Università degli Studi di Roma, La Sapienza
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TESI DI DOTTORATO

QUANTUM STOCHASTIC CALCULUS AND CONTINUAL MEASUREMENTS: THE CASE OF UNBOUNDED COEFFICIENTS

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Todo cuanto existe en el Universo es fruto del azar y la necesidad

Tutto ciò che esiste nell'universo è frutto del caso e della necessità
Everything existing in the Universe is the fruit of chance and necessity

Demócrito, siglo VII antes de la era común.

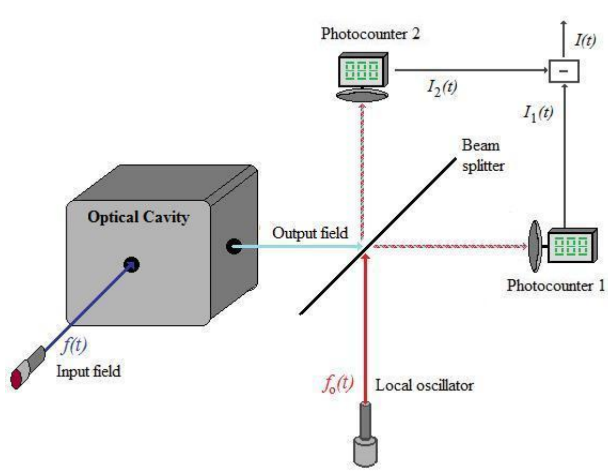
Contents

1 Introduction

1.1 A motivation from the physics

Quantum Open System Theory is nowadays a highly developed field which involves, on the mathematical side, many mathematical concepts such as operator theory, semigroups of operators on certain Banach spaces (algebras), classical and quantum stochastic differential equations...

Inside Quantum Open System Theory there is the theory of quantum continual measurements: a quantum system is monitored with continuity in time. This is surely an open system because, apart from possible interactions with the external environment like a thermal bath, the system is interacting with the measuring apparatus. The typical field in which a theory of continual measurements is needed is quantum optics: when light emitted by some system is observed (by counting of photons – direct detection – or through some interference mechanism – heterodyne and homodyne detection) a continual measurement is performed.



Balanced heterodyne detection

The theory needed for such a kind of measurements has been developed, but with mathematical rigour only in the case in which the system operators involved are bounded. But typical systems in quantum optics are resonant cavities, with inside matter and modes of the electromagnetic field. This system can be stimulated by external lasers and emits light in the free space which can be detected. As soon as modes of the electromagnetic field are involved, creation and annihilation operators, which are unbounded, enter into play. The present doctoral thesis is born motivated by the study of the continual measurement in this type of phenomena.

1.2 Continual Measurements

The theory of Continual Measurements begins with the work of E.B. Davis in the 70's and several authors contributed to its development, as A.S. Holevo, V.P. Belavkin and A. Barchielli. In the work *Continual Measurements in Quantum Mechanics and Quantum Stochastic Calculus* (Lecture Notes in Mathematics, Springer, Berlin 2006) a general account of the theory is presented and an exhaustive study under very general conditions is developed, but always considering bounded operators for the system under measurement. This work will serve as a guide for the development of this thesis; my thesis is about reconstructing the theory in a slightly less general frame (rich enough to contain the physically interesting observables), but now unbounded system operators are permitted for.

One of the possible ways of approaching to the study of continuous measurements is through Quantum Stochastic Calculus, whose bases where developed in the 80's with the classic works of R.L. Hudson and K.R. Parthasarathy, where they obtained the fundamental rules of this calculus. In their theory the coefficients involved in their famous equation (quantum stochastic Schrödinger equation, or Hudson-Parthasarathy equation) are always bounded operators and, therefore, this equation is not applicable directly in the case of our interest. Many authors developed the theory of such an equation to the unbounded case; in particular F. Fagnola and S.J. Wills published *Solving Quantum Stochastic Differential Equations with Unbounded Coefficients* (Journal the Functional Analysis, 2003), giving sufficient conditions to obtain existence, uniqueness and unitarity of the solution of Hudson-Parthasarathy equations with unbounded coefficients. Moreover, in *Quantum Stochastic Differential Equations and Dilation of Completely Positive Semigroup*, published in Lecture Notes in Mathematics (Springer, Berlin 2006), F. Fagnola builds a Quantum Markov Semigroup by $\big\langle U(t)v \otimes \psi(0) \big| (X \otimes \mathbb{1}) U(t)u \otimes \psi(0) \big\rangle$ and studies its infinitesimal generator; here, $U(t)$ is the unitary solution of the Hudson-Parthasarthy equation with unbounded coefficients and $\psi(0)$ represents the vacuum vector in the Fock space. They will be the hypotheses given in these two works which we take as starting point for the development of this thesis. The aim is to combine these results together with the ideas of the theory of continual measurements.

1.3 The main results

The starting point is a quantum system $S_{\mathscr{H}}$ living in a Hilbert space $\mathscr{H}$ in interaction with a Bose field living in a Fock space $\mathscr{F}$; the unitary evolution is governed by an Hudson-Parthasarathy equation with unbounded coefficients.

1. We construct and study the reduced dynamics of $S_{\mathscr{H}}$ starting from a *coherent state* $|\psi(f)\rangle\langle\psi(f)|$ for the field; in other words we study the evolution $\big\langle U(t)v \otimes \psi(f) \big| (X \otimes \mathbb{1}) U(t)u \otimes \psi(f) \big\rangle$, $X \in \mathscr{B}(\mathscr{H})$, $u, v \in \mathscr{H}$. In particular, we consider the case $f =$ constant, for which the previous quantity gives rise to a Quantum Dynamical Semigroup. Unexpectedly, we were compelled to add new hypotheses to the Fagnola-Wills ones in order to obtain a generator with good properties.

2. The reduced dynamics is not a semigroup when f is not constant, but it is possible to obtain it as a uniform limit of an evolutions made up of a time-ordered product of semigroups.

3. In the case of a generic f, a time-dependent form-generator of the reduced dynamics is obtained. Moreover it is shown that the reduced dynamics satisfies a backward master equation.

4. For what concerns the Continual Measurements, from [3] the notion of *Characteristic Operator* $\widehat{\Phi}_{\boldsymbol{k}}(t)$ is taken; it is the Fourier transform of a projection valued measure. Then, the quantity $\big\langle U(t)v \otimes \psi(f) \big| (X \otimes \widehat{\Phi}_{\boldsymbol{k}}(t)) U(t)u \otimes \psi(f) \big\rangle$ defines a reduced characteristic operator and the problem is to obtain an evolution equation for it. In the case in which all the functions involved are constant, again a semigroup of operators is obtained, which is not a Quantum Dynamical Semigroup because it is not positive, but it is the Fourier transform of a positive operator-valued measure.

5. As in the case of the dynamics, also the generic reduced characteristic operator is proved to be the limit of time-ordered products of semigroups.

6. Again a backward evolution equation with a the time-dependent generator is obtained for the reduced characteristic operator.

1.4 Open problems

The main mathematical open problem is the uniqueness of the solution of the two backward equations quoted above, the one for the reduced dynamics and the one for the reduced characteristic operator. By construction we know that both equations have a solution with all the required good properties, but to show uniqueness seems to be a delicate problem, mainly in the second case, where positivity cannot be invoked.

In order to have well defined generators of all the semigroups we had to introduced, the Fagnola-Wills hypotheses were strengthened: was this strictly necessary?

Another natural line of development of the thesis would be the application of our results to concrete physical models as the ones talked about in Section 1.1. Here two kind of problems arise: one is to check if the concrete unbounded operators involved in the model satisfy the abstract hypotheses and the other is to see how to arrive to compute analytically or numerically some of the (stochastic) properties of the output of the continual measurement.

Chapter 2

The Hudson-Parthasarathy Equations

2.1 Fock space and Weyl operators

2.1.1 The symmetric Fock space over $\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$

Let $\mathscr{Z}$ be a complex separable Hilbert space and let us introduce the space $\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$ of the $\mathscr{Z}$-valued square-integrable functions on $\mathbb{R}^+$. For any Hilbert space $\mathscr{K}$, we denoted for $U(\mathscr{K})$ the class of the unitary operators. Let Γ^n be the n-th tensor product of $\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$ with itself, i.e. $\Gamma^n = \bigotimes_{k=1}^{n} \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$. We denote by Γ^n_{symm} the symmetric part of Γ^n ([12] p. 106).

We denote by $\mathscr{F}$ the symmetric Fock space over $\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$:

$$\mathscr{F} = \Gamma^0_{\text{symm}} \oplus \Gamma^1_{\text{symm}} \oplus \Gamma^2_{\text{symm}} \oplus \Gamma^3_{\text{symm}} \oplus \cdots \equiv \Gamma_{\text{symm}}\big(\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})\big). \tag{2.1}$$

The n-th direct summand Γ^n_{symm} is called the *n-particle subspace*. When $n = 0$, it is called *vacuum subspace*. Any element of the n-particle subspace is called an *n-particle vector*. The vector $1 \oplus 0 \oplus 0 \cdots$ is called the *vacuum vector*. The elements of the dense linear manifold generated by all n-particle vector, $n = 0, 1, 2, \ldots$, are called *finite particle vectors*.

Let us consider $f \in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$; we will denote by $\mathrm{e}(f)$ and $\psi(f)$ respectively the *Exponential Vector* and *Coherent Vector* defined in the $\mathscr{F}$ space by

$$\mathrm{e}(f) := 1 \oplus f \oplus \frac{f \otimes f}{\sqrt{2!}} \oplus \cdots \oplus \frac{f^{\otimes n}}{\sqrt{n!}} \oplus \cdots \qquad \text{and} \qquad \psi(f) := \frac{\mathrm{e}(f)}{\|\mathrm{e}(f)\|} . \tag{2.2}$$

The inner product between two exponential vectors or between two coherent vectors results to be given by

$$\big\langle \mathrm{e}(f) \big| \mathrm{e}(g) \big\rangle = \mathrm{e}^{\langle f|g\rangle} , \quad \langle \psi(f)|\psi(g)\rangle = \exp\left\{ -\frac{1}{2}\|f-g\|^2 + \mathrm{i}\,\mathrm{Im}\langle f|g\rangle \right\} . \tag{2.3}$$

If $\mathcal{M}$ is a dense linear manifold in $\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$, then, the linear span $\mathcal{E}(\mathcal{M})$ of the vectors $\mathrm{e}(f)$, with $f \in \mathcal{M}$, is dense in $\mathscr{F}$.

An important feature of the Fock space $\mathscr{F}$ is its structure of continuous tensor product. For any choice of the times $0 \leq s \leq t$ let us introduce the spaces

$$\mathscr{F}_{(s,t)} = \Gamma_{\mathrm{symm}}(\mathrm{L}^2((s,t);\mathscr{Z})) \qquad \text{and} \qquad \mathscr{F}_{(t} = \Gamma_{\mathrm{symm}}(\mathrm{L}^2((t,\infty);\mathscr{Z}));$$

they are defined similarly to $\Gamma_{\mathrm{symm}}\big(\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})\big)$. Then, we have the natural identifications

$$\mathscr{F} \simeq \mathscr{F}_{(0,s)} \otimes \mathscr{F}_{(s,t)} \otimes \mathscr{F}_{(t} \qquad \text{and} \qquad \mathrm{e}(f) \simeq \mathrm{e}(f_{(0,s)}) \otimes \mathrm{e}(f_{(s,t)}) \otimes \mathrm{e}(f_{(t}) , \tag{2.4}$$

where $\quad f_{(s,t)}(x) := 1_{(s,t)}(x) f(x) \quad$ and $\quad f_{(t}(x) := 1_{(t,\infty)}(x) f(x)$.

Similarly, if P is any orthogonal projection, one has the factorization

$$\Gamma = \Gamma_{\mathrm{symm}}\big(PL^2(\mathbb{R}_+;\mathcal{Z})\big) \otimes \Gamma_{\mathrm{symm}}\big((\mathbb{1}-P)L^2(\mathbb{R}_+;\mathcal{Z})\big). \tag{2.5}$$

2.1.2 The Weyl operators

The Weyl operator $W(g;\mathscr{U})$ is the unique unitary operator defined by

$$W(g;\mathscr{U})\mathrm{e}(f) = \exp\Big\{ -\frac{1}{2}\|g\|^2 - \langle g|\mathscr{U} f\rangle \Big\} \mathrm{e}(\mathscr{U} f + g), \qquad \forall f \in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z}), \tag{2.6}$$

with $g \in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$ and $\mathscr{U}$ an unitary operator over $\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$. By using the coherent vectors, one has, equivalently,

$$W(g;\mathscr{U})\psi(f) = \exp\big\{ \mathrm{i}\ \mathrm{Im}\langle \mathscr{U} f|g\rangle \big\} \psi(\mathscr{U} f + g), \qquad \forall f \in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z}). \tag{2.7}$$

From the definition one obtains the relations

$$W(g;\mathscr{U})^{-1} = W(g;\mathscr{U})^* = W(-\mathscr{U}^* g;\mathscr{U}^*) \tag{2.8}$$

and the composition law

$$W(h;\mathscr{V})W(g;\mathscr{U}) = \exp\big\{-\text{i Im}\langle h|\mathscr{V}g\rangle\big\}W(h+\mathscr{V}g;\mathscr{V}\mathscr{U}). \qquad (2.9)$$

We will also use the notation $W(f) = W(f;\mathbb{1})$ and $\Upsilon(\mathscr{U}) = W(0;\mathscr{U})$.

Now, for $f, g \in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$ and $\mathscr{U}, \mathscr{V}$ unitary operators on $\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$, we have the following relations ([12], pg. 136):

$$\begin{aligned}
(i) &= W(f)W(g) = W(g)W(f)\exp\big\{-2\text{i Im}\langle f|g\rangle\big\}, & (2.10)\\
(ii) &= W(f)W(g) = \exp\big\{-\text{i Im}\langle f|g\rangle\big\}W(f+g), & (2.11)\\
(iii) &= \Upsilon(\mathscr{U})\Upsilon(\mathscr{V}) = \Upsilon(\mathscr{U}\mathscr{V}), & (2.12)\\
(iv) &= \Upsilon(\mathscr{U})W(f)\Upsilon(\mathscr{U})^{-1} = W(\mathscr{U}f), & (2.13)\\
(v) &= W(sf)W(tf) = W\big((s+t)f\big), \quad s,t\in\mathbb{R}. & (2.14)
\end{aligned}$$

2.2 Field operators

2.2.1 The operator $Q(f)$

For every element $f \in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$ the map $k \to W(\mathrm{i}kf)$ is a strongly continuous one parameter unitary group and we denote by $Q(f)$ its Stone generator and we write

$$W(\mathrm{i}kf) = \exp\big\{\mathrm{i}kQ(f)\big\}. \qquad (2.15)$$

The generator $Q(f)$ is a selfadjoint operator and it enjoys the properties ([12], Prop. 20.4 and Corol. 20.5):

(i) $Q(af) = aQ(f), \quad \forall a \in \mathbb{R}$,

(ii) $Q(f)$ is essentially selfadjoint in the exponential domain $\mathcal{E}(\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z}))$,

(iii) $\mathcal{E}(\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z}))$ is a core for $Q(f)$, i.e. $\overline{Q(f)\big|_{\mathcal{E}(\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z}))}} = Q(f)$,

(iv) the linear manifold of all finite particle vectors is a core for $Q(f)$,

(v) $\mathcal{E}(\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z}))$ is included in the domain of the product $Q(f_1)\cdots Q(f_n)$, $\forall n$, $\forall f_1,\ldots,f_n \in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$,

(vi) $[Q(h),Q(g)]\mathrm{e}(f) = \big\{2\mathrm{i}\mathrm{Im}\langle h|g\rangle\big\}\mathrm{e}(f)$.

2.2.2 The operator $\lambda(B)$

The operator $\Upsilon(\mathscr{U})$ is called the *second quantization* of $\mathscr{U}$. Let $\mathrm{e}^{-\mathrm{i}kB}$ be a strongly continuous, one parameter unitary group in $\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$ with B a selfadjoint operator in $\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$; its second quantization $\Upsilon(\mathrm{e}^{-\mathrm{i}kB})$ is a strongly continuous, one parameter unitary group too. We denote by $\lambda(B)$ its Stone generator:

$$\Upsilon(\mathrm{e}^{-\mathrm{i}kB}) = \mathrm{e}^{-\mathrm{i}k\lambda(B)}. \tag{2.16}$$

Sometimes $\lambda(B)$ is called "second quantization of B" and λ "conservation map".

Moreover, one has ([12], Theorem 20.7):

(i) $\mathcal{E}\big(\mathrm{Dom}(B)\big)$ is included in the domain of $\lambda(B)$,

(ii) $\mathcal{E}(\mathrm{Dom}(B^2))$ is a core for $\lambda(B)$,

(iii) if B is bounded, then $\lambda(B)$ is essentially selfadjoint in the exponential domain $\mathcal{E}(\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z}))$,

(iv) $\mathrm{i}[\lambda(B_1),\lambda(B_2)]\mathrm{e}(f) = \lambda(\mathrm{i}[B_1,B_2])\mathrm{e}(f)$, $\forall B_1, B_2$ bounded selfadjoint operators and $\forall f \in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$,

(v) $\mathcal{E}(\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z}))$ is included in the domain of the product $\lambda(B_1)\cdots\lambda(B_n)$, where the B_i are bounded selfadjoint operators.

For any $h \in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$ and any selfadjoint operator B in $\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$ let us set ([12], Prop. 21.6)

$$\lambda(B,h) := W(-h)\lambda(B)W(h). \tag{2.17}$$

Then, the operator $\lambda(B,h)$ is the generator of the unitary group

$$k \mapsto W(-h)\Upsilon(\mathrm{i}kB)W(h)$$

and it is essentially selfadjoint on the linear manifold generated by $\big\{\mathrm{e}(f-h) : f \in \mathrm{Dom}(B^2)\big\}$. When B is also bounded, $\mathcal{E}(\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z}))$ is a core for $\lambda(B,h)$ and, on the exponential domain $\mathcal{E}(\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z}))$, one has

$$\lambda(B,h) = \lambda(B) + a(Bh) + a^\dagger(Bh) + \langle h|Bh\rangle \mathbb{1}, \tag{2.18}$$

where the *creation* and *annihilation* operators $a(h)$ and $a^\dagger(h)$ are defined by

$$a(h) := \frac{1}{2}\big(Q(h) + \mathrm{i}Q(\mathrm{i}h)\big), \qquad a^\dagger(h) := \frac{1}{2}\big(Q(h) - \mathrm{i}Q(\mathrm{i}h)\big).$$

2.3 The quantum stochastic integral

Let us fix a complete orthonormal system $\{z_i;\ i \geq 1\}$ in $\mathscr{Z}$. We denote by

$A_i^\dagger(t) := a^\dagger(z_i \otimes 1_{(0,t)}), \quad \Lambda_{ij}(t) := \lambda\big((|z_i\rangle\langle z_j|) \otimes 1_{(0,t)}\big), \quad A_i(t) := a(z_i \otimes 1_{(0,t)})$

the *creation, conservation* and *annihilation processes* respectively ([12], Theorem 20.10); the maps $a^\dagger$, λ and a have been defined in the previous section. The following matrix notation is very useful:

$$\Lambda = (\Lambda_{ij}) = \begin{pmatrix} \mathbb{1}t & A_1 & A_2 & \cdots \\ A_1^\dagger & \Lambda_{11} & \Lambda_{12} & \cdots \\ A_2^\dagger & \Lambda_{21} & \Lambda_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2.19}$$

i.e. $\Lambda_{00} := \mathbb{1}t$, $\Lambda_{0i} := A_i$ and $\Lambda_{i0} := A_i^\dagger$. For these processes one has:

$$\big\langle \mathrm{e}(g)\big|\Lambda_{ij}(t)\mathrm{e}(f)\big\rangle = \int_0^t \overline{g_i(s)} f_j(s)\,\mathrm{d}s\,\langle \mathrm{e}(g)|\mathrm{e}(f)\rangle, \qquad \forall\, i,j \geq 0, \tag{2.20}$$

where we have used the notation

$$f_i(t) := \begin{cases} 1 & \text{if } i = 0 \\ \langle z_i|f(t)\rangle & \text{if } i \geq 1 \end{cases}$$

From now on we fix the sets

$$\mathscr{M} = \Big\{ f \in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z}) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathscr{Z}) \big| f_k(t) \equiv 0\ \ \forall t \text{ and} \\ \text{for all but a finite number of } k\text{'s} \Big\}$$

and

$$\mathcal{E} = \big\{ \text{linear span of all the exponential vectors } \mathrm{e}(f) \text{ with } f \in \mathscr{M} \big\}.$$

Notice that $\mathscr{M}$ is a dense linear manifold in $\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$, and, so, $\mathcal{E} \equiv \mathcal{E}(\mathscr{M})$ is dense in $\mathscr{F}$.

Let us introduce now a complex, separable Hilbert space $\mathscr{H}$, which we call *System Space*. Its role will be to describe a quantum system interacting with the Bose fields Λ.

Definition 2.1 (Adapted Process – [12] pg. 180)**.** Let D be a dense manifold in $\mathscr{H}$. A family $\{L(t), t \geq 0\}$ of operators in $\mathscr{H} \otimes \mathscr{F}$ is an *adapted process* with respect to $(D, \mathscr{M})$ if

(i) $D\underline{\otimes}\mathcal{E} \subset \bigcap_{t\geq 0} \mathrm{Dom}(L(t))$ (where $\underline{\otimes}$ denotes the algebraic tensor product),

(ii) The map $t \mapsto L(t)u \otimes \mathrm{e}(f)$ is strongly measurable, $\forall u \in D,\ f \in \mathscr{M}$,

(iii) $L(t)u \otimes \mathrm{e}(f_{(0,t)}) \in \mathscr{H} \otimes \mathscr{F}_{(0,t)}$ and $L(t)u \otimes \mathrm{e}(f) = \big(L(t)u \otimes \mathrm{e}(f_{(0,t)})\big) \otimes \mathrm{e}(f_{(t}), \forall t \geq 0, u \in D, f \in \mathscr{M}$.

If additionally the map $t \mapsto L(t)u \otimes \mathrm{e}(f)$ is continuous for every $u \in D$ and $f \in \mathscr{M}$ the process is said to be *regular adapted*.

By the factorization property (2.4) and the properties of Λ, we have for any $i, j \geq 0$

$$\big(\Lambda_{ij}(t) - \Lambda_{ij}(s)\big)\mathrm{e}(f) = \mathrm{e}(f_{(0,s)}) \otimes \Big(\big(\Lambda_{ij}(t) - \Lambda_{ij}(s)\big)\mathrm{e}(f_{(s,t)})\Big) \otimes \mathrm{e}(f_{(t})$$

for all $0 \leq s < t < \infty$ and $f \in \mathscr{H}$; here $\big(\Lambda_{ij}(t) - \Lambda_{ij}(s)\big)\mathrm{e}(f_{(s,t)}) \in \mathscr{F}_{(s,t)}$. Notice the similarity with the notion of process with independent increments in classical probability.

By a partition of $\mathbb{R}^+$ we mean a sequence $0 = t_0 < t_1 < \cdots < t_n < \cdots$ where $t_n \to \infty$ as $n \to \infty$. Let $L(t)$ be an adapted process in $\mathscr{H}$ with respect to $(D, \mathscr{M})$, simple with respect to the partition $\{t_n\}$. Let us define for L the stochastic integral

$$I_L(t) := \int_0^t L(s)\mathrm{d}\Lambda_{ij}$$

by putting $\mathrm{Dom}(I_L(t)) = D\underline{\otimes}\mathscr{E}$ and

$$I_L(t)u \otimes \mathrm{e}(f) = \big(L(0)u\big) \otimes \big(\Lambda_{ij}(t)\mathrm{e}(f)\big), \qquad \text{if } 0 \leq t \leq t_1\,,$$

$$\begin{aligned} I_L(t)u \otimes \mathrm{e}(f) &= I_L(t_n)u \otimes \mathrm{e}(f) \\ &\quad + \big(L(t_n)u \otimes \mathrm{e}(f_{(0,t_n)})\big)\big(\Lambda_{ij}(t) - \Lambda_{ij}(t_n)\big)\mathrm{e}(f_{(t_n}), \\ &\qquad\qquad \text{if } t_n < t \leq t_{n+1},\ \ n = 1, 2, \ldots, \end{aligned}$$

where the right hand side is determined inductively in n. I_L possesses the following properties:

- $I_L(t)$ is independent of the partition with respect to which L is simple.
- The map $t \to I_L(t)u \otimes \mathrm{e}(f)$ is continuous for any $u \in D$ and $f \in \mathscr{M}$.
- $\{I_L(t),\, t \geq 0\}$ is a regular adapted process with respect to $(D, \mathscr{M})$.

- $\left\|\int_0^t L(s)\mathrm{d}\Lambda_{ij}(s)u\otimes \mathrm{e}(f)\right\|^2 \leq \int_0^t \|L(s)u\otimes \mathrm{e}(f)\| \left(1+\|f(s)\|^2\right)\mathrm{d}s.$

Making use of these properties we can extend the notion of stochastic integral to adapted processes which are not necessarily simple, by the standard procedures of integration theory.

Definition 2.2 (Stochastically Integrable - [9] Def. 2.1)**.** A family $\{L_i;\, i\geq 0\}$ of $(D,\mathscr{M})$ adapted process is said to be stochastically integrable if, $\forall t\geq 0, u\in D$ and $f\in\mathscr{M}$, one has

$$\sum_{i\geq 0}\int_0^t \|L_i(s)u\otimes \mathrm{e}(f)\|^2\mathrm{d}s<\infty\,. \tag{2.21}$$

We denote by $\mathbb{L}(D,\mathscr{M})$ the class of the stochastically integrable families of $(D,\mathscr{M})$ adapted processes.

Definition 2.3 (Quantum Stochastic Integral)**.** Let I_0 be in $\mathscr{B}(\mathscr{H})$ and $\{F_{ij};\, i,j\geq 0\}$ in $\mathbb{L}(D,\mathscr{M})$, then the operator on $D\underline{\otimes}\mathscr{E}$

$$I_F(t)=I_0+\sum_{i,j\geq 0}\int_0^t F_{ij}(s)\mathrm{d}\Lambda_{ij}(s)$$

is called *Quantum Stochastic Integral*.

The construction of the stochastic integrals is developed in [12] Chapter III Section 25.

Proposition 2.1 (First fundamental formula of quantum stochastic calculus – [12] Corollary 27.2)**.** *Let $I_F(t)$ be as in Definition 2.3, then*

$$\begin{aligned}\big\langle v\otimes \mathrm{e}(g)|(I_F(t)-I_0)u\otimes \mathrm{e}(f)\big\rangle \\ =\sum_{i,j\geq 0}\int_0^t \mathrm{d}s\;\overline{g_i(s)}\big\langle v\otimes \mathrm{e}(g)\big|F_{ij}(s)u\otimes \mathrm{e}(f)\big\rangle f_j(s)\end{aligned} \tag{2.22}$$

for all $v\in\mathscr{H}$, $u\in D$, $f,g\in\mathscr{M}$ and $t>0$.

Proposition 2.2 (Second fundamental formula of quantum stochastic calculus – [12] Corollary 27.2)**.** *Let $I_F(t)$ as in Definition 2.3 and $I_{\widetilde{F}}(t)$ defined in similar* *terms. Then*

$$\begin{aligned}\big\langle I_{\widetilde F}(t)v\otimes \mathrm{e}(g)\big|I_F(t)u\otimes \mathrm{e}(f)\big\rangle - \big\langle \widetilde I_0 v\otimes \mathrm{e}(g)\big|I_0 u\otimes \mathrm{e}(f)\big\rangle \\ = \sum_{i,j\geq 0}\int_0^t \mathrm{d}s\ \overline{g_i(s)}\Big\{\big\langle I_{\widetilde F}(s)v\otimes \mathrm{e}(g)\big|F_{ij}(s)u\otimes \mathrm{e}(f)\big\rangle \\ + \big\langle \widetilde{F_{ji}}(s)v\otimes \mathrm{e}(g)\big|I_F(s)u\otimes \mathrm{e}(f)\big\rangle \\ + \sum_{k\geq 1}\big\langle \widetilde{F_{ki}}(s)v\otimes \mathrm{e}(g)\big|F_{kj}(s)u\otimes \mathrm{e}(f)\big\rangle\Big\} f_j(s)\end{aligned}$$

for all $u, v \in D$, $f, g \in \mathscr{M}$ and $t > 0$.

Proposition 2.3 ([12] Corollary 25.8, Proposition 27.1)**.** *The quantum stochastic integral $I_F(t)$ is a $(D, \mathscr{M})$ regular adapted process.*

Remark 2.1. All these propositions can be translated in the following practical rules for manipulating quantum stochastic integrals:

1. $\mathrm{d}A_i^\dagger(t), \mathrm{d}\Lambda_{ij}(t)$ and $\mathrm{d}A_i(t)$ commute with adapted processes at time t, so that they can be shifted towards the right or the left, according to the convenience.
2. The products of the fundamental differentials satisfy the Itô table: $\mathrm{d}\Lambda_{ik}(t)\mathrm{d}\Lambda_{lj}(t) = \hat\delta_{kl}\mathrm{d}\Lambda_{ij}(t)$, $i, j, k, l \geq 0$, where the symbol $\hat\delta_{kl}$ is defined by $\hat\delta_{kl} = 1$ if $k = l > 0$ and $\hat\delta_{kl} = 0$ otherwise.
3. For $i, j \geq 1$ we have $\mathrm{d}A_i(t)\mathrm{e}(f) = \mathrm{d}t f_i(t)\mathrm{e}(f)$, $\mathrm{d}\Lambda_{ij}(t)\mathrm{e}(f) = \mathrm{d}A_i^\dagger(t) f_j(t)\mathrm{e}(f)$, $\langle \mathrm{e}(f)|\mathrm{d}A_i^\dagger(t) = \overline{f_i(t)}\mathrm{d}t\langle \mathrm{e}(f)|$.

2.4 The Hudson-Parthasarathy equations

Now let K, R_i, N_i, F_{ij}, $\widetilde K$, $\widetilde R_i$, $\widetilde N_i$ and $\widetilde F_{ij}$, with $i, j \geq 1$, be (possibly unbounded) operators on the initial space $\mathscr{H}$, with which we build the two Quantum Stochastic Differential Equations (QSDE), known as Hudson-Parthasarathy(H-P) equations [12]:

Right Equation:

$$\begin{cases}\mathrm{d}U(t) = \Big(\sum_{i\geq 1} R_i \mathrm{d}A_i^\dagger(t) + \sum_{i,j\geq 1} F_{ij}\mathrm{d}\Lambda_{ij}(t) + \sum_{j\geq 1} N_j \mathrm{d}A_j(t) + K\mathrm{d}t\Big)U(t) \\ U(0) = \mathbb{1}\end{cases} \tag{2.23}$$

Left Equation:

$$\begin{cases} \mathrm{d}V(t) = V(t)\Big(\sum_{i\geq 1}\widetilde{N}_i \mathrm{d}A_i^\dagger(t) + \sum_{i,j\geq 1}\widetilde{F}_{ij}\mathrm{d}\Lambda_{ij}(t) + \sum_{j\geq 1}\widetilde{R}_j \mathrm{d}A_j(t) + \widetilde{K}\mathrm{d}t\Big) \\ V(0) = \mathbb{1} \end{cases} \tag{2.24}$$

By setting $F_{00} = K$, $F_{i0} = R_i$, $F_{0j} = N_j$, $\widetilde{F}_{00} = \widetilde{K}$, $\widetilde{F}_{i0} = \widetilde{N}_i$, $\widetilde{F}_{0j} = \widetilde{R}_j$, we can write:

Right Equation: $$\begin{cases} \mathrm{d}U(t) &= \sum_{i,j\geq 0} F_{ij}\ \mathrm{d}\Lambda_{ij}(t)\ U(t) \\ U(0) &= \mathbb{1} \end{cases}$$

Left Equation: $$\begin{cases} \mathrm{d}V(t) &= V(t)\sum_{i,j\geq 0} \widetilde{F}_{ij}\ \mathrm{d}\Lambda_{ij}(t) \\ V(0) &= \mathbb{1} \end{cases}$$

Definition 2.4 (Right Solution - [9] Def. 3.2)**.** Let D be a dense subspace in $\mathscr{H}$. An operator process U is a *solution of the Right QSDE* in $D\underline{\otimes}\mathcal{E}$ for the matrix F if:

(i) Each operator $F_{ij}\otimes\mathbb{1}$ is closable,

(ii) $\bigcup_{t\geq 0} U(t)(D\underline{\otimes}\mathcal{E}) \subset \bigcap_{i,j\geq 0} \mathrm{Dom}(\overline{F_{ij}\otimes\mathbb{1}})$,

(iii) Each process $\overline{F_{ij}\otimes\mathbb{1}}\ U$ is stochastically integrable, i.e.

$$\sum_{j\geq 0}\int_0^t \|\big(\overline{F_{ij}\otimes\mathbb{1}}\ U(s)\big)(u\otimes \mathrm{e}(f))\|^2 \mathrm{d}s < \infty\,, \qquad \forall i\geq 0,\ u\in D,\ f\in\mathscr{M},$$

(iv) $U(t) = \mathbb{1} + \sum_{i,j\geq 0}\int_0^t \overline{F_{ij}\otimes\mathbb{1}}\ U(s)\mathrm{d}\Lambda_{ij}(s)$ on $D\underline{\otimes}\mathcal{E}$, $\forall t\geq 0$.

Let us recall that, if F_{ij} is closable as an operator in $\mathscr{H}$, then $F_{ij}\otimes\mathbb{1}$ is closable in $\mathscr{H}\otimes\mathscr{F}$ ([9] pg. 188).

Definition 2.5 (Left Solution - [9] Def. 3.1)**.** Let $\widetilde{D}$ be a dense subspace in $\mathscr{H}$. An operator process V is a *solution of the Left QSDE* in $\widetilde{D}\underline{\otimes}\mathcal{E}$ for the matrix $\widetilde{F}$ if:

(i) $\widetilde{D}\subset \bigcap_{i,j\geq 0}\mathrm{Dom}(\widetilde{F}_{ij})$,

(ii) The linear manifold $\left(\bigcup_{i,j\geq 0} \widetilde{F}_{ij}\left(\widetilde{D}\right)\right) \underline{\otimes} \mathcal{E}$ is contained in the domain of $V(t)$, $\forall t \geq 0$,

(iii) $\forall t \geq 0$, the processes $\left(V(t)\widetilde{F}_{ij}; t\geq 0\right)$ are stochastically integrable, i.e.

$$\sum_{j\geq 0}\int_0^t \|V(s)\widetilde{F}_{ij}(u\otimes \mathrm{e}(f))\|^2 \mathrm{d}s < \infty, \qquad \forall i\geq 0,\ u\in \widetilde{D},\ f\in \mathscr{M},$$

(iv) $V(t) = \mathbb{1} + \sum_{i,j\geq 0}\int_0^t V(s)\widetilde{F}_{ij}\ \mathrm{d}\Lambda_{ij}(s) \quad$ on $\widetilde{D}\underline{\otimes}\mathcal{E}, \qquad \forall t\geq 0.$

Remark 2.2. By Proposition 2.3 right and left solutions are regular adapted processes with respect to $(D,\mathscr{M})$ or $(\widetilde{D},\mathscr{M})$, respectively (see Definition 2.1).

2.5 The algebraic isometric condition

Let us assume that the process $U(t)$ is a solution of the Right Equation associated to the matrix F as in Definition 2.4 and that it is isometric, which means

$$\langle U(t)v\otimes \mathrm{e}(g)|U(t)u\otimes \mathrm{e}(f)\rangle = \langle v\otimes \mathrm{e}(g)|u\otimes \mathrm{e}(f)\rangle, \qquad \forall u,v\in D\,, \quad \forall f,g\in \mathscr{M}. \tag{2.25}$$

By using the second fundamental formula (Proposition 2.2) and Eqs. (2.25), (2.23), we get

$$\begin{aligned}\sum_{i,j\geq 0}\int_0^t \mathrm{d}s\ \overline{g_i(s)}\Big\{&\big\langle U(s)v\otimes \mathrm{e}(g)\big|\overline{F_{ij}\otimes \mathbb{1}}U(s)u\otimes \mathrm{e}(f)\big\rangle\\ &+\big\langle \overline{F_{ji}\otimes \mathbb{1}}U(s)v\otimes \mathrm{e}(g)\big|U(s)u\otimes \mathrm{e}(f)\big\rangle\\ &+\sum_{k\geq 1}\big\langle \overline{F_{ki}\otimes \mathbb{1}}U(s)v\otimes \mathrm{e}(g)\big|\overline{F_{kj}\otimes \mathbb{1}}U(s)u\otimes \mathrm{e}(f)\big\rangle\Big\}f_j(s) = 0\,.\end{aligned}$$

By choosing $f(t)$ and $g(t)$ continuous, by dividing by t and by taking the limit $t\downarrow 0$, we get, by the arbitrariness of $f(0)$ and $g(0)$, the necessary condition

$$\langle v|\overline{F_{ij}}u\rangle + \langle \overline{F_{ji}}v|u\rangle + \sum_{k\geq 1}\langle \overline{F_{ki}}v|\overline{F_{kj}}u\rangle = 0, \qquad \forall u,v\in D, \quad \forall i,j\geq 0. \tag{2.26}$$

Let us assume that the process $V(t)$ is a solution of the Left Equation associated to the matrix $\widetilde{F}$ as in Definition 2.5 and that it is isometric. By going on as before and recalling that $V(t)$ is an isometry, we get

$$\begin{aligned}
0 = \sum_{i,j\geq 0} \int_0^t \mathrm{d}s\ \overline{g_i(s)} \Big\{ \big\langle V(s)v\otimes \mathrm{e}(g) \big| V(s)\widetilde{F}_{ij}u\otimes \mathrm{e}(f)\big\rangle \\
+ \big\langle V(s)\widetilde{F}_{ji}v\otimes \mathrm{e}(g) \big| V(s)u\otimes \mathrm{e}(f)\big\rangle \\
+ \sum_{k\geq 1} \big\langle V(s)\widetilde{F}_{ki}v\otimes \mathrm{e}(g) \big| V(s)\widetilde{F}_{kj}u\otimes \mathrm{e}(f)\big\rangle \Big\} f_j(s) \\
= \sum_{i,j\geq 0} \int_0^t \mathrm{d}s\ \overline{g_i(s)} \Big\{ \big\langle v\otimes \mathrm{e}(g) \big| \widetilde{F}_{ij}u\otimes \mathrm{e}(f)\big\rangle + \big\langle \widetilde{F}_{ji}v\otimes \mathrm{e}(g) \big| u\otimes \mathrm{e}(f)\big\rangle \\
+ \sum_{k\geq 1} \big\langle \widetilde{F}_{ki}v\otimes \mathrm{e}(g) \big| \widetilde{F}_{kj}u\otimes \mathrm{e}(f)\big\rangle \Big\} f_j(s)\,,
\end{aligned}$$

which gives (see Proposition 8.1 of [9])

$$\langle v|\widetilde{F}_{ij}u\rangle + \langle \widetilde{F}_{ji}v|u\rangle + \sum_{k\geq 1}\langle \widetilde{F}_{ki}v|\widetilde{F}_{kj}u\rangle = 0, \qquad \forall u,v\in \widetilde{D}, \quad \forall i,j\geq 0. \tag{2.27}$$

2.6 The bounded case

Consider $F=(F_{ij})$ with $F_{ij}\in\mathscr{B}(\mathscr{H})$. Let us consider the right equation with matrix $F=(F_{ij})$ and the left equation with matrix $\widetilde{F}=(\widetilde{F}_{ij}):=(F^*_{ji})=F^*$; then we have:

$$\begin{cases} \mathrm{d}U(t) = \Big(\sum_{i\geq 1} R_i \mathrm{d}A_i^\dagger(t) + \sum_{i,j\geq 1} F_{ij}\mathrm{d}\Lambda_{ij}(t) + \sum_{j\geq 1} N_i \mathrm{d}A_i(t) + K\mathrm{d}t\Big) U(t) \\ U(0) = \mathbb{1} \end{cases}$$

$$\begin{cases} \mathrm{d}V(t) = V(t)\Big(\sum_{i\geq 1} N_i^* \mathrm{d}A_i^\dagger(t) + \sum_{i,j\geq 1} F^*_{ji}\mathrm{d}\Lambda_{ij}(t) + \sum_{j\geq 1} R_i^* \mathrm{d}A_i(t) + K^*\mathrm{d}t\Big) \\ V(0) = \mathbb{1} \end{cases}$$

Let us assume the isometry conditions (2.26), (2.27) with $D=\widetilde{D}=\mathscr{H}$, which become

$$F_{ij} + F^*_{ji} + \sum_{k\geq 1} F^*_{ki}F_{kj} = F^*_{ji} + F_{ij} + \sum_{k\geq 1} F_{ik}F^*_{jk} = 0, \qquad \forall i,j\geq 0,$$

or, equivalently,

$$F_{ij} + F_{ji}^* + \sum_{k\geq 1} F_{ki}^* F_{kj} = F_{ji}^* + F_{ij} + \sum_{k\geq 1} F_{ik} F_{jk}^* = 0, \qquad \forall i,j \geq 1, \quad (2.28)$$

$$K + K^* + \sum_{k\geq 1} R_k^* R_k = K^* + K + \sum_{k\geq 1} N_k N_k^* = 0, \quad (2.29)$$

$$N_i + R_i^* + \sum_{k\geq 1} R_k^* F_{ki} = R_i^* + N_i + \sum_{k\geq 1} F_{ki} N_k^* = 0, \qquad \forall i \geq 1. \quad (2.30)$$

Let us define $S_{ij} := F_{ji} + \delta_{ji}$, for $i,j \geq 1$. Then, the conditions above become

$$\sum_{k\geq 1} S_{ki}^* S_{kj} = \sum_{k\geq 1} S_{ik} S_{jk}^* = \delta_{ij}\,,$$

$$K = -\mathrm{i}H - \frac{1}{2}\sum_{k\geq 1} R_k^* R_k, \qquad N_i = -\sum_{k\geq 1} R_k^* S_{ki},$$

where H is a bounded selfadjoint operator in $\mathscr{B}(\mathscr{H})$ and $\sum_{k\geq 1} R_k^* R_k$ converges strongly to a bounded operator.

Therefore we can write the Right Equation as

$$\begin{cases} \mathrm{d}U(t) &= \Big(\sum_{i\geq 1} R_i \mathrm{d}A_i^\dagger(t) + \sum_{i,j\geq 1} \big(S_{ij} - \delta_{ij}\big)\mathrm{d}\Lambda_{ij}(t) - \sum_{i,j\geq 1} R_i^* S_{ij} \mathrm{d}A_j(t) \\ & \qquad + K\mathrm{d}t\Big) U(t) \\ U(0) &= \mathbb{1} \end{cases}$$

Theorem 2.4 ([12] Prop. 27.5, Theo. 27.8, Coro. 3.2)**.** *In the hypotheses above, there exist a unique adapted process $U(t)$ which solves the right equation (2.23) and a unique adapted process $V(t)$ which solves the left equation (2.24). Both processes turn out to be unitary, strongly continuous in t and mutually adjoint, i.e. $V(t) = U^*(t)$*

2.7 The unbounded case

2.7.1 Some conditions on F

From Eqs. (2.26), (2.27) we expect the operators F_{ij} for $i,j \geq 1$ to be related to unitary operators as in the previous section. So, it is natural to ask them to be bounded even in the general case. Moreover, in order not to have too much analytical complications all together, we ask the space $\mathscr{X}$ to be finite-dimensional. In Definition 2.4 the operators F_{ij} are needed to be closable and in what follows only their closures are involved; so, we can ask them to be closed.

Hypothesis 2.1. Let us assume $\dim(\mathscr{X}) = d < \infty$.
Let $F = (F_{ij})$ be a $(d+1) \times (d+1)$-dimensional operator valued matrix with $F_{ij} \in \mathscr{B}(\mathscr{H})$ for all $i, j = 1, \ldots, d$, while F_{i0} and F_{0i}, for $i \geq 0$ are closed operators. Moreover, we consider the right H-P equation with matrix $F = (F_{ij})$ and the left H-P equation with matrix $\widetilde{F} = (\widetilde{F}_{ij}) := (F^*_{ji}) = F^*$.

We also define $\mathrm{Dom}(F) := \bigcap_{i,j \geq 0} \mathrm{Dom}(F_{ij})$, $\quad \mathrm{Dom}(F^*) := \bigcap_{i,j \geq 0} \mathrm{Dom}(F^*_{ij})$ and we assume both $\mathrm{Dom}(F)$ and $\mathrm{Dom}(F^*)$ to be dense in $\mathscr{H}$.

Following [9] pg. 191, we introduce the quadratic form $\vartheta(F, X)$, where $X \in \mathscr{B}(\mathscr{H})$:

$\forall u = (u_i),\ v = (v_i) \in \bigoplus_{k=0}^{d} \mathrm{Dom}(F),$

$$\vartheta(F, X)(u, v) := \sum_{i,j \geq 0} \langle u_i | X F_{ij} v_j \rangle + \sum_{i,j \geq 0} \langle F_{ji} u_i | X v_j \rangle + \sum_{i,j \geq 0} \sum_{k \geq 1} \langle F_{ki} u_i | X F_{kj} v_j \rangle. \tag{2.31}$$

The quadratic form $\vartheta(F^*, X)$ is defined in an analogous way. We define also

$$\vartheta(F) := \vartheta(F, \mathbb{1}), \qquad \vartheta(F^*) := \vartheta(F^*, \mathbb{1}). \tag{2.32}$$

Note that the isometric condition (2.26) is equivalent to $\vartheta(F) = 0$ and the co-isometric condition (2.27) is equivalent to $\vartheta(F^*) = 0$.

2.7.2 Existence of solutions of the H-P equations

The mathematical treatment of the Right Equation is more complicated and the sufficient hypotheses for existence and uniqueness of its solution are stronger. For this reason, we will establish first the conditions for the existence and uniqueness of the solution of the Left Equation.

Theorem 2.5 ([9] Proposition 8.1, Theorem 8.5)**.** *If the matrix F^* satisfies Hypothesis 2.1 and, moreover,*

1. *the operator* $K^* = F^*_{00}$ *is the infinitesimal generator of a strongly continuous contraction semigroup on* $\mathscr{H}$ *and it exists a dense subspace* $\widetilde{D} \subset \mathscr{H}$ *such that* $\widetilde{D}$ *is a core for* K^* *contained in* $\mathrm{Dom}(F^*)$*,*

2. $\vartheta(F^*) \leq 0$ *in* $\widetilde{D}$*,*

then, there exists a unique contractive solution V *on* $\widetilde{D}\underline{\otimes}\mathcal{E}$ *of the left QSDE (2.24) with* $\widetilde{F} = F^*$*.*

If, moreover, V *is an isometric process, then* $\vartheta(F^*) = 0$ *in* $\widetilde{D}$*.*

Remark 2.3. By Remark 2.2 and the fact that $V(t)$ is a contraction, we have that it is strongly continuous in t.

Theorem 2.6 ([9] Theorem 11.1, [10] Theorem 2.3)**.** *Let* U *be a contraction process and let* F *be an operator matrix satisfying Hypotheses 2.1. Let us set* $V(t) = U(t)^*$ *and assume that* V *is the unique contractive solution of the Left Equation for the matrix* F^* *in* $\widetilde{D}\underline{\otimes}\mathcal{E}$*, where* $\widetilde{D}$ *is some dense linear set in* $\mathscr{H}$*.*

Also suppose that C *is a positive self-adjoint operator on* $\mathscr{H}$*, and that* $\delta > 0$ *and* $b_1, b_2 \geq 0$ *are constants such that the following properties hold:*

1. *for each* $\epsilon \in (0, \delta)$*, there exists a dense subspace* $D_\epsilon \subset \widetilde{D}$ *such that* $(C_\epsilon)^{\frac{1}{2}}(D_\epsilon) \subset \widetilde{D}$ *and each operator* $F^*_{ij}(C_\epsilon)^{\frac{1}{2}}|_{D_\epsilon}$ *is bounded, where* $C_\epsilon = \frac{C}{1+\epsilon C}$*;*

2. $\mathrm{Dom}(C^{\frac{1}{2}}) \subset \mathrm{Dom}(F)$*;*

3. *for all* $0 < \epsilon < \delta$*, the form* $\vartheta(F, C_\epsilon)$ *on* $\mathrm{Dom}(F)$ *satisfies the inequality:* $\vartheta(F, C_\epsilon) \leq b_1 \iota(C_\epsilon) + b_2 \mathbb{1}$*, where* $\iota(C_\epsilon)$ *is the* $(d+1) \times (d+1)$ *matrix* $\mathrm{diag}(C_\epsilon)$ *of operators on* $\mathscr{H}$*.*

Then, U *is a solution of the right QSDE (2.23) on* $\mathrm{Dom}(C^{\frac{1}{2}})\underline{\otimes}\mathscr{E}$ *for the operator matrix* F*.*

Chapter 3

Unitary Cocycles and QDS

In quantum mechanics the dynamics of a closed system is represented by a one-parameter group of unitary operators. So, for the applications in quantum mechanics first of all we need $U(t)$ to be a unitary operator. Then, we have also to show that it is strictly related with a unitary group. To give sufficient conditions to guarantee this is the content of this chapter.

3.1 Cocycles

Let us consider now an extension to negative times of the Fock space introduced in Eq. (2.4):

$$\widetilde{\mathscr{F}} := \Gamma_{\text{symm}}(\mathrm{L}^2(\mathbb{R};\mathscr{Z})) = \mathscr{F}_{0)} \otimes \mathscr{F}_{(0}\,. \tag{3.1}$$

With the usual convention of not to write the tensor products with the identity, the solutions $U(t)$ and $V(t)$ of Eqs. (2.23) and (2.24) can be understood as operators on $\mathscr{H} \otimes \widetilde{\mathscr{F}}$.

We introduce the strongly continuous one-parameter unitary group θ of the shift operators on $\mathrm{L}^2(\mathbb{R};\mathscr{Z})$ and its second quantization Θ on $\widetilde{\mathscr{F}}$: for every $t \in \mathbb{R}$

$$(\theta_t f)(x) = f(x+t) \qquad \text{and} \qquad \Theta_t \psi(f) = \psi(\theta_t f), \qquad \forall f \in \mathrm{L}^2(\mathbb{R};\mathscr{Z}). \tag{3.2}$$

Let us note that, for $r < s$, $\left(\theta_t 1_{(r,s)}\right)(x) = 1_{(r,s)}(x+t) = 1_{(r-t,s-t)}(x)$; this implies

$$\Theta_t \mathscr{F}_{(r,s)} \subset \mathscr{F}_{(r-t,s-t)}\,. \tag{3.3}$$

We extend Θ_t to the space $\mathscr{H}\otimes\widetilde{\mathscr{F}}$ by stipulating that it acts as the identity on $\mathscr{H}$.

Definition 3.1 (Right Cocycle)**.** A bounded, adapted operator process $X(t)$ in $\mathscr{H}\otimes\widetilde{\mathscr{F}}$ is called *right cocycle* if for every $s,t\geq 0$ we have $X(t+s) = \Theta_s^* X(t)\Theta_s X(s)$.

Definition 3.2 (Left Cocycle)**.** A bounded, adapted operator process $X(t)$ in $\mathscr{H}\otimes\widetilde{\mathscr{F}}$ is called *left cocycle* if for every $s,t\geq 0$ we have $X(t+s) = X(s)\Theta_s^* X(t)\Theta_s$.

Lemma 3.1 ([9] Lemma 6.2)**.** *Let X be a bounded operator process on $\mathscr{H}\otimes\widetilde{\mathscr{F}}$ such that*

$$X(t) = \int_0^t M(r)\,\mathrm{d}\Lambda_{ij}(r),$$

for some $i,j\geq 0$ and some stochastically integrable operator process $M\in\mathbb{L}(D,\mathscr{M})$. Then, for all $t,s\geq 0$ we have

$$\Theta_s^* X(t)\Theta_s = \int_s^{t+s} \Theta_s^* M(r-s)\Theta_s\ \mathrm{d}\Lambda_{ij}(r)\,.$$

The process $r\mapsto 1_{(s,+\infty)}(r)\Theta_s^* M(r-s)\Theta_s$ belongs to $\mathbb{L}(D,\mathscr{M})$.

Proof. By property (iii) of Definition 2.1 for M, we have for $0\leq s\leq r$, $u\in D$, $f\in\mathscr{M}$,

$$\begin{aligned}
&\Theta_s^* M(r-s)\Theta_s\, u\otimes \mathrm{e}(f)\\
&\quad= \Theta_s^*\mathrm{e}\left((\theta_s f)_{0)}\right)\otimes\left(M(r-s)\,u\otimes\mathrm{e}\left((\theta_s f)_{(0,r-s)}\right)\right)\otimes\mathrm{e}\left((\theta_s f)_{(r-s}\right)\\
&\qquad\qquad= \mathrm{e}\left(f_{s)}\right)\otimes\left(\Theta_{-s}M(r-s)\,u\otimes\mathrm{e}\left((\theta_s f)_{(0,r-s)}\right)\right)\otimes\mathrm{e}\left(f_{(r}\right).
\end{aligned}$$

Together with (3.3), this implies property (iii) of Definition 2.1 for the process $r\mapsto 1_{(s,+\infty)}(r)\Theta_s^* M(r-s)\Theta_s$. Also the other properties in Definitions 2.1 and 2.2 can be shown to hold and the last statement of the Lemma is proved.

By the definition of the shift semigroup Θ and the integral expression of X, one has also

$$\begin{aligned}
&\langle v\otimes\mathrm{e}(g)|\Theta_s^* X(t)\Theta_s u\otimes\mathrm{e}(f)\rangle = \langle v\otimes\mathrm{e}(\theta_s g)|X(t)u\otimes\mathrm{e}(\theta_s f)\rangle\\
&\qquad= \int_0^t \overline{(\theta_s g)_i(r)}\langle v\otimes\mathrm{e}(\theta_s g)|M(r)u\otimes\mathrm{e}(\theta_s f)\rangle(\theta_s f)_j(r)\mathrm{d}r\\
&\qquad= \int_0^t \overline{g_i(r+s)}\langle v\otimes\mathrm{e}(g)|\Theta_s^* M(r)\Theta_s\,u\otimes\mathrm{e}(f)\rangle f_j(r+s)\mathrm{d}r\\
&\qquad\qquad= \left\langle v\otimes\mathrm{e}(g)\middle|\int_s^{t+s}\Theta_s^* M(r-s)\Theta_s\mathrm{d}\Lambda_{ij}(r)\,u\otimes\mathrm{e}(f)\right\rangle
\end{aligned}$$

and this completes the proof. □

Theorem 3.2 ([9] Prop. 6.3)**.** *Let D be a dense subspace of $\mathscr{H}$ and $F = (F_{ij})$ a matrix of operators on $\mathscr{H}$. Suppose that there exists a unique bounded process U solving Eq. (2.23) on $D\underline{\otimes}\mathcal{E}$. Then, U is a right cocycle.*

Proof. Fix $s > 0$ and let X be the bounded processes defined by

$$X(t) = \begin{cases} U(t) & \text{if } t \leq s \\ \Theta_s^* U(t-s)\Theta_s U(s) & \text{if } t > s \end{cases}$$

By Lemma 3.1 and Eq. (2.23), one has

$$\begin{aligned} X(t+s) &= \Theta_s^* U(t)\Theta_s U(s) \\ &= U(s) + \sum_{i,j\geq 0} \int_s^{t+s} \Theta_s^* F_{ij} U(r-s)\Theta_s \mathrm{d}\Lambda_{ij}(r)\, U(s) \\ &= U(s) + \sum_{i,j\geq 0} \int_s^{t+s} F_{ij} \Theta_s^* U(r-s)\Theta_s U(s)\, \mathrm{d}\Lambda_{ij}(r) \\ &= U(s) + \sum_{i,j\geq 0} \int_s^{t+s} F_{ij} X(r)\, \mathrm{d}\Lambda_{ij}(r) \\ &= \mathbb{1} + \sum_{i,j\geq 0} \int_0^{t+s} F_{ij} X(r)\, \mathrm{d}\Lambda_{ij}(r)\,. \end{aligned}$$

Therefore $X(t)$ and $U(t)$ are bounded solutions of the right equation (2.23) and by uniqueness they are equal. □

Theorem 3.3 ([9] Prop. 6.3)**.** *Let $\widetilde{D}$ be a dense subspace of $\mathscr{H}$ and $\widetilde{F} = (\widetilde{F}_{ij})$ a matrix of operators on $\mathscr{H}$. Suppose that there exists a unique bounded process V solving the equation (2.24) on $\widetilde{D}\underline{\otimes}\mathcal{E}$. Then, V is a left cocycle.*

Theorems 3.2 and 3.3 can be modified in a obvious way when the solution is unique among the contractions or among the isometries.

Proposition 3.4. *Let $U(t)$ be a unitary right cocycle and set*

$$\mathscr{U}_t = \begin{cases} \Theta_t U(t) & \text{if } t \geq 0 \\ U(|t|)^* \Theta_t & \text{if } t \leq 0 \end{cases} \tag{3.4}$$

Then, the family of operators $\mathscr{U}_t$ is a one-parameter group.

If, moreover, $U(t)$ is strongly continuous, $\mathscr{U}_t$ is a strongly continuous one-parameter group of unitary operators.

Proof. Let us observe that $\Theta_t^* = \Theta_{-t}$ and $\Theta_{t-s} = \Theta_t\Theta_s^*$.

From the definition one has

$$\mathscr{U}_t^* = \mathscr{U}_{-t}\,, \qquad \forall t \in \mathbb{R}\,. \tag{3.5}$$

From the cocycle property and the fact that Θ is a group one gets, $\forall t, s \geq 0$,

$$\mathscr{U}_t\mathscr{U}_s = \Theta_t U(t)\Theta_s U(s) = \Theta_{t+s}\Theta_s^* U(t)\Theta_s U(s) = \Theta_{t+s}U(t+s) = \mathscr{U}_{t+s}\,. \tag{3.6}$$

All the other combinations of positive and negative times can be examined and give the same result.

Being $\mathscr{U}$ a unitary group, it is enough to prove its strong continuity in 0, which follows from the unitarity and the strong continuity of U and Θ. For $t \geq 0$ and $\Upsilon \in \mathscr{H} \otimes \widetilde{\mathscr{F}}$ we have

$$\begin{aligned}\big\|\big(\mathscr{U}_t - \mathbb{1}\big)\Upsilon\big\| = \big\|\big(\Theta_t U(t) - \mathbb{1}\big)\Upsilon\big\| &\leq \big\|\Theta_t\big(U(t) - \mathbb{1}\big)\Upsilon\big\| + \big\|\big(\Theta_t - \mathbb{1}\big)\Upsilon\big\| \\ &= \big\|\big(U(t) - \mathbb{1}\big)\Upsilon\big\| + \big\|\big(\Theta_t - \mathbb{1}\big)\Upsilon\big\| \to 0 \qquad \text{as } t \downarrow 0\,.\end{aligned}$$

For $t \leq 0$, we have

$$\begin{aligned}\big\|\big(\mathscr{U}_t - \mathbb{1}\big)\Upsilon\big\| &= \big\|\big(U(|t|)^*\Theta_t - \mathbb{1}\big)\Upsilon\big\| \\ &\leq \big\|U(|t|)^*\big(\Theta_t - \mathbb{1}\big)\Upsilon\big\| + \big\|\big(U(|t|)^* - \mathbb{1}\big)\Upsilon\big\| \\ &= \big\|\big(\Theta_t - \mathbb{1}\big)\Upsilon\big\| + \big\|\big(U(|t|) - \mathbb{1}\big)\Upsilon\big\| \to 0 \qquad \text{as } t \uparrow 0\,.\end{aligned}$$

□

Definition 3.3. Let $U(t)$ be a unitary right cocycle and $\mathscr{U}_t$ be the unitary group defined by Eq. (3.4). For $s \leq t$ we define the two-parameter family of unitary operators

$$U(t,s) := \Theta_t^*\mathscr{U}_{t-s}\Theta_s \equiv \Theta_s^* U(t-s)\Theta_s\,. \tag{3.7}$$

Proposition 3.5. *The two-parameter family of unitary operators defined above satisfies the composition law*

$$U(t,r) = U(t,s)U(s,r), \qquad r \leq s \leq t\,.$$

If, moreover, $U(t)$ is strongly continuous, $U(t,s)$ is strongly continuous in t and s.

Proof.

$$\begin{aligned}
U(t,r) &= \Theta_r^* U(t-r)\Theta_r \\
&= \Theta_r^* U\big((t-s)+(s-r)\big)\Theta_r \qquad \text{by the cocycle property} \\
&= \Theta_r^*\Theta_{s-r}^* U(t-s)\Theta_{s-r}U(s-r)\Theta_r \\
&= \underbrace{\Theta_s^* U(t-s)\Theta_s}\,\underbrace{\Theta_r^* U(s-r)\Theta_r} \\
&= \quad U(t,s) \qquad\quad U(s,r).
\end{aligned}$$

The proof of the strong continuity in t and s is similar to the one of the strong continuity of $\mathscr{U}$. It follows from the unitarity and strong continuity of Θ and $\mathscr{U}$ and the decompositions

$$\begin{aligned}
U(t+\epsilon,s)-U(t,s) &= \Theta_t^*\left(\Theta_{-\epsilon}\mathscr{U}_\epsilon - \Theta_{-\epsilon}+\Theta_{-\epsilon}-\mathbb{1}\right)\mathscr{U}_{t-s}\Theta_s\,, \\
U(t,s+\epsilon)-U(t,s) &= \Theta_t^*\mathscr{U}_{t-s}\left(\mathscr{U}_{-\epsilon}\Theta_\epsilon - \mathscr{U}_{-\epsilon}+\mathscr{U}_{-\epsilon}-\mathbb{1}\right)\Theta_s\,.
\end{aligned}$$

□

Remark 3.1 ([3] - pg. 220). Let $U(t,s)$ be as defined in Proposition 3.5.

1. From the definition we have immediately $U(t,0)=U(t)$, $\forall t\geq 0$.

2. The operator $U(t,s)$ is adapted to $\mathscr{H}\otimes\mathscr{F}_{(s,t)}$ in the sense that it acts as the identity on $\mathscr{F}_{s)}\otimes\mathscr{F}_{(t}$ and leaves $\mathscr{H}\otimes\mathscr{F}_{(s,t)}$ invariant. This follows from Eq. (3.3) and the fact that $U(t)$ is adapted to $\mathscr{H}\otimes\mathscr{F}_{(0,t)}$ by the definition of cocycle.

3. From the definition of $U(t,s)$ and the cocycle property we get immediately

$$U(t,s)=U(t)U(s)^*, \qquad \forall s,t \quad 0\leq s\leq t\,. \tag{3.8}$$

4. If, moreover, $U(t)$ is a solution of the right H-P equation (2.23), then, $U(t,s)$ with respect to t satisfies the same equation with initial condition $U(s,s)=\mathbb{1}$, i.e.

$$U(t,s)=\mathbb{1}+\sum_{i,j\geq 0}\int_s^t F_{ij}U(r,s)\mathrm{d}\Lambda_{ij}(r). \tag{3.9}$$

Indeed, from Eq. (2.23) and Lemma 3.1 we get

$$\begin{aligned} U(t,s) &= \Theta_s^* U(t-s)\Theta_s \\ &= \mathbb{1} + \Theta_s^* \sum_{i,j\geq 0} \int_0^{t-s} F_{ij} U(r) \mathrm{d}\Lambda_{ij}(r)\, \Theta_s \\ &= \mathbb{1} + \sum_{i,j\geq 0} \int_s^t \Theta_s^* F_{ij} U(r-s) \Theta_s \, \mathrm{d}\Lambda_{ij}(r)\,, \end{aligned}$$

which gives (3.9).

3.2 Quantum dynamical semigroups

Definition 3.4 (Quantum Dynamical Semigroup)**.** Let us consider a family $\{\mathcal{T}(t), t \geq 0\}$ of bounded operators on $\mathscr{B}(\mathscr{H})$ with the following properties:

1. $\mathcal{T}(0) = \mathbb{1}$;
2. $\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s)$, $\forall s,t \geq 0$;
3. $\mathcal{T}(t)$ is completely positive, $\forall t \geq 0$;
4. $\mathcal{T}(t)$ is a σ-weakly continuous operator on $\mathscr{B}(\mathscr{H})$, $\forall t \geq 0$;
5. For each $X \in \mathscr{B}(\mathscr{H})$ the map $t \mapsto \mathcal{T}(t)[X]$ is continuous with respect to the σ-weak topology of $\mathscr{B}(\mathscr{H})$.

Then, the family of operators $\mathcal{T}(t)$ is called a *quantum dynamical semigroup* (QDS).

Definition 3.5 (Quantum Markov semigroup)**.** A QDS $\mathcal{T}(t)$ is said to be *Markov* or *conservative* if $\mathcal{T}(t)[\mathbb{1}] = \mathbb{1}$.

Definition 3.6 (Infinitesimal Generator)**.** The infinitesimal generator of the quantum dynamical semigroup $\mathcal{T}$ is the operator $\mathcal{L}$ defined as

$$\mathcal{L}[X] = \lim_{t\to 0^+} \frac{\mathcal{T}(t)[X] - X}{t}$$

with

$$\mathrm{Dom}(\mathcal{L}) = \left\{ X \in \mathscr{B}(\mathscr{H}) \ : \ \lim_{t\to 0^+} \frac{\mathcal{T}(t)[X] - X}{t} \text{ exits in the } \sigma\text{-weak topology} \right\}$$

Hypothesis 3.1. Let A and B_k, $k = 1, \ldots, d$, be operators in $\mathscr{H}$ that satisfy the following conditions.

- The operator A is the infinitesimal generator of a strongly continuous contraction semigroup $P(t)$ in $\mathscr{H}$.
- The domain of each operator B_k contains the domain of A and for every $u \in \mathrm{Dom}(A)$ we have

$$\langle u|Au\rangle + \langle Au|u\rangle + \sum_{k\geq 1}\langle B_k u|B_k u\rangle = 0\,. \tag{3.10}$$

For all $X \in \mathscr{B}(\mathscr{H})$, let us consider the quadratic form $\mathscr{L}[X]$ in $\mathscr{H}$ with domain $\mathrm{Dom}(A) \times \mathrm{Dom}(A)$ given by

$$\langle v|\mathscr{L}[X]u\rangle = \langle v|XAu\rangle + \langle Av|Xu\rangle + \sum_{k\geq 1}\langle B_k v|XB_k u\rangle. \tag{3.11}$$

We want to construct a QDS satisfying the equation

$$\langle v|\mathcal{T}(t)[X]u\rangle = \langle v|Xu\rangle + \int_0^t \big\langle v\big|\mathscr{L}\big[\mathcal{T}(s)[X]\big]u\big\rangle \mathrm{d}s. \tag{3.12}$$

Theorem 3.6 ([8] - Theorem 3.22)**.** *Suppose that Hypothesis 3.1 holds. Then, there exists a QDS $\mathcal{T}(t)$ solving Eq. (3.12) with generator (3.11) and such that*

- $\mathcal{T}(t)[\mathbb{1}] \leq \mathbb{1}$, $\forall t \geq 0$*;*
- *for every σ-weakly continuous family $\mathscr{T}(t)$ of positive maps on $\mathscr{B}(\mathscr{H})$ satisfying Eqs. (3.11) and (3.12) we have $\mathcal{T}(t)[X] \leq \mathscr{T}(t)[X]$, $\forall t \geq 0$, for all positive $X \in \mathscr{B}(\mathscr{H})$.*

The QDS $\mathcal{T}(t)$ defined in Theorem 3.6 is called the *Minimal Quantum Dynamical Semigroup* generated by A and B_k.

Proposition 3.7 ([8] - Cor. 3.23)**.** *Suppose that Hypothesis 3.1 holds and that the minimal QDS $\mathcal{T}(t)$ is Markov. Then, it is the unique σ-weakly continuous family of positive maps on $\mathscr{B}(\mathscr{H})$ satisfying Eq. (3.12).*

Theorem 3.8 (Conservativeness conditions – [8] Proposition 3.24, Theorem 3.28, Proposition 3.31)**.** *Suppose that Hypothesis 3.1 holds and define the linear positive map $\mathcal{Q}_\lambda[X] : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ for $\lambda > 0$ by*

$$\langle v|\mathcal{Q}_\lambda[X]u\rangle := \sum_{k\geq 1}\int_0^\infty \mathrm{e}^{-\lambda s}\langle B_k P(s)v|XB_k P(s)u\rangle \mathrm{d}s\,, \qquad \forall u, v \in Dom(A).$$

Then, $\mathcal{Q}_\lambda$ is a normal, completely positive contraction. Moreover, the following conditions are equivalent ($\lambda > 0$ is fixed):

1. *the minimal QDS $\mathcal{T}(t)$ generated by A and B_k is Markov,*
2. *it does not exist $X \neq 0$ in $\mathscr{B}(\mathscr{H})$ such that $\mathcal{Q}_\lambda[X] = X$,*
3. *it does not exist $X \neq 0$ in $\mathscr{B}(\mathscr{H})$ such that $\mathscr{L}[X] = \lambda X$ ($\mathscr{L}$ is defined in (3.11)).*

3.3 Unitary solutions of H-P equations

Let $F = (F_{ij} : i, j \geq 0)$ be a matrix of operators satisfying Hypothesis 2.1, $\vartheta(F) = 0$, $\vartheta(F^*) = 0$, and such that the operators $A = F_{00}^* = K^*$, $B_k = F_{0k}^* = N_k^*$ satisfy Hypothesis 3.1. Then, the hypotheses of Theorem 2.5 hold; let V be the unique contractive solution of the left QSDE (2.24) with $\widetilde{F} = F^*$ on $\widetilde{D}\underline{\otimes}\mathscr{E}$ as defined in Theorem 2.5. By Remark 2.3 $V(t)$ is strongly continuous in t and by Theorem 3.3 it is a left cocycle.

We define the maps $\widetilde{\mathcal{T}^0}(t)$, $t \geq 0$, on $\mathscr{B}(\mathscr{H})$ by

$$\langle v|\widetilde{\mathcal{T}^0}(t)[X]u\rangle = \langle V(t)v \otimes \mathrm{e}(0)|(X \otimes \mathbb{1})V(t)u \otimes \mathrm{e}(0)\rangle, \qquad \forall u, v \in \mathscr{H}, \quad \forall X \in \mathscr{B}(\mathscr{H}). \tag{3.13}$$

It is easy to see that $\widetilde{\mathcal{T}^0}(t)$ is a completely positive, bounded, linear map. Moreover, the cocycle property of V implies that $\widetilde{\mathcal{T}^0}$ is a semigroup [1, 8].

By the second fundamental formula of QSC (Proposition 2.2), we get, $\forall u, v \in \widetilde{D}$,

$$\begin{aligned}
&\big\langle V(t)v \otimes \mathrm{e}(0)\big|(X \otimes \mathbb{1})V(t)u \otimes \mathrm{e}(0)\big\rangle - \langle v|Xu\rangle \\
&\quad = \int_0^t \mathrm{d}s \Big\{ \big\langle V(s)v \otimes \mathrm{e}(0)\big|(X \otimes \mathbb{1})F_{00}^* V(s)u \otimes \mathrm{e}(0)\big\rangle \\
&\qquad + \big\langle F_{00}^* V(s)v \otimes \mathrm{e}(0)\big|(X \otimes \mathbb{1})V(s)u \otimes \mathrm{e}(0)\big\rangle \\
&\qquad\quad + \sum_{k\geq 1} \big\langle F_{0k}^* V(s)v \otimes \mathrm{e}(0)\big|(X \otimes \mathbb{1})F_{0k}V(s)u \otimes \mathrm{e}(0)\big\rangle \Big\}
\end{aligned}$$

and this says that the formal generator of $\widetilde{\mathcal{T}^0}(t)$ is

$$\langle v|\widetilde{\mathscr{L}}[X]u\rangle = \langle F_{00}^* v|Xu\rangle + \sum_{k\geq 1} \langle F_{0k}^* v|XF_{0k}^* u\rangle + \langle v|XF_{00}^* u\rangle.$$

Theorem 3.9 ([9] - Theorems 10.2 and 10.3)**.** *Let $F = (F_{ij} : i, j \geq 0)$ be a matrix of operators such that F^* satisfies the hypotheses of Theorem 2.5, $\mathrm{Dom}(F^*_{00}) \subset \mathrm{Dom}(F^*_{0k})$, $\forall k \geq 1$, and $\vartheta(F^*) = 0$ on $\widetilde{D}$. Then, the unique contraction V solving (2.24) is a strongly continuous left cocycle such that the family of operators $\widetilde{\mathcal{T}^0}(t)$ defined by Eq. (3.13) is the minimal quantum dynamical semigroup generated by F^*_{00} and F^*_{0k}.*

Moreover, the following conditions are equivalent:

(i) The process V is an isometry;

(ii) The minimal QDS associated with F^*_{00} and F^*_{0k} is Markov.

As we take $U = V^*$, if V is an isometry process, U is a coisometry process and vice versa. The following Proposition is a small variation of Corollary 2.4 of [10] or of Corollary 11.2 of [9]. Let us recall that in our hypotheses all the operators F_{ij} are closed.

Proposition 3.10. *Let $F = (F_{ij} : i, j \geq 0)$ be a matrix of operators such that F satisfies the hypotheses of Theorem 2.6 and $\vartheta(F) = 0$ on $\mathrm{Dom}(F)$. Then, the contractive solution U of the right HP equation introduced in Theorem 2.6 is an isometry process. Moreover, if U is unitary, it is the unique bounded solution on $\mathrm{Dom}(C^{1/2})\underline{\otimes}\mathscr{E}$ of such an equation.*

Proof. Let $\widetilde{U}$ be another bounded solution and apply the second fundamental formula of QSC to U, $\widetilde{U}$. We get, $\forall f, g \in \mathscr{M}$, $\forall u, v \in \mathrm{Dom}(C^{1/2})$,

$$\begin{aligned}
&\big\langle \widetilde{U}(t)v \otimes \mathrm{e}(g) \big| U(t)u \otimes \mathrm{e}(f) \big\rangle - \big\langle v \otimes \mathrm{e}(g) \big| u \otimes \mathrm{e}(f) \big\rangle \\
&\qquad = \sum_{i,j \geq 0} \int_0^t \mathrm{d}s \, \overline{g_i(s)} \Big\{ \big\langle \widetilde{U}(s)v \otimes \mathrm{e}(g) \big| F_{ij} \otimes \mathbb{1}\, U(s)u \otimes \mathrm{e}(f) \big\rangle \\
&\qquad\qquad + \big\langle F_{ji} \otimes \mathbb{1}\, \widetilde{U}(s)v \otimes \mathrm{e}(g) \big| U(s)u \otimes \mathrm{e}(f) \big\rangle \\
&\qquad\qquad + \sum_{k \geq 1} \big\langle F_{ki} \otimes \mathbb{1}\, \widetilde{U}(s)v \otimes \mathrm{e}(g) \big| F_{kj} \otimes \mathbb{1}\, U(s)u \otimes \mathrm{e}(f) \big\rangle \Big\} f_j(s) \,.
\end{aligned}$$

But U and $\widetilde{U}$ are solutions on $\mathrm{Dom}(C^{1/2})\underline{\otimes}\mathscr{E}$; by point (ii) of Definition 2.4 and $\vartheta(F) = 0$, we have that the integrand vanishes. Therefore, $\forall f, g \in \mathscr{M}$, $\forall u, v \in \mathrm{Dom}(C^{1/2})$, $\big\langle \widetilde{U}(t)v \otimes \mathrm{e}(g) \big| U(t)u \otimes \mathrm{e}(f) \big\rangle = \big\langle v \otimes \mathrm{e}(g) \big| u \otimes \mathrm{e}(f) \big\rangle$ and this is equivalent to $\widetilde{U}(t)^* U(t) = \mathbb{1}$. This equation for $\widetilde{U} = U$ gives the fact that $U(t)$ is isometric, while for U unitary gives $\widetilde{U}(t)^* = U(t)^*$. □

Therefore, if the conditions of Theorems 3.9 and 3.10 hold and the minimal QDS $\widetilde{\mathcal{T}^0}$ is Markov, then the processes U and V are unitary.

3.4 A summary

3.4.1 Unitarity: the conditions $\vartheta(F) = 0$, $\vartheta(F^*) = 0$

Let us examine more closely the necessary conditions (2.26) and (2.27) found in Section 2.5.

Let Hypothesis 2.1 be satisfied and recall that we use the notation

$$F_{00} = K\,, \qquad F_{i0} = R_i\,, \qquad F_{0j} = N_j\,. \tag{3.14}$$

We assume also $\vartheta(F) = 0$ on a dense set $D \subset \mathrm{Dom}(F)$ and $\vartheta(F^*) = 0$ on a dense set $\widetilde{D} \subset \mathrm{Dom}(F^*)$. Moreover, we want all the operators to be determined by their actions on D or $\widetilde{D}$. So, we take D to be a core for all the operators F_{ij} and $\widetilde{D}$ to be a core for all the operators F^*_{ij}; recall that all the operators F_{ij}, F^*_{ij} are closed.

When both indices are different from 0, the operators F_{ij} are asked to be bounded; by continuity, Eq. (2.26) holds $\forall u, v \in \mathscr{H}$ and it is equivalent to the first of Eqs. (2.28). Analogously, from (2.27), we arrive to the second of Eqs. (2.28). As in the bounded case, conditions (2.28) give

$$F_{ij} = S_{ij} - \delta_{ij}\,, \qquad S_{ij} \in \mathcal{B}(\mathscr{H})\,, \qquad \forall i, j \geq 1\,, \tag{3.15a}$$

$$\sum_{k\geq 1} S^*_{ki} S_{kj} = \delta_{ij}\,, \qquad \sum_{k\geq 1} S_{ik} S^*_{jk} = \delta_{ij}\,, \qquad \forall i, j \geq 1\,. \tag{3.15b}$$

Let us consider now Eq. (2.26) for $i = j = 0$. Without loss of generality, we can take $v = u$; by (3.14) we get

$$2\mathrm{Re}\langle Ku|u\rangle = -\sum_{k\geq 1} \|R_k u\|^2\,, \qquad \forall u \in D\,. \tag{3.16}$$

Analogously, from Eq. (2.27) we get

$$2\mathrm{Re}\langle K^*v|v\rangle = -\sum_{k\geq 1} \|N^*_k v\|^2\,, \qquad \forall v \in \widetilde{D}\,. \tag{3.17}$$

Therefore, $K\big|_D$ and $K^*\big|_{\widetilde{D}}$ are densely defined dissipative operators and, by Theorem A.8, their closures K and K^* are dissipative. By Corollary A.14

K and K^* are infinitesimal generators of strongly continuous semigroups of contractions.

By taking into account Eqs. (3.15), when only one index is different from 0, Eq. (2.26) reduces to

$$\langle v|N_i u\rangle = -\sum_{k\geq 1}\langle R_k v|S_{ki}u\rangle, \qquad \forall i\geq 1, \quad \forall u,v\in D\,. \tag{3.18}$$

Similarly, Eq. (2.27) gives

$$\langle v|R_i^* u\rangle = -\sum_{k\geq 1}\langle N_k^* v|S_{ik}^* u\rangle, \qquad \forall i\geq 1, \quad \forall u,v\in \widetilde{D}\,. \tag{3.19}$$

Let us write $G_i = -\sum_{k\geq 1} S_{ki}^* R_k$; because all operators S_{ki}^* are bounded we have $\mathrm{Dom}(G_i) = \bigcap\limits_{k\geq 1} \mathrm{Dom}(R_k)$ and being D a core for all the operators R_i we get $\mathrm{Dom}(G_i) \supset D$. Then, Eq. (3.18) becomes $\langle v|N_i u\rangle = \langle G_i v|u\rangle$, $\forall u,v\in D$, $\forall i\geq 1$. By Remark A.2, point 1, we get $\mathrm{Dom}(N_i^*)\supset D$, $N_i^* u = -\sum_{k\geq 1} S_{ki}^* R_k u$, $\forall i\geq 1$, $\forall u\in D$.

Similarly, by recalling that in our assumptions $\left(R_i^*\big|_{\widetilde{D}}\right)^* = R_i^{**} = R_i$, we get from Eq. (3.19) $\mathrm{Dom}(R_i)\supset \widetilde{D}$ and $R_i v = -\sum_{k\geq 1} S_{ik} N_k^* v$, $\forall i\geq 1$, $\forall v\in \widetilde{D}$. But, by using Eq. (3.15b), we get N^* in terms of R on $\widetilde{D}$. Together with the previous result we have

$$\mathrm{Dom}(N_i^*)\supset D\cup\widetilde{D}\,, \qquad \mathrm{Dom}(R_i)\supset D\cup\widetilde{D}\,, \qquad \forall i\geq 1\,, \tag{3.20}$$

$$N_i^* u = -\sum_{k\geq 1} S_{ki}^* R_k u\,, \qquad \forall u\in D\cup\widetilde{D}\,, \quad \forall i\geq 1\,. \tag{3.21}$$

The last equation is equivalent to

$$R_i u = -\sum_{k\geq 1} S_{ik} N_k^* u\,, \qquad \forall u\in D\cup\widetilde{D}\,, \quad \forall i\geq 1\,. \tag{3.22}$$

These two equations can be written also as

$$N_i = -\left(\left(\sum_{k\geq 1} S_{ki}^* R_k\right)\Big|_{D\cup\widetilde{D}}\right)^*, \tag{3.23}$$

$$R_i^* = -\left(\left(\sum_{k\geq 1} S_{ik} N_k^*\right)\Big|_{D\cup\widetilde{D}}\right)^*. \tag{3.24}$$

3.4.2 The assumptions and the main statements

Let us collect here the main assumptions introduced up to now.

Hypothesis 3.2 (Hilbert spaces)**.** The Fock space $\mathscr{F} = \Gamma_{\text{symm}}\big(\mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})\big)$ has multiplicity space $\mathscr{Z}$ finite dimensional: $\dim(\mathscr{Z}) = d \in \mathbb{N}$. The initial Hilbert space $\mathscr{H}$ is separable. All the Hilbert spaces are complex.

- Vectors in $\mathscr{Z}$ are identified with their components with respect to the privileged c.o.n.s. $\{z_i,\, i = 1,\dots,d\}$.
- Fock spaces, some notations:

$$\widetilde{\mathscr{F}} = \Gamma_{\text{symm}}\big(\mathrm{L}^2(\mathbb{R};\mathscr{Z})\big), \qquad \mathscr{F}_{(s,t)} = \Gamma_{\text{symm}}\big(\mathrm{L}^2((s,t);\mathscr{Z})\big),$$
$$\mathscr{F}_{t)} = \Gamma_{\text{symm}}\big(\mathrm{L}^2((-\infty,t);\mathscr{Z})\big), \qquad \mathscr{F}_{(t} = \Gamma_{\text{symm}}\big(\mathrm{L}^2((t,+\infty);\mathscr{Z})\big),$$

- The exponential domain:

$$\mathscr{M} = \Big\{ f \in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z}) \cap \mathrm{L}^\infty_{\text{loc}}(\mathbb{R}^+;\mathscr{Z}) \Big\}$$
$$\mathcal{E} = \big\{\text{linear span of all the exponential vectors } \mathrm{e}(f) \text{ with } f \in \mathscr{M}\big\}.$$

Hypothesis 3.3 (U is a cocycle)**.** $U(t)$, $t \geq 0$, is a strongly continuous unitary right cocycle on $\mathscr{H} \otimes \widetilde{\mathscr{F}}$ (see Def. 3.1).

Remark 3.2. Let Hypothesis 3.3 holds and set

$$\mathscr{U}_t := \begin{cases} \Theta_t U(t) & \text{if } \ t \geq 0 \\ U(|t|)^* \Theta_t & \text{if } \ t \leq 0 \end{cases}$$

$$U(t,s) := \Theta_t^* \mathscr{U}_{t-s} \Theta_s \equiv \Theta_s^* U(t-s) \Theta_s\,, \qquad t \geq s\,.$$

By Propositions 3.4, 3.5 and Remark 3.1 we have:

1. $\mathscr{U}_t$, $t \in \mathbb{R}$, is a strongly continuous one-parameter group of unitary operators.
2. $U(t,s)$, $t \geq s$, is a two-parameter family of unitary operators, strongly continuous in t and s, satisfying the composition law

$$U(t,r) = U(t,s)U(s,r), \qquad r \leq s \leq t\,.$$

3. $U(t,0) = U(t)$, $\quad \forall t \geq 0$; $\qquad U(t,s) = U(t)U(s)^*$, $\quad \forall s,t : 0 \leq s \leq t$.

4. The operator $U(t,s)$ is adapted to $\mathscr{H} \otimes \mathscr{F}_{(s,t)}$ in the sense that it acts as the identity on $\mathscr{F}_{s)} \otimes \mathscr{F}_{(t}$ and leaves $\mathscr{H} \otimes \mathscr{F}_{(s,t)}$ invariant.

5. $U(t)^*$ is a strongly continuous unitary left cocycle.

Hypothesis 3.4 (The matrix F)**.** 1. $F = (F_{ij}; 0 \le i,j \le d)$ is a matrix of closed operators in the initial space $\mathscr{H}$. By F^* we denote the adjoint matrix, defined by $(F^*)_{ij} = F^*_{ji}$. We use the notation $F_{00} = K$ and $F_{k0} = R_k$, $F_{0k} = N_k$ for $k \ge 1$.

2. For $1 \le i,j \le d$, we have $F_{ij} = S_{ij} - \delta_{ij}$, where the S_{ij} are bounded operators on $\mathscr{H}$ satisfying $\displaystyle\sum_{k=1}^{d} S^*_{ki} S_{kj} = \sum_{k=1}^{d} S_{ik} S^*_{jk} = \delta_{ij}$.

3. There exist a dense subspace D which is a core for K, R_i, N_i, $i = 1, \ldots, d$, and a dense subspace $\widetilde{D}$ which is a core for K^*, R^*_i, N^*_i, $i = 1, \ldots, d$. Moreover, $\mathrm{Dom}(N^*_i) \supset D \cup \widetilde{D}$, $\mathrm{Dom}(R_i) \supset D \cup \widetilde{D}$, $\forall i \ge 1$.

4. The operators K and K^* are the infinitesimal generators of two strongly continuous contraction semigroups on $\mathscr{H}$. Moreover,

$$2\mathrm{Re}\langle Ku|u\rangle = -\sum_{k\ge 1} \|R_k u\|^2 , \qquad \forall u \in D ;$$

$$2\mathrm{Re}\langle K^* v|v\rangle = -\sum_{k\ge 1} \|N^*_k v\|^2 , \qquad \forall v \in \widetilde{D} .$$

5. $\displaystyle N^*_i u = -\sum_{k\ge 1} S^*_{ki} R_k u , \qquad \forall u \in D \cup \widetilde{D} , \quad \forall i \ge 1.$

6. There exists a positive self-adjoint operator C on $\mathscr{H}$ and the constants $\delta > 0$ and $b_1, b_2 \ge 0$ such that the following properties hold:

   (a) for each $\epsilon \in (0,\delta)$, there exists a dense subspace $D_\epsilon \subset \widetilde{D}$ such that $C_\epsilon^{\frac{1}{2}} D_\epsilon \subset \widetilde{D}$ and each operator $F^*_{ij} C_\epsilon^{\frac{1}{2}}|_{D_\epsilon}$ is bounded, where $C_\epsilon = \frac{C}{1+\epsilon C}$;

   (b) $D = \mathrm{Dom}(C^{\frac{1}{2}}) \subset \mathrm{Dom}(F)$.

   (c) For all $0 < \epsilon < \delta$, the form $\vartheta(F, C_\epsilon)$ on $\mathrm{Dom}(F)$ satisfies the inequality: $\vartheta(F, C_\epsilon) \le b_1 \iota(C_\epsilon) + b_2 \mathbb{1}$, where $\iota(C_\epsilon)$ is the $(d+1) \times (d+1)$ matrix $\mathrm{diag}(C_\epsilon)$ of operators on $\mathscr{H}$.

7. The minimal QDS $\widetilde{\mathcal{T}}^0$ associated with K^* and N^*_k is Markov.

Recall that $\mathrm{Dom}(F) := \bigcap_{i,j\geq 0} \mathrm{Dom}(F_{ij})$ and $\mathrm{Dom}(F^*) := \bigcap_{i,j\geq 0} \mathrm{Dom}(F^*_{ij})$.

Remark 3.3. Let us assume that Hypotheses 3.2 and 3.4 hold true.

1. The left Hudson-Parthasarathy equation with operator matrix $\widetilde{F} = F^*$ admits a unique contractive solution on $\widetilde{D}\underline{\otimes}\mathscr{E}$. Let V be such a solution.
2. V is a strongly continuous unitary left cocycle.
3. The semigroup defined by Eq. (3.13) is the minimal QDS $\widetilde{\mathcal{T}^0}$ associated with K^* and N^*_k and it is identity preserving by assumption.
4. Let us define $U(t) := V(t)^*$, $\forall t \geq 0$. The process U is a strongly continuous unitary right cocycle. This is Hypothesis 3.3.
5. The process U is the unique bounded solution of the right Hudson-Parthasarathy equation with operator matrix F on $\mathrm{Dom}(C^{1/2})\underline{\otimes}\mathscr{E}$.
6. The unitary group $\mathscr{U}_t$ and the evolution $U(t,s)$ can be constructed and all the points in Remark 3.2 hold true.
7. $U(t,s)$ with respect to t satisfies the right Hudson-Parthasarathy equation with initial condition $U(s,s) = \mathbb{1}$, i.e.

$$U(t,s) = \mathbb{1} + \sum_{i,j\geq 0} \int_s^t F_{ij} U(r,s) \mathrm{d}\Lambda_{ij}(r).$$

Some more consequences.

Proposition 3.11. *Under Hypotheses 3.4 also the following properties hold.*

1. $\mathrm{Dom}(R_k) \supset \mathrm{Dom}(K) \cup \mathrm{Dom}(K^*)$, $\quad \mathrm{Dom}(N^*_k) \supset \mathrm{Dom}(K) \cup \mathrm{Dom}(K^*)$, $k = 1, \ldots, d$.

2.

$$2\mathrm{Re}\langle Ku|u\rangle = -\sum_{k\geq 1} \|R_k u\|^2 , \qquad \forall u \in \mathrm{Dom}(K)\,; \tag{3.25}$$

$$2\mathrm{Re}\langle K^* v|v\rangle = -\sum_{k\geq 1} \|R_k v\|^2 , \qquad \forall v \in \mathrm{Dom}(K^*)\,. \tag{3.26}$$

3. $\forall u \in \mathrm{Dom}(K) \cup \mathrm{Dom}(K^*)$, $\quad \forall i, k \geq 1$,

$$N^*_i u = -\sum_{k\geq 1} S^*_{ki} R_k u\,, \qquad R_k u = -\sum_{i\geq 1} S_{ki} N^*_i u\,.$$

Proof. By the first equation in Hypothesis 4 we get, $\forall \phi \in D$, $\|R_k \phi\| \leq |\langle K\phi|\phi\rangle|$. For any $u \in \mathrm{Dom}(K)$ we can find a sequence $u_n \in D$ converging to u; moreover, $\|Ku_n\| \leq c$ by Proposition A.1. Then, we have

$$\|R_k(u_n - u_m)\|^2 \leq 2|\langle K(u_n - u_m)|u_n - u_m\rangle| \leq 4c\|u_n - u_m\|.$$

Therefore, $R_k u_n$ is a Cauchy sequence. Being R_k closed, by Remark A.1 we get $u \in \mathrm{Dom}(R_k)$ and, so, $\mathrm{Dom}(R_k) \supset \mathrm{Dom}(K)$. Again by Remark A.1 we get that the first equation in point 4 can be extended to the whole $\mathrm{Dom}(K)$. Similarly, from the second equation in Hypothesis 4 we get that it can be extended to the whole $\mathrm{Dom}(K^*)$ and that $\mathrm{Dom}(N_k^*) \supset \mathrm{Dom}(K^*)$.

Once again by Remark A.1 and by the unitarity of the operator matrix S, we get that point 5 can be extended to $\mathrm{Dom}(K) \cup \mathrm{Dom}(K^*)$. Therefore, on the same domain, $\sum_{k\geq 1} \|R_k u\|^2 = \sum_{k\geq 1} \|N_k^* u\|^2$.

By exchanging R_k and N_k^* in the two equations in Hypothesis 4 we get also $\mathrm{Dom}(R_k) \supset \mathrm{Dom}(K^*)$ e $\mathrm{Dom}(N_k^*) \supset \mathrm{Dom}(K)$. $\square$

Chapter 4

The Reduced Dynamics

4.1 Reduced dynamics and system-field state

The quantum system described in $\mathscr{H}$ and the fields described in the Fock space $\widetilde{\mathscr{F}}$ form a closed system whose dynamics is given by the unitary group $\mathscr{U}_t$ (3.4). Let us consider as initial state of this composed system $\phi \otimes \psi(f)$, $\phi \in \mathscr{H}$ with $\|\phi\| = 1$, $f \in L^2(\mathbb{R}^+; \mathscr{Z})$. This means that the system and the fields are initially fully uncorrelated and that the fields are in a coherent state; f represents the action of lasers, for instance.

If we are interested in the dynamics of the system in $\mathscr{H}$ we have to study the quantity $\left\langle \mathscr{U}_t\phi \otimes \psi(f) \middle| (X \otimes \mathbb{1})\, \mathscr{U}_t\phi \otimes \psi(f) \right\rangle$ for f fixed, but $\forall \phi \in \mathscr{H}$ with $\|\phi\| = 1$, $\forall X \in \mathscr{B}(\mathscr{H})$, $\forall t \geq 0$. This means that we are considering only "system observables" X: only direct measurements on the system in $\mathcal{H}$ are permitted, the fields are ignored. But Θ_t commutes with $X \otimes \mathbb{1}$ and we have $\Theta_t^* \mathscr{U}_t = U(t)$; so,

$$\left\langle \mathscr{U}_t\phi \otimes \psi(f) \middle| (X \otimes \mathbb{1})\, \mathscr{U}_t\phi \otimes \psi(f) \right\rangle = \left\langle U(t)\phi \otimes \psi(f) \middle| (X \otimes \mathbb{1})\, U(t)\phi \otimes \psi(f) \right\rangle .$$

If the initial state of the subsystem is not a pure one, we need the notion of statistical operator.

Definition 4.1. For any separable Hilbert space $\mathcal{K}$, we shall denote by $\mathcal{T}(\mathcal{K})$ the *trace-class* on $\mathcal{K}$, i.e.

$$\mathcal{T}(\mathcal{K}) = \left\{ \tau \in \mathcal{B}(\mathcal{K}) : \operatorname{Tr}\left\{ \sqrt{\tau^* \tau} \right\} < +\infty \right\},$$

where $\mathrm{Tr}\{a\}$ is the trace of a. Moreover, the set of *statistical operators* is denoted by

$$\mathcal{S}(\mathcal{K}) = \{\rho \in \mathcal{T}(\mathcal{K}) : \rho \geq 0,\ \mathrm{Tr}\{\rho\} = 1\}\,.$$

When the initial time is not 0, but a generic one, say $s \geq 0$, the initial state of the system is $\rho \in \mathcal{S}(\mathscr{H})$ and the state of the field is coherent, say

$$\eta(f) := |\psi(f)\rangle\langle\psi(f)| \in \mathcal{S}(\mathcal{F}), \qquad f \in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z}), \tag{4.1}$$

we have to study the expression

$$\mathrm{Tr}_{\mathscr{H}\otimes\mathscr{F}}\left\{(X\otimes\mathbb{1})U(t,s)\left(\rho\otimes\eta(f)\right)U(t,s)^*\right\}.$$

The map defined by the partial trace

$$\rho \mapsto \mathrm{Tr}_{\mathscr{F}}\left\{U(t,s)\left(\rho\otimes\eta(f)\right)U(t,s)^*\right\}$$

is called the *reduced dynamics* of the subsystem living in $\mathscr{H}$. In this chapter we study such a dynamics or, better, its adjoint action on $\mathscr{B}(\mathscr{H})$, in the case in which we only know that $U(t)$ is a unitary cocycle and in the case we know that it is a unitary solution of the Hudson-Parthasarathy equation.

4.2 General properties of the reduced dynamics

Let us assume that $U(t)$ is a strongly continuous unitary cocycle (Hypothesis 3.3) and define $\mathscr{U}_t$ and $U(t,s)$ by Eqs. (3.4) and (3.7). Then, all the statements of Remark 3.2 hold.

4.2.1 The case of a coherent vector with a generic function

Definition 4.2. Let $f \in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$ be fixed. The *reduced dynamics in the Heisenberg picture* $\mathcal{T}^f(s,t)$, $0 \leq s \leq t$, is the linear map from $\mathscr{B}(\mathscr{H})$ into itself uniquely defined by

$$\big\langle v\big|\mathcal{T}^f(s,t)[X]u\big\rangle := \big\langle U(t,s)v\otimes\psi(f)\big|(X\otimes\mathbb{1})U(t,s)u\otimes\psi(f)\big\rangle, \tag{4.2}$$
$$\forall u,v\in\mathscr{H}, \qquad \forall X\in\mathscr{B}(\mathscr{H}).$$

Theorem 4.1. *In the hypotheses above, the family of linear maps $\mathcal{T}^f(s,t)$, $t \geq s \geq 0$ has the following properties:*

1. $\mathcal{T}^f(s,t)[\mathbb{1}] = \mathbb{1}$;

2. $\mathcal{T}^f(s,s) = \mathbb{1}$;

3. $\mathcal{T}^f(s,t)$ *is completely positive;*

4. $\mathcal{T}^f(s,t)$ *has a pre-adjoint acting on the trace class on* $\mathscr{H}$;

5. $\mathcal{T}^f(s,t)$ *is a* σ*-weakly continuous operator on* $\mathscr{B}(\mathscr{H})$;

6. *For each* $X \in \mathscr{B}(\mathscr{H})$ *the maps* $t \mapsto \mathcal{T}^f(s,t)[X]$ *and* $s \mapsto \mathcal{T}^f(s,t)[X]$ *are continuous with respect to the* σ*-weak topology of* $\mathscr{B}(\mathscr{H})$;

7. $\forall u, v \in \mathscr{H}, \quad \forall X \in \mathscr{B}(\mathscr{H})$,
$$\big\langle v \big| \mathcal{T}^f(s,t)[X]u \big\rangle = \big\langle U(t,s)v \otimes \psi(f_{(s,t)}) \big| (X \otimes \mathbb{1}) U(t,s)u \otimes \psi(f_{(s,t)}) \big\rangle; \tag{4.3}$$

8. $\mathcal{T}^f(r,s)\mathcal{T}^f(s,t) = \mathcal{T}^f(r,t)$, *for* $t \geq s \geq r \geq 0$.

Proof. 1. Let $u, v \in \mathscr{H}$

$$\begin{aligned}\big\langle v \big| \mathcal{T}^f(s,t)[\mathbb{1}]u \big\rangle &= \big\langle U(t,s)v \otimes \psi(f) \big| \big(\mathbb{1} \otimes \mathbb{1}\big) U(t,s)u \otimes \psi(f) \big\rangle \\ &= \big\langle U(t,s)v \otimes \psi(f) \big| U(t,s)u \otimes \psi(f) \big\rangle = \langle v \otimes \psi(f) | u \otimes \psi(f) \rangle \\ &= \langle v|u\rangle \|\psi(f)\|^2 = \langle v|u\rangle\end{aligned}$$

$$\therefore T^f(s,t)[\mathbb{1}] = \mathbb{1}$$

2. Let $u, v \in \mathscr{H}$ and $X \in \mathscr{B}(\mathscr{H})$

$$\begin{aligned}\big\langle v \big| \mathcal{T}^f(s,s)[X]u \big\rangle &= \big\langle U(s,s)v \otimes \psi(f) \big| \big(X \otimes \mathbb{1}\big) U(s,s)u \otimes \psi(f) \big\rangle \\ &= \big\langle v \otimes \psi(f) \big| \big(X \otimes \mathbb{1}\big) u \otimes \psi(f) \big\rangle = \langle u|Xv\rangle \|\psi(f)\|^2 = \langle v|Xu\rangle\end{aligned}$$

$$\therefore \mathcal{T}^f(s,s)[X] = X$$

3. Let $n \in \mathbb{N}$ and $u_i \in \mathscr{H}$, $X_i \in \mathscr{B}(\mathscr{H})$ for all $i = 1, \ldots, n$. Now

$$\begin{aligned}
&\sum_{i,j=1}^{n} \left\langle u_i \middle| \mathcal{T}^f(s,t)[X_i^* X_j] u_j \right\rangle \\
&\qquad = \sum_{i,j=1}^{n} \left\langle U(t,s)u_i \otimes \psi(f) \middle| \big((X_i^* X_j) \otimes \mathbb{1} \big) U(t,s) u_j \otimes \psi(f) \right\rangle \\
&\qquad = \sum_{i,j=1}^{n} \left\langle (X_i \otimes \mathbb{1}) U(t,s) u_i \otimes \psi(f) \middle| (X_j \otimes \mathbb{1}) U(t,s) u_j \otimes \psi(f) \right\rangle \\
&\qquad = \left\langle \sum_{i=1}^{n} (X_i \otimes \mathbb{1}) U(t,s) u_i \otimes \psi(f) \middle| \sum_{j=1}^{n} (X_j \otimes \mathbb{1}) U(t,s) u_j \otimes \psi(f) \right\rangle \\
&\qquad = \left\| \sum_{i=1}^{n} (X_i \otimes \mathbb{1}) U(t,s) u_i \otimes \psi(f) \right\|^2 \geq 0.
\end{aligned}$$

4. Let us remember that we say that $\mathcal{A} \in \mathcal{L}\big(\mathscr{B}(\mathscr{H})\big)$ admits a pre-adjoint if it exists an operator $\mathcal{A}_* \in \mathcal{L}\big(\mathcal{T}(\mathscr{H})\big)$ and

$$\mathrm{Tr}_{\mathscr{H}} \{\mathcal{A}[X]\tau\} = \mathrm{Tr}_{\mathscr{H}} \{\mathcal{A}_*[\tau] X\}, \qquad \forall X \in \mathscr{B}(\mathscr{H}), \quad \forall \tau \in \mathcal{T}(\mathscr{H}).$$

For $u, v \in \mathscr{H}$, we can write $\langle v | \mathcal{T}^f(s,t)[X] u \rangle = \mathrm{Tr}_{\mathscr{H}} \big[\mathcal{T}^f(s,t)[X] |u\rangle\langle v|\big]$. Let us take $\tau = \sum_n |u_n\rangle\langle v_n| \in \mathscr{T}(\mathscr{H})$ and $\eta(f) = |\psi(f)\rangle\langle\psi(f)|$. First we have

$$\begin{aligned}
&\mathrm{Tr}_{\mathscr{H}}[\mathcal{T}^f(s,t)[X]\tau] \\
&= \sum_n \left\langle v_n \otimes \psi(f) \middle| U(t,s)^* (X \otimes \mathbb{1}) U(t,s) u_n \otimes \psi(f) \right\rangle \\
&= \sum_n \mathrm{Tr}_{\mathscr{H} \otimes \mathscr{F}} \big[U(t,s)^* (X \otimes \mathbb{1}) U(t,s) |u_n \otimes \psi(f)\rangle\langle v_n \otimes \psi(f)| \big] \\
&= \mathrm{Tr}_{\mathscr{H} \otimes \mathscr{F}} \big[(X \otimes \mathbb{1}) U(t,s) \tau \otimes \eta(f) U(t,s)^* \big]
\end{aligned}$$

(by the definition of partial trace)

$$= \mathrm{Tr}_{\mathscr{H}} \Big[X \, \mathrm{Tr}_{\mathscr{F}} \big[U(t,s)\tau \otimes \eta(f) U(t,s)^* \big] \Big];$$

then

$$\mathrm{Tr}_{\mathscr{H}} \big[\mathcal{T}^f(s,t)[X]\tau\big] = \mathrm{Tr}_{\mathscr{H}} \Big[X \, \mathrm{Tr}_{\mathscr{F}} \big[U(t,s)\tau \otimes \eta(f) U(t,s)^* \big] \Big].$$

Let us set $\mathcal{T}^f(t,s)_* := \mathrm{Tr}_{\mathscr{F}}\big[U(t,s)\tau\otimes\eta(f)U(t,s)^*\big]$; this structure defines a positive operator and it is easy to check that it is bounded. Therefore, $\mathcal{T}^f(t,s)_*$ is the pre-adjoint of the operator $\mathcal{T}^f(s,t)$.

5. The existence of the pre-adjoint of $\mathcal{T}^f(s,t)$ implies its σ-weak continuity (Corollary of Theorem 1.13.2 in [15], page 29).

6. Note that if $\tau \geq 0$, then $\mathcal{T}^f(s,t)_*[\tau] \geq 0$, by properties 3 and 4. Moreover, any positive trace-class operator can be written as $\tau = \sum_n |u_n\rangle\langle u_n|$, by Corollary 1.15.4 p. 39 of [15]. Then, we have

$$\mathrm{Tr}\left[\mathcal{T}^f(t,s)_*[\tau]\right] = \sum_n \langle u_n\otimes\psi(f)|u_n\otimes\psi(f)\rangle = \sum_n |u_n|^2 = \mathrm{Tr}[\tau].$$

Now, if $X \geq 0, \tau \geq 0$ and $t > s \geq 0$, we have

$$\begin{aligned}
&\mathrm{Tr}_{\mathscr{H}}\left[\mathcal{T}^f(s,t)[X]\tau\right] \\
&\quad = \mathrm{Tr}_{\mathscr{H}\otimes\mathscr{F}}\left[U(t,s)\left(\tau\otimes|\psi(f)\rangle\langle\psi(f)|\right)U(t,s)^*(X\otimes\mathbb{1})\right] \\
&\quad \leq \|X\|\,\mathrm{Tr}_{\mathscr{H}\otimes\mathscr{F}}\left[U(t,s)\tau\otimes|\psi(f)\rangle\langle\psi(f)|U(t,s)^*\right] \\
&\quad = \|X\|\;\|\tau\otimes|\psi(f)\rangle\langle\psi(f)|\;\|_1 = \|X\|\,\|\tau\|_1\,.
\end{aligned}$$

This implies that $\mathcal{T}^f(s,t)[X]$ is bounded uniformly in t and s. By the fact that the σ-weak continuity is equivalent to the weak continuity on the bounded spheres, it is enough to prove the weak continuity.

By setting $\Psi := U(t,s)v$ and $\Phi := U(t,s)u$, we have

$$\begin{aligned}
&\Big|\big\langle v\big|\mathcal{T}^f(s,t+\varepsilon)[X]u\big\rangle - \big\langle v\big|\mathcal{T}^f(s,t)[X]u\big\rangle\Big| \\
&\quad = \Big|\big\langle U(t+\varepsilon,s)v\otimes\psi(f)\big|(X\otimes\mathbb{1})U(t+\varepsilon,s)u\otimes\psi(f)\big\rangle \\
&\qquad - \sum_n \big\langle U(t,s)v\otimes\psi(f)\big|(X\otimes\mathbb{1})U(t,s)u\otimes\psi(f)\big\rangle\Big| \\
&\quad = \Big|\big\langle U(t+\varepsilon,t)\Psi\big|(X\otimes\mathbb{1})U(t+\varepsilon,t)\Phi\big\rangle - \big\langle\Psi\big|(X\otimes\mathbb{1})\Phi\big\rangle\Big| \\
&\quad = \Big|\big\langle U(t+\varepsilon,t)\Psi\big|(X\otimes\mathbb{1})\big(U(t+\varepsilon,t)-\mathbb{1}\big)\Phi\big\rangle \\
&\qquad + \big\langle\big(U(t+\varepsilon,t)-\mathbb{1}\big)\Psi\big|(X\otimes\mathbb{1})\Phi\big\rangle\Big| \\
&\quad \leq \|X\|\Big(\big\|\big(U(t+\varepsilon,t)-\mathbb{1}\big)\Psi\big\|\|v\| + \big\|\big(U(t+\varepsilon,t)-\mathbb{1}\big)\Phi\big\|\|u\|\Big) \to 0
\end{aligned}$$

as $\varepsilon \downarrow 0$, due to the strong continuity in t of $U(t,s)$.

By setting $Y := U(t,s)^*\,(X\otimes \mathbb{1})\,U(t,s)$ we get in a similar way

$$\begin{aligned}
&\Big|\big\langle v\big|\mathcal{T}^f(s,t)[X]u\big\rangle - \big\langle v\big|\mathcal{T}^f(s-\varepsilon,t)[X]u\big\rangle\Big| \\
&\quad = \Big|\big\langle U(s,s-\varepsilon)v\otimes\psi(f)\big|Y\big(\mathbb{1}-U(s,s-\varepsilon)\big)u\otimes\psi(f)\big\rangle \\
&\qquad + \big\langle\big(\mathbb{1}-U(s,s-\varepsilon)\big)v\otimes\psi(f)\big|Yu\otimes\psi(f)\big\rangle\Big| \\
&\qquad \le \|X\|\Big(\big\|\big(\mathbb{1}-U(s,s-\varepsilon)\big)u\otimes\psi(f)\big\|\|u\| \\
&\qquad\qquad + \big\|\big(\mathbb{1}-U(s,s-\varepsilon)\big)v\otimes\psi(f)\big\|\|v\|\Big) \to 0
\end{aligned}$$

as $\varepsilon\downarrow 0$, due to the strong continuity in s of $U(t,s)$.

7. For $f\in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z}))$ we can write

$$\psi(f)\approx\psi(f_{(0,s)})\otimes\psi(f_{(s,t)})\otimes\psi(f_{(t})\in\mathscr{F}_{(0,s)}\otimes\mathscr{F}_{(s,t)}\otimes\mathscr{F}_{(t}\approx\mathscr{F}$$

and correspondingly $\mathbb{1}_{\mathscr{F}}\approx\mathbb{1}_{\mathscr{F}_{(0,s)}}\otimes\mathbb{1}_{\mathscr{F}_{(s,t)}}\otimes\mathbb{1}_{\mathscr{F}_{(t}}$.
Due to the fact that $U(t,s)$ is adapted to $\mathscr{F}_{(s,t)}$ one has that, $\forall u,v\in\mathscr{H}$,

$$\begin{aligned}
\underbrace{U(t,s)v\otimes\psi(f)}_{\in\,\mathscr{H}\otimes\mathscr{F}} &= U(t,s)\Big(v\otimes\psi(f_{(0,s)})\otimes\psi(f_{(s,t)})\otimes\psi(f_{(t)})\Big) \\
&= \Big(\underbrace{U(t,s)v\otimes\psi(f_{(s,t)})}_{\in\,\mathscr{H}\otimes\mathscr{F}_{(s,t)}}\Big)\otimes\underbrace{\psi(f_{(0,s)})}_{\in\,\mathscr{F}_{(0,s)}}\otimes\underbrace{\psi(f_{(t})}_{\in\,\mathscr{F}_{(t}}
\end{aligned}$$

and

$$\begin{aligned}
&(X\otimes\mathbb{1})U(t,s)u\otimes\psi(f) \\
&\qquad = \Big(\big(X\otimes\mathbb{1}_{\mathscr{F}_{(s,t)}}\big)U(t,s)u\otimes\psi(f_{(s,t)})\Big)\otimes\psi(f_{(0,s)})\otimes\psi(f_{(t})
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\big\langle v\big|\mathcal{T}^f(s,t)[X]u\big\rangle = \big\langle U(t,s)\big(v\otimes\psi(f)\big)\big|(X\otimes\mathbb{1})U(t,s)\big(u\otimes\psi(f)\big)\big\rangle \\
= &\ \big\langle U(t,s)\big(v\otimes\psi(f_{(s,t)})\big)\otimes\psi(f_{(0,s)})\otimes\psi(f_{(t)})\big| \\
&\ (X\otimes\mathbb{1}_{\mathscr{F}_{(s,t)}})U(t,s)\big(u\otimes\psi(f_{(s,t)})\big)\otimes\psi(f_{(0,s)})\otimes\psi(f_{(t)})\big\rangle \\
= &\ \big\langle U(t,s)v\otimes\psi(f_{(s,t)})\big|(X\otimes\mathbb{1}_{\mathscr{F}_{(s,t)}})U(t,s)u\otimes\psi(f_{(s,t)})\big\rangle\cdot \\
&\ \|\psi(f_{(0,s)})\|^2\cdot\|\psi(f_{(t)})\|^2 \\
= &\ \big\langle U(t,s)v\otimes\psi(f_{(s,t)})\big|(X\otimes\mathbb{1}_{\mathscr{F}_{(s,t)}})U(t,s)u\otimes\psi(f_{(s,t)})\big\rangle
\end{aligned}$$

Then, $\mathcal{T}^f(s,t)$ is the unique linear operator on $\mathscr{B}(\mathscr{H})$ satisfying (4.3).

8. Let us take $\mu,\nu \in \mathscr{H}\otimes\mathscr{F}_{(r,s)}$, $f \in \mathrm{L}^2\left((0,r)\cup(s,\infty);\mathscr{Z}\right)$ and $\chi \in \mathscr{B}(\mathscr{H}\otimes\mathscr{F}_{(r,s)})$ and define the operator $\mathscr{T}^f(s,t) : \mathscr{B}(\mathscr{H}\otimes\mathscr{F}_{(r,s)}) \to \mathscr{B}(\mathscr{H}\otimes\mathscr{F}_{(r,s)})$ by

$$\left\langle \nu|\mathscr{T}^f(s,t)[\chi]\mu\right\rangle := \left\langle U(t,s)\nu\otimes\psi(f)|(\chi\otimes\mathbb{1})U(t,s)\mu\otimes\psi(f)\right\rangle. \quad (4.4)$$

By the localization property we get

$$\left\langle \nu|\mathscr{T}^f(s,t)[\chi]\mu\right\rangle = \left\langle U(t,s)\nu\otimes\psi(f_{(s,t)})|(\chi\otimes\mathbb{1}_{(s,t)})U(t,s)\mu\otimes\psi(f_{(s,t)})\right\rangle. \quad (4.5)$$

Let us verify the following relation between $\mathscr{T}^f(s,t)$ and $\mathcal{T}^f(s,t)$

$$\mathscr{T}^f(s,t)[X\otimes\mathbb{1}_{(r,s)}] = \mathcal{T}^f(s,t)[X]\otimes\mathbb{1}_{(r,s)}. \quad (4.6)$$

Let $u,v\in\mathscr{H}$, $X\in\mathscr{B}(\mathscr{H})$ and $g,h\in\mathrm{L}^2\big((r,s);\mathscr{Z})\big)$, then for (4.5)

$$\begin{aligned}
& \left\langle v\otimes\psi(g)\middle|\mathscr{T}^f(s,t)[X\otimes\mathbb{1}_{(r,s)}]u\otimes\psi(h)\right\rangle \\
= \; & \big\langle U(t,s)v\otimes\psi(g)\otimes\psi(f_{(s,t)})\big| \\
& \qquad \big((X\otimes\mathbb{1}_{(r,s)})\otimes\mathbb{1}_{(s,t)}\big)U(t,s)u\otimes\psi(h)\otimes\psi(f_{(s,t)})\big\rangle \\
= \; & \big\langle U(t,s)v\otimes\psi(f_{(s,t)})\otimes\psi(g)\big| \\
& \qquad \big(X\otimes\mathbb{1}_{(s,t)}\big)U(t,s)u\otimes\psi(f_{(s,t)})\otimes\psi(h)\rangle \\
= \; & \big\langle U(t,s)v\otimes\psi(f_{(s,t)})\big| \\
& \qquad \big(X\otimes\mathbb{1}_{(s,t)}\big)U(t,s)u\otimes\psi(f_{(s,t))}\big\rangle\langle\psi(g)|\psi(h)\rangle \qquad \text{for (4.3)} \\
= \; & \big\langle v\big|\mathcal{T}^f(s,t)[X]u\big\rangle\langle\psi(g)|\psi(h)\rangle \\
= \; & \langle v\otimes\psi(g)|(\mathcal{T}^f(s,t)[X]\otimes\mathbb{1}_{(r,s)})u\otimes\psi(h)\rangle
\end{aligned}$$

$$\therefore \qquad \mathscr{T}(s,t)[X\otimes\mathbb{1}_{(r,s)}] = \mathcal{T}(s,t)[X]\otimes\mathbb{1}_{(r,s)}$$

Now we can verify the composition law for $\mathcal{T}^f(s,t)$:

$$\mathcal{T}^f(r,t) = \mathcal{T}^f(r,s)\mathcal{T}^f(s,t) \qquad \text{with} \qquad r\le s\le t$$

Let us take $u, v \in \mathscr{H}, X \in \mathscr{B}(\mathscr{H})$ and $f \in \mathrm{L}^2(\mathbb{R};\mathscr{Z})$, then:

$$\begin{aligned}
&\big\langle v\big|\mathcal{T}^f(r,s)\big[\mathcal{T}^f(s,t)[X]\big]u\big\rangle \\
=\ & \big\langle U(s,r)v\otimes\psi(f_{(r,s)})\big|\big(\mathcal{T}^f(s,t)[X]\otimes\mathbb{1}_{(r,s)}\big)U(s,r)u\otimes\psi(f_{(r,s)})\big\rangle \\
=\ & \big\langle \underbrace{U(s,r)v\otimes\psi(f_{(r,s)})}_{\nu\in\mathscr{H}\otimes\mathscr{F}_{(r,s)}}\big|\mathscr{T}^f(s,t) \\
& \Big[\underbrace{X\otimes\mathbb{1}_{(r,s)}}_{\chi\in\mathscr{B}(\mathscr{H}\otimes\mathscr{F}_{(r,s)})}\Big]\underbrace{U(s,r)u\otimes\psi(f_{(r,s)})}_{\mu\in\mathscr{H}\otimes\mathscr{F}_{(r,s)}}\big\rangle \quad \text{for (4.6)} \\
=\ & \big\langle U(t,s)\underbrace{U(s,r)v\otimes\psi(f_{(r,s)})}_{\nu}\otimes\psi(f_{(s,t)})\big| \\
& \big(\underbrace{(X\otimes\mathbb{1}_{(r,s)})}_{\chi}\otimes\mathbb{1}_{(s,t)}\big)U(t,s)\underbrace{U(s,r)u\otimes\psi(f_{(r,s)})}_{\mu}\otimes\psi(f_{(s,t)})\big\rangle \\
=\ & \big\langle U(t,r)v\otimes\psi(f_{(r,t)})\big|(X\otimes\mathbb{1}_{(r,t)})U(t,r)u\otimes\psi(f_{(r,t)})\big\rangle \\
& \text{by Proposition 3.5} \\
=\ & \langle v|\mathcal{T}^f(r,t)[X]u\rangle
\end{aligned}$$

$\therefore \mathcal{T}^f(r,t) = \mathcal{T}^f(r,s)\mathcal{T}^f(s,t)$

□

Remark 4.1. Let us collect here some simple consequences of Definition 4.2 and Theorem 4.1.

1. $\big\|\mathcal{T}^f(s,t)[X]\big\| \le \|X\|$, $\|\mathcal{T}^f(s,t)\| = 1$.
2. $\mathcal{T}^f(s,t)$ is normal.
3. If $f(x) = g(x)$ for all $x \in (s,t)$, then $\mathcal{T}^f(s,t) = \mathcal{T}^g(s,t)$.
4. $\mathcal{T}^f(s,t)$ is well defined for all $f \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R};\mathscr{Z})$.
5. If $g(\xi) = f(\xi+s)$ for all $\xi \in (0,t)$, with $s,t \ge 0$, then $\mathcal{T}^f(s,s+t) = \mathcal{T}^g(0,t)$.

Proof. 1. Let $X \ge 0$, then $X \le \|X\|\mathbb{1}$ and by the positivity of $\mathcal{T}^f(s,t)$ we have $\mathcal{T}^f(s,t)[X] \le \mathcal{T}^f(s,t)\big[\|X\|\mathbb{1}\big] = \|X\|\mathcal{T}^f(s,t)[\mathbb{1}] = \|X\|\mathbb{1}$.

Let $X \in \mathcal{B}(\mathcal{H})$; by the positivity of $\mathcal{T}^f(s,t)$ the following Schwarz inequality holds: $\mathcal{T}^f(s,t)[X^*X] \geq \mathcal{T}^f(s,t)[X^*]\mathcal{T}^f(s,t)[X]$; moreover, $\mathcal{T}^f(s,t)[X]^* = \mathcal{T}^f(s,t)[X^*]$ and $\mathcal{T}^f(s,t)[\mathbb{1}] = \mathbb{1}$. Then,

$$\left\|\mathcal{T}^f(s,t)[X]u\right\|^2 = \langle u|\mathcal{T}^f(s,t)[X^*]\mathcal{T}^f(s,t)[X]u\rangle \leq \langle u|\mathcal{T}^f(s,t)[X^*X]u\rangle \leq \|u\|^2\|X\|^2,$$

which gives the statements.

2. By Theorem 1.13.2 in [15], for positive operators the σ-weak continuity and the normality are equivalent.

3. Immediate by point 7 of Theorem 4.1.

4. Immediate by point 7 of Theorem 4.1 and of the previous point.

5. Notice that $\big(\theta_s f_{(s,t)}\big)(\xi) = (\theta f)_{(0,t-s)}(\xi)$, for any f, and that from Eqs. (3.7) and (4.3) and the fact that Θ_t commutes with $X \otimes \mathbb{1}$ we obtain:

$$\langle v|\mathcal{T}^f(s,t)[X]u\rangle = \langle U(t-s)v \otimes \psi(\theta_s f_{(s,t)})|(X \otimes \mathbb{1}_{(0,t-s)})U(t-s)u \otimes \psi(\theta_s f_{(s,t)})\rangle$$

which gives point 5.

□

4.2.2 The quantum Markov semigroup associated to a coherent vector with a constant function

Always in Hypothesis 3.3, let us define the family $\left\{\mathcal{T}^\lambda(t),\, t \geq 0\right\}$ of bounded linear maps on $\mathscr{B}(\mathscr{H})$, where $\lambda \in \mathscr{Z}$, fixed, by

$$\mathcal{T}^\lambda(t) = \mathcal{T}^f(0,t) \qquad \text{with} \;\; f(x) = 1_{(0,T)}(x)\,\lambda\,, \;\; T \geq t\,. \tag{4.7}$$

By point 3 in Remark 4.1, $\mathcal{T}^\lambda(t)$ does not depend on T.

Proposition 4.2. *In the hypothesis above, $\left\{\mathcal{T}^\lambda(t),\, t \geq 0\right\}$ is a quantum Markov semigroup.*

Proof. All the properties of Quantum Dynamical Semigroup are contained in Theorem 4.1 with exception of the semigroup property. Without loss of generality we can suppose that $0 \leq t+s \leq T$; then

$f_{(0,t)} = f_{(s,t+s)} = \lambda$ and by point 3 in Remark 4.1 $\mathcal{T}^f(0,t) = \mathcal{T}^f(s,t+s)$.

By point 7 in Theorem 4.1 and point 5 in Remark 4.1 we obtain $\mathcal{T}^\lambda(t+s) = \mathcal{T}^f(0,t+s) = \mathcal{T}^f(0,s)\mathcal{T}^f(s,t+s) = \mathcal{T}^f(0,s)\mathcal{T}^f(0,t) = \mathcal{T}^\lambda(s)\mathcal{T}^\lambda(t)$. □

4.2.3 Approximation of a generic reduced evolution by quantum Markov semigroups

Now we want to show how any dynamics $\mathcal{T}^f(s,t)$ can be approximated by time ordered products of quantum Markov semigroups.

Let us consider a coherent vector with a constant function up to some time T, say $f(x) = 1_{(0,T)}(x)\,\lambda$. By applying points 5 and 7 of Theorem 4.1 and point 3 of Remark 4.1, for $0 \le s \le t$, we get

$$\mathcal{T}^f(s,t) = \begin{cases} \mathcal{T}^\lambda(t-s) & \text{if } t \le T \\ \mathcal{T}^0(t-s) & \text{if } s \ge T \\ \mathcal{T}^\lambda(T-s)\mathcal{T}^0(t-T) & \text{if } s \le T \le t \end{cases} \tag{4.8}$$

In an analogous way we can consider the step function

$$f(x) = \sum_{k=1}^{n} 1_{[t_{k-1},t_k)}(x)\,\lambda_k\,, \qquad \lambda_k \in \mathscr{Z}\,, \quad 0 = t_0 < t_1 < \cdots < t_n\,.$$

Then, with the convention that $t_{n+1} = +\infty$, $\lambda_{n+1} = 0$, we have that our evolution $\mathcal{T}^f(0,t)$ for $t \in [t_j, t_{j+1})$ is given by the time-ordered product

$$\mathcal{T}^f(0,t) = \mathcal{T}^{\lambda_1}(t_1-t_0)\,\mathcal{T}^{\lambda_2}(t_2-t_1)\cdots\mathcal{T}^{\lambda_j}(t_j-t_{j-1})\,\mathcal{T}^{\lambda_{j+1}}(t-t_j)\,. \tag{4.9}$$

Proposition 4.3. *For $f, g \in L^2(\mathbb{R};\mathscr{Z})$ we have the estimate*

$$\begin{aligned}&\left\|\mathcal{T}^g(s,t) - \mathcal{T}^f(s,t)\right\| \\ &\quad \le \sqrt{8}\left(1 - \exp\left\{-\frac{1}{2}\left\|g_{(s,t)} - f_{(s,t)}\right\|\right\}\cos\left(\operatorname{Im}\langle g_{(s,t)}|f_{(s,t)}\rangle\right)\right)^{1/2}.\end{aligned} \tag{4.10}$$

Proof. We have

$$\begin{aligned}
&\left|\langle v| \left(\mathcal{T}^g(s,t) - \mathcal{T}^f(s,t)\right)[X]u\rangle\right| \\
&\qquad = \left|\langle U(t,s)v \otimes \psi(g_{(s,t)})|(X \otimes \mathbb{1})U(t,s)u \otimes \psi(g_{(s,t)})\rangle\right. \\
&\qquad \left. - \langle U(t,s)v \otimes \psi(f_{(s,t)})|(X \otimes \mathbb{1})U(t,s)u \otimes \psi(f_{(s,t)})\rangle\right| \\
&= \left|\langle U(t,s)v \otimes \psi(g_{(s,t)})|(X \otimes \mathbb{1})U(t,s)u \otimes \left(\psi(g_{(s,t)}) - \psi(f_{(s,t)})\right)\rangle\right. \\
&\left. + \langle U(t,s)v \otimes \left(\psi(g_{(s,t)}) - \psi(f_{(s,t)})\right) | X \otimes \mathbb{1}U(t,s)u \otimes \psi(f_{(s,t)})\rangle\right| \\
&\leq \left|\langle U(t,s)v \otimes \psi(g_{(s,t)})|(X \otimes \mathbb{1})U(t,s)u \otimes \left(\psi(g_{(s,t)}) - \psi(f_{(s,t)})\right)\rangle\right| \\
&+ \left|\langle U(t,s)v \otimes \left(\psi(g_{(s,t)}) - \psi(f_{(s,t)})\right) |(X \otimes \mathbb{1})U(t,s)u \otimes \psi(f_{(s,t)})\rangle\right| \\
&\qquad \leq 2\,\|v\|\,\|u\|\,\|X\|\,\left\|\psi(g_{(s,t)}) - \psi(f_{(s,t)})\right\|.
\end{aligned}$$

By Eq. (2.3) we have

$$\begin{aligned}
\left\|\psi(g_{(s,t)}) - \psi(f_{(s,t)})\right\|^2 &= \left\|\psi(g_{(s,t)})\right\|^2 + \left\|\psi(f_{(s,t)})\right\|^2 - 2\,\mathrm{Re}\langle\psi(g_{(s,t)})|\psi(f_{(s,t)})\rangle \\
&= 2 - 2\exp\left\{-\frac{1}{2}\left\|g_{(s,t)} - f_{(s,t)}\right\|\right\}\cos\left(\mathrm{Im}\langle g_{(s,t)}|f_{(s,t)}\rangle\right).
\end{aligned}$$

The result follows by recalling that the norm of a linear operator $\mathcal{T}$ from $\mathscr{B}(\mathscr{H})$ into itself is

$$\|\mathcal{T}\| = \sup_{\substack{X \in \mathscr{B}(\mathscr{H}) \\ \|X\|=1}} \sup_{\substack{u,v \in \mathscr{H} \\ \|u\|=\|v\|=1}} |\langle v|\mathcal{T}[X]u\rangle|.$$

□

An immediate consequence of the estimate above is that the L^2-convergence of the functions implies the uniform convergence of the reduced dynamics.

Corollary 4.4. *For f, $f^{(n)} \in L^2(\mathbb{R};\mathscr{Z})$ we have*

$$\lim_{n\to\infty}\left\|f^{(n)}_{(s,t)} - f_{(s,t)}\right\| = 0 \quad \Rightarrow \quad \lim_{n\to\infty}\left\|\mathcal{T}^{f^{(n)}}(s,t) - \mathcal{T}^f(s,t)\right\| = 0\,.$$

By taking into account that step functions are dense in L^2, we have that for any f we can find a sequence of step functions converging to f in $L^2\big((s,t);\mathscr{Z}\big)$ and, so, any $\mathcal{T}^f(s,t)$ can be approximated in the uniform topology by a sequence of products of quantum Markov semigroups of the kind of Eq. (4.9).

4.3 Some contraction semigroups in $\mathscr{H}$

4.3.1 The contraction semigroups $P^\lambda_\gamma(t)$ and $P^{\lambda *}_\gamma(t)$

In order to study the generator of the semigroup $\mathcal{T}^\lambda(t)$, it is useful to introduce some related semigroups of contractions in $\mathscr{H}$.

Let us introduce a notation for the coherent vectors with a constant function up to a certain time:

$$\phi(\lambda;t) := \psi\left(1_{[0,t]}\,\lambda\right), \qquad \lambda \in \mathscr{Z}, \quad t \geq 0\,. \tag{4.11}$$

Note that by (2.3) we have

$$\langle \phi(\gamma;t)|\phi(\lambda;t)\rangle = \exp\left\{-\frac{t}{2}\big(\|\gamma-\lambda\|^2 - 2\mathrm{i}\,\mathrm{Im}\langle\gamma|\lambda\rangle\big)\right\} =: h(-t). \tag{4.12}$$

Theorem 4.5 ([9] pp. 204–205)**.** *For any $t \geq 0$, define the operator $P^\lambda_\gamma(t)$ on $\mathscr{H}$ by: $\forall v, u \in \mathscr{H}$*

$$\big\langle v|P^\lambda_\gamma(t)u\big\rangle = \big\langle v\otimes\phi(\gamma;t)\big|U(t)u\otimes\phi(\lambda;t)\big\rangle. \tag{4.13}$$

Then, $\{P^\lambda_\gamma(t),\ t\geq 0\}$ and $\{P^\lambda_\gamma(t)^*,\ t\geq 0\}$ are strongly continuous semigroups of contractions on $\mathscr{H}$.

Proof. By the normalization of $\phi(\gamma;t)$, $\phi(\lambda;t)$, and the unitarity of $U(t)$, we have $\big|\big\langle v|P^\lambda_\gamma(t)u\big\rangle\big| \leq \|v\|\|u\|$; this implies that $P^\lambda_\gamma(t)$ is a contraction.

By $U(0)=\mathbb{1}$, $\phi(\gamma;0)=\phi(\lambda;0)=\mathrm{e}(0)$, we get from the definition the property $P^\lambda_\gamma(0)=\mathbb{1}$.

Being U a right cocycle, for $s,t\geq 0$ we have

$$\begin{aligned}\langle v|P^\lambda_\gamma(t+s)u\rangle &= \langle v\otimes\phi(\gamma;t+s)|\Theta^*_sU(t)\Theta_sU(s)\left(u\otimes\phi(\lambda;t+s)\right)\rangle\\ &= \langle\Theta^*_sU(t)^*\left(v\otimes\psi(1_{(-s,t)}\gamma)\right)|U(s)\left(u\otimes\phi(\lambda;t+s)\right)\rangle\\ &= \langle\phi(\gamma;s)\otimes\Theta^*_sU(t)^*\left(v\otimes\phi(\gamma;t)\right)|\left(U(s)\left(u\otimes\phi(\lambda;s)\right)\right)\otimes\psi(1_{(s,s+t)}\lambda)\rangle\end{aligned}$$

Let u_j be a c.o.n.s. in $\mathscr{H}$; we get

$$\begin{aligned}\langle v|P_\gamma^\lambda(t+s)u\rangle &= \sum_j \langle \Theta_s^* U(t)^* \left(v\otimes\phi(\gamma;t)\right)|u_j\otimes\psi(1_{(s,s+t)}\lambda)\rangle \\ &\qquad \times \langle u_j\otimes\phi(\gamma;s)|U(s)\left(u\otimes\phi(\lambda;s)\right)\rangle \\ &= \sum_j \langle v\otimes\phi(\gamma;t)|U(t)\left(u_j\otimes\phi(\lambda;t)\right)\rangle\langle u_j\otimes\phi(\gamma;s)|U(s)\left(u\otimes\phi(\lambda;s)\right)\rangle \\ &= \sum_j \langle v|P_\gamma^\lambda(t)u_j\rangle\langle u_j|P_\gamma^\lambda(s)u\rangle = \langle v|P_\gamma^\lambda(t)P_\gamma^\lambda(s)u\rangle,\end{aligned}$$

which is the semigroup property.

Being $P_\gamma^\lambda(t)$ a semigroup, it is enough to prove its strong continuity in 0. By using the function $h(t)$ defined in (4.12) and taking $T>t$, we have

$$\begin{aligned}&\left|\langle v|P_\gamma^\lambda(t)u-u\rangle\right| = |\langle v\otimes\phi(\gamma;t)|U(t)\left(u\otimes\phi(\lambda;t)\right)\rangle - \langle v|u\rangle| \\ &= |h(T-t)\langle v\otimes\phi(\gamma;T)|U(t)\left(u\otimes\phi(\lambda;T)\right)\rangle - h(T)\langle v\otimes\phi(\gamma;T)|u\otimes\phi(\lambda;T)\rangle| \\ &\qquad = \big|\left(h(T-t)-h(T)\right)\langle v\otimes\phi(\gamma;T)|U(t)\left(u\otimes\phi(\lambda;T)\right)\rangle \\ &\qquad\qquad + h(T)\langle v\otimes\phi(\gamma;T)|\left(U(t)-\mathbb{1}\right)u\otimes\phi(\lambda;T)\rangle\big| \\ &\qquad \le |h(T-t)-h(T)|\,\|v\|\,\|u\| + |h(T)|\,\|v\|\,\|\left(U(t)-\mathbb{1}\right)u\otimes\phi(\lambda;T)\|\,.\end{aligned}$$

By the strong continuity of U and the expression of h, we get the strong continuity of P_γ^λ.

By Remark A.7, also $P_\gamma^\lambda(t)^*$ is a strongly continuous semigroup of contractions. □

Remark 4.2. From Eq. (4.13) we get easily

$$\left\langle v|P_\lambda^\gamma(t)^*u\right\rangle = \left\langle v\otimes\phi(\gamma;t)\middle|U(t)^*u\otimes\phi(\lambda;t)\right\rangle. \tag{4.14}$$

4.3.2 The infinitesimal generator of the contraction semigroup $P_\gamma^{\lambda*}$

As we shall see in Section 4.4, it is natural to think the semigroup $\mathcal{T}^\lambda(t)$ as a perturbation of $P_0^\lambda(t)^*\bullet P_0^\lambda(t)$; so, we start by studying the generators of $P_\gamma^{\lambda*}$ and P_γ^λ.

Now Hypotheses 3.2, 3.3, 3.4 are assumed to hold. As for $\mathscr{Z}$-valued functions, we use the convention $\lambda_0=\gamma_0=1$. The two dense sets D and $\widetilde{D}$ are introduced in Hypothesis 3.4.

Theorem 4.6. *Let us denote by $K_\gamma^{\lambda *}$ the infinitesimal generator of the strongly continuous semigroup of contractions $P_\gamma^\lambda(t)^*$. Then,* $\mathrm{Dom}(K_\gamma^{\lambda *}) \supset \widetilde{D}$ *and* $\forall v \in \widetilde{D}$ *we have*

$$\begin{aligned} K_\gamma^{\lambda *} v &= \sum_{i,j\geq 0} \overline{\lambda_j} F_{ij}^* \gamma_i v + \langle \lambda|\gamma\rangle v - \frac{1}{2}(\|\gamma\|^2 + \|\lambda\|^2) v \\ &= K^* v + \sum_{k\geq 1} \left(R_k^* \gamma_k - \zeta_k^* R_k + \zeta_k^* \gamma_k\right) v - \frac{1}{2}\left(\|\lambda\|^2 + \|\gamma\|^2\right) v \qquad (4.15) \end{aligned}$$

and

$$-2\,\mathrm{Re}\langle K_\gamma^{\lambda *} v|v\rangle = \sum_{k\geq 1} \left\|\left(R_k + \zeta_k - \gamma_k\right) v\right\|^2 , \qquad (4.16)$$

where

$$\zeta_k = \sum_{i\geq 1} S_{ki}\lambda_i\,, \qquad k = 1, \ldots, d. \qquad (4.17)$$

Proof. Let us take any $u \in \mathscr{H}$ and any $v \in \widetilde{D}$. By recalling that $U(t)^*$ satisfies the left H-P-equation, we get

$$\begin{aligned} &\langle v|P_\gamma^\lambda(t)u\rangle = \langle v\otimes \phi(\gamma;t)|U(t)\, u\otimes\phi(\lambda;t)\rangle \\ &= \langle v\otimes\phi(\gamma;t)|u\otimes\phi(\lambda;t)\rangle + \sum_{i,j\geq 0}\int_0^t \overline{\gamma_i}\langle F_{ij}^*\, v\otimes\phi(\gamma;t)|U(s)\, u\otimes\phi(\lambda;t)\rangle\lambda_j\,\mathrm{d}s \\ &\qquad = h(-t)\langle v|u\rangle + \int_0^t h(s-t)\langle Av|P_\gamma^\lambda(s)u\rangle \mathrm{d}s\,, \end{aligned}$$

where $A := \sum_{i,j\geq 0} \overline{\lambda_j} F_{ij}^* \gamma_i$.

In the previous formula h is differentiable and the integrand is continuous; so, we can take the time derivative of both sides:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle v|P_\gamma^\lambda(t)u\rangle &= -\frac{1}{2}\left(\|\gamma\|^2 + \|\lambda\|^2 - 2\langle\gamma|\lambda\rangle\right)\bigg(h(-t)\langle v|u\rangle \\ &\quad + \int_0^t h(s-t)\langle Av|P_\gamma^\lambda(s)u\rangle\mathrm{d}s\bigg) + \langle Av|P_\gamma^\lambda(t)u\rangle = \langle Bv|P_\gamma^\lambda(t)u\rangle, \end{aligned}$$

where $B := A - \frac{1}{2}\left(\|\gamma\|^2 + \|\lambda\|^2\right) + \langle\lambda|\gamma\rangle$.

Let us show that on $\widetilde{D}$ the operator B is the generator of $P_\gamma^\lambda(t)^*$. By the previous equation we have

$$\langle v|P_\gamma^\lambda(t)u\rangle - \langle v|u\rangle = \int_0^t \langle Bv|P_\gamma^\lambda(s)u\rangle \mathrm{d}s$$

and

$$\frac{1}{t}\langle\left(P_\gamma^\lambda(t)^* - \mathbb{1}\right)v|u\rangle - \langle Bv|u\rangle = \frac{1}{t}\int_0^t \langle Bv|\left(P_\gamma^\lambda(s) - \mathbb{1}\right)u\rangle \mathrm{d}s.$$

Then, we have

$$\left\|\frac{1}{t}\left(P_\gamma^\lambda(t)^*v - v\right) - Bv\right\| = \sup_{\|u\|=1}\frac{1}{t}\left|\int_0^t \langle Bv|\left(P_\gamma^\lambda(s) - \mathbb{1}\right)u\rangle \mathrm{d}s\right|$$
$$\leq \sup_{\|u\|=1}\frac{1}{t}\int_0^t \left\|\left(P_\gamma^\lambda(s)^* - \mathbb{1}\right)Bv\right\|\mathrm{d}s$$

and the last term goes to zero as t goes to zero by the strong continuity of $P_\gamma^\lambda(s)^*$. This gives $\mathrm{Dom}(K_\gamma^{\lambda*}) \supset \widetilde{D}$ and $K_\gamma^{\lambda*}v = Bv$ for $v \in \widetilde{D}$. By using the explicit expressions of B and F_{ij} we get formula (4.15).

By explicit calculations and Eq. (3.26), we get Eq. (4.16). □

By definition $K_\gamma^{\lambda*}$ is the generator of the strongly continuous semigroup of contractions $P_\gamma^{\lambda*}$, but we do not know if $\widetilde{D}$ is a core for $K_\gamma^{\lambda*}$, in general. We can say something when $\gamma = 0$, which, after all, is the case of main interest for our construction.

Proposition 4.7. *In the hypotheses above, we have* $\mathrm{Dom}(K_0^{\lambda*}) = \mathrm{Dom}(K^*)$ *and, on this domain,*

$$K_0^{\lambda*} = K^* - \sum_{k\geq 1}\zeta_k^* R_k - \frac{1}{2}\|\lambda\|^2,$$

where $\zeta_k = \sum_{i\geq 1} S_{ki}\lambda_i$, $k = 1,\ldots,d$. *Moreover,* $\tilde{D}$ *is a core for* $K_0^{\lambda*}$ *and*

$$-2\,\mathrm{Re}\langle K_0^{\lambda*}v|v\rangle = \sum_{k\geq 1}\left\|\left(R_k + \zeta_k\right)v\right\|^2, \qquad \forall v \in \mathrm{Dom}(K^*).$$

Proof. By following [9] p. 204, we can write

$$\left\|\sum_{ik}\overline{\lambda_i}S_{ki}^*R_kv\right\| \leq \|\lambda\|\left(\sum_i\left\|\sum_k S_{ki}^*R_kv\right\|^2\right)^{1/2} = \|\lambda\|\left(\sum_i\|R_iv\|^2\right)^{1/2}$$
$$= \|\lambda\|\left|2\,\mathrm{Re}\langle K^*v|v\rangle\right|^{1/2} \leq 2\|\lambda\|\|v\|^{1/2}\|K^*v\|^{1/2} \leq \epsilon\|K^*v\| + \epsilon^{-1}\|\lambda\|\|v\|.$$

This holds for all $\epsilon \in (0,1)$.

Let us set $B := -\sum_{k\geq 1} \zeta_k^* R_k - \frac{1}{2}\|\lambda\|^2$. By recalling that $\mathrm{Dom}(R_k) \supset \mathrm{Dom}(K^*)$, we have that $\mathrm{Dom}(B) \supset \mathrm{Dom}(K^*)$ and, by the inequality above, B is relatively bounded with respect to K^*, with relative bound less than 1. By Proposition A.10 and Theorem A.11 the operator $A := K^* + B$ defined on $\mathrm{Dom}(K^*)$ is closed and maximally dissipative and, therefore, it generates a strongly continuous contraction semigroup. Being B relatively bounded with respect to K^* and being $\tilde{D}$ a core for K^*, by Remark (a), p. 205, in [9], $\tilde{D}$ is a core for A.

But $K_0^{\lambda *}$ is closed and on $\tilde{D}$ it is equal to A; so it is an extension of A. Being $K_0^{\lambda *}$ too a generator, by point 2 in Remark A.6, A and $K_0^{\lambda *}$ coincide.

The last statement follows by taking Eq. (4.16) for $\gamma = 0$ and by extending it. □

Obviously, we have $K_0^{\lambda} = K_0^{\lambda **}$, but it seems difficult to say something about $\mathrm{Dom}(K_0^{\lambda})$. A way to overcome this difficulty is to make some more hypotheses on R_k^*, similar to the ones on R_k, as done in the next section.

4.3.3 Additional hypotheses on R_k^* and N_k

Hypothesis 4.1. Let us consider the following additional hypotheses:

1. $\mathrm{Dom}(N_k) \supset \mathrm{Dom}(K) \quad \forall k \geq 1$.
2. $\forall k \geq 1$, $\forall u \in \mathrm{Dom}(K)$: $S_{ki} u \in \mathrm{Dom}(R_k^*)$, $\forall i \geq 1$.
3. For all $z \in \mathbb{C}^d$, $\sum_{k\geq 1} z_k N_k$ is relatively bounded with respect to K with relative bound less than 1.

From now on all Hypotheses 3.2, 3.3, 3.4, 4.1 are assumed to hold.

Proposition 4.8. *We have that*

$$N_i u = -\sum_{k\geq 1} R_k^* S_{ki} u \,, \qquad \forall u \in \mathrm{Dom}(K), \quad \forall i = 1, \dots, d.$$

Proof. By point 3 in Proposition 3.4 we have, $\forall i \geq 1, \forall v \in \mathrm{Dom}(K) \cup \mathrm{Dom}(K^*)$, $N_i^* v = -\sum_{k\geq 1} S_{ki}^* R_k v$, $\forall v \in D$. By using Hypothesis 4.1 we have, $\forall u \in \mathrm{Dom}(K)$,

$\forall v \in \mathrm{Dom}(K) \cup \mathrm{Dom}(K^*)$,

$$\begin{aligned}\langle v|N_i u\rangle &= \langle N_i^* v|u\rangle \\ &= -\sum_{k\geq 1}\left\langle S_{ki}^* R_k v\middle|u\right\rangle = -\sum_{k\geq 1}\left\langle R_k v\middle|S_{ki}u\right\rangle = -\sum_{k\geq 1}\left\langle v\middle|R_k^* S_{ki}u\right\rangle.\end{aligned}$$

By the density in $\mathscr{H}$ of $\mathrm{Dom}(K) \cup \mathrm{Dom}(K^*)$ we have the statement. □

Proposition 4.9. *We have* $\mathrm{Dom}(K_0^\lambda) = \mathrm{Dom}(K)$ *and on this domain*

$$K_0^\lambda = K - \sum_{k\geq 1} R_k^* \zeta_k - \frac{1}{2}\|\lambda\|^2,$$

where $\zeta_k = \sum_{i\geq 1} S_{ki}\lambda_i$, $k = 1, \ldots, d$. *Moreover,* D *is a core for* K_0^λ *and*

$$-2\,\mathrm{Re}\langle K_0^\lambda u|u\rangle = \sum_{k\geq 1}\|(R_k + \zeta_k)\,u\|^2, \qquad \forall u \in \mathrm{Dom}(K). \tag{4.18}$$

Proof. Let us set $B := -\sum_{k\geq 1} R_k^*\zeta_k - \frac{1}{2}\|\lambda\|^2 = \sum_{i\geq 1} N_i\lambda_i - \frac{1}{2}\|\lambda\|^2$; then, we have that $\mathrm{Dom}(B) \supset \mathrm{Dom}(K)$ and B is relatively bounded with respect to K with relative bounded less then 1. By Proposition A.10 $K+B$ is closed in $\mathrm{Dom}(K)$.

Moreover, we have $\forall u \in \mathrm{Dom}(K), \forall t \in [0,1]$

$$\begin{aligned}2\mathrm{Re}\langle u|(K+tB)u\rangle &= -\sum_{k\geq 1}\|R_k u\|^2 + 2t\mathrm{Re}\langle u|Bu\rangle \\ &= -\sum_{k\geq 1}\|(R_k + t\sum_{i\geq 1} S_{ki}\lambda_i)u\|^2 - t(1-t)\|\lambda\|^2\|u\|^2 \\ &\leq 0,\end{aligned}$$

and $K+tB$ is dissipative. Being K m-dissipative, by Theorem A.11, also $K+B$ is m-dissipative and by point 2 in Remark A.6 the first statement is proved.

As in Proposition 4.7 we get the other statements. □

4.4 The generator of the reduced dynamics

4.4.1 The form-generator

Let us consider now the quantum Markov semigroup $\mathcal{T}^\lambda$, defined in Eq. (4.7).

Theorem 4.10. *Let us set* $\zeta_k := \sum_{j\geq 1} \lambda_j S_{kj}$ *and* $B_k = R_k + \zeta_k$. *Then,* $\forall u, v \in D$,

$$\lim_{t\downarrow 0} \frac{1}{t} \left(\langle v | \mathcal{T}^\lambda(t)[X] u \rangle - \langle v | X u \rangle \right) \\ = \langle v | X K_0^\lambda u \rangle + \langle K_0^\lambda v | X u \rangle + \sum_{k\geq 1} \langle B_k v | X B_k u \rangle. \quad (4.19)$$

Moreover, the operators K_0^λ *and* B_k *satisfy the Hypothesis 3.1.*

Proof. From Definition 4.2 and Eq. (4.7) we obtain

$$\langle v | \mathcal{T}^\lambda(t)[X] u \rangle - \langle v | X u \rangle = \langle U(t) v \otimes \psi(f) | (X \otimes \mathbb{1}) U(t) u \otimes \psi(f) \rangle \\ - \langle v \otimes \psi(f) | (X \otimes \mathbb{1}) u \otimes \psi(f) \rangle$$

where $f(x) = 1_{(0,T)}(x)\lambda$ and $U(t)$ satisfies the right HP-equation (2.23)

$$U(t) - \mathbb{1} = \int_0^t \Bigg(\sum_{i\geq 1} R_i \mathrm{d}A_i^\dagger(s) + \sum_{i,j\geq 1} (S_{ij} - \delta_{ij}) \mathrm{d}\Lambda_{ij}(s) \\ + \sum_{j\geq 1} N_j \mathrm{d}A_j(s) + K \mathrm{d}s \Bigg) U(s).$$

By the second fundamental formula of quantum stochastic calculus (Proposition 2.2) and Proposition 4.8, we get, $\forall u, v \in D$,

$$\lim_{t\downarrow 0} \frac{1}{t} \left(\langle v | \mathcal{T}^\lambda(t)[X] u \rangle - \langle v | X u \rangle \right) \\ = \langle v | X K u \rangle + \sum_{i\geq 1} \overline{\lambda_i} \langle v | X R_i u \rangle + \sum_{j\geq 1} \lambda_j \langle v | X N_j u \rangle + \\ + \sum_{i,j\geq 1} \overline{\lambda_i} \lambda_j \langle v | X (S_{ij} - \delta_{ij}) u \rangle + \langle K v | X u \rangle + \sum_{j\geq 1} \overline{\lambda_j} \langle N_j v | X u \rangle \\ + \sum_{j\geq 1} \lambda_j \langle R_j v | X u \rangle + \sum_{i,j\geq 1} \overline{\lambda_i} \lambda_j \langle (S_{ji} - \delta_{ji}) v | X u \rangle \\ + \sum_{i\geq 1} \langle R_i v | X R_i u \rangle + \sum_{ij\geq 1} \overline{\lambda_i} \langle (S_{ij} - \delta_{ij}) v | X R_j u \rangle + \\ + \sum_{ij\geq 1} \lambda_j \langle R_i v | X (S_{ij} - \delta_{ij}) u \rangle + \sum_{i,j,k\geq 1} \overline{\lambda_i} \lambda_j \langle (S_{ki} - \delta_{ki}) v | X (S_{kj} - \delta_{kj}) u \rangle$$

$$
\begin{aligned}
&= \Big\langle v\Big| X\Big(K + \sum_{i\geq 1}\overline{\lambda_i}R_i - \sum_{ij\geq 1}\lambda_j R_i^* S_{ij} + \sum_{ij}\overline{\lambda_i}\lambda_j(S_{ij}-\delta_{ij})\Big)u\Big\rangle \\
&\quad + \Big\langle\Big(K + \sum_{i\geq 1}\overline{\lambda_i}R_i - \sum_{ij\geq 1}\lambda_j R_i^* S_{ij} + \sum_{ij}\overline{\lambda_i}\lambda_j(S_{ij}-\delta_{ij})\Big)v\Big| Xu\Big\rangle \\
&\quad + \sum_{k\geq 1}\Big\langle\Big(R_k + \sum_{i\geq 1}\lambda_i(S_{ki}-\delta_{ki})\Big)v\Big| X\Big(R_k + \sum_{j\geq 1}\lambda_j(S_{kj}-\delta_{kj})\Big)u\Big\rangle \\
&= \langle v\big|XK_0^\lambda u\rangle + \langle K_0^\lambda v\big|Xu\rangle + \sum_{k\geq 1}\Big\langle\Big(R_k + \sum_{i\geq 1}\lambda_i S_{ki}\Big)v\Big| X\Big(R_k + \sum_{j\geq 1}\lambda_j S_{kj}\Big)u\Big\rangle.
\end{aligned}
$$

So, we have Eq. (4.19).

We already know that the operator K_0^λ is the generator of the strongly continuous contraction semigroup P_0^λ defined in Eq. (4.13). So, the first of Hypotheses 3.1 holds.

By Eq. (4.18) also the second of Hypotheses 3.1 holds. □

For all $X \in \mathscr{B}(\mathscr{H})$, let us consider the quadratic form $\mathscr{L}^\lambda[X]$ in $\mathscr{H}$ with domain $\mathrm{Dom}(K) \times \mathrm{Dom}(K)$ defined by

$$
\langle v|\mathscr{L}^\lambda[X]u\rangle = \langle v\big|XK_0^\lambda u\rangle + \langle K_0^\lambda v\big|Xu\rangle + \sum_{k\geq 1}\langle B_k^\lambda v\big|XB_k^\lambda u\rangle, \tag{4.20}
$$

$$
B_k^\lambda = R_k + \sum_{j\geq 1}\lambda_j S_{jk}\,. \tag{4.21}
$$

From Theorem 4.10 we have, $\forall u, v \in D$,

$$
\lim_{t\downarrow 0}\frac{1}{t}\left(\langle v\big|\mathcal{T}^\lambda(t)[X]u\rangle - \langle v\big|Xu\rangle\right) = \langle v|\mathscr{L}^\lambda[X]u\rangle. \tag{4.22}
$$

By the semigroup property and the and continuity properties we get that $\mathcal{T}^\lambda(t)$ satisfies the equation

$$
\langle v|\mathcal{T}^\lambda(t)[X]u\rangle = \langle v|Xu\rangle + \int_0^t \langle v\big|\mathscr{L}^\lambda\big[\mathcal{T}^\lambda(s)[X]\big]u\rangle \mathrm{d}s, \tag{4.23}
$$

$\forall u, v \in \mathrm{Dom}(K)$, $\forall X \in \mathscr{B}(\mathscr{H})$.

If the minimal QDS generated by K_0^λ, $R_k + \zeta_k$ is Markov, then it coincides with $\mathcal{T}^\lambda$ (Proposition 3.7) and $\mathcal{T}^\lambda$is the unique σ-weakly continuous positive solution of Eq. (4.23).

4.4.2 The evolution equation for the reduced dynamics

Theorem 4.11. *Under Hypotheses 3.4 and 4.1 and $f \in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$, the reduced dynamics $\mathcal{T}^f(s,t)$ introduced in Definition 4.2 satisfies the evolution equation:* $\forall u, v \in D$, $\forall X \in \mathscr{B}(\mathscr{H})$,

$$\big\langle v\big|\mathcal{T}^f(s,t)[X]u\big\rangle = \langle v|Xu\rangle + \int_s^t \big\langle v\big|\mathscr{L}^f(r)\big[\mathcal{T}^f(r,t)[X]\big]u\big\rangle \mathrm{d}r\,, \tag{4.24}$$

where

$$\mathscr{L}^f(r) := \mathscr{L}^{f(r)} \tag{4.25}$$

and $\mathscr{L}^\lambda$ is defined by the quadratic form introduced in Eq. (4.20)*.*

Proof. We make the demonstration in two parts, firstly for a step function f and then for a generic one.

Let us consider the sequence of times $0 = t_0 < t_1 < \cdots < t_n = T$ and the step function $f(x) = \sum_{i=1}^n 1_{[t_{i-1},t_i)}(x)\lambda^i$ with $\lambda^i \in \mathscr{Z}$. Let us take now s and s_1 such that $0 \le s < s_1 \le t$ and $t_i \le s < s_1 \le t_{i+1}$ for some i. Then

$$\begin{aligned}\big\langle v\big|\mathcal{T}^f(s,t)[X]u\big\rangle &= \big\langle v\big|\mathcal{T}^f(s,s_1)\mathcal{T}^f(s_1,t)[X]u\big\rangle \\ &= \big\langle v\big|\mathcal{T}^{\lambda^i}(s_1-s)\mathcal{T}^f(s_1,t)[X]u\big\rangle.\end{aligned}$$

To abbreviate the writing let us set $\mathcal{T}_1 = \mathcal{T}^{\lambda^i}$ and $\mathscr{L}_1 = \mathscr{L}^{\lambda^i}$. Now, from here and Eq.(4.23) we get

$$\begin{aligned}&\big\langle v\big|\mathcal{T}^f(s,t)[X]u\big\rangle \\ &\quad= \big\langle v\big|\mathcal{T}^f(s_1,t)[X]u\big\rangle + \int_0^{s_1-s} \mathrm{d}r\big\langle v\big|\mathscr{L}_1\big[\mathcal{T}_1(r)\mathcal{T}^f(s_1,t)[X]\big]u\big\rangle, \\ &\quad= \big\langle v\big|\mathcal{T}^f(s_1,t)[X]u\big\rangle + \int_s^{s_1} \mathrm{d}x\big\langle v\big|\mathscr{L}_1\big[\mathcal{T}^f(x,t)[X]\big]u\big\rangle;\end{aligned}$$

similarly, we have

$$\big\langle v\big|\mathcal{T}^f(s_1,t)[X]u\big\rangle = \big\langle v\big|\mathcal{T}^f(s_2,t)[X]u\big\rangle + \int_{s_1}^{s_2} \mathrm{d}x\big\langle v\big|\mathscr{L}_2\big(\mathcal{T}^f(x,t)[X]\big)u\big\rangle.$$

Let q be such that $t_{i+q-1} < t \le t_{i+q}$ and let us set $s_0 = s$, $s_q = t$, $s_j = t_{i+j}$ for

$1 \le j \le q-1$. By induction we arrive to

$$\begin{aligned}\big\langle v\big|\mathcal{T}^f(s,t)[X]u\big\rangle &= \big\langle v\big|Xu\big\rangle + \sum_{j=1}^{q}\int_{s_{j-1}}^{s_j} \mathrm{d}x\big\langle v\big|\mathscr{L}_j\big[\mathcal{T}^f(x,t)[X]\big]u\big\rangle \\ &= \big\langle v\big|Xu\big\rangle + \int_s^t \mathrm{d}x\big\langle v\big|\mathscr{L}^f(x)\big[\mathcal{T}^f(x,t)[X]\big]u\big\rangle.\end{aligned}$$

Let us stress that, being f a step function, then

$$\mathscr{L}^f(x) = \sum_{j=1}^{q} 1_{[s_{j-1},s_j)}(x)\mathscr{L}^{\lambda^j}\,.$$

Let now $f^{(n)}$ be a sequence of step functions approximating $f \in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$: $\lim_{n\to\infty}\|f^{(n)} - f\|_2 = 0$. To simplify the notation we write $\mathcal{T}^{f^{(n)}} = \mathcal{T}_n$ and $\mathscr{L}^{f^{(n)}} = \mathscr{L}_n$. By Corollary 4.4 $\lim_{n\to\infty}\mathcal{T}_n(s,t) = \mathcal{T}^f(s,t)$ uniformly.

Now, for $u, v \in D$ we have

$$\begin{aligned}&\int_s^t \mathrm{d}x\big\langle v\big|\mathscr{L}_n(x)\mathcal{T}_n(x,t)[X]u\big\rangle - \int_s^t \mathrm{d}x\big\langle v\big|\mathscr{L}^f(x)\mathcal{T}^f(x,t)[X]u\big\rangle \\ &\quad= \int_s^t \mathrm{d}x\langle v\big|\big(\mathscr{L}_n(x) - \mathscr{L}^f(x)\big)\mathcal{T}_n(x,t)[X]u\big\rangle \\ &\qquad+ \int_s^t \mathrm{d}x\big\langle v\big|\mathscr{L}^f(x)\big(\mathcal{T}_n(x,t) - \mathcal{T}^f(x)\big)[X]u\big\rangle.\end{aligned}$$

The first term gives

$$\begin{aligned}&\int_s^t \mathrm{d}x\langle v\big|\big(\mathscr{L}_n(x) - \mathscr{L}^f(x)\big)\mathcal{T}_n(x,t)[X]u\big\rangle \\ &\quad= \int_s^t \mathrm{d}x\bigg(\big\langle v\big|\mathcal{T}_n(x,t)[X]\left(K_0^{f^{(n)}(x)} - K_0^{f(x)}\right)u\big\rangle \\ &\qquad+ \big\langle \left(K_0^{f^{(n)}(x)} - K_0^{f(x)}\right)v\big|\mathcal{T}_n(x,t)[X]u\big\rangle \\ &+ \sum_{k\ge1}\left(\big\langle B_k^{f^{(n)}(x)}v\big|\mathcal{T}_n(x,t)[X]B_k^{f^{(n)}(x)}u\big\rangle\right) - \big\langle B_k^{f(x)}v\big|\mathcal{T}(x,t)[X]B_k^{f(x)}u\big\rangle\bigg)\bigg).\end{aligned}$$

We have on the domain D

$$K_0^{f^{(n)}(x)} - K_0^{f(x)} = \frac{1}{2}\Big(\|f(x)\|^2 - \|f^{(n)}(x)\|^2\Big) + \sum_{k,j\geq 1}\Big(f_j^{(n)}(x) - f_j(x)\Big)R_k^* S_{kj};$$

therefore

$$\begin{aligned}\Big|\int_s^t \mathrm{d}x \big\langle v\big|\mathcal{T}_n(x,t)[X]\big(K_0^{f^{(n)}(x)} - K_0^{f(x)}\big)u\big\rangle\Big| &\leq \frac{1}{2}\int_s^t \mathrm{d}x\Big|\|f(x)\|^2 - \|f^{(n)}(x)\|^2\Big|\Big|\big\langle v\big|\mathcal{T}_n(x,t)[X]u\big\rangle\Big| \\ &\quad + \int_s^t \mathrm{d}x \sum_{k,j\geq 1}\Big|f_j^{(n)}(x) - f_j(x)\Big|\Big|\big\langle v\big|\mathcal{T}_n(x,t)[X]R_k^* S_{kj}u\big\rangle\Big|.\end{aligned}$$

We know from Remark 4.1 point 1 that $\Big|\big\langle v\big|\mathcal{T}_n(x,t)[X]u\big\rangle\Big| \leq \big|\langle v|Xu\rangle\big|$; by using also $\Big|\|f(x)\|^2 - \|f^{(n)}(x)\|^2\Big| \leq \Big\|f(x) - f^{(n)}(x)\Big\|^2$ we get

$$\int_s^t \mathrm{d}x\Big|\|f(x)\|^2 - \|f^{(n)}(x)\|^2\Big|\Big|\big\langle v\big|\mathcal{T}_n(x,t)[X]u\big\rangle\Big| \leq \big|\big\langle v\big|Xu\big\rangle\big|\int_s^t \mathrm{d}x\big\|f(x) - f^{(n)}(x)\big\|^2 \xrightarrow{n\to+\infty} 0.$$

Obviously, $\Big|f_j^{(n)}(x) - f_j(x)\Big| \leq \Big\|f^{(n)}(x) - f(x)\Big\|$; by Hölder inequality, we have

$$\begin{aligned}\int_s^t \mathrm{d}x \sum_{k,j\geq 1}\Big|f_j^{(n)}(x) - f_j(x)\Big|\Big|\big\langle v\big|\mathcal{T}_n(x,t)[X]R_k^* S_{kj}u\big\rangle\Big| \\ \leq \sum_{k,j\geq 1}\big\langle v\big|XR_k^* S_{kj}u\big\rangle\bigg((t-s)\int_s^t \mathrm{d}x\big\|f(x) - f^{(n)}(x)\big\|^2\bigg)^{\frac{1}{2}} \xrightarrow{n\to+\infty} 0.\end{aligned}$$

All the involved terms are analyzed in the same way. Therefore

$$\int_s^t \mathrm{d}x\big\langle v\big|\mathscr{L}_n(x)\mathcal{T}_n(x,t)[X]u\big\rangle \xrightarrow{n\to+\infty} \int_s^t \mathrm{d}x\big\langle v\big|\mathscr{L}^f(x)\mathcal{T}^f(x,t)[X]u\big\rangle$$

and this implies the result. □

Remark 4.3. Equation (4.24) is the integral form of a backward differential equation. Indeed, at least when f is a continuous function, the r.h.s. is differentiable with respect to s and we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\langle v\middle|\mathcal{T}^f(s,t)[X]u\right\rangle = -\left\langle v\middle|\mathscr{L}^f(s)\big[\mathcal{T}^f(s,t)[X]\big]u\right\rangle, \tag{4.26}$$

with final condition $\left\langle v\middle|\mathcal{T}^f(t,t)[X]u\right\rangle = \left\langle v\middle|Xu\right\rangle$.

Chapter 5

The Continual Measurement

When the fields represent pure noise, it is natural to consider system observables as in Section 4.1; then, the reduced dynamics comes out. In other situations, as when the fields are intended to represent the electromagnetic field, the natural observables are field quantities, from which inferences are done on the sub-system in $\mathcal{H}$; let us call the sub-system $S_{\mathcal{H}}$. We are interested in the behaviour of the system $S_{\mathcal{H}}$, but we measure field observables; this scheme is known as *indirect measurement*.

Another way to think to this situation is the following one. We cannot act directly on our system $S_{\mathcal{H}}$, but any action is mediated by some quantum input and output channel. We can think of an atom driven by a laser (input) and emitting fluorescence light (output) or of the light entering (input) and leaving (output) an optical cavity. In these examples the role of input and output channels is played by the electromagnetic field and we can think of approximating it by the Bose fields on which QSC is based.

So, we have to identify the main field observables, which eventually we want to take under measurement with continuity in time. But before to introduce continual measurements we have to recall some notions on commuting self-adjoint operators, projection valued measures and their Fourier transforms.

5.1 Characteristic functional

Let X be a selfadjoint operator and $\mathrm{e}^{\mathrm{i}kX}$ be the group generated by X. Then, there exists a unique *projection-valued measure* (pvm) ξ on $\big(\mathbb{R},\mathscr{B}(\mathbb{R})\big)$ such that

$$\mathrm{e}^{\mathrm{i}kX} = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kx}\xi(\mathrm{d}x), \qquad \forall k \in \mathbb{R}. \tag{5.1}$$

Let $\boldsymbol{X} \equiv (X_1,\dots,X_d)$ be a set of d mutually commuting selfadjoint operators, in the sense that the groups generated by them commute or that the associated pvm ξ_j commute. Then, there exists a unique pvm ξ on $\big(\mathbb{R}^d,\mathscr{B}(\mathbb{R}^d)\big)$ such that

$$\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{X}} \equiv \prod_{j\geq 1} \mathrm{e}^{\mathrm{i}k_jX_j} = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\xi(\mathrm{d}\boldsymbol{x}), \qquad \forall \boldsymbol{k} \in \mathbb{R}^d. \tag{5.2}$$

Moreover, in the state ϱ, the characteristic function (Fourier transform) of the probability law

$$\mathbb{P}_\varrho^{\boldsymbol{X}}(\mathrm{d}\boldsymbol{x}) = \mathrm{Tr}\left\{\varrho\xi(\mathrm{d}\boldsymbol{x})\right\} \tag{5.3}$$

of the observable associated to $\boldsymbol{X}$ is

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathbb{P}_\varrho^{\boldsymbol{X}}(\mathrm{d}\boldsymbol{x}) = \mathrm{Tr}\left\{\varrho\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{X}}\right\}. \tag{5.4}$$

These results extend to "infinitely many" commuting selfadjoint operators; only uniqueness is lost.

Proposition 5.1 ([12] p. 59, [3] Proposition 3.1)**.** *Let $\{\xi_t,\ t \in I\}$ be a family of commuting pvm on $\big(\mathbb{R},\mathscr{B}(\mathbb{R})\big)$ and X_t be the selfadjoint operator associated with ξ_t. Then, there exist a measurable space $(\Omega,\mathcal{F})$, a pvm ξ on $(\Omega,\mathcal{F})$ and a family of real valued measurable functions $\big\{\widetilde{X}_t,\ t \in I\big\}$ on Ω such that*

$$\mathrm{e}^{\mathrm{i}kX_t} = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kx}\xi_t(\mathrm{d}x) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}k\widetilde{X}_t(\omega)}\xi(\mathrm{d}\omega), \qquad \forall k \in \mathbb{R},\ \forall t \in I. \tag{5.5}$$

In the situation described in this proposition, if ϱ is a fixed state and we set

$$\mathbb{P}_\varrho(\mathrm{d}\omega) := \mathrm{Tr}\left\{\varrho\xi(\mathrm{d}\omega)\right\},$$

we have that $(\Omega,\mathcal{F},\mathbb{P}_\varrho)$ is a classical probability space and $\big\{\widetilde{X}_t(\cdot),\ t \in I\big\}$ becomes a classical stochastic process. The characteristic functions of the finite dimensional distributions of the process are given by

$$\int_\Omega \exp\left(\mathrm{i}\sum_{j=1}^n k_j\widetilde{X}_{t_j}(\omega)\right)\mathbb{P}_\varrho(\mathrm{d}\omega) = \mathrm{Tr}\left\{\varrho\exp\left(\mathrm{i}\sum_{j=1}^n k_jX_{t_j}\right)\right\}. \tag{5.6}$$

Let us consider now the case in which the index becomes time plus a discrete label: $I = \{(i,t) \ : \ i = 1, \dots, d, \ 0 < t \leq T\}$. Then, we denote the process by $\widetilde{X}_i(t;\omega)$, the operators by $X_i(t)$ and we assume, for simplicity, $\widetilde{X}_i(0;\omega) = 0$, $X_i(0) = 0$. Instead of considering the finite-dimensional distributions of the process $\widetilde{X}_i(t)$, it is equivalent and simpler to introduce the finite-dimensional distributions of the increments of the original process, whose characteristic functions are

$$\int_\Omega \exp\left(\mathrm{i}\sum_{i=1}^{d}\sum_{j=1}^{n} k_i(t_j)\Big(\widetilde{X}_i(t_j;\omega) - \widetilde{X}_i(t_{j-1};\omega)\Big)\right)\mathbb{P}_\varrho(\mathrm{d}\omega)$$
$$= \operatorname{Tr}\left\{\varrho \exp\left(\mathrm{i}\sum_{i=1}^{d}\sum_{j=1}^{n} k_i(t_j)\Big(X_i(t_j) - X_i(t_{j-1})\Big)\right)\right\}, \quad (5.7)$$
$$0 = t_0 < t_1 < \cdots < t_n \leq T.$$

5.2 The output fields

The fields we have introduced are expressed in the interaction picture. However, in order to construct a theory of continual measurements, based on the usual rules of quantum mechanics, which require the existence of joint spectral measures, we need observables commuting at different times *in the Heisenberg picture.*

Let us call "input fields" the fields before the interaction with the system $S_\mathcal{H}$, i.e. the fields $A_k(t)$, $A_k^\dagger(t)$, $\Lambda_{kl}(t)$ and let us call "output fields" the fields after the interaction with the system $S_\mathcal{H}$ or, in other words, the fields in the Heisenberg picture. We have

$$A_j^{\mathrm{out}}(t) := U(t)^* A_j(t) U(t) \quad (5.8)$$

and similar definitions for $A_j^{\mathrm{out}\,\dagger}(t)$, $\Lambda_{ij}^{\mathrm{out}}(t)$. Note that, if $\mathtt{D}$ is the domain of $A_j(t)$, then $U(t)^*\mathtt{D}$ is the domain of $A_j^{\mathrm{out}}(t)$.

By Proposition 3.5 we have

$$U(T) = U(T,t)U(t)\,, \qquad \forall T \geq t\,, \quad (5.9)$$

with $U(T,t)$ adapted to $\mathcal{H} \otimes \mathcal{F}_{(t,T)}$ and, so, commuting with $A_j(t)$, $A_j^\dagger(t)$, $\Lambda_{ij}(t), \dots$ Therefore, we have

$$A_j^{\mathrm{out}}(t) = U(T)^* A_j(t) U(T)\,, \qquad \forall T \geq t\,. \quad (5.10)$$

This implies immediately that the output fields satisfy the same canonical commutation rules of the input fields: the output fields remain Bose free fields.

As the output fields are Bose free fields the set of operators (field quadratures)

$$A_j^{\mathrm{out}}(t) + A_j^{\dagger\,\mathrm{out}}(t), \qquad t \geq 0, \quad j = 1, \ldots, d,$$

are commuting selfadjoint operators and can be jointly observed. The same is true for families of number operators such that

$$\Lambda_{jj}^{\mathrm{out}}(t), \qquad t \geq 0, \quad j = 1, \ldots, d.$$

By applying the formal rules of QSC we can express the output fields as the quantum stochastic integrals [2]

$$A_j^{\mathrm{out}}(t) = \int_0^t \Big\{ \sum_k U(s)^* S_{jk} U(s) \mathrm{d}A_k(s) + U(s)^* R_j U(s)\,\mathrm{d}s \Big\}, \tag{5.11a}$$

$$A_j^{\mathrm{out}\,\dagger}(t) = \int_0^t \Big\{ \sum_k U(s)^* S_{jk}^* U(s) \mathrm{d}A_k^\dagger(s) + U(s)^* R_j^* U(s)\,\mathrm{d}s \Big\}, \tag{5.11b}$$

$$\begin{aligned} \Lambda_{ij}^{\mathrm{out}}(t) = \int_0^t \Big\{ &\sum_{kl} U(s)^* S_{ik}^* S_{jl} U(s)\,\mathrm{d}\Lambda_{kl}(s) + \sum_k U(s)^* S_{ik}^* R_j U(s)\,\mathrm{d}A_k^\dagger(s) \\ &+ \sum_l U(s)^* R_i^* S_{jl} U(s)\,\mathrm{d}A_l(s) + U(s)^* R_i^* R_j U(s)\,\mathrm{d}s \Big\}. \end{aligned} \tag{5.11c}$$

From these equations one explicitly sees that the output fields carry information on system $S_{\mathcal{H}}$: the quantities R_k, S_{kl} are the system operators appearing in the system–field interaction.

What we have to do now is to put in a rigorous basis the joint observation of the commuting families above, to generalize to other similar families of operators, and to show how to reduce the theory to some "tractable" expressions and evolution equations.

5.3 Characteristic operator and observables

5.3.1 The characteristic operator

In order to construct the joint field observables we start from the Fourier transform of their joint projection valued measure (characteristic operator); this approach has the advantage of giving rise to unitary operators which can be handled with quantum stochastic calculus. We use here a simplified version of the structure given in [3], Theorem 3.1, but rich enough to contain the physically interesting observables.

1. Let $\boldsymbol{P} = \big(P_1, P_2, \ldots, P_m\big)$ be a vector of mutually orthogonal projections in $\mathscr{Z}$, i.e.

$$P_\alpha = P_\alpha^*, \qquad P_\alpha P_\beta = \delta_{\alpha\beta}\, P_\alpha\,, \qquad \forall \alpha, \beta = 1, \ldots, m;$$

   Some of the projections can be zero or the identity.

2. Let us take $\boldsymbol{h} = (h^1, h^2 \ldots, h^m)$ with $h^\alpha \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathscr{Z})$, $\alpha = 1, \ldots, m$, such that $\forall t \geq 0, \quad \forall \alpha, \beta = 1, \ldots, m,$

$$\mathrm{Im}\langle h^\alpha(t) | h^\beta(t)\rangle = 0, \qquad P_\alpha h^\beta(t) = 0.$$

3. For any test function $\boldsymbol{k} = \big(k_1, k_2, \ldots, k_m\big) \in \mathrm{L}^\infty(\mathbb{R}^+; \mathbb{R}^m)$ let us define

   - $\mathsf{S}\big(\boldsymbol{k}(s)\big) = \prod_{\alpha=1}^m \mathrm{e}^{\mathrm{i}k_\alpha(s) P_\alpha}$,
   - $\forall f \in \mathrm{L}^2(\mathbb{R}^+; \mathscr{Z})$,

$$\begin{aligned}\big(\mathsf{S}_t(\boldsymbol{k}) f\big)(s) &= 1_{(0,t)}(s)\mathsf{S}\big(\boldsymbol{k}(s)\big) f(s) + 1_{[t,+\infty)}(s) f(s) \\ &= 1_{(0,t)}(s) \sum_{\alpha=1}^m \left(\mathrm{e}^{\mathrm{i}k_\alpha(s)} - 1\right) P_\alpha f(s) + f(s),\end{aligned} \tag{5.12}$$

   - $r_t(\boldsymbol{k})(s) = 1_{(0,t)}(s)\mathrm{i}\boldsymbol{k}(s) \cdot \boldsymbol{h}(s)$.

   Note that $\mathsf{S}_t(\boldsymbol{k}) \in U\big(\mathrm{L}^2(\mathbb{R}^+; \mathscr{Z})\big)$ and $r_t(\boldsymbol{k}) \in \mathrm{L}^2(\mathbb{R}^+; \mathscr{Z})$.

4. Let us define the *Characteristic Operator* as the Weyl operator

$$\widehat{\Phi}_{\boldsymbol{k}}(t) := W\big(r_t(\boldsymbol{k}); \mathsf{S}_t(\boldsymbol{k})\big). \tag{5.13}$$

Theorem 5.2 ([3] - Theorem. 3.1)**.** *The characteristic operator* (5.13) *has the following properties*

1. *Localization property: for* $0 \leq r < s \leq t$
   $\widehat{\Phi}_{\boldsymbol{k}_{(r,s)}}(t) = \widehat{\Phi}_{\boldsymbol{k}_{(r,s)}}(s) \in U(\mathcal{F}_{(r,s)}), \qquad \boldsymbol{k}_{(r,s)}(t) := 1_{(r,s)}(t)\boldsymbol{k}(t).$

2. *Group property:*
   $\widehat{\Phi}_{\boldsymbol{k}_1}(t)\widehat{\Phi}_{\boldsymbol{k}_2}(t) = \widehat{\Phi}_{\boldsymbol{k}_1+\boldsymbol{k}_2}(t), \quad \forall\ \boldsymbol{k}_1,\, \boldsymbol{k}_2 \in \mathrm{L}^\infty(\mathbb{R}^+; \mathbb{R}^d).$

3. *Matrix elements:*

$$\begin{aligned}\langle \mathrm{e}(g)|\widehat{\Phi}_{\boldsymbol{k}}(t)\mathrm{e}(f)\rangle &= \langle \mathrm{e}(g)|\mathrm{e}(f)\rangle \\ &\times \exp\Bigg\{\sum_{\alpha=1}^{m}\int_0^t \mathrm{d}s\Bigg(-\frac{1}{2}\sum_{\beta=1}^{m} k_\alpha(s)\langle h^\alpha(s)|h^\beta(s)\rangle k_\beta(s) \\ &+ \mathrm{i}k_\alpha(s)\Big(\langle h^\alpha(s)|f(s)\rangle + \langle g(s)|h^\alpha(s)\rangle\Big) \\ &+ \Big(\mathrm{e}^{\mathrm{i}k_\alpha(s)} - 1\Big)\langle g(s)|P_\alpha f(s)\rangle\Bigg)\Bigg\}. \qquad (5.14)\end{aligned}$$

4. $\left\|\left(\widehat{\Phi}_{\boldsymbol{k}'}(t) - \widehat{\Phi}_{\boldsymbol{k}''}(t)\right)\mathrm{e}(f)\right\|^2 = 2\left\|\mathrm{e}(f)\right\|^2(1 - \mathrm{e}^a\cos b),$

$$\begin{aligned}a = -\frac{1}{2}\sum_{\alpha,\beta=1}^{m}\int_0^t k_\alpha(s)\langle h^\alpha(s)|h^\beta(s)\rangle k_\beta(s)\,\mathrm{d}s \\ + \sum_{\alpha=1}^{m}\int_0^t(\cos k_\alpha(s) - 1)\langle f(s)|P_\alpha f(s)\rangle \mathrm{d}s,\end{aligned}$$

$$b = 2\sum_{\alpha=1}^{m}\int_0^t k_\alpha(s)\,\mathrm{Re}\langle h^\alpha(s)|f(s)\rangle \mathrm{d}s + \sum_{\alpha=1}^{m}\int_0^t \sin k_\alpha(s)\,\langle f(s)|P_\alpha f(s)\rangle \mathrm{d}s,$$
$$\boldsymbol{k} = \boldsymbol{k}' - \boldsymbol{k}''.$$

5. *Continuity:*
 $\widehat{\Phi}_{\varkappa\boldsymbol{k}}(t)$ *is strongly continuous in* $\varkappa \in \mathbb{R}$ *and in* $t \geq 0$.

6. *Given the initial condition* $\widehat{\Phi}_{\boldsymbol{k}}(0) = \mathbb{1}$, $\widehat{\Phi}_{\boldsymbol{k}}(t)$ *is the unique unitary solution of the QSDE of Hudson-Parthasarathy left type*

$$\begin{aligned}\mathrm{d}\widehat{\Phi}_{\boldsymbol{k}}(t) = \widehat{\Phi}_{\boldsymbol{k}}(t)\Bigg(\sum_{i\geq 1} G_{i0}(t)\mathrm{d}A_i^\dagger(t) + \sum_{i,j\geq 1} G_{ij}(t)\mathrm{d}\Lambda_{ij}(t) \\ + \sum_{j\geq 1} G_{0j}(t)\mathrm{d}A_j(t) + G_{00}(t)\mathrm{d}t\Bigg) \qquad (5.15)\end{aligned}$$

with

$$
\begin{aligned}
G_{i0}(t) &= \mathrm{i}\sum_{\alpha=1}^{m} k_\alpha(t)\langle z_i|h^\alpha(t)\rangle, && \textit{for } i\geq 1\\
G_{ij}(t) &= \sum_{\alpha=1}^{m}\left(\mathrm{e}^{\mathrm{i}k_\alpha(t)}-1\right)\langle z_i|P_\alpha z_j\rangle, && \textit{for } i,j\geq 1\\
G_{0j}(t) &= -\mathrm{i}\sum_{\alpha=1}^{m} k_\alpha(t)\big\langle h^\alpha(t)\big|z_j\big\rangle, && \textit{for } j\geq 1\\
G_{00}(t) &= -\frac{1}{2}\sum_{\alpha,\beta=1}^{m} k_\alpha(t)\langle h^\alpha(t)|h^\beta(t)\rangle k_\beta(t).
\end{aligned}
$$

7. *Given the initial condition* $\widehat{\Phi}_{\boldsymbol{k}}(0)=\mathbb{1}$, $\widehat{\Phi}_{\boldsymbol{k}}(t)$ *is the unique unitary solution of the QSDE of Hudson-Parthasarathy right type*

$$
\begin{aligned}
\mathrm{d}\widehat{\Phi}_{\boldsymbol{k}}(t) = \Bigg(&\sum_{i\geq 1} G_{i0}(t)\mathrm{d}A_i^\dagger(t) + \sum_{i,j\geq 1} G_{ij}(t)\mathrm{d}\Lambda_{ij}(t)\\
&+\sum_{j\geq 1} G_{0j}(t)\mathrm{d}A_j(t) + G_{00}(t)\mathrm{d}t\Bigg)\widehat{\Phi}_{\boldsymbol{k}}(t) \qquad (5.16)
\end{aligned}
$$

with the same coefficients as above.

8. *There exist a measurable space* $\big(\Omega,\mathcal{F}\big)$, *a pvm* ξ *on* $\big(\Omega,\mathcal{F}\big)$, *a family of real valued measurable functions* $\big\{\widetilde{Y}_\alpha(t;\cdot),\ \alpha=1,\dots,m,\ t\geq 0\big\}$ *on* Ω, *a family of commuting and adapted selfadjoint operator* $\big\{Y_\alpha(t),\ \alpha=1,\dots,m,\ t\geq 0\big\}$ *such that* $\widetilde{Y}_\alpha(0;\omega)=0$, $Y_\alpha(0)=0$ *and, for any choice of* n, $0=t_0,<\dots,<t_n=t$, $\kappa_{\alpha j}\in\mathbb{R}$,

$$
\begin{aligned}
\widehat{\Phi}_{\boldsymbol{k}}(t) &= \exp\left(\mathrm{i}\sum_{j=1}^{n}\sum_{\alpha=1}^{m}\kappa_{\alpha j}\Big(Y_\alpha(t_j)-Y_\alpha(t_{j-1})\Big)\right)\\
&= \int_\Omega \exp\left(\mathrm{i}\sum_{j=1}^{n}\sum_{\alpha=1}^{m}\kappa_{\alpha j}\Big(\widetilde{Y}_\alpha(t_j;\omega)-\widetilde{Y}_\alpha(t_{j-1};\omega)\Big)\right)\xi(\mathrm{d}\omega), \qquad (5.17)
\end{aligned}
$$

where $k_\alpha(s)=\sum_{j=1}^{n} 1_{(t_{j-1},t_j)}(s)\kappa_{\alpha j}$.

9. On the exponential domain one has

$$Y_\alpha(t) = \sum_{k\geq 1} \int_0^t \Big(\langle h^\alpha(t)|z_k\rangle \mathrm{d}A_k(s) + \langle z_k|h^\alpha(s)\rangle \mathrm{d}A_k^\dagger(s)\Big) + \sum_{l,k\geq 1} \int_0^t \langle z_l|P_\alpha z_k\rangle \mathrm{d}\Lambda_{lk}(s). \qquad (5.18)$$

Proof. By the definitions of characteristic operator (5.13) and of Weyl operator (2.6) one has

$$\widehat{\Phi}_{\boldsymbol{k}}(t)\mathrm{e}(f) = \exp\left\{ -\frac{1}{2}\left\|r_t(\boldsymbol{k})\right\|^2 - \langle r_t(\boldsymbol{k})|\mathsf{S}_t(\boldsymbol{k})f\rangle \right\} \mathrm{e}\big(\mathsf{S}_t(\boldsymbol{k})f + r_t(\boldsymbol{k})\big) \qquad (5.19)$$

and, from the properties of the various quantities involved, one gets

$$\left\|r_t(\boldsymbol{k})\right\|^2 = \sum_{\alpha,\beta=1} \int_0^t k_\alpha(s)\langle h^\alpha(s)|h^\beta(s)\rangle k_\beta(s)\,\mathrm{d}s\,, \qquad (5.20)$$

$$-\langle r_t(\boldsymbol{k})|\mathsf{S}_t(\boldsymbol{k})f\rangle = \mathrm{i}\sum_{\alpha=1} \int_0^t k_\alpha(s)\langle h^\alpha(s)|f(s)\rangle \mathrm{d}s\,, \qquad (5.21)$$

$$\begin{aligned}\big(\mathsf{S}_t(\boldsymbol{k})f + r_t(\boldsymbol{k})\big)(s) &= 1_{(0,t)}(s)\left[\mathsf{S}\big(\boldsymbol{k}(s)\big)f(s) + \mathrm{i}\boldsymbol{k}(s)\cdot\boldsymbol{h}(s)\right] + 1_{[t,+\infty)}(s)f(s) \\ &= 1_{(0,t)}(s)\sum_{\alpha=1}^m \left[\left(\mathrm{e}^{\mathrm{i}k_\alpha(s)} - 1\right)P_\alpha f(s) + \mathrm{i}k_\alpha(s)h^\alpha(s)\right] + f(s). \qquad (5.22)\end{aligned}$$

From here one gets immediately the localization property 1 and the matrix elements (5.14).

One can check that the definitions of $\mathsf{S}_t(\boldsymbol{k})$ and $r_t(\boldsymbol{k})$ are such that

$$\mathsf{S}_t(\boldsymbol{k}_1)\mathsf{S}_t(\boldsymbol{k}_2) = \mathsf{S}_t(\boldsymbol{k}_1+\boldsymbol{k}_2)\,, \qquad \mathsf{S}_t(\boldsymbol{k})^{-1} = \mathsf{S}_t(\boldsymbol{k})^* = \mathsf{S}_t(-\boldsymbol{k})\,, \qquad (5.23)$$

$$r_t(\boldsymbol{k}_1) + \mathsf{S}_t(\boldsymbol{k}_1)r_t(\boldsymbol{k}_2) = r_t(\boldsymbol{k}_1+\boldsymbol{k}_2)\,. \qquad (5.24)$$

Together with (2.8), these equations imply the group property 2.

Point 4 follows from the group property and the expression of the matrix elements (5.14).

By the unitarity of $\widehat{\Phi}_{\boldsymbol{k}}(t)$ and the fact that the exponential vectors are total, it is enough to prove the strong continuity on the exponential vectors. By the unitarity and the localization and group properties, the strong continuity on the exponential vectors reduces to the check of the continuity of the matrix elements (5.14).

By checking that the conditions of Definition 2.2 are satisfied, one has that the r.h.s. of (5.15) is well defined. By differentiating the matrix elements (5.14) one gets the QSDE (5.15); by passing to the equation for the matrix elements, which turns out to be a closed ordinary differential equation, one obtains the uniqueness of the solution. Exactly in the same way, the statement in point 7 is proved.

By the group and continuity properties, for any test function $\boldsymbol{k}$, $\left\{\widehat{\Phi}_{\varkappa\boldsymbol{k}}(t),\ \varkappa \in \mathbb{R}\right\}$ is a strongly continuous one-parameter group of unitary operators; so, it is the Fourier transform of a pvm $\xi_{\boldsymbol{k}}$ on $\mathbb{R}$. Any two of such pvm's $\xi_{\boldsymbol{k}_1}$ and $\xi_{\boldsymbol{k}_2}$ commute, again by the group property. Then, the statement in point 8 is an application of Proposition 5.1 to the present case.

From Eq. (5.17) with $\boldsymbol{k} \to \varkappa\boldsymbol{k}$, $k_\beta(s) = \delta_{\alpha\beta}$, we get $\widehat{\Phi}_{\varkappa\boldsymbol{k}}(t) = \exp\{\mathrm{i}\varkappa X(\alpha,t)\}$. By using the matrix elements (5.14) and taking $-\mathrm{i}$ times the derivative with respect to $\varkappa$ in $\varkappa = 0$, we get the statement of point 9. □

5.3.2 Continual measurements and infinite divisibility

It is important to realize that in a coherent state $\psi(f)$ the process $\widetilde{Y}_\alpha(t)$ has independent increments; here we are not considering the interaction with system $\mathcal{S}_{\mathscr{H}}$. Indeed, by Eqs. (5.14), (5.17) and (2.4), one obtains

$$\langle\psi(f)|\widehat{\Phi}_{\boldsymbol{k}}(t)\psi(f)\rangle = \prod_{j=1}^{n}\Big\langle\psi\big(1_{(t_{j-1},t_j)}f\big)\Big|\exp\Big(\mathrm{i}\sum_{\alpha=1}^{m}\kappa_{\alpha j}\big(Y_\alpha(t_j) - Y_\alpha(t_{j-1})\big)\Big)\psi\big(1_{t_{j-1},t_j)}f\big)\Big\rangle. \quad (5.25)$$

By the localization properties, we can reintroduce $\psi(f)$ in every factor and we obtain, again by Eq. (5.17), the independence of the increments

$$\int_\Omega \exp\left(\mathrm{i}\sum_{j=1}^{n}\sum_{\alpha=1}^{m}\kappa_{\alpha j}\Big(\widetilde{Y}_\alpha(t_j;\omega) - \widetilde{Y}_\alpha(t_{j-1};\omega)\Big)\right)\langle\psi(f)|\xi(\mathrm{d}\omega)\psi(f)\rangle$$
$$= \prod_{j=1}^{n}\int_\Omega \exp\left(\mathrm{i}\sum_{\alpha=1}^{m}\kappa_{\alpha j}\Big(\widetilde{Y}_\alpha(t_j;\omega) - \widetilde{Y}_\alpha(t_{j-1};\omega)\Big)\right)\langle\psi(f)|\xi(\mathrm{d}\omega)\psi(f)\rangle. \quad (5.26)$$

This fact implies that the increments follow an infinitely divisible law. The connection between continual measurements and infinitely divisible laws can be used as starting point to arrive to more general expressions for the characteristic operator [3].

5.3.3 The characteristic functional and the finite dimensional laws

Let us consider now the interaction between system $S_{\mathscr{H}}$ and the fields; let $\mathfrak{s}$ be the system-field state. The *characteristic functional* of the process $\tilde{Y}$ (the "Fourier transform" of its probability law) is given by

$$\Phi_{\boldsymbol{k}}(t) = \operatorname{Tr}\left\{\widehat{\Phi}_{\boldsymbol{k}}(t)U(t)\mathfrak{s}U(t)^*\right\} = \operatorname{Tr}\left\{\widehat{\Phi}_{\boldsymbol{k}}^{\text{out}}(t)\mathfrak{s}\right\}, \tag{5.27}$$

where

$$\widehat{\Phi}_{\boldsymbol{k}}^{\text{out}}(t) = U(t)^*\widehat{\Phi}_{\boldsymbol{k}}(t)U(t)\,. \tag{5.28}$$

All the probabilities describing the continual measurement of the observables $Y(\alpha,t)$ are contained in $\Phi_{\boldsymbol{k}}(t)$; let us give explicitly the construction of the joint probabilities for a finite number of increments.

The measurable functions $\left\{\tilde{Y}_\alpha(t;\cdot)\,,\ \alpha=1,\ldots,m,\ t\geq 0\right\}$, introduced in Theorem 5.2, represent the output signal of the continual measurements. Let us denote by $\Delta\boldsymbol{Y}(t_1,t_2) = \left(\tilde{Y}_1(t_2)-\tilde{Y}_1(t_1),\ldots,\tilde{Y}_m(t_2)-\tilde{Y}_m(t_1)\right)$ the vector of the increments of the output in the time interval (t_1,t_2) and by $\xi(\mathrm{d}\boldsymbol{x};t_1,t_2)$ the joint pvm on $\mathbb{R}^m$ of the increments $Y_\alpha(t_2)-Y_\alpha(t_1)$, $\alpha=1,\ldots,m$. Note that, because of the properties of the characteristic operator, not only the different components of an increment are commuting, but also increments related to different time intervals; this implies that the pvm related to different time intervals commute. Moreover, the localization properties of the characteristic operator give

$$\xi(A;t_1,t_2)\in\mathcal{B}(\mathcal{F}_{(t_1,t_2)})\,,\quad \text{for any Borel set } A\subset\mathbb{R}^m\,. \tag{5.29}$$

As in point 8 of Theorem 5.2, let us consider $0=t_0<t_1<\cdots<t_n\leq t$,

$k_\alpha(s) = \sum_{j=1}^{n} 1_{(t_{j-1},t_j)}(s)\,\kappa_{\alpha j}$, $\boldsymbol{\kappa}_j = (\kappa_{1j}, \ldots, \kappa_{mj})$; then we can write

$$\begin{aligned}\Phi_{\boldsymbol{k}}(t) &= \operatorname{Tr}\biggl\{\exp\biggl(\mathrm{i}\sum_{j=1}^{n}\sum_{\alpha=1}^{m}\kappa_{\alpha j}\bigl[Y_\alpha(t_j) - Y_\alpha(t_{j-1})\bigr]\biggr)U(t)\mathfrak{s}U(t)^*\biggr\}\\ &= \int_{\mathbb{R}^{nm}}\biggl(\prod_{j=1}^{n}\mathrm{e}^{\mathrm{i}\boldsymbol{\kappa}_{\cdot j}\cdot\boldsymbol{x}_{\cdot j}}\biggr)\mathbb{P}_{\mathfrak{s}}\bigl[\Delta\boldsymbol{Y}(t_0,t_1)\in \mathrm{d}\boldsymbol{x}_{\cdot 1}, \ldots, \Delta\boldsymbol{Y}(t_{n-1},t_n)\in \mathrm{d}\boldsymbol{x}_n\bigr]\,,\end{aligned} \tag{5.30}$$

where the physical probabilities are given by

$$\begin{aligned}&\mathbb{P}_{\mathfrak{s}}\bigl[\Delta\boldsymbol{Y}(t_0,t_1)\in A_1, \ldots, \Delta\boldsymbol{Y}(t_{n-1},t_n)\in A_n\bigr]\\ &\qquad = \operatorname{Tr}\biggl\{\biggl(\prod_{j=1}^{n}\xi(A_j;t_{j-1},t_j)\biggr)U(t)\mathfrak{s}U(t)^*\biggr\}.\end{aligned} \tag{5.31}$$

Obviously, $\Phi_{\boldsymbol{k}}(t)$ is the characteristic function of the physical probabilities $\mathbb{P}_{\mathfrak{s}}\bigl[\Delta\boldsymbol{Y}(t_0,t_1)\in A_1, \ldots, \Delta\boldsymbol{Y}(t_{n-1},t_n)\in A_n\bigr]$ and it uniquely determines them.

5.4 The reduced description of the continual measurement

5.4.1 The reduced characteristic operator

Now, we trace out the fields in a way similar to the construction of the reduced dynamics in Chapter 4.

Let $U(t)$ a unitary, strongly continuous right cocycle (Hypothesis 3.3), and let us define $U(t,s)$ by Eq. (3.7).

Definition 5.1. Let us take $f \in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$ and $0 \le s \le t$.
Let $\mathscr{G}^f_{\boldsymbol{k}}(s,t) : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ be the unique operator that satisfies, $\forall u, v \in \mathscr{H}$, $\forall X \in \mathscr{B}(\mathscr{H})$,

$$\bigl\langle v\big|\mathscr{G}^f_{\boldsymbol{k}}(s,t)[X]u\bigr\rangle := \bigl\langle U(t,s)v\otimes\psi(f)\big|\bigl(X\otimes\widehat{\Phi}_{\boldsymbol{k}}(t,s)\bigr)U(t,s)u\otimes\psi(f)\bigr\rangle,$$

where $\widehat{\Phi}_{\boldsymbol{k}}(t,s)$ is defined by

$$\widehat{\Phi}_{\boldsymbol{k}}(t,s) := \widehat{\Phi}_{\boldsymbol{k}_{(s}}(t), \qquad \boldsymbol{k}_{(s}(r) = 1_{(s,+\infty)}(r)\boldsymbol{k}(r). \tag{5.32}$$

Remark 5.1. For any $s < t$ we have

1. $\widehat{\Phi}_{\boldsymbol{k}}(s,s) = \mathbb{1}$,
2. $\widehat{\Phi}_{\boldsymbol{k}}(t,s) = \widehat{\Phi}_{\boldsymbol{k}_{(s,t)}}(t,s)$,
3. $\Theta_s \widehat{\Phi}_{\boldsymbol{k}}(t,s)\Theta_s^* = \widehat{\Phi}_{\theta_s \boldsymbol{k}}(t-s,0)\Big|_{\boldsymbol{h}\to\theta_s\boldsymbol{h}}$.

Proof.

1. A trivial consequence of the definition of the characteristic operator and of $r_s\big(\boldsymbol{k}_{(s}\big) = 0$ and $\mathsf{S}_s\big(\boldsymbol{k}_{(s}\big) = \mathbb{1}$.
2. By writing $\boldsymbol{k}_{(s} = \boldsymbol{k}_{(s,t)} + \boldsymbol{k}_{(t}$, the group property (Theorem 5.2, point 2) and the previous point give the statement.
3. It is easy to check that $\theta_s \mathsf{S}_t\big(\boldsymbol{k}_{(s,t)}\big)\theta_{-s} = \mathsf{S}_{t-s}(\theta_s \boldsymbol{k})$ and $\theta_s r_t\big(\boldsymbol{k}_{(s,t)}\big)(x) = 1_{(0,t-s)}(x)\mathrm{i}\boldsymbol{k}(x+s)\cdot\boldsymbol{h}(x+s) = r_{t-s}\big(\theta_s\boldsymbol{k}\big)\Big|_{\boldsymbol{h}\to\theta_s\boldsymbol{h}}$. Then, the statement follows.

□

Theorem 5.3. *In the hypotheses above, the family of linear maps $\mathscr{G}^f_{\boldsymbol{k}}(s,t)$, $t \geq s \geq 0$ has the following properties:*

1. $\mathscr{G}^f_0(s,t) = \mathcal{T}^f(s,t)$*;*
2. $\mathscr{G}^f_{\boldsymbol{k}}(s,s) = \mathbb{1}$*;*
3. $\mathscr{G}^f_{\bullet}(s,t)$ *is* completely positive definite, *i.e., for all integers n, test functions $\boldsymbol{k}^i$, vectors ϕ_i and operators X_i, one has*
$$\sum_{i,j=1}^{n} \big\langle \phi_i \big| \mathscr{G}^f_{\boldsymbol{k}^i-\boldsymbol{k}^j}(s,t)[X_i^* X_j]\phi_j \big\rangle \geq 0\,;$$
4. $\Big\|\mathscr{G}^f_{\boldsymbol{k}}(s,t)\Big\| \leq 1$*;*
5. $\mathscr{G}^f_{\boldsymbol{k}}(s,t)$ *is a σ-weakly continuous operator on $\mathscr{B}(\mathscr{H})$ and it has a pre-adjoint $\mathscr{G}^f_{\boldsymbol{k}}(s,t)_*$ acting on the trace class on $\mathscr{H}$;*

6. *For each* $X \in \mathscr{B}(\mathscr{H})$ *the maps* $t \mapsto \mathscr{G}^f_{\boldsymbol{k}}(s,t)[X]$ *and* $s \mapsto \mathscr{G}^f_{\boldsymbol{k}}(s,t)[X]$ *are continuous with respect to the* σ*-weak topology of* $\mathscr{B}(\mathscr{H})$*;*

7. $\mathscr{G}^f_{\boldsymbol{k}}(r,s)\mathscr{G}^f_{\boldsymbol{k}}(s,t) = \mathscr{G}^f_{\boldsymbol{k}}(r,t), \quad 0 \le r \le s \le t;$

8. $\forall u, v \in \mathscr{H}$, $\forall X \in \mathscr{B}(\mathscr{H})$, $\quad \big\langle v \big| \mathscr{G}^f_{\boldsymbol{k}}(s,t)[X]u \big\rangle =$
$$= \big\langle U(t,s)v \otimes \psi(f_{(s,t)}) \big| \big(X \otimes \widehat{\Phi}_{\boldsymbol{k}}(t,s)\big) U(t,s)u \otimes \psi(f_{(s,t)}) \big\rangle; \tag{5.33}$$

9. *If* $f(x) = g(x)$ *for all* $x \in (s,t)$*, then* $\mathscr{G}^f_{\boldsymbol{k}}(s,t) = \mathscr{G}^g_{\boldsymbol{k}}(s,t)$*;*

10. $\mathscr{G}^f_{\boldsymbol{k}}(s,t)$ *is well defined for all* $f \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R};\mathscr{Z})$*;*

11. *If* $f_s(x) = f(x+s)$*,* $\boldsymbol{k}_s(x) = \boldsymbol{k}(x+s)$*,* $\boldsymbol{h}_s(x) = \boldsymbol{h}(x+s)$*; for all* $x \in (0,t)$*, with* $s, t \ge 0$*, then* $\mathscr{G}^f_{\boldsymbol{k}}(s, s+t) = \mathscr{G}^{f_s}_{\boldsymbol{k}_s}(0,t)\Big|_{\boldsymbol{h}\to\boldsymbol{h}_s}$*.*

Proof.

1. Immediate from the fact that $\widehat{\Phi}_0(t) = \mathbb{1}$.

2. Immediate from the fact that $\widehat{\Phi}_{\boldsymbol{k}}(s,s) = \mathbb{1}$.

3. By using $\widehat{\Phi}_{\boldsymbol{k}^i - \boldsymbol{k}^j}(t,s) = \widehat{\Phi}_{-\boldsymbol{k}^i}(t,s)^* \widehat{\Phi}_{-\boldsymbol{k}^j}(t,s)$ and the definition of $\mathscr{G}^f_{\boldsymbol{k}}$, one gets immediately
$$\sum_{i,j=1}^n \big\langle \phi_i \big| \mathscr{G}^f_{\boldsymbol{k}^i - \boldsymbol{k}^j}(s,t)[X_i^* X_j] \phi_j \big\rangle = \left\| \sum_{j=1}^n X_j \otimes \widehat{\Phi}_{-\boldsymbol{k}^j}(t,s) U(t,s) \phi_j \otimes \psi(f) \right\|^2 .$$

4. One has $\left\| \mathscr{G}^f_{\boldsymbol{k}}(s,t)[X] \right\| \le \left\| X \otimes \widehat{\Phi}_{\boldsymbol{k}}(t,s) \right\| = \|X\|$; the first step is proved exactly as point 1 in Remark 4.1, while the second step is due to the unitarity of the characteristic operator. Then, the statement follows.

5. For $u, v \in \mathscr{H}$, we can write $\langle v | \mathscr{G}^f_{\boldsymbol{k}}(s,t)[X]u \rangle = \mathrm{Tr}_{\mathscr{H}}\big[\mathscr{G}^f_{\boldsymbol{k}}(s,t)[X] |u\rangle\langle v|\big]$.

Let us take $\tau = \sum_n |u_n\rangle\langle v_n| \in \mathscr{T}(\mathscr{H})$. First we have

$$\begin{aligned}
&\mathrm{Tr}_{\mathscr{H}}[\mathscr{G}_{\boldsymbol{k}}^f(s,t)[X]\tau] \\
&= \sum_n \big\langle v_n \otimes \psi(f) \big| U(t,s)^*(X \otimes \widehat{\Phi}_{\boldsymbol{k}}(s,t))U(t,s)u_n \otimes \psi(f)\big\rangle \\
&= \sum_n \mathrm{Tr}_{\mathscr{H}\otimes\mathscr{F}}\big[U(t,s)^*(X \otimes \widehat{\Phi}_{\boldsymbol{k}}(s,t))U(t,s)|u_n \otimes \psi(f)\rangle\langle v_n \otimes \psi(f)|\big] \\
&= \mathrm{Tr}_{\mathscr{H}\otimes\mathscr{F}}\big[(X \otimes \mathbb{1})(\mathbb{1} \otimes \widehat{\Phi}_{\boldsymbol{k}}(s,t))U(t,s)\tau \otimes |\psi(f)\rangle\langle\psi(f)|U(t,s)^*\big]
\end{aligned}$$

(by the definition of partial trace)

$$= \mathrm{Tr}_{\mathscr{H}}\Big[X\mathrm{Tr}_{\mathscr{F}}\big[(\mathbb{1} \otimes \widehat{\Phi}_{\boldsymbol{k}}(s,t))U(t,s)\tau \otimes |\psi(f)\rangle\langle\psi(f)|U(t,s)^*\big]\Big];$$

then

$$\begin{aligned}
&\mathrm{Tr}_{\mathscr{H}}\,\big[\mathscr{G}_{\boldsymbol{k}}^f(s,t)[X]\tau\big] \\
&\quad = \mathrm{Tr}_{\mathscr{H}}\Big[X\,\mathrm{Tr}_{\mathscr{F}}\,\big[(\mathbb{1} \otimes \widehat{\Phi}_{\boldsymbol{k}}(s,t))U(t,s)\tau \otimes |\psi(f)\rangle\langle\psi(f)|U(t,s)^*\big]\Big].
\end{aligned}$$

Then, we have

$$\mathscr{G}_{\boldsymbol{k}}^f(s,t)_*[\tau] = \mathrm{Tr}_{\mathscr{F}}\big[(\mathbb{1} \otimes \widehat{\Phi}_{\boldsymbol{k}}(s,t))U(t,s)\tau \otimes |\psi(f)\rangle\langle\psi(f)|U(t,s)^*\big].$$

The existence of the pre-adjoint of $\mathscr{G}_{\boldsymbol{k}}^f(s,t)$ implies its σ-weak continuity (Corollary of Theorem 1.13.2 in [15], page 29).

6. By point 4, $\mathscr{G}_{\boldsymbol{k}}^f(s,t)$ is bounded uniformly in s and t. Then, the proof of the present statement is a straightforward modification of the proof of point 6 of Theorem 4.1. One has to use the strong continuity in s and t and the unitarity of $U(t,s)$ and $\widehat{\Phi}_{\boldsymbol{k}}(t,s)$.

7. By point 1 in Theorem 5.2, $\widehat{\Phi}_{\boldsymbol{k}}(t,s)$ is adapted to $\mathscr{F}_{(s,t)}$ i.e. $\widehat{\Phi}_{\boldsymbol{k}}(t,s)\psi(f) = \psi(f_{(0,s)}) \otimes \widehat{\Phi}_{\boldsymbol{k}}(t,s)\psi(f_{(s,t)}) \otimes \psi(f_{(t})$. Then, the proof is similar to the one of point 8 in Theorem 4.1. We have only to take into account that the identity on Fock space has to be substituted by $\widehat{\Phi}_{\boldsymbol{k}}(t,s)$ and that one has to use the identifications

$$\widehat{\Phi}_{\boldsymbol{k}}(s,r) \simeq \mathbb{1}_{(0,r)} \otimes \widehat{\Phi}_{\boldsymbol{k}}(s,r) \otimes \mathbb{1}_{(s,+\infty)}\,,$$
$$\mathbb{1}_{(0,r)} \otimes \widehat{\Phi}_{\boldsymbol{k}}(s,r) \otimes \widehat{\Phi}_{\boldsymbol{k}}(t,s) \otimes \mathbb{1}_{(t,+\infty)} \simeq \widehat{\Phi}_{\boldsymbol{k}}(t,r)\,.$$

8. Immediate by using the definition and the identifications introduced in the proof of the previous point.

9. Immediate by the previous point.

10. Immediate by the two previous points.

11. By Definition 3.3, Definition 5.1 and Remark 5.1 item 3 we have

$$
\begin{aligned}
&\left\langle v \middle| \mathscr{G}^f_{\boldsymbol{k}}(s,s+t)[X]u\right\rangle \\
&= \left\langle U(s+t,s)v\otimes\psi(f)\middle|\left(X\otimes\widehat{\Phi}_{\boldsymbol{k}}(s,s+t)\right)U(s+t,s)u\otimes\psi(f)\right\rangle \\
&= \left\langle \Theta^*_s U(t)\Theta_s v\otimes\psi(f)\middle|\left(X\otimes\widehat{\Phi}_{\boldsymbol{k}}(s,s+t)\right)\Theta^*_s U(t)\Theta_s u\otimes\psi(f)\right\rangle \\
&= \left\langle U(t)v\otimes\psi(f_s)\middle|\Theta_s\left(X\otimes\widehat{\Phi}_{\boldsymbol{k}}(s,s+t)\right)\Theta^*_s U(t)u\otimes\psi(f_s)\right\rangle \\
&= \left\langle U(t,0)v\otimes\psi(f_s)\middle|\left(X\otimes\widehat{\Phi}_{\boldsymbol{k}_s}(t,0)\Big|_{\boldsymbol{h}\to\boldsymbol{h}_s}\right)U(t,0)u\otimes\psi(f_s)\right\rangle.
\end{aligned}
$$

□

A semigroup associated to $\mathscr{G}$

Let us define the family $\left\{\mathscr{G}^\lambda_{\boldsymbol{\kappa},\boldsymbol{\ell}}(t),\ t\geq 0\right\}$ of bounded linear maps on $\mathscr{B}(\mathscr{H})$, for $\lambda\in\mathscr{Z}$, $\boldsymbol{\kappa}=(\kappa_1,\dots,\kappa_m)\in\mathbb{R}^m$ and $\boldsymbol{\ell}=(\ell^1,\dots,\ell^m)\in\mathscr{Z}^m$ fixed, by

$$
\begin{aligned}
\mathscr{G}^\lambda_{\boldsymbol{\kappa},\boldsymbol{\ell}}(t) &= \mathscr{G}^f_{\boldsymbol{k}}(0,t)\Big|_{\boldsymbol{h}(\bullet)=\boldsymbol{\ell}} \\
&\text{with}\ \ f(x)=\lambda \text{ and } \boldsymbol{k}(x)=\boldsymbol{\kappa}\,.
\end{aligned}
\tag{5.34}
$$

By point 10 in Theorem 5.3 $\mathscr{G}^\lambda_{\boldsymbol{\kappa},\boldsymbol{\ell}}(t)$ is well defined because a constant f is in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R};\mathscr{Z})$.

Proposition 5.4. *In the hypotheses above, $\left\{\mathscr{G}^\lambda_{\boldsymbol{\kappa},\boldsymbol{\ell}}(t),\ t\geq 0\right\}$ is a one-parameter σ-weakly continuous semigroup of bounded linear operators on $\mathscr{B}(\mathscr{H})$. Moreover, $\mathscr{G}^\lambda_{\boldsymbol{0},\boldsymbol{\ell}}(t)=\mathcal{T}^\lambda(t)$, $\mathscr{G}^\lambda_{\boldsymbol{\kappa},\boldsymbol{\ell}}(0)=\mathbb{1}$, $\left\|\mathscr{G}^\lambda_{\boldsymbol{\kappa},\boldsymbol{\ell}}(t)\right\|\leq 1$, $\mathscr{G}^\lambda_{\boldsymbol{\kappa},\boldsymbol{\ell}}(t)$ has a pre-adjoint acting on the trace class on $\mathscr{H}$, $\mathscr{G}^\lambda_{\bullet,\boldsymbol{\ell}}(t)$ is completely positive definite.*

Proof. All the functions are constants and from point 11 of Theorem 5.3 we get $\mathscr{G}^f_{\boldsymbol{k}}(0,t) = \mathscr{G}^f_{\boldsymbol{k}}(s,t+s)$. By point 7 in Theorem 5.3 we obtain

$$\mathscr{G}^\lambda_{\boldsymbol{\kappa},\boldsymbol{\ell}}(t+s) = \mathscr{G}^f_{\boldsymbol{k}}(0,t+s) = \mathscr{G}^f_{\boldsymbol{k}}(0,s)\mathscr{G}^f_{\boldsymbol{k}}(s,t+s)$$
$$= \mathscr{G}^f_{\boldsymbol{k}}(0,s)\mathscr{G}^f_{\boldsymbol{k}}(0,t) = \mathscr{G}^\lambda_{\boldsymbol{\kappa},\boldsymbol{\ell}}(s)\mathscr{G}^\lambda_{\boldsymbol{\kappa},\boldsymbol{\ell}}(t).$$

All the other statements follow from the properties 1-6 of Theorem 5.3. □

5.4.2 The infinitesimal generator of $\mathscr{G}^\lambda_{\boldsymbol{\kappa},\boldsymbol{\ell}}(t)$

Up to now, we have only made use of the cocycle properties of $U(t)$, but we are interested in finding the infinitesimal generator of the semigroup $\mathscr{G}^\lambda_{\boldsymbol{\kappa},\boldsymbol{\ell}}$ and for that we need also the QSDE for $U(t)$. The semigroup $\mathscr{G}^\lambda_{\boldsymbol{\kappa},\boldsymbol{\ell}}$ is the product of three terms: the operators $X \otimes \widehat{\Phi}_{\boldsymbol{k}}(t)$, $U(t)$ and $U^*(t)$. To compute the differential of this product we have to use two times the second fundamental formula of quantum stochastic calculus (Proposition 2.2).

Let $U(t)$ be the unitary solution of the right H-P equation (2.23), whose matrix of coefficients $F = (F_{ij})$ satisfies Hypotheses 3.4 and 4.1. Moreover, we take $\boldsymbol{k} = 1_{(0,T)}\boldsymbol{\kappa}$ and $\boldsymbol{h} = 1_{(0,T)}\boldsymbol{\ell}$; then, by Theorem 5.2 $\widehat{\Phi}_{\boldsymbol{k}}(t)$ satisfies both the left and the right H-P equations with

$$G_{00} = -\frac{1}{2}\sum_{\alpha,\beta=1}^m \kappa_\alpha \langle \ell^\alpha | \ell^\beta \rangle \kappa_\beta, \qquad G_{i0} = \mathrm{i}\sum_{\alpha=1}^m \kappa_\alpha \langle z_i | \ell^\alpha \rangle = -\overline{G_{0i}}, \quad \text{for } \ i \geq 1$$
$$G_{ij} = \sum_{\alpha=1}^m \left(\mathrm{e}^{\mathrm{i}\kappa_\alpha} - 1\right) \langle z_i | P_\alpha z_j \rangle, \quad \text{for } \ i,j \geq 1.$$

Our first step will be to differentiate the quantity

$$\mathcal{P}^{\boldsymbol{\ell}}_{\boldsymbol{\kappa}}[X](t) := \big(X \otimes \widehat{\Phi}_{\boldsymbol{k}}(t)\big)U(t), \qquad X \in \mathscr{B}(\mathscr{H}). \tag{5.35}$$

Lemma 5.5. *In the hypotheses above, $\mathcal{P}^{\boldsymbol{\ell}}_{\boldsymbol{\kappa}}[X](t)$ can be expressed as the quantum stochastic integral on $D\underline{\otimes}\mathcal{E}$*

$$\mathcal{P}^{\boldsymbol{\ell}}_{\boldsymbol{\kappa}}[X](t) = X \otimes \mathbb{1} + \sum_{i,j\geq 0}\int_0^t \big(X \otimes \widehat{\Phi}_{\boldsymbol{k}}(s)\big) M_{ij} U(s) \mathrm{d}\Lambda_{ij}(s), \tag{5.36}$$

where

$$M_{00} = K + \mathrm{i}\sum_{\alpha=1}^{m}\sum_{r\geq 1}\kappa_\alpha\langle \ell^\alpha|z_r\rangle R_r - \frac{1}{2}\sum_{\alpha,\beta=1}^{m}\kappa_\alpha\langle\ell^\alpha|\ell^\beta\rangle\kappa_\beta\,, \tag{5.37a}$$

$$M_{0j} = N_j + \mathrm{i}\sum_{\alpha=1}^{m}\sum_{r\geq 1}\kappa_\alpha\langle\ell^\alpha|z_r\rangle S_{rj}\,, \qquad j\geq 1\,, \tag{5.37b}$$

$$M_{i0} = \sum_{r\geq 1}\Big\langle z_i\Big|\prod_{\alpha=1}^{m}\mathrm{e}^{\mathrm{i}\kappa_\alpha P_\alpha}\, z_r\Big\rangle R_r + \mathrm{i}\sum_{\alpha=1}^{m}\kappa_\alpha\langle z_i|\ell^\alpha\rangle, \qquad i\geq 1\,, \tag{5.37c}$$

$$M_{ij} = \sum_{r\geq 1}\Big\langle z_i\Big|\prod_{\alpha=1}^{m}\mathrm{e}^{\mathrm{i}\kappa_\alpha P_\alpha}\, z_r\Big\rangle S_{rj} - \delta_{ij}, \qquad i,j\geq 1\,. \tag{5.37d}$$

Moreover, $\forall f,g\in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$ *and* $\forall u,v\in D$*, one has*

$$\begin{aligned}\langle U(t)v\otimes\psi(g)|\mathcal{P}^{\boldsymbol{\ell}}_{\boldsymbol{\kappa}}[X](t)u\otimes\psi(f)\rangle &= \langle v|Xu\rangle\langle\psi(g)|\psi(f)\rangle\\ +\sum_{i,j\geq 0}\int_0^t \mathrm{d}s\,\overline{g_i(s)}\Big\{&\big\langle U(s)v\otimes\psi(g)\big|\big(X\otimes\widehat{\Phi}_{\boldsymbol{k}}(s)\big)M_{ij}U(s)u\otimes\psi(f)\big\rangle\\ &+\langle F_{ji}U(s)v\otimes\psi(g)|\mathcal{P}^{\boldsymbol{\ell}}_{\boldsymbol{\kappa}}[X](s)u\otimes\psi(f)\rangle\\ +\sum_{l\geq 1}\big\langle F_{li}U(s)v\otimes\psi(g)\big|&\big(X\otimes\widehat{\Phi}_{\boldsymbol{k}}(s)\big)M_{lj}U(s)u\otimes\psi(f)\big\rangle\Big\}f_j(s).\end{aligned} \tag{5.38}$$

Let us recall the convention $f_0(s)=g_0(s)=1$.

Proof. By Proposition 2.2 and the fact that $\widehat{\Phi}_{\boldsymbol{k}}(t)^* = \widehat{\Phi}_{-\boldsymbol{k}}(t)$, we get for $f,g\in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$ and $u,v\in D$

$$\begin{aligned}\langle v\otimes\psi(g)|\mathcal{P}^{\boldsymbol{\ell}}_{\boldsymbol{\kappa}}[X](t)u\otimes\psi(f)\rangle &- \langle v|Xu\rangle\langle\psi(g)|\psi(f)\rangle\\ &= \big\langle X^*v\otimes\widehat{\Phi}_{\boldsymbol{k}}(t)^*\psi(g)\big|U(t)u\otimes\psi(f)\big\rangle - \langle v|Xu\rangle\langle\psi(g)|\psi(f)\rangle\\ &= \sum_{i,j\geq 0}\int_0^t \mathrm{d}s\,\overline{g_i(s)}\langle X^*v\otimes\widehat{\Phi}_{\boldsymbol{k}}(t)^*\psi(g)|M_{ij}U(s)u\otimes\psi(f)\rangle f_j(s),\end{aligned} \tag{5.39}$$

where

$$M_{ij} := F_{ij} + G_{ij}\mathbb{1} + \sum_{r\geq 1}G_{ir}F_{rj}\,. \tag{5.40}$$

By inserting the explicit expressions of the elements of the matrices F and G into Eq. (5.40) we get Eqs. (5.37).

Being the processes $F_{ij}U(s)$ stochastically integrable by hypothesis, then it is easy to check that also the processes $\big(X\otimes\widehat{\Phi}_{\boldsymbol{k}}(s)\big)M_{ij}U(s)$ are stochastically integrable in the sense of Definition 2.2. Now Eq. (5.36) follows from Eq. (5.39) and the first fundamental formula of quantum stochastic calculus (Proposition 2.1).

By the second fundamental formula of quantum stochastic calculus (Proposition 2.2) we get immediately Eq. (5.38). □

Definition 5.2. We define $\mathscr{K}^\lambda_{\boldsymbol{\kappa},\boldsymbol{\ell}}$ to be the quadratic form:
$\forall u,v\in D$, $\forall X\in\mathscr{B}(\mathscr{H})$,

$$\begin{aligned}\big\langle v\big|\mathscr{K}^\lambda_{\boldsymbol{\kappa},\boldsymbol{\ell}}[X]u\big\rangle &= \big\langle v\big|\mathscr{L}^\lambda[X]u\big\rangle - \frac{1}{2}\sum_{\alpha,\beta=1}^m \kappa_\alpha\langle\ell^\alpha|\ell^\beta\rangle\kappa_\beta\langle v|Xu\rangle \\ &\quad + \mathrm{i}\sum_{\alpha=1}^m \kappa_\alpha\big(\langle v|XL^\alpha u\rangle + \langle L^\alpha v|Xu\rangle\big) \\ &\quad + \sum_{l,r\geq 1}\sum_{\alpha=1}^m \big(\mathrm{e}^{\mathrm{i}\kappa_\alpha}-1\big)\,\langle z_r|P_\alpha z_l\rangle\big\langle (R_r+\zeta_r)\,v\big|X\,(R_l+\zeta_l)\,u\big\rangle, \end{aligned} \tag{5.41}$$

where $\mathscr{L}^\lambda$ is given by Eq. (4.20), Proposition 4.9 and Theorem 4.10, and

$$\zeta_r = \sum_{l\geq 1} S_{rl}\lambda_l\,, \qquad L^\alpha = \sum_{r\geq 1}\langle\ell^\alpha|z_r\rangle\,(R_r+\zeta_r)\,. \tag{5.42}$$

Theorem 5.6. *Under Hypotheses 3.4 and 4.1, the semigroup defined by Eq.* (5.34) *satisfies the equation:* $\forall u,v\in D$, $\forall X\in\mathscr{B}(\mathscr{H})$,

$$\big\langle v\big|\mathscr{G}^\lambda_{\boldsymbol{\kappa},\boldsymbol{\ell}}(t)[X]u\big\rangle = \langle v|Xu\rangle + \int_0^t \big\langle v\big|\mathscr{K}^\lambda_{\boldsymbol{\kappa},\boldsymbol{\ell}}\big[\mathscr{G}^\lambda_{\boldsymbol{\kappa},\boldsymbol{\ell}}(s)[X]\big]u\big\rangle \mathrm{d}s\,. \tag{5.43}$$

Proof. From Lemma 5.5 with $f(x)=g(x)=1_{[0,T]}(x)\lambda$ we obtain

$$\begin{aligned}\lim_{t\downarrow 0}&\frac{\langle v|\mathscr{G}^\lambda_{\boldsymbol{\kappa},\boldsymbol{\ell}}(t)[X]u\rangle - \langle v|Xu\rangle}{t} \\ &= \sum_{i,j\geq 0}\overline{\lambda_i}\Big\{\langle v|XM_{ij}u\rangle + \langle F_{ji}v|Xu\rangle + \sum_{l\geq 1}\big\langle F_{li}v|XM_{lj}u\big\rangle\Big\}\lambda_j\end{aligned}$$

and by inserting the explicit expressions of the coefficients we get

$$\lim_{t\downarrow 0}\frac{\langle v|\mathscr{G}^\lambda_{\boldsymbol{\kappa},\boldsymbol{\ell}}(t)[X]u\rangle - \langle v|Xu\rangle}{t} = \langle v|\mathscr{K}^\lambda_{\boldsymbol{\kappa},\boldsymbol{\ell}}[X]u\rangle.$$

By this and the semigroup property we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\langle v\middle|\mathscr{G}^{\lambda}_{\boldsymbol{\kappa},\boldsymbol{\ell}}(s)[X]u\right\rangle = \left\langle v\middle|\mathscr{K}^{\lambda}_{\boldsymbol{\kappa},\boldsymbol{\ell}}\left[\mathscr{G}^{\lambda}_{\boldsymbol{\kappa},\boldsymbol{\ell}}(s)[X]\right]u\right\rangle$$

and, then, we have the statement of the theorem. □

5.4.3 Approximation of a generic reduced characteristic operator by a time-ordered product of semigroups

Similarly to what has been done in Section 4.2.3, we want to approximate a generic characteristic operator $\mathscr{G}^{f}_{\boldsymbol{k}}(t)$ by a time-ordered product of terms like $\mathscr{G}^{\lambda}_{\boldsymbol{\kappa},\boldsymbol{\ell}}(t)$.

Let us consider the sequence of times $0 = t_0 < t_1 < \cdots < t_n = T$ and the step functions

$$\begin{aligned} f(x) &= \textstyle\sum_{i=1}^{n} 1_{[t_{i-1},t_i)}(x)\lambda_i\,, \qquad \lambda_i \in \mathscr{Z}\,, \\ \boldsymbol{k}(x) &= \textstyle\sum_{i=1}^{n} 1_{[t_{i-1},t_i)}(x)\boldsymbol{\kappa}_i\,, \qquad \boldsymbol{\kappa}_i \in \mathbb{R}^m\,, \\ \boldsymbol{h}(x) &= \textstyle\sum_{i=1}^{n} 1_{[t_{i-1},t_i)}(x)\boldsymbol{\ell}_i\,, \qquad \boldsymbol{\ell}_i \in \mathscr{Z}^m\,. \end{aligned} \tag{5.44}$$

By point 11 of Theorem 5.3 we have $\mathscr{G}^{f}_{\boldsymbol{k}}(s,t) = \mathscr{G}^{\lambda_i}_{\boldsymbol{\kappa}_i,\boldsymbol{\ell}_i}(t-s)$ when $t_{i-1} \leq s \leq t \leq t_i$. From the composition property (point 9, Theorem 5.3) we get for $t \in [t_{i-1}, t_i)$

$$\mathscr{G}^{f}_{\boldsymbol{k}}(0,t) = \mathscr{G}^{\lambda_1}_{\boldsymbol{\kappa}_1,\boldsymbol{\ell}_1}(t_1 - t_0)\mathscr{G}^{\lambda_2}_{\boldsymbol{\kappa}_2,\boldsymbol{\ell}_2}(t_2 - t_1)\cdots\mathscr{G}^{\lambda_i}_{\boldsymbol{\kappa}_i,\boldsymbol{\ell}_i}(t - t_{i-1}). \tag{5.45}$$

By using step functions we can approximate any f, $\boldsymbol{k}$, $\boldsymbol{h}$ and the following theorem shows that in this way it is possible to approximate $\mathscr{G}^{f}_{\boldsymbol{k}}$ by time ordered products of the type (5.45).

Theorem 5.7. *Let* $f, f^{(n)}, h^{\alpha}, h^{\alpha\,(n)} \in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$, $\alpha = 1,\ldots,m$, *such that* $\lim_{n\to\infty}\left\|f^{(n)}_{(s,t)} - f_{(s,t)}\right\|_2 = 0$, $\lim_{n\to\infty}\left\|h^{\alpha\,(n)}_{(s,t)} - h^{\alpha}_{(s,t)}\right\|_2 = 0$, $\boldsymbol{k}, \boldsymbol{k}^{(n)} \in \mathrm{L}^{\infty}(\mathbb{R}^+;\mathbb{R}^m)$ *such that* $\lim_{n\to\infty}\left\|\boldsymbol{k}^{(n)}_{(s,t)} - \boldsymbol{k}_{(s,t)}\right\|_{\infty} = 0$, $\boldsymbol{h}$ *and* $\boldsymbol{h}^{(n)}$ *satisfy the conditions 2 in Section 5.3.1. Then, we have*

$$\lim_{n\to\infty}\mathscr{G}^{f^{(n)}}_{\boldsymbol{k}^{(n)}}(s,t)\Big|_{\boldsymbol{h}\to\boldsymbol{h}^{(n)}}[X] = \mathscr{G}^{f}_{\boldsymbol{k}}(s,t)[X], \qquad \forall X \in \mathscr{B}(\mathscr{H}),$$

in the weak topology and in the strong operator topology.*

Proof. Let the functions $\boldsymbol{k}, \boldsymbol{h}$ and $\boldsymbol{k}', \boldsymbol{h}'$ satisfy the conditions needed in order to have a characteristic operator; we set $\mathscr{G}'(s,t) := \mathscr{G}_{\boldsymbol{k}'}^{f'}(s,t)\Big|_{\boldsymbol{h}\to\boldsymbol{h}'}$ and $\widehat{\Phi}'(s,t) := \widehat{\Phi}_{\boldsymbol{k}'}(s,t)\Big|_{\boldsymbol{h}\to\boldsymbol{h}'}$. Then, by point 8 of Theorem 5.3 we have

$$\begin{aligned}
&\left|\langle v|\big(\mathscr{G}'(s,t)-\mathscr{G}_{\boldsymbol{k}}^{f}(s,t)\big)[X]u\rangle\right| \\
&\qquad = \Big|\langle U(t,s)v\otimes\psi(f'_{(s,t)})\big|\big(X\otimes\widehat{\Phi}'(s,t)\big)U(t,s)u\otimes\psi(f'_{(s,t)})\rangle \\
&\qquad - \langle U(t,s)v\otimes\psi(f_{(s,t)})\big|\big(X\otimes\widehat{\Phi}_{\boldsymbol{k}}(s,t)\big)U(t,s)u\otimes\psi(f_{(s,t)})\rangle\Big| \\
&= \Big|\langle U(t,s)v\otimes\Big(\psi(f'_{(s,t)})-\psi(f_{(s,t)})\Big)\,\big|\big(X\otimes\widehat{\Phi}'(s,t)\big)U(t,s)u\otimes\psi(f'_{(s,t)})\rangle \\
&+ \langle U(t,s)v\otimes\psi(f_{(s,t)})\big|\big(X\otimes\widehat{\Phi}'(s,t)\big)U(t,s)u\otimes\Big(\psi(f'_{(s,t)})-\psi(f_{(s,t)})\Big)\,\rangle \\
&+ \Big\langle U(t,s)v\otimes\psi(f_{(s,t)})\Big|\Big(X\otimes\big(\widehat{\Phi}'(s,t)-\widehat{\Phi}_{\boldsymbol{k}}(s,t)\big)\Big)U(t,s)u\otimes\psi(f_{(s,t)})\Big\rangle\Big| \\
&\qquad \le 2\,\|v\|\,\|u\|\,\|X\|\,\Big\|\psi(f'_{(s,t)})-\psi(f_{(s,t)})\Big\| \\
&\qquad + \|v\|\,\|X\|\,\Big\|\Big(\mathbb{1}\otimes\big(\widehat{\Phi}'(s,t)-\widehat{\Phi}_{\boldsymbol{k}}(s,t)\big)\Big)U(t,s)u\otimes\psi(f_{(s,t)})\Big\rangle\Big\|.
\end{aligned} \tag{5.46}$$

The inequality is obtained as in the proof of Proposition 4.3. We have also

$$\begin{aligned}
&\left|\langle v|\big(\mathscr{G}'(s,t)-\mathscr{G}_{\boldsymbol{k}}^{f}(s,t)\big)[X]u\rangle\right| \le \|v\|\,\|u\|\,\|X\| \\
&\times\Big(2\Big\|\psi(f'_{(s,t)})-\psi(f_{(s,t)})\Big\|+\Big\|\widehat{\Phi}'(s,t)-\widehat{\Phi}_{\boldsymbol{k}}(s,t)\Big\|\Big) \le 6\,\|v\|\,\|u\|\,\|X\|\,.
\end{aligned} \tag{5.47}$$

Firstly, we have

$$\begin{aligned}
&\Big\|\psi(f'_{(s,t)})-\psi(f_{(s,t)})\Big\|^2 \\
&\qquad = 2-2\exp\left\{-\frac{1}{2}\Big\|f'_{(s,t)}-f_{(s,t)}\Big\|\right\}\cos\Big(\mathrm{Im}\langle f'_{(s,t)}|f_{(s,t)}\rangle\Big)
\end{aligned}$$

and this term goes to zero when f' tends to f.

About the second term in the inequality, we have

$$\Big\|\big(\widehat{\Phi}'(s,t)-\widehat{\Phi}_{\boldsymbol{k}}(s,t)\big)\psi(g)\Big\|^2 = 2-2\exp\left\{-\frac{1}{2}\|g_1-g_2\|\right\}\cos\left(\mathrm{Im}\langle g_2|g_1\rangle\right),$$

$$g_1(x) := 1_{(s,t)}(x)\left[P_0 g(x)+\mathrm{i}\boldsymbol{k}(x)\cdot\boldsymbol{h}(x)+\sum_{\alpha=1}^{m}\mathrm{e}^{\mathrm{i}k_\alpha(x)}P_\alpha g(x)\right],$$

$$g_2(x) := 1_{(s,t)}(x)\left[P_0 g(x) + \mathrm{i}\boldsymbol{k}'(x)\cdot\boldsymbol{h}'(x) + \sum_{\alpha=1}^{m} \mathrm{e}^{\mathrm{i}k'_\alpha(x)} P_\alpha g(x)\right],$$

$$P_0 := \mathbb{1} - \sum_{\alpha=1}^{m} P_\alpha\,.$$

By standard estimates we have also

$$\begin{aligned}\|g_1 - g_2\|^2 \le 3\left\|\boldsymbol{k}_{(s,t)} - \boldsymbol{k}'_{(s,t)}\right\|_\infty^2 \left\|\sum_\alpha h^\alpha_{(s,t)}\right\|^2 \\ + 3\left\|\boldsymbol{k}'_{(s,t)}\right\|_\infty^2 \left\|\sum_\alpha \left(h^\alpha_{(s,t)} - h^{\alpha\prime}_{(s,t)}\right)\right\|^2 \\ + 3\sum_\alpha \left\|\left(\mathrm{e}^{\mathrm{i}(k_\alpha - k'_\alpha)} - 1\right)_{(s,t)}\right\|_\infty^2 \left\|P_\alpha g_{(s,t)}\right\|^2,\end{aligned}$$

$$\begin{aligned}\mathrm{Im}\langle g_2|g_1\rangle = \sum_{\alpha,\beta}\int_s^t k_\alpha(x)k'_\beta(x)\,\mathrm{Im}\langle h^{\beta\prime}(x)|h^\alpha(x)\rangle\mathrm{d}x \\ + \sum_\alpha \int_s^t \sin\left(k_\alpha(x) - k'_\alpha(x)\right)\langle g(x)|P_\alpha g(x)\rangle\mathrm{d}x\,.\end{aligned}$$

Due to the fact that $\mathrm{Im}\langle h^\alpha(x)|h^\beta(x)\rangle = \mathrm{Im}\langle h^{\alpha\prime}(x)|h^{\beta\prime}(x)\rangle = 0$, we have

$$\begin{aligned}\left|\sum_{\alpha\beta}\int_s^t k_\alpha(x)k'_\beta(x)\,\mathrm{Im}\langle h^{\beta\prime}(x)|h^\alpha(x)\rangle\mathrm{d}x\right| &= \left|\mathrm{Im}\langle \boldsymbol{k}'\cdot\boldsymbol{h}'|\boldsymbol{k}\cdot\boldsymbol{h}\rangle_{(s,t)}\right| \\ &= \left|\mathrm{Im}\langle \boldsymbol{k}'\cdot\left(\boldsymbol{h}'-\boldsymbol{h}\right)|\boldsymbol{k}\cdot\boldsymbol{h}\rangle_{(s,t)}\right| \\ &\le \left\|(\boldsymbol{k}\cdot\boldsymbol{h})_{(s,t)}\right\| \sum_\alpha \left\|(k_\alpha)_{(s,t)}\right\|_\infty \left\|(h'_\alpha - h_\alpha)_{(s,t)}\right\|.\end{aligned}$$

Moreover, we have

$$\left|\sum_\alpha \int_s^t \sin\left(k_\alpha(x) - k'_\alpha(x)\right)\langle g(x)|P_\alpha g(x)\rangle\mathrm{d}x\right| \le \|\boldsymbol{k} - \boldsymbol{k}'\|_\infty \|g\|^2.$$

From all these inequalities one deduces that, when the primed quantities go to the unprimed ones, $\forall g \in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$ one has $\left\|\left(\widehat{\Phi}'(s,t) - \widehat{\Phi}_{\boldsymbol{k}}(s,t)\right)\psi(g)\right\|^2 \to$

0. By the fact that the exponential vectors are total, this implies the strong convergence of $\big(\widehat{\Phi}'(s,t) - \widehat{\Phi}_{\boldsymbol{k}}(s,t)\big)$ to zero. Therefore, from the inequality (5.46) we get

$$\lim_{n\to\infty} \mathscr{G}^{f^{(n)}}_{\boldsymbol{k}^{(n)}}(s,t)\Big|_{\boldsymbol{h}\to\boldsymbol{h}^{(n)}}[X] = \mathscr{G}^{f}_{\boldsymbol{k}}(s,t)[X]$$

in the strong operator topology. By the bound (5.47) and Proposition B.4 we get also the weak* convergence. □

5.4.4 The evolution equation for the reduced characteristic operator

Theorem 5.8. *Under Hypotheses 3.4 and 4.1 and $f \in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$, $\boldsymbol{k} \in \mathrm{L}^\infty(\mathbb{R}^+;\mathbb{R}^m)$, $\boldsymbol{h}$ as in point 2 of Section 5.3.1, the reduced characteristic operator introduced in Definition 5.1 satisfies the evolution equation: $\forall u, v \in D$, $\forall X \in \mathscr{B}(\mathscr{H})$,*

$$\big\langle v\big|\mathscr{G}^f_{\boldsymbol{k}}(s,t)[X]u\big\rangle = \langle v|Xu\rangle + \int_s^t \big\langle v\big|\mathscr{K}^f_{\boldsymbol{k},\boldsymbol{h}}(r)\big[\mathscr{G}^f_{\boldsymbol{k}}(r,t)[X]\big]u\big\rangle \mathrm{d}r\,, \tag{5.48}$$

where

$$\mathscr{K}^f_{\boldsymbol{k},\boldsymbol{h}}(r) := \mathscr{K}^{f(r)}_{\boldsymbol{k}(r),\boldsymbol{h}(r)}\,. \tag{5.49}$$

Proof. The proof of the this theorem will follow the same strategy of the Theorem 4.11.

Let us consider the sequence of times $0 = t_0 < t_1 < \cdots < t_n = T$ and the step functions of equations (5.44). Let us take now s and s_1 such that $0 \le s < s_1 \le t$ and $t_i \le s < s_1 \le t_{i+1}$ for some i. Then

$$\begin{aligned}\big\langle v\big|\mathscr{G}^f_{\boldsymbol{k}}(s,t)[X]u\big\rangle &= \big\langle v\big|\mathscr{G}^f_{\boldsymbol{k}}(s,s_1)\mathscr{G}^f_{\boldsymbol{k}}(s_1,t)[X]u\big\rangle \\ &= \big\langle v\big|\mathscr{G}^{\lambda_i}_{\boldsymbol{\kappa}_i,\boldsymbol{\ell}_i}(s_1-s)\mathscr{G}^f_{\boldsymbol{k}}(s_1,t)[X]u\big\rangle.\end{aligned}$$

To abbreviate the writing let us set $\mathscr{G}_1 = \mathscr{G}^{\lambda_i}_{\boldsymbol{\kappa}_i,\boldsymbol{\ell}_i}$ and $\mathscr{K}_1 = \mathscr{K}^{\lambda_i}_{\boldsymbol{\kappa}_i,\boldsymbol{\ell}_i}$. Now, from here and from Eq. (5.43) we get

$$\begin{aligned}&\big\langle v\big|\mathscr{G}^f_{\boldsymbol{k}}(s,t)[X]u\big\rangle \\ &\quad= \big\langle v\big|\mathscr{G}^f_{\boldsymbol{k}}(s_1,t)[X]u\big\rangle + \int_0^{s_1-s} \mathrm{d}r\big\langle v\big|\mathscr{K}_1\big[\mathscr{G}_1(r)\mathscr{G}^f_{\boldsymbol{k}}(s_1,t)[X]\big]u\big\rangle, \\ &\quad= \big\langle v\big|\mathscr{G}^f_{\boldsymbol{k}}(s_1,t)[X]u\big\rangle + \int_s^{s_1} \mathrm{d}x\big\langle v\big|\mathscr{K}_1\big[\mathscr{G}^f_{\boldsymbol{k}}(x,t)[X]\big]u\big\rangle.\end{aligned}$$

Similarly, we have

$$\big\langle v\big|\mathscr{G}_{\boldsymbol{k}}^{f}(s_1,t)[X]u\big\rangle = \big\langle v\big|\mathscr{G}_{\boldsymbol{k}}^{f}(s_2,t)[X]u\big\rangle + \int_{s_1}^{s_2} \mathrm{d}x \big\langle v\big|\mathscr{K}_2\big(\mathscr{G}_{\boldsymbol{k}}^{f}(x,t)[X]\big)u\big\rangle.$$

Let q be such that $t_{i+q-1} < t \leq t_{i+q}$ and let us set $s_0 = s$, $s_q = t$, $s_j = t_{i+j}$ for $1 \leq j \leq q-1$. By induction we arrive to

$$\begin{aligned}\big\langle v\big|\mathscr{G}_{\boldsymbol{k}}^{f}(s,t)[X]u\big\rangle &= \big\langle v\big|Xu\big\rangle + \sum_{j=1}^{q}\int_{s_{j-1}}^{s_j} \mathrm{d}x \big\langle v\big|\mathscr{K}_j\big[\mathscr{G}_{\boldsymbol{k}}^{f}(x,t)[X]\big]u\big\rangle \\ &= \big\langle v\big|Xu\big\rangle + \int_{s}^{t} \mathrm{d}x \big\langle v\big|\mathscr{K}_{\boldsymbol{k},\boldsymbol{h}}^{f}(x)\big[\mathscr{G}_{\boldsymbol{k}}^{f}(x,t)[X]\big]u\big\rangle.\end{aligned}$$

Let us stress that, being f, $\boldsymbol{k}$, $\boldsymbol{h}$ step functions, then

$$\mathscr{K}_{\boldsymbol{k},\boldsymbol{h}}^{f}(x) = \sum_{j=1}^{q} 1_{[s_{j-1},s_j)}(x)\mathscr{K}_{\boldsymbol{\kappa}_j,\boldsymbol{\ell}_j}^{\lambda_j}\,.$$

Let now $f^{(n)}$, $\boldsymbol{h}^{(n)}$ and $\boldsymbol{k}^{(n)}$ be three sequences of step functions approximating $f \in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z})$, $\boldsymbol{h} \in \mathrm{L}^2(\mathbb{R}^+;\mathscr{Z}^m)$, $\boldsymbol{k} \in \mathrm{L}^\infty(\mathbb{R}^+;\mathbb{R}^m)$ respectively. To simplify the notation we write $\mathscr{G}_{\boldsymbol{k}^{(n)}}^{f^{(n)}}(s,t)\Big|_{\boldsymbol{h}\to\boldsymbol{h}^{(n)}} = \mathscr{G}_n(s,t)$, $\mathscr{K}_{\boldsymbol{k}^{(n)},\boldsymbol{h}^{(n)}}^{f^{(n)}}(x) = \mathscr{K}_n(x)$, $\mathscr{L}^{f^{(n)}}(x) = \mathscr{L}_n(x)$.

By Theorem 5.7, $\forall X \in \mathscr{B}(\mathscr{H})$, $\lim_{n\to\infty} \mathscr{G}_n(s,t)[X] = \mathscr{G}_{\boldsymbol{k}}^{f}(s,t)[X]$ weakly* and strongly and by Theorem 5.3, point 4, $\|\mathscr{G}_n(s,t)[X]\| \leq \|X\|$.

Now, for $u, v \in D$ we have

$$\begin{aligned}&\int_s^t \mathrm{d}x \big\langle v\big|\mathscr{K}_n(x)\mathscr{G}_n(x,t)[X]u\big\rangle - \int_s^t \mathrm{d}x \big\langle v\big|\mathscr{K}_{\boldsymbol{k},\boldsymbol{h}}^{f}(x)\mathscr{G}_{\boldsymbol{k}}^{f}(x,t)[X]u\big\rangle \\ &\qquad = \int_s^t \mathrm{d}x \big\langle v\big|\mathscr{K}_{\boldsymbol{k},\boldsymbol{h}}^{f}(x)\big(\mathscr{G}_n(x,t) - \mathscr{G}_{\boldsymbol{k}}^{f}(x)\big)[X]u\big\rangle \\ &\qquad + \int_s^t \mathrm{d}x \langle v\big|\big(\mathscr{L}_n(x) - \mathscr{L}^f(x)\big)\mathscr{G}_n(x,t)[X]u\big\rangle \\ &\quad + \int_s^t \mathrm{d}x \langle v\big|\big(\mathscr{K}_n(x) - \mathscr{L}_n(x) - \mathscr{K}_{\boldsymbol{k},\boldsymbol{h}}^{f}(x) + \mathscr{L}^f(x)\big)\mathscr{G}_n(x,t)[X]u\big\rangle.\end{aligned}$$

The first term goes to zero by Theorem 5.7. The proof that the second term goes to zero is similar to the one of Theorem 4.11; one has to use Theorem

5.3, point 4. It remains to check the last term, which is composed by various sub-terms, for which the same techniques apply. Therefore we analyze only the most interesting terms.

About the first term we have

$$\begin{aligned}\Bigg| \int_s^t \mathrm{d}x \sum_{\alpha,\beta=1}^m k_\alpha^{(n)}(x)\big\langle h^{(n)\alpha}(x)\big|h^{(n)\beta}(x)\big\rangle k_\beta^{(n)}(x)\langle v|\mathscr{G}_n(x,t)[X]u\rangle \\ - \int_s^t \mathrm{d}x \sum_{\alpha,\beta=1}^m k_\alpha(x)\big\langle h^\alpha(x)\big|h^\beta(x)\big\rangle k_\beta(x)\big\langle v|\mathscr{G}_n(x,t)[X]u\big\rangle\Bigg| \\ \leq \int_s^t \mathrm{d}x \left\|\boldsymbol{k}^{(n)}(x)\cdot\boldsymbol{h}^{(n)}(x) - \boldsymbol{k}(x)\cdot\boldsymbol{h}(x)\right\|^2 \left|\big\langle v|\mathscr{G}_n(x,t)[X]u\big\rangle\right| \\ \leq \|v\|\,\|X\|\,\|u\| \int_s^t \mathrm{d}x \left\|\boldsymbol{k}^{(n)}(x)\cdot\boldsymbol{h}^{(n)}(x) - \boldsymbol{k}(x)\cdot\boldsymbol{h}(x)\right\|^2 \xrightarrow{n\to+\infty} 0\end{aligned}$$

by the uniform-convergence of $\boldsymbol{k}^{(n)}$ to $\boldsymbol{k}$ and L^2-convergence of $\boldsymbol{h}^{(n)}$ to $\boldsymbol{h}$. Another interesting term is

$$\sum_{\alpha=1}^m k_\alpha^n(x)\big\langle v\big|\mathscr{G}_n(x,t)[X]L_n^\alpha(x)u\big\rangle - \sum_{\alpha=1}^m k_\alpha(x)\big\langle v\big|\mathscr{G}_n(x,t)[X]L^\alpha u\big\rangle,$$

where $L_n^\alpha = \sum_{r\geq 1}\big\langle h^{n,\alpha}(x)\big|z_r\big\rangle\Big(R_r + \sum_{j\geq 1} f_j^{(n)}(x)S_{rj}\Big)$; let us consider only the contribution of the term with the sums over j. Now

$$
\begin{aligned}
&\Big| \int_s^t \mathrm{d}x \sum_{\alpha=1}^m k_\alpha^{(n)}(x) \sum_{r,j\geq 1} \big\langle h^{(n)\alpha}(x) \big| z_r \big\rangle \langle v | \mathscr{G}_n(x,t)[X] S_{rj} u \rangle f_j^{(n)}(x) \\
&\qquad - \int_s^t \mathrm{d}x \sum_{\alpha=1}^m k_\alpha(x) \sum_{r,j\geq 1} \big\langle h^\alpha(x) \big| z_r \big\rangle \langle v | \mathscr{G}_{\boldsymbol{k}}^f(x,t)[X] S_{rj} u \rangle f_j(x) \Big| \\
&\quad \leq \int_s^t \mathrm{d}x \sum_{r,j\geq 1} \Big| \big\langle \boldsymbol{k}^{(n)}(x)\cdot \boldsymbol{h}^{(n)}(x) \big| z_r \big\rangle f_j^{(n)}(x) - \big\langle \boldsymbol{k}(x)\cdot \boldsymbol{h}(x) \big| z_r \big\rangle f_j(x) \Big| \\
&\qquad\qquad\qquad \times \big| \big\langle v \big| \mathscr{G}_n(x,t)[X] S_{rj} u \big\rangle \big| \\
&\qquad\qquad\qquad \leq \sum_{r,j\geq 1} \|v\|\, \|X\|\, \|S_{rj} u\| \\
&\qquad \times \int_s^t \mathrm{d}x \Big| \big\langle \boldsymbol{k}^{(n)}(x)\cdot \boldsymbol{h}^{(n)}(x) \big| z_r \big\rangle f_j^{(n)}(x) - \big\langle \boldsymbol{k}(x)\cdot \boldsymbol{h}(x) \big| z_r \big\rangle f_j(x) \Big| \\
&\leq \sum_{r,j\geq 1} \|v\|\, \|X\|\, \|S_{rj} u\| \int_s^t \mathrm{d}x \, \Big| \big\langle \boldsymbol{k}^{(n)}(x)\cdot \boldsymbol{h}^{(n)}(x) - \boldsymbol{k}(x)\cdot \boldsymbol{h}(x) \big| z_r \big\rangle f_j^{(n)}(x) \Big| \\
&\quad + \sum_{r,j\geq 1} \|v\|\, \|X\|\, \|S_{rj} u\| \int_s^t \mathrm{d}x \, \big| \big\langle \boldsymbol{k}(x)\cdot \boldsymbol{h}(x) \big| z_r \big\rangle \big| \, \Big| \big\langle z_j \big| f^{(n)}(x) - f(x) \big\rangle \Big| .
\end{aligned}
$$

By Hölder's inequality these two terms go to zero.

All the remaining terms can be analyzed in a similar way; therefore

$$
\int_s^t \mathrm{d}x \big\langle v \big| \mathscr{K}_n(x) \mathscr{G}_n(x,t)[X] u \big\rangle \xrightarrow{n\to+\infty} \int_s^t \mathrm{d}x \big\langle v \big| \mathscr{K}_{\boldsymbol{k},\boldsymbol{h}}^f(x) \mathscr{G}_{\boldsymbol{k}}^f(x,t)[X] u \big\rangle,
$$

which gives Eq. (5.43). □

Remark 5.2. Equation (5.48) is the integral form of a backward differential equation. Indeed, at least when $f, \boldsymbol{k}$ and $\boldsymbol{h}$ are continuous function, the r.h.s. is differentiable with respect to s and we get

$$
\frac{\mathrm{d}}{\mathrm{d}s} \big\langle v \big| \mathscr{G}_{\boldsymbol{k}}^f(s,t)[X] u \big\rangle = -\big\langle v \big| \mathscr{K}_{\boldsymbol{k},\boldsymbol{h}}^f(s) \big[\mathscr{G}_{\boldsymbol{k}}^f(s,t)[X] \big] u \big\rangle , \tag{5.50}
$$

with final condition $\big\langle v \big| \mathscr{G}_{\boldsymbol{k}}^f(t,t)[X] u \big\rangle = \big\langle v \big| X u \big\rangle$.

5.5 Instruments and finite-dimensional laws

In the quantum theory of measurement an important notion is that of *instrument* and the operational approach to continual measurements, mentioned in the Introduction, is based on such a notion. Here we recall a few facts, without developing in full this side of the theory.

By using the joint pvm $\xi(\mathrm{d}\boldsymbol{x}; s, t)$ of the increments $Y_\alpha(t) - Y_\alpha(s)$, $\alpha = 1, \ldots, m$, we define the map–valued measure $\mathcal{I}^f(s, t; \cdot)$, $0 \le s < t$, $f \in \mathrm{L}^2(\mathbb{R}^+; \mathscr{Z})$, by: $\forall X \in \mathscr{B}(\mathscr{H})$, $\forall \varrho \in \mathcal{T}(\mathscr{H})$,

$$\operatorname{Tr}_{\mathscr{H}} \left\{ \varrho\, \mathcal{I}^f(s, t; A)[X] \right\} = \operatorname{Tr}_{\mathscr{H} \otimes \mathscr{F}} \left\{ X \otimes \xi(A; s, t)\, U(t, s)\, \varrho \otimes \eta(f)\, U(t, s)^* \right\}, \tag{5.51}$$

where A is a Borel set in $\mathbb{R}^d$; by the factorization properties of Γ and $\eta(f)$, only $f_{(s,t)}$, the part of f in (s, t), is relevant for the definition of $\mathcal{I}^f(s, t; A)$. The family of maps $\mathcal{I}^f(s, t; \cdot)$ is a completely positive *instrument* [5, 11], whose characterizing properties are

1. $\mathcal{I}^f(s, t; A) \in \mathscr{B}(\mathscr{H})$
2. $\operatorname{Tr} \left\{ \varrho\, \mathcal{I}^f(s, t; \mathbb{R}^m)[\mathbb{1}] \right\} = \operatorname{Tr} \{\varrho\}$, $\forall \varrho \in \mathcal{T}(\mathscr{H})$;
3. $\displaystyle\sum_{i,j=1}^{n} \left\langle \psi_i \middle| \mathcal{I}^f(s, t; A)[X_i^* X_j] \psi_j \right\rangle \ge 0$, $\forall n$, $\forall \psi_j \in \mathscr{H}$, $\forall X_j \in \mathscr{B}(\mathscr{H})$;
4. for any finite or countable (Borel) partition $A_1, A_2, \ldots$ of a Borel set A one has $\sum_j \operatorname{Tr} \left\{ \varrho\, \mathcal{I}^f(s, t; A_j)[X] \right\} = \operatorname{Tr} \left\{ \varrho\, \mathcal{I}^f(s, t; A)[X] \right\}$, $\forall \varrho \in \mathcal{T}(\mathscr{H})$, $\forall X \in \mathscr{B}(\mathscr{H})$.

Let us consider now a constant test function $\boldsymbol{k}(t) = \boldsymbol{\kappa}$; we have

$$\begin{aligned} \int_{\mathbb{R}^m} \mathrm{e}^{\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{x}} \operatorname{Tr}_{\mathscr{H}} \left\{ \varrho\, \mathcal{I}^f(s, t; \mathrm{d}\boldsymbol{x})[X] \right\} &= \operatorname{Tr}_{\mathscr{H} \otimes \mathscr{F}} \left\{ X \otimes \widehat{\Phi}_{\boldsymbol{k}}(t, s) U(t, s) \varrho \otimes \eta(f) U(t, s)^* \right\} \\ &= \operatorname{Tr}_{\mathscr{H}} \left\{ \varrho\, \mathscr{G}^f_{\boldsymbol{k}}(s, t)[X] \right\}. \end{aligned} \tag{5.52}$$

Therefore, $\mathscr{G}^f_{\boldsymbol{k}}(s, t)$ with a constant test function, is the Fourier transform of the instrument $\mathcal{I}^f(s, t; \mathrm{d}\boldsymbol{x})$ and this instrument is the anti-Fourier transform of the reduced characteristic operator $\mathscr{G}^f_{\boldsymbol{k}}(s, t)$.

Let us note that the finite dimensional laws of Section 5.3.3, with initial state $\mathfrak{s} = \rho_0 \otimes \eta(f)$ at time t_0, can be written as

$$\begin{aligned} &\mathbb{P}_{\rho_0} \left[\Delta\boldsymbol{Y}(t_0, t_1) \in A_1, \ldots, \Delta\boldsymbol{Y}(t_{n-1}, t_n) \in A_n \right] \\ &\qquad = \operatorname{Tr}_{\mathscr{H}} \left\{ \rho_0 \mathcal{I}^f(t_0, t_1; A_1) \circ \cdots \circ \mathcal{I}^f(t_{n-1}, t_n; A_n)[\mathbb{1}] \right\}. \end{aligned} \tag{5.53}$$

5.6 Field Observables

Let us finish by particularizing our characteristic operator to two cases of observables typical of what can be continually observed in quantum optics.

Counts of quanta

Let $P \in \mathcal{B}(\mathcal{Z})$ be an orthogonal projection; for any $t \geq 0$, we introduce the operator

$$N(P;t) := \lambda\left(P \otimes 1_{(0,t)}\right) = \sum_{kl} \langle z_k | P z_l \rangle \Lambda_{kl}(t)\,. \tag{5.54}$$

By propriety of the equation (2.15), this operator is essentially selfadjoint on $\mathcal{E}$ and its domain includes also the finite particle number vectors. From point 3 in Remark 2.1 we have

$$\langle e(g)|N(P;t)e(f)\rangle = \exp\left\{\langle g_{(0,t)}|(\mathbb{1}-P)f_{(0,t)}\rangle + \langle g_{(t}|f_{(t}\rangle\right\} \times \sum_{n=0}^{\infty} \frac{n}{n!}\left(\langle g_{(0,t)}|Pf_{(0,t)}\rangle\right)^n\,; \tag{5.55}$$

by taking into account the factorization (2.5), one sees that the eigenvalues of $N(P;t)$ are the integers $n = 0, 1, \ldots$ and that the eigenspace corresponding to n is the "n-particle sector of $\Gamma_{\text{symm}}\big((P\mathcal{Z}) \otimes \left(1_{(0,t)}L^2(\mathbb{R}_+)\right)\big)$" tensor $\Gamma_{\text{symm}}\big(\left(\mathbb{1} - P \otimes 1_{(0,t)}\right)\left(\mathcal{Z} \otimes L^2(\mathbb{R}_+)\right)\big)$. Therefore, we can interpret $N(P;t)$ as the number operator which counts the quanta injected in the system up to time t with state in $P\mathcal{Z}$. Another way to see that $N(P;t)$ is a number operator is to use the heuristic rules of QSC; by (5.54), point 2 in Remark 2.1 and the fact that P is a projection, we have immediately

$$\left(\mathrm{d}N(P;t)\right)^2 = \mathrm{d}N(P;t)\,, \tag{5.56}$$

which shows that an infinitesimal increment has eigenvalues 0 and 1.

By (2.16) and (5.54), we have

$$\exp\{\mathrm{i}\kappa N(P;t)\} = W\left(0; \exp\{\mathrm{i}\kappa P \otimes 1_{(0,t)}\}\right) \tag{5.57}$$

and by (2.9) one sees that the unitary groups generated by $N(P;t)$ and $N(P;s)$ commute; therefore, $\{N(P;t),\, t \geq 0\}$ is a set of jointly diagonalizable selfadjoint operators, or, in physical terms, of *compatible observables*. The same is true for

$$\left\{N(P_\alpha;t) \quad t \geq 0\,, \quad \alpha = 1, 2, \ldots\right\} \quad \text{with } P_\alpha P_\beta = \delta_{\alpha\beta}\, P_\alpha = \delta_{\alpha\beta}\, P_\alpha^*\,, \tag{5.58}$$

i.e. P_1, P_2, ... are mutually orthogonal projections.

In the case of photons the measurement of number operators can be experimentally realized through the so called *direct detection*.

Let us consider the family of compatible observables $\{N(P_\alpha;t),\ t \geq 0,\ \alpha = 1,2,\dots,m\}$; $P_1,\dots,P_d$ are mutually orthogonal projections on $\mathcal{Z}$. According to the discussion above and Proposition 5.1, we can handle the stochastic process associated to these operators by means of the finite–dimensional characteristic functions for the increments, which in turn can be summarized in a *characteristic functional*, which is suggested by the structure of equation (5.7) and which now we construct.

Let us introduce the *test functions* $\boldsymbol{k} \in L^\infty(\mathbb{R}_+;\mathbb{R}^m)$, the unitary operators $\mathsf{S}_t(\boldsymbol{k})$ on $L^2(\mathbb{R}^+;\mathscr{Z})$ by

$$\mathsf{S}_t(\boldsymbol{k}) := \exp\left[\mathrm{i}\sum_{\alpha=1}^m P_\alpha \otimes 1_{(0,t)}k_\alpha\right] \tag{5.59a}$$

or by

$$\begin{aligned}\big(\mathsf{S}_t(k)f\big)(s) &= \exp\left[\mathrm{i}1_{(0,t)}(s)\sum_{\alpha=1}^m k_\alpha(s)P_\alpha\right]f(s)\\ &\equiv 1_{(0,t)}(s)\sum_{\alpha=1}^m\left[\exp^{\mathrm{i}k_\alpha(s)}-1\right]P_\alpha f(s)+f(s). \end{aligned}\tag{5.59b}$$

Then the characteristic operator

$$\widehat{\Phi}_{\boldsymbol{k}}(t) = W\big(0;\mathsf{S}_t(\boldsymbol{k})\big)\,. \tag{5.60}$$

By (2.16) and (5.59a) the generator of the unitary group, $\kappa \mapsto \widehat{\Phi}_{\kappa\boldsymbol{k}}(t)$ is

$$\begin{aligned}\lambda\bigg(\sum_\alpha P_\alpha\otimes 1_{(0,t)}k_\alpha\bigg) &= \sum_{kl}\int_0^t \langle z_k|\sum_\alpha k_\alpha(s)P_\alpha z_l\rangle \mathrm{d}\Lambda_{kl}(s)\\ &\equiv \sum_\alpha\int_0^t k_\alpha(s)\,\mathrm{d}N(P_\alpha;s) \end{aligned}\tag{5.61}$$

and, so, we can write

$$\begin{cases}\mathrm{d}\widehat{\Phi}_{\boldsymbol{k}}(t) &= \widehat{\Phi}_{\boldsymbol{k}}(t)\displaystyle\sum_{i,j,\alpha\geq 1}\left(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_\alpha(t)}-1\right)\langle z_i|P_\alpha zj\rangle \mathrm{d}\Lambda_{ij}\\ \widehat{\Phi}_{\boldsymbol{k}}(0) &= \mathbb{1}\end{cases}$$

and

$$\widehat{\Phi}_{\boldsymbol{k}}(t) = \exp\Big[\mathrm{i}\sum_{\alpha}\int_0^t k_\alpha(s)\,\mathrm{d}N(P_\alpha;s)\Big]. \tag{5.62}$$

Measurements of field quadratures

Let us consider now the field quadratures

$$Q(h;t) := Q\left(h_{(0,t)}\right) = \sum_k \left\{\int_0^t \overline{h_k(s)}\,\mathrm{d}A_k(s) + \int_0^t h_k(s)\,\mathrm{d}A_k^\dagger(s)\right\}, \tag{5.63}$$

which are essentially selfadjoint operators on $\mathcal{E}$ (propriety of the equation (2.15)). The spectrum of $Q(h;t)$ is the whole real axis, because $(\sqrt{2}\,\|h\|)^{-1} \times Q(h;t)$ and $(\sqrt{2}\,\|h\|)^{-1}\ Q(ih;t)$ form a couple of canonically conjugated selfadjoint operators (the commutator gives i). By (2.10), (2.15), we have that

$$\big\{Q(h_\alpha;t)\,,\ t\geq 0\,,\ \alpha = 1,2,\dots\big\}\,,\quad \text{with } \langle h_\alpha(s)|h_\beta(s)\rangle = \delta_{\alpha\beta}\,\|h_\alpha(s)\|^2\,, \tag{5.64}$$

is a family of compatible observables.

In the case of photons the measurement of field quadratures can be experimentally realized through the so called *heterodyne* or *homodyne detection* schemes.

Let us consider the family of compatible observables $\{Q(h_\alpha;t),\ t\geq 0,\ \alpha = 1,2,\dots,m\}$, with $\langle h_\alpha(s)|h_\beta(s)\rangle = \delta_{\alpha\beta}\,\|h_\alpha(s)\|^2$; we can repeat the construction of the previous subsection. By (5.63) we have

$$\sum_{\alpha=1}^m \int_0^t k_\alpha(s)\,\mathrm{d}Q(h_\alpha;s) = Q\Big(\sum_\alpha k_\alpha h_\alpha;t\Big) \tag{5.65}$$

and, by taking into account (2.15), we can write the characteristic operator as an adapted Weyl operator again:

$$\begin{aligned}\widehat{\Phi}_{\boldsymbol{k}}(t) &= \exp\Big\{\mathrm{i}\sum_{\alpha=1}^m\int_0^t k_\alpha(s)\,\mathrm{d}Q(h_\alpha;s)\Big\}\\ &= \exp\Big\{\mathrm{i}Q\Big(\sum_\alpha k_\alpha h_\alpha;t\Big)\Big\} = W\Big(\mathrm{i}\sum_\alpha k_\alpha h_\alpha 1_{(0,t)};\mathbb{1}\Big)\,. \end{aligned} \tag{5.66}$$

Then, the characteristic functional of the process $\tilde{Q}_\alpha(s)$ associated with the selfadjoint operators $Q(h_\alpha;s)$ is given by (5.27) again or by

$$\Phi_{\boldsymbol{k}}(t) = \mathrm{Tr}\left\{\exp\Big[\mathrm{i}\sum_\alpha\int_0^t k_\alpha(s)\,\mathrm{d}Q^{\mathrm{out}}(h_\alpha;s)\Big]\mathfrak{s}\right\}. \tag{5.67}$$

Appendix A

Some notions of operator theory

Here we collect a few facts from operator theory and some notions on semigroups of operators.

A first useful result is the following one.

Proposition A.1 ([14] p. 112)**.** *In a Banach space every weakly convergent sequence is bounded.*

For a linear operator T we denote by $\mathrm{Dom}(T)$ its domain, by $\mathrm{Ker}(T)$ its kernel, i.e. the set of vectors $x \in \mathrm{Dom}(T)$ such that $Tx = 0$, and by $\mathrm{Ran}(T)$ its range.

A.1 Properties of single operators

A.1.1 Closed operators

Here $\mathcal{X}$ is a complex Banach space. This material is taken mainly from [14] pp. 250–252.

Let T be a linear operator in $\mathcal{X}$ with domain $\mathrm{Dom}(T)$.

Definition A.1 (Closed operator)**.** The *graph* of T is the set of pairs $\Gamma(T) = \{(x, Tx) : x \in \mathrm{Dom}(T)\} \subset \mathcal{X} \times \mathcal{X}$. The operator T is said to be *closed* if $\Gamma(T)$

is a closed subset of $\mathcal{X} \times \mathcal{X}$.

Remark A.1 ([7] Definition B.1 p. 515). The operator T is closed if and only if

$$x_n \in \mathrm{Dom}(T), \quad \lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} Tx_n = y$$
$$\Downarrow$$
$$x \in \mathrm{Dom}(T) \text{ and } Tx = y\,.$$

Definition A.2 (Extension of an operator)**.** The operator T_1 is said to be an *extension* of T if $\mathrm{Dom}(T_1) \supset \mathrm{Dom}(T)$ and $T_1 x = Tx$ for all $x \in \mathrm{Dom}(T)$. In this case we write $T_1 \supset T$.

Definition A.3 (Closable operator)**.** An operator T is *closable* if it has a closed extension. Every closable operator has a smallest closed extension, called its *closure*, which we denote by $\overline{T}$.

Proposition A.2. *If T is closable, then $\Gamma(\overline{T}) = \overline{\Gamma(T)}$.*

Definition A.4 (Core of an operator)**.** Let T be a closed linear operator in $\mathcal{X}$, $D \subset \mathrm{Dom}(T)$ be a linear manifold and $T|_D$ be the restriction of T to D. If the closure of $T|_D$ is T, then D is called a *core* for T.

Definition A.5 (Resolvent)**.** Let T be a closed operator on $\mathcal{X}$. A complex number λ is in the *resolvent set* $\rho(T)$ if $\lambda\mathbb{1} - T$ is a bijection of $\mathrm{Dom}(T)$ onto $\mathcal{X}$ with a bounded inverse. If $\lambda \in \rho(T)$, the operator $R(\lambda; A) := (\lambda\mathbb{1} - T)^{-1}$ is called the *resolvent* of T at λ.

A.1.2 The adjoint

Here $\mathscr{H}$ is a complex separable Hilbert space. This material is taken from [14] pp. 252–256.

Definition A.6 (Adjoint operator)**.** Let T be a densely defined linear operator in $\mathscr{H}$. Let $\mathrm{Dom}(T^*)$ be the set of $\phi \in \mathscr{H}$ for which there is an $\eta \in \mathscr{H}$ with $\langle\phi|T\psi\rangle = \langle\eta|\psi\rangle$ for all $\psi \in \mathrm{Dom}(T)$. For each such $\phi \in \mathrm{Dom}(T^*)$, we define $T^*\phi = \eta$. The operator T^* is called the *adjoint* of T.

Remark A.2.

1. By the Riesz lemma, $\phi \in \mathrm{Dom}(T^*)$ if and only if $|\langle\phi|T\psi\rangle| \le C\|\psi\|$ for all $\psi \in \mathrm{Dom}(T)$.
2. $S \subset T$ implies $T^* \subset S^*$.
3. T^* is closed.
4. T is closable if and only if $\mathrm{Dom}(T^*)$ is dense in which case $\overline{T} = T^{**}$.

5. If T is closable, then $\left(\overline{T}\right)^* = T^*$.

Definition A.7 (Symmetric and selfadjoint operators)**.** A densely defined operator T in $\mathscr{H}$ is called *symmetric* if $T \subset T^*$, that is, if $\mathrm{Dom}(T) \subset \mathrm{Dom}(T^*)$ and $T\phi = T^*\phi$ for all $\phi \in \mathrm{Dom}(T)$.

T is called *selfadjoint* if $T = T^*$, that is, if and only if T is symmetric and $\mathrm{Dom}(T) = \mathrm{Dom}(T^*)$.

Remark A.3.

1. A symmetric operator T is closable and $T \subset \overline{T} = T^{**} \subset T^*$.
2. For a closed symmetric operator T we have $T = T^{**} \subset T^*$.
3. For a selfadjoint operator T we have $T = T^{**} = T^*$.

Definition A.8 (Essentially selfadjoint operator)**.** A symmetric operator T is called *essentially selfadjoint* if its closure is selfadjoint.

Remark A.4. An essentially selfadjoint operator has one and only one selfadjoint extension and, conversely, an operator possessing one and only one selfadjoint extension is essentially selfadjoint. The operator T is essentially selfadjoint if and only if $T \subset T^{**} = T^*$.

A.1.3 Isometries

[14] pp. 197, 297–298.

Definition A.9. An operator $U \in \mathscr{B}(\mathscr{H})$ is called an *isometry* if $\|Ux\| = \|x\|$ for all $x \in \mathscr{H}$. U is called a *partial isometry* if U is an isometry when restricted to the closed subspace $\mathrm{Ker}(U)^\perp$.

If U is a partial isometry, $\mathscr{H}$ can be written as $\mathscr{H} = \mathrm{Ker}(U) \oplus \mathrm{Ker}(U)^\perp$ and as $\mathscr{H} = \mathrm{Ran}(U) \oplus \mathrm{Ran}(U)^\perp$; then, U is a unitary operator between $\mathrm{Ker}(U)^\perp$, the *initial subspace* of U, and $\mathrm{Ran}(U)$, the *final subspace* of U. Moreover, U^* is a partial isometry from $\mathrm{Ran}(U)$ to $\mathrm{Ker}(U)^\perp$ which acts as the inverse of the map $U : \mathrm{Ker}(U)^\perp \to \mathrm{Ran}(U)$.

Proposition A.3. *Let U be a partial isometry. Then, $P_{\mathrm{i}} = U^*U$ and $P_{\mathrm{f}} = UU^*$ are respectively the projections onto the initial and the final subspaces of U. Conversely, if $U \in \mathscr{B}(\mathscr{H})$ with U^*U and UU^* projections, then U is a partial isometry.*

Theorem A.4 (The polar decomposition)**.** *Let T be an arbitrary closed operator on a Hilbert space $\mathscr{H}$. Then, there is a positive self-adjoint operator $|T|$, with $\mathrm{Dom}(|T|) = \mathrm{Dom}(T)$ and a partial isometry U with initial space $\mathrm{Ker}(T)^\perp$, and final space $\overline{\mathrm{Ran}(T)}$, so that $T = U|T|$. $|T|$ and U are uniquely determined by these properties together with the additional condition $\mathrm{Ker}(|T|) = \mathrm{Ker}(T)$.*

A.2 Strongly continuous semigroups of operators

We take this material mainly from [13] pp. 1–16, 81–82. Here $\mathcal{X}$ is a complex Banach space.

Definition A.10. A one-parameter family $T(t)$, $0 \leq t < +\infty$, of bounded linear operators from $\mathcal{X}$ into $\mathcal{X}$ is a *semigroup of bounded linear operators on* $\mathcal{X}$ if (i) $T(0) = \mathbb{1}$, (ii) $T(t+s) = T(t)T(s)$, $\forall t, s \geq 0$.

A semigroup $T(t)$ of bounded linear operators is *uniformly continuous* if $\lim_{t \downarrow 0} \|T(t) - \mathbb{1}\| = 0$.

A semigroup $T(t)$ of bounded linear operators is *strongly continuous* if $\lim_{t \downarrow 0} T(t)x = x$, $\forall x \in \mathcal{X}$.

Remark A.5. For a semigroup, uniform or strong continuity in 0 implies the same continuity for all t. Moreover, for a semigroup strong continuity is equivalent to weak continuity ([7] Theorem 5.8).

The linear operator A defined by

$$\mathrm{Dom}(A) = \left\{ x \in \mathcal{X} : \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \quad \text{exists} \right\}$$

and

$$Ax = \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \equiv \left. \frac{\mathrm{d}^+ T(t)x}{\mathrm{d}t} \right|_{t=0} \qquad \text{for} \quad x \in \mathrm{Dom}(A)$$

is the *infinitesimal generator* of the semigroup $T(t)$; $\mathrm{Dom}(A)$ is the domain of A.

Theorem A.5. *A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator. Moreover, a uniformly continuous semigroup is uniquely determined by its generator.*

Theorem A.6. *Let A be the infinitesimal generator of a strongly continuous semigroup $T(t)$ of bounded linear operators, then A is a densely defined, closed linear operator. Moreover, the semigroup $T(t)$ is uniquely determined by its generator A.*

When $\|T(t)\| \leq 1$ we have a semigroup of *contractions*.

Theorem A.7 (Hille-Yosida)**.** *A linear (unbounded) operator A is the infinitesimal generator of a strongly continuous semigroup of contractions $T(t)$, $t \geq 0$,* *if and only if (i) A is densely defined and closed, (ii) the resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and for every $\lambda > 0$ one has $\|R(\lambda; A)\| \leq 1/\lambda$.*

Definition A.11. A linear operator A is *dissipative* if $\|(\lambda \mathbb{1} - A)x\| \geq \lambda \|x\|$ for all $x \in \mathrm{Dom}(A)$ and $\lambda > 0$.

Theorem A.8. *Let A be a dissipative operator.*

1. *If for some $\lambda_0 > 0$ one has $\mathrm{Ran}(\lambda_0 \mathbb{1} - A) = \mathcal{X}$, then $\mathrm{Ran}(\lambda \mathbb{1} - A) = \mathcal{X}$ for all $\lambda > 0$.*
2. *If A is closable, also its closure $\overline{A}$ is dissipative.*
3. *If $\overline{\mathrm{Dom}(A)} = \mathcal{X}$, then A is closable.*

Theorem A.9 (Lumer-Phillips)**.** *Let A be a linear operator with dense domain $\mathrm{Dom}(A)$ in $\mathcal{X}$.*

1. *If A is dissipative and there is a $\lambda_0 > 0$ such that $\mathrm{Ran}(\lambda_0 \mathbb{1} - A) = \mathcal{X}$, then $\overline{A}$ is the infinitesimal generator of a strongly continuous semigroup of contractions on $\mathcal{X}$.*
2. *If A is the infinitesimal generator of a strongly continuous semigroup of contractions on $\mathcal{X}$, then $\mathrm{Ran}(\lambda \mathbb{1} - A) = \mathcal{X}$ for all $\lambda > 0$ and A is dissipative.*

Definition A.12 (Maximally dissipative operators)**.** A dissipative operator A for which $\mathrm{Ran}(\mathbb{1} - A) = \mathcal{X}$ is called *m-dissipative*.

Remark A.6.

1. The closure of a densely defined operator A is the infinitesimal generator of a strongly continuous semigroup of contractions if and only if A is m-dissipative.
2. Let A and B be the generators of two strongly continuous contraction semigroups with $Ax = Bx$, $\forall x \in \mathrm{Dom}(A) \subset \mathrm{Dom}(B)$. Then, $\mathrm{Dom}(B) = \mathrm{Dom}(A)$ and $B = A$. See [7] p. 75.

Definition A.13 ([7] Definition 2.1 p. 169)**.** Let A and B be two operators in $\mathcal{X}$. The operator B is said to be *(relatively) A-bounded* if $\mathrm{Dom}(A) \subset \mathrm{Dom}(B)$ and if there exist two non-negative constants α and β such that

$$\|Bx\| \leq \alpha \|Ax\| + \beta \|x\|, \qquad \forall x \in \mathrm{Dom}(A). \tag{A.1}$$

The *A-bound* of B is $\alpha_0 := \inf\{\alpha \geq 0\colon$ there exists $\beta \geq 0$ such that (A.1) holds$\}$.

Proposition A.10 ([7] Lemma 2.4 p. 171)**.** *If $\big(A, \mathrm{Dom}(A)\big)$ is closed and $\big(B, \mathrm{Dom}(B)\big)$ is A-bounded with A-bound $\alpha_0 < 1$, then $\big(A+B, \mathrm{Dom}(A)\big)$ is a closed operator.*

Theorem A.11. *Let A and B be linear operators in $\mathcal{X}$ such that $\mathrm{Dom}(B) \supset \mathrm{Dom}(A)$ and $A + tB$ is dissipative for $0 \leq t \leq 1$. If B is A-bounded with A-bound $\alpha_0 < 1$ and for some $t_0 \in [0, 1]$ the operator $A + t_0 B$ is m-dissipative, then $A + tB$ is m-dissipative for all $t \in [0, 1]$.*

Particularization to the Hilbert space case.

Remark A.7. By Remark A.5, if $P(t)$ is a strongly continuous semigroup of contractions in $\mathscr{H}$, then also $P(t)^*$ is a strongly continuous semigroup of contractions in $\mathscr{H}$.

Theorem A.12. *A linear operator A in $\mathscr{H}$ is dissipative if and only if*

$$\mathrm{Re}\langle\phi|A\phi\rangle \leq 0\,, \qquad \forall\phi \in \mathrm{Dom}(A).$$

Theorem A.13. *Let A be a dissipative operator. If $\mathrm{Ran}(\mathbb{1} - A) = \mathscr{H}$, then $\overline{\mathrm{Dom}(A)} = \mathscr{H}$.*

Proposition A.14. *Let A be a densely defined closed linear operator in $\mathscr{H}$. If both A and A^* are dissipative, then A is the infinitesimal generator of a strongly continuous semigroup of contractions.*

Appendix B

The algebra of bounded operators on $\mathscr{H}$

B.1 Basic Definition

Definition B.1 (Involution)**.** Let $\mathscr{A}$ be a Banach algebra. A mapping $x \to x^*$ of $\mathscr{A}$ into itself is called an *involution* if the following conditions are satisfied

1. $(x^*)^* = x$,
2. $(x + y)^* = x^* + y^*$,
3. $(xy)^* = y^* x^*$,
4. $(\lambda x)^* = \overline{\lambda} x^*, \qquad \lambda \in \mathbb{C}$.

A Banach algebra with an involution * is called a Banach **-algebra.*

Definition B.2 (C*-algebra)**.** A Banach *-algebra $\mathscr{A}$ is called a *C*-algebra* if it satisfies $\|x^* x\| = \|x\|^2$ for all $x \in \mathscr{A}$.

Definition B.3 (W*-algebra)**.** A C*-algebra $\mathscr{M}$ is called a *W*-algebra* (or Von Neumann algebra) if it is a dual space as a Banach space (i.e if there exists a Banach space $\mathscr{M}_*$ such that $(\mathscr{M}_*)^* = \mathscr{M}$, where $(\mathscr{M}_*)^*$ is the dual Banach space of $\mathscr{M}_*$). We shall call such a Banach space $\mathscr{M}_*$ the *predual* of $\mathscr{M}$.

Definition B.4 (Uniform topology)**.** The topology defined by the norm $\| \ \|$ on a Banach space $C*-algebra$ $\mathscr{A}$ is called the *uniform topology.*

Definition B.5 (σ-topology)**.** The weak*-topology $\sigma(\mathscr{M}, \mathscr{M}_*)$ on a W*-algebra $\mathscr{M}$ is called *weak topology* or *σ-topology* on $\mathscr{M}$.

Let us recall the meaning of the weak*-topology: the sequence $\{a_n\} \subset \mathscr{M}$ converges in the weak* sense to a if $\lim_{n\to+\infty}\langle a_n, x\rangle = \langle a, x\rangle$ for all $x \in \mathscr{M}_*$.

Definition B.6 (C*-subalgebra)**.** A subset V of a C*-algebra $\mathscr{A}$ is called *selfadjoint* if $x \in V$ implies $x^* \in V$. A selfadjoint uniformly closed subalgebra of $\mathscr{A}$ is also a C*-algebra and it is called a C*-subalgebra of $\mathscr{A}$.

Definition B.7 (W*-subalgebra)**.** A selfadjoint σ-closed subalgebra $\mathscr{N}$ of a W*-algebra $\mathscr{M}$ is also a W*-algebra and it is called a W*-subalgebra.

Definition B.8 (Positivity)**.** An element a in a C*-algebra $\mathscr{A}$ is said to be positive if $a = b^*b$ for some $b \in \mathscr{A}$. A linear functional φ on a $\mathscr{A}$ is called *positive* if $\phi(a^*a) \geq 0$ for all $a \in \mathscr{A}$.

Definition B.9 (Normal)**.** A positive linear functional φ on a W*-algebra $\mathscr{M}$ is said to be normal if it satisfies $\varphi(\text{l.u.b. } x_\alpha) = \text{l.u.b. } \varphi(a_\alpha)$ for every uniformly bounded increasing directed set $\{a_\alpha\}$ of positive elements in $\mathscr{M}$.

Definition B.10 (Completely positive linear maps)**.** Let $\mathscr{A}$ and $\mathscr{B}$ be two *-algebras with unit. The linear map $T : \mathscr{A} \to \mathscr{B}$ is called

1. *n-positive* if for every family $a_1, \dots, a_n$ of element of $\mathscr{A}$ and every family $b_1, \dots, b_n$ of elements of $\mathscr{B}$ we have
$$\sum_{i,j=1}^{n} b_i^* T(a_i^* a_j) b_j \geq 0$$

2. *completely positive* if it is n-positive for every integer $n \geq 1$.

Theorem B.1 (Theorem 1.13.2. in [15])**.** *Let φ be a positive linear functional on a W*-algebra $\mathscr{M}$. Then the following conditions are equivalent:*

1. *φ is normal,*

2. *φ is $\sigma(\mathscr{M}, \mathscr{M}_*)$-continuous.*

Let $\mathscr{H}$ be a complex Hilbert space and let $\mathscr{B}(\mathscr{H})$ be the algebra of all bounded linear operators on $\mathscr{H}$. We can define various topologies in $\mathscr{B}(\mathscr{H})$.

Theorem B.2 (Proposition 2.9 in [8], page 18)**.** *Let $\mathscr{A}$ be a C*-algebra and T a linear map from $\mathscr{A}$ to $\mathscr{B}(\mathscr{H})$. T is completely positive if and only if for any $n \geq 1$, $a_1, \dots a_n \in \mathscr{A}$, $u_1, \dots, u_n \in \mathscr{H}$, one has*
$$\sum_{1\leq i,j\leq n} \left\langle u_i \middle| T[a_i^* a_j] u_j \right\rangle \geq 0.$$

B.2 Topologies in $\mathscr{B}(\mathscr{H})$

1. **The uniform topology in $\mathscr{B}(\mathscr{H})$.** The uniform topology in $\mathscr{B}(\mathscr{H})$ is given by the operator norm $\|a\|$ $(a \in \mathscr{B}(\mathscr{H}))$, where $\|a\| = \sup_{\|\xi\|\leq 1} \|a\xi\|$.

   $\mathscr{B}(\mathscr{H})$ is a Banach algebra with this norm. We shall take the adjoint operation $a \to a^*$ as involution * on $\mathscr{B}(\mathscr{H})$ (i.e., $\langle a\xi|\eta\rangle = \langle \xi|a^*\eta\rangle$) for $\xi, \eta \in \mathscr{H}$ where $\langle\ |\ \rangle$ is the scalar product of $\mathscr{H}$. Then

$$\|a\|^2 = \sup_{\|\xi\|=1} \|a\xi\|^2 = \sup_{\|\xi\|=1} \langle a\xi|a\xi\rangle = \sup_{\|\xi\|=1} \langle a^*a\xi|\xi\rangle \leq \|a^*a\|.$$

   Since $\|a^*\| = \|a\|$, then $\|a\|^2 = \|a^*a\|$. Therefore, $\mathscr{B}(\mathscr{H})$ is a C*-algebra, and so any uniformly closed selfadjoint subalgebra of $\mathscr{B}(\mathscr{H})$ is also a C*-algebra.

2. **The strong operator topology in $\mathscr{B}(\mathscr{H})$.** Let $\xi \in \mathscr{B}(\mathscr{H})$. The function $a \to \|a\xi\|$ is a semi-norm on $\mathscr{B}(\mathscr{H})$. The set of all such semi-norms $\{\|a\xi\| \mid \xi \in \mathscr{B}(\mathscr{H})\}$ defines a Hausdorff locally convex topology in $\mathscr{B}(\mathscr{H})$. This is the *strong operator topology.*

3. **The strongest operator topology.** Let $(\xi_i) \subset \mathscr{H}$ be any sequence of elements in $\mathscr{H}$ such that $\sum_{i=1}^{\infty} \|\xi_i\|^2 < +\infty$. The function

$$\left(\sum_{i=1}^{\infty} \|a\xi_i\|^2\right)^{\frac{1}{2}} \qquad a \in \mathscr{B}(\mathscr{H})$$

   defines a semi-norm on $\mathscr{B}(\mathscr{H})$. The set of all such semi-norms

$$\left\{\left(\sum_{i=1}^{\infty} \|a\xi_i\|^2\right)^{\frac{1}{2}} \middle| (\xi_i) \subset \mathscr{H}, \sum_{i=1}^{\infty} \|\xi_i\|^2 < +\infty\right\}$$

   defines a Hausdorff locally convex topology in $\mathscr{B}(\mathscr{H})$. This is the *strongest operators topology.*

4. **The weak operator topology.** For $\xi, \eta \in \mathscr{H}$, the function $|\langle a\xi|\eta\rangle|$ is a semi-norm on $\mathscr{B}(\mathscr{H})$. The set of all semi-norms $\{|\langle a\xi|\eta\rangle| \big| \xi, \eta \in \mathscr{H}\}$ defines a Hausdorff locally convex topology. This is the *weak operator topology.*

5. **The σ-weak operator topology.** For (ξ_n), $(\eta_n) \subset \mathscr{H}$ such that $\sum_{i=1}^{\infty} \|\xi_n\|^2 < +\infty$, $\sum_{i=1}^{\infty} \|\eta_n\|^2 < +\infty$, consider the semi-norm $\left|\sum_{i=1}^{\infty} \langle a\xi_n|\eta_n\rangle\right|$

on $\mathscr{B}(\mathscr{H})$. The set of all such semi-norms will defined a Hausdorff locally convex topology, the *σ-weak operator topology* on $\mathscr{B}(\mathscr{H})$.

Let $\mathcal{C}(\mathscr{H})$ be the linear space of all continuous linear functionals on $\mathscr{B}(\mathscr{H})$ with respect to the weak operator topology. Then the weak operator topology is equivalent to the $\sigma\big(\mathscr{B}(\mathscr{H}),\mathcal{C}(\mathscr{H})\big)$ topology. One can easily see that the unit sphere S of $\mathscr{B}(\mathscr{H})$ is $\sigma\big(\mathscr{B}(\mathscr{H}),\mathcal{C}(\mathscr{H})\big)$-compact.

Proposition B.3 (Proposition 1.15.1 in [15]). *Let* $\mathscr{N}$ *be a selfadjoint subalgebra of* $\mathscr{B}(\mathscr{H})$. *Then following condition are equivalent.* $\mathscr{N}$ *is closed in*

1. *the weak operator topology,*
2. *the σ-weak operator topology,*
3. *the stronger operator topology,*
4. *the strongest operator topology and*
5. $\sigma\big(\mathscr{B}(\mathscr{H}),\mathscr{B}(\mathscr{H})_*\big)$.

Proposition B.4 (proposition 1.15.2 in [15]). *Let* $\mathscr{N}$ *be a weakly closed selfadjoint subalgebra of* $\mathscr{B}(\mathscr{H})$, *and let* $\mathscr{N}_*$ *be the predual of* $\mathscr{N}$. *Then*

1. *the weak operator topology, the σ-weak operator topology and* $\sigma\big(\mathscr{N},\mathscr{N}_*\big)$ *are equivalent on bounded spheres;*
2. *the strong operator topology, the strongest operators topology and the* $s\big(\mathscr{N},\mathscr{N}_*\big)$ *topology are equivalent on bounded spheres.*

B.3 Operators of trace class

Definition B.11 (Trace of an operator). Let (ξ_α) be a complete orthonormal system of $\mathscr{H}$. For $h(\geq 0) \in \mathscr{B}(\mathscr{H})$, put $\mathrm{Tr}(h) = \sum_\alpha \langle h\xi_\alpha|\xi_\alpha\rangle$; then $\mathrm{Tr}(h)$ does not depend on the choice of (ξ_α). We will call *trace of the operator* h the quantity $\mathrm{Tr}(h)$.

Definition B.12 (The trace class of $\mathscr{H}$). An element $a \in \mathscr{B}(\mathscr{H})$ is called an *operator of the trace class* if $\mathrm{Tr}(|a|) \equiv \mathrm{Tr}\left\{\sqrt{a^*a}\right\} < +\infty$. The set of all operators of the trace class is called the *trace class of* $\mathscr{H}$ and we will denote it by $\mathscr{T}(\mathscr{H})$.

Now, for $a \in \mathscr{T}(\mathscr{H})$ let us put $\|a\|_1 = \mathrm{Tr}(|a|)$, then $\mathscr{T}(\mathscr{H})$ is a normed linear space under $\| \ \|_1$, moreover $\mathscr{T}(\mathscr{H})$ is a Banach space.

Theorem B.5 (Theorem 1.15.3. in [15]). *The predual $\mathscr{B}(\mathscr{H})_*$ of $\mathscr{B}(\mathscr{H})$ may be identified with the Banach space of all trace class operators on $\mathscr{H}$ under the isometric linear mapping $a \to \varphi_a$, where $\varphi_a(x) = \mathrm{Tr}(xa)$, with $x \in \mathscr{B}(\mathscr{H})$ and $a \in \mathscr{T}(\mathscr{H})$. Moreover, under this identification, positive elements in $\mathscr{B}(\mathscr{H})_*$ are identified with positive elements in $\mathscr{T}(\mathscr{H})$.*

Corollary B.6 (Corollary 1.15.5. in [15]). *Let f be a linear functional on $\mathscr{B}(\mathscr{H})$. Then the following conditions are equivalent:*

1. *$f \in \mathscr{B}(\mathscr{H})_*$;*
2. *There exist two sequences $(\xi_n), (\eta_n)$ of elements in $\mathscr{H}$ such that for all $x \in \mathscr{B}(\mathscr{H})$*
$$\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty, \ \sum_{n=1}^{\infty} \|\eta_n\|^2 < \infty \text{ and } f(x) = \sum_{n=1}^{\infty} \langle x\xi_n | \eta_n \rangle < \infty.$$

Corollary B.7 (Corollary 1.15.6. in [15]). *The strongest operator topology in $\mathscr{B}(\mathscr{H})$ is equivalent to the $s\big(\mathscr{B}(\mathscr{H}), \mathscr{B}(\mathscr{H})_*\big)$ one. The σ-weak operator topology on $\mathscr{B}(\mathscr{H})$ is equivalent to $\sigma\big(\mathscr{B}(\mathscr{H}), \mathscr{B}(\mathscr{H})_*\big)$.*

Let $\mathscr{C}(\mathscr{H})$ the algebra of all compact linear operators on $\mathscr{H}$. $\mathscr{T}(\mathscr{H})$ can be identified with the predual $\mathscr{B}(\mathscr{H})_*$ of $\mathscr{B}(\mathscr{H})$. For $x \in \mathscr{C}(\mathscr{H})$ and $a \in \mathscr{T}(\mathscr{H})$, let $\psi_a(x) = \mathrm{Tr}(xa)$. Then ψ_a is bounded linear functional on $\mathscr{C}(\mathscr{H})$.

Theorem B.8 (Proposition 1-19.1 in [15]). *The mapping $a \to \psi_a$ of $\mathscr{T}(\mathscr{H})$ into $\mathscr{C}(\mathscr{H})^*$ is an isometric linear mapping of $\mathscr{T}(\mathscr{H})$ onto $\mathscr{C}(\mathscr{H})^*$. Therefore, under the mapping $a \to \psi_a$, $\mathscr{T}(\mathscr{H})$ can be identified with $\mathscr{C}(\mathscr{H})^*$. Hence, we have $\mathscr{C}(\mathscr{H})^* = \mathscr{T}(\mathscr{H})$ and $\mathscr{T}(\mathscr{H})^* = \mathscr{B}(\mathscr{H})$.*

Bibliography

[1] L. Accardi, *On the quantum Feynman-Kac formula*, Rend. Sem. Mat. Fis. Milano **XLVIII** (1978) 135-179.

[2] A. Barchielli, *Measurement theory and stochastic differential equations in quantum mechanics*, Phys. Rev. A **34** (1986) 1642–1649.

[3] A. Barchielli, *Continual Measurements in Quantum Mechanics and Quantum Stochastic Calculus*. In S. Attal, A. Joye, C.-A. Pillet (eds.), *Open Quantum Systems III*, Lecture Notes in Mathematics **1882** (Springer, Berlin, 2006), pp. 207–291.

[4] A. Barchielli, G. Lupieri, *Convolution semigroups in quantum stochastic calculus* in L. Accardi, W. von Waldenfels (eds.), *Quantum Probability and Applications IV*. Lecture Notes in Mathematics, Vol. 1396 (Springer, Berlin, 1989); pp. 107–127.

[5] E. B. Davies, *Quantum Theory of Open Systems*. Academic Press, London, 1976.

[6] N.Dunford, J.T. Schwartz, *Linear Operators. I. General Theory.* Pure and Applied Mathematics, Vol 7 Interscience Publishers, Inc., New York; Interscience Publisher, Ltd., London 1958.

[7] K.-J. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations* (Springer, Berlin, 2000).

[8] F. Fagnola, *Quantum Markov Semigroups and Quantum Flows* Proyecciones 18, n.3 (1999).

[9] F. Fagnola, *Quantum Stochastic Differential Equations and Dilation of Completely Positive Semigroups*. In S. Attal, A. Joye, C.-A. Pillet (eds.), *Open Quantum Systems II*, Lecture Notes in Mathematics **1881** (Springer, Berlin, 2006), pp. 183–220.

[10] F. Fagnola, S. J. Wills, *Solving quantum stochastic differential equations with unbounded coefficients*, J. Funct. Anal. **198** (2003) 279–310.

[11] A. S. Holevo, *Statistical Structure of Quantum Theory*, Lecture Notes in Physics m 67. Springer, Berlin, 2001.

[12] K.R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus* (Birkhäuser, Basel, 1992).

[13] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations* (Springer, Berlin, 1983).

[14] M. Reed, B. Simon, *Methods of Modern Mathematical Physics: I Functional Analysis* (Academic Press, 1980).

[15] S. Sakai, *C^*-Algebras and W^*-Algebras* (Springer, Berlin 1971).