

UNIVERSITÀ DEGLI STUDI DI ROMA
"LA SAPIENZA"
FACOLTÀ DI SCIENZE M.F.N.

Tesi di Dottorato in Matematica
di
Daniele Castorina

Properties of solutions of elliptic problems with critical growth

ANNO ACCADEMICO 2004 - 2005
XVII CICLO

Classificazione AMS : 36J65, 35B33, 35J20.

Parole Chiave : Critical exponent, Maximum principle, Nonlinear elliptic.

Contents

| | |
|----------------------------------------------------------------|----|
| Chapter 1. Introduction | 3 |
| Chapter 2. Maximum Principle and Symmetry | 9 |
| 1. Sufficient conditions for the Maximum Principle | 9 |
| 2. Symmetry and monotonicity via the Maximum Principle | 14 |
| Chapter 3. The almost critical problem in an annulus - Part I | 21 |
| 1. Introduction | 21 |
| 2. Preliminaries and notations | 24 |
| 3. Proof of Theorem 3.1 | 26 |
| 4. Proof of Theorem 3.2 | 28 |
| Chapter 4. The almost critical problem in an annulus - Part II | 39 |
| 1. Introduction | 39 |
| 2. Preliminaries | 40 |
| 3. Proof of Theorem 4.1 | 43 |
| Chapter 5. The singularly perturbed critical problem | 55 |
| 1. Introduction | 55 |
| 2. L^2 global concentration | 59 |
| 3. Pohozaev identity and "reverse" L^2 concentration | 62 |
| Bibliography | 69 |

CHAPTER 1

Introduction

In the last decades there has been considerable interest in the study of qualitative properties of positive solutions to nonlinear elliptic Dirichlet problems such as the following

$$(P) \begin{cases} -\Delta u + \lambda u = u^p & x \in \Omega \\ u > 0 & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

where Ω is an open bounded subset of \mathbb{R}^N , $\lambda \in \mathbb{R}$ is a parameter and p is either the critical Sobolev growth $2^* - 1 = \frac{N+2}{N-2}$ or a slightly subcritical exponent.

Problem (P) is a model case for many problems in differential geometry (Yamabe problem, the scalar curvature problem, H-bubbles, harmonic maps, minimal surfaces), physics (Yang-Mills connections, liquid crystals, Ginzburg-Landau model for superconductivity) or even biology (Keller-Segel aggregation model, "shadow system" for some activator-inhibitor models).

The existence of solutions for problem (P) can be studied through a variational approach, i.e. solutions can be seen as critical points $u \in H_0^1(\Omega)$ of the energy functional

$$E_\lambda(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 + \lambda u^2 - \frac{1}{p} \int_\Omega |u|^p$$

However, classical variational tools such as the Mountain Pass, which guarantee existence in the subcritical case, completely fail in the critical case.

The main difficulty in dealing with problem (P) is that the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ is continuous but not compact.

The existence of noncompact invariant symmetry groups for such PDEs is responsible for the noncompactness, i.e. weak but not strong convergence, of Palais-Smale sequences at certain energy levels. In this case the phenomenon of "blow-up" solutions appears, and the study of such solutions has been a major research issue in Nonlinear Analysis in recent years.

Historically, the study of problem (P) begins in 1960 when Yamabe in [42] addresses the following question: is it possible, given a compact Riemannian manifold (M, g) of dimension $N \geq 3$, to find a metric which is conformal to g and has constant scalar curvature?

Even if this conjecture was true, Yamabe's original proof contained a gap: this is why such question has become famous as the "Yamabe problem" (see [37]).

The link between this geometrical issue and problem (P) is that Yamabe's conjecture is equivalent to an elliptic problem with critical Sobolev growth on manifolds.

Eventually Yamabe problem was completely solved, especially through the works of Aubin [5] and Schoen [39], but problems such as (P) began to raise considerable interest in research.

The first result regarding this problem is negative and is a famous nonexistence theorem due to Pohozaev [32] in 1965: if p is critical and the domain Ω is starshaped, problem (P) has no solutions for any $\lambda \geq 0$.

In 1975 Kazdan and Warner [28] observe a singular circumstance: in striking contrast to Pohozaev result, problem (P) is solvable if the domain Ω is an annulus. Both results suggested a link between the topology of the domain and the solvability of problem (P) , but this connection has not been fully understood yet.

In 1976 Talenti [41] proved that the radial functions

$$U_{\varepsilon, y}(x) = \frac{1}{\varepsilon^{\frac{2-N}{2}}} \left[\frac{N(N-2)}{N(N-2) + |x-y|^2} \right]^{\frac{N-2}{2}}$$

for any $y \in \mathbb{R}^N$ and $\varepsilon > 0$ are the unique minimizers of the Sobolev quotient in \mathbb{R}^N .

In fact, defining the best Sobolev constant S as

$$S = \inf_{H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2}{\left(\int_{\Omega} u^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}}}$$

this infimum is attained in \mathbb{R}^N by all the Talenti functions and this constant is independent of the domain Ω . In particular, S is never attained if the domain Ω is either a bounded domain or a half space.

In 1983 Brezis and Nirenberg [11] prove that for some values of λ there is compactness of any Palais-Smale sequence for (P) exactly below the energy threshold $\frac{S^{N/2}}{N}$. Furthermore, taking

$$S_{\lambda} = \inf_{H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 + \lambda u^2}{\left(\int_{\Omega} u^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}}}$$

this infimum is attained (and thus (P) has at least a minimal solution) if and only if $S_{\lambda} < S$.

Their beautiful result is the following

Theorem 1.1. *Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a domain (in particular bounded). Let λ_1 be the first eigenvalue of the Laplacian $-\Delta$ with homogeneous Dirichlet condition on the boundary $\partial\Omega$. Then if $N \geq 4$ for any $\lambda \in (-\lambda_1, 0)$ we have $S_{\lambda} < S$; if $N = 3$ there exists $\lambda_* \in (-\lambda_1, 0]$ such that for any $\lambda \in (-\lambda_1, \lambda_*)$ we have $S_{\lambda} < S$.*

Another fundamental existence result, based on a more topological approach, is the following theorem proved in 1988 by Bahri and Coron [6]

Theorem 1.2. *Assume Ω has "nontrivial" topology (for example, Ω noncontractible). Then (P) has a solution.*

Anyhow we will not be interested in existence theory for problems such as (P) whereas we will mainly focus our attention in answering questions concerning the qualitative properties of positive solutions of problem (P) such as:

1. Suppose the domain has any symmetry, do solutions of problem (P) inherit this symmetry? And where are the critical points located?
2. Suppose problem (P) is singularly perturbed, do bounded energy solutions cease to exist if the perturbation parameter goes to infinity? And where are the blow-up points located?

Let us describe the outline of this Thesis.

In chapter 2 we will present a survey about all the different formulations of the Maximum Principle that will be applied to prove the results in the subsequent two chapters, as well as some applications.

In chapter 3, we will consider the slightly subcritical problem

$$(I) \begin{cases} -\Delta u = N(N-2)u^{p-\varepsilon} & \text{in } A \\ u > 0 & \text{in } A \\ u = 0 & \text{on } \partial A \end{cases}$$

where A is an annulus in \mathbb{R}^N , $N \geq 3$, $p+1 = \frac{2N}{N-2}$ is the critical Sobolev exponent and $\varepsilon > 0$ is a small parameter. We will prove that solutions of (I) which concentrate at one or two points are axially symmetric by means of the Maximum Principle and the method of rotating hyperplanes (see [15]).

In chapter 4, we will continue our investigation of problem (I) and extend the symmetry results of the preceding chapter to solutions with any number of peaks. Through a geometrical approach we prove that solutions of (I) which concentrate at k points, $3 \leq k \leq N$, have these points all lying in the same $(k-1)$ -dimensional hyperplane Π_k passing through the origin and are symmetric with respect to any $(N-1)$ -dimensional hyperplane containing Π_k (see [16]).

In chapter 5 we will consider a singular perturbation of problem (P) with critical growth

$$\begin{cases} -\Delta u + \lambda u = u^{\frac{N+2}{N-2}} & x \in \Omega \\ u > 0 & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

where λ is a large parameter. Following a previous work of Druet-Hebey-Vaugon (see [17]), we prove that the energy of positive solutions of this problem tends to infinity as $\lambda \rightarrow +\infty$. We also prove, extending and simplifying recent results, that bounded energy solutions to the corresponding mixed B.V.P. have at least one blow-up point on the Neumann component of the boundary $\partial\Omega$ as $\lambda \rightarrow +\infty$. In other words, for large λ we prove the nonexistence of bounded energy solutions for the Dirichlet problem and the nonexistence of solutions concentrating only in the interior of the domain for the mixed problem (see [14]).

CHAPTER 2

Maximum Principle and Symmetry

In this chapter we would like to collect a survey of all the different formulations of the Maximum Principle for linear elliptic partial differential equations we will use and give sufficient conditions for this important tool to hold. We will also consider a few interesting applications of the Maximum Principle, especially those regarding the symmetry and monotonicity of solutions of elliptic PDEs. This will allow us to show how the Maximum Principle is a simple and yet powerful tool in proving the partial symmetry of solutions to problem (P) with almost critical growth in an annulus, which will be the object of the next two chapters.

1. Sufficient conditions for the Maximum Principle

Let $\Omega \subset \mathbb{R}^N$ be open, smooth, bounded and connected and let us consider a general second order partial differential operator $L : C^2(\Omega) \rightarrow C(\Omega)$. L can be written in the form

$$L \equiv \sum_{i,j=1}^N a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i^N b_i(x) \frac{\partial}{\partial x_i} + c(x)$$

where all the coefficients $a_{ij}(x)$, $b_i(x)$, $c(x) \in C(\Omega)$, $i, j = 1, \dots, N$.

Consider the matrix A associated to the principal part of L , $A_{ij}(x) = a_{ij}(x)$, $i, j = 1, \dots, N$, and let $\lambda(x)$ and $\Lambda(x)$ be respectively its smallest and largest eigenvalues for $x \in \Omega$. We have the following

Definition 2.1. L is **elliptic** if $A(x) > 0$ for any $x \in \Omega$, **strictly elliptic** if $\lambda(x) > 0$ for any $x \in \Omega$, **uniformly elliptic** if the ratio $\frac{\Lambda(x)}{\lambda(x)}$ is bounded in Ω .

Let us begin by recalling three classical versions of the Maximum Principle for elliptic operators.

Theorem 2.2 (Weak Maximum Principle). *Let L be elliptic in Ω with $c(x) \leq 0$. Suppose $Lu \geq 0$ and $u \leq 0$ on $\partial\Omega$. Then $u \leq 0$ in Ω .*

Theorem 2.3 (Hopf Lemma). *Let L be uniformly elliptic in Ω . Suppose $Lu \geq 0$ and that there exists $x_0 \in \partial\Omega$ such that $u(x) < u(x_0)$ for $x \in \Omega$. Then $\frac{\partial u}{\partial \nu}(x_0) > 0$.*

Theorem 2.4 (Strong Maximum Principle). *Let L be uniformly elliptic in Ω . Suppose $Lu \geq 0$ and $u \leq 0$ in Ω . Then either $u < 0$ in Ω or $u \equiv 0$ in Ω .*

For our purposes the classical versions of the Maximum Principle would not be sufficient, that is why we will need a slightly more general point of view on the topic. To be more specific let us give the following

Definition 2.5. *Let $L : C^2(\Omega) \cap C(\bar{\Omega}) \rightarrow C(\Omega)$ be a linear partial differential operator. L satisfies the Maximum Principle in Ω if for any $u \in C^2(\Omega) \cap C(\bar{\Omega})$ such that $Lu \geq 0$ in Ω and $u \leq 0$ on $\partial\Omega$ we have $u \leq 0$ in Ω .*

We easily deduce from the classical Weak Maximum Principle that being elliptic with nonnegative order zero coefficient is sufficient for an operator L to satisfy the maximum principle. However, this is not always the case for most elliptic operators considered, especially those related to the critical growth. We will state and prove several different conditions which guarantee the validity of the Maximum Principle for general L .

One of the main conditions for the Maximum Principle to hold is the existence of at least one strictly positive test function for L for which L is nonpositive. As we will see, this is related to the first eigenvalue and the correspondent eigenfunction of L . As we will see later on, this condition can be relaxed by considering an even more general formulation of the Maximum Principle (see [9]).

Proposition 2.6. *Suppose that there exists $g > 0$ in $\bar{\Omega}$ such that $Lg \leq 0$. Then L satisfies the Maximum Principle.*

Proof: Let u be any function such that $Lu \geq 0$ in Ω and $u \leq 0$ on $\partial\Omega$ and consider $v \equiv \frac{u}{g}$.

Let $L_0 = L - c(x)$. By simple calculations we get

$$0 \leq Lu = L_0v + (Lg)v \equiv L_1v$$

Since L_1 is elliptic with nonpositive order zero coefficient $c_1(x) = Lg$, L_1 satisfies the Maximum Principle.

Thus from $v \leq 0$ on $\partial\Omega$ and $L_1 \geq 0$ in Ω we obtain $v = \frac{u}{g} \leq 0$ in Ω . Since $g > 0$ in $\bar{\Omega}$, we get $u \leq 0$ in Ω , as we wanted to prove. \blacksquare

The Maximum Principle is somewhat related also to the size, especially the "width" of the domain Ω , as we will see in the next corollary of the previous proposition. Heuristically, the Maximum Principle holds if $c(x) \leq \lambda_1(L)$, the first eigenvalue. If the domain is very narrow λ_1 tends to infinity and $c(x)$ becomes necessarily smaller than the first eigenvalue. To be more precise we have the following

Corollary 2.7. *There exists $\varepsilon = \varepsilon(L)$ such that L satisfies the Maximum Principle in Ω if $\bar{\Omega}$ is contained in a strip $S_\varepsilon = \{x \in \mathbb{R}^N : a - \varepsilon < x_1 < a\}$ for some $a \in \mathbb{R}$.*

Proof: We will construct a function g such that $g > 0$ in $\bar{\Omega}$ and $Lg \leq 0$.

Take $g(x) = e^{\alpha a} - e^{\alpha x_1}$, with α to be chosen later.

Obviously $g > 0$ in S_ε , so $g > 0$ in $\bar{\Omega}$ for any $\varepsilon > 0$.

Suppose $a_{11}(x) \geq m > 0$, $b_1(x) \geq -b$ and $c(x) \leq c$ for some positive constants m, b, c . We have

$$\begin{aligned} Lg &= -a_{11}\alpha^2 e^{\alpha x_1} - \alpha b_1 e^{\alpha x_1} + c(x) (e^{\alpha a} - e^{\alpha x_1}) \leq \\ &\leq -m\alpha^2 e^{\alpha x_1} + b\alpha e^{\alpha x_1} + ce^{\alpha x_1} (e^{\alpha(a-x_1)} - 1) = \\ &= e^{\alpha x_1} (-m\alpha^2 + b\alpha + c(e^{\alpha(a-x_1)} - 1)) \leq \\ &\leq e^{\alpha x_1} (-m\alpha^2 + b\alpha + c(e^{\alpha\varepsilon} - 1)) \end{aligned}$$

Taking $\alpha = \frac{2b}{m}$ and noticing that $e^{\alpha\varepsilon} - 1 = O(\varepsilon)$, we obtain

$$Lg \leq e^{\alpha x_1} \left(-\frac{2b^2}{m} + O(\varepsilon) \right) < 0$$

for sufficiently small ε , and this concludes the proof. \blacksquare

Not only the width, but also the measure of the domain Ω is responsible for the Maximum Principle to hold for an operator $L = \Delta + c(x)$: if the measure of the domain is small enough the first eigenvalue of L becomes positive. This is the so called "Maximum Principle in small domains" and is the content of the following

Theorem 2.8. *Let $Lu = \Delta u + c(x)u$. There exists $\delta = \delta(N, \text{diam } \Omega) > 0$ such that L satisfies the Maximum Principle in any $\Omega' \subset \Omega$ such that $|\Omega'| \leq \delta$.*

Proof: The proof is based on the following result by Alexandrov-Bakelman-Pucci: suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$, $f \in L^N(\Omega)$ and $Lu = \Delta u + c(x)u \geq f$ in Ω with $c(x) \leq 0$ and $u \leq 0$ on $\partial\Omega$. Then

$$(2.1.1) \quad \sup_{\Omega} u \leq C(\text{diam } \Omega, N) |f|_{L^N(\Omega)}$$

Now if $c^+(x) = \max\{c(x), 0\}$ and $c^-(x) = -\min\{c(x), 0\}$ we have

$$0 \leq Lu = \Delta u + c(x)u = \Delta u + c^+u - c^-u$$

which implies

$$\Delta u - c^-u \geq -c^+u \geq -c^+u^+$$

From (2.1.1) we get

$$\sup_{\Omega} u \leq C |c^+u^+|_{L^N(\Omega)}$$

for some fixed constant C .

Suppose by contradiction $u \not\leq 0$ in Ω , so that $\sup_{\Omega} u = \sup_{\Omega} u^+$. Then

$$\sup_{\Omega} u^+ = \sup_{\Omega} u \leq C |c(x)|_{L^\infty(\Omega)} \sup_{\Omega} u^+ |\Omega|^{\frac{1}{N}}$$

Choosing $|\Omega| < \delta(N, d) = (C^N |c(x)|_{L^\infty})^{-1}$ in this inequality we reach a contradiction. Then $u \leq 0$ in Ω and L satisfies the Maximum Principle.

■

We would like to recall that there is a generalized formulation of the Maximum Principle and a more general definition of the first eigenvalue $\lambda_1(L, \Omega)$ when the boundary $\partial\Omega$ is not smooth due to Berestycki, Nirenberg and Varadhan. Even if their definitions are much more general than the ones we intend to use in proving our partial symmetry results, they are worth mentioning for two reasons: first of all, with this "refined" definitions the link between the validity of the Maximum Principle and the positivity of the principal eigenvalue are much clearer; furthermore, some of the conditions for the validity of the Maximum Principle can be relaxed.

The definition is the following

$$\lambda_1(L, \Omega) = \sup\{\lambda : \text{there exists } \phi > 0 \text{ in } \Omega \text{ satisfying } (L + \lambda)\phi \leq 0\}$$

In [9] they show that even with this definition all the main properties of the "classical" principal eigenvalue continue to hold. In particular we have

Proposition 2.9. *The principal eigenvalue $\lambda_1(L, \Omega)$ is strictly decreasing in its dependence on Ω and on the coefficient $c(x)$. Moreover the "refined" Maximum Principle holds for L if and only if $\lambda_1(L, \Omega)$ is positive.*

It is important to notice that, by using this generalized definition of the first eigenvalue, it is possible to prove that also the following condition, which is slightly weaker than proposition 2.6, is sufficient for the Maximum Principle to hold.

Proposition 2.10. *Suppose there exists $g \in W_{loc}^{2,N} \cap C(\bar{\Omega})$, $g > 0$ in Ω such that $Lg \leq 0$ in Ω but $g \not\equiv 0$ on some regular part of $\partial\Omega$. Then L satisfies the Maximum Principle in Ω .*

To conclude this section we would like to mention an interesting and easy condition, due to Grossi and Molle (see [23]), which relates the positivity of the first eigenvalue of an operator $L = \Delta + c(x)$ to the $L^{\frac{N}{2}}(\Omega)$ of the coefficient $c(x)$. Define the best Sobolev constant as

$$S = \inf_{u \in H_0^1(\Omega), u \neq 0} \frac{\int_{\Omega} |\nabla u|^2}{\left(\int_{\Omega} |u|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}}}$$

The result is the following

Proposition 2.11. *Suppose $|c(x)|_{L^{\frac{N}{2}}(\Omega)} < S$. Then $\lambda_1(L, \Omega) > 0$.*

Proof: Suppose by contradiction that there exists $\phi \in H_0^1(\Omega)$ such that $|\phi|_{L^{\frac{2N}{N-2}}(\Omega)} = 1$ and $L\phi \leq 0$.

By Holder and Sobolev inequalities and the hypothesis we have that

$$\begin{aligned} \int_{\Omega} |\nabla \phi|^2 &\leq \int_{\Omega} c(x) \phi^2 \leq |c(x)|_{L^{\frac{N}{2}}(\Omega)} \left(\int_{\Omega} |\phi|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}} \leq \\ &\leq |c(x)|_{L^{\frac{N}{2}}(\Omega)} \frac{1}{S} \int_{\Omega} |\nabla \phi|^2 < \int_{\Omega} |\nabla \phi|^2 \end{aligned}$$

which is a contradiction. ■

2. Symmetry and monotonicity via the Maximum Principle

The Maximum Principle is a simple and yet powerful tool to prove qualitative properties of solutions of second order elliptic equations, especially the symmetry and monotonicity of such solutions. Probably the best example of this fact is the beautiful paper of Gidas, Ni and Nirenberg [24]: in this paper the authors combine the classical Maximum Principle with the Method of Moving Hyperplanes developed by Alexandrov and Serrin (see [38]) to prove the symmetry and monotonicity of solutions of the problem

$$(P) \begin{cases} -\Delta u = f(u) & x \in \Omega \\ u > 0 & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

where Ω is symmetric and convex in one direction and f is a locally Lipschitz nonlinearity. The beauty of their result can be found in many aspects, especially in the fact that it is very general and the proofs are elegant and easy to read.

Let us state their main result

Theorem 2.12. *Suppose Ω is convex and symmetric in the x_1 direction. Let $u \in C^2(\overline{\Omega})$ be a solution of problem (P). Then u is symmetric in x_1 and $\frac{\partial u}{\partial x_1} < 0$ for $x_1 > 0$.*

Simple but important consequences of this theorem are the following

- If $\Omega = B_R(0)$ a ball, all the solutions of (P) are radially symmetric and radially decreasing with a unique critical point in the origin
- If $\Omega = A_{r,R}$ an annulus, all the solutions are radially decreasing (and so have no critical points) in the outer shell $|x| > \frac{r+R}{2}$

We will present a simplified proof by Beresticki and Nirenberg (see [8]) which makes use of the Maximum Principle in small domains. This proof requires a weaker regularity for the solution, namely $u \in C^2(\Omega) \cap C(\bar{\Omega})$ instead of $C^2(\bar{\Omega})$.

Proof: Let $x = (x_1, y) \in \mathbb{R} \times \mathbb{R}^{N-1} = \mathbb{R}^N$ and suppose a is the first value for which the hyperplanes $T_\lambda = \{x \in \mathbb{R}^N : x_1 = \lambda\}$ hit the domain Ω , namely $-a = \inf_{x \in \Omega} x_1$.

We will prove that $\frac{\partial u}{\partial x_1} < 0$ for $x_1 > 0$ and that $u(x_1, y) < u(x'_1, y)$ for $x_1 + x'_1 < 0$ and $x_1 < x'_1$.

Then taking the limit for $x'_1 \uparrow -x_1$ we find by continuity that $u(x_1, y) \leq u(-x_1, y)$ for $x_1 < 0$.

Exchanging x_1 with $-x_1$, we obtain the reverse inequality which gives the symmetry of u with respect to the x_1 variable.

For $-a < \lambda < 0$, let

$$w_\lambda(x) \equiv v_\lambda(x) - v(x) \equiv u(2\lambda - x_1, y) - u(x)$$

and

$$\Sigma_\lambda = \{x \in \Omega : x_1 < \lambda\}$$

Clearly

$$\begin{cases} \Delta w_\lambda(x) + c_\lambda(x)w_\lambda(x) = 0 & x \in \Sigma_\lambda \\ w_\lambda(x) \geq 0 & x \in \partial\Sigma_\lambda \end{cases}$$

where

$$c_\lambda(x) = \frac{f(v_\lambda(x)) - f(u(x))}{v_\lambda(x) - u(x)}$$

We want to prove that $w_\lambda > 0$ in Σ_λ .

If $\lambda \sim -a$, the measure of Σ_λ is small. From the Maximum Principle in small domains we get $w_\lambda \geq 0$ in Σ_λ , and from the Strong Maximum Principle $w_\lambda > 0$ in Σ_λ .

Define

$$\mu \equiv \sup\{\lambda > -a : w_\lambda(x) > 0 \text{ in } \Sigma_\lambda\}$$

Arguing by contradiction, suppose $\mu < 0$ and consider $w_\mu(x)$: clearly $w_\mu \geq 0$ in Σ_μ and from the Strong Maximum Principle $w_\mu > 0$ in Σ_μ . This means we can continue this process, considering $\Sigma_{\mu+\varepsilon}$ for small positive ε .

Take $K \subset \Sigma_\mu$ compact such that $|\Sigma_\mu \setminus K| \leq \frac{\delta}{2}$ for small positive δ . In particular, $w_\mu(x) > 0$ in K .

For $\varepsilon > 0$ sufficiently small, $|\Sigma_{\mu+\varepsilon} \setminus K| \leq \delta$. Then

$$\begin{cases} \Delta w_{\mu+\varepsilon}(x) + c_{\mu+\varepsilon}(x)w_{\mu+\varepsilon}(x) = 0 & x \in (\Sigma_{\mu+\varepsilon} \setminus K) \\ w_{\mu+\varepsilon}(x) \not\geq 0 & x \in \partial(\Sigma_{\mu+\varepsilon} \setminus K) \end{cases}$$

since $w_{\mu+\varepsilon} > 0$ on ∂K .

From the Maximum Principle in small domains, $w_{\mu+\varepsilon} \geq 0$ in $(\Sigma_{\mu+\varepsilon} \setminus K)$. Furthermore $w_{\mu+\varepsilon} > 0$ in K , so that $w_{\mu+\varepsilon} \geq 0$ in the whole $\Sigma_{\mu+\varepsilon}$, but this contradicts the maximality of μ , so $\mu = 0$ as we wanted to prove.

The monotonicity of the solution u comes from the Hopf Lemma applied on the boundary $\partial\Sigma_\lambda$ of every cap Σ_λ : in fact on $T_\lambda \cap \Omega$ we have

$$0 > \frac{\partial w_\lambda}{\partial \nu}(x) = -\frac{\partial u}{\partial x_1}(x)$$

and this concludes the proof. ■

Let us now recall a proposition contained in [31] to show how the Maximum Principle can be effectively applied, as we will see in the next two chapters, to investigate symmetry properties of classical $C^2(\Omega) \cap C(\overline{\Omega})$ solutions of problems of the type

$$\begin{cases} -\Delta u = f(x, u) & x \in \Omega \\ u = g(x) & x \in \partial\Omega \end{cases}$$

where Ω is a bounded, somehow symmetric domain in \mathbb{R}^N not necessarily convex, $N \geq 2$, $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function of class C^1 with respect to the second variable, g is continuous and both functions have some symmetry in x .

The new and simple idea contained in the paper [31] to study the symmetry of the solutions of problem (2), which works efficiently when $f(x, s)$ is convex in the s -variable, is to prove the non-negativity of the first eigenvalue of the linearized operator in both caps determined by a symmetry hyperplane for the domain. To be more precise let us fix some notations.

Let us assume that Ω contains the origin and is symmetric with respect to the hyperplane

$$T = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 = 0\}$$

and denote by Ω^- and Ω^+ the caps to the left and right of T , i.e.

$$\Omega^- = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 < 0\}$$

$$\Omega^+ = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 > 0\}$$

Let u be a solution of (2) and let us consider the linearized operator at u , that is

$$L = -\Delta - f'(x, u)$$

where f' denotes the derivative of $f(x, s)$ with respect to s .

We denote by $\lambda_1(L, D)$ be the first eigenvalue of L in a subdomain $D \subset \Omega$ with zero Dirichlet boundary conditions. We can now recall Proposition 1.1 of [31]:

Proposition 2.13. *Suppose $f(x, s)$ and $g(x)$ are even in x_1 , f is strictly convex in s and suppose that $\lambda_1(L, \Omega^-)$ and $\lambda_1(L, \Omega^+)$ are both non-negative. Then u is symmetric with respect to the x_1 -variable i.e. $u(x_1, \dots, x_N) = u(-x_1, \dots, x_N)$. The same result holds if f is only convex but $\lambda_1(L, \Omega^-)$ and $\lambda_1(L, \Omega^+)$ are both positive.*

Proof: Let us denote by v^- and v^+ the reflected functions of u in the domains Ω^- and Ω^+ respectively:

$$\begin{aligned} v^-(x) &= u(-x_1, x_2, \dots, x_N), & x \in \Omega^- \\ v^+(x) &= u(-x_1, x_2, \dots, x_N), & x \in \Omega^+ \end{aligned}$$

Assume f is strictly convex. In this case we have

$$\begin{aligned} f(x, v^-(x)) - f(x, u(x)) &\geq f'(x, u(x))(v^-(x) - u(x)) & \text{in } \Omega^- \\ f(x, v^+(x)) - f(x, u(x)) &\geq f'(x, u(x))(v^+(x) - u(x)) & \text{in } \Omega^+ \end{aligned}$$

and the strict inequality holds whenever $v^-(x) \neq u(x)$ (respectively $v^+(x) \neq u(x)$).

Hence by (2), using the symmetry of f and g in the x_1 -variable and considering the functions $w^- = v^- - u$ and $w^+ = v^+ - u$ we have

$$(2.2.2) \quad -\Delta w^- - f'(x, u)w^- \geq 0$$

$$(2.2.3) \quad -\Delta w^+ - f'(x, u)w^+ \geq 0$$

with the strict inequality whenever $w^-(x) \neq 0$ or $w^+(x) \neq 0$ and

$$(2.2.4) \quad w^-(x) = 0 \text{ (resp. } w^+(x) = 0) \text{ on } \partial\Omega^- \text{ (resp. } \partial\Omega^+)$$

If w^- and w^+ are both nonnegative in the respective domains Ω^- and Ω^+ then $w^- \equiv w^+ \equiv 0$ by the very definition, and hence u is symmetric with respect to x_1 .

Therefore arguing by contradiction, we can assume that one among the two functions, say w^+ , is negative somewhere in Ω^+ . Then, considering a connected component D in Ω^+ of the set where $w^+ < 0$, multiplying the equation (2.2.3) by w^+ , integrating and using (2.2.4) and the strict convexity of f we get

$$\int_D |\nabla w^+|^2 - \int_D f'(x, u)(w^+)^2 < 0$$

which implies that $\lambda_1(L, \Omega^+) < 0$ against the hypothesis. Hence u is symmetric.

Now we assume that f is only convex but $\lambda_1(L, \Omega^-) > 0$ and $\lambda_1(L, \Omega^+) > 0$. Then the maximum principle holds both in Ω^- and Ω^+ . Therefore by (2.2.2)-(2.2.4) we immediately get $w^- \geq 0$ and $w^+ \geq 0$ which imply the symmetry of u . \blacksquare

In our partial symmetry results we will actually use a slight variation of the previous proposition which is the following

Proposition 2.14. *If either $\lambda_1(L, \Omega_\nu^-)$ or $\lambda_1(L, \Omega_\nu^+)$ is non-negative and u has a critical point on $T_\nu \cap \Omega$ then u is symmetric with respect to the hyperplane T_ν .*

Proof: Assume that ν is the direction of the x_1 -axis in \mathbb{R}^N and that $\lambda_1(L, \Omega_\nu^-) \geq 0$. Denote by v^- the reflection of the function u in the domain Ω_ν^- , that is $v^-(x) = u(-x_1, x_2, \dots, x_N)$ for $x \in \Omega_\nu^-$.

Hence the function $w^- = v^- - u$ satisfies

$$(2.2.5) \quad \begin{cases} L(w^-) \geq 0 \text{ (} > 0 \text{ if } w^-(x) \neq 0 \text{) in } \Omega_\nu^- \\ w^- \equiv 0 \text{ on } \partial\Omega_\nu^- \end{cases}$$

by the strict convexity of the function $f(s)$.

Since $\lambda_1(L, \Omega_\nu^-) \geq 0$, by (2.2.5) we have that $w^- \geq 0$ in Ω_ν^- .

If $w^- \neq 0$ in Ω_ν^- , by the strong maximum principle we would have $w^- > 0$ in Ω_ν^- .

Then, applying Hopf Lemma to any point of $T_\nu \cap \Omega$ we would have $\frac{\partial w^-}{\partial x_1} < 0$ on $T_\nu \cap \Omega$, which would imply $|\nabla u| > 0$ on $T_\nu \cap \Omega$, contradicting the hypothesis that u has a critical point on $T_\nu \cap \Omega$. ■

CHAPTER 3

The almost critical problem in an annulus - Part I

1. Introduction

In this chapter we will discuss the results contained in [15]. We consider the problem

$$(3.1.6) \quad \begin{cases} -\Delta u = N(N-2)u^{p-\varepsilon} & \text{in } A \\ u > 0 & \text{in } A \\ u = 0 & \text{on } \partial A \end{cases}$$

where A is an annulus centered at the origin in \mathbb{R}^N , $N \geq 3$, $p+1 = \frac{2N}{N-2}$ is the critical Sobolev exponent and $\varepsilon > 0$ is a small parameter.

It is well known that the study of (3.1.6) is strictly related to the limiting problem ($\varepsilon = 0$) which exhibits a lack of compactness and gives rise to solutions of (3.1.6) which blow up as $\varepsilon \rightarrow 0$.

Several authors have studied the existence and the behaviour of solutions of (3.1.6) which blow-up in a general bounded domain Ω ([7], [10], [19], [21], [27], [29], [33], [34]).

A deep analysis of solutions of (3.1.6) which blow up at k points has been done in [7] and [34]. In [7] the authors completely characterize the blow-up points of solutions of (3.1.6) (as $\varepsilon \rightarrow 0$) in terms of the critical points of some functions which naturally arise in the study of these problems.

We are interested in the geometrical properties of the solutions of (3.1.6) which blow-up and, more precisely, on their symmetry and on the location of their blow-up points, as $\varepsilon \rightarrow 0$.

It is obvious that solutions of (3.1.6) which concentrate in a finite number of points cannot be radially symmetric. Nevertheless it is natural

to expect a partial symmetry of the solutions, as well as a symmetric location of the concentration points.

In order to state precisely our result we need some notations.

We say that a family of solutions $\{u_\varepsilon\}$ of (3.1.6) has $k \geq 1$ concentration points at $\{P_\varepsilon^1, P_\varepsilon^2, \dots, P_\varepsilon^k\} \subset A$ if the following holds

$$(3.1.7) \quad P_\varepsilon^i \neq P_\varepsilon^j, i \neq j \text{ and each } P_\varepsilon^i \text{ is a strict local maximum for } u_\varepsilon$$

$$(3.1.8) \quad u_\varepsilon \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \text{ locally uniformly in } A \setminus \{P_\varepsilon^1, P_\varepsilon^2, \dots, P_\varepsilon^k\}$$

Notice that in this first definition we do not require that $u_\varepsilon(P_\varepsilon^i) \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

Theorem 3.1. *Let u_ε be a family of solutions of (3.1.6) with one concentration point $P_\varepsilon \in A$. Then, for ε small, u_ε is symmetric with respect to any hyperplane passing through the axis r connecting the origin with the point P_ε . Moreover, all the critical points of u_ε belong to the symmetry axis r and*

$$(3.1.9) \quad \frac{\partial u_\varepsilon}{\partial \nu_T}(x) > 0 \quad \forall x \in T \cap A$$

where T is any hyperplane passing through the origin but not containing P_ε and ν_T is the normal to T , oriented towards the half space containing P_ε . The same holds if u_ε has $k > 1$ concentration points, all located on the same half line passing through the origin.

Remark 1. If the solution u_ε has Morse index one then its axial symmetry, for any $\varepsilon > 0$, is a consequence of a general result of [31]. However, if u_ε has more than one concentration point then its Morse index must be greater than one ([19]). Finally, even if u_ε has only one concentration point, its Morse index could be larger than one since it is related to the index of the critical points of the auxiliary function considered in [7].

Remark 2. Though for solutions of (3.1.6) we have that $u_\varepsilon(P_\varepsilon^i) \rightarrow \infty$, as $\varepsilon \rightarrow 0$, whenever u_ε concentrates at P_ε^i , we have stated Theorem 3.1 requiring only (3.1.7), (3.1.8) (and not (3.1.10) as below), because

the result of Theorem 3.1, with exactly the same proof, holds for any solution u_ε of a problem of the type

$$\begin{cases} -\Delta u = f_\varepsilon(u) & \text{in } A \\ u > 0 & \text{in } A \\ u = 0 & \text{on } \partial A \end{cases}$$

with $f_\varepsilon \in C^1(\mathbb{R})$ strictly convex and which concentrates at k points all located on the same half line passing through the origin as $\varepsilon \rightarrow 0$. For example f_ε could be $f_\varepsilon = \frac{1}{\varepsilon}u_\varepsilon + u_\varepsilon^q$ with $1 < q < \frac{N+2}{N-2}$ and in this case u_ε does not blow up at the concentration points.

Now we add the condition that solutions blow-up at their concentration points

$$(3.1.10) \quad u_\varepsilon(P_\varepsilon^i) \rightarrow \infty \text{ as } \varepsilon \rightarrow 0$$

For solutions which blow up at two points we have the following result

Theorem 3.2. *Let $\{u_\varepsilon\}$ be a family of solutions to (3.1.6) with two blow-up points, P_ε^1 and P_ε^2 , belonging to A . Then, for ε small, the points P_ε^i lay on the same line passing through the origin and u_ε is axially symmetric with respect to this line.*

The proof of the above theorems is based on the procedure developed in [31] to prove the axial symmetry of solutions of index one in the presence of a strictly convex nonlinearity. As already mentioned in Chapter 2, the main idea is to evaluate the sign of the first eigenvalue of the linearized operator in the half domains determined by the symmetry hyperplanes. This procedure is not too difficult in the case of solutions with one concentration point (Theorem 3.1) but requires a careful analysis of the limiting problem in the case of more blow-up points. To do this some results of [7] and [29] are also used.

Let us conclude by observing that the same results hold, with the same proofs, for positive solutions of $-\Delta u = u^{p-\varepsilon} - \lambda u$ in A , $u = 0$ on ∂A for any $\lambda > 0$.

The outline of this chapter is the following: in section 2 we recall some preliminary results, section 3 is devoted to the proof of Theorem 3.1 and in section 4 we prove Theorem 3.2.

2. Preliminaries and notations

Let A be the annulus defined as $A \equiv \{x \in \mathbb{R}^N : 0 < R_1 < |x| < R_2\}$ and T_ν be the hyperplane passing through the origin defined by $T_\nu \equiv \{x \in \mathbb{R}^N : x \cdot \nu = 0\}$, ν being a direction in \mathbb{R}^N . We denote by A_ν^- and A_ν^+ the caps in A determined by T_ν : $A_\nu^- \equiv \{x \in A : x \cdot \nu < 0\}$ and $A_\nu^+ \equiv \{x \in A : x \cdot \nu > 0\}$.

In A we consider problem (3.1.6) and denote by L_ε the linearized operator at a solution u_ε of (3.1.6):

$$(3.2.11) \quad L_\varepsilon = -\Delta - N(N-2)(p-\varepsilon)u_\varepsilon^{p-\varepsilon-1}$$

Let $\lambda_1(L_\varepsilon, D)$ be the first eigenvalue of L_ε in a subdomain $D \subset A$ with zero Dirichlet boundary conditions. By Proposition 1.1 of [31] we have the following

Proposition 3.3. *If $\lambda_1(L_\varepsilon, A_\nu^-)$ and $\lambda_1(L_\varepsilon, A_\nu^+)$ are both non-negative, then u_ε is symmetric with respect to the hyperplane T_ν .*

A slight variation of the previous result is the following

Proposition 3.4. *If either $\lambda_1(L_\varepsilon, A_\nu^-)$ or $\lambda_1(L_\varepsilon, A_\nu^+)$ is non-negative and u_ε has a critical point on $T_\nu \cap A$ then u_ε is symmetric with respect to the hyperplane T_ν .*

The proof of both propositions has been discussed in Section 2

Let $\{u_\varepsilon\}$ be a family of solutions of (3.1.6) with k blow up points P_ε^i , $i = 1, \dots, k$. Then we have

Proposition 3.5. *There exist constants $\alpha_0 > 0$ and $\alpha_{ij} > 0$, $i, j = 1, \dots, k$ such that as $\varepsilon \rightarrow 0$*

$$(3.2.12) \quad |P_\varepsilon^i - P_\varepsilon^j| > \alpha_0 \quad i \neq j$$

$$(3.2.13) \quad \frac{u_\varepsilon(P_\varepsilon^i)}{u_\varepsilon(P_\varepsilon^j)} \rightarrow \alpha_{ij} \text{ for any } i, j \in \{1, \dots, k\}$$

Moreover

$$(3.2.14) \quad (u_\varepsilon(P_\varepsilon^i))^\varepsilon \rightarrow 1$$

Proof: Formulas (3.2.12), (3.2.13) are due to Schoen and can be found in [29] (see also [7]). For (3.2.14) see again [29]. ■

In the sequel we will often use the classical result that for $N \geq 3$ the problem

$$(3.2.15) \quad \begin{cases} -\Delta u = N(N-2)u^p & \text{in } \mathbb{R}^N \\ u(0) = 1 \end{cases}$$

has a unique classical solution which is

$$U(y) = \frac{1}{(1 + |y|^2)^{\frac{N-2}{2}}}$$

Moreover, all non trivial solutions of the linearized problem of (3.2.15) at the solution U , i.e.

$$(3.2.16) \quad -\Delta v = N(N-2)pU^{p-1}v \text{ in } \mathbb{R}^N$$

are linear combinations of the functions

$$(3.2.17) \quad V_0 = \frac{1 - |y|^2}{(1 + |y|^2)^{\frac{N}{2}}}, \quad V_i = \frac{\partial U}{\partial y_i}, \quad i = 1, \dots, N$$

In particular the only non-trivial solutions of the problem

$$(3.2.18) \quad \begin{cases} -\Delta v = N(N-2)pU^{p-1}v & \text{in } \mathbb{R}_-^N = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 < 0\} \\ v = 0 & \text{on } \partial\mathbb{R}_-^N = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 = 0\} \end{cases}$$

are the functions $kV_1 = k\frac{\partial U}{\partial y_1}$, $k \in \mathbb{R}$.

3. Proof of Theorem 3.1

First we consider the case of one concentration point.

Since the solutions can be rotated, without loss of generality, we can assume that the concentration points P_ε belong to the same semi-axis passing through the origin, say the x_N -axis, so that $P_\varepsilon = (0, \dots, 0, t_\varepsilon)$ with $t_\varepsilon > 0$. Let us fix a hyperplane T passing through the x_N -axis and denote by A^- and A^+ the two open caps determined by T . To prove the symmetry of u_ε with respect to T , using Proposition 3.4, we have to show that $\lambda_1(L_\varepsilon, A^-) \geq 0$ for any ε sufficiently small.

For sake of simplicity suppose that T is the hyperplane $T = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 = 0\}$ and let T_0 be the hyperplane defined as $T_0 = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_N = 0\}$.

We claim that

$$(3.3.19) \quad \lambda_1(L_\varepsilon, A_0^-) > 0 \quad \text{for } \varepsilon \text{ small}$$

where $A_0^- = \{x \in \mathbb{R}^N : x_N < 0\} \cap A$.

Indeed, since u_ε concentrates at P_ε and $|P_\varepsilon - x| \geq R_1 > 0$ for any $x \in \overline{A_0^-}$, we have that $u_\varepsilon \rightarrow 0$ uniformly in $\overline{A_0^-}$. Therefore the term $(p - \varepsilon)u_\varepsilon^{p-\varepsilon-1}$ in the expression (3.2.11) of the linearized operator L_ε can be made as small as we like as $\varepsilon \rightarrow 0$.

In particular, for ε sufficiently small, we have that $(p - \varepsilon)u_\varepsilon^{p-\varepsilon-1} < \lambda_1(-\Delta, A_0^-)$, which is the first eigenvalue of the Laplace operator in A_0^- with zero boundary conditions. This implies (3.3.19).

Let us denote by T_ϑ the hyperplane $T_\vartheta = \{x \in \mathbb{R}^N : x_1 \sin \vartheta + x_N \cos \vartheta = 0\}$, with $\vartheta \in [0, \frac{\pi}{2}]$. For $\vartheta = 0$ this hyperplane is T_0 while for $\vartheta = \frac{\pi}{2}$ it coincides with T .

As before, we set $A_{\vartheta}^- = \{x \in \mathbb{R}^N : x_1 \sin \vartheta + x_N \cos \vartheta < 0\}$ and $A_{\vartheta}^+ = \{x \in \mathbb{R}^N : x_1 \sin \vartheta + x_N \cos \vartheta > 0\}$. Because of (3.3.19), for any fixed ε sufficiently small, we can define

$$\tilde{\vartheta} = \sup\{\vartheta \in [0, \frac{\pi}{2}] : \lambda_1(L_\varepsilon, A_{\vartheta}^-) \geq 0\}$$

We would like to prove that $\tilde{\vartheta} = \frac{\pi}{2}$.

If $\tilde{\vartheta} < \frac{\pi}{2}$ then $P_\varepsilon \notin A_{\tilde{\vartheta}}^-$ and $\lambda_1(L_\varepsilon, A_{\tilde{\vartheta}}^-) = 0$, by the definition of $\tilde{\vartheta}$.

Thus, arguing as in Proposition 3.4, we have that

$$w_{\varepsilon, \tilde{\vartheta}}(x) = v_{\varepsilon, \tilde{\vartheta}}(x) - u_\varepsilon(x) \geq 0 \text{ in } A_{\tilde{\vartheta}}^-$$

where $v_{\varepsilon, \tilde{\vartheta}}$ is defined as the reflection of u_ε with respect to $T_{\tilde{\vartheta}}$. Since $u_\varepsilon(P_\varepsilon) > u_\varepsilon(x)$ for any $x \in A_{\tilde{\vartheta}}^-$ we have, by the strong maximum principle, that $w_{\varepsilon, \tilde{\vartheta}} > 0$ in $A_{\tilde{\vartheta}}^-$.

Hence, denoting by P'_ε the point in $A_{\tilde{\vartheta}}^-$ which is given by the reflection of P_ε with respect to $T_{\tilde{\vartheta}}$, we have that

$$(3.3.20) \quad w_{\varepsilon, \tilde{\vartheta}}(x) > \eta > 0 \text{ for } x \in \overline{B(P'_\varepsilon, \delta)} \subset A_{\tilde{\vartheta}}^-$$

where $B(P'_\varepsilon, \delta)$ is the ball with center in P'_ε and radius $\delta > 0$ suitably chosen. Thus

$$(3.3.21) \quad w_{\varepsilon, \tilde{\vartheta} + \sigma}(x) > \frac{\eta}{2} > 0 \text{ for } x \in \overline{B(P''_\varepsilon, \delta)} \subset A_{\tilde{\vartheta} + \sigma}^-$$

for $\sigma > 0$ sufficiently small, where P''_ε is the reflection of P_ε with respect to $T_{\tilde{\vartheta} + \sigma}$.

On the other side, by the monotonicity of the eigenvalues with respect to the domain, we have that $\lambda_1(L_\varepsilon, A_{\tilde{\vartheta}}^- \setminus \overline{B(P'_\varepsilon, \delta)}) > 0$ and hence $\lambda_1(L_\varepsilon, A_{\tilde{\vartheta} + \sigma}^- \setminus \overline{B(P'_\varepsilon, \delta)}) > 0$, for σ sufficiently small.

This implies, by the maximum principle and (3.3.21), that

$$(3.3.22) \quad w_{\varepsilon, \tilde{\vartheta} + \sigma}(x) > 0 \text{ for } x \in A_{\tilde{\vartheta} + \sigma}^-$$

Since $L_\varepsilon(w_{\varepsilon, \tilde{\vartheta}+\sigma}) \geq 0$ in $A_{\tilde{\vartheta}+\sigma}^-$ (by the convexity of the function $u_\varepsilon^{p-\varepsilon}$), the inequality (3.3.22) implies that $\lambda_1(L_\varepsilon, D) > 0$ in any subdomain D of $A_{\tilde{\vartheta}+\sigma}^-$, and so $\lambda_1(L_\varepsilon, A_{\tilde{\vartheta}+\sigma}^-) \geq 0$ for σ positive and sufficiently small. Obviously this contradicts the definition of $\tilde{\vartheta}$ and proves that $\tilde{\vartheta} = \frac{\pi}{2}$, i.e. $\lambda_1(L_\varepsilon, A^-) \geq 0$ as we wanted to show.

We have thus established the symmetry of u_ε with respect to any hyperplane passing through the x_N -axis. The second part of the statement of Theorem 3.1 is merely a consequence of Hopf's lemma.

Indeed, the previous proof shows that $\lambda_1(L_\varepsilon, A_\vartheta^-) \geq 0$ for any $\vartheta \in [0, \frac{\pi}{2})$. This readily implies that the function $w_{\varepsilon, \vartheta} = v_{\varepsilon, \vartheta} - u_\varepsilon$, $v_{\varepsilon, \vartheta}$ being the reflection with respect to T_ϑ , is positive in A_ϑ^- , since $P_\varepsilon \notin A_\vartheta^-$. Thus, applying Hopf's lemma to $w_{\varepsilon, \vartheta}$ (which solves a linear elliptic equation) at any point on $T_\vartheta \cap A$ we get (3.1.9).

Finally it is easy to see that the same proof applies if u_ε has $k > 1$ concentration points all located on the same half-line passing through the origin. ■

4. Proof of Theorem 3.2

In this section we consider solutions of (3.1.6) with two blow-up points, P_ε^1 and P_ε^2 .

Lemma 3.6. *Let $\{u_\varepsilon\}$ be a family of solutions of (3.1.6) with two blow-up points P_ε^1 and P_ε^2 . Then, for ε small, both points P_ε^i , $i = 1, 2$, lay on the same line passing through the origin.*

The proof of this lemma is rather long and will be given later.

Proof of Theorem 3.2 : The first part of the statement is exactly Lemma 3.6. Hence we only have to prove that u_ε is symmetric with respect to any hyperplane passing through the axis containing P_ε^1 and P_ε^2 . Assume that this axis is the x_N -axis and that P_ε^1 and P_ε^2 lay on different sides of this axis with respect to the origin, otherwise the proof is the same as in Theorem 3.1.

First of all let us observe that because the solutions have two blow-up points we have (see [7], [29], [34])

$$(3.4.23) \quad \frac{\int_A |\nabla u_\varepsilon|^2}{\left(\int_A u_\varepsilon^{p-\varepsilon+1}\right)^{\frac{2}{p-\varepsilon+1}}} \xrightarrow{\varepsilon \rightarrow 0} 2^{\frac{2}{N}} S$$

where S is the best Sobolev constant for the embedding of $H_0^1(\mathbb{R}^N)$ in $L^{p+1}(\mathbb{R}^N)$.

Let us fix a hyperplane \mathbb{T} passing through the x_N -axis and, for simplicity, assume that $T = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 = 0\}$, so that $A^- = \{x \in A : x_1 < 0\}$ and $A^+ = \{x \in A : x_1 > 0\}$.

Let us consider in A^- the function

$$w_\varepsilon(x) = v_\varepsilon(x) - u_\varepsilon(x), \quad x \in A^-$$

where v_ε is the reflection of u_ε , i.e. $v_\varepsilon(x_1, \dots, x_N) = u_\varepsilon(-x_1, \dots, x_N)$.

We would like to prove that $w_\varepsilon \equiv 0$ in A^- , for ε small.

Assume, by contradiction, that for a sequence $\varepsilon_n \rightarrow 0$, $w_{\varepsilon_n} = w_n \not\equiv 0$. Let us consider the rescaled functions around $P_n^1 = P_{\varepsilon_n}^1$ and $P_n^2 = P_{\varepsilon_n}^2$:

$$(3.4.24) \quad \tilde{w}_n^1(y) \equiv \frac{1}{\beta_n^1} w_n(P_n^1 + \delta_n y) \quad \tilde{w}_n^2(y) \equiv \frac{1}{\beta_n^2} w_n(P_n^2 + \delta_n y)$$

defined on the rescaled domains $A_{i,n}^- = \frac{A^- - P_n^i}{\delta_n}$, with $\delta_n = (u_n(P_n^1))^{\frac{1-p_n}{2}}$, $p_n = p - \varepsilon_n$ and $\beta_n^i = \|\tilde{w}_n^i\|_{L^{2^*}(A_{i,n}^-)}$, $\tilde{w}_n^i = w_n(P_n^i + \delta_n y)$, $i = 1, 2$.

Notice that, by (3.2.13), both functions are rescaled by the same factor δ_n .

We claim that \tilde{w}_n^i converge in C_{loc}^2 to a function w satisfying

$$(3.4.25) \quad \begin{cases} -\Delta w = N(N-2)pU^{p-1}w & \text{in } \mathbb{R}_-^N = \{y = (y_1, \dots, y_N) \in \mathbb{R}^N : y_1 < 0\} \\ w = 0 & \text{on } \{y = (y_1, \dots, y_N) \in \mathbb{R}^N : y_1 = 0\} \\ \|w\|_{L^{2^*}} \leq 1 \end{cases}$$

where U is the solution of (3.2.15).

Let us prove the claim for \tilde{w}_n^1 , the same proof will apply to \tilde{w}_n^2 , because of (3.2.13).

We have that the functions \tilde{w}_n^1 solve the following problem:

$$(3.4.26) \quad \begin{cases} -\Delta \tilde{w}_n^1 = c_n \tilde{w}_n^1 & \text{in } A_{1,n}^- \\ \tilde{w}_n^1 = 0 & \text{on } \partial A_{1,n}^- \end{cases}$$

where

$$c_n(y) = N(N-2)p_n \int_0^1 \left[t \left(\frac{1}{u_n(P_n^1)} u_n(P_n^1 + \delta_n y) \right) + (1-t) \left(\frac{1}{u_n(P_n^1)} v_n(P_n^1 + \delta_n y) \right) \right]^{p_n-1} dt$$

One can observe that the functions $\tilde{u}_n^1 = \frac{1}{u_n(P_n^1)} u_n(P_n^1 + \delta_n y)$ and $\tilde{v}_n^1 = \frac{1}{u_n(P_n^1)} v_n(P_n^1 + \delta_n y)$ which appear in the definition of $c_n(y)$ are uniformly bounded by (3.2.13) and hence $c_n(y)$ is uniformly bounded too. Thus c_n is locally in any L^q space (in particular $q > \frac{N}{2}$) and hence \tilde{w}_n^1 is locally uniformly bounded.

Then, by standard elliptic estimates and by the convergence in $C_{loc}^2(\mathbb{R}^N)$ of $\tilde{u}_n^1, \tilde{v}_n^1$ to the solution U of (3.2.15), we get the $C_{loc}^2(\bar{\mathbb{R}}_-^N)$ -convergence of \tilde{w}_n^1 to a solution w of (3.4.25).

Let us evaluate the $L^{\frac{N}{2}}$ -norm of c_n :

$$\begin{aligned} \int_{A_{1,n}^-} |c_n(y)|^{\frac{N}{2}} dy &\leq C_N \left[\int_{A_{1,n}^-} \left| \frac{1}{u_n(P_n^1)} u_n(P_n^1 + \delta_n y) \right|^{(p_n-1)\frac{N}{2}} dy \right] + \\ &C_N \left[\int_{A_{1,n}^-} \left| \frac{1}{u_n(P_n^1)} v_n(P_n^1 + \delta_n y) \right|^{(p_n-1)\frac{N}{2}} dy \right] \end{aligned}$$

where C_N is a constant which depends only on N .

For the first integral in the previous formula we have

$$\int_{A_{1,n}^-} \left| \frac{1}{u_n(P_n^1)} u_n(P_n^1 + \delta_n y) \right|^{(p_n-1)\frac{N}{2}} dy = \int_{A^-} |u_n(x)|^{2^* - \frac{N\varepsilon_n}{2}} dx \leq B_N$$

by (3.4.23) and (3.1.6), B_N being a constant depending only on N .

An analogous estimate holds for the second integral.

Hence the $L^{\frac{N}{2}}$ -norm of c_n is uniformly bounded and we have

$$(3.4.27) \quad \int_{A_{1,n}^-} |c_n(y)|^{\frac{N}{2}} dy \leq 2C_N B_N$$

Then multiplying (3.4.26) by \tilde{w}_n^1 and integrating we have that

$$(3.4.28) \quad \begin{aligned} & \int_{A_{1,n}^-} |\nabla \tilde{w}_n^1|^2 dy = \int_{A_{1,n}^-} c_n (\tilde{w}_n^1)^2 dy \leq \\ & \leq \left(\int_{A_{1,n}^-} |c_n|^{\frac{N}{2}} dy \right)^{\frac{2}{N}} \left(\int_{A_{1,n}^-} |\tilde{w}_n^1|^{2^*} dy \right)^{\frac{2}{2^*}} \leq (2C_N B_N)^{\frac{2}{N}} \end{aligned}$$

Then by (3.2.16) - (3.2.18) we get $w = kV_1 = k\frac{\partial U}{\partial y_1}$, $k \in \mathbb{R}$, since, by (3.4.28) $w \in D^{1,2}(\mathbb{R}_-^N) = \{\varphi \in L^{2^*}(\mathbb{R}_-^N) : |\nabla \varphi| \in L^2(\mathbb{R}_-^N)\}$.

Let us first assume that for one of the two sequences $\{\tilde{w}_n^i\}$, say $\{\tilde{w}_n^1\}$, the limit is $w = k\frac{\partial U}{\partial y_1}$ with $k \neq 0$.

Then, since the points P_n^1 are on the reflection hyperplane T and $\nabla u_n(P_n^1) = 0$ we have that $\frac{\partial \tilde{w}_n^1}{\partial y_1}(0) = 0$. This implies that $\frac{\partial w}{\partial y_1}(0) = k\frac{\partial^2 U}{\partial y_1^2}(0) = 0$ with $k \neq 0$, which is a contradiction since for the function $U(y) = \frac{1}{(1+|y|^2)^{\frac{N-2}{2}}}$ we have $\frac{\partial^2 U}{\partial y_1^2}(0) < 0$.

So we are left with the case when both sequences \tilde{w}_n^i converge to zero in C_{loc}^2 .

Then, for any fixed R and for n sufficiently large in the domains $E_{i,n}(R) = B(0, R) \cap A_{i,n}^-$ we have the following estimates

$$(3.4.29) \quad |\tilde{w}_n^i(y)| \leq \frac{S}{4(2C_N B_N)^2 |B(0, R)|^{\frac{2}{2^*}}} \quad i = 1, 2$$

where $|B(0, R)|$ is the measure of the ball $B(0, R)$.

Now we focus only on the rescaling around P_n^1 and observe that the domains $E_{2,n}(R)$, under the rescaling around P_n^1 , correspond to domains $F_{2,n}(R)$ contained in $A_{1,n}^-$ which are translations of $E_{1,n}(R)$ by the vector $\frac{P_n^2 - P_n^1}{\delta_n}$ and also the function \tilde{w}_n^2 is the translation of \tilde{w}_n^1 by the same vector, indeed $\tilde{w}_n^2 = \tilde{w}_n^1 \left(y + \frac{P_n^2 - P_n^1}{\delta_n} \right)$

Hence from (3.4.29) we have

$$(3.4.30) \quad |\tilde{w}_n^1(y)| \leq \frac{S}{4(2C_N B_N)^2 |B(0, R)|^{\frac{2}{2^*}}} \quad \text{in } (E_{1,n}(R) \cup F_{2,n}(R))$$

Now let us choose R sufficiently large such that

$$(3.4.31) \quad \int_{B(0,R)} |U|^{2^*} > \left(\frac{15}{16}S\right)^{\frac{N}{2}}$$

where U is the solution of (3.2.15). Then, since both functions \tilde{u}_n^i ($i = 1, 2$) which appear in the definition of c_n converge to the function U and the function \tilde{u}_n^1 is just the translation of the function $\tilde{u}_n^2 = \frac{1}{u_n(P_n^1)} u_n(P_n^2 + \delta_n y)$ by the vector $\frac{P_n^2 - P_n^1}{\delta_n}$, we have by (3.4.31)

$$(3.4.32) \quad \int_{B(0,R) \cup B(\frac{P_n^2 - P_n^1}{\delta_n}, R)} |\tilde{u}_n^1|^{p_n+1} > \left(\frac{7}{4}S\right)^{\frac{N}{2}}$$

for n sufficiently large. This implies, by (3.4.23)

$$(3.4.33) \quad \int_{A_{1,n}^- \setminus (E_{1,n}(R) \cup F_{2,n}(R))} |c_n|^{\frac{N}{2}} < \left(\frac{1}{4}S\right)^{\frac{N}{2}}$$

Since the functions \tilde{w}_n^1 solve (3.4.26), multiplying (3.4.26) by \tilde{w}_n^1 and integrating we get

$$\begin{aligned} & \int_{A_{1,n}^-} |\nabla \tilde{w}_n^1|^2 dy = \int_{A_{1,n}^-} c_n (\tilde{w}_n^1)^2 dy = \\ & \int_{A_{1,n}^- \setminus (E_{1,n}(R) \cup F_{2,n}(R))} c_n (\tilde{w}_n^1)^2 dy + \int_{(E_{1,n}(R) \cup F_{2,n}(R))} c_n (\tilde{w}_n^1)^2 dy \leq \\ & \left(\int_{A_{1,n}^- \setminus (E_{1,n}(R) \cup F_{2,n}(R))} |c_n|^{\frac{N}{2}} dy \right)^{\frac{2}{N}} \left(\int_{A_{1,n}^- \setminus (E_{1,n}(R) \cup F_{2,n}(R))} |\tilde{w}_n^1|^{2^*} dy \right)^{\frac{2}{2^*}} + \\ & + \left(\int_{(E_{1,n}(R) \cup F_{2,n}(R))} |c_n|^{\frac{N}{2}} dy \right)^{\frac{2}{N}} \left(\int_{(E_{1,n}(R) \cup F_{2,n}(R))} |\tilde{w}_n^1|^{2^*} dy \right)^{\frac{2}{2^*}} \leq \frac{S}{2} \end{aligned}$$

because $\|\tilde{w}_n^1\|_{L^{2^*}(A_{1,n}^-)} = 1$, the $L^{\frac{N}{2}}$ -norm of c_n is uniformly bounded by (3.4.27), (3.4.33) and (3.4.30) hold.

On the other hand, by the Sobolev inequality, we have

$$\int_{A_{1,n}^-} |\nabla \tilde{w}_n^1|^2 dy > S$$

which gives a contradiction.

Hence the sequences \tilde{w}_n^i cannot converge both to zero, so that $w_\varepsilon \equiv 0$ for ε small, as we wanted to prove. \blacksquare

Finally we prove Lemma 3.6 in several steps

Proof of Lemma 3.6: Let us assume that the line connecting P_ε^1 with the origin is the x_N -axis. We would like to show that also the point P_ε^2 belongs to the same axis. So we assume by contradiction that for a sequence $\varepsilon_n \rightarrow 0$ the points $P_{\varepsilon_n}^2 = P_n^2$ are given by $P_n^2 = (\alpha_n, x_2^n, \dots, x_N^n)$, $\alpha_n > 0$, where the first coordinate α_n represents the distance of P_n^2 from the x_N -axis. As before we define $\delta_n = (u_n(P_n^1))^{\frac{1-p_n}{2}}$ where $p_n = p - \varepsilon_n$.

Claim 1 *It is not possible that*

$$(3.4.34) \quad \frac{\alpha_n}{\delta_n} \xrightarrow{n \rightarrow \infty} \infty$$

Assume that (3.4.34) holds and consider the hyperplane $T = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 = 0\}$ which obviously passes through the x_N -axis and does not contain the point P_n^2 . We claim that, for n sufficiently large,

$$(3.4.35) \quad \lambda_1(L_n, A^-) \geq 0$$

where, as before, $L_n \equiv L_{\varepsilon_n}$ denotes the linearized operator and $A^- = \{x = (x_1, \dots, x_N) \in A : x_1 < 0\}$. To prove (3.4.35) let us take the two balls $B(P_n^i, R\delta_n)$ centered at the two points P_n^i and with radius $R\delta_n$, $R > 1$ to be fixed later.

By (3.4.34) and (3.2.13) we have that $B(P_n^2, R\delta_n)$ does not intersect A^- , for large n . Moreover if we take $\vartheta_0 \in [0, \frac{\pi}{2}]$ and we consider the hyperplane $T_{\vartheta_0} = \{x = (x_1, \dots, x_N) : x_1 \sin \vartheta_0 + x_N \cos \vartheta_0 = 0\}$, by (3.4.34), (3.2.13) and the fact that P_n^1 belongs to $T = T_{\frac{\pi}{2}}$ we can choose $\vartheta_{0,n} < \frac{\pi}{2}$ and close to $\frac{\pi}{2}$ such that both balls $B(P_n^i, R\delta_n)$ do not intersect the cap $A_{\vartheta_{0,n}}^- = \{x = (x_1, \dots, x_N) : x_1 \sin \vartheta_{0,n} + x_N \cos \vartheta_{0,n} < 0\}$ for n large enough.

Then, arguing as in [23] (see also [19]), it is easy to see that it is possible to choose R such that $\lambda_1(L_n, A_{\vartheta_{0,n}}^-) > 0$ for n large, because $B(P_n^i, R\delta_n) \cap A_{\vartheta_{0,n}}^- = \emptyset$, $i = 1, 2$ and u_n concentrates only at P_n^i .

Hence, fixing n sufficiently large, we set

$$\tilde{\vartheta}_n \equiv \sup\{\vartheta \in [\vartheta_{0,n}, \frac{\pi}{2}] : \lambda_1(L_n, A_{\vartheta}^-) \geq 0\}$$

and, repeating the same procedure as in the proof of Theorem 3.1, we get that $\tilde{\vartheta}_n = \frac{\pi}{2}$ and hence (3.4.35) holds.

So, by Proposition 3.4, since $P_n^1 \in T = T_{\frac{\pi}{2}}$, we get that u_n is symmetric with respect to the hyperplane T , which is not possible, since P_n^2 does not belong to T . Hence (3.4.34) cannot hold.

Claim 2 *It is not possible that*

$$(3.4.36) \quad \frac{\alpha_n}{\delta_n} \xrightarrow{n \rightarrow \infty} l > 0$$

Assume that (3.4.36) holds and, as before, denote by T the hyperplane $T = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 = 0\}$ to which P_n^1 belongs while $P_n^2 \notin T$.

We would like to prove as in Claim 1 that

$$(3.4.37) \quad \lambda_1(L_n, A^-) \geq 0$$

If the points P_n^1 and P_n^2 have the N -th coordinate of the same sign, i.e. they lay on the same side with respect to the hyperplane $\{x_N = 0\}$, then it is obvious that we can argue exactly as for the first claim and

choose $\vartheta_0 \in [0, \frac{\pi}{2}]$ such that both balls $B(P_n^i, R\delta_n)$, R as before, do not intersect the cap $A_{\vartheta_0}^-$. Then the proof is the same as before.

Hence we assume that P_n^1 and P_n^2 lay on different sides with respect to the hyperplane $\{x_N = 0\}$. Let us then consider $\vartheta_n \in [0, \frac{\pi}{2}]$ such that the points P_n^1 and P_n^2 have the same distance $d_n > 0$ from the hyperplane

$$T_{\vartheta_n} = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 \sin \vartheta_n + x_N \cos \vartheta_n = 0\}$$

Of course, because of (3.4.36), we have

$$(3.4.38) \quad \frac{d_n}{\delta_n} \xrightarrow{n \rightarrow \infty} l_1 > 0$$

Then, choosing $R > 0$ such that $\lambda_1(L_n, D_n^R) > 0$, for n large, $D_n^R = A \setminus [B(P_n^1, R\delta_n) \cup B(P_n^2, R\delta_n)]$ (see [23]), either both balls $B(P_n^i, R\delta_n)$ do not intersect the cap $A_{\vartheta_n}^-$, for n large enough, or they do. In the first case we argue as for the first claim. In the second case we observe that in each set $E_{\vartheta_n}^{n,i} = A_{\vartheta_n}^- \cap B(P_n^i, R\delta_n)$, $i = 1, 2$, we have, for n large

$$(3.4.39) \quad u_n(x) \leq v_n^{\vartheta_n}(x) \quad x \in E_{\vartheta_n}^{n,i} \quad i = 1, 2$$

where $v_n^{\vartheta_n}(x) = u_n(x^{\vartheta_n})$, x^{ϑ_n} being the reflection of x with respect to $T_{\vartheta_n}^n$.

In fact if (3.4.39) were not true we could construct a sequence of points $x_{n_k} \in E_{\vartheta_{n_k}}^{n_k,i}$, $i = 1$ or 2 , such that

$$(3.4.40) \quad u_{n_k}(x_{n_k}) > v_{n_k}^{\vartheta_{n_k}}(x_{n_k})$$

Then there would exist a sequence of points $\xi_{n_k} \in E_{\vartheta_{n_k}}^{n_k,i}$ such that

$$(3.4.41) \quad \frac{\partial u_{n_k}}{\partial \vartheta_{n_k}}(\xi_{n_k}) < 0$$

Thus, by rescaling u_{n_k} in the usual way around $P_{n_k}^1$ or $P_{n_k}^2$, and using (3.4.38) we would get a point $\xi \in (E_{\vartheta_0}^{l_1})^- = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N :$

$x_1 \sin \vartheta_0 + x_N \cos \vartheta_0 < -l_1 < 0$ such that $\frac{\partial U}{\partial \vartheta_0}(\xi) \leq 0$ while $\frac{\partial U}{\partial \vartheta_0} > 0$ in $(E_{\vartheta_0}^{l_1})^-$, ϑ_0 being the limit of ϑ_{n_k} .

Hence (3.4.39) holds.

Now, arguing again as in [23] and [19], in the set $(F_{\vartheta_n}^n)^- = A_{\vartheta_n}^- \setminus (B(P_n^1, R\delta_n) \cup B(P_n^2, R\delta_n))$ we have that $\lambda_1(L_n, (F_{\vartheta_n}^n)^-) \geq 0$.

Hence, by (3.4.39), applying the maximum principle, we have that $w_{n, \vartheta_n}(x) \geq 0$ in $(F_{\vartheta_n}^n)^-$, and, again by (3.4.39) and the strong maximum principle

$$(3.4.42) \quad w_{n, \vartheta_n}(x) > 0 \quad \text{in } A_{\vartheta_n}^-$$

As in the proof of Theorem 3.2, this implies that $\lambda_1(L_n, A_{\vartheta_n}^-) \geq 0$.

Then, arguing as for the first claim we get (3.4.37), which gives the same kind of contradiction because P_n^2 does not belong to T .

Claim 3 *It is not possible that*

$$(3.4.43) \quad \frac{\alpha_n}{\delta_n} \xrightarrow{n \rightarrow \infty} 0$$

Let us argue by contradiction and assume that (3.4.43) holds. As before we denote by T the hyperplane $T = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 = 0\}$. Since the points P_n^2 are in the domain $A_n^+ = \{x = (x_1, \dots, x_N) \in A : x_1 > 0\}$, we have that the function

$$w_n(x) = v_n(x) - u_n(x), \quad x \in A_n^+$$

where v_n is the reflection of u_n , i.e. $v_n(x_1, \dots, x_N) = u_n(-x_1, x_2, \dots, x_N)$, is not identically zero.

Then, as in the proof of Theorem 3.2, rescaling the function w_n around P_n^1 or P_n^2 and using (3.2.13) we have that the functions

$$(3.4.44) \quad \tilde{w}_n^i(y) \equiv \frac{1}{\beta_n^i} w_n(P_n^i + \delta_n y), \quad i = 1, 2$$

defined in the rescaled domain $A_{i,n}^+ = \frac{A^+ - P_n^i}{\delta_n}$, converge both, by (3.4.43) and standard elliptic estimates, in C_{loc}^2 to a function w_i satisfying (3.4.25) but in the half space $\mathbb{R}_+^N = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 > 0\}$.

Again by (3.2.16) - (3.2.18) we have that $w_i = k_i \frac{\partial U}{\partial y_1}$, $k_i \in \mathbb{R}$, where U is the solution of (3.2.15).

Exactly as in the proof of Theorem 3.2 we can exclude the case that both sequences \tilde{w}_n^i converge to zero in C_{loc}^2 .

Hence for at least one of the two sequences \tilde{w}_n^i we have that the limit is $w_i = k_i \frac{\partial U}{\partial y_1}$ with $k_i \neq 0$.

If this happens for \tilde{w}_n^1 then, since the points P_n^1 are on the reflection hyperplane T , arguing exactly as in the proof of Theorem 3.2, we get a contradiction.

So we are left with the case when $\tilde{w}_n^1 \rightarrow k_1 \frac{\partial U}{\partial y_1}$, $k_1 = 0$ and $\tilde{w}_n^2 \rightarrow k_2 \frac{\partial U}{\partial y_1}$ with $k_2 \neq 0$ in C_{loc}^2 .

At the points P_n^2 , obviously we have that $\frac{\partial u_n}{\partial y_1}(P_n^2) = 0$.

Let us denote by \tilde{P}_n^2 the reflection of P_n^2 with respect to T .

Hence, for the function \tilde{w}_n^2 we have, applying the mean value theorem

$$\begin{aligned} \frac{\partial \tilde{w}_n^2}{\partial y_1}(0) &= \frac{\delta_n^{\frac{2}{1-p_n}}}{\beta_n^2} \left(\frac{\partial \tilde{u}_n}{\partial y_1}(0) + \frac{\partial \tilde{u}_n}{\partial y_1} \left(\frac{\tilde{P}_n^2 - P_n^2}{\delta_n} \right) \right) = \\ &= \frac{\delta_n^{\frac{2}{1-p_n}}}{\beta_n^2} \left(\frac{\partial \tilde{u}_n}{\partial y_1} \left(\frac{\tilde{P}_n^2 - P_n^2}{\delta_n} \right) - \frac{\partial \tilde{u}_n}{\partial y_1}(0) \right) = \\ &= - \frac{\delta_n^{\frac{2}{1-p_n}}}{\beta_n^2} \frac{\partial^2 \tilde{u}_n}{\partial y_1^2}(\xi_n) \frac{2\alpha_n}{\delta_n} \end{aligned}$$

where $\tilde{u}_n(y) = \delta_n^{\frac{2}{p_n-1}} u_n(P_n^2 + \delta_n y)$ and ξ_n belongs to the segment joining the origin with the point $\frac{\tilde{P}_n^2 - P_n^2}{\delta_n}$ in the rescaled domain $A_{2,n}^+$.

Since $\frac{\partial \tilde{w}_n^2}{\partial y_1}(0) \rightarrow k_2 \frac{\partial^2 U}{\partial y_1^2}(0)$ and $\frac{\partial^2 \tilde{u}_n}{\partial y_1^2}(\xi_n) \rightarrow \frac{\partial^2 U}{\partial y_1^2}(0)$, with $k_2 \neq 0$ and $\frac{\partial^2 U}{\partial y_1^2}(0) < 0$ we get

$$(3.4.45) \quad \frac{\alpha_n \delta_n^{\frac{2}{1-p_n}}}{\beta_n^2 \delta_n} \rightarrow \gamma \neq 0$$

Our aim is now to prove that (3.4.45) implies that $k_1 \neq 0$ which will give a contradiction.

Let us observe that if the function w_n does not change sign near P_n^1 , then, since $w_n \not\equiv 0$, we would get a contradiction, applying Hopf's lemma to w_n (which solves a linear elliptic equation) at the point P_n^1 , because $\nabla u_n(P_n^1) = 0$.

Then in any ball $B(P_n^1, \alpha_n)$, α_n as in (3.4.43), there are points Q_n^1 such that $\frac{\partial u_n}{\partial y_1}(Q_n^1) = 0$ and $Q_n^1 \notin T$. Indeed, since w_n changes sign near P_n^1 , in any set $B(P_n^1, \alpha_n) \cap A^+$ there are points where w_n is zero, i.e. u_n coincides with the reflection v_n . This implies that there exist points Q_n^1 in $B(P_n^1, \alpha_n)$ where $\frac{\partial u_n}{\partial y_1}(Q_n^1) = 0$, and by Hopf's lemma applied to the points of the hyperplane T we have that $Q_n^1 \notin T$. Let us denote by \tilde{Q}_n^1 the reflection of Q_n^1 with respect to T .

Assume that $Q_n^1 \in A^-$ (the argument is the same if $Q_n^1 \in A^+$). Then as before we have

$$\begin{aligned} \frac{\partial \tilde{w}_n^1}{\partial y_1} \left(\frac{Q_n^1 - P_n^1}{\delta_n} \right) &= \frac{\delta_n^{\frac{2}{1-p_n}}}{\beta_n^1} \left(\frac{\partial \tilde{u}_n}{\partial y_1} \left(\frac{\tilde{Q}_n^1 - P_n^1}{\delta_n} \right) - \frac{\partial \tilde{u}_n}{\partial y_1} \left(\frac{Q_n^1 - P_n^1}{\delta_n} \right) \right) = \\ &= -\frac{\delta_n^{\frac{2}{1-p_n}}}{\beta_n^1} \frac{\partial^2 \tilde{u}_n}{\partial y_1^2}(\xi_n) \frac{2\alpha_n}{\delta_n} \end{aligned}$$

where ξ_n belongs to the segment joining $\frac{\tilde{Q}_n^1 - P_n^1}{\delta_n}$ and $\frac{Q_n^1 - P_n^1}{\delta_n}$ in the rescaled domain $A_{1,n}^+$.

Since $\frac{\partial \tilde{w}_n^1}{\partial y_1} \left(\frac{\tilde{Q}_n^1 - P_n^1}{\delta_n} \right) \rightarrow k_1 \frac{\partial^2 U}{\partial y_1^2}(0)$, $\frac{\partial^2 \tilde{u}_n}{\partial y_1^2}(\xi_n) \rightarrow \frac{\partial^2 U}{\partial y_1^2}(0) < 0$ and using (3.4.45), we get $k_1 \neq 0$ and hence a contradiction.

So also the third claim is true and the proof of Lemma 3.6 is complete.

■

CHAPTER 4

The almost critical problem in an annulus - Part II

1. Introduction

In this chapter we will discuss the results contained in [16]. We continue the study of the symmetry of solutions of the problem

$$(4.1.46) \quad \begin{cases} -\Delta u = N(N-2)u^{p-\varepsilon} & \text{in } A \\ u > 0 & \text{in } A \\ u = 0 & \text{on } \partial A \end{cases}$$

where A is an annulus centered at the origin in \mathbb{R}^N , $N \geq 3$, $p+1 = \frac{2N}{N-2}$ is the critical Sobolev exponent for the embedding of $H_0^1(A)$ into $L^{p+1}(A)$ and $\varepsilon > 0$ is a small parameter.

In Chapter 3 we analyzed the symmetry of solutions to (3.1.6) which concentrate at one or two points, as $\varepsilon \rightarrow 0$. Indeed it is well known that the study of (4.1.46) is strictly related to the limiting problem ($\varepsilon = 0$) which exhibits a lack of compactness and gives rise to solutions of (4.1.46) which concentrate and blow up as $\varepsilon \rightarrow 0$ ([7], [10], [27], [33], [34]). Obviously, solutions of (4.1.46) which blow-up at a finite number of points cannot be radially symmetric. Nevertheless in [15] we proved that solutions that concentrate at one or two points are axially symmetric with respect to an axis passing through the origin which contains the concentration points.

In this chapter we consider the case of solutions which concentrate at $k \geq 3$, $k \leq N$, points in A and prove a partial symmetry result.

To be more precise we need some notations.

We say that a family of solutions $\{u_\varepsilon\}$ of (4.1.46) has $k \geq 1$ concentration points at $\{P_\varepsilon^1, P_\varepsilon^2, \dots, P_\varepsilon^k\} \subset A$ if the following holds

$$(4.1.47) \quad P_\varepsilon^i \neq P_\varepsilon^j, i \neq j \text{ and each } P_\varepsilon^i \text{ is a strict local maximum for } u_\varepsilon$$

$$(4.1.48) \quad u_\varepsilon \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \text{ locally uniformly in } \Omega \setminus \{P_\varepsilon^1, P_\varepsilon^2, \dots, P_\varepsilon^k\}$$

$$(4.1.49) \quad u_\varepsilon(P_\varepsilon^i) \rightarrow \infty \text{ as } \varepsilon \rightarrow 0$$

Our result is the following

Theorem 4.1. *Let $\{u_\varepsilon\}$ be a family of solutions to (4.1.46) which concentrate at k points $P_\varepsilon^j \in A$, $j = 1, \dots, k$, $k \geq 3$ and $k \leq N$. Then, for ε small, the points P_ε^j lie on the same $(k-1)$ -dimensional hyperplane Π_k passing through the origin and u_ε is symmetric with respect to any $(N-1)$ -dimensional hyperplane containing Π_k .*

As in [15] the proof of the above theorem is based on the procedure developed in [31] to prove the symmetry of solutions of semilinear elliptic equations in the presence of a strictly convex nonlinearity. The main idea is to evaluate the sign of the first eigenvalue of the linearized operator in the half domains determined by the symmetry hyperplanes. To carry out this procedure we also use results of [7] and [29].

The outline of the chapter is the following: in Section 2 we recall some preliminary results and prove a geometrical lemma, while in Section 3 we prove Theorem 4.1.

2. Preliminaries

Let A be the annulus defined as $A \equiv \{x \in \mathbb{R}^N : 0 < R_1 < |x| < R_2\}$ and T_ν be the hyperplane passing through the origin defined by $T_\nu \equiv \{x \in \mathbb{R}^N : x \cdot \nu = 0\}$, ν being a direction in \mathbb{R}^N . We denote by A_ν^- and A_ν^+ the caps in A determined by T_ν : $A_\nu^- \equiv \{x \in A : x \cdot \nu < 0\}$ and $A_\nu^+ \equiv \{x \in A : x \cdot \nu > 0\}$.

In A we consider problem (4.1.46) and denote by L_ε the linearized operator at a solution u_ε of (4.1.46):

$$(4.2.50) \quad L_\varepsilon = -\Delta - N(N-2)(p-\varepsilon)u_\varepsilon^{p-\varepsilon-1}$$

Let $\lambda_1(L_\varepsilon, D)$ be the first eigenvalue of L_ε in a subdomain $D \subset A$ with zero Dirichlet boundary conditions.

In [15] the following proposition, which is a variant of a result of [31], was proved

Proposition 4.2. *If either $\lambda_1(L_\varepsilon, A_\nu^-)$ or $\lambda_1(L_\varepsilon, A_\nu^+)$ is non-negative and u_ε has a critical point on $T_\nu \cap A$ then u_ε is symmetric with respect to the hyperplane T_ν .*

Let us recall some results about solutions of (4.1.46), proved in [29] and [7].

Let $\{u_\varepsilon\}$ be a family of solutions of (4.1.46) with k blow up points P_ε^i , $i = 1, \dots, k$. Then we have

Proposition 4.3. *There exist constants $\alpha_0 > 0$ and $\alpha_{ij} > 0$, $i, j = 1, \dots, k$ such that as $\varepsilon \rightarrow 0$*

$$(4.2.51) \quad |P_\varepsilon^i - P_\varepsilon^j| > \alpha_0 \quad i \neq j$$

$$(4.2.52) \quad \frac{u_\varepsilon(P_\varepsilon^i)}{u_\varepsilon(P_\varepsilon^j)} \rightarrow \alpha_{ij} \text{ for any } i, j \in \{1, \dots, k\}$$

Moreover

$$(4.2.53) \quad (u_\varepsilon(P_\varepsilon^i))^\varepsilon \rightarrow 1$$

In the sequel we will often use the classical result that for $N \geq 3$ the problem

$$(4.2.54) \quad \begin{cases} -\Delta u = N(N-2)u^p & \text{in } \mathbb{R}^N \\ u(0) = 1 \end{cases}$$

has a unique classical solution which is

$$(4.2.55) \quad U(y) = \frac{1}{(1 + |y|^2)^{\frac{N-2}{2}}}$$

Moreover, all non trivial solutions of the linearized problem of (4.2.54) at the solution U , i.e.

$$(4.2.56) \quad -\Delta v = N(N-2)pU^{p-1}v \text{ in } \mathbb{R}^N$$

are linear combinations of the functions

$$(4.2.57) \quad V_0 = \frac{1 - |y|^2}{(1 + |y|^2)^{\frac{N}{2}}}, \quad V_i = \frac{\partial U}{\partial y_i}, \quad i = 1, \dots, N$$

In particular the only non-trivial solutions of the problem

$$(4.2.58) \quad \begin{cases} -\Delta v = N(N-2)pU^{p-1}v & \text{in } \mathbb{R}_-^N = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 < 0\} \\ v = 0 & \text{on } \partial\mathbb{R}_-^N = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 = 0\} \end{cases}$$

are the functions $kV_1 = k\frac{\partial U}{\partial y_1}$, $k \in \mathbb{R}$.

We conclude this section with a geometrical lemma that will be used in the proof of Theorem 4.1.

Lemma 4.4. *Let $\{P_1, \dots, P_k\}$, $2 \leq k \leq N$, be k points in \mathbb{R}^N , $P_i \neq 0 \in \mathbb{R}^N$. Then*

(i) *if the line passing through 0 and P_1 does not contain any P_i , $i \neq 1$, then there exist two $(N-1)$ -dimensional parallel hyperplanes T and Σ with T passing through the origin 0 such that $P_1 \in T$ and $P_i \in \Sigma$, for any $i \in \{2, \dots, k\}$;*

(ii) *if the line passing through 0 and P_1 contains some P_i 's, $i \neq 1$, then there exists a $(k-1)$ -dimensional hyperplane Π passing through the origin containing all points P_i , $i = 1, \dots, k$.*

Proof: In the case (i) let us consider the vectors $v_1 = P_1 - 0$, $v_2 = P_2 - P_3, \dots, v_{k-1} = P_{k-1} - P_k$, $v_k = P_k - 0$.

The vectors $\{v_1, \dots, v_{k-1}\}$ obviously span a $(k-1)$ -dimensional vector space. Let us consider any $(N-1)$ -dimensional subspace V containing $\{v_1, \dots, v_{k-1}\}$ and not containing v_k and let us define $T = V$ and $\Sigma = v_k + V$. Then the first assertion is proved.

In the case (ii) $\{v_1, \dots, v_k\}$ are linearly dependent and so they are contained in a $(k-1)$ -dimensional hyperplane Π passing through the origin.

■

3. Proof of Theorem 4.1

We start by stating a lemma, whose proof will be given later

Lemma 4.5. *Let $\{u_\varepsilon\}$ be a family of solutions of (4.1.46) with k blow-up points P_ε^i , $i = 1, \dots, k$, $3 \leq k \leq N$. Then, for ε small, all points P_ε^i , $i = 1, \dots, k$, lie on the same $(k-1)$ -dimensional hyperplane Π_k passing through the origin.*

Proof of Theorem 4.1: The proof is similar to that of Theorem 2 of [15], we will write the details for the reader's convenience. The first part of the statement is exactly Lemma 4.5. Hence we only have to prove that u_ε is symmetric with respect to any hyperplane containing Π_k . For simplicity let us assume that $\Pi_k = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 = 0, \dots, x_{N-(k-1)} = 0\}$.

Let us observe that because the solutions have k blow-up points we have (see [7], [29], [34])

$$(4.3.59) \quad \frac{\int_A |\nabla u_\varepsilon|^2}{\left(\int_A u_\varepsilon^{p-\varepsilon+1}\right)^{\frac{2}{p-\varepsilon+1}}} \xrightarrow{\varepsilon \rightarrow 0} k^{\frac{2}{N}} S$$

where S is the best Sobolev constant for the embedding of $H_0^1(\mathbb{R}^N)$ in $L^{p+1}(\mathbb{R}^N)$.

Let us fix a $(N-1)$ -hyperplane T containing Π_k and, for simplicity, assume that $T = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 = 0\}$, so that $A^- = \{x \in A : x_1 < 0\}$ and $A^+ = \{x \in A : x_1 > 0\}$.

Let us consider in A^- the function

$$w_\varepsilon(x) = v_\varepsilon(x) - u_\varepsilon(x), \quad x \in A^-$$

where v_ε is the reflection of u_ε , i.e. $v_\varepsilon(x_1, \dots, x_N) = u_\varepsilon(-x_1, \dots, x_N)$.

We would like to prove that $w_\varepsilon \equiv 0$ in A^- , for ε small.

Assume, by contradiction, that for a sequence $\varepsilon_n \rightarrow 0$, $w_{\varepsilon_n} = w_n \not\equiv 0$. Let us consider the rescaled functions around $P_n^i = P_{\varepsilon_n}^i$, $i = 1, \dots, k$:

$$(4.3.60) \quad \tilde{w}_n^i(y) \equiv \frac{1}{\beta_n^i} w_n(P_n^i + \delta_n y)$$

defined on the rescaled domains $A_{i,n}^- = \frac{A^- - P_n^i}{\delta_n}$, with $\delta_n = (u_n(P_n^1))^{\frac{1-p_n}{2}}$, $p_n = p - \varepsilon_n$ and $\beta_n^i = \|\tilde{w}_n^i\|_{L^{2^*}(A_{i,n}^-)}$, $\tilde{w}_n^i = w_n(P_n^i + \delta_n y)$, $i = 1, \dots, k$.

Notice that, by (4.2.52), all functions are rescaled by the same factor δ_n .

We claim that \tilde{w}_n^i converge in C_{loc}^2 to a function w satisfying

$$(4.3.61) \quad \begin{cases} -\Delta w = N(N-2)pU^{p-1}w & \text{in } \mathbb{R}_-^N = \{y = (y_1, \dots, y_N) \in \mathbb{R}^N : y_1 < 0\} \\ w = 0 & \text{on } \partial\mathbb{R}_-^N = \{y = (y_1, \dots, y_N) \in \mathbb{R}^N : y_1 = 0\} \\ \|w\|_{L^{2^*}} \leq 1 \end{cases}$$

where U is defined in (4.2.55).

Let us prove the claim for \tilde{w}_n^1 , the same proof will apply to any \tilde{w}_n^i , because of (4.2.52).

We have that the functions \tilde{w}_n^1 solve the following problem:

$$(4.3.62) \quad \begin{cases} -\Delta \tilde{w}_n^1 = c_n \tilde{w}_n^1 & \text{in } A_{1,n}^- \\ \tilde{w}_n^1 = 0 & \text{on } \partial A_{1,n}^- \end{cases}$$

where

$$c_n(y) = N(N-2)p_n \int_0^1 \left[t \left(\frac{1}{u_n(P_n^1)} u_n(P_n^1 + \delta_n y) \right) + (1-t) \left(\frac{1}{u_n(P_n^1)} v_n(P_n^1 + \delta_n y) \right) \right]^{p_n-1} dt$$

One can observe that the functions $\tilde{u}_n^1 = \frac{1}{u_n(P_n^1)} u_n(P_n^1 + \delta_n y)$ and $\tilde{v}_n^1 = \frac{1}{u_n(P_n^1)} v_n(P_n^1 + \delta_n y)$ which appear in the definition of $c_n(y)$ are uniformly bounded by (4.2.52) and hence $c_n(y)$ is uniformly bounded too. Thus

c_n is locally in any L^q space (in particular $q > \frac{N}{2}$) and hence \tilde{w}_n^1 is locally uniformly bounded.

Then, by standard elliptic estimates and by the convergence in $C_{loc}^2(\mathbb{R}^N)$ of $\tilde{u}_n^1, \tilde{v}_n^1$ to the solution U of (4.2.54), we get the $C_{loc}^2(\bar{\mathbb{R}}_-^N)$ -convergence of \tilde{w}_n^1 to a solution w of (4.3.61).

Let us evaluate the $L^{\frac{N}{2}}$ -norm of c_n :

$$\begin{aligned} \int_{A_{1,n}^-} |c_n(y)|^{\frac{N}{2}} dy &\leq C_N \left[\int_{A_{1,n}^-} \left| \frac{1}{u_n(P_n^1)} u_n(P_n^1 + \delta_n y) \right|^{(p_n-1)\frac{N}{2}} dy \right] + \\ &C_N \left[\int_{A_{1,n}^-} \left| \frac{1}{u_n(P_n^1)} v_n(P_n^1 + \delta_n y) \right|^{(p_n-1)\frac{N}{2}} dy \right] \end{aligned}$$

where C_N is a constant which depends only on N .

For the first integral in the previous formula we have

$$\int_{A_{1,n}^-} \left| \frac{1}{u_n(P_n^1)} u_n(P_n^1 + \delta_n y) \right|^{(p_n-1)\frac{N}{2}} dy = \int_{A^-} |u_n(x)|^{2^* - \frac{N\varepsilon_n}{2}} dx \leq B_N$$

by (4.3.59) and (4.1.46), B_N being a constant depending only on N .

An analogous estimate holds for the second integral.

Hence the $L^{\frac{N}{2}}$ -norm of c_n is uniformly bounded and we have

$$(4.3.63) \quad \int_{A_{1,n}^-} |c_n(y)|^{\frac{N}{2}} dy \leq 2C_N B_N$$

Then multiplying (4.3.62) by \tilde{w}_n^1 and integrating we have that

$$\begin{aligned} (4.3.64) \quad &\int_{A_{1,n}^-} |\nabla \tilde{w}_n^1|^2 dy = \int_{A_{1,n}^-} c_n (\tilde{w}_n^1)^2 dy \leq \\ &\leq \left(\int_{A_{1,n}^-} |c_n|^{\frac{N}{2}} dy \right)^{\frac{2}{N}} \left(\int_{A_{1,n}^-} |\tilde{w}_n^1|^{2^*} dy \right)^{\frac{2}{2^*}} \leq (2C_N B_N)^{\frac{2}{N}} \end{aligned}$$

Then by (4.2.56) - (4.2.58) we get $w = kV_1 = k\frac{\partial U}{\partial y_1}$, $k \in \mathbb{R}$, since, by (4.3.64) $w \in D^{1,2}(\mathbb{R}_-^N) = \{\varphi \in L^{2^*}(\mathbb{R}_-^N) : |\nabla \varphi| \in L^2(\mathbb{R}_-^N)\}$.

Let us first assume that for one of the sequences $\{\tilde{w}_n^i\}$, say $\{\tilde{w}_n^1\}$, the limit is $w = k \frac{\partial U}{\partial y_1}$ with $k \neq 0$.

Then, since the points P_n^1 are on the reflection hyperplane T and $\nabla u_n(P_n^1) = 0$ we have that $\frac{\partial \tilde{w}_n^1}{\partial y_1}(0) = 0$. This implies that $\frac{\partial w}{\partial y_1}(0) = k \frac{\partial^2 U}{\partial y_1^2}(0) = 0$ with $k \neq 0$, which is a contradiction since for the function $U(y) = \frac{1}{(1+|y|^2)^{\frac{N-2}{2}}}$ we have $\frac{\partial^2 U}{\partial y_1^2}(0) < 0$.

So we are left with the case when all sequences \tilde{w}_n^i converge to zero in C_{loc}^2 .

Then, for any fixed R and for n sufficiently large in the domains $E_{i,n}(R) = B(0, R) \cap A_{i,n}^-$ we have the following estimates

$$(4.3.65) \quad |\tilde{w}_n^i(y)| \leq \frac{S}{4(2C_N B_N)^2 |B(0, R)|^{\frac{2}{2^*}}} \quad i = 1, \dots, k$$

where $|B(0, R)|$ is the measure of the ball $B(0, R)$.

Now we focus only on the rescaling around P_n^1 and observe that the domains $E_{i,n}(R)$, $i \geq 2$, under the rescaling around P_n^1 , correspond to domains $F_{i,n}(R)$ contained in $A_{1,n}^-$ which are translations of $E_{1,n}(R)$ by the vector $\frac{P_n^i - P_n^1}{\delta_n}$ and also the functions \tilde{w}_n^i are the translation of \tilde{w}_n^1 by the same vector, indeed $\tilde{w}_n^i = \tilde{w}_n^1 \left(y + \frac{P_n^i - P_n^1}{\delta_n} \right)$

Hence from (4.3.65) we have

$$(4.3.66) \quad |\tilde{w}_n^1(y)| \leq \frac{S}{4(2C_N B_N)^2 |B(0, R)|^{\frac{2}{2^*}}} \quad \text{in } (E_{1,n}(R) \cup (\cup_{i \geq 2} F_{i,n}(R)))$$

Now let us choose R sufficiently large such that

$$(4.3.67) \quad \int_{B(0,R)} |U|^{2^*} > \left(\frac{4k-1}{4k} S \right)^{\frac{N}{2}}$$

where U is, as usual, the function defined in (4.2.55). Then, since both functions which appear in the definition of c_n converge to the function U and the function \tilde{u}_n^1 is just the translation of the function $\tilde{u}_n^i = \frac{1}{u_n(P_n^1)} u_n(P_n^i + \delta_n y)$ by the vector $\frac{P_n^i - P_n^1}{\delta_n}$, we have by (3.4.31)

$$(4.3.68) \quad \int_{B(0,R) \cup (\cup_{i \geq 2} B(\frac{P_n^i - P_n^1}{\delta_n}, R))} |\tilde{w}_n^1|^{p_n+1} > \left(\frac{4k-1}{4} S \right)^{\frac{N}{2}}$$

for n sufficiently large. This implies, by (4.3.59)

$$(4.3.69) \quad \int_{A_{1,n}^- \setminus (E_{1,n}(R) \cup (\cup_{i \geq 2} F_{i,n}(R)))} |c_n|^{\frac{N}{2}} < \left(\frac{1}{4} S \right)^{\frac{N}{2}}$$

Since the functions \tilde{w}_n^1 solve (4.3.62), multiplying (4.3.62) by \tilde{w}_n^1 and integrating we get

$$\begin{aligned} & \int_{A_{1,n}^-} |\nabla \tilde{w}_n^1|^2 dy = \int_{A_{1,n}^-} c_n (\tilde{w}_n^1)^2 dy = \\ & \int_{A_{1,n}^- \setminus (E_{1,n}(R) \cup (\cup_{i \geq 2} F_{i,n}(R)))} c_n (\tilde{w}_n^1)^2 dy + \int_{(E_{1,n}(R) \cup (\cup_{i \geq 2} F_{i,n}(R)))} c_n (\tilde{w}_n^1)^2 dy \leq \\ & \left(\int_{A_{1,n}^- \setminus (E_{1,n}(R) \cup (\cup_{i \geq 2} F_{i,n}(R)))} |c_n|^{\frac{N}{2}} dy \right)^{\frac{2}{N}} \left(\int_{A_{1,n}^- \setminus (E_{1,n}(R) \cup (\cup_{i \geq 2} F_{i,n}(R)))} |\tilde{w}_n^1|^{2^*} dy \right)^{\frac{2}{2^*}} + \\ & + \left(\int_{(E_{1,n}(R) \cup (\cup_{i \geq 2} F_{i,n}(R)))} |c_n|^{\frac{N}{2}} dy \right)^{\frac{2}{N}} \left(\int_{(E_{1,n}(R) \cup (\cup_{i \geq 2} F_{i,n}(R)))} |\tilde{w}_n^1|^{2^*} dy \right)^{\frac{2}{2^*}} \leq \frac{S}{2} \end{aligned}$$

because $\|\tilde{w}_n^1\|_{L^{2^*}(A_{1,n}^-)} = 1$, the $L^{\frac{N}{2}}$ -norm of c_n is uniformly bounded by (4.3.63), (4.3.69) and (4.3.66) hold.

On the other hand, by the Sobolev inequality, we have

$$\int_{A_{1,n}^-} |\nabla \tilde{w}_n^1|^2 dy > S$$

which gives a contradiction.

Hence the sequences \tilde{w}_n^i cannot all converge to zero, so that $w_\varepsilon \equiv 0$ for ε small, as we wanted to prove. \blacksquare

Finally we prove Lemma 4.5.

Proof of Lemma 4.5: As for the proof of Theorem 4.1 we adapt the proof of Lemma 6 of [15] to the case of k blow-up points, $k \geq 3$. Let us consider the line r_ε connecting P_ε^1 with the origin. By the

second statement of Lemma 4.4 if, for ε small, r_ε contains any other point P_ε^i , $i \neq 1$, then all points P_ε^i , $i = 1, \dots, k$, belong to the same $(k-1)$ -dimensional hyperplane Π passing through the origin and hence the assertion is proved. Therefore let us assume that for a sequence $\varepsilon_n \rightarrow 0$ the line $r_n = r_{\varepsilon_n}$ does not contain any point $P_n^i = P_{\varepsilon_n}^i$, $i \neq 1$. Then, again by Lemma 4.4, we have that there exist two $(N-1)$ -dimensional parallel hyperplanes T_n and Σ_n , with T_n passing through the origin, such that $P_n^1 \in T_n$ and $P_n^i \in \Sigma_n$, for any $i \in \{2, \dots, k\}$. By rotating we can always assume that $T_n = T = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 = 0\}$ and $\Sigma_n = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 = \alpha_n\}$ with $\alpha_n > 0$. In this way $P_n^1 = (0, y_2^n, \dots, y_N^n)$ while $P_n^i = (\alpha_n, x_{i,2}^n, \dots, x_{i,N}^n)$ for $i = 2, \dots, k$.

As before we define $\delta_n = (u_n(P_n^1))^{\frac{1-p_n}{2}}$ where $p_n = p - \varepsilon_n$.

Claim 1 *It is not possible that*

$$(4.3.70) \quad \frac{\alpha_n}{\delta_n} \xrightarrow{n \rightarrow \infty} \infty$$

Assume, by contradiction, that (4.3.70) holds. We claim that, for n sufficiently large,

$$(4.3.71) \quad \lambda_1(L_n, A^-) \geq 0$$

where $L_n \equiv L_{\varepsilon_n}$ denotes the linearized operator and, as before, $A^- = \{x = (x_1, \dots, x_N) \in A : x_1 < 0\}$. To prove (4.3.71) let us take the balls $B(P_n^i, R\delta_n)$ centered at the points P_n^i , $i = 1, \dots, k$, and with radius $R\delta_n$, $R > 1$ to be fixed later.

By (4.3.70) and (4.2.52) we have that $B(P_n^i, R\delta_n)$ does not intersect A^- for $i \geq 2$ and for large n . Moreover if we take $\vartheta_0 \in [0, \frac{\pi}{2}]$ and we consider the hyperplane $T_{\vartheta_0} = \{x = (x_1, \dots, x_N) : x_1 \sin \vartheta_0 + x_N \cos \vartheta_0 = 0\}$, by (4.3.70), (4.2.52) and the fact that P_n^1 belongs to $T = T_{\frac{\pi}{2}}$ we can choose $\vartheta_{0,n} < \frac{\pi}{2}$ and close to $\frac{\pi}{2}$ such that all balls $B(P_n^i, R\delta_n)$ do not intersect the cap $A_{\vartheta_{0,n}}^- = \{x = (x_1, \dots, x_N) : x_1 \sin \vartheta_{0,n} + x_N \cos \vartheta_{0,n} < 0\}$ for n large enough.

Then, arguing as in [23] (see also [19]), it is easy to see that it is possible to choose R such that $\lambda_1(L_n, A_{\vartheta_{0,n}}^-) > 0$ for n large, because u_n concentrates only at the points P_n^i , $i = 1, \dots, k$.

Then, fixing n sufficiently large, we set

$$\tilde{\vartheta}_n \equiv \sup\{\vartheta \in [\vartheta_{0,n}, \frac{\pi}{2}] : \lambda_1(L_n, A_{\vartheta}^-) \geq 0\}$$

We would like to prove that $\tilde{\vartheta}_n = \frac{\pi}{2}$.

If $\tilde{\vartheta}_n < \frac{\pi}{2}$ then $P_n^i \notin A_{\tilde{\vartheta}_n}^-$, $i = 1, \dots, k$, and $\lambda_1(L_n, A_{\tilde{\vartheta}_n}^-) = 0$, by the definition of $\tilde{\vartheta}_n$.

Thus considering the functions

$$w_{n, \tilde{\vartheta}_n}(x) = v_{n, \tilde{\vartheta}_n}(x) - u_n(x) \text{ in } A_{\tilde{\vartheta}_n}^-$$

where $v_{n, \tilde{\vartheta}_n}$ is defined as the reflection of u_n with respect to $T_{\tilde{\vartheta}_n}$, we have, by the strict convexity of f , that

$$\begin{cases} L_n(w_{n, \tilde{\vartheta}_n}) \geq 0 \text{ (} > 0 \text{ if } w_{n, \tilde{\vartheta}_n}(x) \neq 0 \text{) in } A_{\tilde{\vartheta}_n}^- \\ w_{n, \tilde{\vartheta}_n} \equiv 0 \text{ on } \partial A_{\tilde{\vartheta}_n}^- \end{cases}$$

Since $\lambda_1(L_n, A_{\tilde{\vartheta}_n}^-) = 0$, by the maximum principle, we have that $w_{n, \tilde{\vartheta}_n} \geq 0$ in $A_{\tilde{\vartheta}_n}^-$ and, since $u_n(P_n^1) > u_n(x)$ for any $x \in A_{\tilde{\vartheta}_n}^-$ we have, by the strong maximum principle, that $w_{n, \tilde{\vartheta}_n} > 0$ in $A_{\tilde{\vartheta}_n}^-$.

Hence, denoting by $(P_n^1)'$ the point in $A_{\tilde{\vartheta}_n}^-$ which is given by the reflection of P_n^1 with respect to $T_{\tilde{\vartheta}_n}$, we have that

$$(4.3.72) \quad w_{n, \tilde{\vartheta}_n}(x) > \eta > 0 \quad \text{for } x \in \overline{B((P_n^1)', \delta)} \subset A_{\tilde{\vartheta}_n}^-$$

where $B((P_n^1)', \delta)$ is the ball with center in $(P_n^1)'$ and radius $\delta > 0$ suitably chosen. Thus

$$(4.3.73) \quad w_{n, \tilde{\vartheta}_n + \sigma}(x) > \frac{\eta}{2} > 0 \quad \text{for } x \in \overline{B((P_n^1)'' , \delta)} \subset A_{\tilde{\vartheta}_n + \sigma}^-$$

for $\sigma > 0$ sufficiently small, where $(P_n^1)''$ is the reflection of P_n^1 with respect to $T_{\tilde{\vartheta}_n + \sigma}$.

On the other side, by the monotonicity of the eigenvalues with respect to the domain, we have that $\lambda_1(L_n, A_{\tilde{\vartheta}_n}^- \setminus \overline{B((P_n^1)', \delta)}) > 0$ and hence $\lambda_1(L_n, A_{\tilde{\vartheta}_n + \sigma}^- \setminus \overline{B((P_n^1)'' , \delta)}) > 0$, for σ sufficiently small.

This implies, by the maximum principle and (4.3.73), that

$$(4.3.74) \quad w_{n, \tilde{\vartheta}_n + \sigma}(x) > 0 \text{ for } x \in A_{\tilde{\vartheta}_n + \sigma}^-$$

Since $L_n(w_{n, \tilde{\vartheta}_n + \sigma}) \geq 0$ in $A_{\tilde{\vartheta}_n + \sigma}^-$ (by the convexity of the function $u_\varepsilon^{p-\varepsilon}$), the inequality (4.3.74) implies that $\lambda_1(L_n, D) > 0$ in any subdomain D of $A_{\tilde{\vartheta}_n + \sigma}^-$, and so $\lambda_1(L_n, A_{\tilde{\vartheta}_n + \sigma}^-) \geq 0$ for σ positive and sufficiently small. Obviously this contradicts the definition of $\tilde{\vartheta}_n$ and proves that $\tilde{\vartheta}_n = \frac{\pi}{2}$, i.e. (4.3.71) holds.

So, by Proposition 4.2, since $P_n^1 \in T = T_{\frac{\pi}{2}}$, we get that u_n is symmetric with respect to the hyperplane T , which is not possible, since P_n^i do not belong to T , for $i = 2, \dots, k$. Hence (4.3.70) cannot hold.

Claim 2 *It is not possible that*

$$(4.3.75) \quad \frac{\alpha_n}{\delta_n} \xrightarrow{n \rightarrow \infty} l > 0$$

Assume that (4.3.75) holds and, as before, denote by T the hyperplane $T = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 = 0\}$ to which P_n^1 belongs while $P_n^i \notin T$, $i \geq 2$.

We would like to prove as in Claim 1 that

$$(4.3.76) \quad \lambda_1(L_n, A^-) \geq 0$$

If the points P_n^1 and all the P_n^i have the N -th coordinate of the same sign, i.e. they lie on the same side with respect to the hyperplane $\{x_N = 0\}$, then it is obvious that we can argue exactly as for the first claim and choose $\vartheta_0 \in [0, \frac{\pi}{2}]$ such that all the balls $B(P_n^i, R\delta_n)$, R as before, do not intersect the cap $A_{\vartheta_0}^-$. Then the proof is the same as before.

Hence we assume that P_n^1 and some P_n^i , $i \neq 1$, lie on different sides with respect to the hyperplane $\{x_N = 0\}$. Let us then consider $\vartheta_n \in [0, \frac{\pi}{2}]$ such that the points P_n^1 and some of the P_n^i , say P_n^2, \dots, P_n^j , $j \leq k$, have the same distance $d_n > 0$ from the hyperplane T_{ϑ_n}

$$T_{\vartheta_n} = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 \sin \vartheta_n + x_N \cos \vartheta_n = 0\}$$

while the other points P_n^{j+1}, \dots, P_n^k have distance bigger than d_n from T_{ϑ_n} .

Of course, because of (4.3.75), we have

$$(4.3.77) \quad \frac{d_n}{\delta_n} \xrightarrow{n \rightarrow \infty} l_1 > 0$$

Then, choosing $R > 0$ such that $\lambda_1(L_n, D_n^R) > 0$, for n large, where $D_n^R = A \setminus [B(P_n^1, R\delta_n) \cup (\cup_{i \geq 2} B(P_n^i, R\delta_n))]$ (see [23]), either all balls $B(P_n^i, R\delta_n)$, $i = 1, \dots, k$ do not intersect the cap $A_{\vartheta_n}^-$, for n large enough, or they do. In the first case we argue as for the first claim. In the second case we observe that in each set $E_{\vartheta_n}^{n,i} = A_{\vartheta_n}^- \cap B(P_n^i, R\delta_n)$, $i = 1, \dots, k$, we have, for n large, and whenever the intersection is not empty,

$$(4.3.78) \quad u_n(x) \leq v_n^{\vartheta_n}(x) \quad x \in E_{\vartheta_n}^{n,i} \quad i = 1, \dots, k$$

where $v_n^{\vartheta_n}(x) = u_n(x^{\vartheta_n})$, x^{ϑ_n} being the reflection of x with respect to T_{ϑ_n} .

In fact if (4.3.78) were not true we could construct a sequence of points $x_{n_k} \in E_{\vartheta_{n_k}}^{n_k,i}$, for some $i = 1, \dots, k$, such that

$$(4.3.79) \quad u_{n_k}(x_{n_k}) > v_{n_k}^{\vartheta_{n_k}}(x_{n_k})$$

Then there would exist a sequence of points $\xi_{n_k} \in E_{\vartheta_{n_k}}^{n_k,i}$ such that

$$(4.3.80) \quad \frac{\partial u_{n_k}}{\partial \vartheta_{n_k}}(\xi_{n_k}) < 0$$

Thus, by rescaling u_{n_k} in the usual way around the $P_{n_k}^i$ and using (4.3.77) we would get a point $\xi \in (E_{\vartheta_0}^{l_1})^- = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 \sin \vartheta_0 + x_N \cos \vartheta_0 < -l_1 < 0\}$ such that $\frac{\partial U}{\partial \vartheta_0}(\xi) \leq 0$ while $\frac{\partial U}{\partial \vartheta_0} > 0$ in $(E_{\vartheta_0}^{l_1})^-$, ϑ_0 being the limit of ϑ_{n_k} .

Hence (4.3.78) holds.

Now, arguing again as in [23] and [19], in the set $(F_{\vartheta_n}^n)^- = A_{\vartheta_n}^- \setminus (\cup_{i \geq 1} B(P_n^i, R\delta_n))$ we have that $\lambda_1(L_n, (F_{\vartheta_n}^n)^-) \geq 0$.

Hence, by (4.3.78), applying the maximum principle, we have that $w_{n,\vartheta_n}(x) \geq 0$ in $(F_{\vartheta_n}^n)^-$, and, again by (4.3.78) and the strong maximum principle

$$(4.3.81) \quad w_{n,\vartheta_n}(x) > 0 \quad \text{in } A_{\vartheta_n}^-$$

As in the proof of Claim 1, this implies that $\lambda_1(L_n, A_{\vartheta_n}^-) \geq 0$.

Then, arguing again as for the first claim we get (4.3.76), which gives the same kind of contradiction because P_n^i , $i \geq 2$, do not belong to T .

Claim 3 *It is not possible that*

$$(4.3.82) \quad \frac{\alpha_n}{\delta_n} \xrightarrow{n \rightarrow \infty} 0$$

Let us argue by contradiction and assume that (4.3.82) holds. As before we denote by T the hyperplane $T = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 = 0\}$. Since the points P_n^i , $i \geq 2$, are in the domain $A_n^+ = \{x = (x_1, \dots, x_N) \in A : x_1 > 0\}$, we have that the function

$$w_n(x) = v_n(x) - u_n(x), \quad x \in A_n^+$$

where v_n is the reflection of u_n , i.e. $v_n(x_1, \dots, x_N) = u_n(-x_1, x_2, \dots, x_N)$, is not identically zero.

Then, as in the proof of Theorem 4.1, rescaling the function w_n around P_n^1 or P_n^i , $i \geq 2$, and using (4.2.52) we have that the functions

$$(4.3.83) \quad \tilde{w}_n^i(y) \equiv \frac{1}{\beta_n^i} w_n(P_n^i + \delta_n y), \quad i = 1, \dots, k$$

defined in the rescaled domain $A_{i,n}^+ = \frac{A^+ - P_n^i}{\delta_n}$, converge both, by (4.3.82) and standard elliptic estimates, in C_{loc}^2 to a function w_i satisfying (4.3.61) but in the half space $\mathbb{R}_+^N = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 > 0\}$.

Again by (4.2.56) - (4.2.58) we have that $w_i = k_i \frac{\partial U}{\partial y_1}$, $k_i \in \mathbb{R}$, where U is the function defined by (4.2.54).

Exactly as in the proof of Theorem 4.1 we can exclude the case that all sequences \tilde{w}_n^i converge to zero in C_{loc}^2 .

Hence for at least one of the sequences \tilde{w}_n^i we have that the limit is $w_i = k_i \frac{\partial U}{\partial y_1}$ with $k_i \neq 0$.

If this happens for \tilde{w}_n^1 then, since the points P_n^1 are on the reflection hyperplane T , arguing exactly as in the proof of Theorem 4.1, we get a contradiction.

So we are left with the case when $\tilde{w}_n^1 \rightarrow k_1 \frac{\partial U}{\partial y_1}$, $k_1 = 0$ and $\tilde{w}_n^i \rightarrow k_i \frac{\partial U}{\partial y_1}$ with $k_i \neq 0$ for some $i \geq 2$ in C_{loc}^2 . For the sake of simplicity let us suppose that $i = 2$.

At the points P_n^2 , obviously we have that $\frac{\partial u_n}{\partial y_1}(P_n^2) = 0$.

Let us denote by \tilde{P}_n^2 the reflection of P_n^2 with respect to T .

Hence, for the function \tilde{w}_n^2 we have, applying the mean value theorem

$$\begin{aligned} \frac{\partial \tilde{w}_n^2}{\partial y_1}(0) &= \frac{\delta_n^{\frac{2}{1-p_n}}}{\beta_n^2} \left(\frac{\partial \tilde{u}_n}{\partial y_1}(0) + \frac{\partial \tilde{u}_n}{\partial y_1} \left(\frac{\tilde{P}_n^2 - P_n^2}{\delta_n} \right) \right) = \\ &= \frac{\delta_n^{\frac{2}{1-p_n}}}{\beta_n^2} \left(\frac{\partial \tilde{u}_n}{\partial y_1} \left(\frac{\tilde{P}_n^2 - P_n^2}{\delta_n} \right) - \frac{\partial \tilde{u}_n}{\partial y_1}(0) \right) = \\ &= -\frac{\delta_n^{\frac{2}{1-p_n}}}{\beta_n^2} \frac{\partial^2 \tilde{u}_n}{\partial y_1^2}(\xi_n) \frac{2\alpha_n}{\delta_n} \end{aligned}$$

where $\tilde{u}_n(y) = \delta_n^{\frac{2}{p_n-1}} u_n(P_n^2 + \delta_n y)$ and ξ_n belongs to the segment joining the origin with the point $\frac{\tilde{P}_n^2 - P_n^2}{\delta_n}$ in the rescaled domain $A_{2,n}^+$.

Since $\frac{\partial \tilde{w}_n^2}{\partial y_1}(0) \rightarrow k_2 \frac{\partial^2 U}{\partial y_1^2}(0)$ and $\frac{\partial^2 \tilde{u}_n}{\partial y_1^2}(\xi_n) \rightarrow \frac{\partial^2 U}{\partial y_1^2}(0)$, with $k_2 \neq 0$ and $\frac{\partial^2 U}{\partial y_1^2}(0) < 0$ we get

$$(4.3.84) \quad \frac{\alpha_n \delta_n^{\frac{2}{1-p_n}}}{\beta_n^2 \delta_n} \rightarrow \gamma \neq 0$$

Our aim is now to prove that (4.3.84) implies that $k_1 \neq 0$ which will give a contradiction.

Let us observe that if the function w_n does not change sign near P_n^1 , then, since $w_n \not\equiv 0$, we would get a contradiction, applying Hopf's lemma to w_n (which solves a linear elliptic equation) at the point P_n^1 , because $\nabla u_n(P_n^1) = 0$.

Then in any ball $B(P_n^1, \alpha_n)$, α_n as in (4.3.82), there are points Q_n^1 such that $\frac{\partial u_n}{\partial y_1}(Q_n^1) = 0$ and $Q_n^1 \notin T$. Indeed, since w_n changes sign near P_n^1 , in any set $B(P_n^1, \alpha_n) \cap A^+$ there are points where w_n is zero, i.e. u_n coincides with the reflection v_n . This implies that there exist points Q_n^1 in $B(P_n^1, \alpha_n)$ where $\frac{\partial u_n}{\partial y_1}(Q_n^1) = 0$, and by Hopf's lemma applied to the points of the hyperplane T we have that $Q_n^1 \notin T$. Let us denote by \tilde{Q}_n^1 the reflection of Q_n^1 with respect to T .

Assume that $Q_n^1 \in A^-$ (the argument is the same if $Q_n^1 \in A^+$). Then as before we have

$$\begin{aligned} \frac{\partial \tilde{w}_n^1}{\partial y_1} \left(\frac{Q_n^1 - P_n^1}{\delta_n} \right) &= \frac{\delta_n^{\frac{2}{1-p_n}}}{\beta_n^1} \left(\frac{\partial \tilde{u}_n}{\partial y_1} \left(\frac{\tilde{Q}_n^1 - P_n^1}{\delta_n} \right) - \frac{\partial \tilde{u}_n}{\partial y_1} \left(\frac{Q_n^1 - P_n^1}{\delta_n} \right) \right) = \\ &= -\frac{\delta_n^{\frac{2}{1-p_n}}}{\beta_n^1} \frac{\partial^2 \tilde{u}_n}{\partial y_1^2}(\xi_n) \frac{2\alpha_n}{\delta_n} \end{aligned}$$

where ξ_n belongs to the segment joining $\frac{\tilde{Q}_n^1 - P_n^1}{\delta_n}$ and $\frac{Q_n^1 - P_n^1}{\delta_n}$ in the rescaled domain $A_{1,n}^+$.

Since $\frac{\partial \tilde{w}_n^1}{\partial y_1} \left(\frac{\tilde{Q}_n^1 - P_n^1}{\delta_n} \right) \rightarrow k_1 \frac{\partial^2 U}{\partial y_1^2}(0)$, $\frac{\partial^2 \tilde{u}_n}{\partial y_1^2}(\xi_n) \rightarrow \frac{\partial^2 U}{\partial y_1^2}(0) < 0$ and using (4.3.84), we get $k_1 \neq 0$ and hence a contradiction.

So also the third claim is true and the proof of Lemma 4.5 is complete.

■

CHAPTER 5

The singularly perturbed critical problem

1. Introduction

In this chapter we will discuss the results contained in [14]. Let Ω be a smooth bounded domain in \mathbb{R}^N , $N \geq 3$, $\lambda \geq 0$. We consider the problem

$$(D)_\lambda \begin{cases} -\Delta u + \lambda u = u^p & x \in \Omega \\ u > 0 & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

where $p = \frac{N+2}{N-2} = 2^* - 1$ is the critical Sobolev exponent. In case Ω is starshaped $(D)_\lambda$ has no solutions: an obstruction to existence is given by the well known Pohozaev identity. However, $(D)_\lambda$ might have solutions for any λ : this is the case if Ω is an annulus.

In [17], Druet-Hebey-Vaugon, investigating the role of Pohozaev type identities in a Riemannian context, discovered that such identities still provide some kind of obstruction to existence: if u_j solve $(D)_{\lambda_j}$ on some compact (conformally flat) Riemannian manifold M and $\lambda_j \rightarrow +\infty$, then $\int_M u_j^{p+1} \rightarrow +\infty$ (see [20] for extensions to fourth order elliptic PDE's and [26] for more questions).

In other words, there are no positive solutions with energy below some given bound, if λ is too large.

This result does not carry over to manifolds with boundary under general boundary conditions (e.g. homogeneous Neumann boundary conditions, see [1], [2] for this non trivial fact.)

The main purpose of this chapter is to show that the result quoted above is indeed true for the homogeneous Dirichlet B.V.P. $(D)_\lambda$; see Theorem 5.1 below.

Our approach is as in [17]: to show that a sequence of solutions cannot blow-up at a finite number of points (as it should be assuming a bound on the energy). The obstruction found in [17] is given by local L^2 estimates. In turn, these estimates are based on inequalities obtained localizing the standard Pohozaev identity on balls centered at blow-up points (see 5.3.97 below).

Now, differently from [17], where there is no boundary, we have to take into account possible blow-up at boundary points. Since Pohozaev type inequalities on balls centered at boundary points do not hold, in general, the main issue here is to get local L^2 estimates at boundary points.

Notice that a more or less straightforward application of arguments from [17] would only lead to the statement: bounded energy solutions have to blow up at least at one boundary point, which is the (quite interesting in itself) correct statement for the Neumann problem (see Theorem 5.2 below) but which is not the result we are looking for in the Dirichlet problem.

Since it does not seem easy to rule out blow up at boundary points, we stick to the approach in [17], but we have to deal with the new difficulty coming from possible blow up at boundary points.

To handle this difficulty, we will establish Pohozaev-type inequalities *suitably localized at interior points* (see Lemma 5.6 below), which, used in a clever way, will allow to obtain *estimates up to boundary points* (see Lemma 5.7, which also provides a simple adaptation and a self contained exposition of the main arguments in [17]).

This is the main technical contribution of the paper [14], and we believe that such estimates might be of interest by themselves.

Our first result is

Theorem 5.1. *Let $N \geq 3$. Let u_j be solutions of $(D)_{\lambda_j}$, with $\lambda_j \rightarrow +\infty$. Then $\int_{\Omega} u_j^{p+1} \rightarrow +\infty$.*

Remark: Such a result is quite obvious if p is subcritical, i.e. $p < \frac{N+2}{N-2}$: a simple scaling argument implies $\int_{\Omega} u_j^{p+1}$ blows (at least) like λ_j^γ , $\gamma = \frac{p+1}{p-1} - \frac{N}{2}$ as $\lambda_j \rightarrow +\infty$. In this case, in contrast with the critical case, it holds true also for the homogeneous Neumann B.V.P. (see [30])

for an explicit lower bound on the energy of ground state solutions).

A second question we address in [14] is concerned with the mixed boundary value problem

$$(M)_\lambda \begin{cases} -\Delta u + \lambda u = u^{\frac{N+2}{N-2}} & x \in \Omega \\ u > 0 & x \in \Omega \\ u = 0 & x \in \Gamma_0 \\ \frac{\partial u}{\partial \nu} = 0 & x \in \Gamma_1 \end{cases}$$

Here $\partial\Omega = \Gamma_0 \cup \Gamma_1$ with Γ_i disjoint components.

As for the Neumann problem, $(M)_\lambda$ possesses low energy solutions for any λ positive, at least if the mean curvature of Γ_1 is somewhere positive, and hence non existence of bounded energy solutions for λ large is false, in general. Indeed, several existence results for (bounded energy) solutions blowing up at (one or several) boundary points are known (see [35] for an extensive bibliography).

Nothing is known, to our best knowledge, about existence of solutions blowing up both at interior and at boundary points. On the other hand, the extreme case of purely interior blow-up has been widely investigated:

(Q) Are there solutions which blow up only at interior points?

Let us review the known results, all of them actually concerning the homogeneous Neumann problem (i.e. $\Gamma_0 = \emptyset$).

A first, negative, answer has been given in [13] for $N \geq 5$: $(M)_\lambda$ has no solutions of the form

$$u_\lambda = w_\lambda + \sum_{j=1}^k U_{\mu_\lambda^j, y_\lambda^j}, \quad w_\lambda \rightarrow 0 \quad \text{in } H^1(\Omega)$$

with $\mu_\lambda^j \rightarrow +\infty$, $y_\lambda^j \rightarrow y^j \in \Omega$ as λ goes to infinity and $y^i \neq y^j$ for any $i \neq j$. Here $U(x) = \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}$ and $U_{\mu,y} = \mu^{\frac{N-2}{2}} U(\mu(x-y))$.

Now, bounded energy solutions u_λ (with λ going to infinity) are known to be of the form given above, apart from the property $y^i \neq y^j$ for all $i \neq j$, which has to be regarded as a "no multiple concentration at a single point" assumption.

Under the even more restrictive assumption $k = 1$, a corresponding nonexistence result has been proved in [22] in any dimension $N \geq 3$.

To our best knowledge, the only result fully answering (Q) is due to Rey [35], but it is limited to the dimension $N = 3$. According to Rey, "the main difficulty is to eliminate the possibility of multiple interior peaks", and he accomplishes this task through a very careful expansion of solutions blowing up at interior points: this is the basic tool to obtain a negative answer to (Q) in dimension $N = 3$.

It is henceforth remarkable that, as a by-product of our L^2 estimates, we can bypass this difficulty and easily prove

Theorem 5.2. *Let $N \geq 3$. Let u_j be solutions of $(M)_{\lambda_j}$, with $\lambda_j \rightarrow +\infty$. If $\sup_j \int_\Omega u_j^{p+1} < \infty$, then u_j has at least one concentration point which lies on the Neumann component Γ_1 .*

Actually, we expect that solutions for the mixed problem, with a uniform bound on the energy, should not even exist for λ large, if the mean curvature of the Neumann component is strictly negative. This is fairly obvious in the case of one peak solutions, which, by the above, should blow-up at one boundary point. However, in such a point the mean curvature should be non negative (see [3], [4] or [36]).

We would like to mention a very interesting result due to Esposito (see [18]), who was able to answer the full question (Q) in high dimension. The result is the following

Theorem 5.3. *Let $N > 6$. Suppose $\lambda_n \rightarrow \infty$ and let u_n be a sequence of solutions to*

$$\begin{cases} -\Delta u_n + \lambda_n u_n = u_n^{\frac{N+2}{N-2}} & x \in \Omega \\ u_n > 0 & x \in \Omega \\ \frac{\partial u}{\partial \nu} = 0 & x \in \partial\Omega \end{cases}$$

with uniformly bounded energy

$$\sup_{n \in \mathbb{N}} \int_{\Omega} u_n^{\frac{2N}{N-2}} < \infty$$

Then, for any compact set K in Ω there exist C_K such that:

$$\max_{x \in K} u_n(x) \leq C_K$$

for any $n \in \mathbb{N}$.

The proof of this theorem is based on a local description of possible compactness loss which makes use of inequality 5.3.98 and does not need any boundary condition. In fact, this theorem is a particular case of a general interior compactness result: Esposito proved that blow-up solutions with bounded energy remain bounded in $L_{loc}^{\infty}(\Omega)$ for $\lambda \rightarrow \infty$ without any boundary condition. Thus in the Neumann case all the blow-up points have to lie on the boundary $\partial\Omega$.

The outline of this chapter is the following: in section 2 we recall some well known facts and prove the L^2 global concentration of the solutions, in section 3 we prove Theorem 5.1 and 5.2 by contradiction through a localized Pohozaev identity and a "reverse" L^2 concentration.

2. L^2 global concentration

To be self contained, we review in this section some essentially well known facts. Let u_n be solutions of $(D)_{\lambda_n}$ (but also homogeneous Neumann, or mixed, boundary conditions might be allowed, with minor changes, here). Multiplying the equation by u_n , integrating by parts and using Sobolev inequality, we see that

$$(5.2.85) \quad \int_{\Omega} |\nabla u_n|^2 \geq S^{\frac{N}{2}}, \quad \int_{\Omega} u_n^{p+1} \geq S^{\frac{N}{2}}$$

where S denotes the best Sobolev constant. We start recalling concentration properties of u_n , assuming $u_n \rightharpoonup 0$ in H_0^1 .

Lemma 5.4. *Let u_n be solutions of $(D)_{\lambda_n}$. Assume $u_n \rightharpoonup 0$ in H_0^1 . Then there is a finite set $C \subset \bar{\Omega}$ such that $u_n \rightarrow 0$ in $H_{loc}^1(\bar{\Omega} \setminus C)$ and in $C_{loc}^0(\bar{\Omega} \setminus C)$.*

Proof: Let $C := \{x \in \bar{\Omega} : \limsup_{n \rightarrow \infty} \int_{B_r(x) \cap \Omega} |\nabla u_n|^2 > 0, \forall r > 0\}$.

Because of (5.2.85) and compactness of $\bar{\Omega}$, C cannot be empty. We claim that

$$(5.2.86) \quad \forall x \in C, \forall r > 0, \quad \limsup_{n \rightarrow \infty} \int_{B_r(x) \cap \Omega} u_n^{p+1} \geq S^{\frac{N}{2}}$$

To prove the claim, let $\varphi \in C_0^\infty(B_{2r}(x))$, $\varphi \equiv 1$ on $B_r(x)$, $0 \leq \varphi \leq 1$. Notice that $-\int_{\Omega} u_n \varphi^2 \Delta u_n = \int_{\Omega} |\nabla u_n|^2 \varphi^2 + o(1) = \int_{\Omega} |\nabla u_n \varphi|^2 + o(1)$ because $u_n \rightharpoonup 0$

From the equation, using Holder and Sobolev inequalities, we get

$$(5.2.87) \quad \int |\nabla u_n \varphi|^2 + o(1) \leq \int u_n^{\frac{4}{N-2}} (u_n \varphi)^2 \leq \frac{1}{S} \left(\int_{B_{2r}(x)} u_n^{\frac{2N}{N-2}} \right)^{\frac{2}{N}} \int |\nabla u_n \varphi|^2$$

For $x \in C$, $\int |\nabla u_n \varphi|^2$ is bounded away from zero along some subsequence, and then (5.2.86) follows by (5.2.87). Also, if $x_1, \dots, x_k \in C$, choosing $B_r(x_j)$ disjoint balls, and eventually passing to a subsequence, we get by (5.2.86)

$$k S^{\frac{N}{2}} \leq \sum_j \int_{B_r(x_j) \cap \Omega} u_n^{p+1} \leq \sup_n \int_{\Omega} u_n^{p+1} < +\infty$$

Thus C is finite. Also, from the very definition of C , it follows that $\nabla u_n \rightarrow 0$ in $L_{loc}^2(\Omega \setminus C)$, and, by Sobolev inequality, $u_n \rightarrow 0$ in $L_{loc}^{p+1}(\Omega \setminus C)$ as well.

Finally, $C_{loc}^0(\bar{\Omega} \setminus C)$ convergence will follow by standard elliptic theory once one has proved that $u_n \rightarrow 0$ in $L_{loc}^q(\Omega \setminus C) \forall q$. In turn, this fact readily follows iterating the (Moser type) scheme

$$(5.2.88) \quad q \geq 2, \int_{B_{2r}(x)} u^{p+1} \leq \left(\frac{S}{q}\right)^{\frac{N}{2}} \Rightarrow \left(\int_{B_r(x)} u^{sq}\right)^{\frac{1}{s}} \leq \frac{8}{Sr^2} \int_{B_{2r}(x)} u^q, s := \frac{p+1}{2}$$

To prove (5.2.88), we can proceed as for (5.2.87), choosing now as test function $\varphi^2 u^{q-1}$, $\|\nabla\varphi\|_\infty \leq \frac{2}{r}$. We now obtain

$$(5.2.89) \quad \int \nabla u \nabla (u^{q-1} \varphi^2) \leq \int u^{p-1} (\varphi u^{\frac{q}{2}})^2 \leq \frac{1}{S} \left(\int_{B_{2r}(x)} u^{p+1} \right)^{\frac{2}{N}} \int |\nabla \varphi u^{\frac{q}{2}}|^2$$

On the other hand

$$(5.2.90) \quad \int \nabla u \nabla (u^{q-1} \varphi^2) = \int (q-1) \varphi^2 u^{q-2} |\nabla u|^2 + 2 u^{q-1} \varphi \nabla u \nabla \varphi$$

$$(5.2.91) \quad \frac{2}{q} \int |\nabla(\varphi u^{\frac{q}{2}})|^2 = \int \frac{q}{2} \varphi^2 u^{q-2} |\nabla u|^2 + 2 u^{q-1} \varphi \nabla u \nabla \varphi + \frac{2}{q} |\nabla \varphi|^2 u^q$$

Subtracting (5.2.90) from (5.2.91) and then using (5.2.89), we obtain

$$(5.2.92) \quad \begin{aligned} \frac{2}{q} \int |\nabla(\varphi u^{\frac{q}{2}})|^2 &\leq \int \nabla u \nabla (u^{q-1} \varphi^2) + \frac{2}{q} \int |\nabla \varphi|^2 u^q \leq \\ &\leq \frac{1}{S} \left(\int_{B_{2r}(x)} u^{p+1} \right)^{\frac{2}{N}} \int |\nabla(\varphi u^{\frac{q}{2}})|^2 + \frac{8}{qr^2} \int_{B_{2r}(x)} u^q \end{aligned}$$

Hence, using the assumption $\int_{B_{2r}(x)} u^{p+1} \leq \left(\frac{S}{q}\right)^{\frac{N}{2}}$ and Sobolev inequality, we get (5.2.88). Now, iterating (5.2.88) with $x \in \bar{\Omega} \setminus C_\delta$, $2r < \delta$, $u = u_n$ and using elliptic estimates ([25], page 194), we obtain

$$(5.2.93) \quad \sup_{\Omega \setminus C_\delta} u_n \leq \frac{c_N}{\delta^{\frac{N}{2}}} \left(\int_{\Omega \setminus C_{\frac{\delta}{2}}} u_n^2 \right)^{\frac{1}{2}} \rightarrow 0$$

■

Points in C are called "geometrical" concentration points and C is the concentration set. A crucial observation is that L^2 norm concentrates around C (see [17]):

Lemma 5.5. *Let u_n be solutions of $(D)_{\lambda_n}$. Assume $u_n \rightharpoonup 0$, and let $C := \{x_1, \dots, x_m\}$ be its concentration set. Let $C_\delta := \cup_{j=1}^m B_\delta(x_j)$, $B_\delta(x_j)$ disjoint closed balls. Then, for n large,*

$$\int_{\Omega \setminus C_\delta} u_n^2 \leq \frac{16}{\delta^2 \lambda_n} \int_{\Omega} u_n^2$$

In particular, if $\lambda_n \rightarrow +\infty$, then

$$(5.2.94) \quad \frac{\int_{\Omega \setminus C_\delta} u_n^2}{\int_{C_\delta} u_n^2} \rightarrow 0$$

Proof: Let $\varphi \in C^\infty(\mathbb{R}^N)$, $0 \leq \varphi \leq 1$, $\varphi \equiv 0$ in $C_{\frac{\delta}{2}}$, $\varphi \equiv 1$ in $\mathbb{R}^N \setminus C_\delta$, $\|\nabla \varphi\|_\infty < \frac{4}{\delta}$. Multiplying the equation by $u_n \varphi^2$ and integrating, we get

$$(5.2.95) \quad \int |\nabla u_n|^2 \varphi^2 + 2 \int u_n \varphi \nabla u_n \nabla \varphi + \lambda_n \int u_n^2 \varphi^2 = o(1) \int u_n^2 \varphi^2$$

because $u_n^{p-1}(x) \rightarrow 0$ uniformly in $\bar{\Omega} \setminus C_{\frac{\delta}{2}}$ by Lemma 5.4. After setting $\gamma_n^2 := \frac{\int |\nabla u_n|^2 \varphi^2}{\int_{\Omega \setminus C_{\frac{\delta}{2}}} u_n^2}$ we get from (5.2.95) the desired inequality

$$\lambda_n \frac{\int_{\Omega \setminus C_\delta} u_n^2}{\int_{\Omega \setminus C_{\frac{\delta}{2}}} u_n^2} \leq o(1) + 2\|\nabla \varphi\|_\infty \gamma_n - \gamma_n^2 \leq o(1) + \|\nabla \varphi\|_\infty^2 \leq \frac{16}{\delta^2}$$

■

3. Pohozaev identity and "reverse" L^2 concentration

In this Section, after briefly recalling the Pohozaev identity, we first derive suitably localized Pohozaev inequalities which will allow to get uniform L^2 -local estimates up to boundary points. These up-to-the-boundary estimates are the main novelty with respect to [17]: combined with the L^2 global concentration reviewed in Section 2, they will readily imply Theorems 5.1 and 5.2.

Given any $v \in C^2(\Omega)$, $x_0 \in \mathbb{R}^N$, an elementary computation (see [40]) gives

$$\langle x - x_0, \nabla v \rangle \Delta v - \frac{N-2}{2} |\nabla v|^2 = \operatorname{div} \left(\langle x - x_0, \nabla v \rangle \nabla v - \frac{|\nabla v|^2}{2} (x - x_0) \right)$$

If in addition $v \in C_0^1(\bar{\Omega})$, so that $\nabla v(x) = \frac{\partial v}{\partial \nu} \nu(x)$ for any $x \in \partial\Omega$, where $\nu(x)$ is the exterior unit normal at $x \in \partial\Omega$, an integration by parts yields

$$(5.3.96) \quad \int_{\Omega} \langle x - x_0, \nabla v \rangle \Delta v + \frac{N-2}{2} v \Delta v = \frac{1}{2} \int_{\partial\Omega} \langle x - x_0, \nu(x) \rangle |\nabla v|^2 d\sigma$$

If furthermore $-\Delta v = g(x, v)$, $g(x, t) \equiv b(x)|t|^{p-1}t - \lambda a(x)t$, through another integration by parts we obtain

$$\begin{aligned} \frac{1}{N} \int_{\Omega} \langle x - x_0, v^{p+1} \frac{\nabla b}{p+1} - \frac{\lambda}{2} v^2 \nabla a \rangle - \frac{\lambda}{N} \int_{\Omega} a v^2 &= \\ &= \frac{1}{2} \int_{\partial\Omega} \langle x - x_0, \nu \rangle |\nabla v|^2 d\sigma \end{aligned}$$

A straightforward and well known consequence of this identity is that v has to be identically zero if Ω is starshaped with respect to some x_0 and $\langle x - x_0, \nabla b \rangle \leq 0 \leq \langle x - x_0, \nabla a \rangle$. However, this conclusion is false, in general. The idea, following [17], would be to localize (5.3.96) to obtain, for every given $x_0 \in \bar{\Omega}$ and some $\delta = \delta_{x_0} > 0$ and for all $\varphi \in C_0^\infty(B_{4\delta}(x_0))$, inequalities of the form

$$(5.3.97) \quad \int_{B_{4\delta}(x_0) \cap \Omega} \langle x - x_0, \nabla(\varphi v) \rangle \Delta(\varphi v) + \frac{N-2}{2} \varphi v \Delta(\varphi v) \geq 0,$$

However, while this can be done at interior points (e.g. with $4\delta = d(x_0, \partial\Omega)$), (5.3.97) is in general false, $\forall \delta$ small, if $x_0 \in \partial\Omega$. So, we have to localize (5.3.96) at interior points but in a careful way, to cover, in some sense, also boundary points. Our basic observation is that (5.3.97) holds true for x_0 as long as $d(x_0, \partial\Omega) \geq \delta$ if δ is sufficiently small, and this will be enough to get control up to the boundary. The first statement is the content of the following simple but crucial lemma.

Lemma 5.6. *There is $\bar{\delta} = \bar{\delta}(\partial\Omega)$ such that, if $0 < \delta \leq \bar{\delta}$, $v \in C^2(\Omega) \cap C^1(\bar{\Omega})$, $v \equiv 0$ on $\partial\Omega$ and $x_0 \in \Omega$ with $d(x_0, \partial\Omega) \geq \delta$, then (5.3.97) holds true.*

Proof: Let $\bar{\delta}$ be such that, for every $z \in \partial\Omega$, any $x \in \partial\Omega \cap B_{8\bar{\delta}}(z)$ can be uniquely written in the form

(i) $x = z + \eta + \gamma^z(\eta)\nu(z)$, $\langle \eta, \nu(z) \rangle = 0$, with $|\gamma^z(\eta)| \leq c(\partial\Omega)|\eta|^2$, for some smooth γ^z , with $\gamma^z(0) = 0$, $\nabla\gamma^z(0) = 0$, $|\eta| \leq 8\bar{\delta}$ and some

constant c only depending on $\partial\Omega$. We will also require

- (ii) $|\nu(z') - \nu(z'')| \leq \frac{1}{8}$ for any $z', z'' \in \partial\Omega$ with $|z' - z''| \leq 8\bar{\delta}$
- (iii) $\bar{\delta} < \frac{1}{128c}$, $c = c(\partial\Omega)$.

Let $0 < \delta \leq \bar{\delta}$. We are going to apply (5.3.96) with Ω replaced by $B_{4\delta}(x_0) \cap \Omega$ and v by φv , $\varphi \in C_0^\infty(B_{4\delta}(x_0))$.

If $d(x_0, \partial\Omega) \geq 4\delta$, then equality holds in (5.3.97), so, let us assume $0 < \delta \leq d(x_0, \partial\Omega) \leq 4\delta$.

We can write $x_0 = z - \tau\nu(z)$ for some $z \in \partial\Omega$ and $\delta \leq \tau \leq 4\delta$.

For $x \in B_{4\delta}(x_0)$ we have $|x - z| \leq 8\delta$ and hence, (i) holds: $x = z + \eta + \gamma^z(\eta)\nu(z)$, $|\eta| \leq 8\delta$.

Now, using (ii) – (iii), we see that $\langle x - x_0, \nu(x) \rangle = \langle x - x_0, \nu(x) - \nu(z) \rangle + \langle z + \eta + \gamma^z(\eta)\nu(z) - (z - \tau\nu(z)), \nu(z) \rangle \geq \frac{\delta}{2} - 64c\delta^2 \geq 0$.

Hence the r.h.s. in (5.3.96) (with Ω replaced by $\Omega \cap B_{4\delta}(x_0)$ and v by φv) is nonnegative and the Lemma is proved. \blacksquare

In the Lemma 5.7 below we will show how Pohozaev inequalities lead to "reverse L^2 concentration" of solutions at any blow up point. We will adapt arguments from [17], where, however, it is made a crucial use of the validity of (5.3.97) at any point, which is not the case here. Still, a clever use of Lemma 5.6, i.e. of (5.3.97) limited to points which are δ -away from the boundary, will enable us to get the desired estimates up to boundary points.

Lemma 5.7. *There is a constant $c = c_N$, only depending on N , such that if u_λ is a solution of $(D)_\lambda$ and $0 < \delta \leq \bar{\delta}(\partial\Omega) \leq 1$, then*

$$(5.3.98) \quad \lambda \int_{B_\delta(x)} u_\lambda^2 \leq \frac{c}{\delta^3} \int_{B_{4\delta}(x) \setminus B_\delta(x)} (u_\lambda^2 + u_\lambda^{p+1}), \quad \forall x \in \bar{\Omega}$$

Proof: We are going to apply (5.3.97) with $x_0 = x$ if $d(x, \partial\Omega) \geq \delta$, while, if $d(x, \partial\Omega) < \delta$, $x = z - \tau\nu(z)$ for some $z \in \partial\Omega$ and $0 \leq \tau < \delta$, we will choose $x_0 = z - \delta\nu(z)$. Let, without loss of generality, $x_0 = 0$. By Lemma 5 we have

$$\int_{B_{4\delta} \cap \Omega} \langle x, \nabla(\varphi u_\lambda) \rangle \Delta(\varphi u_\lambda) + \frac{N-2}{2} \varphi u_\lambda \Delta(\varphi u_\lambda) \geq 0$$

i.e. (dropping subscript $B_{4\delta} \cap \Omega$)

$$(5.3.99) \quad \int \left[\langle x, \nabla(\varphi u_\lambda) \rangle + \frac{N-2}{2} \varphi u_\lambda \right] \varphi \Delta u_\lambda + 2\varphi \langle x, \nabla u_\lambda \rangle \langle \nabla \varphi, \nabla u_\lambda \rangle + R(\lambda) \geq 0,$$

where $R(\lambda) := R_1(\lambda) + R_2(\lambda)$, R_1, R_2 given by

$$R_1(\lambda) := \int 2u_\lambda \langle x, \nabla \varphi \rangle \langle \nabla \varphi, \nabla u_\lambda \rangle + \langle x, \nabla(\varphi u_\lambda) \rangle u_\lambda \Delta \varphi$$

$$R_2(\lambda) := \frac{N-2}{2} \int \varphi u_\lambda [u_\lambda \Delta \varphi + 2 \langle \nabla u_\lambda, \nabla \varphi \rangle]$$

In what follows we properly adapt and simplify arguments from [17]. Taking φ radially symmetric and radially decreasing, we have $\nabla \varphi = \langle \nabla \varphi, \frac{x}{|x|} \rangle \frac{x}{|x|}$, with $\langle \nabla \varphi(x), x \rangle \leq 0$.

In particular

$$\begin{aligned} & \langle \nabla \varphi(x), \nabla u_\lambda(x) \rangle \langle x, \nabla u_\lambda(x) \rangle = \\ & = \langle \nabla \varphi(x), \frac{x}{|x|^2} \rangle \langle x, \nabla u_\lambda(x) \rangle^2 \leq 0 \end{aligned}$$

and hence (5.3.99) yields

$$(5.3.100) \quad R(\lambda) + \int \varphi \langle x, \nabla(\varphi u_\lambda) \rangle \Delta u_\lambda + \frac{N-2}{2} \int \varphi^2 u_\lambda \Delta u_\lambda \geq 0$$

Now, let us write $g(t) := \lambda t - |t|^{p-1}t$, $G(t) := \frac{\lambda}{2}t^2 - \frac{u^{p+1}}{p+1}$. We first rewrite, integrating by parts, the second term in (5.3.100) as follows:

$$\begin{aligned} \int \varphi \langle x, \nabla(\varphi u_\lambda) \rangle \Delta u_\lambda &= \int \varphi \langle x, \nabla \varphi \rangle u_\lambda g(u_\lambda) + \sum_{j=1}^N \int \varphi^2 x_j g(u_\lambda) \frac{\partial u_\lambda}{\partial x_j} = \\ &= \int \varphi \langle x, \nabla(\varphi) \rangle [u_\lambda g(u_\lambda) - 2G(u_\lambda)] - N \int \varphi^2 G(u_\lambda) = \\ (5.3.101) \quad &= -\frac{2}{N} \int \varphi u_\lambda^{p+1} - N \int \varphi^2 G(u_\lambda) \end{aligned}$$

because $2G(u) - ug(u) = -\frac{2}{N}u^{p+1}$. Since $NG(u_\lambda) - \frac{N-2}{2}u_\lambda g(u_\lambda) = -\lambda u_\lambda^2$, (5.3.100) gives

$$(5.3.102) \quad R(\lambda) - \frac{2}{N} \int \langle x, \nabla \varphi \rangle \varphi u_\lambda^{p+1} \geq -\lambda \int u_\lambda^2$$

Let us now transform, integrating by parts, $R(\lambda)$ as an integral against $u_\lambda^2 dx$.

$$\begin{aligned} R_1(\lambda) &= \int \sum_{j=1}^N \left[\langle x, \nabla \varphi \rangle \frac{\partial \varphi}{\partial x_j} + \frac{1}{2} x_j \varphi \Delta \varphi \right] \frac{\partial^2 u_\lambda^2}{\partial x_j^2} + \langle x, \nabla \varphi \rangle u_\lambda^2 \Delta \varphi = \\ &= - \int u_\lambda^2 \sum_{j=1}^N \left[\langle x, \nabla \varphi \rangle \frac{\partial^2 \varphi}{\partial x_j^2} + \frac{\partial}{\partial x_j} \langle x, \nabla \varphi \rangle \frac{\partial \varphi}{\partial x_j} + \frac{1}{2} \varphi \Delta \varphi + \frac{1}{2} x_j \frac{\partial}{\partial x_j} (\varphi \Delta \varphi) \right] + \\ &+ u_\lambda^2 \langle x, \nabla \varphi \rangle \Delta \varphi = - \int u_\lambda^2 \left[\langle \nabla \varphi, \nabla (\langle x, \nabla \varphi \rangle) \rangle + \frac{N}{2} \varphi \Delta \varphi + \frac{1}{2} \langle x, \nabla (\varphi \Delta \varphi) \rangle \right] \\ R_2(\lambda) &= \frac{N-2}{2} \left[\int u_\lambda^2 \varphi \Delta \varphi - \frac{1}{2} \int u_\lambda^2 \Delta \varphi^2 \right] = -\frac{N-2}{2} \int u_\lambda^2 |\nabla \varphi|^2 \end{aligned}$$

and thus

$$R(\lambda) = - \int u_\lambda^2 \left[\langle \nabla \varphi, \nabla \langle x, \nabla \varphi \rangle \rangle + \frac{N}{2} \varphi \Delta \varphi + \frac{1}{2} \langle x, \nabla (\varphi \Delta \varphi) \rangle + \frac{N-2}{2} |\nabla \varphi|^2 \right]$$

Now, assuming $\varphi \equiv 1$ on $B_{2\delta}$, $\varphi \equiv 0$ outside $B_{3\delta}$, we obtain

$$R(\lambda) \leq \frac{c}{\delta^3} \int_{B_{3\delta} \setminus B_{2\delta}} u_\lambda^2$$

for some $c = c(N)$, and hence, by (5.3.102),

$$\lambda \int_{B_{2\delta}} u_\lambda^2 \leq \frac{c}{\delta^3} \int_{B_{3\delta} \setminus B_{2\delta}} u_\lambda^2 + u_\lambda^{p+1}$$

Since $B_\delta(x) \subset B_{2\delta}$ and $B_{3\delta} \setminus B_{2\delta} \subset B_{4\delta}(x) \setminus B_\delta(x)$, the Lemma is proved. \blacksquare

Proof of Theorem 5.1. Lemma 5.5 and 5.6 provide the tools for the proof, which, at this stage, goes like in [17]. We briefly sketch the argument.

We have to prove that if u_n are solutions of $(D)_{\lambda_n}$ with $\sup_n \int u_n^{p+1} < +\infty$, then $\sup_n \lambda_n < +\infty$. This is clear if u_n has a non zero weak limit, so we can assume $u_n \rightharpoonup 0$.

According to Lemma 5.4, there are $x_1, \dots, x_k \in \bar{\Omega}$ such that $u_n \rightarrow 0$ in $C_{loc}^0(\mathbb{R}^N \setminus C)$ with $C \equiv \{x_1, \dots, x_k\}$.

Let $0 < \delta < \min\{\bar{\delta}(\partial\Omega), \frac{1}{8}d(x_i, x_j), i \neq j\}$, so that (5.3.98) in Lemma 5.7 holds for all $x_j \in C$:

$$\lambda_n \int_{B_\delta(x_j)} u_n^2 \leq \frac{2c_N}{\delta^3} \int_{B_{4\delta}(x_j) \setminus B_\delta(x_j)} u_n^2 \quad \forall x_j \in C$$

Since the balls $B_{4\delta}(x_j)$ are taken disjoint, we get

$$(5.3.103) \quad \lambda_n \int_{C_\delta} u_n^2 \leq \frac{2c}{\delta^3} \int_{\Omega \setminus C_\delta} u_n^2$$

which, jointly with (5.2.94), implies λ_n remains bounded. \blacksquare

Proof of Theorem 5.2. The proof is by contradiction: we assume that there is a sequence u_λ of bounded energy solutions with $\lambda \rightarrow +\infty$ and no blow-up points on Γ_1 . Hence, for this sequence, (5.3.98) holds true at any blow-up point.

In addition, Lemma 5.5 holds true for the problem $(M)_\lambda$. In fact, arguments in the proof of Lemma 5.5 are not affected by the presence of Neumann boundary conditions, and the concentration behaviour assumed therein follows by a simple adjustment in the proof of Lemma 5.4: in (5.2.86) the term $S^{\frac{N}{2}}$ becomes $\frac{S^{\frac{N}{2}}}{2}$ as it follows by replacing in (5.2.87), the Sobolev inequality with the Cherrier inequality (see [12]): for any $\delta > 0$ there exists $C(\delta) > 0$ such that for any $u \in H^1(\Omega)$

$$\left(\frac{S}{2^{\frac{2}{N}}} - \delta\right) \left(\int_{\Omega} |u|^{p+1}\right)^{\frac{2}{p+1}} \leq \int_{\Omega} |\nabla u|^2 + C(\delta) \int_{\Omega} u^2$$

and hence Lemma 5.4 holds for $(M)_\lambda$ as well, thanks to this inequality, to the fact that $\int_{\Omega} u_n^2 \rightarrow 0$ (so that $C(\delta)|u_n|_2^2 = o(1)$) and since, because of the null boundary conditions, there are no boundary contributions in the estimates. We use Cherrier inequality also in the Moser-type scheme to obtain $C_{loc}^0(\bar{\Omega} \setminus C)$ convergence.

Since (5.2.94) and (5.3.98) are satisfied, the same argument as in the proof of Theorem 5.1 applies, giving a contradiction. ■

Bibliography

- [1] Adimurthi, G.Mancini, *The Neumann problem for elliptic equations with critical non-linearity, A tribute in honor to G. Prodi*, Scuola Normale Superiore di Pisa, (1991), pag. 9-25.
- [2] Adimurthi, G. Mancini, S.L. Yadava, *The role of the mean curvature in semilinear Neumann problem involving critical exponent*, Communications in Partial Differential Equations, Vol. 20 no 3/4 (1995), pag. 591-631.
- [3] Adimurthi, F. Pacella, S.L. Yadava, *Interaction between the geometry of the boundary and positive solutions of a semilinear Neumann problem with critical non linearity*, Journal of Functional Analysis, Vol. 113 (1993), pag. 318-350.
- [4] Adimurthi, F. Pacella, S.L. Yadava, *Characterization of concentration points and L^∞ estimates for solutions of a semilinear Neumann problem involving the critical Sobolev exponent*, Differential and Integral Equations, Vol. 8 no 1 (1995), pag. 41-68.
- [5] T. Aubin, *Equation differentielle non lineaires et probleme de Yamabe concernant la courbure scalaire*, Journal de Mathematique Pure et Appliques, Vol. 55 (1976), pag. 269-296.
- [6] A. Bahri, M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain*, Communications on Pure and Applied Mathematics (41), 1988, 253-290.
- [7] A. Bahri, Y. Li, O. Rey, *On a variational problem with lack of compactness: the topological effect of critical points at infinity*, Calculus of Variations and Partial Differential Equations, Vol. 3 no. 1 (1995), pag. 67-93.
- [8] H. Beresticki, L. Nirenberg, *On the method of moving planes and the sliding method*, Bol. Soc. Bras. Mat, Vol. 22 (1991), pag. 1-37.
- [9] H. Berestycki, L. Nirenberg, S.N.S. Varadhan, *The principal eigenvalues and maximum principle for second order elliptic operators in general domains*, Communications on Pure and Applied Mathematics, Vol. 47 (1994), pag. 47-92.
- [10] H. Brezis, L.A. Peletier, *Asymptotics for elliptic equations involving critical growth*, Partial Differential Equations and Calculus of Variations, Progr. No.D.E.A. 1, Birkhauser Boston, MA, Vol. 1 (1989), pag. 149-192.
- [11] H. Brezis, L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Communications on Pure and Applied Mathematics, Vol. 36 (1983), pag. 437-477.
- [12] P. Cherrier, *Problems de Neumann nonlineaires sur les varietes riemanniennes*, Journal of Functional Analysis, Vol. 57 (1984), pag. 154-206.

- [13] D. Cao, E.S. Noussair, S. Yan, *Existence and nonexistence of interior peaked solutions for a nonlinear Neumann problem*, Pacific Journal of Mathematics, Vol. 200 no. 1 (2001), pag. 19-41.
- [14] D. Castorina, G. Mancini, *Non-existence of bounded energy solutions for some semilinear elliptic equations with a large parameter*, Rendiconti del Seminario Matematico dell' Università di Padova, Vol. 101 (2003), pag. 147-160.
- [15] D. Castorina, F. Pacella, *Symmetry of positive solutions of an almost-critical problem in an annulus*, Calculus of Variations and Partial Differential Equations, Vol. 23 no. 2 (2005), pag. 125-138.
- [16] D. Castorina, F. Pacella, *Symmetry of solutions of a semilinear elliptic problem in an annulus*, Progress in Nonlinear Differential Equations and Their Applications, Vol. 66 (2005), pag. 135-148.
- [17] O. Druet, E. Hebey, M. Vaugon, *Pohozaev type obstructions and solutions of bounded energy for quasilinear elliptic equations with critical Sobolev growth. The conformally flat case*, Nonlinear Analysis Theory Methods and Applications, Vol. 51 no.1 (2002), pag. 79-94.
- [18] P. Esposito, *Interior estimates for some semilinear elliptic problems with critical nonlinearity*, preprint.
- [19] K. El Medhi, F. Pacella, *Morse index and blow-up points of solutions of some nonlinear problems*, Atti Acc. Naz. Lincei Cl. Sc. Fis. Mat. Natur. Rend. Lincei (9), Mat. Appl., Vol. 13 no. 2 (2002), pag. 101-105.
- [20] V. Felli, E. Hebey, and F. Robert, *Fourth order equations of critical Sobolev growth. Energy function and solutions of bounded energy in the conformally flat case*, preprint, (2002).
- [21] F. Gladiali, M. Grossi, *On the curvature of the level sets of solutions of some nonlinear elliptic equations*, preprint, (2003).
- [22] N. Ghoussub, C. Gui, M. Zhu, *On a singularly perturbed Neumann problem with the critical exponent*, Communications in Partial Differential Equations, Vol. 6 no. 11-12 (2001), pag. 1929-1946
- [23] M. Grossi, R. Molle, *On the shape of the solutions of some semilinear elliptic problems*, Communications in Contemporary Mathematics, Vol. 5 no. 1 (2003), pag. 85-99.
- [24] B. Gidas, W.M. Ni, L. Nirenberg, *Symmetry and related properties via the Maximum Principle*, Communications in Mathematical Physics, Vol. 68 (1979), pag. 209-243.
- [25] D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 1983.
- [26] E. Hebey, *Nonlinear elliptic equations of critical Sobolev growth from a dynamical point of view*, preprint.
- [27] Z. C. Han., *Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent*, Annales Institute Henri Poincaré, Vol. 8 no. 2 (1991), pag. 159-174
- [28] J. Kazdan, F. Warner, *Remarks on some quasilinear elliptic equations*, Communications on Pure and Applied Mathematics, Vol. 28 (1975), pag. 567-597.

- [29] Y. Li, *Prescribing scalar curvature on S^N and related problems - Part I*, Journal Differential Equations, Vol. 120 (1995), pag. 319-410
- [30] G. Mancini, R. Musina, *The role of the boundary in some semilinear Neumann problems*, Rendiconti del Seminario Matematico della Università di Padova, Vol. 88 (1992), pag. 127-138.
- [31] F. Pacella, *Symmetry results for solutions of semilinear elliptic equations with convex nonlinearities*, Journal of Functional Analysis, Vol. 192 no. 1 (2002), pag. 271-282.
- [32] S. Pohozaev, *Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$* , Soviet Mathematical Doklady, Vol 6 (1965), pag. 1408-1411.
- [33] O. Rey, *A multiplicity result for a variational problem with lack of compactness*, Nonlinear Analysis Theory Methods and Applications, Vol. 13 no. 10 (1989), pag. 1241-1249.
- [34] O. Rey, *Blow-up points of solutions to elliptic equations with limiting nonlinearities*, Differential and Integral Equations, Vol. 4 no. 6 (1991), pag. 1157-1167.
- [35] O. Rey, *The question of interior blow-up points for an elliptic problem: the critical case*, Journal de Mathematiques pures et appliquees, Vol. 81 (2002), pag. 655-696
- [36] O. Rey, *Boundary effect for an elliptic Neumann problem with critical nonlinearity*, Communications in Partial Differential Equations, Vol. 22 no. 7-8 (1997), pag. 1055-1139.
- [37] N. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures of compact manifolds*, Annali della Scuola Normale Superiore di Pisa, Vol. 22 (1968), pag. 265-274.
- [38] J. Serrin, *A symmetry problem in potential theory*, Archives Rational Mechanics, Vol. 43 (1971), pag. 304-318.
- [39] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, Journal of Differential Geometry, Vol. 20 (1984), pag. 479-495.
- [40] M. Struwe, *Variational Methods, applications to nonlinear partial differential equations and Hamiltonian systems*, Springer, 1990.
- [41] G. Talenti, *Best constants in Sobolev inequalities*, Annali di Matematica Pura ed Applicata, Vol. 110 (1976), pag. 353-372.
- [42] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka Mathematical Journal, Vol. 12 (1960), pag. 21-37