)\bar{u}(t,x)]\}$$

on Ω and we obtain, for all $t \in \mathbb{R}$,

$$\begin{aligned}\left|\tfrac{d}{dt}\int_\Omega \chi(x)|u(t,x)|^2\,dx\right| &\lesssim \left|\int_\Omega \nabla\chi(x)\cdot \Im[a(x)\nabla^b u(t,x)\bar{u}(t,x)]\,dx\right| \\ &\lesssim \|u(t,\cdot)\|_{L^2(\Omega)}\|\nabla^b u(t,\cdot)\|_{L^2(\Omega)} \\ &\leqslant \|u(0,\cdot)\|_{L^2(\Omega)}\sup_{t\in\mathbb{R}}\|\nabla u(t,\cdot)\|_{L^2(\Omega)} =: \overline{C} < +\infty,\end{aligned} \tag{2.7.14}$$

where we used (2.2.15). This implies

$$\int_{Q_2(x_k)}|u(t,x)|^2\,dx \geqslant \int_{Q_1(x_k)}|u(t_k,x)|^2\,dx - \overline{C}|t-t_k|,$$

whence (2.7.13) follows provided that we choose $\bar{t} > 0$ such that $\epsilon_1^2 - \overline{C}\bar{t} > \epsilon_1^2/4$. Note, by passing to a subsequence, we can assume the intervals $(t_k, t_k + \bar{t})$ to be disjoint.

If $n \geqslant 4$, we get

$$\int_{\mathbb{R}}\int_{\Omega\times\Omega}\frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^3}dxdydt \gtrsim \sum_k \int_{t_k}^{t_k+\bar{t}}\int_{Q_2(x_k)\times Q_2(x_k)}|u(t,x)|^2|u(t,y)|^2dxdydt = \infty.$$

but this is in contradiction with (2.1.23), since $u \in L^\infty(\mathbb{R}, H_0^1(\Omega))$, and this concludes the proof in this case. On the other hand, if $n = 3$, from (2.7.13) we get that

$$\|u\|^4_{L^4((t_k,t_k+\bar{t})\times Q_2(x_k))} \geqslant C\epsilon_1^4\bar{t},$$

which is in contradiction with (2.1.24). □

By fairly standard arguments, property (2.7.8) implies that the Strichartz norms of a global H^1 solutions are bounded, and scattering follows. The only limitation here is the requirement $q_1 < n$ in Assumption (S), which is effective only in dimension $n = 3, 4$. We sketch the arguments for the sake of completeness:

Proposition 2.7.4. *Let $u \in C \cap L^\infty(\mathbb{R}; H_0^1(\Omega))$ be a solution to Problem (2.7.1) under the assumptions of Theorem 2.1.6 if $n \geqslant 4$ and under the assumptions of Theorem 2.1.7 if $n = 3$. Moreover, assume that (S) holds and that $\gamma > 1 + \frac{4}{n}$. Then for every admissible pair (p,q) we have $u \in L^pL^q$, and for every admissible pair (p,q) with $q < n$ we have $\nabla u \in L^pL^q$.*

Proof. We consider in detail the case $n \geqslant 4$, where $\gamma + 1 < n$. For the case $n = 3$ in the range $2 \leqslant \gamma < 6$, the following arguments can be easily modified as in the last part of the proof in Theorem 2.7.2. Note that we know that the Strichartz norms are finite on bounded time intervals, and we only need to prove an uniform bound as the time interval invades $\mathbb{R}$.

We use the notation $L^p_{T,t}L^q := L^p(T,t;L^q(\Omega))$ for $t > T$. By Strichartz estimates on the time interval $[T,t]$ for the admissible couple $(p,\gamma+1)$ where $p = \frac{4}{n}\frac{\gamma+1}{\gamma-1}$ we have

$$\begin{aligned}\|u\|_{L^p_{T,t}L^{\gamma+1}} &\lesssim \|u(T)\|_{L^2} + \|f(u)\|_{L^{p'}_{T,t}L^{(\gamma+1)'}}\\ &\lesssim \|u(T)\|_{L^2} + \|\|u\|^{\gamma}_{L^{\gamma+1}}\|_{L^{p'}_{T,t}}\end{aligned}$$

since $|f(u)| \lesssim |u|^\gamma$ and $(\gamma+1)'\gamma = \gamma+1$. The condition $\gamma > 1 + \frac{4}{n}$ is equivalent to $\gamma > \frac{p}{p'}$, thus we can continue the estimate as follows:

$$\begin{aligned}&\lesssim \|u(T)\|_{L^2} + \|u\|^{\gamma-\frac{p}{p'}}_{L^\infty_{T,t}L^{\gamma+1}}\|\|u\|^{\frac{p}{p'}}_{L^{\gamma+1}}\|_{L^{p'}_{T,t}}\\ &\leqslant \|u(T)\|_{L^2} + \|u\|^{\gamma-\frac{p}{p'}}_{L^\infty_{T,\infty}L^{\gamma+1}}\|u\|^{\frac{p}{p'}}_{L^{p'}_{T,t}L^{\gamma+1}}.\end{aligned}$$

By Proposition 2.7.3 we know that $o(T) = \|u\|_{L^\infty_{T,\infty}L^{\gamma+1}} \to 0$ as $T \to \infty$. Thus the function $\phi(t) := \|u\|_{L^p_{T,t}L^{\gamma+1}}$ satisfies an inequality of the form $\phi(t) \leqslant C + o(T)\phi(t)^{\frac{p}{p'}}$. Taking T large enough, an easy continuity argument shows that $\phi(t)$ is bounded for all $t > T$. This proves that $u \in L^pL^{\gamma+1}$. Now we notice that in the previous computations we have also proved that $f(u) \in L^{p'}L^{(\gamma+1)'}$, and using again Strichartz estimates we conclude that $u \in L^rL^q$ for all admissible (r,q).

The estimate of ∇u is similar:

$$\begin{aligned}\|\nabla u\|_{L^p_{T,t}L^{\gamma+1}} &\lesssim \|\nabla u(T)\|_{L^2} + \|\nabla f(u)\|_{L^{p'}_{T,t}L^{(\gamma+1)'}}\\ &\lesssim \|\nabla u(T)\|_{L^2} + \|\|u\|^{\gamma-1}_{L^{\gamma+1}}\|\nabla u\|_{L^{\gamma+1}}\|_{L^{p'}_{T,t}}\end{aligned}$$

since $|f'(u)| \lesssim |u|^{\gamma-1}$, and as before, using Hölder inequality,

$$\begin{aligned}&\lesssim \|\nabla u(T)\|_{L^2} + \|u\|^{\gamma-\frac{p}{p'}}_{L^\infty_{T,\infty}L^{\gamma+1}}\|\|u\|^{\frac{p}{p'}-1}_{L^{\gamma+1}}\|\nabla u\|_{L^{\gamma+1}}\|_{L^{p'}_{T,t}}\\ &\lesssim \|\nabla u(T)\|_{L^2} + \|u\|^{\gamma-\frac{p}{p'}}_{L^\infty_{T,\infty}L^{\gamma+1}}\|u\|^{\frac{p}{p'}-1}_{L^p_{T,t}L^{\gamma+1}}\|\nabla u\|_{L^p_{T,t}L^{\gamma+1}}.\end{aligned}$$

By the bound already proved, this implies

$$\|\nabla u\|_{L^p_{T,t}L^{\gamma+1}} \lesssim \|\nabla u(T)\|_{L^2} + o(T)\|\nabla u\|_{L^p_{T,t}L^{\gamma+1}}$$

and taking T large enough we obtain the claim. □

We can now conclude the proof of Theorem 2.1.9. Part (i) is Theorem 2.7.2. Scattering is an immediate consequence of the a priori bounds of the Strichartz norms proved in Proposition 2.7.4. We briefly sketch the main steps of the proof which are completely standard, in the case $t \to +\infty$; the case $t \to -\infty$ is identical.

To construct the wave operator (claim (ii) of the Theorem), given $u_+ \in H^1_0(\Omega)$, we consider the integral equation

$$u(t) := e^{-itL}u_+ + i\int_t^\infty e^{-i(t-s)}f(u(s))ds \tag{2.7.15}$$

and we look for a solution defined on $[T, \infty)$, for T sufficiently large. Using Strichartz estimates with the same choice of indices as in the proof of local existence, and noticing that the Strichartz norms of $e^{-itL}u_+$ are arbitrarily small for T large, by a fixed point approach we construct a solution $u \in C \cap L^\infty([T, +\infty), H_0^1(\Omega))$ to (2.7.15). This is also a solution to the Schrödinger equation in (2.7.1), and thanks to the global existence result, u can be extended to a solution $u \in C \cap L^\infty(\mathbb{R}, H_0^1(\Omega))$ defined for all $t \in \mathbb{R}$. We can then choose $u_0 = u(0)$. Uniqueness follows by a similar argument: if two solutions u_1, u_2 of (2.7.1), with possibly different data, have the same asymptotic behaviour i.e. $\|u_1(t) - u_2(t)\|_{H^1} \to 0$ as $t \to +\infty$, then they both solve (2.7.15), and the previous fixed point argument implies $u_1(t) = u_2(t)$ for t large. Then $u_1 \equiv u_2$ by global uniqueness.

To prove asymptotic completeness (claim (iii) of the Theorem), we fix a $u_0 \in H_0^1(\Omega)$ and let $u(t)$ be the corresponding global solution to Problem (2.7.1). Then we define $v(t) = e^{itL}u(t)$ and note that

$$v(t) = u_0 + i \int_0^t e^{isL} f(u(s)) ds.$$

Note that $\|e^{itL}\phi\|_{L2} = \|\phi\|_{L^2}$ by the unitarity of e^{itL}; moreover, since $(-L\phi, \phi)_{L^2} \simeq \|\phi\|^2_{\dot{H}^1}$, we have $\|e^{itL}\phi\|^2_{\dot{H}^1} \simeq (-Le^{itL}\phi, e^{itL}\phi)_{L^2} \simeq \|\phi\|_{\dot{H}^1}$, and in conclusion we get

$$\|e^{itL}\phi\|_{H^1} \simeq \|\phi\|_{H^1} \qquad \forall \phi \in H_0^1(\Omega)$$

with constants uniform in t. Thus for $0 < \tau < t$ we can write

$$\|v(t) - v(\tau)\|_{H^1} \simeq \left\|e^{-itL}(v(t) - v(\tau))\right\|_{H^1} = \left\| \int_\tau^t e^{-i(t-s)L} f(u)\, ds \right\|_{L_t^\infty H^1}$$

and by Strichartz estimates, Hölder inequality and interpolation, we get

$$\|v(t) - v(\tau)\|_{H^1} \lesssim \|f(u)\|_{L^{p'}_{\tau,t} W^{1,(\gamma+1)''}}$$

where $p = \frac{4}{n}\frac{\gamma+1}{\gamma-1}$; this choice is always possible in dimension $n \geqslant 4$; in dimension $n = 3$ for the range $2 \leqslant \gamma < 6$ one needs to modify the choice as in the proof of Theorem 2.7.2. By Proposition 2.7.4 we know that the Strichartz norms of u are bounded, and by the same argument used in that proof we see that $f(u) \in L^{p'} W^{1,(\gamma+1)'}$. As a consequence, the right hand side of the previous inequality can be made arbitrarily small provided t, τ are large enough. We deduce that $v(t)$ converges in $H_0^1(\Omega)$ as $t \to +\infty$ to a limit u_+, and finally

$$\|u(t) - e^{-itL}u_+\|_{H^1} \simeq \|v(t) - u_+\|_{H^1} \to 0$$

as claimed.

2.8 Strichartz estimates

Throughout this section, $\Omega = \mathbb{R}^n$ and L is the selfadjoint operator on $L^2(\mathbb{R}^n)$ defined in Proposition 2.6.1. We look for sufficient conditions on the coefficients a, b, c in order to have Strichartz estimates on $\mathbb{R}^n$ for the flow e^{itL}

$$\|e^{itL}u_0\|_{L^{p_1}L^{q_1}} \lesssim \|u_0\|_{L^2}, \tag{2.8.1}$$

$$\| \int_0^t e^{i(t-s)L} F ds \|_{L^{p_1} L^{q_1}} \lesssim \|F\|_{L^{p_2'} L^{q_2'}} \tag{2.8.2}$$

and for the derivative of the flow ∇e^{itL}

$$\|\nabla e^{itL} u_0\|_{L^{p_1} L^{q_1}} \lesssim \|\nabla u_0\|_{L^2}, \tag{2.8.3}$$

$$\|\nabla \int_0^t e^{i(t-s)L} F ds \|_{L^{p_1} L^{q_1}} \lesssim \|\nabla F\|_{L^{p_2'} L^{q_2'}} \tag{2.8.4}$$

for admissible couples of indices (p_j, q_j). Recall that *admissible couples* (p, q) satisfy $p \in [2, \infty]$, $q \in [2, \frac{2n}{n-2}]$ with $\frac{2}{p} + \frac{n}{q} = \frac{n}{2}$ and the *endpoint* is the couple $(2, \frac{2n}{n-2})$.

We shall derive the estimates of the first kind by combining Tataru's results in [84] with our smoothing estimates. On the other hand, in order to deduce (2.8.3), (2.8.4) we shall use the equivalence of Sobolev norms proved in Proposition 2.6.3. The following result is a direct application of [84]:

Theorem 2.8.1. *Let $n \geqslant 3$. Assume the coefficients a, b, c of L satisfy*

$$|a - I| + \langle x \rangle (|a'| + |b|) + \langle x \rangle^2 (|a''| + |b'| + |c|) \leqslant \epsilon \langle x \rangle^{-\delta} \tag{2.8.5}$$

for some $\epsilon, \delta > 0$. If ϵ is sufficiently small, the flow e^{itL} satisfies the Strichartz estimates (2.8.1), (2.8.2) for all admissible couples (p_j, q_j), $j = 1, 2$, including the endpoint.

Proof. We rewrite L as the sum of $Au = \nabla \cdot (a \nabla u)$ plus lower order terms

$$Lu = Au + 2ia(\nabla u, b) + i\partial_j (a_{jk} b_k) u - a(b, b) u - c(x) u.$$

Define the norm

$$\|v\|_Z = \|v\|_{L^\infty(|x| \leqslant 1)} + \sum_{j \geqslant 1} \|v\|_{L^\infty(2^{j-1} \leqslant |x| \leqslant 2^j)}.$$

By Theorem 4 and Remarks 6 and 7 in [84], if a, b, c satisfy

$$\|\langle x \rangle^2 |a''(x)|\|_Z + \|\langle x \rangle |a'(x)|\|_Z + \||a(x) - I|\|_Z \leqslant \epsilon, \tag{2.8.6}$$

$$\|\langle x \rangle^2 \partial_m (a_{jk} b_k)\|_Z + \|\langle x \rangle a_{jk} b_k\|_Z \leqslant \epsilon, \tag{2.8.7}$$

$$\|\langle x \rangle^2 [|\partial_j (a_{jk} b_k)| + |a(b, b)| + |c(x)|]\|_Z \leqslant \epsilon \tag{2.8.8}$$

for ϵ small enough, then the linear flow e^{itL} satisfies the full set of Strichartz estimates (2.8.1), (2.8.2). It is immediate to check that condition (2.8.5) implies (2.8.6)–(2.8.8). □

Combining the previous Theorem with our smoothing estimate (Corollary 2.1.5) we cover the case of repulsive electric potentials with a large positive part:

Theorem 2.8.2. *Let $n \geqslant 3$. Assume the coefficients a, b of L satisfy*

$$|a - I| + \langle x\rangle(|a'| + |b|) + \langle x\rangle^2(|a''| + |b'|) + \langle x\rangle^3|a'''| \leqslant \epsilon\langle x\rangle^{-\delta} \tag{2.8.9}$$

while the potential $c(x)$ satisfies

$$-\epsilon\langle x\rangle^{-2} \leqslant c(x) \leqslant C_+^2\langle x\rangle^{-2}, \qquad \langle x\rangle^{1+\delta}c \in L^n \tag{2.8.10}$$

and the repulsivity condition

$$a(x)x \cdot \nabla c(x) \leqslant \epsilon|x|^{-1}\langle x\rangle^{-1-\delta} \tag{2.8.11}$$

for some $\epsilon, \delta, C_+ > 0$. If ϵ is sufficiently small, the flow e^{itL} satisfies the homogeneous Strichartz estimates (2.8.1) for all admissible couples, and the inhomogeneous estimates (2.8.2) for all couples with the exception of the endpoint-endpoint case.

Proof. By Theorem 2.8.1, Strichartz estimates are valid for the flow e^{itL_0} with $c = 0$. The complete flow $u = e^{itL}u_0$ satisfies the equation $iu_t + L_0u = cu$, hence it can be written

$$u = e^{itL}u_0 = e^{itL_0}u_0 - i\int_0^t e^{i(t-s)L_0}(cu)ds$$

so that, by the previous result,

$$\|u\|_{L^pL^q} \lesssim \|u_0\|_{L^2} + \|cu\|_{L^2L^{\frac{2n}{n+2}}}$$

for all admissible couples (p, q). By Hölder inequality we have

$$\|cu\|_{L^2L^{\frac{2n}{n+2}}} \lesssim \|\langle x\rangle^{1+\delta}c\|_{L^n}\|\langle x\rangle^{-1-\delta}u\|_{L^2L^2}$$

and the homogeneous estimate will be proved if we can prove the estimate

$$\|\langle x\rangle^{-1-\delta}u\|_{L^2L^2} \lesssim \|u_0\|_{L^2}. \tag{2.8.12}$$

Indeed, the assumptions of Corollary 2.1.5 are satisfied by L; in particular, the gaussian upper bound for the heat flow e^{itL} is valid for general L^∞ coefficients (see Theorem 5.4 in [70] or [69]). Thus (2.8.12) follows from inequality (2.1.21) and we obtain the full set of homogeneous Strichartz estimates for the flow e^{itL}.

To prove inhomogeneous estimates it is sufficient to apply a standard TT^* argument combined with the Christ-Kiselev lemma, and this gives (2.8.2) with the exception of the endpoint-endpoint case. □

We conclude the section by proving the estimates for the flow ∇e^{itL}, which are now a straightforward consequence of the previous results. Note that the application of Proposition 2.6.3 imposes an additional condition $q_1 < n$, which is restrictive only in dimensions $n = 3$ and 4.

Corollary 2.8.3. *Let $n \geqslant 3$. Estimates (2.8.3), (2.8.4) hold for the flow ∇e^{itL}, for all admissible couples (p_j, q_j), $j = 1, 2$, provided $q_1 < n$ and the coefficients a, b, c of L satisfy either assumption (2.8.5), or assumptions (2.8.9), (2.8.10), (2.8.11), provided ϵ is small enough.*

Proof. In both cases we see that the assumptions of Proposition 2.6.3 are satisfied. In particular, in the second case the smallness of the $L^{\frac{n}{2},1}$ norm of c_- follows from the fact that the L^n norm of $\langle x\rangle^{1+\delta}c$ is arbitrarily small outside a sufficiently large ball, and inside the ball we have $|c_-| \leqslant \epsilon$ by condition (2.8.10).

Now in the first case the assumptions of Theorem 2.8.1 are satisfied and we can write

$$\begin{aligned}\|\nabla e^{itL}u_0\|_{L^{p_1}L^{q_1}} &\simeq \|(-L)^{\frac{1}{2}}e^{itL}u_0\|_{L^{p_1}L^{q_1}} = \|e^{itL}(-L)^{\frac{1}{2}}u_0\|_{L^{p_1}L^{q_1}} \\ &\lesssim \|(-L)^{\frac{1}{2}}u_0\|_{L^2} \simeq \|\nabla u_0\|_{L^2}\end{aligned}$$

by a repeated application of (2.6.8) for $\sigma = \frac{1}{2}$. The proof of the remaining claims is identical. □

2.9 Definitions and basic results on Lorentz spaces

For the convenience of the reader, we recall here the definitions and the main properties of the Lorentz spaces $L^{p,q}$, in view of the applications needed in the proof of our results.

For any measurable function $f\colon \mathbb{R}^n \to \mathbb{C}$ and any $s \geqslant 0$ we define the *upper-level* set E^f_s and the *non-increasing rearrangement* f^* of f:

$$\begin{aligned}E^f_s &:= \{x \mid |f(x)| > s\},\\ f^*(t) &:= \inf\{s > 0\colon |E^f_s| \leqslant t\}, \quad t \in (0, +\infty).\end{aligned}$$

Moreover, we consider the average of f^*, defined by

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(r)\,dr.$$

The Lorentz spaces are hence defined as follows:

Definition 2.9.1. For any $1 \leqslant p < \infty$ and $1 \leqslant q < \infty$ we define the quasinorm $\|f\|_{L^{p,q}}$ as follows:

$$\|f\|_{L^{p,q}} = \begin{cases}\left[\int_0^\infty \left(t^{1/p}f^*(t)\right)^q\,dt/t\right]^{1/q}, & 1 \leqslant q < \infty,\\ \sup_{t>0} t^{1/p}f^*(t), & q = \infty.\end{cases} \tag{2.9.1}$$

When $p \neq 1$, if we replace f^* with f^{**} in the above definitions we obtain an equivalent quasinorm that is in fact a norm (see [4], [9]). The *Lorentz space* $L^{p,q}$ is defined by

$$L^{p,q} := \{f\colon \|f\|_{L^{p,q}} < \infty\}.$$

Moreover we define

$$L^{1,1} := L^1, \quad L^{\infty,\infty} := L^\infty.$$

The spaces $L^{\infty,q}$ for $1 \leqslant q < \infty$ are usually left undefined (although $L^{\infty,1}$ is defined in [9] as the closure of L^∞ compactly supported functions in the L^∞ norm).

With the above definitions, one obtains immediately the elementary properties

$$L^{p,p} = L^p, \quad 1 \leqslant p \leqslant \infty;$$
$$L^{p,q_1} \subseteq {}^{p,q_2}, \quad 1 < p < \infty, \, 1 \leqslant q_1 \leqslant q_2 \leqslant \infty,$$

with continuous embedding. For $1 \leqslant p \leqslant \infty$, the spaces $L^p_w := L^{p,\infty}$ are usually called *weak Lebesgue spaces* or *Marcinkiewicz spaces*.

It is remarkable that the Lorentz spaces can be obtained by an equivalent construction using real interpolation:

$$L^{p,q} := (L^{p_0}, L^{p_1})_{\theta,q}, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

provided

$$p_0 < p_1, \quad p_0 < q \leqslant \infty, \quad 0 < \theta < 1.$$

The Hölder, Young and Sobolev inequalities can be established in the framework of the Lorentz spaces, where they turn out to be stronger than in the Lorentz spaces: we collect them here in the followin Theorems, referring to [68] for the proofs.

Theorem 2.9.2 (Hölder inequality)**.** *Let $f \in L^{p_1,q_1}$, $g \in L^{p_2,q_2}$. The following estimates hold:*

- *if $p_1, p_2, p \in (1, \infty)$, $q_1, q_2, q \in [1, \infty]$, then*
$$\|fg\|_{L^{p,q}} \leqslant C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}, \quad p^{-1} = p_1^{-1} + p_2^{-1}, \, q^{-1} \leqslant q_1^{-1} + q_2^{-1};$$
- *if $p_1, p_2 \in (1, \infty)$, $q_1, q_2 \in [1, \infty]$, then*
$$\|fg\|_{L^1} \leqslant C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}, \quad 1 = p_1^{-1} + p_2^{-1}, \, 1 \leqslant q_1^{-1} + q_2^{-1}.$$

We remark that the above statement does not cover the inequality

$$\|fg\|_{L^{p,q}} \leqslant \|f\|_{L^\infty}\|g\|_{L^{p,q}},$$

that clearly holds whenever $L^{p,q}$ is defined.

Theorem 2.9.3 (Young inequality)**.** *Let $f \in L^{p_1,q_1}$, $g \in L^{p_2,q_2}$. The following estimates hold:*

- *if $p_1, p_2, p \in (1, \infty)$, $q_1, q_2, q \in [1, \infty]$, then*
$$\|f * g\|_{L^{p,q}} \leqslant C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}, \quad p^{-1} = p_1^{-1} + p_2^{-1}, \, q^{-1} \leqslant q_1^{-1} + q_2^{-1};$$
- *if $p_1, p_2 \in (1, \infty)$, $q_1, q_2 \in [1, \infty]$, then*
$$\|f * g\|_{L^1} \leqslant C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}, \quad 1 = p_1^{-1} + p_2^{-1}, \, 1 \leqslant q_1^{-1} + q_2^{-1}.$$

As before, we remark that the above statement does not cover the inequality

$$\|f * g\|_{L^{p,q}} \leqslant \|f\|_{L^1}\|g\|_{L^{p,q}},$$

that can be shown to hold whenever $L^{p,q}$ is defined by real interpolation.

Theorem 2.9.4 (Gagliardo-Nirenberg-Sobolev inequality)**.** *Let* $0 < s < n$*,* $1 < q \leqslant r < \infty$ *and* $f \in \dot{H}^{s,q}$*. Then the following holds:*

$$\|f\|_{L^{r,q}} \leqslant C\||D|^s f\|_{L^q}, \qquad \textit{with } s - \frac{n}{q} = -\frac{n}{r}.$$

We conclude this survey showing some example of functions in Lorentz spaces. We state an aternative and useful characterization of the Lorentz space norm, given in the following Lemma from [82] (see also [52, Lemma 5.1]).

Lemma 2.9.5. *Let* $f\colon \mathbb{R}^n \to \mathbb{C}$ *be a measurable function and let* $1 < p, q < \infty$*. Then* $f \in L^{p,q}$ *if and only if there exist a sequence of sets* $(E_j)_{j\in\mathbb{Z}}$ *and a sequence of numbers* $a = (a_j)_{j\in\mathbb{Z}}$ *such that* $|E_j| = O(2^j)$*,* $a \in l^q$ *and the following estimate holds, for some constant* $C > 0$*:*

$$|f(x)| \leqslant C \sum_{j\in\mathbb{Z}} a_j 2^{-j/p} \chi_{E_j}(x).$$

With $\beta \geqslant 0$, denote

$$w_\beta(x) := |x|(|\log|x|| + 1)^\beta.$$

If $\beta = 0$, then it is easy to see from the definition that $w_\beta^{-s} \in L^{n/s,\infty}$ for any $s > 0$. In the case $\beta > 0$, for any $s > 0$ and $q \in [1, \infty]$, we have $w_\beta^{-s} \in L^{n/s,q}$, provided $\beta > 1/sq$. We will use Lemma 2.9.5: for any $j \in \mathbb{Z}$ consider the ball $B^j := B_{2^{j/n}} = \{x\colon |x| \leqslant 2^{j/n}\}$ and the rings $E_j := B^{j+1} \setminus B^j$; it is clear that $|E_j| = C_n 2^j$, where C_n depends only on the dimension n. Then, for all $x \in \mathbb{R}^n$ we have the estimate

$$|w_\beta^{-s}(x)| = \sum_{j\in\mathbb{Z}} \frac{1}{|x|^s(|\log|x|| + 1)^{\beta s}} \chi_{E_j}(x) \leqslant C \sum j \in \mathbb{Z}(|j| \log 2{+}1)^{-\beta s} 2^{-js/n} \chi_{E_j}(x).$$

We get immediately the claim, remarking that $a_j = (|j| \log 2 + 1)^{-\beta s}$ is in l^q if and only if $\beta > 1/sq$.

Chapter 3

Sharp Hardy uncertainty principle and gaussian decay properties of covariant Schrödinger evolutions

In this chapter we prove a sharp version of the Hardy uncertainty principle for Schrödinger equations with external bounded electromagnetic potentials, based on logarithmic convexity properties of Schrödinger evolutions. We provide, in addition, an example of a real electromagnetic potential which produces the existence of solutions with critical gaussian decay, at two distinct times. The results in the present chapter are proved in [11]: since this chapter is somehow apart from the previous ones, we prefer to keep the notations used in the paper and in the literature on this topic even if they are slightly different from the ones used until now.

3.1 Introduction

This chapter is concerned with the sharpest possible gaussian decay, at two distinct times, of solutions to Schrödinger equations of the type

$$\partial_t u = i(\Delta_A + V)u, \tag{3.1.1}$$

where $u = u(x,t) : \mathbb{R}^n \times [0,1] \to \mathbb{C}$, and

$$V = V(x,t) \colon \mathbb{R}^n \times [0,1] \to \mathbb{C},$$
$$\Delta_A := \nabla_A^2, \quad \nabla_A := \nabla - iA, \quad A = A(x) \colon \mathbb{R}^n \to \mathbb{R}^n.$$

The results in this chapter follow a program which has been developed in the magnetic free case $A \equiv 0$, in the recent years, by Escauriaza, Kenig, Ponce, and Vega in the sequel of papers [33, 31, 32, 34, 35], and with Cowling in [21]. One of the main motivations is the connection with the *Hardy uncertainty principle*, which can be stated as follows:

if $f(x) = O\left(e^{-|x|^2/\beta^2}\right)$ *and its Fourier transform* $\hat{f}(\xi) = O\left(e^{-4|\xi|^2/\alpha^2}\right)$*, then*

$$\alpha\beta < 4 \Rightarrow f \equiv 0$$

$$\alpha\beta = 4 \Rightarrow f \text{ is a constant multiple of } e^{-\frac{|x|^2}{\beta^2}}.$$

The solving formula for solutions to the free Schrödinger equation with initial datum f in L^2, namely

$$u(x,t) := e^{it\Delta}f(x) = (2\pi it)^{-\frac{n}{2}} e^{i\frac{|x|^2}{4t}} \mathcal{F}\left(e^{i\frac{|\cdot|^2}{4t}} f\right)\left(\frac{x}{2t}\right),$$

gives a hint of the following PDE's-version of the Hardy uncertainty principle:

if $u(x,0) = O\left(e^{-|x|^2/\beta^2}\right)$ *and* $u(x,T) := e^{iT\Delta}u(x,0) = O\left(e^{-|x|^2/\alpha^2}\right)$, *then*

$$\alpha\beta < 4T \Rightarrow u \equiv 0$$

$$\alpha\beta = 4T \Rightarrow u(x,0) \text{ is a constant multiple of } e^{-\left(\frac{1}{\beta^2}+\frac{i}{4T}\right)|x|^2}.$$

The corresponding L^2-versions of the previous results were proved in [77] and affirm the following:

$$e^{|x|^2/\beta^2} f \in L^2,\ e^{4|\xi|^2/\alpha^2}\hat{f} \in L^2,\ \alpha\beta \leqslant 4 \Rightarrow f \equiv 0$$

$$e^{|x|^2/\beta^2} u(x,0) \in L^2,\ e^{|x|^2/\alpha^2} e^{iT\Delta} u(x,0) \in L^2,\ \alpha\beta \leqslant 4T \Rightarrow u \equiv 0.$$

We mention [5, 39, 80] as interesting surveys about this topic. In the sequel of papers [21, 33, 31, 32, 34, 35], the authors investigated the validity of the previous statements for zero-order perturbations of the Schrödinger equation of the form

$$\partial_t u = i(\Delta + V(t,x))u. \tag{3.1.2}$$

An interesting contribution of the above papers is that a purely real analytical proof of the uncertainty principle is provided, based on the logarithmic convexity properties of weighted L^2-norms of solutions to (3.1.2). Namely, norms of the type $H(t) := \|e^{a(t)|x+b(t)|^2}u(t)\|_{L^2(\mathbb{R}^n)}$, where $a(t)$ is a suitable bounded function, and $b(t)$ is a curve in $\mathbb{R}^n$, are logarithmically convex in time. The interest of these results relies on various motivations. First, since just real analytical techniques are involved, rough potentials $V \in L^\infty$ can be considered, which are usually difficult to handle by Fourier techniques. In addition, in [34] it is shown that a gaussian decay at times 0 and T of solutions to (3.1.2) is not only preserved, but also improved, in some sense, for intermediate times, up to suitably move the center of the gaussian. A consequence of Theorem 1 in [34] is the following: if $V(t,x) \in L^\infty$ is the sum of a real-valued potential V_1 and a sufficiently regular complex-valued potential V_2, and $\|e^{|x|^2/\beta^2}u(0)\|_{L^2} + \|e^{|x|^2/\alpha^2}u(T)\|_{L^2} < +\infty$, with $\alpha\beta < 4T$, then $u \equiv 0$. Moreover, the result is sharp in the class of complex potentials: indeed, Theorem 2 in [34] provides an example of a (complex) potential V for which there exists a non-trivial solution $u \neq 0$ with the above gaussian decay properties, with $\alpha\beta = 4T$.

The fact that the potential in [34] is complex-valued might have an appealing connection with the examples by Meshkov and Cruz-Sampedro in [61, 22] about unique continuation at infinity for stationary Schrödinger equations. In particular, an interesting question is still open, concerning with the possibility or not of providing analogous real-valued examples.

Our first result states the following: if one introduces a magnetic potential in the hamiltonian, then real-valued examples in the spirit of Theorem 2 in [34] can be found.

Theorem 3.1.1. *Let $n = 3$, $k > 3/2$, and define $A = A(x, y, z, t) : \mathbb{R}^{3+1} \to \mathbb{R}^3$ and $V = V(x, y, z, t) : \mathbb{R}^{3+1} \to \mathbb{R}$ as follows:*

$$A(x,y,z,t) = \frac{2kt}{1+t^2} \cdot \frac{z}{(x^2+y^2)(1+r^2)} \left(xz, yz, -x^2-y^2\right), \tag{3.1.3}$$

$$V(x,y,z,t) = \frac{k}{1+r^2}\left(\frac{2}{1+t^2} + 6 - \frac{4(1+k)r^2}{1+r^2}\right) - |A(x,y,z,t)|^2, \tag{3.1.4}$$

where $r^2 := x^2 + y^2 + z^2$. Then the function

$$u = u(r,t) = (1+it)^{2k-\frac{n}{2}}(1+r^2)^{-k} e^{-\frac{(1-it)}{4(1+t^2)}r^2} \tag{3.1.5}$$

is a solution to

$$i\partial_t u + \Delta_A u = Vu$$

satisfying $\left\|e^{\frac{r^2}{8}} u(-1)\right\|_{L^2} + \left\|e^{\frac{r^2}{8}} u(1)\right\|_{L^2} < \infty$.

Remark 3.1.2. The choice of the time interval $[-1, 1]$ instead of $[0, T]$ does not led the generality of the result, since by scaling one can always reduce matters to this case (see also Remark 3.1.5 below). Notice that both A and V are real-valued, and this is (at our knowledge) a novelty. Observe moreover that A is time-dependent, and singular all over the z-axis $x = y = 0$, with Coulomb-type singularity $(x^2 + y^2)^{-\frac{1}{2}}$. We finally remark that we are not able to generalize the above example to any dimension $n \neq 3$, and it is unclear to us if this is an intrinsic obstruction or not. The main idea relies in the expansion

$$\Delta_A = \Delta - 2iA \cdot \nabla - i\mathrm{div}\, A - |A|^2.$$

Applying this operator to the function u in (3.1.5), one notice that the first order term $2iA \cdot \nabla u$ vanishes, since u is radial and we choose the Crönstrom gauge $A \cdot x \equiv 0$; on the other hand a purely imaginary, non null zero-order term $i\mathrm{div}\, A$ naturally appears, being A real valued. We refer to section 3.2 below for the details of the proof, which is a quite simple computation.

Theorem 3.1.1 motivates us to think to electromagnetic Schrödinger evolutions as a natural setting for the study of Hardy uncertainty principles. We also need to keep in mind the well known fact that the magnetic ground states (and hence the corresponding standing waves) have gaussian decay (see [30] and the references therein).

In the recent years, some results in the spirit of the Hardy principle appeared, concerning with generic first-order perturbations of Schrödinger operators. Among the others, Dong and Staubach in [27] proved that an uncertainty property holds, under suitable assumptions on the lower order terms; nevertheless, a quantitative knowledge of the critical constant in the gaussian weights seems to be difficult to be found, due to the generality of the model. The paper [27] generalize a previous result by Ionescu and Kenig in [47], in which unique continuation from the exterior of a ball is proved, in the same setting.

We stress that an electromagnetic field is not any first-order perturbation of a Schrödinger operator, since it has a peculiar intrinsic algebra which has to be taken into account. The feeling is that quantitative results could be obtained for such operators, under geometric assumptions on the magnetic field. As an example, we mention [2], where a non-sharp version of the Hardy uncertainty principle (with $\alpha\beta < 2T$) in presence of (possibly large) magnetic fields has been recently proved, inspired to the techniques in [32]. The last result of this chapter improves the ones in [2], covering the sharp range $\alpha\beta < 4T$. In order to settle the theorem, we need to introduce a few notations.

In the sequel, we denote by $A = (A^1(x), \dots, A^n(x)) \colon \mathbb{R}^n \to \mathbb{R}^n$ a real vector field (magnetic potential). The magnetic field, denoted by $B \in M_{n\times n}(\mathbb{R})$ is the antisymmetric gradient of A, namely

$$B = B(x) = DA(x) - DA^t(x), \qquad B_{jk}(x) = A^k_j(x) - A^j_k(x).$$

In dimension $n = 3$, B is identified with the vector field $\operatorname{curl} A$, by the elementary properties of antisymmetric matrices. Finally, in the following we will denote by f_t the time derivative $\partial_t f$ of any function f. We can now state the last result of this chapter.

Theorem 3.1.3. *Let $n \geqslant 3$, and let $u \in \mathcal{C}([0,1]; L^2(\mathbb{R}^n))$ be a solution to*

$$\partial_t u = i\left(\Delta_A + V_1(x) + V_2(x,t)\right) u \tag{3.1.6}$$

in $\mathbb{R}^n \times [0,1]$, with $A = (A^1(x), \dots, A^n(x)) \in \mathcal{C}^{1,\varepsilon}_{loc}(\mathbb{R}^n;\mathbb{R}^n)$, $V_1 = V_1(x) : \mathbb{R}^n \to \mathbb{R}$, $V_2 = V_2(x,t) : \mathbb{R}^{n+1} \to \mathbb{C}$. Moreover, denote by $B = B(x) = DA - DA^t$, $B_{jk} = A^k_j - A^j_k$ and assume that there exists a unit vector $\xi \in \mathbb{S}^{n-1}$ such that

$$\xi^t B(x) \equiv 0. \tag{3.1.7}$$

Finally, assume that

$$\|x^t B\|^2_{L^\infty} < \infty \tag{3.1.8}$$

$$\|V_1\|_{L^\infty} < \infty \tag{3.1.9}$$

$$\sup_{t\in[0,1]} \left\| e^{\frac{|\cdot|^2}{(\alpha t+\beta(1-t))^2}} V_2(\cdot,t)\right\|_{L^\infty} e^{\sup_{t\in[0,1]}\|\Im V_2(\cdot,t)\|_{L^\infty}} < \infty \tag{3.1.10}$$

$$\left\| e^{\frac{|\cdot|^2}{\beta^2}} u(\cdot,0)\right\|_{L^2} + \left\| e^{\frac{|\cdot|^2}{\alpha^2}} u(\cdot,1)\right\|_{L^2} < \infty, \tag{3.1.11}$$

for some $\alpha, \beta > 0$.

The following holds:

- *if $\alpha\beta < 4$ then $u \equiv 0$.*
- *if $\alpha\beta \geqslant 4$ then*

$$\sup_{t\in[0,1]} \left\| e^{a(t)|\cdot|^2} u(t)\right\|_{L^2(\mathbb{R}^n)} + \left\| \sqrt{t(1-t)}\nabla_A (e^{a(t)+\frac{i\dot a(t)}{8a(t)}|\cdot|^2} u)\right\|_{L^2(\mathbb{R}^n\times[0,1])} \leqslant N\left[\left\| e^{\frac{|\cdot|^2}{\beta^2}} u(0)\right\|_{L^2(\mathbb{R}^n)} + \left\| e^{\frac{|\cdot|^2}{\alpha^2}} u(1)\right\|_{L^2(\mathbb{R}^n)}\right], \tag{3.1.12}$$

with

$$a(t) = \frac{\alpha\beta R}{2(\alpha t + \beta(1-t))^2 + 2R^2(\alpha t - \beta(1-t))^2},$$

where R is the smallest root of the equation

$$\frac{1}{2\alpha\beta} = \frac{R}{4(1+R^2)},$$

and $N > 0$ is a constant depending on α, β and $\|V\|_{L^\infty(\mathbb{R}^n\times[0,1])}$, $\|x^t B\|_{L^\infty(\mathbb{R}^n)}$.

Remark 3.1.4. Notice that, apart from the local regularity assumption $A \in \mathcal{C}^{1,\varepsilon}_{\text{loc}}(\mathbb{R}^n)$, which is the minimal request in order to justify an approximation argument in Lemma 3.3.3 below, all the hypotheses of the theorem are in terms of B and V, respecting the gauge invariance of the result.

Remark 3.1.5. The choice of the time interval $[0,1]$ does not led the generality of the results. Indeed, $v \in C([0,T], L^2(\mathbb{R}^n))$ is solution to (3.1.1) in $\mathbb{R}^n \times [0,T]$ if and only if $u\colon [0,1] \to \mathbb{C}$, $u(x,t) = T^{\frac{n}{4}} v(\sqrt{T}x, Tt)$ is solution to

$$\partial_t u = i(\Delta_{A_T} u + V_T(x,t)u), \text{ in } [0,1]\times\mathbb{R}^n,$$

where

$$A_T(x,t) = \sqrt{T}A(\sqrt{T}x, Tt), \quad V_T(x) = TV(\sqrt{T}x, Tt).$$

Moreover, observe that

$$\|e^{\frac{|x|^2}{\beta^2}} v(0)\| = \|e^{\frac{|x|^2}{\beta'^2}} u(0)\|, \quad \|e^{\frac{|x|^2}{\alpha^2}} v(T)\| = \|e^{\frac{|x|^2}{\alpha'^2}} u(1)\|,$$
$$\sup_{t\in[0,T]} \|e^{\frac{T^2|x|^2}{(\alpha t + \beta(T-t))^2}} v(t)\| = \sup_{t\in[0,1]} \|e^{\frac{|x|^2}{(\alpha' t + \beta'(1-t))^2}} u(t)\|.$$

with $\beta' = T^{-\frac{1}{2}}\beta$, $\alpha' = T^{-\frac{1}{2}}\alpha$.

Remark 3.1.6. The magnetic field in Theorem 3.1.3 does not depend on time, differently from the example in Theorem 3.1.1. Nevertheless, it could be probably possible to generalize the result to the case of time dependent magnetic fields, by assuming the existence of the purely magnetic flow and the L^2-preservation, but this will not be an object of study in the present work.

Remark 3.1.7. Assumption (3.1.7) is fundamental in our strategy of proof, and it does not allow to include the 2D-case in the statement of Theorem 3.1.3, due to elementary properties of antisymmetric matrices. We mention [2] for some explicit examples of magnetic fields satisfying (3.1.7). It is an interesting open question if there exist examples of magnetic fields which do not satisfy (3.1.7), for which the Hardy uncertainty holds with different quantitative constants or different exponential decays. Observe that the example in (3.1.3) satisfies (3.1.7), with $\xi = (0,0,1)$. Indeed, an explicit computation shows that

$$B = \operatorname{curl} A = \frac{2kt}{1+t^2} \cdot \frac{2z}{(x^2+y^2)(1+r^2)^2}(-y, x, 0).$$

The strategy of the proof of Theorem 3.1.3 is the following:

1. first we reduce to the Crönstrom gauge $x \cdot A \equiv 0$ (see Section 3.4.1), which turns out to be a helpful choice;
2. by conformal (or Appell) transformation (see Lemma 3.4.3), we reduce to the case $\alpha = \beta$, and we perform a time scaling to reduce to the time interval $[-1, 1]$ (see Section 3.4.2);
3. we prove Theorem 3.1.3 in the case $\mu := \alpha = \beta$ (see Section 3.4.3);
4. we translate the result in terms of the original solution, by inverting the transformations at step 2, obtaining the final result (see the end of the proof).

The key ingredient is Lemma 3.3.3 below, which comes into play in the proof of step 3. This is based on an iteration scheme, introduced in [34]: by successive approximations, we can start an iterative improvement of the decay assumption (3.1.11), by suitably moving the center of the gaussian weight. In the limit, this argument leads to an optimal choice of the function $a = a(t) : [-1, 1] \to \mathbb{R}$ for which the estimate

$$\|e^{a(t)|x|^2} u(x,t)\|_{L^\infty_t([-1,1])L^2_x(\mathbb{R}^n)} \leqslant C(\alpha, \beta, T, \|V\|_{L^\infty}, \|x^t B\|_{L^\infty}) \tag{3.1.13}$$

holds. The presence of a magnetic fields makes things quite more complicate, once the iteration starts, as wee see in the sequel. The rest of the chapter is devoted to the proofs of our main theorems.

3.2 Proof of Theorem 3.1.1

The proof of Theorem 3.1.1 is a straightforward computation. First, we expand the magnetic Laplace operator and rewrite

$$(i\partial_t + \Delta_A)\, u = (i\partial_t + \Delta)\, u - 2iA \cdot \nabla_x u - i(\mathrm{div}_x A) u - |A|^2 u.$$

Now we compute

$$\begin{aligned}(i\partial_t + \Delta)\, u &= \frac{1}{1+r^2}\left(\frac{2k}{1+it} + 6k - \frac{4k(k+1)r^2}{1+r^2}\right)\\ &= \frac{1}{1+r^2}\left(-\frac{2ikt}{1+t^2} + \frac{2k}{1+t^2} + 6k - \frac{4k(k+1)r^2}{1+r^2}\right),\end{aligned}$$

where u is given by (3.1.5). Observe that, since u is radial and $A \cdot x \equiv 0$ by the definition (3.1.3), we have $A \cdot \nabla_x u \equiv 0$. Finally, another direct computation gives

$$i\mathrm{div}_x A = -\frac{2ikt}{1+t^2} \cdot \frac{1}{1+r^2}.$$

In conclusion,

$$(i\partial_t + \Delta_A)\, u = \left[\frac{k}{1+r^2}\left(\frac{2}{1+t^2} + 6 - \frac{4(k+1)r^2}{1+r^2}\right) - |A|^2\right] u = Vu,$$

by the definition (3.1.4), which completes the proof.

The rest of the chapter is devoted to the proof of Theorem 3.1.3.

3.3 Some preliminary lemmata

Let us fix some notations and recall some results from [34] and [2]. We denote

$$(f,g) := \int_{\mathbb{R}^n} f\bar{g}\,dx, \quad H(f) = \|f\|^2 := (f,f),$$

for $f, g \in L^2(\mathbb{R}^n)$.

Lemma 3.3.1 ([34], Lemma 2)**.** *Let $\mathcal{S}$ be a symmetric operator, $\mathcal{A}$ a skew-symmetric one, both allowed to depend on the time variable, and f a smooth enough function. Moreover let $\gamma\colon [c,d] \to (0,+\infty)$ and $\psi\colon [c,d] \to \mathbb{R}$ be smooth functions. If*

$$(\gamma\,\mathcal{S}_t f(t) + \gamma\,[\mathcal{S},\mathcal{A}]f(t) + \dot{\gamma}\,\mathcal{S}f(t), f(t)) \geqslant -\psi(t)H(t), \quad t \in [c,d] \tag{3.3.1}$$

then, for all $\varepsilon > 0$,

$$H(t) + \varepsilon \leqslant e^{2T(t)+M_\varepsilon(t)+2N_\varepsilon(t)}(H(c)+\varepsilon)^{\theta(t)}(H(d)+\varepsilon)^{1-\theta(t)}, \quad t \in [c,d]$$

where T and M_ε verify

$$\begin{cases} \partial_t(\gamma\partial_t T) = -\psi, & t \in [c,d] \\ T(c) = T(d) = 0, \end{cases} \qquad \begin{cases} \partial_t(\gamma\partial_t M_\varepsilon) = -\gamma\frac{\|\partial_t f - \mathcal{S}f - \mathcal{A}f\|^2}{H+\varepsilon} & t \in [c,d] \\ M_\varepsilon(c) = M_\varepsilon(d) = 0, \end{cases}$$

$$N_\varepsilon = \int_c^d \left| \Re \frac{((\partial_s - \mathcal{S} - \mathcal{A})f(s), f(s))}{H(s)+\varepsilon} \right| ds, \quad \theta(t) = \frac{\int_t^d \frac{ds}{\gamma}}{\int_c^d \frac{ds}{\gamma}}.$$

Moreover

$$\begin{aligned} &\partial_t(\gamma\,\partial_t H - \gamma\,\Re(\partial_t f - \mathcal{S}f - \mathcal{A}f, f)) + \gamma\,\|\partial_t f - \mathcal{S}f - \mathcal{A}f\|^2 \\ &\quad \geqslant 2(\gamma\,\mathcal{S}_t f + \gamma\,[\mathcal{S},\mathcal{A}]f + \dot{\gamma}\,\mathcal{S}f, f). \end{aligned} \tag{3.3.2}$$

For $\varphi = \varphi(x,t) : \mathbb{R}^{n+1} \to \mathbb{R}$, we can write

$$e^{\varphi(x,t)}(\partial_t - i\Delta_A)e^{-\varphi(x,t)} = \partial_t - \mathcal{S} - \mathcal{A}$$

where

$$\mathcal{S} = -i\,(\Delta_x\varphi + 2\nabla_x\varphi\cdot\nabla_A) + \varphi_t \tag{3.3.3}$$

$$\mathcal{A} = i\left(\Delta_A + |\nabla_x\varphi|^2\right) \tag{3.3.4}$$

(see [2]). Observe that $\mathcal{S}$ and $\mathcal{A}$ are respectively a symmetric and a skew-symmetric operator. Our first goal is to apply Lemma 3.3.1 with a suitable choice of $\mathcal{S}, \mathcal{A}$. In order to do this, we need to obtain the lower bound (3.3.1) when $\mathcal{S}$ and $\mathcal{A}$ are given by (3.3.3) and (3.3.4), respectively: this is done in the following lemma, analogous to [34], Lemma 3.

Lemma 3.3.2. *Let*

$$\begin{aligned} &\varphi(x,t) = a(t)|x+\mathbf{b}(t)|^2, \\ &a = a(t)\colon \mathbb{R}\to\mathbb{R}, \quad \mathbf{b} = \mathbf{b}(t) = b(t)\xi\colon \mathbb{R}\to\mathbb{R}^n, \quad \xi \in \mathbb{S}^{n-1}, \end{aligned} \tag{3.3.5}$$

and $\mathcal{S}, \mathcal{A}$ *be defined as in* (3.3.3) *and* (3.3.4). *Assume that*

$$\begin{aligned} x \cdot A_t(x) &= 0, \\ \mathbf{b} \cdot A_t(x) &= 0, \end{aligned} \tag{3.3.6}$$

for all $x \in \mathbb{R}^n$ *and assume* (3.1.7). *Assume moreover*

$$F(a, \gamma) = \left(\ddot{a} + 32a^3 - \frac{3\dot{a}^2}{2a} - \frac{a}{2}\left(\frac{\dot{a}}{a} + \frac{\dot{\gamma}}{\gamma}\right)^2 \right) \gamma > 0 \ \textit{in} \ [c, d]. \tag{3.3.7}$$

Then, for a smooth enough function f,

$$((\gamma \mathcal{S}_t + \gamma[\mathcal{S}, \mathcal{A}] + \dot{\gamma}\mathcal{S})f, f) \geqslant -\left(\left(\frac{\gamma^2 a^2 |\ddot{\mathbf{b}}|^2}{F(a,\gamma)} + 2\gamma a \|x^t B\|^2_{L^\infty} \right) f, f \right), \qquad \textit{for all } t \in [c, d]. \tag{3.3.8}$$

Proof. The proof is analogous to the one of Lemma 3 in [34], with some additional magnetic terms to be considered. Explicit computations (see Lemma 2.9 in [2]) give:

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{S} f \bar{f} \, dx &= \int_{\mathbb{R}^n} \left[-i(2na|f|^2 + 4a(x + \mathbf{b}) \cdot \nabla_A f \bar{f}) \right] dx \\
&\quad + \int_{\mathbb{R}^n} \left[\dot{a}|x + \mathbf{b}|^2 |f|^2 + 2a\dot{\mathbf{b}} \cdot (x + \mathbf{b})|f|^2 \right] dx \\
\int_{\mathbb{R}^n} \mathcal{A} f \bar{f} \, dx &= \int_{\mathbb{R}^n} \left[(i\Delta_A f + 4ia^2|x + \mathbf{b}|^2 f)\bar{f} \right] dx \\
\int_{\mathbb{R}^n} [\mathcal{S}, \mathcal{A}] f \bar{f} \, dx &= \int_{\mathbb{R}^n} \left[8a|\nabla_A f|^2 + 32a^3 |x + \mathbf{b}|^2 |f|^2 \right] dx \\
&\quad - \int_{\mathbb{R}^n} \left[4\Im[f \, 2a(x + \mathbf{b})^t B \overline{\nabla_A f}] \right] dx \\
&\quad + \int_{\mathbb{R}^n} \left[4\Im[\bar{f}\dot{a}(x + \mathbf{b}) \cdot \nabla_A f + \bar{f} a \dot{\mathbf{b}} \cdot \nabla_A f] \right] dx
\end{aligned}$$

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{S}_t f \bar{f} \, dx &= \int_{\mathbb{R}^n} \left[2\Im[(2\dot{a}(x + \mathbf{b}) + 2a\dot{\mathbf{b}}) \cdot \nabla_A f]\bar{f} \right] dx \\
&\quad + \int_{\mathbb{R}^n} \left[\ddot{a}|x + \mathbf{b}|^2 - 4a(x + \mathbf{b}) \cdot A_t \right] |f|^2 \, dx \\
&\quad + \int_{\mathbb{R}^n} \left[4\dot{a}\dot{\mathbf{b}} \cdot (x + \mathbf{b}) + 2a\ddot{\mathbf{b}} \cdot (x + \mathbf{b}) + 2a|\dot{\mathbf{b}}|^2 \right] |f|^2 \, dx.
\end{aligned}$$

Summing up we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} (\gamma \mathcal{S}_t + \gamma[\mathcal{S},\mathcal{A}] + \dot\gamma \mathcal{S}) f \bar f \, dx \\
&\quad = \int_{\mathbb{R}^n} (\ddot a \gamma + 32 a^3 \gamma + \dot\gamma \dot a) |x+\mathbf{b}|^2 |f|^2 \, dx \\
&\quad\quad + \int_{\mathbb{R}^n} [(4\gamma \dot a \dot{\mathbf{b}} + 2\gamma a \ddot{\mathbf{b}} + 2\dot\gamma a \dot{\mathbf{b}}) \cdot (x+\mathbf{b}) + 2\gamma a |\dot{\mathbf{b}}|^2] |f|^2 \, dx \\
&\quad\quad + \int_{\mathbb{R}^n} 8\gamma a |\nabla_A f|^2 + 2\Re(-i\nabla_A f) \cdot \overline{(4\gamma a \dot{\mathbf{b}} f)} \, dx \\
&\quad\quad + \int_{\mathbb{R}^n} 2\Re(-i\nabla_A f) \cdot \overline{((2\dot\gamma a + 4\gamma \dot a)(x+\mathbf{b}) f)} \, dx \\
&\quad\quad - \int_{\mathbb{R}^n} 4\Im(\gamma f 2a (x+\mathbf{b})^t B \overline{\nabla_A f}) \, dx \\
&\quad\quad - \int_{\mathbb{R}^n} 4a\gamma \, (x+\mathbf{b}) \cdot A_t |f|^2 \, dx.
\end{aligned}$$

The last term in the previous equation vanishes, because of (3.3.6). Completing the squares in the previous equation we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} (\gamma \mathcal{S}_t + \gamma[\mathcal{S},\mathcal{A}] + \dot\gamma \mathcal{S}) f \bar f \, dx \\
&\quad = \int_{\mathbb{R}^n} 8\gamma a \left| -i\nabla_A f + \frac{\dot{\mathbf{b}}}{2} f + \left(\frac{\dot a}{2a} + \frac{\dot\gamma}{4\gamma} \right) (x+\mathbf{b}) f \right|^2 dx \\
&\quad\quad + \int_{\mathbb{R}^n} F(a,\gamma) \left| x + \mathbf{b} + \frac{a\gamma \ddot{\mathbf{b}}}{F(a,\gamma)} \right|^2 |f|^2 \, dx - \frac{\gamma^2 a^2 |\ddot{\mathbf{b}}|^2}{F(a,\gamma)} \int_{\mathbb{R}^n} |f|^2 \, dx \\
&\quad\quad - 8\gamma a \int_{\mathbb{R}^n} \Im(f (x+\mathbf{b})^t B \overline{\nabla_A f}) \, dx.
\end{aligned} \tag{3.3.9}$$

Thanks to hypotesis (3.1.7) and the fact that B is anti-symmetric we have

$$f(x+\mathbf{b})^t B \overline{\nabla_A f} = f x^t B \overline{\nabla_A f} = f x^t B \overline{\left(\nabla_A f + i \frac{\dot{\mathbf{b}}}{2} f + i \left(\frac{\dot a}{2a} + \frac{\dot\gamma}{4\gamma} \right) (x+\mathbf{b}) f \right)},$$

for almost all $x \in \mathbb{R}^n, t \in [0,1]$.

By use of the elementary inequality $ab \leqslant \delta a^2 + b^2/4\delta$, with $\delta = 1/4$, we can finally estimate

$$\begin{aligned}
&8\gamma a \, \Im \int_{\mathbb{R}^n} f(x+\mathbf{b})^t B \overline{\nabla_A f} \, dx \\
&\quad = 8\gamma a \Re \int_{\mathbb{R}^n} f x^t B \overline{\left(-i\nabla_A f + \frac{\dot{\mathbf{b}}}{2} f + \left(\frac{\dot a}{2a} + \frac{\dot\gamma}{4\gamma} \right) (x+\mathbf{b}) f \right)} \, dx \\
&\quad \leqslant 2\gamma a \|x^t B\|^2_{L^\infty} \int_{\mathbb{R}^n} |f|^2 \, dx \\
&\quad\quad + 8\gamma a \int_{\mathbb{R}^n} \left| -i\nabla_A f + \frac{\dot{\mathbf{b}}}{2} f + \left(\frac{\dot a}{2a} + \frac{\dot\gamma}{4\gamma} \right) (x+\mathbf{b}) f \right|^2 dx,
\end{aligned}$$

which proves the result. □

We now choose

$$\gamma := a^{-1},$$

hence

$$F(a) := F(a,\gamma) = \frac{1}{a}\left(\ddot a + 32a^3 - \frac{3\dot a^2}{2a}\right).$$

The next result is the key ingredient in the proof of our main Theorem 3.1.3. Its magnetic-free version $B \equiv 0$ has been proved in [34].

Lemma 3.3.3 (improved decay). *Let $u \in C([-1,1], L^2(\mathbb{R}^n))$ be a solution to*

$$\partial_t u = i(\Delta_A u + V(x,t)u) \qquad \text{in } \mathbb{R}^n \times [-1,1],$$

with V a bounded complex-valued potential and $A \in \mathcal{C}^{1,\varepsilon}_{loc}(\mathbb{R}^n)$. Assume that, for some $\mu > 0$,

$$\sup_{t\in[-1,1]} \|e^{\mu|x|^2}u(t)\| < +\infty, \tag{3.3.10}$$

and, for $a\colon [-1,1] \to (0,+\infty)$, smooth, even and such that $\dot a \leqslant 0$, $a(1) = \mu$, $a \geqslant \mu$, and $F(a) > 0$ in $[-1,1]$, we have

$$\sup_{t\in[-1,1]} \|e^{(a(t)-\varepsilon)|x|^2}u(t)\| < +\infty \qquad \text{for all } \varepsilon > 0. \tag{3.3.11}$$

Then, for $\mathbf{b} = \mathbf{b}(t) = b(t)\xi\colon [-1,1] \to \mathbb{R}^n$, smooth, such that $\mathbf{b}(-1) = \mathbf{b}(1) = 0$,

$$\|e^{a(t)|x+\mathbf{b}(t)|^2}u(t)\| \leqslant e^{T(t)+2\|V\|_{L^\infty}+\frac{\|V\|^2_{L^\infty}}{4}} \sup_{s\in[-1,1]} \|e^{\mu|x|^2 u}(s)\|, \quad -1 \leqslant t \leqslant 1, \tag{3.3.12}$$

where T is defined by

$$\begin{cases} \partial_t\left(\frac{1}{a}\partial_t T\right) = -\left(\frac{|\ddot{\mathbf{b}}|^2}{F(a)} + 2\|x^t B\|^2_{L^\infty}\right) & \text{in } [-1,1] \\ T(-1) = T(1) = 0. \end{cases}$$

Moreover, there is $C_a > 0$ such that

$$\begin{aligned} &\|\sqrt{1-t^2}\nabla_A(e^{a+\frac{i\dot a}{8a}|x|^2}u)\|_{L^2(\mathbb{R}^n\times[-1,1])} \\ &+ C_a\|\sqrt{1-t^2}e^{a(t)|x|^2}\nabla_A u\|_{L^2(\mathbb{R}^n\times[-1,1])} \\ &\leqslant e^{2\|V\|_{L^\infty}+\frac{\|V\|^2_{L^\infty}}{4}} \sup_{t\in[-1,1]} \|e^{\mu|x|^2}u(t)\|. \end{aligned} \tag{3.3.13}$$

Proof. Extend u to $\mathbb{R}^{n+1}$ as $u \equiv 0$ when $|t| > 1$ and, for $\varepsilon > 0$, set

$$a_\varepsilon(t) := a(t) - \varepsilon, \quad g_\varepsilon(x,t) = e^{a_\varepsilon(t)|x|^2}u(x,t), \quad f_\varepsilon(x,t) = e^{a_\varepsilon(t)|x+\mathbf{b}(t)|^2}u(x,t).$$

The function f_ε is in $L^\infty([-1,1], L^2(\mathbb{R}^n))$ and satisfies

$$\partial_t f_\varepsilon - \mathcal{S}_\varepsilon f_\varepsilon - \mathcal{A}_\varepsilon f_\varepsilon = iV(x,t)f_\varepsilon$$

in the sense of distribution, i.e.

$$\int_{\mathbb{R}^n} f_\varepsilon \overline{(-\partial_s \zeta - \mathcal{S}_\varepsilon \zeta + \mathcal{A}_\varepsilon \zeta)}\, dyds = i \int_{\mathbb{R}^n} V f_\varepsilon \bar{\zeta}\, dyds$$

for all $\zeta \in C_0^\infty(\mathbb{R}^n \times (-1,1))$, where $\mathcal{S}_\varepsilon$ and $\mathcal{A}_\varepsilon$ are defined as $\mathcal{S}$ and $\mathcal{A}$ are in (3.3.3), (3.3.4) with a_ε in place of a. We denote here $\mathcal{S}_\varepsilon^{x,t}$, $\mathcal{A}_\varepsilon^{x,t}$ and $\mathcal{S}_\varepsilon^{y,s}$, $\mathcal{A}_\varepsilon^{y,s}$ the operators acting on the variables x, t and y, s respectively.

Since all the previous results make sense for regular functions, the strategy is to mollify the function f_ε, obtain results for the new regular function, and uniformly control the errors. Let then $\theta \in C^\infty(\mathbb{R}^{n+1})$ be a standard mollifier supported in the unit ball of $\mathbb{R}^{n+1}$ and, for $0 < \delta \leqslant \frac{1}{4}$ set $g_{\varepsilon,\delta} = g_\varepsilon * \theta_\delta$, $f_{\varepsilon,\delta} = f_\varepsilon * \theta_\delta$, and

$$\theta_\delta^{x,t}(y,s) = \delta^{-n-1}\theta\left(\frac{x-y}{\delta}, \frac{t-s}{\delta}\right).$$

The functions $f_{\varepsilon,\delta}$ and $g_{\varepsilon,\delta}$ are in $C^\infty([-1,1], \mathcal{S}(\mathbb{R}^n))$.

By continuity, there exists $\varepsilon_a > 0$ such that

$$F(a_\varepsilon) \geqslant \frac{F(a)}{2}, \qquad \text{in } [-1,1],$$

when $0 < \varepsilon \leqslant \varepsilon_a$, and for such an $\varepsilon > 0$ it is possible to find $\delta_\varepsilon > 0$, with δ_ε approaching zero as ε tends to zero, such that

$$\left(a(t) - \frac{\varepsilon}{2}\right)|x|^2 \leqslant \mu|x|^2, \quad \left(a(t) - \frac{\varepsilon}{2}\right)|x + \mathbf{b}(t)|^2 \leqslant \mu|x|^2,$$

when $x \in \mathbb{R}^n$, $1 - \delta_\varepsilon \leqslant |t| \leqslant 1$. In the following we assume $0 < \varepsilon \leqslant \varepsilon_a$ and $0 < \delta \leqslant \delta_\varepsilon$.

We can apply Lemma 3.3.1 to $f_{\varepsilon,\delta}$, with $H_{\varepsilon,\delta}(t) = \|f_{\varepsilon,\delta}(t)\|^2$, $[c,d] = [-1+\delta_\varepsilon, 1-\delta_\varepsilon]$, $\gamma = a_\varepsilon^{-1}$, $\mathcal{S} = \mathcal{S}_\varepsilon$ and $\mathcal{A} = \mathcal{A}_\varepsilon$: it turns out that

$$H_{\varepsilon,\delta}(t) \leqslant \left(\sup_{t\in[-1,1]} \|e^{\mu|x|^2}u(t)\| + \varepsilon\right)^2 e^{2T_\varepsilon(t) + M_{\varepsilon,\delta}(t) + 2N_{\varepsilon,\delta}(t)} \tag{3.3.14}$$

when $|t| \leqslant 1 - \delta_\varepsilon$, and where T_ε, $M_{\varepsilon,\delta}$ and $N_{\varepsilon,\delta}$ are defined by

$$\begin{cases} \partial_t(\frac{1}{a_\varepsilon}\partial_t T_\varepsilon) = -\frac{|\ddot{\mathbf{b}}|^2}{F(a_\varepsilon)} - 2\|x^t B\|_{L^\infty}^2, & t \in [-1+\delta_\varepsilon, 1-\delta_\varepsilon] \\ T(-1+\delta_\varepsilon) = T(1-\delta_\varepsilon) = 0 \end{cases}, \tag{3.3.15}$$

$$\begin{cases} \partial_t(\frac{1}{a_\varepsilon}\partial_t M_{\varepsilon,\delta}) = -\frac{1}{a_\varepsilon}\frac{\|\partial_t f_{\varepsilon,\delta} - \mathcal{S}_\varepsilon f_{\varepsilon,\delta} - \mathcal{A}_\varepsilon f_{\varepsilon,\delta}\|^2}{H_{\varepsilon,\delta}+\varepsilon}, & t \in [-1+\delta_\varepsilon, 1-\delta_\varepsilon] \\ M_{\varepsilon,\delta}(-1+\delta_\varepsilon) = M_{\varepsilon,\delta}(1-\delta_\varepsilon) = 0 \end{cases}, \tag{3.3.16}$$

$$N_{\varepsilon,\delta} = \int_{-1+\delta_\varepsilon}^{1-\delta_\varepsilon} \frac{\|(\partial_s - \mathcal{S}_\varepsilon - \mathcal{A}_\varepsilon)f_{\varepsilon,\delta}(s)\|}{\sqrt{H_{\varepsilon,\delta}(s)+\varepsilon}}\, ds, \tag{3.3.17}$$

In view to let $\delta \to 0$ in (3.3.14), (3.3.15), (3.3.16), (3.3.17), we compute

$$\begin{aligned} &(\partial_t f_{\varepsilon,\delta} - \mathcal{S}_\varepsilon^{x,t} f_{\varepsilon,\delta} - \mathcal{A}_\varepsilon^{x,t} f_{\varepsilon,\delta})(x,t) \\ &= \int_{\mathbb{R}^n} f_\varepsilon(y,s)\overline{(-\partial_s\theta_\delta^{x,t}(y,s))}\, dyds + \int_{\mathbb{R}^n} (-\mathcal{S}_\varepsilon^{x,t} - \mathcal{A}_\varepsilon^{x,t}) f_\varepsilon(y,s)\theta_\delta^{x,t}(y,s)\, dyds \\ &= \int_{\mathbb{R}^n} f_\varepsilon(y,s)\overline{(-\partial_s - \mathcal{S}_\varepsilon^{y,s} + \mathcal{A}_\varepsilon^{y,s})\theta_\delta^{x,t}(y,s)}\, dyds \\ &\quad + \int_{\mathbb{R}^n} f_\varepsilon(y,s)(-\mathcal{S}_\varepsilon^{x,t} - \mathcal{A}_\varepsilon^{x,t} + \overline{\mathcal{S}_\varepsilon^{y,s} - \mathcal{A}_\varepsilon^{y,s}})\theta_\delta^{x,t}(y,s)\, dyds. \end{aligned}$$

Expliciting the term in the previous relation, we get

$$\begin{aligned}
&(\partial_t f_{\varepsilon,\delta} - \mathcal{S}_\varepsilon^{x,t} f_{\varepsilon,\delta} - \mathcal{A}_\varepsilon^{x,t} f_{\varepsilon,\delta})(x,t)\\
&= \int_{\mathbb{R}^n} f_\varepsilon(y,s)\overline{(-\partial_s - \mathcal{S}_\varepsilon^{y,s} + \mathcal{A}_\varepsilon^{y,s})\theta_\delta^{x,t}(y,s)}\, dyds\\
&\quad + \int_{\mathbb{R}^n} f_\varepsilon(y,s)[(\dot a_\varepsilon(s) + 4ia_\varepsilon^2(s))|y+\mathbf{b}(s)|^2 - (\dot a_\varepsilon(t) + 4ia_\varepsilon^2(t))|x+\mathbf{b}(t)|^2]\theta_\delta^{x,t}(y,s)\, dyds\\
&\quad + \int_{\mathbb{R}^n} f_\varepsilon(y,s)[2a_\varepsilon(s)\dot{\mathbf{b}}(s)\cdot(y+\mathbf{b}(s)) - 2a_\varepsilon(t)\dot{\mathbf{b}}(t)\cdot(x+\mathbf{b}(t))]\theta_\delta^{x,t}(y,s)\, dyds\\
&\quad + 4i\int_{\mathbb{R}^n} f_\varepsilon(y,s)[a_\varepsilon(s)(y+\mathbf{b}(s))\cdot\overline{\nabla_{A,y}} + a_\varepsilon(t)(x+\mathbf{b}(t))\cdot\nabla_{A,x}]\theta_\delta^{x,t}(y,s)\, dyds\\
&\quad + \int_{\mathbb{R}^n} 2in f_\varepsilon(y,s)[a_\varepsilon(s) + a_\varepsilon(t)]\theta_\delta^{x,t}(y,s)\, dyds\\
&\quad - i\int_{\mathbb{R}^n} f_\varepsilon(y,s)[\Delta_{A,x} - \overline{\Delta_{A,y}}]\theta_\delta^{x,t}(y,s)\, dyds =: I + II + II + IV + V + VI.
\end{aligned} \tag{3.3.18}$$

Since $\nabla_x\theta_\delta^{x,t}(y,s) = -\nabla_y\theta_\delta^{x,t}(y,s)$, we have

$$\begin{aligned}
IV =& 4i\int_{\mathbb{R}^n} f_\varepsilon(y,s)[a_\varepsilon(s)(y+\mathbf{b}(s)) - a_\varepsilon(t)(x+\mathbf{b}(t))]\cdot\nabla_y\theta_\delta^{x,t}(y,s)]\, dyds\\
&+ 4\int_{\mathbb{R}^n} f_\varepsilon(y,s)[-a_\varepsilon(s)(y+\mathbf{b}(s))\cdot A(y) + a_\varepsilon(t)(x+\mathbf{b}(t))\cdot A(x)]\theta_\delta^{x,t}(y,s)\, dyds.
\end{aligned} \tag{3.3.19}$$

Moreover, recalling that

$$\Delta_A f = \nabla_A^2 f = \Delta f - i(\nabla\cdot A)f - 2iA\cdot\nabla f - |A|^2 f,$$

and $\Delta_y\theta_\delta^{x,t}(y,s) = \Delta_x\theta_\delta^{x,t}(y,s)$, we obtain

$$\begin{aligned}
VI = \int_{\mathbb{R}^n} f_\varepsilon(y,s)\Big[&-(\nabla_x\cdot A(x) + \nabla_y\cdot A(y)) + 2(A(x) - A(y))\cdot\nabla_y\\
&+ i(|A(x)|^2 - |A(y)|^2)\Big]\theta_\delta^{x,t}(y,s)\, dyds
\end{aligned} \tag{3.3.20}$$

By (3.3.18), (3.3.19), (3.3.20) we can hence write

$$(\partial_t f_{\varepsilon,\delta} - \mathcal{S}_\varepsilon^{x,t} f_{\varepsilon,\delta} - \mathcal{A}_\varepsilon^{x,t} f_{\varepsilon,\delta})(x,t) = i(Vf_\varepsilon)*\theta_\delta(x,t) + A_{\varepsilon,\delta}(x,t) + B_{\varepsilon,\delta}(x,t), \tag{3.3.21}$$

where

$$\begin{aligned}
&A_{\varepsilon,\delta}(x,t)\\
&= \int_{\mathbb{R}^n} f_\varepsilon(y,s)[(\dot a_\varepsilon(s) + 4ia_\varepsilon^2(s))|y+\mathbf{b}(s)|^2 - (\dot a_\varepsilon(t) + 4ia_\varepsilon^2(t))|x+\mathbf{b}(t)|^2]\theta_\delta^{x,t}(y,s)\, dyds\\
&\quad + \int_{\mathbb{R}^n} f_\varepsilon(y,s)[2a_\varepsilon(s)\dot{\mathbf{b}}(s)\cdot(y+\mathbf{b}(s)) - 2a_\varepsilon(t)\dot{\mathbf{b}}(t)\cdot(x+\mathbf{b}(t))]\theta_\delta^{x,t}(y,s)\, dyds\\
&\quad + 4\int_{\mathbb{R}^n} f_\varepsilon(y,s)[a_\varepsilon(t)(x+\mathbf{b}(t))\cdot A(x) - a_\varepsilon(s)(y+\mathbf{b}(s))\cdot A(y)]\theta_\delta^{x,t}(y,s)\, dyds\\
&\quad + i\int_{\mathbb{R}^n} f_\varepsilon(y,s)[|A(x)|^2 - |A(y)|^2]\theta_\delta^{x,t}(y,s)\, dyds,
\end{aligned}$$

and

$$\begin{aligned}
&B_{\varepsilon,\delta}(x,t)\\
&= 4i\int_{\mathbb{R}^n} f_\varepsilon(y,s)[4i(a_\varepsilon(s)(y+\mathbf{b}(s)) - a_\varepsilon(t)(x+\mathbf{b}(t))) + 2(A(x)-A(y))]\cdot\nabla_y\theta_\delta^{x,t}(y,s)\,dyds\\
&\quad+\int_{\mathbb{R}^n} f_\varepsilon(y,s)[2in(a_\varepsilon(s)+a_\varepsilon(t)) - (\nabla_x\cdot A(x)+\nabla_y\cdot A(y))]\theta_\delta^{x,t}(y,s)\,dyds.
\end{aligned}$$

Since a_ε, $\mathbf{b}$ are smooth, and $A\in\mathcal{C}^{1,\varepsilon}_{\rm loc}(\mathbb{R}^n)$, there is a $N_{a,\mathbf{b},A,\varepsilon}>0$ such that

$$\|A_{\varepsilon,\delta}\|_{L^2(\mathbb{R}^n\times[-1+\delta,1-\delta])} \leqslant \delta N_{a,\mathbf{b},A,\varepsilon}\sup_{t\in[-1,1]}\|e^{(a(t)-\frac{\varepsilon}{2})|x|^2}u(t)\|, \tag{3.3.22}$$

$$\|B_{\varepsilon,\delta}\|_{L^2(\mathbb{R}^n\times[-1+\delta,1-\delta])} \leqslant N_{a,\mathbf{b},A,\varepsilon}\sup_{t\in[-1,1]}\|e^{(a(t)-\frac{\varepsilon}{2})|x|^2}u(t)\|. \tag{3.3.23}$$

Moreover

$$\sup_{t\in[-1,1]}\|(Vf_\varepsilon)*\theta_\delta(t)\| \leqslant \|V\|_{L^\infty(\mathbb{R}^n\times[-1,1])}\sup_{t\in[-1,1]}\|e^{(a(t)-\frac{\varepsilon}{2})|x|^2}u(t)\|. \tag{3.3.24}$$

The function $g_{\varepsilon,\delta}$ verifies analogous relations, obtained setting $\mathbf{b}\equiv 0$ in the previous equations.

Since the $f_{\varepsilon,\delta}$ and $g_{\varepsilon,\delta}$ are now regular, (3.3.2) holds. Therefore,

$$\begin{aligned}
&\partial_t\left(\frac{1}{a_\varepsilon}\,\partial_t H_{\varepsilon,\delta} - \frac{1}{a_\varepsilon}\,\Re(\partial_t g_{\varepsilon,\delta} - \mathcal{S}_\varepsilon g_{\varepsilon,\delta} - \mathcal{A}_\varepsilon g_{\varepsilon,\delta}, g_{\varepsilon,\delta})\right)\\
&\quad+\frac{1}{a_\varepsilon}\,\|\partial_t g_{\varepsilon,\delta} - \mathcal{S}_\varepsilon g_{\varepsilon,\delta} - \mathcal{A}_\varepsilon g_{\varepsilon,\delta}\|^2\\
&\geqslant 2\left(\frac{1}{a_\varepsilon}\mathcal{S}_{\varepsilon t} g_{\varepsilon,\delta} + \frac{1}{a_\varepsilon}\,[\mathcal{S}_\varepsilon,\mathcal{A}_\varepsilon]g_{\varepsilon,\delta} - \frac{\dot a_\varepsilon}{a_\varepsilon^2}\,\mathcal{S}_\varepsilon g_{\varepsilon,\delta}, g_{\varepsilon,\delta}\right).
\end{aligned} \tag{3.3.25}$$

Moreover, from (3.3.9) in Lemma 3.3.2, with $\gamma = 1/a_\varepsilon$ and $\mathbf{b}\equiv 0$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\frac{1}{a_\varepsilon}\mathcal{S}_{\varepsilon t} + \frac{1}{a_\varepsilon}[\mathcal{S}_\varepsilon,\mathcal{A}_\varepsilon] - \frac{\dot a_\varepsilon}{a_\varepsilon^2}\mathcal{S}_\varepsilon\right) g_{\varepsilon,\delta}\bar g_{\varepsilon,\delta}\,dx\\
&= \int_{\mathbb{R}^n} 8\left|-i\nabla_A g_{\varepsilon,\delta} + \left(\frac{\dot a_\varepsilon}{4a_\varepsilon}\right) x g_{\varepsilon,\delta}\right|^2 dx + \int_{\mathbb{R}^n} F(a_\varepsilon)|x|^2|g_{\varepsilon,\delta}|^2\,dx\\
&\quad - 8\int_{\mathbb{R}^n}\Im(g_{\varepsilon,\delta}\,x^t B\overline{\nabla_A g_{\varepsilon,\delta}})\,dx.
\end{aligned} \tag{3.3.26}$$

Since $F(a_\varepsilon)>0$, there exists a constant $C>0$ depending on a, such that we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} 8\left|-i\nabla_A g_{\varepsilon,\delta} + \left(\frac{\dot a_\varepsilon}{4a_\varepsilon}\right) x g_{\varepsilon,\delta}\right|^2 dx + \int_{\mathbb{R}^n} F(a_\varepsilon)|x|^2|g_{\varepsilon,\delta}|^2\,dx\\
&\geqslant \int_{\mathbb{R}^n}\left|\nabla_A\left(e^{\frac{i\dot a_\varepsilon}{8a_\varepsilon}|x|^2} g_{\varepsilon,\delta}\right)\right|^2 dx + C_a\int_{\mathbb{R}^n}|\nabla_A g_{\varepsilon,\delta}|^2 + |x|^2|g_{\varepsilon,\delta}|^2\,dx.
\end{aligned} \tag{3.3.27}$$

Moreover there exists an arbitrarily small $\eta>0$ such that

$$-8\int_{\mathbb{R}^n}\Im(g_{\varepsilon,\delta}\,x^t B\overline{\nabla_A g_{\varepsilon,\delta}})\,dx \geqslant -\frac{16}{\eta}\|x^t B\|^2_{L^\infty}\int_{\mathbb{R}^n}|g_{\varepsilon,\delta}|^2\,dx - \eta\int_{\mathbb{R}^n}|\nabla_A g_{\varepsilon,\delta}|^2\,dx. \tag{3.3.28}$$

By (3.3.25), (3.3.26), (3.3.27), (3.3.28), we get

$$\begin{aligned}\int_{\mathbb{R}^n} \left|\nabla_A \left(e^{\frac{i\dot{a}_\varepsilon}{8a_\varepsilon}|x|^2} g_{\varepsilon,\delta}\right)\right|^2 \, dx + C \int_{\mathbb{R}^n} |\nabla_A g_{\varepsilon,\delta}|^2 + |x|^2 |g_{\varepsilon,\delta}|^2 \, dx \\ \leqslant \partial_t \left(\frac{1}{a_\varepsilon} \partial_t H_{\varepsilon,\delta} - \frac{1}{a_\varepsilon} \Re(\partial_t g_{\varepsilon,\delta} - \mathcal{S}_\varepsilon g_{\varepsilon,\delta} - \mathcal{A}_\varepsilon g_{\varepsilon,\delta}, g_{\varepsilon,\delta})\right) \\ + \frac{1}{a_\varepsilon} \|\partial_t g_{\varepsilon,\delta} - \mathcal{S}_\varepsilon g_{\varepsilon,\delta} - \mathcal{A}_\varepsilon g_{\varepsilon,\delta}\|^2 + D\|x^t B\|_{L^\infty}^2 H_{\varepsilon,\delta},\end{aligned} \tag{3.3.29}$$

for some constants $C, D > 0$ depending on a. Multiplying the last inequality by $(1-\delta_\varepsilon)^2 - t^2$, and integrating by part in time, we get

$$\|\sqrt{(1-\delta_\varepsilon)^2 - t^2} \nabla_A g_{\varepsilon,\delta}\|_{L^2(\mathbb{R}^n \times [-1+\delta_\varepsilon, 1-\delta_\varepsilon])} \leqslant N_{a,B,\varepsilon},$$

and analogously

$$\|\sqrt{(1-\delta_\varepsilon)^2 - t^2} \nabla_A f_{\varepsilon,\delta}\|_{L^2(\mathbb{R}^n \times [-1+\delta_\varepsilon, 1-\delta_\varepsilon])} \leqslant N_{a,\mathbf{b},B,\varepsilon},$$

thanks to (3.3.21), (3.3.22), and (3.3.23). Letting δ tend to zero, we find that

$$\|\sqrt{(1-\delta_\varepsilon)^2 - t^2} \nabla_A f_\varepsilon\|_{L^2(\mathbb{R}^n \times [-1+\delta_\varepsilon, 1-\delta_\varepsilon])} \leqslant N_{a,\mathbf{b},B,\varepsilon},$$

which makes possibile to integrate in time by parts the first term in $B_{\varepsilon,\delta}$, obtaining

$$\begin{aligned}B_{\varepsilon,\delta}(x,t) = &- \int_{\mathbb{R}^n} \nabla_y f_\varepsilon(y,s) \cdot [4i(a_\varepsilon(s)(y + \mathbf{b}(s)) - a_\varepsilon(t)(x + \mathbf{b}(t)))]\theta_\delta^{x,t}(y,s) \, dyds \\ &- \int_{\mathbb{R}^n} \nabla_y f_\varepsilon(y,s) \cdot [2(A(x) - A(y))]\theta_\delta^{x,t}(y,s) \, dyds \\ &+ \int_{\mathbb{R}^n} f_\varepsilon(y,s)[2in(a_\varepsilon(t) - a_\varepsilon(s)) + (\nabla_y \cdot A(y) - \nabla_x \cdot A(x))]\theta_\delta^{x,t}(y,s) \, dyds.\end{aligned}$$

This, together with the fact that $A \in \mathcal{C}^{1,\varepsilon}_{\text{loc}}(\mathbb{R}^n)$, allows to get finally

$$\|B_{\varepsilon,\delta}\|_{L^2(\mathbb{R}^n, [-1+\delta_\varepsilon, 1-\delta_\varepsilon])} \leqslant \delta N_{a,\mathbf{b},A,\varepsilon}, \tag{3.3.30}$$

when $0 < \delta \leqslant \delta_\varepsilon$, which improves (3.3.23).

Thanks to the above convergence results, we have that f_ε is in $C^\infty((-1,1), L^2(\mathbb{R}^n))$ and that $H_{\varepsilon,\delta}$ converges uniformly on compact sets of $(-1,1)$ to $H_\varepsilon(t) = \|f_\varepsilon(t)\|^2$. Letting δ and ε tend to zero, we get finally

$$\|e^{a(t)|x+\mathbf{b}(t)|^2} u(t)\|^2 \leqslant \sup_{t \in [-1,1]} \|e^{\mu|x|^2} u(t)\| e^{2T(t) + M(t) + 4\|V\|_{L^\infty}}$$

when $|t| \leqslant 1$, with

$$\begin{cases} \partial_t \left(\frac{1}{a} \partial_t M\right) = -\frac{1}{a}\|V\|_{L^\infty}^2 \\ M(-1) = M(1) = 0. \end{cases}$$

Notice that M is even, and

$$M(t) = \|V\|_{L^\infty}^2 \int_t^1 \int_0^s \frac{a(s)}{a(\tau)} \, d\tau ds, \quad \text{in } [0,1],$$

and, since a is monotone in $[0,1]$, we get the (3.3.12). Using again (3.3.27), analogously we have

$$\begin{aligned}
&\|\sqrt{(1-\delta_\varepsilon)^2-t^2}\nabla_A(e^{\frac{i\dot{a}_\varepsilon}{8a_\varepsilon}|x|^2}g_{\varepsilon,\delta})\|_{L^2(\mathbb{R}^n\times[-1+\delta_\varepsilon,1-\delta_\varepsilon])}\\
&+C_a\|\sqrt{(1-\delta_\varepsilon)^2-t^2}\nabla_A g_{\varepsilon,\delta}\|_{L^2(\mathbb{R}^n\times[-1+\delta_\varepsilon,1-\delta_\varepsilon])}\\
&+C_a\|\sqrt{(1-\delta_\varepsilon)^2-t^2}x g_{\varepsilon,\delta}\|_{L^2(\mathbb{R}^n\times[-1+\delta_\varepsilon,1-\delta_\varepsilon])}\\
&\leqslant Ce^{2\|V\|_{L^\infty}+\frac{\|V\|^2_{L^\infty}}{4}}\sup_{s\in[-1,1]}\|e^{\mu|x|^2}u(s)\|+\delta N_{a,\varepsilon,A,B},
\end{aligned}$$

for $C=C(\|V\|_\infty,\|x^tB\|_\infty)$. Letting δ and ε go to zero, we get (3.3.13) and we conclude the proof. □

3.4 Proof of Theorem 3.1.3

For convenience, we will denote by

$$M_B:=2\|x^tB\|^2_{L^\infty}<+\infty, \tag{3.4.1}$$

$$M_V:=2\|V\|_{L^\infty}+\frac{\|V\|^2}{4}<+\infty. \tag{3.4.2}$$

The proof is divided into several steps.

3.4.1 Crönstrom gauge

The first step consists in reducing to the Crönostrom gauge

$$x\cdot A(x)=0 \qquad \text{for all } x\in\mathbb{R}^n,$$

by means of the following result.

Lemma 3.4.1. *Let $A=A(x)=(A^1(x),\dots,A^n(x)):\mathbb{R}^n\to\mathbb{R}^n$, for $n\geqslant 2$ and denote by $B=DA-DA^t\in\mathcal{M}_{n\times n}(\mathbb{R})$, $B_{jk}=A^k_j-A^j_k$, and $\Psi(x):=x^tB(x)\in\mathbb{R}^n$. Assume that the two vector quantities*

$$\int_0^1 A(sx)\,ds\in\mathbb{R}^n, \qquad \int_0^1\Psi(sx)\,ds\in\mathbb{R}^n \tag{3.4.3}$$

are finite, for almost every $x\in\mathbb{R}^n$; moreover, define the (scalar) function

$$\varphi(x):=x\cdot\int_0^1 A(sx)\,ds\in\mathbb{R}. \tag{3.4.4}$$

Then, the following two identities hold:

$$\widetilde{A}(x):=A(x)-\nabla\varphi(x)=-\int_0^1\Psi(sx)\,ds \tag{3.4.5}$$

$$x^tD\widetilde{A}(x)=-\Psi(x)+\int_0^1\Psi(sx)\,ds. \tag{3.4.6}$$

Remark 3.4.2. Notice that

$$x \cdot \widetilde{A}(x) \equiv 0, \qquad x \cdot x^t D\widetilde{A}(x) \equiv 0. \tag{3.4.7}$$

From now on, we will hence assume, without loss of generality, that (3.4.7) are satisfied by A. Observe moreover that assumption (3.1.7) in Theorem 3.1.3 is preserved by the above gauge transformation, and we have in addition that $A \cdot \xi \equiv 0$. We also remark that

$$\|\tilde{A}\|_{L^\infty}^2 + \|x^t B\|_{L^\infty}^2 \leqslant M_B.$$

Finally notice that the first condition in (3.4.3) is guaranteed by the assumption $A \in \mathcal{C}^{1,\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$ in Theorem 3.1.3.

We mention [48] for the proof of the previous Lemma; see alternatively Lemma 2.2 in [2].

3.4.2 Appell Transformation

Following the strategy in [32, 34, 2], the second step is to reduce assumption (3.1.11) to the case $\alpha = \beta$, by pseudoconformal transformation (Appell transformation).

Lemma 3.4.3 ([2], Lemma 2.7)**.** *Let $A = A(y,s) = (A^1(y,s),\dots,A^n(y,s)) : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $V = V(y,s)$, $F = F(y,s) : \mathbb{R}^n \to \mathbb{C}$, $u = u(y,s) : \mathbb{R}^n \times [0,1] \to \mathbb{C}$ be a solution to*

$$\partial_s u = i\left(\Delta_A u + V(y,s)u + F(y,s)\right), \tag{3.4.8}$$

and define, for any $\alpha, \beta > 0$, the function

$$\widetilde{u}(x,t) := \left(\frac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}\right)^{\frac{n}{2}} u\left(\frac{x\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}, \frac{t\beta}{\alpha(1-t)+\beta t}\right) e^{\frac{(\alpha-\beta)|x|^2}{4i(\alpha(1-t)+\beta t)}}. \tag{3.4.9}$$

Then $\widetilde{u}$ is a solution to

$$\partial_t \widetilde{u} = i\left(\Delta_{\widetilde{A}}\widetilde{u} + \frac{(\alpha-\beta)\widetilde{A}\cdot x}{(\alpha(1-t)+\beta t)}\widetilde{u} + \widetilde{V}(x,t)\widetilde{u} + \widetilde{F}(x,t)\right), \tag{3.4.10}$$

where

$$\widetilde{A}(x,t) = \frac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t} A\left(\frac{x\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}, \frac{t\beta}{\alpha(1-t)+\beta t}\right) \tag{3.4.11}$$

$$\widetilde{V}(x,t) = \frac{\alpha\beta}{(\alpha(1-t)+\beta t)^2} V\left(\frac{x\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}, \frac{t\beta}{\alpha(1-t)+\beta t}\right) \tag{3.4.12}$$

$$\widetilde{F}(x,t) = \left(\frac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}\right)^{\frac{n}{2}+2} F\left(\frac{x\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}, \frac{t\beta}{\alpha(1-t)+\beta t}\right) e^{\frac{(\alpha-\beta)|x|^2}{4i(\alpha(1-t)+\beta t)}}. \tag{3.4.13}$$

Remark 3.4.4. The term containing $\tilde{A} \cdot x$ vanishes (see Remark 3.4.2 above). Moreover, assumptions (3.4.1) and (3.4.2) still hold for $\tilde{B}$ and $\tilde{V}$. We finally remark that $\widetilde{A}$ is time-dependent. Nevertheless, notice that

$$x \cdot \widetilde{A}_t(x) = 0, \quad \xi \cdot \tilde{A}_t(x) = 0, \tag{3.4.14}$$

for all $x \in \mathbb{R}^n, t \in [0,1]$.

By direct computations, we have

$$\left\|e^{\frac{|\cdot|^2}{\alpha\beta}}\widetilde{u}(0)\right\|_{L^2}=\left\|e^{\frac{|\cdot|^2}{\beta^2}}u(0)\right\|_{L^2},\quad \left\|e^{\frac{|\cdot|^2}{\alpha\beta}}\widetilde{u}(1)\right\|_{L^2}=\left\|e^{\frac{|\cdot|^2}{\alpha^2}}u(1)\right\|_{L^2},$$
$$\sup_{t\in[0,1]}\left\|e^{\frac{|\cdot|^2}{\alpha\beta}}\widetilde{u}(t)\right\|_{L^2}=\sup_{t\in[0,1]}\left\|e^{\frac{|\cdot|^2}{(\alpha t+\beta(1-t))^2}}u(t)\right\|_{L^2}.$$

For convenience, we change the time interval in $[-1,1]$: let $v(x,t)=2^{-\frac{n}{4}}\widetilde{u}\left(\frac{x}{\sqrt{2}},\frac{1+t}{2}\right)$. The function v is a solution to

$$\partial_t v=i(\Delta_{\mathcal{A}}v+\mathcal{V}v),\qquad \text{in } \mathbb{R}^n\times[-1,1],$$

with

$$\mathcal{A}(x,t)=\frac{1}{\sqrt{2}}A\left(\frac{x}{\sqrt{2}},\frac{1+t}{2}\right),\quad \mathcal{V}(x,t)=\frac{1}{2}V\left(\frac{x}{\sqrt{2}},\frac{1+t}{2}\right).$$

The assumptions of Theorem 3.1.3 still hold (up to a change of the constants) and moreover

$$\left\|e^{\frac{|\cdot|^2}{2\alpha\beta}}v(-1)\right\|_{L^2}=\left\|e^{\frac{|\cdot|^2}{\alpha\beta}}\widetilde{u}(0)\right\|_{L^2}=\left\|e^{\frac{|\cdot|^2}{\beta^2}}u(0)\right\|_{L^2},$$
$$\left\|e^{\frac{|\cdot|^2}{2\alpha\beta}}v(1)\right\|_{L^2}=\left\|e^{\frac{|\cdot|^2}{\alpha\beta}}\widetilde{u}(1)\right\|_{L^2}=\left\|e^{\frac{|\cdot|^2}{\alpha^2}}u(1)\right\|_{L^2},$$
$$\sup_{t\in[-1,1]}\left\|e^{\frac{|\cdot|^2}{2\alpha\beta}}v(t)\right\|_{L^2}=\sup_{t\in[0,1]}\left\|e^{\frac{|\cdot|^2}{\alpha\beta}}\widetilde{u}(t)\right\|_{L^2}=\sup_{t\in[0,1]}\left\|e^{\frac{|\cdot|^2}{(\alpha t+\beta(1-t))^2}}u(t)\right\|_{L^2}.$$

We set

$$\mu:=\frac{1}{2\alpha\beta}. \tag{3.4.15}$$

The basic ingredient of our proof is the following logarithmic convexity estimate:

$$\begin{aligned}\sup_{t\in[-1,1]}\left\|e^{\mu|\cdot|^2}v(t)\right\|_{L^2(\mathbb{R}^n)}&=\sup_{t\in[0,1]}\left\|e^{\frac{|\cdot|^2}{(\alpha t+\beta(1-t))^2}}u(t)\right\|_{L^2(\mathbb{R}^n)}\\&\leqslant C\sup_{t\in[0,1]}\left\|e^{\frac{|\cdot|^2}{\beta^2}}u(\cdot,0)\right\|_{L^2}^{\frac{\beta(1-t)}{\alpha t+\beta(1-t)}}\left\|e^{\frac{|\cdot|^2}{\alpha^2}}u(\cdot,1)\right\|_{L^2}^{\frac{\alpha t}{\alpha t+\beta(1-t)}}\\&\leqslant C\left(\left\|e^{\frac{|\cdot|^2}{\beta^2}}u(\cdot,0)\right\|_{L^2}+\left\|e^{\frac{|\cdot|^2}{\alpha^2}}u(\cdot,1)\right\|_{L^2}\right)<+\infty,\end{aligned} \tag{3.4.16}$$

with

$$C=C\left(\alpha,\beta,\|x^tB\|_{L^\infty},\|V_1\|_{L^\infty},\sup_{t\in[0,1]}\left\|e^{\frac{|\cdot|^2}{(\alpha t+\beta(1-t))^2}}V_2(\cdot,t)\right\|_{L^\infty}e^{\sup_{t\in[0,1]}\|\Im V_2(\cdot,t)\|_{L^\infty}}\right).$$

For the proof of (3.4.16) see Theorem 1.5 in [2]. From now on, we denote v, $\mathcal{A}$ and $\mathcal{V}$ by u, A and V.

We follow the same strategy as in [34], which is based on an iteration scheme. The argument here is a bit more delicate, due to the presence of additional terms involving the magnetic field.

3.4.3 Conclusion of the Proof

We now apply an iteration scheme which is completely analogous to the one performed in [34]. The idea is to get the best possible choice for $a(t)$ such that an estimate like

$$\|e^{a(t)|x|^2}u(x,t)\|_{L_t^\infty([-1,1])L_x^2(\mathbb{R}^n)} \leqslant C(\alpha,\beta,T,\|V\|_{L^\infty},\|x^tB\|_{L^\infty}). \tag{3.4.17}$$

holds. In order to do this, we will construct a as the limit of an appropriate sequence $a_j(t)$, having in mind the improvement result of Lemma 3.3.3. At each step of the procedure, assumptions (3.3.10) and (3.3.11) have to be checked. Also the curve $\mathbf{b}(t) = b(t)\xi$, with $\xi \in \mathbb{S}^{n-1}$ as in (3.1.7) is naturally involved in the following argument.

Iteration scheme

Let us first construct the iteration scheme. Assume that k even and smooth functions $a_j\colon [-1,1] \to (0,+\infty)$ and $C_{a_j} > 0$, $j = 1,\dots,k$ have been generated, such that

$$\begin{cases} \mu \equiv a_1 < a_2 < \cdots < a_k \quad \text{in } (-1,1), \\ \dot a_j \leqslant 0 \text{ in } [0,1], \quad F(a_j) > 0 \text{ in } [-1,1], \quad a_j(\pm 1) = \mu, \\ \sup_{t\in[-1,1]}\|e^{a_j(t)|\cdot|^2}u(\cdot,t)\| \leqslant e^{M_B\int_0^1 s a_j(s)\,ds}e^{M_V}\sup_{t\in[-1,1]}\|e^{\mu|\cdot|^2}u(\cdot,t)\|, \\ \|\sqrt{1-t^2}\nabla_A(e^{a_j+\frac{i\dot a_j}{8a_j}|x|^2}u)\|_{L^2(\mathbb{R}^n\times[-1,1])} + C_{a_j}\|\sqrt{1-t^2}e^{a_j(t)|x|^2}\nabla_A u\|_{L^2(\mathbb{R}^n\times[-1,1])} \\ \quad \leqslant Ce^{M_V}\sup_{t\in[-1,1]}\|e^{\mu|\cdot|^2}u(\cdot,t)\|, \end{cases} \tag{3.4.18}$$

where $C = C(\|V\|_\infty, \|x^tB\|_\infty) > 0$, for all $j = 1,\dots,k$.

The construction is identical to the one in [34]; we repeat it here for the sake of completeness. In order to simplify notations, set $c_k := a_k^{-\frac{1}{2}}$. Let $b_k\colon [-1,1] \to \mathbb{R}$ be the solution to

$$\begin{cases} \ddot b_k = -\frac{F(a_k)}{a_k} = -2c_k(16c_k^{-3} - \ddot c_k) \\ b_k(\pm 1) = 0. \end{cases} \tag{3.4.19}$$

Observe that b_k is even and

$$b_k(t) = \int_t^1\int_0^s \frac{F(a_k(\tau))}{a_k(\tau)}\,d\tau ds \quad \text{in } [-1,1]; \tag{3.4.20}$$

moreover $\dot b_k < 0$ in $(0,1]$. Apply now (3.3.12) in Lemma 3.3.3 with $a = a_k$ and $\mathbf{b} = b_k\eta$, for $\eta \in \mathbb{R}\xi = \{p\xi \mid p \in \mathbb{R}\}$: we get

$$\|e^{a_k(t)|\cdot + b_k(t)\eta|^2}u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leqslant e^{T_k(t)+M_V}\sup_{s\in[-1,1]}\|e^{\mu|\cdot|^2}u(\cdot,s)\|_{L^2(\mathbb{R}^n)}, \tag{3.4.21}$$

with

$$\begin{cases} \partial_t\left(\frac{1}{a}\partial_t T_k\right) = -\left(\frac{|\ddot b_k|^2|\eta|^2}{F(a_k)} + M_B\right) = -\left(\frac{F(a_k)|\eta|^2}{a_k^2} + M_B\right) \quad \text{in } [-1,1] \\ T_k(\pm 1) = 0. \end{cases}$$

T_k is even and, remembering that $a_k(s) \leqslant a_k(\tau)$ if $\tau \leqslant s$,

$$\begin{aligned} T_k(t) &= \int_t^1 \int_0^s \left(\frac{a_k(s)}{a_k(\tau)} \frac{F(a_k(\tau))|\eta|^2}{a_k(\tau)} + a_k(s) M_B \right) d\tau ds \\ &\leqslant |\eta|^2 \int_t^1 \int_0^s \frac{F(a_k(\tau))}{a_k(\tau)} \, d\tau ds + M_B \int_t^1 s a_k(s) \, ds \\ &= b_k(t)|\eta|^2 + M_B \int_t^1 s a_k(s) \, ds, \end{aligned}$$

for $t \in (-1, 1)$. Therefore the right hand side of (3.4.21) can be estimated as follows:

$$\int_{\mathbb{R}^n} e^{2a_k(t)|x+b_k(t)\eta|^2} |u(t)|^2 \, dx \leqslant e^{b_k(t)|\eta|^2} e^{M_B \int_t^1 s a_k(s) \, ds} e^{M_V} \sup_{s \in [-1,1]} \|e^{\mu|\cdot|^2} u(\cdot, s)\|.$$

Consequently we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} e^{2a_k(t)|x|^2 - 2|\eta|^2 b_k(t)(1-a_k(t)b_k(t)) + 4a_k(t)b_k(t)x\cdot\eta} |u(t)|^2 \, dx \\ &\qquad \leqslant e^{M_B \int_t^1 s a_k(s) \, ds} e^{M_V} \sup_{s \in [-1,1]} \|e^{\mu|\cdot|^2} u(\cdot, s)\|. \end{aligned} \tag{3.4.22}$$

Notice that, since a_k is continuous in $[-1, 1]$, we can estimate

$$e^{M_B \int_t^1 s a_k(s) \, ds} \leqslant C_k < +\infty.$$

By (3.4.22), the check to be performed is concerned with the sign of $1 - a_k(0)b_k(0)$. If $1 - a_k(0)b_k(0) \leqslant 0$ then by (3.4.22) $u \equiv 0$ and the scheme stops: indeed (3.4.22) forces $u(x, 0) = 0$ for almost all $x \in \mathbb{R}^n$, and $u \equiv 0$ in $\mathbb{R}^n_x \times [-1, 1]_t$ thanks to the uniqueness of the solution to (3.1.1).

If $1 - a_k(0)b_k(0) > 0$, then $1 - a_k(t)b_k(t) > 0$ for all $t \in [-1, 1]$, because of the monotonicity of a_k and b_k. In this case, we define the $(k + 1)-$th functions a_{k+1} and c_{k+1} as follows:

$$a_{k+1} = \frac{a_k}{1 - a_k b_k}, \quad c_{k+1} = a_{k+1}^{-\frac{1}{2}}. \tag{3.4.23}$$

We prove that the new defined a_{k+1} verifies the requests (3.4.18). Indeed it is easily seen that a_{k+1} is even, $a_{k+1}(\pm) = \mu$, $a_k < a_{k+1}$ in $(-1, 1)$, $\dot{a}_{k+1} \leqslant 0$ in $[0, 1]$. The proof that $F(a_{k+1}) > 0$ in $[-1, 1]$ deserves some comment: recall that

$$F(a_{k+1}) = 2c_{k+1}^{-1}(16c_{k+1}^{-3} - \ddot{c}_{k+1}),$$

moreover, from (3.4.23),

$$\begin{aligned} c_{k+1} &= (c_k^2 - b_k)^{\frac{1}{2}}, \\ \ddot{c}_{k+1} &= c_{k+1}^{-3} \left(16 - \frac{\dot{b}_k^2}{4} + c_k \dot{c}_k \dot{b}_k - \dot{c}_k^2 b_k - 16 c_k^{-2} b_k \right). \end{aligned}$$

From (3.4.18) and (3.4.20), we get $\dot{c}_k \dot{b}_k \leqslant 0$ and $16 b_k c_k^{-2} + b_k^2 > 0$ in $[-1, 1]$, hence $16 c_{k+1}^{-3} - \ddot{c}_{k+1} > 0$.

Multiplying (3.4.22) by $\exp(-2\varepsilon b_k(t)|\eta|^2)$, $\varepsilon > 0$ and integrating the corresponding inequality on the line $\mathbb{R}\xi$, with respect to η, we get

$$\sup_{t\in[-1,1]} \|e^{a^{\varepsilon}_{k+1}(t)|\cdot|^2} u(\cdot,t)\| \leqslant C_k(1+\varepsilon^{-1})^{\frac{n}{4}} e^{M_V} \sup_{s\in[-1,1]} \|e^{\mu|\cdot|^2} u(\cdot,s)\|, \tag{3.4.24}$$

with

$$a^{\varepsilon}_{k+1} = \frac{(1+\varepsilon)a_k}{1+\varepsilon - a_k b_k}.$$

Thanks to (3.4.24), we have

$$\sup_{t\in[-1,1]} \|e^{(a_{k+1}(t)-\varepsilon)|\cdot|^2} u(\cdot,t)\| < +\infty, \qquad \text{for all } \varepsilon > 0, \tag{3.4.25}$$

indeed

$$a^{\epsilon}_{k+1} - a_{k+1} = -\frac{a_k^2 b_k\, \epsilon}{(1-a_k b_k)(1+\epsilon - a_k b_k)} \geqslant -C_k \epsilon,$$

and (3.4.25) follows from the arbitrariness of ε in the definition of a^{ε}_{k+1}. Thanks to (3.4.25) and Lemma 3.3.3, we can conclude that (3.4.18) holds up to $j = k+1$.

Application of the iteration scheme

Let us describe the first step of the iteration. We choose $a_1(t) \equiv \mu$, for all $t \in [-1,1]$: obviously (3.4.18) hold. We set b_1 to be the solution to (3.4.19), that is

$$b_1(t) = 16\mu(1-t^2), \quad t \in [-1,1].$$

We need the following preliminary result, already proved in [2], which will be useful in the sequel.

Lemma 3.4.5 ([2], Theorem 1.1)**.** *In the hypoteses of Theorem 3.1.3, if $\alpha\beta \leqslant 2$ then $u \equiv 0$.*

Proof. By direct computation, we see that the condition $\alpha\beta \leqslant 2$, namely $\mu \geqslant \frac{1}{4}$ by (3.4.15), is equivalent to $1 - a_1(0)b_1(0) \leqslant 0$. Then $u \equiv 0$ by the above arguments based on (3.4.22), and the proof is complete. □

By means of the previous Lemma, we only need to consider the range $\alpha\beta > 2$, i.e. $\mu < \frac{1}{4}$.

We apply the above described iteration procedure. If there exists $k \in \mathbb{N}$ such that $1 - a_k(0)b_k(0) \leqslant 0$, then $u \equiv 0$ and the procedure stops. If for all $k \geqslant 1$ we have $1 - a_k(0)b_k(0) > 0$, the above described iteration produces an increasing sequence $(a_k)_{k\geqslant 1}$ of functions verifying (3.4.18). Set

$$a(t) := \lim_k a_k(t), \quad t \in [-1,1].$$

We now need to distinguish two cases.

Case 1: $\lim_{\mathbf{k}} \mathbf{a_k(0)} < +\infty$. In this case, from (3.4.18) we have

$$\sup_{t\in[-1,1]} \|e^{a(t)|\cdot|^2} u(\cdot,t)\| \leqslant e^{M_B \int_0^1 s a(s)\, ds} e^{M_V} \sup_{t\in[-1,1]} \|e^{\mu|\cdot|^2} u(\cdot,t)\|. \tag{3.4.26}$$

$$\|\sqrt{1-t^2}\nabla_A(e^{(a+\frac{i\dot a}{8a})|x|^2}u)\|_{L^2(\mathbb{R}^n\times[-1,1])}+C_a\|\sqrt{1-t^2}e^{(a(t)-\varepsilon)|x|^2}\nabla_A u\|_{L^2(\mathbb{R}^n\times[-1,1])}$$
$$\leqslant C\sup_{s\in[-1,1]}\|e^{\mu|x|^2}u(s)\|,$$

for some $C=C(\|V\|_\infty,\|x^tB\|_\infty)>0$.

Moreover, a can be determined as the solution to a suitable ordinary differential equation (see [34] for details). One has

$$a(t)=\frac{R}{4(1+R^2t^2)},$$

where $R>0$ is such that

$$\mu=\frac{R}{4(1+R^2)}.$$

This forces $\mu\leqslant\frac{1}{8}$. Estimate (3.1.12) hence immediately follows after inverting the changes in Section 3.4.2 (see below).

Case 2: $\lim_{\mathbf{k}}\mathbf{a_k(0)}=+\infty$**.** In this case, if $\int_0^1 sa(s)\,ds<+\infty$, then (3.4.18) forces $u\equiv 0$. If otherwise $\int_0^1 sa(s)\,ds=+\infty$, we need a more detailed analysis. For all $k\geqslant 1$, let s_k be the maximum point of $sa_k(s)$ in $[0,1]$: from (3.4.18) we have

$$\infty>e^{2\|V\|_{L^\infty}+\frac{\|V\|^2}{4}}\sup_{t\in[-1,1]}\|e^{\mu|\cdot|^2}u(\cdot,t)\|\geqslant\int_{\mathbb{R}^n}e^{2a_k(0)|x|^2-M_B\int_0^1 sa_k(s)\,ds}|u(x,0)|^2\,dx$$
$$\geqslant\int_{\mathbb{R}^n}e^{2a_k(0)|x|^2-M_Bs_ka_k(s_k)}|u(x,0)|^2\,dx\geqslant\int_{\mathbb{R}^n}e^{2a_k(0)\left(|x|^2-M_B\frac{s_k}{2}\right)}|u(x,0)|^2\,dx.$$

If there exists a subsequence $(s_{k_h})_h$ such that $s_{k_h}\to 0$, then the previous inequality implies that $u(0)\equiv 0$ in $\mathbb{R}^n$, i.e. $u\equiv 0$. If no subsequences of s_k accumulate in 0, take $\bar s>0$ a limit point of $(s_k)_k$: the previous inequality implies that $u(0)\equiv 0$ in the complementary of the ball centered in the origin of radius $((M_B\bar s)/2)^{1/2}$. As a consequence, by (3.1.11), one can take $\beta>0$ arbitrarily small: then, by Lemma 3.4.5, we conclude that $u\equiv 0$ in this case.

In conclusion, we summarize the above argument as follows: if $\mu>\frac{1}{8}$, then necessarily we are either in the case 2 or in the case in which the scheme stops in a finite number of steps. In both cases, we proved that $u\equiv 0$; if $\mu\leqslant\frac{1}{8}$, one can prove the logarithmic convexity estimates in (3.1.12), by the arguments described in the case 1 above and the inversion of the changes of variables of Section 3.4.2. In detail, we performed this change of variables:

$$(x,t)\mapsto(y,s)=\left(\frac{\sqrt{2\alpha\beta}x}{\alpha(1-t)+\beta(1+t)},\frac{\beta(1+t))}{\alpha(1-t)+\beta(1+t)}\right),$$

whose inverse is

$$(y,s)\mapsto(x,t)=\left(\frac{\sqrt{2\alpha\beta}y}{\alpha s+\beta(1-s)},\frac{\alpha s-\beta(1-s)}{\alpha s+\beta(1-s)}\right).$$

We call now u the solution of (3.1.1) and v the function obtained after the change of variables (as we did in Section 3.4.2): the relation between the two functions is the following.

$$v(x,t)=$$
$$\left(\frac{\sqrt{2\alpha\beta}}{\alpha(1-t)+\beta(1+t)}\right)^{\frac{n}{2}}e^{\frac{(\alpha-\beta)|x|^2}{4i(\alpha(1-t)+\beta(1+t))}}u\left(\frac{\sqrt{2\alpha\beta}x}{\alpha(1-t)+\beta(1+t)},\frac{\beta(1+t)}{\alpha(1-t)+\beta(1+t)}\right).$$

It is clear now that if $\alpha\beta < 4$ then $u \equiv 0$.

If $\alpha\beta \geqslant 4$, that is $\mu \leqslant 1/8$, than (3.4.26) reads now

$$\sup_{t\in[0,1]} \|e^{\tilde{a}(t)|\cdot|^2} u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leqslant (1+R^2)^{\frac{M_B}{8R}} e^{M_V} \sup_{t\in[-1,1]} \|e^{\mu|\cdot|^2} v(\cdot,t)\|,$$

with

$$\tilde{a}(t) = \frac{\alpha\beta R}{2(\alpha t + \beta(1-t))^2 + 2R^2(\alpha t - \beta(1-t))^2}.$$

Analogously we get

$$\begin{aligned} &\|\sqrt{1-t^2}\nabla_A(e^{a+\frac{i\dot{a}}{8a}|x|^2}u)\|_{L^2(\mathbb{R}^n)\times[-1,1]} \\ &+ C_\varepsilon\|\sqrt{1-t^2}e^{(a(t)-\varepsilon)|x|^2}\nabla_A u\|_{L^2(\mathbb{R}^n\times[-1,1]} \\ &\qquad \leqslant e^{M_V} \sup_{t\in[-1,1]} \|e^{\mu|x|^2}v(t)\|. \end{aligned}$$

The proof of Theorem 3.1.3 is then completed.

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