SAPIENZA
UNIVERSITÀ DI ROMA

The rigorous derivation of the Linear Landau equation from a particle system in a weak-coupling limit

Nicolò Catapano

Thesis Advisor

Prof. M. Pulvirenti

Facoltà di Scienze Matematiche Fisiche e Naturali

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Indice

1 Introduzione

Nel 1872 Boltzmann derivò in maniera euristica la prima equazione di evoluzione per descrivere il comportamento di un gas rarefatto modellizzato da un sistema di sfere dure. L'idea di partenza di Boltzmann è che non è possibile studiare l'evoluzione di un gas guardando ogni singola particella, ovvero studiare l'evoluzione dinamica di un sistema di N sfere dure interagenti per mezzo di collisioni elastiche. Risulta invece conveniente studiare l'evoluzione statistica del gas studiando la funzione di densità di una particella fissata $f(x, v)$. Il gas è pensato essere formato da N sfere dure interagenti per mezzo di collisioni eleatiche. Nella figura 1 diamo una rappresentazione di una collisione elastica tra due particelle con velocità v_i e v_j.

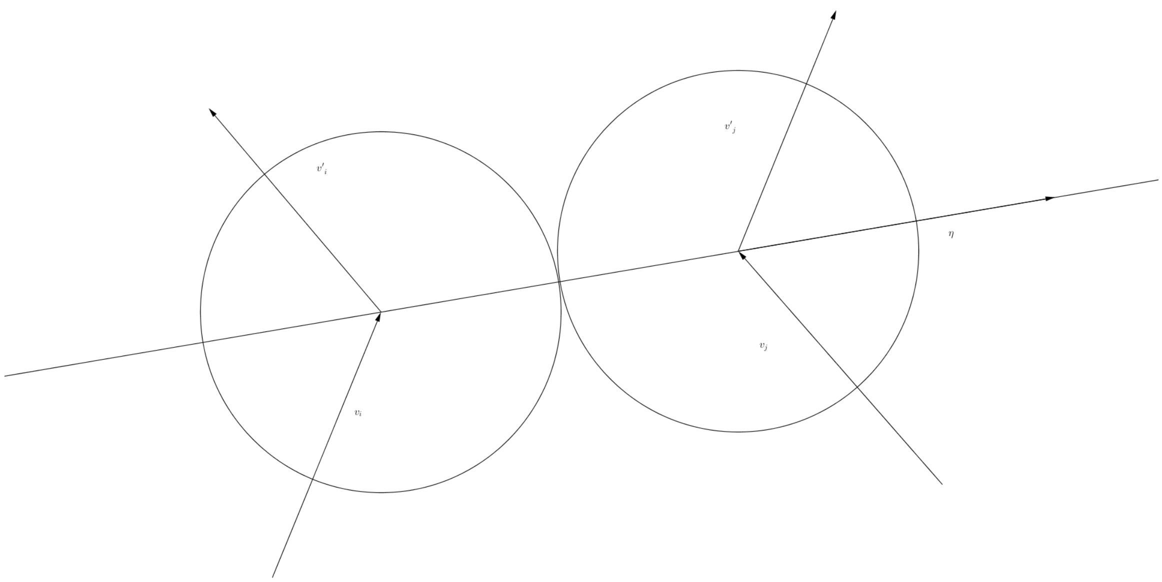

Figura 1: Nella figura $\eta \in S^2$ è il versore che punta dalla particella x_i sul punto di collisione, mentre $v_i^{'}$ e $v_j^{'}$ sono le velocità post-collisionali date rispettivamente da $v_i^{'} = v_i - \eta\,[\eta \cdot (v_i - v_j)]$ e $v_j^{'} = v_j + \eta\,[\eta \cdot (v_i - v_j)]$.

La variazione nel tempo della f è dovuta al moto libero delle particelle e alle interazioni tra queste. La variazione causata dal moto libero viene rappresentato dall'operatore di trasporto $(\partial_t + v \cdot \nabla)$. Se non ci sono collisioni abbiamo che

$$\partial_t f + v \cdot \nabla f = 0 \tag{1}$$

ovvero l'equazione del trasporto per la densità di probabilità di una particella. La probabilità rimane costante nel tempo lungo le caratteristiche $x + vt$. Se invece avvengono delle collisioni tra le particelle la variazione di f dovuta a queste interazione viene rappresentata da un operatore Q, detto operatore di collisione. L'operatore Q viene derivato euristicamente da Boltzmann con il seguente argomento. Esso può essere considerato come somma di due termini

$$Q = G - L \tag{2}$$

Il termine G viene detto termine di guadagno (gain) mentre L è il termine di perdita (loss). Il termine G rappresenta il contributo positivo alla variazione di f dovuta alle collisioni, ovvero la probabilità che la particella test appaia in una cella dello spazio delle fasi $dxdv$ intorno al punto x, v in un intervallo temporale $(t, t + dt)$ a causa di una collisione è data da $Gdxdvdt$. Analogamente L rappresenta la variazione negativa , la probabilità che la particella scompaia dalla cella $dxdv$ a causa di una collisione nell'intervallo $(t, t + dt)$ è data da $Ldxdvdt$. Queste quantità possono essere calcolate esplicitamente assumendo che le interazioni tra le particelle siano binarie. Si ottiene che

$$G(f_2) = (N-1)\epsilon^2 \int dv_1 \int\limits_{\eta\cdot(v-v_1)<0} d\eta |\eta \cdot (v - v_1)| f_2(x, v, x + \epsilon\eta, v_1) \tag{3}$$

$$L(f_2) = (N-1)\epsilon^2 \int dv_1 \int\limits_{\eta\cdot(v-v_1)>0} d\eta |\eta \cdot (v - v_1)| f_2(x, v, x + \epsilon\eta, v_1) \tag{4}$$

dove ϵ è il diametro delle particelle e $\eta \in S^2$. L'equazione ottenuta è la seguente

$$\partial_t f + v \cdot \nabla f = Q(f_2) \tag{5}$$

Osserviamo che l'equazione (5) non è una equazione chiusa, infatti il termine di destra dipende dalla distribuzione a due particelle f_2. Volendo scrivere una equazione per la f_2 risulterebbe che questa dipende da f_3, e così via. Per chiudere l'equazione Boltzmann introduce l'ipotesi della propagazione del caos, nota nella letteratura come "Stosszahlansatz", che afferma che se le distribuzioni della particella test e di un'altra particella sono scorrelate al tempo zero allora lo sono per ogni tempo t. In altre parole la f_2 può essere fattorizzata nella seguente maniera

$$f_2(x, v, x_1, v_1) = f_1(x, v) f_1(x_1, v_1) \tag{6}$$

Questa ipotesi non sembra naturale, difatti la collisione tra due particelle crea correlazione , però prima della collisione sembra ragionevole assumere che le particelle abbiano distribuzioni scorrelate. Inoltre la collisione tra due particelle fissate in un gas rarefatto è un evento di probabilità piccola. Consideriamo infatti il comportamento asintotico dei parametri per un numero molto grande di particelle, ovvero per $N \to \infty$. Abbiamo che per un gas rarefatto risulta $N\epsilon^2 = O(1)$, e dunque $\epsilon^2 = O(N^{-1})$. Questo implica che la probabilità che due particelle fissate collidano è $O(\epsilon^2)$ e quindi molto piccola, mentre la probabilità che due particelle qualsiasi collidano è $O(N\epsilon^2) = O(1)$. Dato che l'equazione (6) si riferisce a due particelle ben specifiche, scrivendo il termine di destra in funzione delle variabili pre-collisionali e considerando il regime asintotico per N grande l'ipotesi della propagazione del caos risulta essere accettabile. Vogliamo applicare l'assunzione di propagazione del caos nell'equazione (5) per farla così diventare una equazione chiusa. Dobbiamo però scrivere l'operatore di collisione in funzione delle variabili pre-collisionali. Riscriviamo quindi l'operatore G attraverso le sostituzioni $v' = v - \eta\,[\eta \cdot (v - v_1)]$ e $v_1' = v_1 + \eta\,[\eta \cdot (v - v_1)]$ ottenendo

$$G(f) = (N-1)\epsilon^2 \int dv_1 \int_{\eta\cdot(v-v_1)>0} d\eta |\eta \cdot (v - v_1)| f(x, v') f(x + \epsilon\eta, v_1') \tag{7}$$

In questo modo il dominio di integrazione è cambiato dalle velocità post-collisionali a quelle pre-collisionali. Con questa osservazione e ricordando che in un regime di gas rarefatto risulta $N\epsilon^2 = O(1)$ e $\epsilon^2 = o(N^{-1})$, per N grandi l'equazione (5) diventa la seguente equazione di Boltzmann

$$\partial_t f + v \cdot \nabla f = (N-1)\epsilon^2 \int dv_1 \int_{\eta\cdot(v-v_1)>0} d\eta |\eta \cdot (v - v_1)| \left[f(x, v') f(x, v_1') - f(x, v) f(x, v_1) \right] \tag{8}$$

Nel lavoro in cui Boltzmann da la derivazione euristica della sua equazione viene anche dimostrata una importante proprietà dell'equazione di Boltzmann, che verrà in seguito chiamata teorema H. Il teorema H stabilisce che il funzionale

$$H(f) = \int \int dx dv\, f(x, v,) \log f(x, v) \tag{9}$$

detto entropia, è decrescente lungo le soluzioni della equazione 8. Il minimo del funzionale H è dato dalla distribuzione Maxwelliana definita come

$$M_\beta(v) = C_\beta e^{-\beta \frac{|v|^2}{2}} \tag{10}$$

dove C_β è scelto in modo tale che $\int M_\beta(v)dv = 1$. Questo intuitivamente ci dice che la soluzione dell'equazione di Boltzmann nel tempo tende a raggiungere la distribuzione Maxwelliana, che si può facilmente dimostrare essere una soluzione stazionaria della equazione di Boltzmann. Da una parte il teorema H trova coerenza con l'evoluzione di un gas e rafforza l'attitudine dell'equazione di Boltzmann nel rappresentarla. Dall'altra però, a prima vista, il teorema H indebolisce le speranze che l'equazione di Boltzmann possa essere direttamente derivabile dalle equazioni di Newton, come fortemente sostenuto da Boltzmann stesso. Un sistema governato dalle equazioni di newton è temporalmente reversibile, mentre l'esistenza di un funzionale decrescente nel tempo lungo le soluzione rende impossibile questa proprietà. Per questi motivi ci fu una forte opposizione all'equazione di Boltzmann e nacque la necessita di provarne una derivazione rigorosa partendo dall'evoluzione di un sistema dinamico di sfere dure. Il problema venne ripreso da H. Grad nel 1949 il quale teorizzò che per riuscire a dimostrare la validità rigorosa della equazione di Boltzmann si doveva considerare il comportamento asintotico della equazione per $N \to \infty$, $\epsilon \to 0$ e $N\epsilon^2 \to 1$, questo limite prese il nome di limite di Boltzmann-Grad. Nel 1972 C. Cercignani dimostrò in maniera formale che partendo dalla gerarchia BBGKY per sfere dure e considerando il limite indicato da Grad si otteneva una gerarchia di infinite equazioni, detta gerarchia di Boltzmann. La prima equazione di questa infinita gerarchia era proprio l'equazione di Boltzmann. L'argomento di Cercignani mostrava come da un punto di vista formale la derivazione dell'equazione di Boltzmann non fosse problematica. Finalmente nel 1975 O. Lanford dimostrò la derivazione rigorosa dell'equazione di Boltzmann partendo da un sistema di sfere dure. La derivazione di Lanford era possibile solo per tempi piccoli. Questo risultato è stato in seguito migliorato nel 1989 da R. Illner e M. Pulvirenti nel caso di una nube di gas rarefatto che si espande nel vuoto, per cui è stato possibile dimostrare la derivazione per un tempo arbitrario.

2 Equazione di Boltzmann

In questa sezione daremo una prova della derivazione formale dell'equazione di Boltzmann partendo da un sistema di sfere dure riportando l'argomento di C. Cercignani. Consideriamo quindi un sistema di N sfere dure di diametro ϵ interagenti tramite collisioni elastiche. Indichiamo con $\boldsymbol{z}_N = (x_1, v_1, ..., x_N, v_N)$ una configurazione delle N particelle. Lo spazio delle fasi è dato da

$$\Gamma_{N,\epsilon} := \left\{ \boldsymbol{z}_N \in \mathbb{R}^{3N} \times \mathbb{R}^{3N} \,|\, v_i \in \mathbb{R}^3,\, |x_i - x_j| > \epsilon\,,\, \forall i,j \leq N \ i \neq j \right\} \tag{11}$$

La condizione $|x_i - x_j| > \epsilon$ corrisponde al fatto che le sfere non possono compenetrarsi. Il bordo dello spazio delle fasi corrisponde ad urto elastico. Se $\boldsymbol{z}_N \in \partial\Gamma_{N,\epsilon}$ devono esistere due particelle x_ie x_j tali che $|x_i - x_j| = \epsilon$.

L'equazione di evoluzione per una densità di probabilità per le N particelle $W^N(t)$ con dato iniziale $W^N(0) = W_0^N$ è data dalla equazione di Liouville

$$\partial_t W^N + \sum_{k=1}^{N} v_k \cdot \nabla_{x_k} W^N = 0 \tag{12}$$

dove le condizione sul bordo sono date dalla risoluzione dell'urto elastico, ovvero se $\boldsymbol{z}_N \in \partial\Gamma_N$ e quindi $|x_i - x_j| = \epsilon$ abbiamo che

$$W^N\left(x_1, v_1, .., x_i, v_i, .., x_j, v_j, .., x_N, v_N\right) = W^N\left(x_1, v_1, .., x_i, v_i^{'}, .., x_j, v_j^{'}, .., x_N, v_N\right) \tag{13}$$

con

$$v_i^{'} = v_i - \eta\left[\eta \cdot (v_i - v_j)\right] \tag{14}$$

$$v_j^{'} = v_j + \eta\left[\eta \cdot (v_i - v_j)\right] \tag{15}$$

e $\eta = \frac{x_j - x_i}{|x_j - x_i|}$. Sul dato iniziale W_0^N facciamo l'ipotesi che sia simmetrico nello scambio di particelle e continuo sulle collisioni.

Consideriamo ora le marginali per le prime $\boldsymbol{z}_j$ variabili. Queste si ottengono integrando la densità W^N nelle restanti $N - j$ variabili, che indichiamo con $\boldsymbol{z}_{j+1,N}$. Abbiamo che

$$f_j^N(\boldsymbol{z}_j) = \int d\boldsymbol{z}_{j+1,N} W^N(\boldsymbol{z}_N) \tag{16}$$

Vogliamo scrivere una equazione per le marginali partendo dalla equazione di Liouville, per farlo quindi integriamo in $d\boldsymbol{z}_{j+1,N}$ l'equazione (12) e otteniamo

$$\partial_t f_j^N + \sum_{n=1}^{j} \int v_n \cdot \nabla_{x_n} W^N(\boldsymbol{z}_N) d\boldsymbol{z}_{j+1,N} + \sum_{m=j+1}^{N} \int v_m \cdot \nabla_{x_m} W^N d\boldsymbol{z}_{j+1,N} = 0 \tag{17}$$

Consideriamo la seguente identità, ottenuta tramite una integrazione per parti nel caso $n \leq j$

$$\int v_n \cdot \nabla_{x_n} W^N(\boldsymbol{z}_N) d\boldsymbol{z}_{j+1,N} = v_n \cdot \nabla_{x_n} f_j^N(\boldsymbol{z}_j) - \sum_{k=j+1}^{N} \int f_{j+1}^N v_n \cdot \eta_{nk} d\sigma_{nk} dv_k \tag{18}$$

dove η_{nk} è la normale esterna alla sfera centrata in x_k di raggio ϵ e $d\sigma_{nk}$ è l'elemento di superficie della stessa sfera. Applicando la (18) nella (17) otteniamo

$$\partial_t f_j^N + \sum_{n=1}^{j} v_n \cdot \nabla_{x_n} f_j^N(\boldsymbol{z}_j) = \sum_{k=j+1}^{N} \int f_{j+1}^N v_n \cdot \eta_{nk} d\sigma_{nk} dv_k - \sum_{m=j+1}^{N} \int v_m \cdot \nabla_{x_m} W^N d\boldsymbol{z}_{j+1,N} \quad (19)$$

Abbiamo così ricostruito il trasporto libero nel termine di sinistra. Nel caso $m > j$ utilizzando il teorema di Gauss invece otteniamo la seguente identità

$$\int v_m \cdot \nabla_{x_m} W^N d\boldsymbol{z}_{j+1,N} = \sum_{n=1}^{j} \int f_{j+1}^N v_m \cdot \eta_{nm} d\sigma_{nm} dv_m + \sum_{\substack{n=j+1 \\ n\neq m}}^{N} \int f_{j+2}^N v_m \cdot \eta_{nm} d\sigma_{nm} dv_m dx_n dv_n \quad (20)$$

Questa identità inserita nell'equazione (19) ci da

$$\partial_t f_j^N + \sum_{n=1}^{j} v_n \cdot \nabla_{x_n} f_j^N(\boldsymbol{z}_j) = \sum_{n=1}^{j} \sum_{m=j+1}^{N} f_{j+1}^N (v_n - v_m) \eta_{nm} d\sigma_{nm} dv_m +$$

$$\frac{1}{2} \sum_{\substack{n=j+1 \\ n\neq m}}^{N} \int f_{j+2}^N (v_m - v_n) \cdot \eta_{nm} d\sigma_{nm} dv_m dx_n dv_n \quad (21)$$

dove nell'ultimo integrale abbiamo usato che $v_m \cdot \eta_{nm}$ può essere sostituito con $\frac{1}{2}(v_m - v_n) \cdot \eta_{nm}$, dato che $\eta_{nm} = -\eta_{mn}$. Si dimostra facilmente che l'ultimo integrale è zero sfruttando l'assunzione di continuità sulle collisioni della marginale e separando l'integrale sulla sfera nell'emisfero delle velocità pre-collisionali e post-collisionali. Osserviamo che nel primo integrale l'indice m è muto e quindi può essere fissato $m = j + 1$. Eseguendo la sommatoria otteniamo

$$\partial_t f_j^N + \sum_{n=1}^{j} v_n \cdot \nabla_{x_n} f_j^N(\boldsymbol{z}_j) = (N - j) \sum_{n=1}^{j} \int f_{j+1}^N (v_n - v_{j+1}) \cdot \eta_n d\sigma_n dv_{j+1} \quad (22)$$

dove con η_n e $d\sigma_n$ stiamo indicando rispettivamente la normale esterna e la misura di superficie della sfera centrata in x_{j+1} di raggio ϵ.

Separiamo ora l'integrale di destra in due termini uno relativo alla semisfera $(v_n - v_{j+1}) \cdot \eta_n > 0$ e uno alla semisfera $(v_n - v_{j+1}) \cdot \eta_n < 0$. Ricordando che $d\sigma_n = \epsilon^2 d\eta_n$ abbiamo

$$\partial_t f_j^N + \sum_{n=1}^{j} v_n \cdot \nabla_{x_n} f_j^N(\boldsymbol{z}_j) = (N - j)\epsilon^2 \sum_{n=1}^{j}$$

$$\left(\int_{\mathbb{R}^3}\int_{(v_n-v_{j+1})\cdot\eta_n>0} f^N_{j+1}|(v_n-v_{j+1})\cdot\eta_n|d\eta_n dv_{j+1}-\int_{\mathbb{R}^3}\int_{(v_n-v_{j+1})\cdot\eta_n<0} f^N_{j+1}|(v_n-v_{j+1}\cdot\eta_n|d\eta_n dv_{j+1}\right) \tag{23}$$

Nell'integrale nelle variabili pre-collisionali, ovvero $(v_n-v_{j+1})\cdot\eta_n>0$, usiamo nuovamente la continuità sulle collisioni per ottenere

$$\partial_t f^N_j+\sum_{n=1}^{j} v_n\cdot\nabla_{x_n} f^N_j(\boldsymbol{z}_j)=(N-j)\epsilon^2$$

$$\sum_{n=1}^{j}\int_{\mathbb{R}^3}\int_{(v_n-v_{j+1})\cdot\eta_n>0} f'^N_{j+1}|(v_n-v_{j+1})\cdot\eta_n|d\eta_n dv_{j+1}-\int_{\mathbb{R}^3}\int_{(v_n-v_{j+1})\cdot\eta_n<0} f^N_{j+1}|(v_n-v_{j+1})\cdot\eta_n|d\eta_n dv_{j+1} \tag{24}$$

dove

$$f'^N_{j+1}=f^N_{j+1}(x_1,v_1,...,x_n,v'_n,...,x_j,v_j,x_{j+1},v'_{j+1}) \tag{25}$$

Nel secondo integrale poniamo $\eta_n=-\eta_n$ e osserviamo che $x_{j+1}=x_n-\epsilon\eta$, segue che

$$\partial_t f^N_j+\sum_{n=1}^{j} v_n\cdot\nabla_{x_n} f^N_j(\boldsymbol{z}_j)=(N-j)\epsilon^2\sum_{n=1}^{j}\int_{\mathbb{R}^3} dv_{j+1}\int_{(v_n-v_{j+1})\cdot\eta>0} d\eta|(v_n-v_{j+1})\cdot\eta|$$

$$\Big[f^N_{j+1}(x_1,v_1,...,x_n,v'_n,...,x_j,v_j,x_n-\epsilon\eta,v'_{j+1})-f^N_{j+1}(x_1,v_1,...,x_n,v_n,...,x_j,v_j,x_n+\epsilon\eta,v_{j+1})\Big] \tag{26}$$

Abbiamo così ottenuto una gerarchia di equazioni conosciuta come gerarchia BBGKY per sfere dure. Osserviamo che il termine di sinistra della (26) rappresenta l'interazione interna tra le j particelle, mentre il termine di destra rappresenta l'interazione delle j particelle con le restanti N-j.

Passando formalmente al limite di Boltzmann-Grad, ovvero $N\to\infty$ $\epsilon\to 0$ $N\epsilon^2\to 1$, otteniamo la seguente gerarchia di infinite equazioni, detta gerarchia di Boltzmann

$$\partial_t f_j+\sum_{n=1}^{j} v_n\cdot\nabla_{x_n} f_j(\boldsymbol{z}_j)=\sum_{n=1}^{j}\int_{\mathbb{R}^3} dv_{j+1}\int_{(v_n-v_{j+1})\cdot\eta>0} d\eta|(v_n-v_{j+1})\cdot\eta|$$

$$\Big[f_{j+1}(x_1,v_1,...,x_n,v'_n,...,x_j,v_j,x_n,v'_{j+1})-f_{j+1}(x_1,v_1,...,x_n,v_n,...,x_j,v_j,x_n,v_{j+1})\Big] \tag{27}$$

A questo punto dobbiamo introdurre l'ipotesi del caos molecolare, o propagazione del caos. Se se il dato iniziale della gerarchia di Boltzmann si può fattorizzare, ovvero

$$f_j(x_1, v_1, ..., x_j, v_j, 0) = f_{j,0}(x_1, v_1, ..., x_j, v_j) = \prod_{i=1}^{j} f_{1,0}(x_i, v_i) \tag{28}$$

allora la soluzione per ogni tempo può essere fattorizzata. Questa ipotesi, assunta a priori, si recupera poi nella derivazione rigorosa. Fatta questa assunzione la prima equazione della gerarchia di Boltzmann diventa l'equazione di Boltzmann

$$\partial_t f + \sum_{n=1}^{j} v_n \cdot \nabla_{x_n} f = \int_{\mathbb{R}^3} dv_1 \int_{(v-v_1)\cdot\eta>0} d\eta |(v - v_1) \cdot \eta| \left[f(x, v') f(x, v_1') - f(x, v) f(x, v_1) \right] \tag{29}$$

Come detto questo approccio formale fu reso rigoroso da Lanford nel 1975. La tecnica di Lanford è la seguente, si costruisce una serie per la soluzione della gerarchia BBGKY (26) iterando il principio di Duhamel ottenendo

$$f_j^N(t) = \sum_{n=0}^{N-j} \epsilon^{2n} (N-j)...(N-j-n+1) \int_0^t dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{n-1}} dt_n S^\epsilon(t-t_1) Q_{j+1}^\epsilon ... S^\epsilon(t_{n-1}-t_n) Q_{j+n}^\epsilon S^\epsilon(t_n) f_{0,j+n}^N \tag{30}$$

dove $S^\epsilon(t)$ è l'operatore di trasporto libero sullo spazio $\Gamma_{N,\epsilon}$ e Q_j^ϵ è l'operatore di collisione definito nella parte di destra della (26), che riscriviamo per comodità

$$Q_j^\epsilon = (N - j)\epsilon^2 \sum_{k=1}^{j} \int_{\mathbb{R}^3} dv_{j+1} \int_{(v_k - v_{j+1})\cdot\eta>0} d\eta |(v_k - v_{j+1}) \cdot \eta|$$

$$\left[f_{j+1}^N(x_1, v_1, ..., x_k, v_k', ..., x_j, v_j, x_k - \epsilon\eta, v_{j+1}') - f_{j+1}^N(x_1, v_1, ..., x_k, v_k, ..., x_j, v_j, x_k + \epsilon\eta, v_{j+1}) \right] \tag{31}$$

Similmente si costruisce la stessa serie per la soluzione della gerarchia di Boltzmann ottenendo

$$f_j(t) = \sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{n-1}} dt_n$$

$$S(t - t_1) Q_{j+1} ... S(t_{n-1} - t_n) Q_{j+n} S(t_n) f_{0,j+n} \tag{32}$$

dove $S(t)$ è il trasporto libero su Γ_N, ovvero lo spazio delle fasi senza frontiera dovuta alle collisioni elastiche, e Q_j è l'operatore di collisione della gerarchia di Boltzmann.

Come primo passo per la dimostrazione del teorema di Lanford si dimostra che grazie a stime di continuità sugli operatori di collisione, le serie (30) e (32) sono convergenti in norma $\| \cdot \|_\infty$ per un tempo abbastanza piccolo. Il secondo passo consiste nel dimostrare che l'insieme su cui le due soluzioni sono differenti è piccolo per $\epsilon \to 0$. In particolare la maggiore differenza tra la soluzione della gerarchia BBGKY e la soluzione della gerarchia di Boltzmann è data dalle ricollisioni. Le ricollisioni sono collisioni tra due particelle che non sono date da una creazione di una nuova particella, ovvero da un operatore Q_j^ϵ nella serie (30) o Q_j nella serie (32). Per capire meglio analizziamo la struttura delle serie. Partiamo dalla serie (30), si procede indietro nel tempo per un tempo t_1 con il flusso a j particelle. Al tempo $t - t_1$ si aggiunge una nuova particella in configurazione pre o post-collisionale con una delle j particelle e si prosegue nuovamente indietro nel tempo per un tempo t_2. Si procede analogamente fino ad aver aggiunto n particelle. Gli operatori di collisioni vengono chiamati anche operatori di creazione perché di fatto aggiungono una nuova particella nella struttura della serie. La serie (32) funziona nello stesso modo tranne che per il flusso libero dato dall'operatore S che non conta eventuali collisioni interne tra le particelle presenti. Una ricollisione è quindi data da una collisione dovuta al flusso S^ϵ, le ricollisioni sono presenti nella serie (30) mentre non ci sono nella serie (32).

Dimostrare che le ricollisioni sono un insieme che tende a zero nel limite di Boltzmann-Grad è forse il passo più significativo nella derivazione rigorosa dell'equazione di Boltzmann, si va con mano a dimostrare che le differenze tra la dinamica di particelle e quella data dall'equazione di Boltzmann sono piccole nel limite di bassa densità.

La derivazione rigorosa dell'equazione di Boltzmann è stata recentemente provata anche nel caso di particelle interagenti tramite un potenziale regolare da Gallagher, Saint-Raymond e Texier [4] e da Pulvirenti, Saffirio e Simonella [5]. Si assume che il potenziale sia regolare, decrescente, radiale e a supporto compatto. Per semplicità supponiamo che $\phi(|x|) = 0$ se $|x| > 0$. L'interazione tra due particelle x_i e x_j può essere ridotta a un problema di forze centrali ponendo il centro di massa nell'origine delle coordinate e considerando la velocità relativa $V = v_i - v_j$.

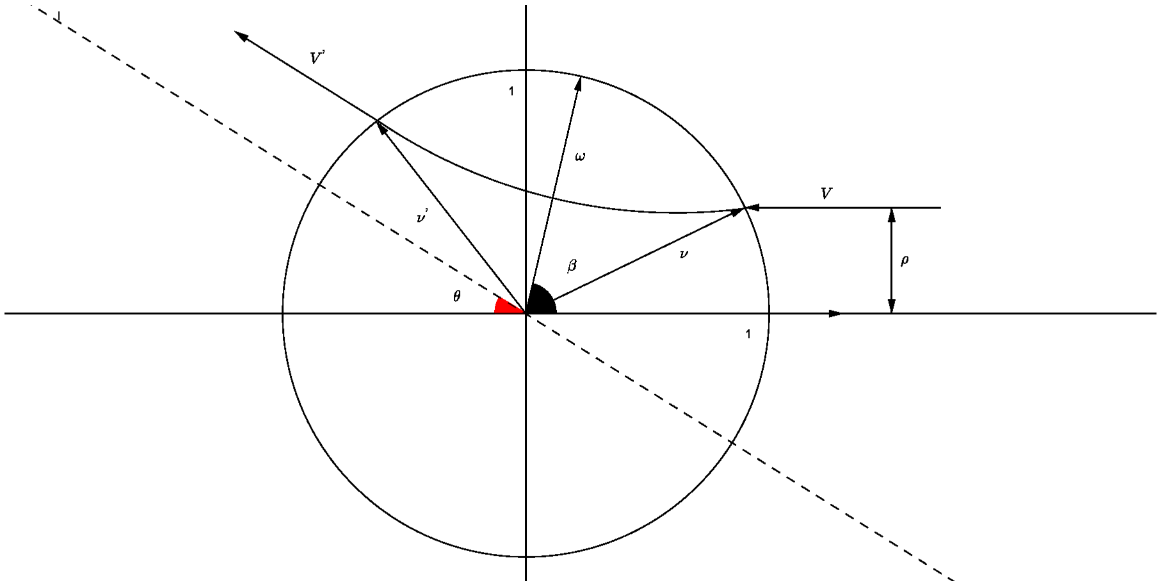

Figura 2: L'interazione tra due corpi ridotta a un problema di forze centrali. La velocità relativa post-collisionale è data da $V^{'} = v_i^{'} - v_j^{'}$, ρ è detto parametro di impatto mentre θè l'angolo di deflessione. Il versore ω divide a metà l'angolo tra $-V$ e $V^{'}$.

Nel caso di particelle interagenti tramite un potenziale la gerarchia di equazioni che si considera per l'evoluzione delle densità a j particelle non è la gerarchia BBGKY ma la gerarchia di Grad. Questa gerarchia viene ottenuta a partire dalla equazione di Liouville considerando invece delle usuali marginali altre quantità dette marginali ridotte. Queste sono definite come

$$\widetilde{f_j^N}(\boldsymbol{z}_j) = \int W^N(z_1,..,z_N) \prod_{\substack{1\leq i\leq j \\ j+1\leq k\leq N}} \chi\left\{|x_i - x_k| > \epsilon\right\} dz_{j+1}...dz_N \tag{33}$$

dove χ è la funzione caratteristica dell'insieme nelle parentesi graffe.

La gerarchia di Grad risulta essere asintotica nel limite di Boltzmann-Grad alla gerarchia BBGKY. Rimandiamo a [5] per una trattazione più esauriente dell'argomento.

3 Equazione di Landau

Nel 1936 Landau introdusse una nuova equazione cinetica per un gas denso debolmente interagente. Landau derivò l'equazione per particelle interagenti tramite un potenziale Coulombiano partendo dall'operatore di collisione di Boltzmann e mettendo alcuni cutoff per controllare gli insiemi patologici. L'idea è quella di considerare una espansione di Taylor rispetto al momento trasferito in una collisione, che risulta essere piccolo data la debole interazione tra le particelle. Da un punto di vista rigoroso si sa molto poco sulla equazione di Landau. Recentemente A.V. Boblylev, M. Pulvirenti e C. Saffirio in [1] hanno ottenuto un risultato di consistenza per la derivazione rigorosa dell'equazione di Landau a partire da un sistema di particelle.

Diamo ora una derivazione euristica dell'operatore di Landau partendo dall'operatore di Boltzmann. Consideriamo un operatore di collisione della seguente forma

$$Q(f) = \int dv_1 \int dp\, w(p)\delta(p^2 + (v - v_1) \cdot p) \left[f(x, v^{'}) f(x, v_1^{'}) - f(x, v) f(x, v_1) \right] \tag{34}$$

dove p rappresenta il momento trasferito nello scattering e $v^{'} = v + p$, $v_1^{'} = v - p$. Assumiamo che la funzione $w(p)$ sia a simmetria sferica e regolare quanto serve e osserviamo che la funzione δ ci assicura la conservazione dell'energia, mentre la conservazione del momento è automatica. Per rappresentare il fatto che il gas è denso riscaliamo l'operatore nella seguente maniera

$$Q(f) \to \frac{1}{\epsilon} Q(f) \tag{35}$$

Inoltre vogliamo che solamente le collisioni radenti siano rilevanti, le collisioni radenti corrispondo a collisioni in cui le velocità post-collisionali sono vicine alle velocità pre-collisionali. Per questo motivo riscaliamo nella seguente maniera la funzione $w(p)$

$$w(p) \to \frac{1}{\epsilon^3} w\left(\frac{p}{\epsilon}\right) \tag{36}$$

Il nostro operatore diventa così

$$Q_\epsilon(f) = \frac{1}{\epsilon^4} \int dv_1 \int dp\, w(\frac{p}{\epsilon})\delta(p^2 + (v - v_1) \cdot p) \left[f(x, v^{'}) f(x, v_1^{'}) - f(x, v) f(x, v_1) \right] =$$

$$\frac{1}{2\pi\epsilon^2} \int dv_1 \int dp\, w(p) \int_{-\infty}^{+\infty} ds\, e^{is(p^2\epsilon + (v - v_1)\cdot p)} \left[f(x, v + p) f(x, v_1 - p) - f(x, v) f(x, v_1) \right] =$$

$$\frac{1}{2\pi\epsilon} \int dv_1 \int dp\, w(p) \int_0^1 d\lambda \int_{-\infty}^{+\infty} ds\, e^{is(p^2\epsilon + (v - v_1)\cdot p)} \frac{d}{d\lambda} f(x, v + \epsilon\lambda p) f(x, v_1 - \epsilon\lambda p) =$$

$$\frac{1}{2\pi\epsilon} \int dv_1 \int dp\, w(p) \int_0^1 d\lambda \int_{-\infty}^{+\infty} ds\, e^{is(p^2\epsilon + (v - v_1)\cdot p)} p \cdot (\nabla_v - \nabla_{v_1}) f(x, v + \epsilon\lambda p) f(x, v_1 - \epsilon\lambda p) \tag{37}$$

Vogliamo studiare il comportamento asintotico dell'operatore per ϵ piccoli. Dimentichiamoci della dipendenza spaziale della f e consideriamo una funzione test φ, abbiamo che

$$(\varphi, Q_\epsilon(f)) = \frac{1}{2\pi\epsilon}\int dv \int dv_1 \int dp\, w(p) \int_0^1 d\lambda \int_{-\infty}^{+\infty} ds\, e^{is(p^2(\epsilon-2\epsilon\lambda)+(v-v_1)\cdot p)}\varphi(v-\epsilon\lambda p)p\cdot(\nabla_v-\nabla_{v_1})f(v)f(v_1)$$

$$= \frac{1}{2\pi\epsilon}\int dv \int dv_1 \int dp\, w(p) \int_0^1 d\lambda \int_{-\infty}^{+\infty} ds\, e^{is(v-v_1)\cdot p}[\varphi(v) - \epsilon p\cdot\nabla_v\varphi(v)]p\cdot(\nabla_v-\nabla_{v_1})f(v)f(v_1)+$$

$$\frac{1}{2\pi}\int dv \int dv_1 \int dp\, w(p) \int_{-\infty}^{+\infty} ds\, e^{is(v-v_1)\cdot p}\varphi(v)\, isp^2 \int_0^1 d\lambda(1-2\lambda)p\cdot(\nabla_v-\nabla_{v_1})f(v)f(v_1)+o(\epsilon) \quad (38)$$

Il termine divergente in ϵ è uguale a zero grazie alla parità di w, mentre dato che $\int_0^1 d\lambda(1-2\lambda)=0$ il termine di ordine 1 scompare. Rimane

$$(\varphi, Q_\epsilon(f)) = -\frac{1}{2\pi}\int dv \int dv_1 \int dp\, w(p) \int_{.-\infty}^{\infty} ds\, e^{is(v-v_1)\cdot p}(p\cdot\nabla_v\varphi)p\cdot(\nabla_v-\nabla_{v_1})f(v)f(v_1)+o(\epsilon) \quad (39)$$

Considerando il comportamento per $\epsilon \to 0$ abbiamo che

$$Q_L(f) = \int dv_1 \nabla_v a(\nabla_v - \nabla_{v_1})f(v)f(v_1) \quad (40)$$

dove a è la seguente matrice

$$a_{i,j}(v-v_1) = \int dp w(p)\delta(p\cdot(v-v_1))p_i p_j \quad (41)$$

La matrice si può anche scrivere nella seguente forma, indicando con $V = v - v_1$,

$$a_{i,j} = \frac{B}{|V|}\int d\hat{p}\,\delta(\hat{V}\cdot\hat{p})\,\hat{p}_i\hat{p}_j \quad (42)$$

dove $\hat{V} = \frac{V}{|V|}$, $\hat{p} = \frac{p}{|p|}$ e

$$B = \int_0^\infty dr\, r^3 w(r) \quad (43)$$

Un ultimo conto ci da che

$$a_{i,j} = \frac{B}{|V|}\left(\delta_{i,j} - \hat{V}_i\hat{V}_j\right) = \frac{B}{|V|}P_V^\perp \quad (44)$$

con $P_V^{\perp}$ il proiettore sull'ortogonale al sottospazio generato da V. Abbiamo così derivato l'equazione di Landau partendo dall'equazione di Boltzmann per un particolare operatore di collisione. Nel caso generale di particelle interagenti per mezzo di un potenziale ϕ, che assumiamo sia radiale, a supporto compatto e regolare, l'equazione di Landau è la seguente

$$\partial_t f + v \cdot \nabla f = \int dv_1 \nabla_v a(\nabla_v - \nabla_{v_1}) f(v) f(v_1) \tag{45}$$

dove anche in questo caso la matrice a è definita come

$$a_{i,j} = \frac{B}{|V|} P_V^{\perp} \tag{46}$$

ma

$$B = \frac{1}{8\pi} \int_0^{+\infty} dr\, r^3 \hat{\phi}^2(r) \tag{47}$$

dove $\hat{\phi}$ è la trasformata di Fourier di ϕ. Osserviamo che nel coefficiente B rimane l'unica informazione della interazione tra le particelle.

Una derivazione rigorosa per l'equazione di Landau non è ancora presente in letteratura. Nel 2001 L. Desvilettes e V. Ricci [3] hanno dimostrato una derivazione rigorosa di una equazione di Landau lineare partendo da un gas di Lorentz, ovvero una particella che si muove in un background di ostacoli fissi. Come vederemo l'equazione che si ottiene nel caso di un gas di Lorentz è diversa da quella ottenuta nel caso di un sistema di particelle interagenti.

Il punto di partenza per una derivazione rigorosa dell'equazione di Landau è uno scaling diverso da quello di bassa densità, detto weak-coupling. Consideriamo N particelle identiche che si muovono in un dominio di $\mathbb{R}^3$ e interagiscono attraverso un potenziale ϕ con le stesse ipotesi descritte prime. Le coordinate microscopiche delle particelle nello spazio delle fasi sono date dal vettore $\boldsymbol{q}_N = (q_1, ..., q_N)$ per le posizioni, e dal vettore $\boldsymbol{v}_N(v_1, ..., v_N)$ per le velocità, mentre con τ indichiamo il tempo. Riscaliamo il sistema nella seguente maniera

$$\boldsymbol{q}_N \to \frac{\boldsymbol{x}_N}{\epsilon} \quad \tau \to \frac{t}{\epsilon} \tag{48}$$

Dato che siamo interessati nel comportamento per un regime di alta densità poniamo

$$N\epsilon^2 = \epsilon^{-1} \tag{49}$$

Infine per concentrare le interazioni sulle collisioni radenti riscaliamo anche il potenziale nella seguente maniera

$$\phi(q) = \frac{1}{\sqrt{\epsilon}}\phi\left(\frac{x}{\epsilon}\right) \tag{50}$$

Vediamo intuitivamente perché uno scaling del genere dovrebbe portarci a una equazione di diffusione nelle velocità come quella di Landau. La forza è $O(\frac{1}{\sqrt{\epsilon}})$ e agisce in un intervallo di tempo $o(\epsilon)$, quindi la variazione di momento dovuta a uno scattering è $o(\sqrt{\epsilon})$. Una particella test incontra $O(\frac{1}{\epsilon})$ particelle, e quindi la variazione totale del momento in un unità di tempo è $O(\frac{1}{\sqrt{\epsilon}})$, ma ci aspettiamo che nel caso di un gas omogeneo e di una forza simmetrica questa variazione sia in media zero. La varianza invece è dell'ordine $\frac{1}{\epsilon}o(\sqrt{\epsilon})^2 = O(1)$. Da questa analisi euristica con un argomento del tipo limite centrale ci aspettiamo una diffusione in velocità.

Esistono scaling intermedi tra la bassa densità e il weak-coupling che danno come risultato l'equazione di Landau. Consideriamo lo scaling in spazio e tempo della formula (48) ma scaliamo la densità nella seguente maniera

$$N\epsilon^2 = \epsilon^{-\gamma} \tag{51}$$

e il potenziale come

$$\phi(q) \to \epsilon^{\frac{\gamma}{2}}\phi\left(\frac{x}{\epsilon}\right) \tag{52}$$

con $\gamma \in (0, 1]$. Osserviamo che per $\gamma = 0$ si ottiene lo scaling di bassa densità, mentre per $\gamma = 1$ otteniamo nuovamente il weak-coupling.

4 Equazioni lineari

Una delle difficoltà principali nella derivazione rigorosa delle equazioni di Boltzmann e di Landau risiede nel fatto che si tratta di equazioni non lineari. Per questo motivo in molti casi è utile introdurre una versione lineare della equazione che risulta essere più maneggevole e comunque in grado di descrivere fenomeni fisici interessanti. Per poter definire l'equazione di Boltzmann e di Landau lineare dobbiamo prima definire le soluzioni stazionarie di queste equazioni. In realtà la soluzione stazionaria è la stessa per entrambe le equazioni ed è data dalla distribuzione di Maxwell, detta anche Maxwelliana

$$M_\beta(v) = \frac{1}{C_\beta}e^{-\beta|v|^2} \tag{53}$$

dove $\beta > 0$ è un parametro legato alla temperatura del gas e C_β è scelta in modo tale che

$$\int dv\, M_\beta(v) = 1 \tag{54}$$

Consideriamo quindi la gerarchia di Boltzmann (27) con dato iniziale dato da una perturbazione nella prima particella $g_0(x_1, v_1)$ della Maxwelliana stazionaria

$$f_{0,j} = g_0(x_1, v_1) M^j_\beta(\boldsymbol{v}_j) \tag{55}$$

dove abbiamo definito

$$M^j_\beta(\boldsymbol{v}_j) = \prod_{i=1}^{j} M_\beta(v_i) \tag{56}$$

Supponiamo che la soluzione a un tempo t sia data dall'evoluzione della perturbazione iniziale moltiplicata per la Maxwelliana, ovvero

$$f_j(t) = g(x_1, v_1, t) M^j_\beta(\boldsymbol{v}_j) \tag{57}$$

Imponendo che f_j sia soluzione della gerarchia di Boltzmann otteniamo che g deve risolvere la seguente equazione, detta equazione di Boltzmann lineare

$$\partial_t g + v \cdot \nabla_x g = \int dv_1 M_\beta(v_1) \int\limits_{\nu \cdot V > 0} d\nu |\nu \cdot V| \left[g(x, v^{'}) - g(x, v) \right] \tag{58}$$

Il termine di destra della (58) rappresenta l'interazione tra la particella perturbata e il background all'equilibrio. L'equazione di Boltzmann lineare perde le proprietà di conservazione del momento e di conservazione dell'energia, mentre rimane valida la conservazione della massa.

Grazie alla linearità del problema risulta più semplice una derivazione rigorosa dell'equazione di Boltzmann lineare a partire da un sistema di particelle interagenti. Ciò è stato fatto nel caso di sfere rigide da T.Bodineau, I. Gallagher e L. Saint-Raymond in [2]. In particolare uno dei punti fondamentali è l'esistenza di una stima a priori della soluzione nel caso lineare per particelle che si muovono in un dominio $D \subset \mathbb{R}^3$ limitato.

Anche nel caso dell'equazione di Landau è possibile derivare una equazione lineare considerando una perturbazione dell'equilibrio. Partendo dall'equazione di Landau

$$\partial_t f + v \cdot \nabla f = \int dv_1 \nabla_v a(\nabla_v - \nabla_{v_1}) f(v) f(v_1) \tag{59}$$

per rappresentare la perturbazione in un background stazionario poniamo

$$f(v) = M_\beta(v) h(v) \tag{60}$$

$$f(v_1) = M_\beta(v_1) \tag{61}$$

Esplicitando gli operatori differenziali si arriva alla seguente equazione lineare per la perturbazione h

$$\partial_t h + v \cdot \nabla_x h = A \int dv_1 M_\beta(v_1) \frac{1}{|V|^3} \left[|V|^2 \triangle h(v) - \big(V, D^2 V\big) - 4V \cdot \nabla_v h(v) \right] \tag{62}$$

dove $V = v - v_1$, A è una opportuna costante e D^2 è la matrice Hessiana di h rispetto alla velocità v.

Come abbiamo già detto una equazione lineare del tipo Landau è stata derivata da L. Desvilettes e V. Ricci nel caso di un gas di Lorentz in due dimensioni. In questo caso si ottiene una equazione diversa dalla (62) in quanto il background in cui si muove la particella test è formato da ostacoli fermi e non da particelle interagenti tra di loro. Questo fa si che la velocità della particella test rimanga sempre costante in modulo e quindi si può assumere che v appartenga alla sfera unitaria $S^1 \subset \mathbb{R}^2$. L'equazione ottenuta nel caso di un gas di Lorentz è la seguente

$$\partial_t f + v \cdot \nabla f = \xi \triangle_v f(v) \tag{63}$$

dove ξ è un parametro di diffusione legato alla distribuzione degli ostacoli e $\triangle_v$ è l'operatore di Laplace-Beltrami su S^1.

5 Derivazione rigorosa dell'equazione di Landau lineare

Lo scopo del lavoro presentato è quello di dimostrare la derivazione rigorosa dell'equazione di Landau lineare (62) partendo dalla dinamica di particelle. Nel nostro caso consideriamo N particelle che si muovono in un toro $\Gamma = [0, \frac{1}{\epsilon})^3 \subset \mathbb{R}^3$ con $\epsilon > 0$. Le particelle interagiscono tra loro per mezzo di un potenziale Φ. Le ipotesi che facciamo sul potenziale sono che $\Phi(q) = 0$ se $|q| > 1$, $\Phi \in C^2(\mathbb{R}^3)$ e

$\|\Phi\|_\infty < +\infty$. Supponiamo inoltre che sia radiale, positivo e non decrescente.

Come abbiamo visto per sperare di ottenere l'equazione di Landau partendo dalla dinamica di particelle dobbiamo riscalare il sistema attraverso lo scaling di weak-coupling o utilizzando uno degli scaling intermedi. Nel nostro caso riscaliamo nella seguente maniera

$$\boldsymbol{q}_N \to \frac{\boldsymbol{x}_N}{\epsilon} \quad \tau \to \frac{t}{\epsilon} \tag{64}$$

$$N\epsilon^2 = \alpha \tag{65}$$

$$\phi(q) = \frac{1}{\sqrt{\alpha}}\phi\left(\frac{x}{\epsilon}\right) \tag{66}$$

con α divergente come $\log\log N$. Questo scaling intermedio tra weak-coupling e bassa densità risulta molto vicino alla bassa densità e ci consente di utilizzare le tecniche del lavoro di controllo delle ricollisioni per dimostrare la convergenza della soluzione di particelle a quella dell'equazione di Boltzmann lineare.

La dimostrazione della derivazione rigorosa si basa su due passi fondamentali, e cioè dimostrare prima la derivazione rigorosa dell'equazione di Boltzmann lineare dalla dinamica di particelle per ogni tempo e successivamente dimostrare che la soluzione dell'equazione di Boltzmann lineare converge per ogni tempo alla soluzione di Landau Lineare.

Per dimostrare il primo passo utilizziamo la tecnica del lavoro [2] adattandola al caso di interazione dovuta a un potenziale e non a urti elastici. L'idea nel caso di sfere dure è di scegliere un T arbitrario e dividere l'intervallo $[0, T]$ in k parti di lunghezza s. Si considera la soluzione per serie della gerarchia BBGKY data dalla (30) ma tornando indietro fino al tempo $t-s$ invece che fino al tempo 0, esprimendo così la soluzione al tempo t come l'evoluzione della soluzione al tempo $t-s$. Abbiamo

$$f_j^N(t) = \sum_{n=0}^{N-j} \epsilon^{2n}(N-j)...(N-j-n+1)\int_0^t dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{n-1}} dt_n$$

$$S^\epsilon(t-t_1)Q^\epsilon_{j+1}...S^\epsilon(t_{n-1}-t_n)Q^\epsilon_{j+n}S^\epsilon(t_n)f^N_{j+n}(t-s) \tag{67}$$

A questo punto si spezza sulle creazioni limitandole con un parametro $P \geq 2$. Si ottiene

$$f_j^N(t) = \sum_{n=0}^{P} \epsilon^{2n}(N-j)...(N-j-n+1)\int_0^t dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{n-1}} dt_n$$

$$S^\epsilon(t-t_1)Q^\epsilon_{j+1}...S^\epsilon(t_{n-1}-t_n)Q^\epsilon_{j+n}S^\epsilon(t_n)f^N_{j+n}(t-s)+R(t-s) \tag{68}$$

dove il termine di resto $R(t-s)$ è la coda della serie. Si può iterare il procedimento k volte arrivando fino al tempo 0 considerando per il j-esimo intervallo un numero di creazioni limitato da P^j. Otteniamo così

$$f^N_j(t)=\sum_{j_1=0}^{P^1}\sum_{j_2=0}^{P^2}...\sum_{j_k=0}^{P^k}A^\epsilon_{j,j_1}(s)A^\epsilon_{j+j_1,j_2}(s)....A^\epsilon_{j+j_1+..+j_{h-1},j_h}(s)f^N_{0,j+j_1+..+j_h}+R \tag{69}$$

dove abbiamo definito per comodità

$$A^\epsilon_{j,n}(h)=\epsilon^{2n}(N-j)...(N-j-n+1)\int_0^t dt_1\int_0^{t_1}dt_2...\int_0^{t_{n-1}}dt_n$$

$$S^\epsilon(t-t_1)Q^\epsilon_{j+1}...S^\epsilon(t_{n-1}-t_n)Q^\epsilon_{j+n}S^\epsilon(t_n)f^N_{j+n}(t-h) \tag{70}$$

e R è un complicato termine di resto che tiene con dell'evoluzione all'indietro nel tempo di tutte le code delle serie. Le code delle serie rappresentano tutte le traiettorie che hanno un numero di creazioni super-esponenziale. Con una scelta opportuna del parametro k e con un dato iniziale dato da

$$f^N_{0,n}=g_0(x_1,v_1)\prod_{i=1}^n M_\beta(v_1) \tag{71}$$

si dimostra che il termine di resto è arbitrariamente piccolo, grazie anche alla stima a priori. In questo modo si estende la convergenza della serie ad ogni tempo. Nel nostro caso la serie che prendiamo in considerazione non è quella derivante dalla gerarchia BBGKY ma quella relativa alla gerarchia di Grad. Si costruisce poi una serie simile per la gerarchia di Boltzmann e si prova anch'essa essere convergente per ogni tempo.

Per dimostrare che le due soluzioni sono una convergente all'altra dobbiamo dimostrare che le ricollisioni tendono a zero. Per fare questo adattiamo la tecnica di [5] per particelle su tutto lo spazio nel caso sul toro. Il controllo delle ricollisioni è particolarmente tecnico, l'idea generale è che se due particelle ricollidono dopo una creazione è possibile trovare un elemento di piccolezza, che va a zero con opportunamente conϵ, nella storia all'indietro delle due particelle.

La dimostrazione della convergenza dell'equazione di Boltzmann lineare a l'equazione di Landau lineare si basa sull'idea di Landau. Dal riscalamento effettuato sul potenziale segue la seguente formula per l'angolo di deflessione θ

$$\theta(\rho,\alpha) = \frac{-2}{|V|^2\sqrt{\alpha}}\gamma(\rho,\alpha) + \frac{1}{|V|^4\alpha}M(\rho,\alpha) \tag{72}$$

dove M e γ sono opportune funzioni limitate. Quando $\alpha \to \infty$ l'angolo θ tende a zero e di conseguenza la differenza tra le velocità post-collisionali e pre-collisionali va zero. Partendo dall'equazione di Boltzmann lineare consideriamo quindi uno sviluppo in serie di Taylor per momenti trasferiti piccoli. In particolare, riscrivendo l'equazione di Boltzmann lineare per chiarezza,

$$\partial_t g + v \cdot \nabla_x g = \int dv_1 M_\beta(v_1) \int_{\nu \cdot V > 0} d\nu |\nu \cdot V| \left[g(x, v^{'}) - g(x,v) \right] \tag{73}$$

sviluppiamo in serie di Taylor il termine $g(x, v^{'}) - g(x,v)$. Il primo ordine diverso da zero ottenuto in questo modo è formalmente asintotico all'equazione di Landau lineare

$$\partial_t g + v \cdot \nabla_x g = A \int dv_1 M_\beta(v_1) \frac{1}{|V|^3} \left[|V|^2 \triangle g(v) - \left(V, D^2 V\right) - 4V \cdot \nabla_v g(v) \right] \tag{74}$$

Incidentalmente osserviamo che l'equazione di Landau lineare emerge come sviluppo al secondo ordine, lo sviluppo al primo ordine si dimostra essere zero grazie alla simmetria del potenziale.

Attraverso tecniche di analisi funzionale e di teoria dei semigruppi, ambientando il problema in un opportuno spazio di Hilbert, otteniamo la convergenza della soluzione della equazione (73) alla soluzione della equazione (74) per ogni tempo.

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The rigorous derivation of the Linear Landau equation from a particle system in a weak-coupling limit

N. Catapano*

*Dipartimento di Matematica, Università di Roma “La Sapienza”, P.le A. Moro, 5, 00185 Roma, Italy.

E-mail: catapano@mat.uniroma1.it

Abstract

We consider a system of N particles interacting via a short-range smooth potential, in a weak-coupling regime. This means that the number of particles N goes to infinity and the range of the potential ϵ goes to zero in such a way that $N\epsilon^2 = \alpha$, with α diverging in a suitable way in this regime. We provide a rigorous derivation of the Linear Landau equation from this particle system. The strategy of the proof consist in showing the asymptotic equivalence between the one-particle marginal and the solution of the linear Boltzmann equation with vanishing mean free path. This point follows [1] and makes use of technicalities developed in [2]. Then, following the ideas of Landau, we prove the asympotic equivalence between the solutions of the Boltzmann and Landau linear equation in the grazing collision limit.

Contents

1 Introduction

1.1 The Boltzmann-Grad limit

In kinetic theory a gas is described by a system of small indistinguishable interacting particles. The evolution of this system is quite complicated since the order of particles involved is quite large. For this reason it is interesting to consider the system from a statistical point of view. The starting point is a system of N particles having unitary mass and moving in a domain $D \subseteq \mathbb{R}^3$. These particles can interact by means of a short-range radial potential Φ. The microscopic state of the system is given by the position and velocity variables denoted by $\boldsymbol{q}_N = (q_1, q_2, ..., q_N)$ and $\boldsymbol{v}_N = (v_1, v_2, ..., v_N)$, where q_i, v_i are respectively position and velocity of the i-th particle. The time is denoted by τ. Throughout the paper we will use bold letters for vectors of variables.

Let $\epsilon > 0$ be a parameter denoting the ratio between typical macroscopic and microscopic scales, say the inverse of the number of atomic diameters necessary to fill a centimeter. If we want a macroscopid description of the system it is natural to introduce macroscopic variables defined by

$$\boldsymbol{x}_N = \boldsymbol{q}_N \epsilon \quad t = \tau\epsilon \tag{1.1}$$

where $\boldsymbol{x}_N = (x_1, x_2, ..., x_N)$ are the macroscopic position and t is the macroscopic time variable. From the Liouville equation for the dynamics of particles it is possible to derive a hierarchy of equations for the j-particles marginal probability density function, with $j \leq N$. In the case of hard spheres we found the following BBGKY hierarchy

$$\left(\partial_t + \boldsymbol{v}_j \cdot \nabla_{\boldsymbol{x}_j}\right) f_j^N = (N-j)\epsilon^2 \sum_{k=1}^{j} \int_{\mathbb{R}^3} dv_{j+1} \int_{\nu\cdot(v_k - v_{j+1})\geq 0}$$

$$|\nu \cdot (v_k - v_{j+1})| \left[f_{j+1}^N(x_1, v_1, ..., x_k, v_k^{'}, ..., x_j, v_j, x_k - \eta\epsilon, v_{j+1}^{'}) - f_{j+1}^N(x_1, v_1, ..., x_j, v_j, x_k + \eta\epsilon, v_{j+1}) \right] \tag{1.2}$$

where $\nu = \frac{x_{j+1} - x_k}{|x_{j+1} - x_k|}$ and $v_k^{'} = v_k - \nu\left[\nu \cdot (v_k - v_{j+1})\right]$, $v_{j+1}^{'} = v_{j+1} + \nu\left[\nu \cdot (v_k - v_{j+1})\right]$. Equations 1.2 were first formally derived by [3], then a rigorous analysis has been done by [4, 5, 6, 7].

Scaling according to $N \to \infty$ and $\epsilon \to 0$, in such a way that $N\epsilon^2 \cong 1$, we are in a low-density regime suitable for the description of a rarified gas. This kind of scaling is usually called the Boltzmann-Grad limit. The formal Boltzmann-Grad limit in the BBGKY gives a new hierarchy of equations called the Boltzmann hierarchy. The central idea in kinetic theory is the concept of propagation of chaos, namely, if the initial datum factorizes, i.e. $f_{0,j}(\boldsymbol{x}_j, \boldsymbol{v}_j) = \prod_{i=1}^{j} f_{0,1}(x_i, v_i)$, then also the solution at time t factorizes:

$$f_j(\boldsymbol{x}_j, \boldsymbol{v}_j) = \prod_{i=1}^{j} f_1(x_i, v_i). \tag{1.3}$$

Actually the Boltzmann hierarchy admits factorized solutions so that it is compatible with the propagation of chaso and under this hypothesis, which however must be proved froma rigorous view point, the first equation of this hierarchy is the Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = \int d\nu dv_1 B(\nu, v - v_1) \left[f(x, v^{'}) f(x, v_1^{'}) - f(x, v) f(x, v_1) \right]. \tag{1.4}$$

However, as soon as $\epsilon > 0$ propagation of chaos does not hold because the evolution creates correlation between particles so that we cannot describe the system in terms of a single equation for the one-particle marginal and is the reason why the Boltzmann equation can describe in a more handable way the statistical evolution of a gas.

The validity of the Boltzmann equation is a fundamental problem in the kinetic theory. It consists in proving that the solution of the BBGKY hierarchy for hard spheres converge in the Boltzmann-Grad limit to the solution of the Boltzmann hierarchy. This means that the propagation of chaos is recovered in the limit.

The rigorous derivation of the Boltzmann equation was first proved by Lanford in 1975 [8] in the case of an hard spheres system for a small time. The main idea of the Lanford work is to write the solution of the BBGKY hierarchy for hard spheres and of the Boltzmann hierarchy as a perturbative series of the free evolution and then prove that the series solution of the BBGKY converge to the series solution of the Boltzmann hierarchy.

More recently Gallagher, Saint-Raymond and Texier [9] and Pulvirenti, Saffirio and Simonella [2] proved the rigorous derivation of the Boltzmann equation, for a small time, starting from a system of particle interacting by means of a short-range potential providing an explicit rate of convergence. In the case of a short-range potential the starting hierarchy is no more the BBGKY hierarchy but the Grad hierarchy, that was developed by Grad in [10].

1.2 The linear case

The linear Boltzmann equation describes the evolution of a tagged particle in a random stationary background at equlibrium and reads as follows

$$\partial_t g^\alpha + v \cdot \nabla_x g^\alpha = \alpha \int dv_1 M_\beta(v_1) \int d\nu B(\nu, v - v_1) \left[g^\alpha(x, v^{'}) - g^\alpha(x, v) \right] \tag{1.5}$$

where $M_\beta(v_1) = \frac{1}{C_\beta} e^{-\beta |v_1|^2}$ and C_β is chosen in such a way that $\int dv_1 M_\beta(v_1) = 1$. The linear Boltzmann equation can be obtained from the equation (1.4) setting $f(x, v) = g^\alpha(x, v) M_\beta(v)$ and $f(x, v_1) = M_\beta(v_1)$, g^α is the evolution of the perturbation in the stationary background given by $M_\beta(v)$.

The derivation of the linear Boltzmann equation from an hard spheres system has been proved for an arbitrary time by Spohn, Lebowitz [11] and more recently quantitative estimates on the rate of convergence have been obtained by Bodineau, Gallagher and Saint-Raymond [1]. A different type of linear Boltzmann equation has been derived in the case of a Lorentz gas by [12, 13].

1.3 A different scaling

A different scaling can be used to study a different regime from the low density. In case of particles interacting by means of a short-range radial potential Φ, we rescale position and time as in (1.1) but we set $N\epsilon^2 \cong \epsilon^{-1}$ and $\Phi(q) = \epsilon^{-\frac{1}{2}} \Phi(\frac{x}{\epsilon})$. This scaling is called the weak-coupling limit since the density of the particle is diverging in the limit but this is balanced by the interaction that becomes weaker. This weak interaction between particles is called also a "grazing collision" since it changes only slightly the velocity of a particle. The kinetic equation derived from this scaling is the Landau equation

$$\partial_t f + v \cdot \nabla_x f == \int dv_1 \nabla_v \cdot \left[\frac{A}{|v - v_1|} P^\perp_{(v-v_1)} \left(\nabla_v - \nabla_{v_1} \right) f(v) f(v_1) \right] \tag{1.6}$$

where A is a suitable constant and $P^\perp_{(v-v_1)}$ is the projector on the orthogonal subspace to that generated by $v - v_1$.

The Landau equation was derived in a formal way by Landau in [14] starting from the Boltzmann equation in the so-called grazing collision limit. It rules the dynamics of a dense gas with weak interaction between particles. Recently Boblylev, Pulvirenti and Saffirio proved in [15] a result of consistency, but the problem of the rigorous derivation of Landau equation is still open even for short times.

Also in the case of the Landau equation it is possible to consider the evolution of a perturbation of the stationary solution. This evolution is given by the following linear Landau equation

$$\partial_t g + v \cdot \nabla_x g =$$

$$A \int dv_1 M_\beta(v_1) \frac{1}{|V|^3} \left[|V|^2 \triangle g(v) - \big(V, D^2(g)V\big) - 4V \cdot \nabla_v g(v) \right] \tag{1.7}$$

where $D^2(g)$ is the hessian matrix of g with respect to the velocity variables and A is a suitable constant.

Recently Desvillettes and Ricci [16] and Kirkpatrick [17]proved a rigorous derivation for a type of linear Landau equation in two dimensions starting from a Lorentz gas. In this case the velocity of the test particles does not change and the equation obtained is a diffusion of the velocity on the unitary sphere.

1.4 Main theorem

In this paper we prove the rigorous derivation of the linear Landau equation starting from a system of particles. These particles interact by means of a two body short-range smooth potential and we consider an initial datum given which is a perturbation of the equlibrium. We rescale the variables describing the particles system according to (1.1). Simultanously we set $N\epsilon^2 \cong \alpha$ and $\Phi(q) = \frac{1}{\sqrt{\alpha}}\Phi(\frac{x}{\epsilon})$. This gives us an intermediate scaling between the low density and the weak-coupling and allows us to use the properties of both. Thanks to the low density properties of the scaling as first step we prove that the dynamics of the particles system is near to the solution of the linear Boltzmann equation. In a second step using the weak-coupling properties of the scaling we show that the solution of the linear Landau equation is near to the solution of the linear Boltzmann equation. More precisely let $\overline{f_1^N}$ be the one particle marginal distribution and let g^α be the solution of the linear Boltzmann equation, then we are able to prove that

$$\|\overline{f_1^N}(x,v) - g^\alpha(x,v)M_\beta(v)\|_\infty \to 0. \tag{1.8}$$

Then, denoting with g the solution of the linear Landau equation, it results that

$$\|g(x,v) - g^{\alpha}(x,v)\|_{\mathbf{H}} \to 0 \tag{1.9}$$

where $\mathbf{H} = L^2\left(\Gamma \times \mathbb{R}^3, dx d\mu\right)$, with $d\mu = M_\beta(v)dv$.

2 Dynamics and statistical description of the motion

2.1 Hamiltonian system

We consider a system of N indistinguishable particles with unitary mass moving in a torus $\Gamma_\epsilon = [0, \frac{1}{\epsilon})^3 \subset \mathbb{R}^3$ with $\epsilon > 0$. The particles interact by means of a two body positive, radial and not increasing potential $\Phi : \mathbb{R}^3 \to \mathbb{R}$. We assume also that Φ is short-range, namely $\Phi(q) = 0$ if $|q| > 1$, moreover $\Phi \in C^2(\mathbb{R}^3)$. The Hamiltonian of the system is given by

$$H = \frac{1}{2}\sum_{i=1}^{N} |v_i|^2 + \frac{1}{2}\sum_{i,j=1,\, i\neq j}^{N} \Phi(q_i - q_j) \tag{2.1}$$

where q_i, v_i are respectively position and velocity of the i-th particle.

The Newton equations are the following

$$\frac{d^2 q_i}{d\tau^2}(\tau) = \sum_{i\neq j} F\left(q_i(\tau) - q_j(\tau)\right) \tag{2.2}$$

for $i = 1, ..., N$, where $F\left(q_i - q_j\right) = -\nabla\Phi\left(q_i - q_j\right)$ and τ is the time variable. The hypothesis that we made on the potential ensure the existence and uniqueness of the solution of the (2.2).

2.2 Scaling

We rescale the system from microscopic coordinates (q, τ) to macroscopic ones in the following way. We set

$$x = \epsilon q \quad t = \epsilon\tau \tag{2.3}$$

where x, t are respectively the macroscopic position variable and the macroscopic time variable. We set $N\epsilon^2 \cong \alpha$, with $\alpha \cong (\log\log N)^{\frac{1}{2}}$, and we also assume that $|N\epsilon^2 - \alpha| \to 0$. With this scaling the density of the gas and the inverse of the mean free path are diverging in the limit. This means that a given particle experiences an high number of interaction per unit time. To balance this divergence we rescale also the potential in the following way

$$\Phi \to \alpha^{-\frac{1}{2}}\Phi \tag{2.4}$$

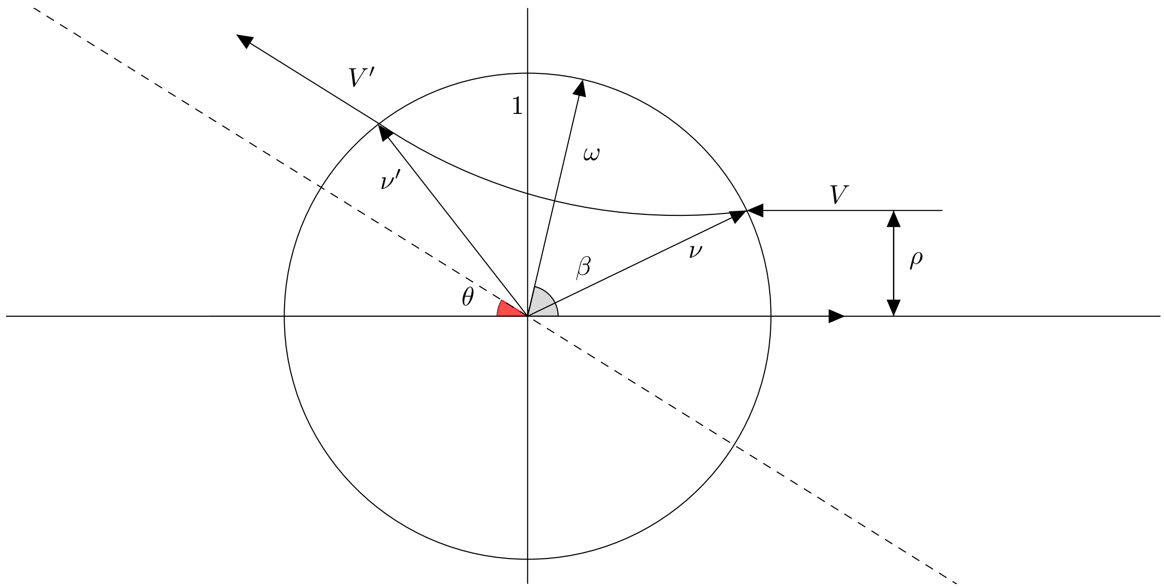

Figure 1: Here $\omega = \omega(\nu, V)$ is the unit vector bisecting the angle between $-V$ and V', ν is the unit vector pointing from the particle with velocity v_1 to the particle with velocity v_2 when they are about to collide. We denote with β the angle between $-V$ and ω, with φ the angle between $-V$ and ν, with $\rho = \sin\varphi$ the impact parameter and with θ the deflection angle. It results that $\theta = \pi - 2\beta$

In the microscopic variables the equations of motion read as

$$\frac{d^2 x_i}{dt^2}(\tau) = \frac{1}{\epsilon\sqrt{\alpha}} \sum_{i \neq j} -\nabla\Phi\left(\frac{x_i(t) - x_j(t)}{\epsilon}\right). \tag{2.5}$$

From now we shall work in macroscopic variables unless explicitely indicated.

2.3 The scattering of two particles

In this section we want to give a picture of the scattering between two particles. We turn back to microscopic variables where the potential is assumed to have range one. Let q_1, v_1, q_2, v_2 be positions and velocities of two particles which are performing a collision. This two-body problem can be reduced to a central-force problem if we set the origin of the coordinates c in the center of mass

$$c = \frac{q_1 + q_2}{2} \tag{2.6}$$

Thanks to the conservation of the angular momentum we have that the scattering takes place on a plane. We define $V = v_1 - v_2$ as the incoming relative velocity and $V^{'} = v_1^{'} - v_2^{'}$ as the outgoing relative velocity with

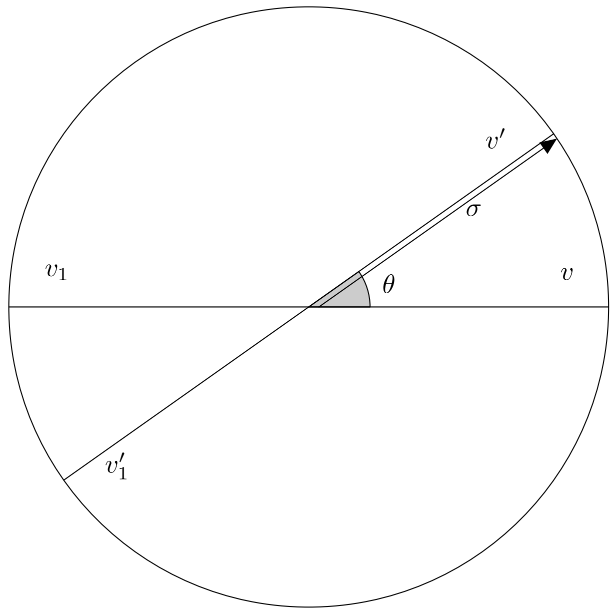

Figure 2: We denote with $\sigma \in S^2\left(\frac{v_1+v_2}{2}\right)$ the direction of $V^{'}$ and with θ the angle between V and $V^{'}$.

$$\begin{cases} v_1^{'} = v_1 - \omega\left[\omega \cdot V\right] \\ v_2^{'} = v_2 + \omega\left[\omega \cdot V\right] \end{cases} \tag{2.7}$$

Another useful way to represent the collision between two particles is the so called σ-representation (Figure 2). With this notation the post collisional velocities can be written as follow

$$\begin{cases} v^{'} = \frac{v+v_1}{2} + \frac{|v-v_1|}{2}\sigma \\ v_1^{'} = \frac{v+v_1}{2} - \frac{|v-v_1|}{2}\sigma \end{cases} \tag{2.8}$$

We can now define the scattering operator I, a map defined over

$$\left\{(\nu, V) \in S^2 \times \mathbb{R}^3 \backslash \{0\}\ s.t.\, V \cdot \nu \leq 0\right\} \tag{2.9}$$

by

$$I(\nu, V) = \left(\nu^{'}, V^{'}\right) \tag{2.10}$$

$$\begin{cases} V^{'} = V - 2\omega\,(\omega \cdot V) \\ \nu^{'} = -\nu + 2\omega\,(\omega \cdot \nu) \end{cases} \tag{2.11}$$

From the definition of $\nu^{'}$ and $V^{'}$ we have that $\nu \cdot V = -\nu^{'} \cdot V^{'}$. It follows that I sends incoming configuration in outgoing configuration. The main property of I is given by the following lemma, proved in [2].

Lemma 2.1. *I is an invertible transformation that preserves the Lebesgue measure.*

We conclude this section with an estimate for the angle θ, for which a complete proof can be found in [16]

Lemma 2.2. *Let Φ be a potential satisfying our assumption and let $\theta(\rho, \alpha)$ be the scattering angle in function of the impact parameter ρ. Then the following estimate holds true:*

$$\theta(\rho, \alpha) \leq \frac{-2}{|V|^2\sqrt{\alpha}}\gamma(\rho) + \frac{1}{|V|^4\alpha}M(\rho, \alpha) \tag{2.12}$$

where

$$\gamma(\rho) = \int_{|\rho|}^{1} \frac{\rho}{u}\Phi^{'}\left(\frac{|\rho|}{u}\right)\frac{du}{\sqrt{1-u^2}} \tag{2.13}$$

and $M(\rho, \alpha)$ is positive bounded functions.

Remark 2.3. Formula (2.12) points out that when $\alpha \to \infty$ the collision becomes grazing.

2.4 Statistical description

Now we want to describe our system from a statistical point of view. We will denote the phase space as

$$\Lambda_N = \left\{ \boldsymbol{z}_N \in \left(\Gamma \times \mathbb{R}^3\right)^N \right\} \tag{2.14}$$

where $\boldsymbol{z}_N = (z_1, z_2, .., z_N)$, $z_i = (x_i, v_i)$ and Γ is the torus of unitary side.

We consider a probability density function $W_{0,N}$ defined on Λ_N. The time evolution of $W_{0,N}$ is given by the solution W_N of the following Liouville equation

$$\begin{cases} \partial_t W_N + \mathcal{L}_N W_N = 0 \\ W_N(0) = W_{0,N} \end{cases} \tag{2.15}$$

where $\mathcal{L}_N = \mathcal{L}_N^0 + \mathcal{L}_N^I$ with

$$\mathcal{L}_N^0 = \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} \tag{2.16}$$

$$\mathcal{L}_N^I = \frac{1}{\epsilon} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} F_{i,j} \cdot \nabla_{v_i} \tag{2.17}$$

and $F_{i,j} = -\frac{1}{\sqrt{\alpha}} \nabla\Phi\left(\frac{x_i(t)-x_j(t)}{\epsilon}\right)$. We suppose that $W_{0,N}$ is symmetric in the exchange of particles, and hence $W_N(t)$ is still symmetric for any positive times.

The marginals distribution of the measure $W_N(t)$ are defined as

$$\overline{f_j^N}(\boldsymbol{z}_j, t) = \int dz_{j+1}...dz_N \, W_N(\boldsymbol{z}_N, t) \tag{2.18}$$

Nevertheless, it is more convenient to work with the reduced marginals $\widetilde{f_j^N}(\boldsymbol{z}_j, t)$ that read as follow

$$\widetilde{f_j^N}(\boldsymbol{z}_j, t) = \int_{S(\boldsymbol{x}_j)^{N-j}} dz_{j+1}...dz_N \, W_N(\boldsymbol{z}_N, t) \tag{2.19}$$

where

$$S(\boldsymbol{x}_j)^{N-j} = \left\{ z = (x, v) \in \Gamma \times \mathbb{R}^3 \,|\, |x - x_k| > \epsilon \; \forall \, 1 \leq k \leq j \right\} \tag{2.20}$$

As can be easily seen the reduced marginals are asymptotically equivalent (for $\epsilon \to 0$) to the standard marginals.

For the reduced marginals it is possible to derive from the Liouville equation the following hierarchy of equations, called the Grad hierarchy (GH),

$$(\partial_t + \mathcal{L}_j) \, f_j^N = \sum_{m=0}^{N-j-1} A_{j+1+m}^{\epsilon} f_{j+1+m}^N \quad 0 \leq j \leq N \tag{2.21}$$

where

$$A_{j+1+m}^{\epsilon} f_{j+1+m}^N (z_j, t) = \binom{N-j-1}{m} (N-j)\epsilon^2 \sum_{i=1}^{j} \int_{S^2} d\nu \chi_{\{min_{l=1,\dots,j; l \neq i} |x_i + \nu\epsilon - x_l > \epsilon|\}}(\nu)$$

$$\int_{\mathbb{R}^3} dv_{j+1} \, (v_{j+1} - v_j) \cdot \nu \int_{\Delta_m(\boldsymbol{x}_{j+1})} dz_{j+1}...dz_{j+1+m} f_{j+1+m}^N (\boldsymbol{z}_j, x_i + \nu\epsilon, v_{j+1}, \boldsymbol{z}_{j+1,m}, t) \tag{2.22}$$

and $\boldsymbol{z}_{j+1,m} = (z_{j+1}, ..., z_{j+1+m})$. The set $\Delta_m(x_{j+1})$ is defined as follows

$$\begin{aligned}\Delta_m(\boldsymbol{x}_{j+1}) = \{&\boldsymbol{z}_{j+1,m} \subset S(x_j)^m \text{ such that } \forall\, l = j+2, ..., j+1+m, \text{ there exists} \\ &\text{a choice of index } h_1, ..., h_r \in \{j+2, ..., j+1+m\} \\ &\text{such that } |x_l - x_{h_1}| \leq \epsilon,\ |x_{h_{k}-1} - x_{h_k}| \leq \epsilon \text{ for } k = 2, ..., r \\ &\text{and } \min_{i \in \{l, h_1, ..., h_r\}} |x_i - x_{j+1}| \leq \epsilon \Big\}\end{aligned} \tag{2.23}$$

This hierarchy was first introduced by Grad [10]. Actually in views of the Boltzmann-Grad limit only the first equation of this hierarchy was considered. The full hierarchy was introduced and derived by King in [18]. A complete derivation of this hierarchy can also be found in [9] adn [2].

It is possible to represent the solution of the Grad hierarchy as a series obtained by iterating the Duhamel formula. It results that

$$\widetilde{f_j^N}(t) = \sum_{n=0}^{\infty} G_{j,n}^{\epsilon}(t) f_{0,j}^N \tag{2.24}$$

where

$$f_{0,j}^N = \int_{S(\boldsymbol{x}_j)^{N-j}} dz_{j+1} ... dz_N\, W_{0,N}(\boldsymbol{z}_N) \tag{2.25}$$

and $G_{j,n}^{\epsilon}(t)$ is defined for $n \leq N-j$ as

$$G_{j,n}^{\epsilon}(t) = \sum_{\substack{m_1,...,m_n \geq 0 \\ j+n+\sum_{i=1}^n m_i \leq N}} \int_0^t dt_1 ... \int_0^{t_{n-1}} dt_n$$

$$S_j^{\epsilon}(t-t_1) A_{j+1+m_1}^{\epsilon} S_{j+1+m_1}^{\epsilon}(t_1 - t_2) ... A_{j+n+\sum_{i=1}^n m_i}^{\epsilon} S_{j+n+\sum_{i=1}^n m_i}^{\epsilon}(t_n) f_{0,j+n+\sum_{i=1}^n m_i}^N \tag{2.26}$$

and it is identically equal to zero for $n > N-j$. The operator $S_j^{\epsilon}(t)$ is the interacting flow operator:

$$S_j^{\epsilon}(t) g(\boldsymbol{z}_j) = g\left(T_j^{\epsilon}(-t)\boldsymbol{z}_j\right), \tag{2.27}$$

where $T_j^{\epsilon}(t)$ is the solution of the Newton equation (2.5). We call this series the Grad series solution (GSS).

Next we introduce the following hierarchy of equations, called the intermediate hierarchy (IH)

$$(\partial_t + \mathcal{L}_j)\, f_j^N = (N-j)\epsilon^2 C^\epsilon_{j+1}(f^N_{j+1}) \tag{2.28}$$

$$C^\epsilon_{j+1}(f^N_{j+1}) = \sum_{k=1}^{j} \int_{\mathbb{R}^3} dv_{j+1} \int_{\nu\cdot(v_k - v_{j+1})\geq 0} d\nu |\nu\cdot(v_k - v_{j+1})|$$

$$\Big[f^N_{j+1}(x_1, v_1, ..., x_k, v_k^{'}, ..., x_j, v_j, x_k - \eta\epsilon, v_{j+1}^{'}) - f^N_{j+1}(x_1, v_1, ..., x_j, v_j, x_k + \eta\epsilon, v_{j+1})\Big] \tag{2.29}$$

This hierarchy is formally similar to the BBGKY hierarchy for hard spheres but the collision operator appearing in IH is different. Indeed, in the IH we have that the trasfered momentum is

$$p = (\mathrm{V}\cdot\omega)\,\omega \tag{2.30}$$

while in hard spheres it is

$$p = (\mathrm{V}\cdot\nu)\,\nu. \tag{2.31}$$

Note that it may be convenient to express ν in terms of ω, which is the parameter appearing in the expression of the outgoing velocities. However, as described in [2], this is a delicate point and we prefer to avoid it, working as much as possible with formula (2.29). We want to notice also that $A^\epsilon_{j+1} f^N_{j+1} = C^\epsilon_{j+1}(f^N_{j+1})$, i.e. the first term in the sum on the right hand side of equation (2.24) is the collision term that arise in the IH case. As we will see this will be the only $O(1)$ term as $\epsilon \to 0$.

Also for IH we can write the following formal series for the solution, that we will call intermediate series solution (ISS)

$$f_j^N(t) = \sum_{n=0}^{\infty} Q^\epsilon_{j,n}(t) f^N_{0,j} \tag{2.32}$$

where the operator $Q^\epsilon_{j,n}(t)$ is defined for $n \leq N-j$ as

$$Q^\epsilon_{j,n}(t) = (N-j)\dots(N-j-n+1)\,\epsilon^{2n}$$
$$\int_0^t dt_1 \dots \int_0^{t_{n-1}} dt_n S^\epsilon_j(t-t_1) C^\epsilon_{j+1} S^\epsilon_{j+1+m_1}(t_1 - t_2) \dots C^\epsilon_{j+n} S^\epsilon_{j+n}(t_n) f^N_{0,j+n} \tag{2.33}$$

and it is identically equal to zero for $n > N-j$.

Finally we observe that by sending $\epsilon \to 0$, $N \to \infty$, $N\epsilon^2 \to \alpha$ in the IH we obtain, formally, the following hierarchy, called the Boltzmann hierarchy (BH)

$$(\partial_t + v \cdot \nabla_x)\, f_j = \alpha C_{j+1}(f_j) \;\; 0 \leq j \tag{2.34}$$

$$C_{j+1}(f^j) = \sum_{k=1}^{j} \int_{\mathbb{R}^3} dv_{j+1} \int_{\nu\cdot(v_k - v_{j+1})\geq 0} d\nu |\nu \cdot (v_k - v_{j+1})|$$

$$\Big[f_{j+1}(x_1, v_1, ..., x_k, v_k^{'}, ..., x_j, v_j, x_k, v_{j+1}) - f_{j+1}(x_1, v_1, ..., x_j, v_j, x_k, v_{j+1})\Big] \tag{2.35}$$

If we assume the propagation of chaos, i.e. that $f_j = f_1^{\otimes j}$, the first equation of this infinite hierarchy becomes the Boltzmann equation.

The series solution for the Boltzmann hierarchy (BSS) is the following

$$f_j^{\alpha}(t) = \sum_{n=0}^{\infty} Q_{j,n}^{\alpha}(t) f_{0,j+n} \tag{2.36}$$

where $f_{0,j+n}$ is the $j+n$ particles initial datum and $Q_{j,n}^{\alpha}(t)$ is defined as follows

$$Q_{j,n}^{\alpha}(t) = \alpha^n \int_0^t dt_1 ... \int_0^{t_{n-1}} dt_n S_j(t-t_1) C_{j+1} S_{j+1+m_1}(t_1 - t_2) ... C_{j+n} S_{j+n}(t_n) f_{0,j+n} \tag{2.37}$$

where $S_j(t)$ is the free flow operator, i.e.

$$S_j(t) g^j(\boldsymbol{z}_j) = g(\boldsymbol{x}_j - \boldsymbol{v}_j t). \tag{2.38}$$

3 Linear regime

In this section we formally derive the linear Boltzmann and Landau equations. First we define the Gibbs measure defined by

$$M_{N,\beta}(\boldsymbol{z}_n) = C_{N,\beta} e^{-\beta H_N(\boldsymbol{z}_n)} \tag{3.1}$$

where $\beta > 0$ and $C_{N,\beta}$ is chosen so that

$$\int_{\Lambda_N} M_{N,\beta}(\boldsymbol{z}_n) d\boldsymbol{z}_n = 1 \tag{3.2}$$

The Gibbs measure is an invariant measure for the gas dynamics and (3.1) is a stationary solution of the Liouville equation.

In case of the Boltzmann and Landau equations, a stationary solution is given by the Maxwellian distribution (free gas)

$$M_\beta(v) = C_\beta e^{-\frac{\beta}{2}|v|^2}, \tag{3.3}$$

where $\beta > 0$ and C_β is such that

$$\int_{\Gamma \times \mathbb{R}^3} M_\beta(v) dx dv = 1. \tag{3.4}$$

Moreover a stationary solution of the Boltzmann hierarchy is

$$M_\beta^{\otimes j}(\boldsymbol{v}_j) = \prod_{i=1}^{j} M_\beta(v_i). \tag{3.5}$$

Now we consider the Liouville equation (2.15) with initial datum given by

$$W_{0,N}(\boldsymbol{z}_N) = M_{N,\beta}(\boldsymbol{z}_N) g_0(x_1, v_1) \tag{3.6}$$

where $g_0 \in L^\infty\left(\Gamma \times \mathbb{R}^3\right)$ is a perturbation on the first particle such that $\int d\boldsymbol{z}_1 M_{N,\beta}(\boldsymbol{z}_1) g_0(x_1, v_1) = 1$.

Theorem 3.1. *Let W^N be the solution of the Liouville equation (2.15) with initial datum (3.6) and let f_j^N be the j-particles reduced marginal. Then for any $1 \leq j \leq N$ the following bound holds*

$$\sup_t f_j^N(\boldsymbol{z}_j, t) \leq M_{N,\beta}(\boldsymbol{z}_j) \|g_0\|_\infty \leq M_\beta^{\otimes j}(\boldsymbol{z}_j) \|g_0\|_\infty \tag{3.7}$$

Proof. From the choice of the initial datum we have that

$$f_0^N(\boldsymbol{z}_N) \leq M_{N,\beta}(\boldsymbol{z}_N)\|g_0\|_\infty \tag{3.8}$$

Since the maximum principle holds for the Liouville equation and $M_{N,\beta}(\boldsymbol{z}_N)$ is a stationary solution we have that

$$W^N(\boldsymbol{z}_N, t) \leq M_{N,\beta}(\boldsymbol{z}_N)\|g_0\|_\infty \tag{3.9}$$

This implies the (3.7) since $M_{N,\beta}(\boldsymbol{z}_j) \leq M_\beta^{\otimes j}(\boldsymbol{z}_j)$ by the positivity of the interaction. □

3.1 Linear Boltzmann equation and asymptotics

In this section we derive the linear Boltzmann equation from the non linear one and study its asymptotic behavior for $\alpha \to \infty$. Suppose that the initial datum of the Boltzmann hierarchy (2.35) is

$$f_{0,j}(x_1, v_1, ..., x_j, v_j) = M_\beta(v_1)...M_\beta(v_j)g_0(x_1, v_1) \tag{3.10}$$

with $g_0(x_1, v_1) \in L^\infty(\Gamma)$. Since the Maxwellian distribution is a stationary solution of the equations we look for a solution at time t given by

$$f_j^\alpha(\boldsymbol{z}_j, t) = M_\beta(v_1)...M_\beta(v_j)g^\alpha(x_1, v_1, t). \tag{3.11}$$

From (3.10) and (2.35) we have that (3.11) is a solution of the Boltzmann hierarchy if g^α satisfies the following equation

$$M_\beta(v)\left(\partial_t g^\alpha + v \cdot \nabla_x g^\alpha\right) = \alpha \int dv_1 \int\limits_{\nu \cdot V > 0} d\nu |\nu \cdot V| \left[M_\beta(v^{'})M_\beta(v_1^{'})g^\alpha(x, v^{'}) - M_\beta(v)M_\beta(v_1)g^\alpha(x, v)\right] \tag{3.12}$$

Since $M_\beta(v^{'})M_\beta(v_1^{'}) = M_\beta(v)M_\beta(v_1)$ the equation (3.12) becomes the Linear Boltzmann equation

$$\partial_t g^\alpha + v \cdot \nabla_x g^\alpha = Q_B(g^\alpha), \tag{3.13}$$

where

$$Q_B(g^\alpha) = \alpha \int dv_1 M_\beta(v_1) \int\limits_{\nu \cdot V > 0} d\nu |\nu \cdot V| \left[g^\alpha(x, v^{'}) - g^\alpha(x, v)\right]. \tag{3.14}$$

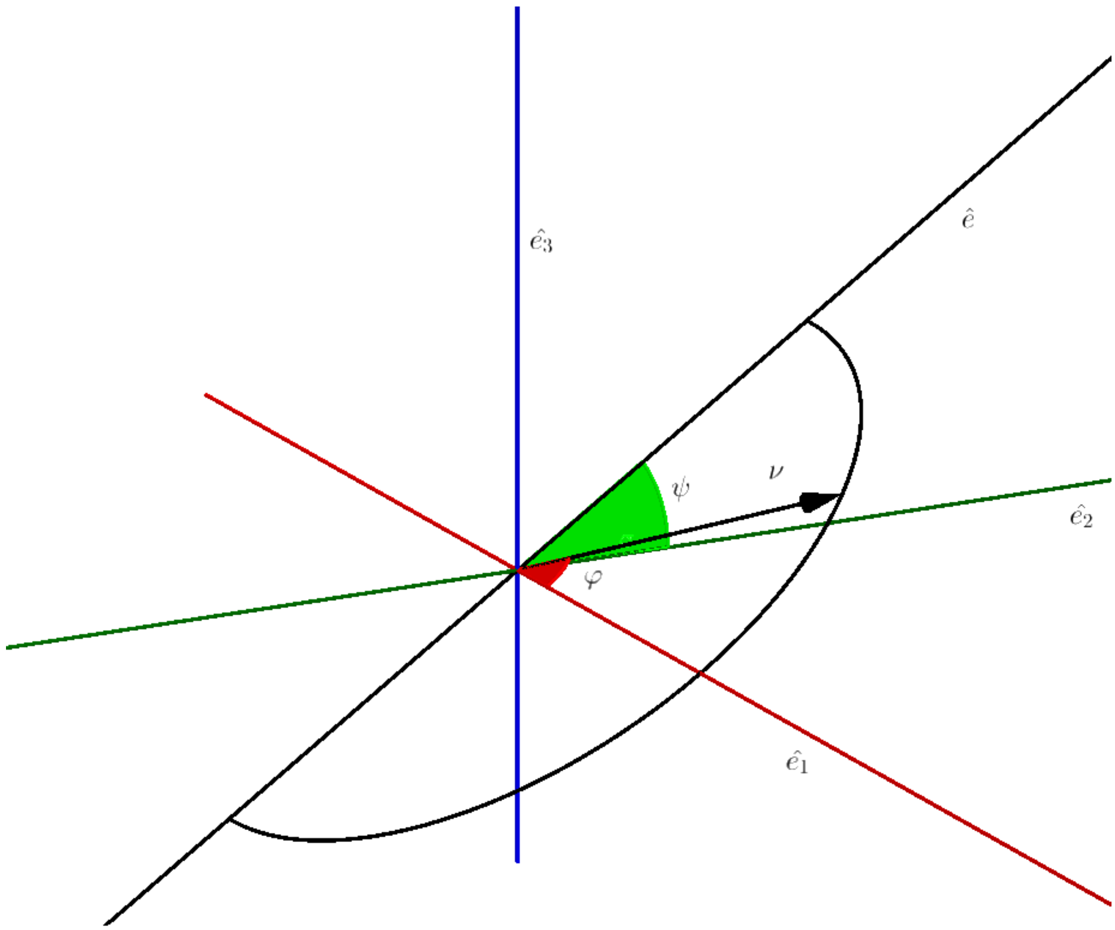

Figure 3: A representation of a three dimensional scattering.

We are interested to investigate the behavior of Q_B when $\alpha \to \infty$. We denote with $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ an orthonormal base of $\mathbb{R}^3$ such that $\hat{e}_1 = \frac{\mathrm{V}}{|\mathrm{V}|}$. Now we consider the semispehere $S^2_+ = \left\{\nu \in s^2 \,|\, \nu \cdot \mathrm{V} > 0\right\}$. For a fixed ν in this semispehere the scattering takes place in the plane generated by $\hat{e}_1$ and ν. An orthonormal base of the scattering plane is given by the vectors $\hat{e}_1$ and $\hat{e}\,(\psi) = \hat{e}_2 \cos\psi + \hat{e}_3 \sin\psi$, calling with ψ the angle between $\hat{e}_2$ and $\hat{e}$. We also denote with φ the angle between $\hat{e}_1$ and ν.

From the $\sigma - representation$ (2.8) we have that

$$v^{'} = c + r\sigma \tag{3.15}$$

where $r = \frac{|\mathrm{V}|}{2}$ and $c = \frac{v+v_1}{2}$. Notice that in our coordinates it results that

$$\sigma = \cos\theta \hat{e_1} - \sin\theta \hat{e}(\psi) \tag{3.16}$$

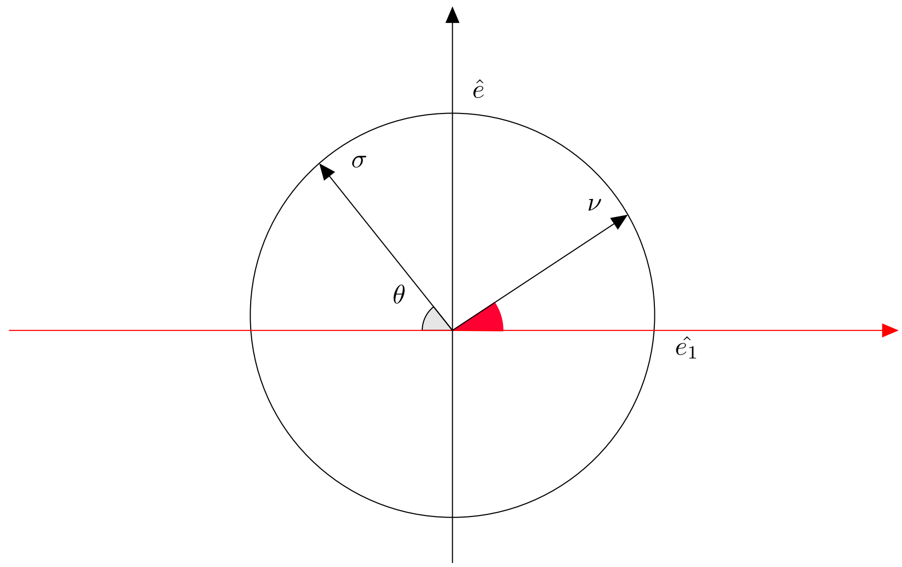

Figure 4:

We denote with $v^{'}(\theta)$ the post collisional velocity in function of the scattering angle θ

$$v^{'}(\theta) = c + r\cos\theta\hat{e}_1 - r\sin\theta\hat{e}(\psi) \tag{3.17}$$

This implies that

$$g^{\alpha}(v^{'}(\theta)) = g^{\alpha}(c + r\cos\theta\hat{e}_1 - r\sin\theta\hat{e}(\psi)) \tag{3.18}$$

For sake of brevity we will not take care of the dependence of g from the spatial variable. Let us consider the Taylor expansion of g with respect to θ up to the second order. We have

$$\begin{aligned} g^{\alpha}(v^{'}) - g^{\alpha}(v) =& g^{\alpha}(v^{'}(\theta)) - g^{\alpha}(v^{'}(0)) \\ =& \theta\nabla_v g^{\alpha}(v)\cdot\frac{dv^{'}}{d\theta}(0) + \frac{\theta^2}{2}\left[\nabla_v g^{\alpha}\cdot\frac{d^2v^{'}}{d\theta^2}(0) + \left(\frac{dv^{'}}{d\theta}(0), D_v^2(g^{\alpha})\frac{dv^{'}}{d\theta}(0)\right)\right] + o(\theta^2) \end{aligned} \tag{3.19}$$

where $D_v^2(g^{\alpha})$ is the hessian matrix of g^{α} with respect to the velocity. A simple calculation gives us that

$$\frac{dv^{'}}{d\theta}(0) = -r\hat{e}(\psi) \tag{3.20}$$

$$\frac{d^2 v^{'}}{d\theta^2}(0) = -r\hat{e}_1 \tag{3.21}$$

It can be easily seen that the integration of the first term is zero by symmetry. Moreover from Lemma 2.2 we have that

$$\theta^2(\rho,\alpha) \leq \frac{4}{|\mathrm{V}|^4 \alpha}\gamma^2(\rho) + o(\alpha^{-1}) \tag{3.22}$$

From this remark and by equations (3.19) and (3.14) we have that

$$Q_B = \int dv_1 M_\beta(v_1) \int\limits_{\nu\cdot V>0} d\nu |\nu\cdot V| \frac{2}{|V|^4}\gamma(\rho)^2$$
$$\left[-\frac{1}{2} V\cdot\nabla_v g^\alpha(v) + \frac{|V|^2}{4}\left(\hat{e}(\psi), D^2\hat{e}(\psi)\right)\right] + o(\alpha^{-1}) \tag{3.23}$$

From the change of variables $\nu \to \psi, \varphi$, since $d\nu = \sin\varphi d\varphi d\psi$, we have that

$$Q_B = \int dv_1 M_\beta(v_1) \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi |\nu\cdot V| \sin\varphi \frac{2}{|\mathrm{V}|^4}\gamma(\rho)^2$$
$$\left[-\frac{1}{2} V\cdot\nabla_v g^\alpha(v) + \frac{|V|^2}{4}\left(\hat{e}(\psi), D^2\hat{e}(\psi)\right)\right] + o(\alpha^{-1}) \tag{3.24}$$

Since $|\nu\cdot V| = |V|\cos\varphi$ and $\rho = \sin\varphi$, it results that $\cos\varphi d\varphi = d\rho$ and so

$$Q_B = \int dv_1 M_\beta(v_1) \int\limits_{-1}^{1} d\rho \frac{1}{|V|^3}\rho\gamma(\rho)^2 \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \left[-V\cdot\nabla_v g^\alpha(v) + \frac{|V|^2}{2}\left(\hat{e}(\psi), D^2\hat{e}(\psi)\right)\right] + o(\alpha^{-1}) =$$

$$\int dv_1 M_\beta(v_1) \frac{1}{|V|^3} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \left[\frac{|V|^2}{2}\left(\hat{e}(\psi), D^2\hat{e}(\psi)\right) - V\cdot\nabla_v g^\alpha(v)\right] \int\limits_{-1}^{1} d\rho\rho\gamma(\rho)^2 + o(\alpha^{-1}) \tag{3.25}$$

From the definition of $\hat{e}(\psi)$, and since $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\psi d\psi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\psi d\psi = \frac{\pi}{2}$ and $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\psi\cos\psi d\psi = 0$, we have that

$$Q_B = \int dv_1 M_\beta(v_1) \frac{1}{|V|^3}\left[|V|^2\left(\hat{e}_2, D^2\hat{e}_2\right) + |V|^2\left(\hat{e}_3, D^2\hat{e}_3\right) - 4V\cdot\nabla_v g^\alpha(v)\right] \frac{\pi}{4}\int\limits_{-1}^{1} d\rho\rho\gamma(\rho)^2 + o(\alpha^{-1}) \tag{3.26}$$

Now since the laplacian is the trace of the Hessian matrix and it is invariant under changes of coordinates we have that

$$\triangle g(v) = \left(\hat{e}_1, D^2\hat{e}_1\right) + \left(\hat{e}_2, D^2\hat{e}_2\right) + \left(\hat{e}_3, D^2\hat{e}_3\right) \tag{3.27}$$

and so

$$|V|^2 \left(\hat{e}_2, D^2\hat{e}_2\right) + |V|^2 \left(\hat{e}_3, D^2\hat{e}_3\right) = |V|^2\triangle g^\alpha(v) - \left(V, D^2V\right) \tag{3.28}$$

Thanks to (3.28) we finally arrive to

$$\begin{aligned} Q_B(g) =& B\int dv_1 M_\beta(v_1)\frac{1}{|V|^3}\left[|V|^2\triangle g^\alpha(v) - \left(V, D^2V\right) - 4V\cdot\nabla_v g^\alpha(v)\right] + o(\alpha) \\ =& Q_L(g) + o(\alpha) \end{aligned} \tag{3.29}$$

where

$$B = \frac{\pi}{4}\int_{-1}^{1} d\rho\rho\gamma(\rho)^2 \tag{3.30}$$

3.2 Linear Landau equation

In this subsection we will show that the linear operator $Q_L(g)$ is indeed the linear Landau operator obtained by the full nonlinear equatios.. Consider

$$\begin{cases} \partial_t f + v\cdot\nabla_x f = C_L(f) \\ f(x,v,0) = f_0(x,v) \end{cases}$$

whit

$$C_L(f) = A\int dv_1 \nabla_v\cdot\left[\frac{1}{|v-v_1|}P^\perp_{(v-v_1)}\left(\nabla_v - \nabla_{v_1}\right)f(v)f(v_1)\right] \tag{3.31}$$

where $A > 0$ is a suitable constant and $P^\perp_{(v-v_1)}$ is the projector on the orthogonal subspace to $v - v_1$.

Also in this case we consider a perturbation of the stationary state. We set $f(v) = M_\beta(v)g(v)$ and $f(v_1) = M_\beta(v_1)$ in (3.31). This represents a single particle perturbed in a stationary background. With this choice equation (3.31) becomes

$$M_\beta(v)\left(\partial_t g + v\cdot\nabla_x g\right) = K(g)$$

$$K(g) = A\int dv_1\nabla_v\cdot\left[\frac{1}{|v-v_1|}P^\perp_{(v-v_1)}\left(\nabla_v - \nabla_{v_1}\right)M_\beta(v)M_\beta(v_1)g(v)\right]$$

We suppose to have all the necessary regularity to give sense to the following calculations. We start from the gradient term which leads to

$$K(g) = A\int dv_1 \nabla_v \cdot \left[\frac{1}{|\mathrm{V}|} P_{\mathrm{V}}^{\perp}\left(M_\beta(v)M_\beta(v_1)\nabla_v g(v) - 2v\beta M_\beta(v)M_\beta(v_1)h(v) + 2v_1\beta M_\beta(v)M_\beta(v_1)g(v)\right)\right]$$

$$= A\int dv_1 M_\beta(v_1)\nabla_v \cdot \left[\frac{1}{|\mathrm{V}|} P_{\mathrm{V}}^{\perp}\left(M_\beta(v)\nabla_v g(v) - 2\beta M_\beta(v)g(v)\left(V\right)\right)\right] \tag{3.32}$$

Notice that $P_V^{\perp}\left(2\beta M_\beta(v)g(v)\left(V\right)\right) = 0$, this yields

$$K(g) = A\int dv_1 M_\beta(v_1)\nabla_v \cdot \left[\frac{1}{|V|} P_{\mathrm{V}}^{\perp}\left(M_\beta(v)\nabla_v g(v)\right)\right] \tag{3.33}$$

We also notice that $\nabla_v \frac{1}{|V|}$ is parallel to V , we calculate the divergence and obtain

$$\nabla_v \cdot \left[\frac{1}{|V|} P_{(\mathrm{V})}^{\perp}\left(M_\beta(v)\nabla_v g(v)\right)\right] = \nabla_v \frac{1}{|V|} \cdot P_{(\mathrm{V})}^{\perp}\left(M_\beta(v)\nabla_v g(v)\right) + \frac{1}{|V|}\nabla_v \cdot P_{(\mathrm{V})}^{\perp}\left(M_\beta(v)\nabla_v g(v)\right) =$$

$$\frac{1}{|V|}\nabla_v \cdot P_V^{\perp}\left(M_\beta(v)\nabla_v g(v)\right)$$

Therefore by (3.33) we have

$$K(g) = A\int dv_1 M_\beta(v_1)\frac{1}{|V|}\nabla_v \cdot \left[M_\beta(v) P_V^{\perp}\left(\nabla_v g(v)\right)\right] \tag{3.34}$$

We calculate again the divergence

$$\nabla_v \cdot \left[M_\beta(v) P_V^{\perp}\left(\nabla_v g(v)\right)\right] = -2\beta v M_\beta(v) \cdot P_V^{\perp}\left(\nabla_v g(v)\right) + M_\beta(v)\nabla_v \cdot \left[P_V^{\perp}\left(\nabla_v g(v)\right)\right] =$$

$$-2\beta\left(v - v_1\right) M_\beta(v) \cdot P_V^{\perp}\left(\nabla_v g(v)\right) - 2\beta v_1 M_\beta(v) \cdot P_V^{\perp}\left(\nabla_v g(v)\right) + M_\beta(v)\nabla_v \cdot \left[P_V^{\perp}\left(\nabla_v g(v)\right)\right] =$$

$$-2\beta v_1 M_\beta(v) \cdot P_V^{\perp}\left(\nabla_v g(v)\right) + M_\beta(v)\nabla_v \cdot \left[P_V^{\perp}\left(\nabla_v g(v)\right)\right] \tag{3.35}$$

From (3.35) and (3.34) we arrive to

$$K(g) = A\int dv_1 M_\beta(v_1)\frac{1}{|V|}\left\{-2\beta v_1 M_\beta(v) \cdot P_V^{\perp}\left(\nabla_v g(v)\right) + M_\beta(v)\nabla_v \cdot \left[P_V^{\perp}\left(\nabla_v g(v)\right)\right]\right\} =$$

$$A\int dv_1 M_\beta(v_1)\frac{1}{|V|}\left[-2\beta v_1 M_\beta(v) \cdot P_V^{\perp}\left(\nabla_v g(v)\right)\right] +$$

$$A\int dv_1 M_\beta(v_1)\frac{1}{|V|}M_\beta(v)\nabla_v\cdot\left[P_V^\perp\left(\nabla_v g(v)\right)\right] \tag{3.36}$$

Now we work on the first term of the right hand side of (3.36). Since $-2\beta v_1 M_\beta(v_1)=\nabla_{v_1}M_\beta(v_1)$, by means of the divergence Theorem we have that

$$A\int dv_1 M_\beta(v_1)\frac{1}{|V|}\left[-2\beta v_1 M_\beta(v)\cdot P_V^\perp\left(\nabla_v g(v)\right)\right]=M_\beta(v)A\int dv_1\left(-2\beta v_1\right)M_\beta(v_1)\cdot\frac{P_V^\perp\left(\nabla_v g(v)\right)}{|V|}=$$

$$-M_\beta(v)A\int dv_1 M_\beta(v_1)\nabla_{v_1}\cdot\left[\frac{P_V^\perp\left(\nabla_v h(v)\right)}{|V|}\right]=-M_\beta(v)A\int dv_1 M_\beta(v_1)\frac{1}{|V|}\nabla_{v_1}\cdot\left[P_V^\perp\left(\nabla_v g(v)\right)\right] \tag{3.37}$$

From (3.37) and (3.36) we arrive to

$$K(g)=M_\beta(v)A\int dv_1 M_\beta(v_1)\frac{1}{|V|}\left(\nabla_v-\nabla_{v_1}\right)\cdot\left[P_V^\perp\left(\nabla_v g(v)\right)\right] \tag{3.38}$$

Now we want to calculate $\nabla_v\cdot\left[P_V^\perp\left(\nabla_v g(v)\right)\right]$ and $\nabla_{v_1}\cdot\left[P_V^\perp\left(\nabla_v g(v)\right)\right]$. First we observe that

$$P_V^\perp\left(\nabla_v g(v)\right)=\nabla_v g(v)-\frac{\left(V,\nabla_v g(v)\right)V}{|V|^2} \tag{3.39}$$

and so

$$\nabla_{v_1}\cdot\left[P_V^\perp\left(\nabla_v g(v)\right)\right]=\nabla_{v_1}\cdot\left[\nabla_v g(v)-\frac{\left(V,\nabla_v g(v)\right)V}{|V|^2}\right]=-\nabla_{v_1}\cdot\left[\frac{\left(V,\nabla_v g(v)\right)V}{|V|^2}\right]=$$

$$2\frac{\left(V,\nabla_v g(v)\right)}{|V|^2} \tag{3.40}$$

For the other term we have that

$$\nabla_v\cdot\left[P_V^\perp\left(\nabla_v g(v)\right)\right]=\nabla_v\cdot\left[\nabla_v g(v)-\frac{\left(V,\nabla_v g(v)\right)V}{|V|^2}\right]=\triangle g(v)-\nabla_v\cdot\left[\frac{\left(V,\nabla_v g(v)\right)V}{|V|^2}\right]=$$

$$\triangle g(v)-\left[2\frac{\left(V,\nabla_v g(v)\right)}{|V|^2}+\frac{\left(V,D^2V\right)}{|V|^2}\right] \tag{3.41}$$

We now use (3.40) and (3.41) together with (3.38) to get

$$K(g)=M_\beta(v)A\int dv_1 M_\beta(v_1)\frac{1}{|V|^3}\left[|V|^2\triangle g(v)-\left(V,D^2V\right)-4V\cdot\nabla_v g(v)\right] \tag{3.42}$$

Finally we can define the linear Landau equation

$$\partial_t g + v \cdot \nabla_x g = \tilde{Q}_L(g) \tag{3.43}$$

where $\tilde{Q}_L$ is the linear Landau operator defined as

$$\tilde{Q}_L(g) = A \int dv_1 M_\beta(v_1) \frac{1}{|V|^3} \left[|V|^2 \triangle g(v) - \big(V, D^2 V\big) - 4V \cdot \nabla_v g(v) \right] \tag{3.44}$$

Notice that $\tilde{Q}_L$ and Q_L are the same operator if $A = B$. The constant A is precisally characterized by the formal derivation of the Landau equation from a system of particles and it has the following value

$$A = \frac{1}{8\pi} \int_0^{+\infty} dr \, r^3 \hat{\Phi}(r)^2 \tag{3.45}$$

where $\hat{\Phi}(|k|) = \int dx \, \Phi(|x|) e^{-ik \cdot x}$. It can be easily proved that $A = B$ by following the calculations made in [17] and, therefore, that $\tilde{Q}_L = Q_L$.

4 Continuity estimates

In this section we will prove some useful estimates for the operators arising in the series solution of the hierarchies. Observe that in the case of $\alpha = 1$ these estimates are enough to prove the convergence of the series solution for a small time. In our case since $\alpha \to \infty$ the time of the convergence of the series is going to zero. As we will see in the next section we can still use these estimates in the linear case thanks to the a priori estimate .

We define the following norm

$$\|f_j(\boldsymbol{z}_j)\|_\beta = \sup_{\boldsymbol{z}_j \in \Lambda_j} \left(e^{\beta H(\boldsymbol{z}_j)} f_j(\boldsymbol{z}_j) \right) \tag{4.1}$$

where the hamiltonian $H(\boldsymbol{z}_j)$ in macroscopic variables reads as

$$H(\boldsymbol{z}_j) = \frac{1}{2} \sum_{i=1}^{j} |v_i|^2 + \frac{1}{2\sqrt{\alpha}} \sum_{i,k=1,\, i\neq k}^{j} \Phi(\frac{x_i - x_k}{\epsilon}) \tag{4.2}$$

For sake of simplicity we don't indicate the dependence from j in the definition of $\| \cdot \|_\beta$. Notice also that the norm depends on α but not in a harmful way.

Since we are interested in the linear regime we will take as initial datum a perturbation of the stationary state, as we have seen in section 3.1 and 3.2. We assume that the initial datum of GH and IH has the form

$$f^N_{j,0}(\boldsymbol{z}_j) = M_{N,\beta}(\boldsymbol{z}_j) g_0(x_1, v_1) \tag{4.3}$$

We assume also that the initial data for the Boltzmann hierarchy is

$$f^\alpha_0(\boldsymbol{z}_j) = M_\beta^{\otimes j}(v_j) g_0(x_1, v_1) \tag{4.4}$$

Notice that the estimates that we will prove work also in case of a general f_0 with $\|f_0\|_\beta < \infty$ for a $\beta > 0$.

4.1 Estimates of the operators

We start by estimating the operator appearing in GSS

Lemma 4.1. *Let g_j^N be a sequence of continuous functions with $g_j^N = 0$ for $j > N$ and suppose that*

$$\|g_j^N\|_\beta \leq C^j \tag{4.5}$$

Then for $\beta^{'} < \beta$ there exist a constant $C_1 = C_1(\beta, \beta^{'}, g_j^N)$ such that for ϵ small enough and $\forall j \geq 0$

$$\|G_{j,n}^\epsilon(t) g_j^N(z_j)\|_{\beta'} \leq (C_1 \alpha t)^n \tag{4.6}$$

Proof. From the definition of the operator $A_{j+1+m}^\epsilon g_{j+1+m}^N$ we have that

$$e^{\beta^{'} H(z_j)} |A_{j+1+m}^\epsilon g_{j+1+m}^N(\boldsymbol{z}_j)| \leq C^{j+1} C^m \epsilon^{3m} \epsilon^2 N^{m+1} \sum_{i=1}^{j} \int dv_{j+1} \left(|v_i| + |v_{j+1}|\right) e^{-\frac{\beta-\beta^{'}}{2} \sum_{i=1}^{j} v_i^2} e^{-\frac{\beta}{2} v_{j+1}^2} \tag{4.7}$$

since

$$\int\limits_{\Delta_m(\boldsymbol{x}_{j+1})} d\boldsymbol{z}_{j+1,m} f_{j+1+m}^N(\boldsymbol{z}_j, x_i + \nu\epsilon, v_{j+1}, \boldsymbol{z}_{j+1,m}, t) \leq C^m \epsilon^{3m} \tag{4.8}$$

and

$$\|g_{j+1+m}^N\|_\beta \leq C^{j+1} C^m \tag{4.9}$$

Now since $\epsilon^2 N \cong \alpha$ we have that

$$\epsilon^{3m} \epsilon^2 N^{m+1} \leq \alpha (C\epsilon\alpha)^m \tag{4.10}$$

and so

$$e^{\beta^{'} H(z_j)} |A_{j+1+m}^\epsilon g_{j+1+m}^N(\boldsymbol{z}_j)| \leq n\alpha (C\epsilon\alpha)^m \tag{4.11}$$

We can choose ϵ small enough, since $\alpha \cong \sqrt{\log\log N}$, to have that $C\epsilon\alpha < 1$. We performe the sum over m to obtain

$$\sum_{m \geq 0} (C\epsilon\alpha)^m \leq C \tag{4.12}$$

that leads us to

$$\|A_{j+1+m}^\epsilon g_{j+1+m}^N(\boldsymbol{z}_j)\|_{\beta'} \leq n\alpha C \tag{4.13}$$

Now since for any $\beta > 0$ it results that

$$\|S_j^\epsilon(t) g_j^N\|_\beta = \|g_j^N\|_\beta \tag{4.14}$$

we can alternate estimate (4.13) and (4.14) and performe the time integrals in (2.26). This gives us that

$$\|G^{\epsilon}_{j,n}(t)g^N_j(\boldsymbol{z}_j)\|_{\beta'} \leq (C_1\alpha t)^n \tag{4.15}$$

□

In the same way we can estimate the operators $Q^{\epsilon}_{j,n}(t)$ and $Q^{\alpha}_{j,n}(t)$ and prove the following lemma

Lemma 4.2. *Let $Q^{\epsilon}_{j,n}(t)$ and $Q^{\alpha}_{j,n}(t)$ be defined respectively as in (2.33) and in (2.37). Let also g^N_j, g_j be sequence of continuous functions with $g^N_j = 0$ for $j > N$ suppose that*

$$\|g^N_j\|_{\beta} \leq C^j \tag{4.16}$$

$$\|g_j\|_{\beta} \leq C^j \tag{4.17}$$

then there exist constants C_2 and C_3 such that for ϵ small enough and $\beta^{'} < \beta$

$$\|Q^{\epsilon}_{j,n}(t)g^N_j\|_{\beta'} \leq (C_2\alpha t)^n \tag{4.18}$$

$$\|Q^{\alpha}_{j,n}(t)g_j\|_{\beta'} \leq (C_3\alpha t)^n \tag{4.19}$$

4.2 Estimates for an arbitrary time

Now we want to use the a priori estimate to prove the convergence of the series solution for an arbitrary time. The main idea is to separate the interval $[0,t]$ in $s \in \mathbb{N}$ parts of length h such that

$$t = sh \tag{4.20}$$

and write $\widetilde{f^N_1}(t)$, $f^{\alpha}_{1,s}(t)$ and $f^N_{1,s}(t)$ in terms of a finite sum plus a remainder. We use the technique used by Bodineau, Gallagher and Saint-Raymond [1]. It consists in bounding the number of interactions in an interval $[ih,(i+1)h]$ $0 \leq i < s$ by $2^i - 1$ and send the time h to zero in a suitable way.

In literature there is another method, which is employed by Colangeli, Pezzoti and Pulvirenti in [19], that consists in taking h smaller than the Lanford time of the convergence of the series solutions and then bounding in a suitable way the number of creations in each interval. We cannot use this method since in our case the time of the convergence of the series is going to zero.

We can write the solution at time t of the GH as the evolution of a time h of the solution at time $t-h$

$$\widetilde{f_1^N}(t) = \sum_{j_1=0}^{\infty} G^{\epsilon}_{1,j_1}(h)\widetilde{f^N_{j_1+1}}(t-h) \tag{4.21}$$

We introduce the Grad truncated series solution (GTS) by truncating the series (4.21) at $j_1 = 2^1 - 1 = 1$. We obtain

$$\widetilde{f_1^N}(t) = \sum_{j_1=0}^{1} G^{\epsilon}_{1,j_1}(h)\widetilde{f^N_{j_1+1}}(t-h) + \widetilde{R_{1,1}}(t-h,t) \tag{4.22}$$

$$\widetilde{R_{1,1}}(t-h,t) = \sum_{j_1=2}^{\infty} G^{\epsilon}_{1,j_1}(h)\widetilde{f^N_{j_1+1}}(t-h) \tag{4.23}$$

Now we can iterate this procedure on $\widetilde{f^N_{j_1+1}}(t-h)$. We have that

$$\widetilde{f^N_{j_1+1}}(t-h) = \sum_{j_2=0}^{\infty} G^{\epsilon}_{j_1+1,j_2}(h)\widetilde{f^N_{j_2+1}}(t-h) \tag{4.24}$$

We truncate again the series at $j_2 = 2^2 - 1$ and we arrive to

$$\widetilde{f^N_{j_1+1}}(t-h) = \sum_{j_2=0}^{2^2-1} G^{\epsilon}_{j_1+1,j_2}(h)\widetilde{f^N_{j_2+1}}(t-2h) + \widetilde{R_{j_1+1,2}}(t-2h,t-h) \tag{4.25}$$

where

$$\widetilde{R_{j_1+1,2}}(t-2h,t-h) = \sum_{p=4}^{N-j_1-1} G^{\epsilon}_{j_1+1,2}(h)\widetilde{f^N_{j_{1+1}+p}}(t-2h) \tag{4.26}$$

From (4.25) and (4.22) we have

$$\widetilde{f_1^N}(t) = \sum_{j_1=0}^{1}\sum_{j_2=0}^{2^2-1} G^{\epsilon}_{1,j_1}(h)G^{\epsilon}_{j_1+1,j_2}(h)\widetilde{f^N_{j_2+1}}(t-2h) + \widetilde{R^2_N}(t) \tag{4.27}$$

where $\widetilde{R^2_N}(t)$ takes into account the evolution of the remainders of each truncation and reads as follows

$$\widetilde{R^2_N}(t) = \widetilde{R_{1,1}}(t-h,t) + \sum_{j_1=0}^{1} G^{\epsilon}_{1,j_1}(h)\widetilde{R_{j_1+1,2}}(t-2h,t-h) \tag{4.28}$$

We iterate this procedure with a sequence of cutoffs $2^i - 1$, this leads to

$$\widetilde{f_1^N}(t) = \widetilde{f^N_{1,s}}(t) + \widetilde{R^s_N}(t) \tag{4.29}$$

where, denoting with $P_i = 1 + \sum_{k=1}^{i} j_k$ the number of particles after i iterations,

$$\widetilde{f_{1,s}^N}(t) = \sum_{j_1=0}^{1} ... \sum_{j_s=0}^{2^s-1} G_{1,j_1}^{\epsilon}(h) G_{P_1,j_2}^{\epsilon}(h) ... G_{P_{s-1},j_S}^{\epsilon}(h) f_0^N \tag{4.30}$$

$$\widetilde{R_N^s}(t) = \sum_{i=1}^{s} \sum_{j_1=0}^{1} ... \sum_{j_{i-1}=0}^{2^{i-1}-1} G_{1,j_1}^{\epsilon}(h) G_{P_1,j_2}^{\epsilon}(h) ... G_{P_{i-2},j_{i-1}}^{\epsilon}(h) \tilde{R}_{P_{i-1},i}\left(t - ih, t-(i-1)h\right) \tag{4.31}$$

$$\tilde{R}_{P_{i-1},i}\left(t - ih, t-(i-1)h\right) = \sum_{p=2^i}^{N-P_{i-1}} G_{P_{i-1},p}^{\epsilon}(h) \tilde{f}_{P_{i-1}+p}^N \tag{4.32}$$

We use the same procedure for the series solution of the Boltzmann hierarchy and we obtain the truncated Boltzmann solution (BTS)

$$f_{1,s}^{\alpha}(t) = \sum_{j_1=0}^{1} ... \sum_{j_s=0}^{2^s-1} Q_{1,j_1}^{\alpha}(h) Q_{P_1,j_2}^{\alpha}(h) ... Q_{P_{s-1},j_s}^{\alpha}(h) f_0^N \tag{4.33}$$

$$R^s(t) = \sum_{i=1}^{s} \sum_{j_1=0}^{1} ... \sum_{j_{i-1}=0}^{2^{i-1}-1} Q_{1,P_1}^{\alpha}(h) Q_{P_1,j_2}^{\alpha}(h) ... Q_{P_{i-2},j_{i-1}}^{\alpha}(h) R_{P_{i-1},i}\left(t - ih, t-(i-1)h\right) \tag{4.34}$$

$$R_{P_{i-1},i}\left(t - ih, t-(i-1)h\right) = \sum_{p=2^i}^{N-P_{i-1}} Q_{P_{i-1},p}^{\alpha}(h) f_{J_{i-1}+p}^N \tag{4.35}$$

We also define the intermediate truncated solution (ITS)

$$f_{1,s}^N(t) = \sum_{j_1=0}^{1} ... \sum_{j_s=0}^{2^s-1} Q_{1,j_1}^{\epsilon}(h) Q_{P_1,j_2}^{\epsilon}(h) ... Q_{P_{s-1},j_s}^{\epsilon}(h) f_0^N \tag{4.36}$$

Now we want to prove an estimate for the remainder term.

Theorem 4.3. *Let $\widetilde{R_N^s}(t)$,$R^s(t)$ be defined respectively as in (4.31) and (4.34). Then the following estimate holds*

$$\|\widetilde{R_N^s}(t)\|_\infty + \|R^s(t)\|_\infty \leq \|g_0\|_\infty \left(\frac{C\left(\alpha t\right)^2}{s} \right)^2 \tag{4.37}$$

Proof. Thanks to the semigroup property we have that

$$\widetilde{R_s^N}(t) = \sum_{i=1}^{s} \sum_{j_1=0}^{1} \dots \sum_{j_{i-1}=0}^{2^{i-1}-1} G^{\epsilon}_{1,P_{i-1}-1}((i-1)h)\tilde{R}_{P_{i-1},i}\left(t-ih, t-(i-1)h\right) \tag{4.38}$$

Now from the steps of Lemma 4.1 it follows that

$$\|G^{\epsilon}_{1,P_{i-1}-1}((i-1)h)\tilde{R}_{P_{i-1},i}\left(t-ih, t-(i-1)h\right)\|_\infty \leq$$

$$\left(C\alpha(i-1)h\right)^{P_{i-1}-1} \|\tilde{R}_{P_{i-1},i}\left(t-ih, t-(i-1)h\right)\|_{\frac{\beta}{2}} \tag{4.39}$$

Furthermore we have that

$$\|\tilde{R}_{P_{i-1},i}\left(t-ih, t-(i-1)h\right)\|_{\frac{\beta}{2}} \leq \sum_{p=2^i}^{N-P_{i-1}} (C\alpha h)^p \, \|\widetilde{f^N_{P_{i-1}+p}}\|_{\frac{\beta}{2}} \leq \|g_0\|_\infty \sum_{p=2^i}^{N-P_{i-1}} (C\alpha h)^p \tag{4.40}$$

We use together the last two estimates and that

$$C\alpha h < \frac{1}{2}$$

and we arrive to

$$\|\widetilde{R_s^N}(t)\|_\infty \leq \|g_0\|_\infty \sum_{i=1}^{s} \sum_{j_1=0}^{1} \dots \sum_{j_{i-1}=0}^{2^{i-1}-1} (C\alpha t)^{P_{i-1}-1} \left(C\alpha h\right)^{2^i} \leq$$

$$\|g_0\|_\infty \sum_{i=1}^{s} \sum_{j_1=0}^{1} \dots \sum_{j_{i-1}=0}^{2^{i-1}-1} \left(C\alpha t\right)^{2^i} \left(C\alpha h\right)^{2^i} \leq \|g_0\|_\infty \sum_{i=1}^{s} 2^{i(i-1)} \left(C\left(\alpha t\right)\alpha h\right)^{2^i}$$

$$\leq \|g_0\|_\infty \sum_{i=1}^{s} \left(\frac{C\left(\alpha t\right)^2}{s}\right)^{2^i} \tag{4.41}$$

In the last step we used that $h = \frac{t}{s}$ and that $i(i-1) \leq 2^i$. Now we assume also that

$$\frac{C\left(\alpha t\right)^2}{s} < \frac{1}{2} \tag{4.42}$$

and we finally arrive to

$$\|\widetilde{R_N^s}(t)\|_\infty \leq \|g_0\|_\infty \left(\frac{C\left(\alpha t\right)^2}{s}\right)^2 \tag{4.43}$$

The estimate for $R^s(t)$ can be obtained in the same way. $\square$

Thanks to Theorem 4.3 we can work directly on the truncated series since we have an estimate on the remainders. We want to prove that the GTS is close to the ITS as $\epsilon \to 0$. We have

Theorem 4.4. *Let $\widetilde{f_{1,s}^N}(t)$, $f_{1,s}^N(t)$ be respectively the solution of the first equation of GH and IH. Then the following estimate holds for all $t \geq 0$*

$$\|\widetilde{f_{1,s}^N}(t) - f_{1,s}^N(t)\|_\infty \leq \|g_0\|_\infty 2^{s(s+1)} \left(C\alpha t\right)^{2^{s+1}} \epsilon \tag{4.44}$$

Proof. The definition of the truncated solution series leads to

$$\widetilde{f_{j,s}^N}(t) - f_{j,s}^N(t) = \sum_{j_1=0}^{2} \dots \sum_{j_s=0}^{2^s-1} \left[G_{1,j_1}^\epsilon(h) G_{P_1,j_2}^\epsilon(h) \dots G_{P_{s-1},j_S}^\epsilon(h) - Q_{1,j_1}^\epsilon(h) Q_{P_1,j_2}^\epsilon(h) \dots Q_{P_{s-1},j_s}^\epsilon(h)\right] f_0^N \tag{4.45}$$

Now from the semigroup property and the identity

$$a^n - b^n = \sum_{i=1}^{n} a^{i-1}(a-b)b^{n-i} \tag{4.46}$$

we have

$$\begin{gathered} G_{1,j_1}^\epsilon(h) G_{P_1,j_2}^\epsilon(h) \dots G_{P_{s-1},j_S}^\epsilon(h) - Q_{1,j_1}^\epsilon(h) Q_{P_1,j_2}^\epsilon(h) \dots Q_{P_{s-1},j_s}^\epsilon(h) = \\ \sum_{l=1}^{s} G_{1,P_{l-1}-1}^\epsilon((l-1)h) \left[G_{P_{l-1},j_l}^\epsilon(h) - Q_{P_{l-1},j_l}^\epsilon(h)\right] Q_{P_l,P_s-P_l}^\epsilon((s-l)h) \end{gathered} \tag{4.47}$$

Since the operator $Q_{P_{l-1},j_l}^\epsilon(h)$ is the first term not equal to zero in the asymptothic of the operator $G_{P_{l-1},j_l}^\epsilon(h)$ we obtain that

$$\begin{gathered} G_{P_{l-1},j_l}^\epsilon(h) - Q_{P_{l-1},j_l}^\epsilon(h) f^N(0) = \sum_{\substack{m_1,\dots,m_{j_l} \geq 0,\, \sum_{i=1}^{j_l} m_i \neq 0 \\ P_{l-1}+j_l+\sum_{i=1}^{j_l} m_i \leq N}} \int_0^h dt_1 \dots \int_0^{t_{j_l-1}} dt_{j_l} \\ S_{P_{l-1}}^\epsilon(h-t_1) A_{P_{l-1}+1+m_1}^\epsilon S_{P_{l-1}+1+m_1}^\epsilon(t_1-t_2) \dots A_{P_{l-1}+j_l+\sum_{i=1}^{j_l} m_i}^\epsilon S_{P_{l-1}+j_l+\sum_{i=1}^{j_l} m_i}^\epsilon(t_{j_l}) f_{0,P_{l-1}+j_l+\sum_{i=1}^{j_l} m_i}^N \end{gathered} \tag{4.48}$$

The same steps of Lemma 4.1 lead to

$$\|G_{P_{l-1},j_l}^\epsilon(h) - Q_{P_{l-1},j_l}^\epsilon(h) f^N(0)\|_{\beta'} \leq (C\alpha h)^{j_l} \|g_0\|_\infty \epsilon \tag{4.49}$$

From (4.49) and (4.47) we arrive to

$$\sum_{l=1}^{s} \|G^{\epsilon}_{1,P_{l-1}-1}((l-1)h)\left[G^{\epsilon}_{P_{l-1},j_l}(h) - Q^{\epsilon}_{P_{l-1},j_l}(h)\right] Q^{\epsilon}_{P_l,P_s-P_l}((s-l)h) f_0^{N,\epsilon}\|_\infty \le \|g_0\|_\infty \epsilon \, (C\alpha t)^{P_s-1} \tag{4.50}$$

We perform the sum over $j_1, ..., j_s$ and we finally have that

$$\|\widetilde{f^N_{1,s}}(t) - f^N_{1,s}(t)\|_\infty \le \|g_0\|_\infty 2^{s(s+1)} \epsilon \, (C\alpha t)^{2^{s+1}} \tag{4.51}$$

□

Thanks to this theorem we can reduce us to study only the convergence of the ITS to the BTS.

5 Convergence to Linear Boltzmann equation

5.1 The Boltzmann backward flow and the Interacting backward flow

In this section we will represent in a convenient way the series (2.32) and (2.36) for the first-particle marginal. These series solutions can be represented graphically as a trees expansion. We define a n-collision tree graph as the following collection of integer

$$\Gamma(n) = \{(i_1, ..., i_n) \in \mathbb{N}^n \mid i_k \leq k\} \tag{5.1}$$

Roughly speaking, this integer represent the label of the particle that creates a new particle in a creation term. In Figure (5) we give a picture of the tree $(1, 1, 2)$.

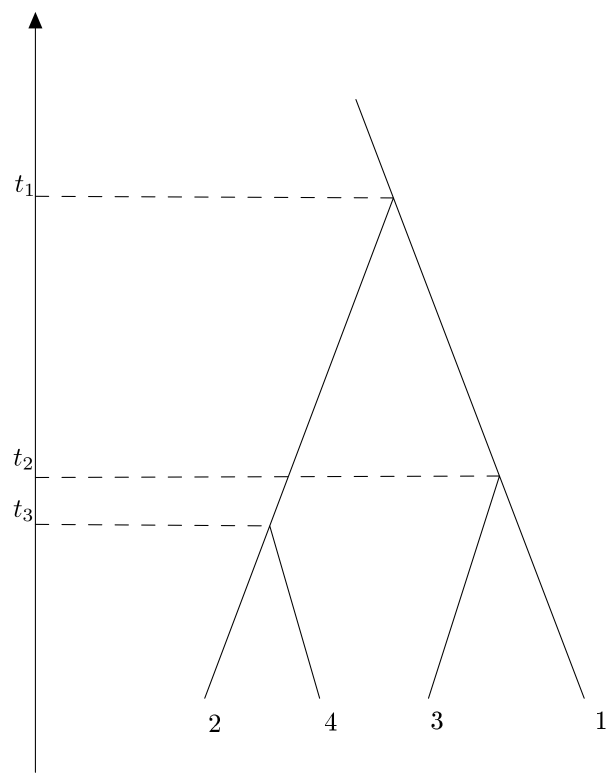

Figure 5: A representation of the tree graph $(1, 1, 2)$. At the time t_1 we create the particle 2 on the particle 1. Then at time t_2 we create the particle3 on the particle 1. Finally at time t_3 the particle 4 is created on the particle 2.

We define the following collections of variables for the ITS

$$\boldsymbol{\sigma}_n = (\sigma_1, ..., \sigma_n) \quad \sigma_i = \pm 1 \quad \boldsymbol{\sigma}_n = \prod_{i=1}^{n} \sigma_i \tag{5.2}$$

$$\boldsymbol{t}_n = (t_1, ..., t_n) \tag{5.3}$$

$$\boldsymbol{w}_n = (w_1, ..., w_n) \tag{5.4}$$

$$\boldsymbol{\nu}_n = (\nu_1, ..., \nu_n) \tag{5.5}$$

Here $t_1, ..., t_n$ are the time variables appearing in the time integrals, while w_i and ν_i are the velocity and the impact parameter that appears in the creation of the (i+1)-th particle. Fixed these variables we can construct the interacting backwards flow (IBF). We define the IBF at time $s \in (t_k, t_{k+1})$ as

$$\zeta^\epsilon(s) = \left(r_1^\epsilon(s), \xi_1^\epsilon(s), ..., r_{1+k}^\epsilon(s), \xi_{1+k}^\epsilon(s)\right) \tag{5.6}$$

where $r_i(s), \xi_i(s)$ are respectively position and velocity of the i-Th particle at time s. At time t we have that $\zeta^\epsilon(t) = (x_1, v_1)$, then we go back in time with the interacting flow defined as the solution of equation (2.5). Between time t and time $t - t_1$ we set $\zeta^\epsilon(s) = T_1^\epsilon(-s)\,(r_1^\epsilon(t), \xi_1^\epsilon(t))$. Then at time $t - t_1$ we create a new particle in position $r_2(t - t_1) = r_{i_1}^\epsilon(t - t_1) + \epsilon\nu_1$ with velocity $\xi_2(t - t_1) = w_1$ in a pre-collisional state if $\sigma_1 = +1$ or in post collisional one if $\sigma_1 = -1$. Between time $t - t_1$ and $t - t_1 - t_2$ we set the IBF as $\zeta^\epsilon(s) = T_2^\epsilon(-t + t_1 + s)\,(r_1^\epsilon(t - t_1), \xi_1^\epsilon(t - t_1), r_2^\epsilon(t - t_1), \xi_2^\epsilon(t - t_1))$. In this way we create a new particle at time $t - t_1 - t_2$ in position $r_{i_2}^\epsilon(t - t_1 - t_2) + \epsilon\nu_2$ with velocity w_2 in pre-collisional or post-collisional configuration that depends on σ_2. We iterate this procedure and we define the IBF up to time 0 by alternating the creation of new particles with the interacting flow T_j^ϵ . For sake of simplicity we define the following time variables

$$\tau_k = t - \sum_{i=1}^{k} t_i \tag{5.7}$$

With this definition τ_k are the backward times of a creation.

We can write the one particle marginal in a more manageable way thanks to the IBF

$$f^N_{1,s}(t) = \sum_{j_1=0}^{1} ... \sum_{j_s=0}^{2^s-1} (N-1)...(N-P_s-1)\left(\epsilon^2\right)^{P_s-1} \sum_{\Gamma(P_s-1)} \sum_{\sigma_{P_s-1}} \boldsymbol{\sigma}_{P_s-1} I^{\epsilon}_{\sigma_{P_s-1}}(z_j,t) \tag{5.8}$$

with

$$I^{\epsilon}_{\sigma_{P_s-1}}(z_j,t) = \int d\boldsymbol{t}_{P_{s-1}} d\boldsymbol{w}_{P_{s-1}} d\boldsymbol{\nu}_{P_{s-1}} \prod_{k=1}^{P_s-1} B^{\epsilon}\left(\nu_k, w_{1+k} - \xi^{\epsilon}_{i_k}(\tau_k)\right) f^N_{0,P_s}(\zeta^{\epsilon}(0)) \tag{5.9}$$

and

$$B^{\epsilon}\left(\nu_k, w_{1+k} - \xi^{\epsilon}_{i_k}(\tau_k)\right) = |\nu_k \cdot \left(w_{1+k} - \xi^{\epsilon}_{i_k}(\tau_k)\right)| \chi \left\{|r_{k+1}(\tau_k) - r_{i_k}(\tau_k)| > \epsilon\right\} \cdot$$

$$\chi\left\{\sigma_k \nu_k \cdot \left(w_{1+k} - \xi^{\epsilon}_{i_k}(\tau_k)\right) \geq 0\right\} \tag{5.10}$$

With a similar procedure we can build the Boltzmann backward flow (BBF) but we have to take into account the following difference:

- The flow between two creation is the free flow and not the interacting flow;
- The new particle in each creation is created in the position of his progenitor, i.e. $r_{i_k}(\tau_k) = r_{k+1}(\tau_k)$;
- There is no constraint on ν_k other than the one implied by the value of σ_k;
- if $\sigma_k = +1$ before to go back in time we have to change the velocities from post collisional in pre-collisional according to the scattering rules.

Taking into account these differences, we define the BBF at time $s \in (t_k, t_{k+1})$ as

$$\zeta(s) = (r_1(s), \xi_1(s), ..., r_{1+k}(s), \xi_{1+k}(s)) \tag{5.11}$$

We use the BBF to write the one particle marginal of the Boltzmann equation as

$$f_{1,s}(t) = \sum_{j_1=0}^{1} ... \sum_{j_s=0}^{2^s-1} (\alpha)^{P_s-1} \sum_{\Gamma(P_s-1)} \sum_{\sigma_{P_s-1}} \boldsymbol{\sigma}_{P_s-1} I_{\sigma_{P_s-1}}(z_j,t) \tag{5.12}$$

$$I_{\sigma_{P_s-1}}(z_j,t) = \int d\boldsymbol{t}_{P_{s-1}} d\boldsymbol{w}_{P_{s-1}} d\boldsymbol{\nu}_{P_{s-1}} \prod_{k=1}^{P_s-1} B\, f_{0,P_s}(\zeta(0)) \tag{5.13}$$

We also define the vectors of the only velocities at time $s \in (t_k, t_{k+1})$ as

$$\xi^\epsilon(s) = \left(\xi_1^\epsilon(s), ..., , \xi_{1+k}^\epsilon(s)\right) \tag{5.14}$$

$$\xi(s) = (\xi_1(s), ..., , \xi_{1+k}(s)) \tag{5.15}$$

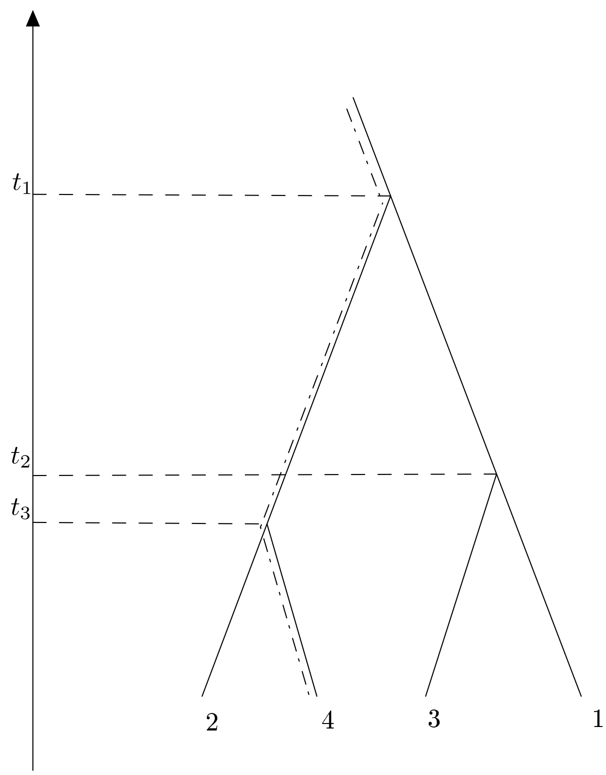

Figure 6: We used a dashed line to evidence the virtual trajectory of the first particle.

Now we define the virtual trajectory of the particle i in the BBF in an inductive way. We set $\zeta^1(s) = (r_1(s), \xi_1(s))$, then we define the inductive step

$$\zeta^i(s) = \left(r^i(s), \xi^i(s)\right) = \begin{cases} (r_i(s), \xi_i(s)) & s \in [0, t_{i-1}] \\ \left(r^j(s), \xi^j(s)\right) & s \in (t_{i-1}, t] \end{cases} \tag{5.16}$$

where $j \in \{1, ..., i-1\}$ is the progenitor of the particle i, i.e. the particle where we create the particle i. With this definition the virtual trajectory of a particle i is its backward trajectory until his creation, before of his creation it is the virtual trajectory of its progenitors.

5.2 Estimate of the recolission

We want to take advantage of the tree expansion to estimate the difference between the intermediate truncated solution and the Boltzmann truncated solution by estimating the difference between the IBF and the BBF. The main difference between the IBF and the BBF are the recolission, i.e. an interaction between particles which is not a creation. This can happen only in the IBF and creates correlations between particles.

First we consider some cutoffs on the integration variables and estimate the complementary term, denoting the various cutoff with an apex Err_i. We establish some obvious estimates useful in the following.

$$\sum_{j_1=0}^{1} \dots \sum_{j_s=0}^{2^s-1} 1 \leq 2^{s(s+1)} \tag{5.17}$$

$$P_s - 1 \leq \sum_{i=1}^{s} 2^i \leq 2^{s+1} \tag{5.18}$$

$$|f^N_{0,P_s}(\zeta^\epsilon(0))| \leq C e^{-\frac{\beta}{2}|\xi^\epsilon(0)|^2} \tag{5.19}$$

$$||f_{0,P_s}(\zeta(0))| \leq C e^{-\frac{\beta}{2}|\xi(0)|^2} \tag{5.20}$$

We also denote with $d\Lambda_{P_{s-1}} = d\boldsymbol{t}_{P_{s-1}} d\boldsymbol{w}_{P_{s-1}} d\boldsymbol{\nu}_{P_{s-1}}$.

First we estimate the error coming from the difference $|\alpha - N\epsilon^2|$.

Lemma 5.1. *Suppose that* $|\alpha - N\epsilon^2| \leq \epsilon$ *and let*

$$f^{N,Err_1}_{1,s} = \sum_{j_1=0}^{1} \dots \sum_{j_s=0}^{2^s-1} (\alpha)^{P_s-1} \sum_{\Gamma(P_s-1)} \sum_{\sigma_{P_s-1}} \boldsymbol{\sigma}_{P_s-1} \int d\Lambda_{P_{s-1}} \prod_{k=1}^{P_s-1} B^\epsilon f^N_{0,P_s}(\zeta^\epsilon(0)) \tag{5.21}$$

Then

$$\|f^N_{1,s} - f^{N,Err_1}_{1,s}\|_\infty \leq \|g_0\|_\infty (Ct)^{2^{s+1}} 2^{s(s+1)} \epsilon \tag{5.22}$$

Proof. We recall that

$$f^N_{1,s} = \sum_{j_1=0}^{1} \dots \sum_{j_s=0}^{2^s-1} (N-1) \dots (N-P_s-1) \left(\epsilon^2\right)^{P_s-1} \sum_{\Gamma(P_s-1)} \sum_{\sigma_{P_s-1}} \boldsymbol{\sigma}_{P_s-1} \int d\Lambda_{P_{s-1}} \prod_{k=1}^{P_s-1} B^\epsilon f^N_{0,P_s}(\zeta^\epsilon(0))$$

and since

$$|(N-1)\dots(N-P_s-1)\left(\epsilon^2\right)^{P_s-1}-(\alpha)^{P_s-1}|\leq 2^{s+1}\alpha^{2^{s+1}}|\alpha-N\epsilon^2|$$

it results that

$$\|f_{1,s}^{N}-f_{1,s}^{N,Err_1}\|_\infty\leq\|g_0\|_\infty 2^{s(s+1)^2}\left(C\alpha t\right)^{2^{s+1}}\epsilon \tag{5.23}$$

□

Next we control the terms $\prod_{k=1}^{P_s-1}B^\epsilon$ and $\prod_{k=1}^{P_s-1}B$. For $\lambda\in(0,1)$ we define the indicator function

$$\chi_\lambda^\epsilon=\chi\left\{\prod_{k=1}^{P_s-1}B^\epsilon\leq\epsilon^{-\lambda}\right\} \tag{5.24}$$

and

$$\chi_\lambda=\chi\left\{\prod_{k=1}^{P_s-1}B\leq\epsilon^{-\lambda}\right\}. \tag{5.25}$$

The following lemma gives us an estimate for the complementary terms.

Lemma 5.2. *Let*

$$f_{1,s}^{N,Err_2}=\sum_{j_1=0}^{1}\dots\sum_{j_s=0}^{2^s-1}(\alpha)^{P_s-1}\sum_{\Gamma(P_s-1)}\sum_{\sigma_{P_s-1}}\boldsymbol{\sigma}_{P_s-1}\int d\Lambda_{P_{s-1}}\chi_\lambda^\epsilon\prod_{k=1}^{P_s-1}B^\epsilon\, f_{0,P_s}^{N}(\zeta^\epsilon(0)) \tag{5.26}$$

and

$$f_{1,s}^{\alpha,Err_2}=\sum_{j_1=0}^{1}\dots\sum_{j_s=0}^{2^s-1}(\alpha)^{P_s-1}\sum_{\Gamma(P_s-1)}\sum_{\sigma_{P_s-1}}\boldsymbol{\sigma}_{P_s-1}\int d\Lambda_{P_{s-1}}\chi_\lambda^\epsilon\prod_{k=1}^{P_s-1}B\, f_{0,P_s}(\zeta(0)) \tag{5.27}$$

Then

$$\|f_{1,s}^{N,Err_1}-f_{1,s}^{N,Err_2}\|_\infty+\|f_{1,s}^{\alpha}-f_{1,s}^{\alpha,Err_2}\|_\infty\leq\epsilon^\lambda\|g_0\|_\infty(C\alpha t)^{2^{(s+1)}}2^{s(s+1)} \tag{5.28}$$

Proof. We prove the estimate only for B^ϵ, the one for B can be obtained along the same lines. We have that

$$|f_{1,s}^{N}-f_{1,s}^{N,\lambda}|\leq\sum_{j_1=0}^{1}\dots\sum_{j_s=0}^{2^s-1}(C\alpha)^{P_s-1}\sum_{\Gamma(P_s-1)}\int d\Lambda_{P_{s-1}}\chi\left\{\prod_{k=1}^{P_s-1}B^\epsilon>\epsilon^{-\lambda}\right\}|\prod_{k=1}^{P_s-1}B^\epsilon\, f_{0,P_s}^{N}(\zeta^\epsilon(0))|\leq$$

$$\epsilon^{\lambda}\|g_0\|_{\infty}\sum_{j_1=0}^{1}...\sum_{j_s=0}^{2^s-1}(C\alpha)^{P_s-1}\sum_{\Gamma(P_s-1)}\int d\Lambda_{P_{s-1}}|\prod_{k=1}^{P_s-1}B^{\epsilon}|^2\, e^{-\frac{\beta}{2}|\xi^{\epsilon}(0)|^2} \tag{5.29}$$

where we used that $1=\epsilon^{-\lambda}\epsilon^{\lambda}\le\epsilon^{\lambda}\prod_{k=1}^{P_s-1}B^{\epsilon}$. Now we observe that

$$\sum_{\Gamma(P_s-1)}|\prod_{k=1}^{P_s-1}B^{\epsilon}|^2\le 2^{P_s-1}\prod_{k=1}^{P_s-1}(P_s)\,v_{k+1}^2+\sum_{i=1}^{P_s}v_i^2 \tag{5.30}$$

Therefore:

$$\begin{aligned}|f_{1,s}^N-f_{1,s}^{N,\lambda}|\le&\epsilon^{\lambda}\|g_0\|_{\infty}(C\alpha)^{2^{(s+1)}}\sum_{j_1=0}^{1}...\sum_{j_s=0}^{2^s-1}\int d\Lambda_{P_{s-1}}e^{-\frac{\beta}{4}|\xi^{\epsilon}(0)|^2}\\&\prod_{k=1}^{P_s-1}\left[(P_s)\,v_{k+1}^2e^{-\frac{\beta}{4}|v_{k+1}|^2}+\sum_{i=1}^{P_s}v_i^2e^{-\frac{\beta}{4P_s}|v_i|^2}\right]\\\le&\epsilon^{\lambda}\|g_0\|_{\infty}(C\alpha t)^{2^{(s+1)}}2^{s(s+1)}.\end{aligned} \tag{5.31}$$

□

The next step is to consider an energy cutoff. We define

$$\chi^{\epsilon}_{\lambda,E}=\chi\left\{\prod_{k=1}^{P_s-1}B^{\epsilon}\le\epsilon^{-\lambda}\right\}\chi\left\{|\xi^{\epsilon}(0)|\le 2E\right\} \tag{5.32}$$

and

$$\chi_{\lambda,E}=\chi\left\{\prod_{k=1}^{P_s-1}B\le\epsilon^{-\lambda}\right\}\chi\left\{|\xi(0)|\le 2E\right\} \tag{5.33}$$

The following estimate holds true

Lemma 5.3. *Let*

$$f_{1,s}^{N,Err_3}=\sum_{j_1=0}^{1}...\sum_{j_s=0}^{2^s-1}(\alpha)^{P_s-1}\sum_{\Gamma(P_s-1)}\sum_{\sigma_{P_s-1}}\boldsymbol{\sigma}_{P_s-1}\int d\Lambda_{P_{s-1}}\chi^{\epsilon}_{\lambda,E}\prod_{k=1}^{P_s-1}B^{\epsilon}\,f^N_{0,P_s}(\zeta^{\epsilon}(0)) \tag{5.34}$$

and let

$$f_{1,s}^{\alpha,Err_3}=\sum_{j_1=0}^{1}...\sum_{j_s=0}^{2^s-1}(\alpha)^{P_s-1}\sum_{\Gamma(P_s-1)}\sum_{\sigma_{P_s-1}}\boldsymbol{\sigma}_{P_s-1}\int d\Lambda_{P_{s-1}}\chi_{\lambda,E}\prod_{k=1}^{P_s-1}B\,f_{0,P_s}(\zeta(0)) \tag{5.35}$$

Then it results:

$$\|f_{1,s}^{N,Err_2} - f_{1,s}^{N,Err_3}\|_\infty + \|f_{1,s}^{\alpha,Err_2} - f_{1,s}^{\alpha,Err_3}\|_\infty \leq \|g_0\|_\infty e^{-\beta E^2} \left(C\alpha t\right)^{2^{s+1}} 2^{s(s+1)} \tag{5.36}$$

Proof. We give a proof only for $\|f_{1,s}^{N,Err_2} - f_{1,s}^{N,Err_3}\|_\infty$, the other one can be proved in the same way. We have that

$$|f_{1,s}^{N,Err_2} - f_{1,s}^{N,Err_3}| \leq \|g_0\|_\infty \sum_{j_1=0}^{1} ... \sum_{j_s=0}^{2^s-1} (C\alpha)^{P_s-1} \sum_{\Gamma(P_s-1)} \int d\Lambda_{P_{s-1}} \prod_{k=1}^{P_s-1} B^\epsilon \, \chi\left\{|\xi^\epsilon(0)| > 2E\right\} e^{-\frac{\beta}{2}|\xi^\epsilon(0)|^2}$$

$$\leq \|g_0\|_\infty e^{-\beta E^2} \left(C\alpha t\right)^{2^{s+1}} 2^{s(s+1)} \tag{5.37}$$

where we used that $e^{-\frac{\beta}{2}|\xi^\epsilon(0)|^2} \leq e^{-\frac{\beta}{4}|\xi^\epsilon(0)|^2} e^{-\beta E^2}$. □

The next cutoff regards the time variables. We want to separate enough the time between two creation, i.e. we want that $t_i - t_{i-1} > \delta \; \forall\, 0 < i \leq P_s - 1$. We define

$$\chi_{\lambda,E,\delta}^\epsilon = \chi\left\{\prod_{k=1}^{P_s-1} B^\epsilon \leq \epsilon^{-\lambda}\right\} \chi\left\{|\xi^\epsilon(0)| \leq 2E\right\} \chi\left\{t_i - t_{i-1} > \delta,\, 0 < i \leq P_s - 1\right\} \tag{5.38}$$

and

$$\chi_{\lambda,E,\delta} = \chi\left\{\prod_{k=1}^{P_s-1} B \leq \epsilon^{-\lambda}\right\} \chi\left\{|\xi(0)| \leq 2E\right\} \chi\left\{t_i - t_{i-1} > \delta,\, 0 < i \leq P_s - 1\right\} \tag{5.39}$$

For the complementary set we have the following lemma

Lemma 5.4. *Let*

$$f_{1,s}^{N,Err_4} = \sum_{j_1=0}^{1} ... \sum_{j_s=0}^{2^s-1} (\alpha)^{P_s-1} \sum_{\Gamma(P_s-1)} \sum_{\boldsymbol{\sigma}_{P_s-1}} \boldsymbol{\sigma}_{P_s-1} \int d\Lambda_{P_{s-1}} \chi_{\lambda,E,\delta}^\epsilon \prod_{k=1}^{P_s-1} B^\epsilon \, f_{0,P_s}^N(\zeta^\epsilon(0)) \tag{5.40}$$

and let

$$f_{1,s}^{\alpha,Err_4} = \sum_{j_1=0}^{1} ... \sum_{j_s=0}^{2^s-1} (\alpha)^{P_s-1} \sum_{\Gamma(P_s-1)} \sum_{\boldsymbol{\sigma}_{P_s-1}} \boldsymbol{\sigma}_{P_s-1} \int d\Lambda_{P_{s-1}} \chi_{\lambda,E,\delta} \prod_{k=1}^{P_s-1} B \, f_{0,P_s}(\zeta(0)) \tag{5.41}$$

Then the following estimate holds

$$\|f_{1,s}^{N,Err_3} - f_{1,s}^{N,Err_4}\|_\infty + \|f_{1,s}^{\alpha,Err_3} - f_{1,s}^{\alpha,Err_5}\|_\infty \leq \epsilon^{-\lambda}\|g_0\|_\infty \left(C\alpha t\right)^{2^{s+1}} 2^{(s+2)(s+1)}\frac{\delta}{t} \tag{5.42}$$

Proof. As the other lemma we give a proof only for the term $\|f_{1,s}^{N,Err_3} - f_{1,s}^{N,Err_4}\|_\infty$ since for the other one the proof is similar. We have that

$$|f_{1,s}^{N,Err_3} - f_{1,s}^{N,Err_4}| \leq \epsilon^{-\lambda}\|g_0\|_\infty \sum_{j_1=0}^{1} ... \sum_{j_s=0}^{2^s-1} (C\alpha)^{P_s-1} \sum_{\Gamma(P_s-1)} \int d\Lambda_{P_{s-1}} \left(\chi_{\lambda,E}^{\epsilon} - \chi_{\lambda,E,\delta}^{\epsilon}\right) e^{-\frac{\beta}{2}|\xi^\epsilon(0)|^2} \tag{5.43}$$

There are $P_s - 1$ choices of time variables such that $t_i - t_{i-1} \leq \delta$, this gives us that

$$|f_{1,s}^{N,Err_3} - f_{1,s}^{N,Err_4}| \leq \epsilon^{-\lambda}\|g_0\|_\infty \left(C\alpha t\right)^{2^{s+1}} 2^{(s+2)(s+1)}\frac{\delta}{t} \tag{5.44}$$

□

Finally we introduce the last cutoff in the integrals. We define the indicator function

$$\chi_{\lambda,E,\delta,q}^{\epsilon} = \chi_{\lambda,E,\delta}^{\epsilon}\chi\left\{|\omega_k \cdot (w_{k+1} - \xi_{i_k}^{\epsilon}(\tau_k))| \geq \epsilon^q,\ |\rho_k| \geq \epsilon^q,\ 1 \leq k \leq P_s - 1\right\} \tag{5.45}$$

and

$$\chi_{\lambda,E,\delta,q} = \chi_{\lambda,E,\delta}\chi\left\{|\omega_k \cdot (w_{k+1} - \xi_{i_k}(\tau_k))| \geq \epsilon^q,\ |\rho_k| \geq \epsilon^q,\ 1 \leq k \leq P_s - 1\right\} \tag{5.46}$$

With this cutoff we are neglecting the grazing and the central velocities in the creation of new particles. We have the following estimate:

Lemma 5.5. *Let*

$$f_{1,s}^{N,Err_5} = \sum_{j_1=0}^{1} ... \sum_{j_s=0}^{2^s-1} (\alpha)^{P_s-1} \sum_{\Gamma(P_s-1)} \sum_{\sigma_{P_s-1}} \boldsymbol{\sigma}_{P_s-1} \int d\Lambda_{P_{s-1}} \chi_{\lambda,E,\delta,q}^{\epsilon} \prod_{k=1}^{P_s-1} B^\epsilon\, f_{0,P_s}^{N}(\zeta^\epsilon(0)) \tag{5.47}$$

and let

$$f_{1,s}^{\alpha,Err_5} = \sum_{j_1=0}^{1} ... \sum_{j_s=0}^{2^s-1} (\alpha)^{P_s-1} \sum_{\Gamma(P_s-1)} \sum_{\sigma_{P_s-1}} \boldsymbol{\sigma}_{P_s-1} \int d\Lambda_{P_{s-1}} \chi_{\lambda,E,\delta,q} \prod_{k=1}^{P_s-1} B\, f_{0,P_s}(\zeta(0)) \tag{5.48}$$

Then:

$$\|f_{1,s}^{N,Err_4} - f_{1,s}^{N,Err_5}\|_\infty + \|f_{1,s}^{\alpha,Err_4} - f_{1,s}^{\alpha,Err_5}\|_\infty \le \|g_0\|_\infty \epsilon^{\frac{q}{2}-\lambda} \left(C\alpha t\right)^{2^{s+1}} 2^{(s+2)(s+1)} \tag{5.49}$$

with $0 < q < 1$.

Proof. We have that

$$|f_{1,s}^{N,Err_4} - f_{1,s}^{N,Err_5}| \le \epsilon^{-\lambda}\|g_0\|_\infty \sum_{j_1=0}^{1} \dots \sum_{j_s=0}^{2^s-1} (C\alpha)^{P_s-1} \sum_{\Gamma(P_s-1)} \int d\Lambda_{P_{s-1}} \left(\chi_{\lambda,E,\delta}^{\epsilon} - \chi_{\lambda,E,\delta,q}^{\epsilon}\right) e^{-\frac{\beta}{2}|\xi^\epsilon(0)|^2} \tag{5.50}$$

This means that there exist a k such that $|\omega_k \cdot (v_{k+1} - \xi_{i_k}^\epsilon(\tau_k)| \le \epsilon^q$. If $|(v_{k+1} - \xi_{i_k}^\epsilon(\tau_k)| \le \epsilon^{\frac{q}{2}}$ then we simply have that

$$|f_{1,s}^{N,Err_4} - f_{1,s}^{N,Err_5}| \le \|g_0\|_\infty \epsilon^{\frac{q}{2}-\lambda} \left(C\alpha t\right)^{2^{s+1}} 2^{(s+2)(s+1)} \tag{5.51}$$

Otherwise if $|(v_{k+1} - \xi_{i_k}^\epsilon(\tau_k)| > \epsilon^{\frac{q}{2}}$ it results that $|\cos\gamma| \le \epsilon^{\frac{q}{2}}$, where γ is the angle between $v_{k+1} - \xi_{i_k}^\epsilon(\tau_k)$ and ω_k. Therefore $|\frac{\pi}{2} - \gamma| \le C\epsilon^{\frac{q}{2}}$ and, fixed $v_{k+1} - \xi_{i_k}^\epsilon(\tau_k)$, ω_k must be in a set of measure bounded by $C\epsilon^q$. The case $\rho_k \le \epsilon^q$ can be easily estimated, since $d\nu_k = \rho_k d\rho_k d\psi$. We have that

$$|f_{1,s}^{N,Err_4} - f_{1,s}^{N,Err_5}| \le \|g_0\|_\infty \epsilon^{q-\lambda} \left(C\alpha t\right)^{2^{s+1}} 2^{(s+2)(s+1)} \tag{5.52}$$

From (5.51) and (5.52) we arrive to

$$\|f_{1,s}^{N,Err_4} - f_{1,s}^{N,Err_5}\|_\infty \le \|g_0\|_\infty \epsilon^{\frac{q}{2}-\lambda} \left(C\alpha t\right)^{2^{s+1}} 2^{(s+2)(s+1)} \tag{5.53}$$

□

We are now in position to estimate the difference between the BBF and the IBF.

We define the following set

$$N^{P_s}(\epsilon_0) = \left\{ (\boldsymbol{t}_{P_s-1}, \boldsymbol{\nu}_{P_s-1}, \boldsymbol{w}_{P_s-1}) \in \mathbb{R}^{P_s-1} \times S^{2(P_s-1)} \times \mathbb{R}^{3(P_s-1)} \mid \min_{i<k} \min_{\tau\in[0,t_{i-1}]} d(r_i(\tau), r_k(\tau)) < \epsilon_0 \right\} \tag{5.54}$$

where $d(\cdot,\cdot)$ denotes the distance over the torus Γ. This set is completely defined via the BBF and it is the set of variables for which a recollision can appear. At this point we need to prove that the measure of the set $N^{P_s}(\epsilon_0)$ is small, taking into account also the constraints given by $\chi_{\lambda,E,\delta,q}^{\epsilon}$ and $\chi_{\lambda,E,\delta,q}$. This smallness is proved in [2] in the case of particles moving in the whole $\mathbb{R}^3$ instead that in a torus. In the following lemma we adapt this result to the present context by using also some geometrical estimate proved in [1].

Lemma 5.6. *Let $\chi_{\lambda,E,\delta,q}$ be defined as in (5.46) and let $\chi\left\{N^{P_s}(\epsilon_0)\right\}$ be the characteristic function of the set (5.54). Then it results that*

$$\sum_{j_1=0}^{1}...\sum_{j_s=0}^{2^s-1}(\alpha)^{P_s-1}\sum_{\Gamma(P_s-1)}\sum_{\sigma_{P_s-1}}\boldsymbol{\sigma}_{P_s-1}\int d\Lambda_{P_s-1}\chi_{\lambda,E,\delta,q}\chi\left\{N^{P_s}(\epsilon_0)\right\}\prod_{k=1}^{P_s-1}B\, f^N_{0,P_s}(\zeta(0))\leq$$

$$\|g_0\|_\infty\,(C\alpha t)^{2^{s+1}}\,E^8 2^{(s+4)(s+1)}\left(\epsilon_0^{\frac{2}{5}-\lambda}+\frac{\epsilon_0^{\frac{4}{5}-\lambda}}{\delta^2}+\epsilon_0^{\frac{4}{5}-\lambda}\right)\tag{5.55}$$

We leave the proof of this lemma in the appendix II.

Thanks to these estimates we can now give a proof of the convergence of the IBF to the BBF and then of the one particle marginal of the GH to the solution of the Boltzmann equation. First we choose the magnitude of the parameters in the following way

$$\alpha\cong C\left(\log\log N\right)^{\frac{1}{2}}\;\; s\cong\frac{\log\log N}{2\log 2}\tag{5.56}$$

Furthermore we have that

$$2^{s+1}\leq 2\left(\log N\right)^{\frac{1}{2}}\tag{5.57}$$

$$2^{(s+2)(s+1)}\leq 2\left(\log N\right)^{\frac{\log\log N}{2\log 2}}\tag{5.58}$$

$$(C\alpha t)^{2^{s+1}}\leq\left(C\log\log N\right)^{\sqrt{\log N}}\tag{5.59}$$

$$N\epsilon^2\leq C\left(\log\log N\right)^{\frac{1}{2}}\tag{5.60}$$

$$\epsilon\leq C\frac{\left(\log\log N\right)^{\frac{1}{4}}}{N^{\frac{1}{2}}}\tag{5.61}$$

We also set $\epsilon_0=\epsilon^{\frac{5}{6}}$, $\delta=\epsilon^{\frac{1}{8}}$, $E=\frac{\sqrt{\log N}}{\beta}$ and we fix $q=\frac{1}{8}$ and $\lambda=\frac{1}{32}$. We have the following theorem

Theorem 5.7. *Let $\widetilde{f_1^N}(t)$ be the one particle marginal of the Grad hierarchy with initial datum as (3.6) and let $f_1^\alpha(t)$ be the solution of the Boltzmann equation with initial datum as (3.10). Then $\forall t\in[0,T]$ it results that*

$$\|\widetilde{f_1^N}(t)-f_1^\alpha(t)\|_\infty\to 0\tag{5.62}$$

for $N \to \infty$, $\epsilon \to 0$, $|N\epsilon^2 - \alpha| \leq \epsilon$.

Proof. We have

$$\|\widetilde{f_1^N}(t) - f_1^\alpha(t)\|_\infty \leq \|\widetilde{f_{1,s}^N}(t) - f_{1,s}^N(t)\|_\infty + \|f_{1,s}^N(t) - f_{1,s}^\alpha(t)\|_\infty + \|\widetilde{R_N^s}(t)\|_\infty + \|R^s(t)\|_\infty \tag{5.63}$$

From Theorems 4.3 and 4.4 it results that

$$\|\widetilde{f_{1,s}^N}(t) - f_{1,s}^N(t)\|_\infty \leq \|g_0\|_\infty 2^{s(s+1)} \epsilon \left(C\alpha t\right)^{2^{s+1}} \leq \|g_0\|_\infty \left(C \log\log N\right)^{\sqrt{\log N}} \left(\log N\right)^{\frac{\log\log N}{4\log 2}} \frac{\left(\log\log N\right)^{\frac{1}{4}}}{N^{\frac{1}{2}}} \tag{5.64}$$

$$\|\widetilde{R_N^s}(t)\|_\infty + \|R^s(t)\|_\infty \leq \|g_0\|_\infty \left(\frac{C\left(\alpha t\right)^2}{s}\right)^2 \leq \frac{C\|g_0\|_\infty}{\log\log N} \tag{5.65}$$

We have to work on the term $\|f_{1,s}^N(t) - f_{1,s}^\alpha(t)\|_\infty$. First it results that

$$\begin{aligned}\|f_{1,s}^N(t) - f_{1,s}^\alpha(t)\|_\infty \leq & \sum_{l=1}^{5} \|f_{1,s}^{N,Err_{l-1}} - f_{1,s}^{N,Err_l}\|_\infty + \sum_{l=0}^{5} \|f_{1,s}^{\alpha,Err_{l-1}}(t) - f_{1,s}^{\alpha,Err_l}\|_\infty \\ & + \|f_{1,s}^{N,Err_5}(t) - f_{1,s}^{\alpha,Err_5}(t)\|_\infty\end{aligned} \tag{5.66}$$

where $f_{1,s}^{N,Err_0}(t) = f_{1,s}^N(t)$ and $f_{1,s}^{\alpha,Err_0}(t) = f_{1,s}^\alpha(t)$. We focus on the last term, it results that

$$\begin{aligned}|f_{1,s}^{N,Err_5}(t) - f_{1,s}^{\alpha,Err_5}(t)| \leq & \sum_{j_1=0}^{1} \dots \sum_{j_s=0}^{2^s-1} \left(C\alpha\right)^{P_s-1} \sum_{\sigma_{P_s-1}} \sum_{\Gamma(P_s-1)} \\ & \sum_{\Gamma(P_s-1)} \int d\Lambda_{P_{s-1}} |\chi_{\lambda,E,\delta,q}^\epsilon f_{0,P_s}^N(\zeta^\epsilon(0)) - \chi_{\lambda,E,\delta,q} f_{0,P_s}(\zeta(0))|\end{aligned} \tag{5.67}$$

We split the integral in $1 - \chi\left\{N^{P_s}(\epsilon_0)\right\}$ and $\chi\left\{N^{P_s}(\epsilon_0)\right\}$. In the first case since we are outside the set $N^{P_s}(\epsilon_0)$ the particles must be at a distance greater than ϵ_0, this implies that $M_{N,\beta}(\boldsymbol{z}_N) = C_{N,\beta} e^{-\frac{\beta}{2}|\boldsymbol{v}_N|^2}$ and that $\chi_{\lambda,E,\delta,q}^\epsilon = \chi_{\lambda,E,\delta,q}$. Then we have that

$$\begin{aligned}&\int d\Lambda_{P_{s-1}} \left(1 - \chi\left\{N^{P_s}(\epsilon_0)\right\}\right) \chi_{\lambda,E,\delta,q} |f_{0,P_s}^N(\zeta^\epsilon(0)) - f_{0,P_s}(\zeta(0))| \leq \\ &\int d\Lambda_{P_{s-1}} \left(1 - \chi\left\{N^{P_s}(\epsilon_0)\right\}\right) \chi_{\lambda,E,\delta,q} \left[|f_{0,P_s}^N(\zeta^\epsilon(0)) - f_{0,P_s}(\zeta^\epsilon(0))| + |f_{0,P_s}(\zeta^\epsilon(0)) - f_{0,P_s}(\zeta(0))|\right]\end{aligned} \tag{5.68}$$

From the definition of the initial datum it turns out that

$$|f^N_{0,P_s}(\zeta^\epsilon(0)) - f_{0,P_s}(\zeta^\epsilon(0))| \le \|g_0\|_\infty |C_{P_s,\beta} - C^{P_s}_\beta| \tag{5.69}$$

A straightforward calculation from the definition (3.2) and (3.3) gives us that

$$|C_{P_s,\beta} - C^{P_s}_\beta| \le 2^{2(s+1)}\epsilon^3 \tag{5.70}$$

Moreover outside the set $N^{P_s}(\epsilon_0)$ the velocities of the BBF and of the IBF are the same and also $p^\epsilon_1(s) = p_1(s)\, 0 \le s \le t$, it follows that

$$|f_{0,P_s}(\zeta^\epsilon(0)) - f_{0,P_s}(\zeta(0))| = |C_{N,\beta} e^{-\frac{\beta}{2}|\xi^\epsilon(0)|^2} g_0(p^\epsilon_1(0), \xi^\epsilon_1(0)) - C_{N,\beta} e^{-\frac{\beta}{2}|\xi(0)|^2} g_0(r^\epsilon_1(0), \xi^\epsilon_1(0))| = 0 \tag{5.71}$$

Finally we have that

$$\sum_{j_1=0}^{1} \dots \sum_{j_s=0}^{2^s-1} (C\alpha)^{P_s-1} \sum_{\sigma_{P_s-1}} \sum_{\Gamma(P_s-1)} \int d\Lambda_{P_{s-1}} \left(1 - \chi\left\{N^{P_s}(\epsilon_0)\right\}\right) \chi_{\lambda,E,\delta,q}\, |\, f^N_{0,P_s}(\zeta^\epsilon(0)) - \, f_{0,P_s}(\zeta(0))| \le$$

$$(C\alpha t)^{2^{s+1}}\, 2^{2s(s+1)}\epsilon^{3-\lambda} \tag{5.72}$$

In the second case we use the estimates of Lemma 5.6 to obtain that

$$\sum_{j_1=0}^{1} \dots \sum_{j_s=0}^{2^s-1} (C\alpha)^{P_s-1} \sum_{\sigma_{P_s-1}} \sum_{\Gamma(P_s-1)} \int d\Lambda_{P_{s-1}}\, \chi\left\{N^{P_s}(\epsilon_0)\right\} |\chi^\epsilon_{\lambda,E,\delta,q}\, f^N_{0,P_s}(\zeta^\epsilon(0)) - \chi_{\lambda,E,\delta,q}\, f^\alpha_{0,P_s}(\zeta(0))|$$

$$\le \epsilon^{-\lambda} \|g_0\|_\infty (C\alpha t)^{2^{s+1}} E^8 2^{(s+4)(s+1)} \left(\epsilon_0^{\frac{2}{5}} + \frac{\epsilon_0^{\frac{4}{5}}}{\delta^2} + \epsilon_0^{\frac{4}{5}}\right) \le$$

$$\|g_0\|_\infty \epsilon^{\frac{1}{20}} \left(C \log\log N\right)^{\sqrt{\log N}} (\log N)^{4 \log\log N} \tag{5.73}$$

We have proved that

$$\|f^{N,Err_5}_{1,s}(t) - f^{\alpha,Err_5}_{1,s}(t)\|_\infty \le \|g_0\|_\infty \epsilon^{\frac{1}{20}} \left(C \log\log N\right)^{\sqrt{\log N}} (\log N)^{4 \log\log N} \tag{5.74}$$

The remainders can be easily handled with the estimates proved in Lemmas 5.1-5.5. It follows that

$$\sum_{l=1}^{5} \|f_{1,s}^{N,Err_{l-1}} - f_{1,s}^{N,Err_l}\|_\infty + \sum_{l=0}^{5} \|f_{1,s}^{\alpha,Err_{l-1}}(t) - f_{1,s}^{\alpha,Err_l}\|_\infty \leq \|g_0\|_\infty \left(C\alpha t\right)^{2^{s+1}} 2^{(s+2)(s+1)}$$

$$\left(\epsilon^{\frac{q}{2}-\lambda} + \epsilon^{\lambda} + \epsilon^{-\lambda}\delta + e^{-\beta E^2}\right) \leq \|g_0\|_\infty \left(C \log\log N\right)^{\sqrt{\log N}} \left(\log N\right)^{\frac{\log\log N}{\log 2}} \left(\epsilon^{\frac{1}{32}} + \frac{1}{N}\right) \tag{5.75}$$

Summarizing, we have that

$$\|\widetilde{f_1^N}(t) - f_1^\alpha(t)\|_\infty \leq \|\widetilde{f_{1,s}^N}(t) - f_{1,s}^N(t)\|_\infty + \|f_{1,s}^N(t) - f_{1,s}^\alpha(t)\|_\infty + \|\widetilde{R_N^s}(t)\|_\infty + \|R^s(t)\|_\infty \leq$$

$$\frac{C\|g_0\|_\infty}{\log\log N} + \|g_0\|_\infty \left(C \log\log N\right)^{\sqrt{\log N}} \left(\log N\right)^{\left(\frac{\log\log N}{\log 2}\right)^3} \left[\epsilon^{\frac{1}{20}} + \frac{1}{N}\right]$$

If we send $N \to \infty$, $\epsilon \to 0$ with $N\epsilon^2 \cong C\left(\log\log N\right)^{\frac{1}{2}}$ we obtain the proof of the theorem. □

6 From Linear Boltzmann to Linear Landau

6.1 Existence of semigroups

In this section we want to prove that the solution of the Linear Boltzmann equation converges as $\alpha \to \infty$ to the solution of the Linear Landau equation. For this purpose we rewrite in the following way the linear Boltzmann and Landau equations

$$\begin{cases} \partial_t f = G_\alpha(f) \\ f(x,v,0) = f_0(x,v) \end{cases} \tag{6.1}$$

$$\begin{cases} \partial_t f = G(f) \\ f(x,v,0) = f_0(x,v) \end{cases} \tag{6.2}$$

where

$$G_\alpha(f) = Q_B(f) - v \cdot \nabla_x f \tag{6.3}$$

and

$$G(f) = Q_L(f) - v \cdot \nabla_x f. \tag{6.4}$$

Now we want to set the problem in the Hilbert space $\mathbf{H} = L^2\left(\Gamma \times \mathbb{R}^3, dxd\mu\right)$ where $d\mu = M_\beta(v)dv$. This space arises naturally from the definition of the operators G and G_α. Indeed, we have that G_α and G are unbounded linear operators densely defined respectively on $D(G_\alpha) = H^1(\Gamma, dx) \times L^2(\mathbb{R}^3, d\mu)$ and $D(G) = H^1(\Gamma, dx) \times H^2(\mathbb{R}^3, d\mu)$, where H^1 and H^2 denote the usual Sobolev spaces.

The main motivation to introduce $\mathbf{H}$ is the following lemma:

Lemma 6.1. *The operators* $Q_B(f)$ and $Q_L(f)$ are well defined as *self-adjoint operators* on $L^2(\mathbb{R}^3, d\mu)$ and $H^2(\mathbb{R}^3, d\mu)$ respectively. *Moreover for the operators G and G_α, defined in (6.3) and (6.4), we have that* $\forall f \in \mathbf{H}$ *and* $\forall g \in D(G)$

$$(G_\alpha^* f, f) = (f, G_\alpha f) \leq 0 \tag{6.5}$$

and

$$(G^* g, g) = (g, Gg) \leq 0 \tag{6.6}$$

i.e. G_α and G are dissipative operators. Furthermore G_α and G are closed operators.

We gives the proof of this lemma in the appendix I.

Thanks to these properties of the operators we can use the following theorem.

Theorem 6.2. *([20]) Let A be a linear operator densely defined on a linear subspace $D(A)$ of the Hilbert space* $\mathbf{H}$*. If both A and A^* are dissipative operators then $\overline{A}$ generate a contraction semigroup on* $\mathbf{H}$*.*

This theorem ensures the existence of $T_\alpha(t)$ and $T(t)$, the semigroups with infinitesimal generator given by G_α and G respectively. Indeed, from Lemma 6.1 we have that G_α and G are closed operators and that G_α^* and G^* are dissipative operators, then we have the existence of $T_\alpha(t)$ and $T(t)$.

6.2 Convergence of the semigroups

The last step of our proof is to show that the semigroup generated by $G_\alpha(f)$ strongly converges to the semigroup generated by $G(f)$ in the limit $\alpha \to 0$. We use the following theorem, that gives necessary and sufficient conditions for the convergence.

Theorem 6.3. *(Trotter-Kato). Let A and A_n be the generators of the contraction semigroups $T(t)$ and $T_n(t)$ respectively. Let D be a core for A. Suppose that $D \subseteq D(A_n)$ $\forall n$ and that $\forall f \in D$ $A_n f \to Af$. Then*

$$\|T_n f - Tf\|_H \to 0 \quad as\ n \to +\infty \tag{6.7}$$

$\forall f \in \mathbf{H}$ and uniformly for $t \in [0, T]$ for any $T > 0$.

A proof of this theorem can be found in [20].

We want to apply this theorem to prove that $T_\alpha f \to Tf$. We choose as a core $D = C_p^\infty(\Gamma) \times C_0^\infty(\mathbb{R}^3)$ and we use the steps of section 3.2 to prove the strongly convergence of the operators on this set.

Theorem 6.4. *Let G_α and G be defined as in (6.3) and (6.4). Then $\forall f \in D$ it results that*

$$\| (G_\alpha - G) f \|_{\mathbf{H}} \underset{\alpha \to \infty}{\longrightarrow} 0$$

Proof. First we define the following operator

$$Q_B^c(f) = \alpha \int dv_1 M_\beta(v_1) \int\limits_{\nu \cdot V > 0} d\nu |\nu \cdot V| \chi \left\{ |V| \geq \alpha^{-\frac{4}{15}} \right\} \left[f(v') - f(v) \right] \tag{6.8}$$

This is the Linear Boltzmann operator with a α-depending cutoff on the small relative velocities. Observe that Q_B^c and Q_B are asymptotically equivalent as $\alpha \to \infty$. Indeed, we have that $\forall f \in D_0$

$$
\begin{aligned}
\| \left(Q_B^c - Q_B\right) f\|_{\mathbf{H}}^2 = & \int dx \int d\mu(v) \left| \alpha \int d\mu(v_1) \int_{\nu \cdot V > 0} \chi \left\{ |V| < \alpha^{-\frac{4}{15}} \right\} |\nu \cdot V| \left[f(x, v^{'}) - f(x,v) \right] d\nu \right|^2 \\
\leq & C \|f\|_\infty^2 \int dx d\mu(v) \left| \int dv_1 \alpha^{\frac{11}{15}} M_\beta(v_1) \chi \left\{ |V| < \alpha^{-\frac{4}{15}} \right\} \right|^2 \\
\leq & C \|f\|_\infty^2 \int dx d\mu(v) \left(\alpha^{\frac{11}{15}} \int \chi \left\{ |V| < \alpha^{-\frac{4}{15}} \right\} dv_1 \right)^2
\end{aligned}
\tag{6.9}
$$

Since

$$
\int \chi \left\{ |V| < \alpha^{-\frac{4}{15}} \right\} dv_1 \leq C \alpha^{-\frac{12}{15}}, \tag{6.10}
$$

we arrive to

$$
\| \left(Q_B^c - Q_B\right) f\|^2 \leq C \alpha^{-\frac{1}{15}}. \tag{6.11}
$$

We put the same cutoff on the operator Q_L and we define

$$
Q_L^c = A \int dv_1 M_\beta(v_1) \frac{1}{|V|^3} \left[|V|^2 \triangle f(v) - \left(V, D^2 V\right) - 4V \cdot \nabla_v f(v) \right] \left\{ |V| \geq \alpha^{-\frac{4}{15}} \right\} \tag{6.12}
$$

Then $\forall f \in D$ we have

$$
\begin{aligned}
\| \left(Q_L^c - Q_L\right) f\|_{\mathbf{H}}^2 = & \int dx d\mu \left| \int dv_1 \frac{M_\beta(v_1) A}{|V|^3} \left[|V|^2 \triangle f(v) - \left(V, D^2 V\right) - 4V \cdot \nabla_v f(v) \right] \left\{ |V| \leq \alpha^{-\frac{4}{15}} \right\} \right|^2 \\
\leq & C(A, f) \alpha^{-\frac{8}{15}}
\end{aligned}
\tag{6.13}
$$

Now we want to prove that Q_B^c converges strongly to Q_L^c when $\alpha \to +\infty$. We have that for all $f \in D$

$$
\begin{aligned}
& \| \left(Q_B^c - Q_L^C\right) f\|_{\mathbf{H}}^2 = \int dx \int d\mu(v) \\
& \left| \int d\mu(v_1) \left\{ \int_{\nu \cdot \mathrm{V} > 0} \alpha \chi \left\{ |V| \geq \alpha^{-\frac{4}{15}} \right\} |v \cdot V| \left[f(x, v^{'}) - f(x,v) \right] d\nu - \right. \right. \\
& \left. \left. \frac{A}{|V|^3} \left[|V|^2 \triangle_v f(x,v) - \left(V, D_v^2 (f(x,v)) V\right) - 4\mathrm{V} \cdot \nabla_v f(x,v) \right] \right\} \chi \left\{ |V| \geq \alpha^{-\frac{4}{15}} \right\} \right|^2
\end{aligned}
\tag{6.14}
$$

We perform the same steps of section 3 to obtain:

$$\| \left(Q_B^c - Q_L^c\right) f\|_{\mathbf{H}}^2 \leq$$

$$C \int dx \int d\mu(v) \left| \int d\mu(v_1) \int_{\nu\cdot V>0} \alpha\, \chi \left\{ |V| \geq \alpha^{-\frac{4}{15}} \right\} |\nu \cdot V| o(\alpha^{-1}) \right|^2 \tag{6.15}$$

For the second term we have to go further in the Taylor expansion and use the Lagrange form for the remainder term. From Lemma (2.2) it results that

$$o(\alpha^{-1}) = \frac{M^2(\rho,\alpha)}{|V|^8\alpha^2} + \frac{\theta^3}{3!} f^{'''}(\xi) \tag{6.16}$$

for a certain $\xi \in [0, \theta]$. Therefore

$$\int dx \int d\mu(v) \left| \int d\mu(v_1) \int_{\nu\cdot V>0} \alpha\, \chi \left\{ |V| \geq \alpha^{-\frac{1}{3}} \right\} |\nu \cdot V| o(\alpha^{-1}) \right|^2 \leq$$

$$\int dx \int d\mu(v) \left| \int d\mu(v_1) \int_{\nu\cdot \mathrm{V}>0} \alpha\, \chi \left\{ |V| \geq \alpha^{-\frac{4}{15}} \right\} |\nu \cdot V| \left[\frac{M^2(\rho,\alpha)}{|V|^8\alpha^2} + \frac{\theta^3}{3!} f^{'''}(\xi) \right] \right|^2 \tag{6.17}$$

Thanks to formula (2.12) we have that

$$|\theta^3(\rho,\alpha)| \leq C \left(\alpha^{-\frac{3}{2}} \frac{\gamma^3(\rho)}{|V|^6} + \alpha^{-3} \frac{M^3(\rho,\alpha)}{|V|^{12}} \right) \tag{6.18}$$

Furthermore from (3.18) it follows that

$$|f^{'''}(\xi)| \leq C(f)|V| \tag{6.19}$$

and then we can write

$$\begin{aligned} \| \left(Q_B^c - Q_L\right) f\|_{\mathbf{H}}^2 \leq & C_1(f,\gamma,M) \int dx \int d\mu(v) \\ & \left[\int d\mu(v_1) \left(\frac{\alpha^{-1}}{|V|^6} + \frac{\alpha^{-\frac{1}{2}}}{|V|^4} + \frac{\alpha^{-2}}{|V|^{10}} \right) \chi \left\{ |V| \geq \alpha^{-\frac{4}{15}} \right\} \right]^2 \end{aligned} \tag{6.20}$$

A change of variables on the right hand side of (6.20) gives us

$$\int d\mu(v_1)\left(\frac{\alpha^{-1}}{|V|^6}+\frac{\alpha^{-\frac{1}{2}}}{|V|^4}+\frac{\alpha^{-2}}{|V|^{10}}\right)\chi\left\{|V|\geq\alpha^{-\frac{1}{3}}\right\}\leq C\int_{\alpha^{-\frac{4}{15}}}^{\infty}dr\left(\frac{\alpha^{-1}}{r^4}+\frac{\alpha^{-\frac{1}{2}}}{r^2}+\frac{\alpha^{-2}}{r^8}\right)\leq C\alpha^{\frac{1}{3}} \tag{6.21}$$

From formula (6.20) and (6.21) we have

$$\|\left(Q_B^c-Q_L\right)f\|^2_{\mathbf{H}}\leq C_2(\gamma M)\alpha^{-\frac{2}{15}} \tag{6.22}$$

Then we have

$$\|\left(Q_B-Q_L\right)f\|^2_{\mathbf{H}}\leq C\alpha^{-\frac{1}{15}} \tag{6.23}$$

and this proves our theorem. □

Finally we use Theorem 6.3 and Theorem 6.4 to prove that the solution of the linear Boltzmann equation converge to the solution of the linear Landau equation.

Theorem 6.5. *Let $g^\alpha(x,v,t)$ be the solution of the linear Boltzmann equation and let $g(x,v,t)$ be the solution of the linear Landau equation. Suppose that the initial datum of both equations is given by $g_0(x,v)$. Then it results that*

$$\|g^\alpha(x,v)-g(x,v)\|_{\mathbf{H}}\to 0 \tag{6.24}$$

when $\alpha\to 0$.

Proof. Since $g^\alpha(t)=T_\alpha(t)g_0(x,v)$ and $g(t)=T(t)g_0(x,v)$ we have that

$$\|g^\alpha(x,v)-g(x,v)\|_{\mathbf{H}}=\|T_\alpha(t)g_0(x,v)-T(t)g_0(x,v)\|_{\mathbf{H}} \tag{6.25}$$

From Theorem 6.3 and Theorem 6.4 we have that the right hand side of (6.25) goes to zero when $\alpha\to 0$ and the theorem is proved. □

7 Proof of the main theorem

In this section we summarize all the estimates obtained and we finally give a proof that the solution of the first equation of the Grad hierarchy converge to the solution of the linear Landau equation in the scaling $N\epsilon^2 \to \alpha$ with $\alpha \cong C\,(\log\log N)^{\frac{1}{2}}$.

Theorem 7.1. *Let $\overline{f_1^N}(t)$ be the first-particle marginal of the solution of the Liouville equation with initial datum given by $W_{0,N}(\boldsymbol{z}_N) = M_{N,\beta}(\boldsymbol{z}_N)g_0(x_1, v_1)$, and let $g(t)$ be the solution of the linear Landau equation with initial datum given by $g(x, v, 0) = g_0(x, v)$. Then $\forall t > 0$*

$$\|\overline{f_1^N}(x, v, t) - M_\beta(v)g(x, v, t)\|_{\mathbf{H}} \to 0$$

when $N \to \infty$, with $N\epsilon^2 \cong (\log\log N)^{\frac{1}{2}}$.

Proof. First we want to estimate the following difference

$$\|\widetilde{f_1^N}(t) - f_1^\alpha(t)\|_{\mathbf{H}} \tag{7.1}$$

Since $\int dx \int d\mu(v)|\widetilde{f_1^N}(t) - f_1^\alpha(t)|^2 \leq C\|\widetilde{f_1^N}(t) - f_1^\alpha(t)\|_\infty^2$ it results that

$$\|\widetilde{f_1^N}(t) - f_1^\alpha(t)\|_{\mathbf{H}} \leq C\|\widetilde{f_1^N}(t) - f_1^\alpha(t)\|_\infty \tag{7.2}$$

From Theorem 5.7 we have that

$$\|\widetilde{f_1^N}(t) - f_1^\alpha(t)\|_\infty \to 0 \tag{7.3}$$

and then

$$\|\widetilde{f_1^N}(t) - f_1^\alpha(t)\|_{\mathbf{H}} \to 0 \tag{7.4}$$

As we have seen the solution of the first equation of the BH with initial data given by (3.10) has the form

$$f_1^\alpha(x, v, t) = M_\beta(v)g^\alpha(x, v, t) \tag{7.5}$$

where $g^\alpha(x, v)$ is the solution of the Linear Boltzmann equation. Then we have proved that

$$\|\widetilde{f_1^N}(t) - M_\beta(v)g^\alpha(x, v, t)\|_{\mathbf{H}} \to 0 \tag{7.6}$$

Since

$$\|M_\beta(v)g^\alpha(x,v,t) - M_\beta(v)g(x,v,t)\|_{\mathbf{H}} \leq C\|g^\alpha(x,v) - g(x,v,t)\|_{\mathbf{H}} \tag{7.7}$$

Thanks to Theorem (6.4) we have that

$$\|g^\alpha(x,v,t) - g(x,v,t)\|_{\mathbf{H}} \to 0 \tag{7.8}$$

From (7.7) and (7.4) we arrive to

$$\|\widetilde{f_1^N}(x,v,t) - M_\beta(v)g(x,v,t)\|_{\mathbf{H}} \leq \|\widetilde{f_1^N}(t) - f_1^\alpha(t)\|_{\mathbf{H}} + C\|g^\alpha(x,v,t) - g(x,v,t)\|_{\mathbf{H}} \tag{7.9}$$

Finally we estimate the difference between the reduced marginal and the standard marginal. We have

$$\begin{aligned}
|\widetilde{f_1^N}(x,v,t) - \overline{f_1^N}(x,v,t)| =& |\int dz_{j+1}...dz_N \, W_N(\boldsymbol{z}_N,t) \left(1 - \chi\left\{S(x_1)^{N-1}\right\}\right)| \\
\leq& N| \int\limits_{|x-x_1|\leq\epsilon} dx_1 dv_1 \overline{f_2^N}(x,v,x_1,v_1,t)| \\
\leq& CN\epsilon^3
\end{aligned} \tag{7.10}$$

Then it results

$$\|\widetilde{f_1^N}(x,v,t) - \overline{f_1^N}(x,v,t)\|_\infty \tag{7.11}$$

We send $\alpha \to \infty$, $N \to \infty$ with $N\epsilon^2 \cong \alpha \cong (\log\log N)^{\frac{1}{2}}$ and we obtain the proof of the theorem. □

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8 Appendix I, Proof of Lemma 6.1

Here we gives a proof of the Lemma 6.1.

Proof. First we want to prove that the operator Q_L is self-adjoint. It results that

$$(f, Q_L(g))_{L^2(d\mu)} = \int dv \int dw M_\beta(v) M_\beta(w) \frac{1}{|V|^3} \left[|V|^2 \triangle g - \left(V, D_v^2(g) V\right) - 4V \cdot \nabla_v g \right] f(v) \tag{8.1}$$

We integrate by parts the first term. We have

$$\int dv \int dw M_\beta(v) M_\beta(w) \frac{1}{|V|^3} |V|^2 f \triangle g =$$

$$\int dv \int dw M_\beta(v) M_\beta(w) \left[-\frac{\nabla f \cdot \nabla g}{|V|} + 2\beta \frac{v \cdot \nabla g}{|V|} f + \frac{V \cdot \nabla g}{|V|^3} f \right] \tag{8.2}$$

For the second term it results that

$$-\int dv \int dw M_\beta(v) M_\beta(w) \frac{1}{|V|^3} \left(V, H_v(g) V\right) =$$

$$\int dv \int dw M_\beta(v) M_\beta(w) \left[\frac{V \cdot \nabla g}{|V|^3} f + \frac{(V \cdot \nabla g)(V \cdot \nabla f)}{|V|^3} - 2\beta \frac{(v \cdot V)(V \cdot \nabla g)}{|V|^3} f \right] \tag{8.3}$$

We put together these two terms with the last one, this gives us

$$(f, Q_L(g))_{L^2(d\mu)} = \int dv \int dw M_\beta(v) M_\beta(w) \left[-\frac{\nabla f \cdot \nabla g}{|V|} + \frac{(V \cdot \nabla g)(V \cdot \nabla f)}{|V|^3} - 2\frac{V \cdot \nabla g}{|V|^3} f \right] +$$

$$\int dv \int dw M_\beta(v) M_\beta(w) \left[2\beta \frac{v \cdot \nabla g}{|V|} f - 2\beta \frac{(v \cdot V)(V \cdot \nabla g)}{|V|^3} f \right] \tag{8.4}$$

Now we observe that $v = w + V$ and that $2\beta w M_\beta(w) = -\nabla_w M_\beta(w)$, we also integrate by parts with respect to the variable w in the second terms of (7.4) and we arrive to

$$\int dv \int dw M_\beta(v) M_\beta(w) \left[2\beta \frac{v \cdot \nabla g}{|V|} f - 2\beta \frac{(v \cdot V)(V \cdot \nabla g)}{|V|^3} f \right] =$$

$$\int dv \int dw M_\beta(v) \nabla_w M_\beta(w) \cdot \left[-\frac{\nabla g}{|V|} f + \frac{V(V \cdot \nabla g)}{|V|^3} f \right] =$$

$$\int dv \int dw M_\beta(v) M_\beta(w) 2 \frac{V \cdot \nabla g}{|V|^3} f \tag{8.5}$$

This yields

$$(f, Q_L(g))_{L^2(d\mu)} = \int dv \int dw M_\beta(v) M_\beta(w) \left[-\frac{\nabla f \cdot \nabla g}{|V|} + \frac{(V \cdot \nabla g)\,(V \cdot \nabla f)}{|V|^3} \right] \tag{8.6}$$

Another integration by parts leads to

$$(f, Q_L(g))_{L^2(d\mu)} = \int dv \int dw M_\beta(v) M_\beta(w)$$

$$\left[\frac{g \triangle f}{|V|} - 2\beta \frac{(v \cdot \nabla f)\, g}{|V|} - \frac{(V \cdot \nabla f)\, g}{|V|^3} - \frac{(V \cdot \nabla f)\, g}{|V|^3} - \frac{\big(V, D^2(f) V\big)\, g}{|V|^3} + 2\beta \frac{(v \cdot V)\,(V \cdot \nabla f)\, g}{|V|^3} \right] =$$

$$\int dv \int dw M_\beta(v) M_\beta(w) \left[\frac{g \triangle f}{|V|} - \frac{\big(V, D^2(f) V\big)\, g}{|V|^3} - 2 \frac{(V \cdot \nabla f)\, g}{|V|^3} \right] +$$

$$\int dv \int dw M_\beta(v) M_\beta(w) \left[-2\beta \frac{(v \cdot \nabla f)\, g}{|V|} + 2\beta \frac{(v \cdot V)\,(V \cdot \nabla f)\, g}{|V|^3} \right] \tag{8.7}$$

We integrate by parts the last term with respect to w, it gives us

$$\int dv \int dw M_\beta(v) M_\beta(w) \left[-2\beta \frac{(v \cdot \nabla f)\, g}{|V|} + 2\beta \frac{(v \cdot V)\,(V \cdot \nabla f)\, g}{|V|^3} \right] =$$

$$\int dv \int dw M_\beta(v) M_\beta(w) - 2 \frac{V \cdot \nabla f}{|V|^3} g \tag{8.8}$$

We use toghether (8.8) and (8.7) and we finally arrive to

$$\begin{aligned} (f, Q_L(g))_{L^2(d\mu)} &= \int dv \int dw M_\beta(v) M_\beta(w) \left[\frac{g \triangle f}{|V|} - \frac{\big(V, D^2(f) V\big)\, g}{|V|^3} - 4 \frac{(V \cdot \nabla f)\, g}{|V|^3} \right] \\ &= (Q_L(f), g)_{L^2(d\mu)} \end{aligned} \tag{8.9}$$

Obviously $D(Q_L) = D(Q_L^*)$ and so Q_L is self-adjoint.

We now prove that the linear Boltzmann operator Q_B is self-adjoint. We have

$$(f, Q_B(g))_{L^2(d\mu)} = \int dv \int dw \int d\nu M_\beta(v) M_\beta(w) |\nu \cdot V| f(v) \left[g(v^{'}) - g(v) \right] =$$

$$\int dv \int dw \int d\nu M_\beta(v) M_\beta(w) |\nu \cdot V| f(v) g(v^{'}) - \int dv \int dw \int d\nu M_\beta(v) M_\beta(w) |\nu \cdot V| f(v) g(v) \tag{8.10}$$

In the first term of the sum we change variables in the integration by using the map defined in formula

(2.10). This gives us

$$\int dv \int dw \int d\nu M_\beta(v) M_\beta(w) |\nu \cdot V| f(v) g(v^{'}) = \int dv^{'} \int dw^{'} \int d\nu^{'} M_\beta(v) M_\beta(w) |\nu \cdot V| f(v^{'}) g(v) \tag{8.11}$$

Now we use Lemma 2.1 that gives us that $dv^{'} dw^{'} d\nu^{'} = dv dw d\nu$, this with (8.10) leads to

$$(f, Q_B(g))_{L^2(d\mu)} = \int dv \int dw \int d\nu M_\beta(v) M_\beta(w) |\nu \cdot V| g(v) \left[f(v^{'}) - f(v) \right] = (Q_B(f), g)_{L^2(d\mu)} \tag{8.12}$$

Formula (6.5) and (6.6) can be proved simply with some integration by parts in the definition of the operators Q_B and Q_L. This leads to

$$(f, Q_B f) \leq 0 \tag{8.13}$$

$$(f, Q_L f) \leq 0 \tag{8.14}$$

$$(f, Q_B(f))_{L^2(d\mu)} = \int dv \int dw \int d\nu M_\beta(v) M_\beta(w) |\nu \cdot V| \left[f(v) f(v^{'}) - f^2(v) \right] \tag{8.15}$$

Another change of variables in the integration gives us

$$(f, Q_B(f))_{L^2(d\mu)} = \int dv \int dw \int d\nu M_\beta(v) M_\beta(w) |\nu \cdot V| \left[f(v) f(v^{'}) - f^2(v^{'}) \right] \tag{8.16}$$

We sum together these two equality, this leads to

$$2 \left(f, Q_B(f) \right)_{L^2(d\mu)} \leq - \int dv \int dw \int d\nu M_\beta(v) M_\beta(w) |\nu \cdot V| \left[f(v^{'}) - f(v) \right]^2 \leq 0 \tag{8.17}$$

From formula (8.6) we have

$$(f, Q_L(f))_{L^2(d\mu)} = \int dv \int dw M_\beta(v) M_\beta(w) \left[\frac{\left(\hat{V} \cdot \nabla f \right)^2 - |\nabla f|^2}{|V|} \right] \tag{8.18}$$

and, since $\left(\hat{V} \cdot \nabla f \right)^2 - |\nabla f|^2 \leq 0$, it results that

$$(f, Q_L(f))_{L^2(d\mu)} \leq 0 \tag{8.19}$$

Now we observe that

$$(f, -v \cdot \nabla_x f) = (v \cdot \nabla_x f, f) = (f, v \cdot \nabla_x f) = 0 \tag{8.20}$$

and we arrive to

$$(f, G_\alpha f) = (G_\alpha^* f, f) = (Lf, f) + ((v \cdot \nabla_x f, f)) = (Lf, f) \leq 0 \tag{8.21}$$

With similar steps it is possible to prove the (6.5).

Since $D(G)$ and $D(G_\alpha)$ are dense in **H** by the Von Neumann Theorem we have that

$$G_\alpha^{**} = \overline{G_\alpha} \tag{8.22}$$

but $G_\alpha^{**} = G_\alpha$ and so G_α is closed. This can be proved in the same way for G. □

9 Appendix II, estimate of the recolission set

Lemma 9.1. *Let* $\chi_{\lambda,E,\delta,q}$ *be defined as in (5.46) and let* $\chi\left\{N^{P_s}(\epsilon_0)\right\}$ *be the characteristic function of the set (5.54). Then it results that*

$$\sum_{j_1=0}^{1} ... \sum_{j_s=0}^{2^s-1} (\alpha)^{P_s-1} \sum_{\Gamma(P_s-1)} \sum_{\sigma_{P_s-1}} \boldsymbol{\sigma}_{P_s-1} \int d\Lambda_{P_{s-1}} \chi_{\lambda,E,\delta,q} \chi\left\{N^{P_s}(\epsilon_0)\right\} \prod_{k=1}^{P_s-1} B\, f^N_{0,P_s}(\zeta(0)) \leq$$

$$\|g_0\|_\infty (C\alpha t)^{2^{s+1}} E^8 2^{(s+4)(s+1)} \left(\epsilon_0^{\frac{2}{5}-\lambda} + \frac{\epsilon_0^{\frac{4}{5}-\lambda}}{\delta^2} + \epsilon_0^{\frac{4}{5}-\lambda} \right) \tag{9.1}$$

Proof. First we observe that

$$N^{P_s}(\epsilon_0) = \bigcup_{i=1}^{k} \bigcup_{k=2}^{P_S} N^{P_s}_{i,k}(\epsilon_0) \tag{9.2}$$

where

$$N^{P_s}_{i,k}(\epsilon_0) =$$

$$\left\{ (\boldsymbol{t}_{P_s-1}, \boldsymbol{\nu}_{P_s-1}, \boldsymbol{w}_{P_s-1}) \in \mathbb{R}^{P_s-1} \times S^{2(P_s-1)} \times \mathbb{R}^{3(P_s-1)} \mid \min_{i<k} \min_{\tau\in[0,t_{i-1}]} d(r_i(\tau), r_k(\tau)) < \epsilon_0 \right\} \tag{9.3}$$

We also define a subsequence t^q of the times $t_1...t_n$ associated to the virtual trajectory of particles i and k. We put t^0 as the time in which the two virtual trajectory merge, then we consider the ordered union of the times of creations in the virtual trajectory of particles i and k (Figure 7).

For a point in $N^{P_s}_{i,k}(\epsilon_0)$ there exist

$$\tau^\star = \max\left\{\tau \in [0, t_{i-1}] \mid d(p^i(\tau), p^k(\tau)) < \epsilon_0\right\} \tag{9.4}$$

It must be $\tau^\star \in [t^l, t^{l+1})$ for some $l \geq 0$. With this definition l represents the total number of creation after the time t^0 in the virtual trajectory of the particles i and k. For $q \in [0, l]$ we define

$$Y^q = r^k(t^q) - r^i(t^q) \tag{9.5}$$

$$\xi_i^q = \xi_i(\tau) \;\; \xi_k^q = \xi_k(\tau) \;\; for\, \tau \in (t^{q+1}, t^q) \tag{9.6}$$

$$W^q = \xi_k^q - \xi_i^q \tag{9.7}$$

Observe that, since we are considering only one tree, it will be always $Y^0 = 0$.

First suppose that $l = 0$, this means that the particles i and k have a recolission after the creation

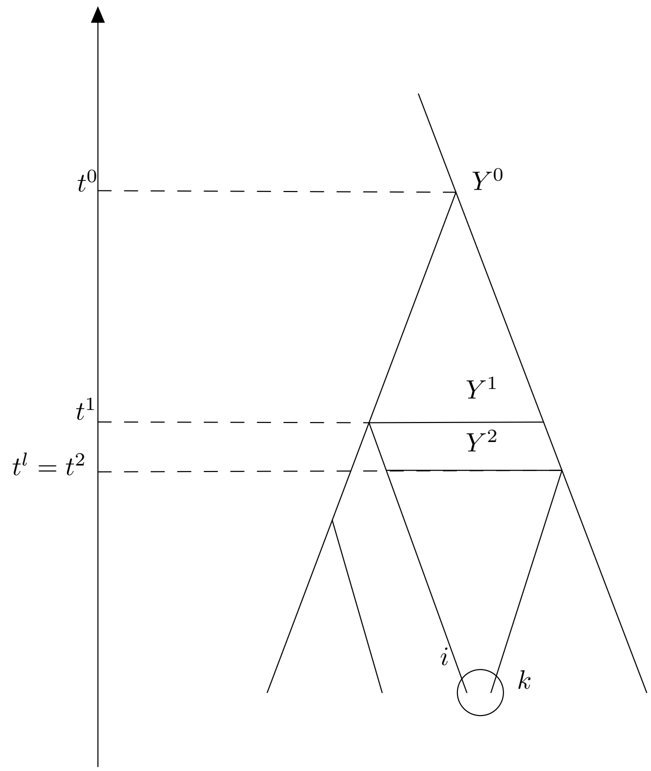

Figure 7: The virtual trajectory of the praticles i and k and their backward history

of the particle k. This can happen in two cases. In the first case the particles i and h do not separate enough after the creation. In the second case the particles, after being separated enough, perform a recolission since the trajectory on the torus have no dispersive properties.

In the first case it must be

$$|W^0|(t^1 - t^0) \leq \epsilon_0 \tag{9.8}$$

We recall that the cutoff (5.46) implies that $(t^1 - t^0) \geq \delta$ and that $|W^0| \geq \epsilon^q$. Then the particles must be separated at least by a distance of $\delta\epsilon^q$. We choose the parameters in such a way that

$$\epsilon_0 \leq \delta\epsilon^q \tag{9.9}$$

and this gives us that the (9.8) cannot happen.

In the second case we prove that W^0 must be in a set of small measure. There exist a $\tau > \delta$ such that

$$d(r^i(t^0-\tau), r^k(t^0-\tau)) \leq \epsilon_0 \tag{9.10}$$

We use the correspondence between the torus and the whole space with periodic structure. We have that

$$\left(r^i(t^0) - \tau\xi^i(t^0)\right) - \left(r^k(t^0) - \tau\xi^k(t^0)\right) \in \bigcup_{p\in\mathbb{Z}^3} B_{\epsilon_0}(p) \tag{9.11}$$

Thanks to the energy cutoff we have that $|W^0| \leq 4E$ and so

$$\tau W^0 \in \left(\bigcup_{p\in\mathbb{Z}^3} B_{\epsilon_0}(r^i(t^0) - r^k(t^0) + p)\right) \bigcap B_{4Et}(0) \tag{9.12}$$

Suppose that $|r^i(t^0) - r^k(t^0) + p| < \frac{1}{4}$. This can happen only for a value of p since the distance between the centers of the spheres is 1. Taking $\hat{v}$ a unit vector normal to $r^i(t^0) - r^k(t^0) + p$ it results that

$$\tau|W^0 \cdot \hat{v}| \leq \epsilon_0 \tag{9.13}$$

and then

$$|W^0 \cdot \hat{v}| \leq \frac{\epsilon_0}{\delta} \tag{9.14}$$

This implies that W^0 is in the intersection of $B_{4E}(0)$ and a cylinder of radius $\frac{\epsilon_0}{\delta}$ and so in a set of measure bounded by $CE\frac{\epsilon_0^2}{\delta^2}$. Suppose now $|r^i(t^0) - r^k(t^0) + p| \geq \frac{1}{4}$ and that ϵ is small enough, then W^0 is in the intersection of $B_{4E}(0)$ and some cone of vertex 0 and solid angle $C\epsilon_0^2$ and these cones are at most $(8Et)^3$. Finally putting together these two estimates gives us that W^0 must be in a suitable set $B_{k,0}$ such that

$$|B_{k,0}| \leq C\left(E\frac{\epsilon_0^2}{\delta^2} + (Et)^3\epsilon_0^2\right) \tag{9.15}$$

We can now suppose that $l \geq 1$. The ϵ_0-overlap is verified only if

$$Y^l - \tau W^l \in \bigcup_{p\in\mathbb{Z}^3} B_{\epsilon_0}(p) \tag{9.16}$$

for some $\tau \in [0, t^l)$. Moreover it results that

$$Y^l = \hat{p} - \sum_{q=0}^{l-1} W^q \left(t^q - t^{q+1}\right) = \hat{p} - W^0 t^0 + \sum_{q=1}^{l} \left(W^{q-1} - W^q\right) t^q \tag{9.17}$$

where $\hat{p} \in \mathbb{Z}^3$ is chosen in such a way that the right hand side of equation (9.17) is a point in the torus.

Now we prove that it must be

$$\sum_{q=1}^{l} |W^q - W^{q-1}| > \epsilon_0^{\frac{2}{5}} \tag{9.18}$$

Otherwise it would be $|W^q - W^0| \leq \epsilon_0^{\frac{2}{5}}$ for all q, then using (9.17) and (9.16) it results that

$$W^0(\tau + t^0 - t^l) \in \bigcup_{p \in \mathbb{Z}^3} B_{(\epsilon_0 + \epsilon_0^{\frac{2}{5}} t)}(p) \tag{9.19}$$

Since $\tau + t^0 - t^l \geq \delta$ we can perform the same steps of estimate (9.15) to prove that in this case W^0must be in a set $B_{k,1}$ of measure bounded by

$$C\left(E\frac{\epsilon_0^{\frac{4}{5}}}{\delta^2} + (Et)^3\epsilon_0^{\frac{4}{5}}\right) \tag{9.20}$$

Condition (9.16) implies that

$$|\left(Y^l + \hat{p}\right) \wedge \hat{W^l}| \leq \epsilon_0 \tag{9.21}$$

with $\hat{W^l} = \frac{W^l}{|W^l|}$. Then from (9.17) we have that

$$|\left(\hat{p} - W^0 t^0\right) \wedge \hat{W^l} - \sum_{q=1}^{l}\left[\left(W^q - W^{q-1}\right) \wedge \hat{W^l}\right] t^q| \leq \epsilon_0 \tag{9.22}$$

Now suppose that

$$\sum_{q=1}^{l} |\left(W^q - W^{q-1}\right) \wedge \hat{W^l}| \leq \epsilon_0^{\frac{3}{5}} \tag{9.23}$$

from (9.18) it must exist a $\bar{q} \in \{1, ..., ..., l\}$ such that

$$U = U^{\bar{q}} = W^{\bar{q}} - W^{\bar{q}-1} \tag{9.24}$$

has modulus

$$|U| > \frac{\epsilon_0^{\frac{2}{5}}}{l} \tag{9.25}$$

Moreover from (9.21) it results that

$$|U \wedge \hat{W^l}| \leq \epsilon_0^{\frac{3}{5}} \tag{9.26}$$

We set $\hat{U} = \frac{U}{|U|}$, this gives us

$$|\hat{U} \wedge \hat{W}^l| \leq (P_s - 1)\epsilon_0^{\frac{1}{5}} \tag{9.27}$$

Thanks to cutoff (5.46) it results that $|W^0| > \epsilon^q$, that with (9.23) gives us that

$$|\hat{W}^0 \wedge \hat{W}^l| \leq \epsilon_0^{\frac{3}{5} - q} \tag{9.28}$$

This with (9.27), assuming $q = \frac{1}{8}$, finally gives

$$|\hat{W}^0 \wedge \hat{U}| \leq C\epsilon_0^{\frac{1}{5}}(P_s - 1) \tag{9.29}$$

We have two cases, if $\sum_{q=1}^{l} |\left(W^q - W^{q-1}\right) \wedge \hat{W}^l| \leq \epsilon_0^{\frac{3}{5}}$ then it results that $|\hat{W}^0 \wedge \hat{U}| \leq C\epsilon_0^{\frac{1}{5}}(P_s - 1)$. Otherwise we have that $\sum_{q=1}^{l} |\left(W^q - W^{q-1}\right) \wedge \hat{W}^l| > \epsilon_0^{\frac{3}{5}}$. This implies that for some $q^\star$ we have

$$|\left(W^{q^\star} - W^{q^\star - 1}\right) \wedge \hat{W}^l| > \frac{\epsilon_0^{\frac{3}{5}}}{l} \tag{9.30}$$

From (9.17) it follows that

$$|\left(\hat{p} - W^0 t^0\right) \wedge \hat{W}^l - \sum_{q=1}^{l} \left[\left(W^q - W^{q-1}\right) \wedge \hat{W}^l\right] t^q| \leq \epsilon_0 \tag{9.31}$$

and then

$$|\left(\hat{p} - W^0 t^0\right) \wedge \hat{W}^l - |\left(W^{q^\star} - W^{q^\star - 1}\right) \wedge \hat{W}^l| t^{q^\star} - \sum_{q=1, q \neq q^\star}^{l} \left[\left(W^q - W^{q-1}\right) \wedge \hat{W}^l\right] t^q| \leq \epsilon_0 \tag{9.32}$$

This last formula implies that, for a fixed $\hat{p}$, $t^{q^\star}$ must be in a interval of length smaller than

$$\epsilon_0|\left(W^{q^\star} - W^{q^\star - 1}\right) \wedge W^l|^{-1} \tag{9.33}$$

that from (9.29) is bounded by $\epsilon_0^{\frac{2}{5}}(P_s - 1)$. Since the possible choices of $\hat{p}$ are at most $(CEt)^3$ it results that $t^{q^\star}$ is in a set of measure bounded by

$$\epsilon_0^{\frac{2}{5}}(P_s - 1)(CEt)^3 \tag{9.34}$$

We summarize as follows. We denote with V_{r_1} and $V_{r_1}^{'}$ respectively the outgoing and incoming relative velocities of the collision at time τ_{r_1} in the BBF. Let $t^{\bar{q}} = t_{r_2}$ and $U^{\bar{q}} = U_{r_2}$that is a function of V_{r_2}, ν_{r_2} only. We have that

$$\chi_{\lambda,E,\delta,q}\chi\left\{N^{P_s}(\epsilon_0)\right\} \leq \chi_{\lambda,E,\delta,q}\sum_{r=1}^{P_s-1}\chi\left\{V_r \in B_{r,0}\bigcup B_{r,1}\right\} +$$

$$\sum_{i,k}\sum_{l=1}^{n_{ik}}\sum_{q^\star=1}^{l}\chi_{\lambda,E,\delta,q}\chi\left\{N_{ik}^{l,q^\star}(\epsilon_0)\right\} + \sum_{r_1=1}^{P_s-1}\sum_{r_2=r_1+1}^{P_s-1}\chi_{\lambda,E,\delta,q}\chi\left\{|\hat{V_{r_1}'} \wedge \hat{U}_{r_2}| \leq C\epsilon_0^{\frac{1}{5}}(P_s-1)\right\} \tag{9.35}$$

where n_{ik} are the total number of creation in the virtual trajectory of the particles i and k between the time t^0 and the time t.

$$N_{ik}^{l,q^\star}(\epsilon_0) = \{\boldsymbol{t}_{P_s-1}, \boldsymbol{\nu}_{P_s-1}, \boldsymbol{w}_{P_s-1}| \text{the virtual trajectories of i and k satisfies (9.16)}, \tag{9.36}$$

$$\text{with } |W^{q^\star} - W^{q^\star-1} \wedge \hat{W}^l| \geq \frac{\epsilon_0^{\frac{3}{5}}}{l}\Bigg\}$$

We now estimate the three terms in the right hand side of (9.35). For the first term by a simple change of variables we have that

$$\sum_{j_1=0}^{1}\cdots\sum_{j_s=0}^{2^s-1}(\alpha)^{P_s-1}\sum_{\Gamma(P_s-1)}\sum_{\sigma_{P_s-1}}\boldsymbol{\sigma}_{P_s-1}$$

$$\int d\Lambda_{P_{s-1}}\chi_{\lambda,E,\delta,q}\sum_{r=1}^{P_s-1}\chi\left\{V_r \in B_{r,0}\bigcup B_{r,1}\right\}\prod_{k=1}^{P_s-1}B\, f_{0,P_s}^N(\zeta(0)) \leq$$

$$\epsilon^{-\lambda}\|g_0\|_\infty (C\alpha t)^{2^{s+1}}\, C\left(E\frac{\epsilon_0^{\frac{4}{5}}}{\delta^2} + (Et)^3\epsilon_0^{\frac{4}{5}}\right) \tag{9.37}$$

For the second term from (9.34) it follows that

$$\sum_{j_1=0}^{1}\cdots\sum_{j_s=0}^{2^s-1}(\alpha)^{P_s-1}\sum_{\Gamma(P_s-1)}\sum_{\sigma_{P_s-1}}$$

$$\boldsymbol{\sigma}_{P_s-1}\int d\Lambda_{P_{s-1}}\chi_{\lambda,E,\delta,q}\sum_{i,k}\sum_{l=1}^{n_{ik}}\sum_{q^\star=1}^{l}\chi_{\lambda,E,\delta,q}\chi\left\{N_{ik}^{l,q^\star}(\epsilon_0)\right\}\prod_{k=1}^{P_s-1}B\, f_{0,P_s}^N(\zeta(0)) \leq$$

$$\epsilon^{-\lambda}\|g_0\|_\infty (C\alpha t)^{2^{s+1}}\, E^3 2^{(s+4)(s+1)}\epsilon_0^{\frac{2}{5}} \tag{9.38}$$

The last term to be estimated is

$$\sum_{j_1=0}^{1}\cdots\sum_{j_s=0}^{2^s-1}(\alpha)^{P_s-1}\sum_{\Gamma(P_s-1)}\sum_{\sigma_{P_s-1}}\boldsymbol{\sigma}_{P_s-1}$$

$$\int d\Lambda_{P_{s-1}}\chi_{\lambda,E,\delta,q}\sum_{k_1=1}^{P_s-1}\sum_{k_2=k_1+1}^{P_s-1}\chi\left\{|\hat{V_{k_1}'}\wedge\hat{U}_{k_2}|\leq C\epsilon_0^{\frac{1}{5}}(P_s-1)\right\}\prod_{k=1}^{P_s-1}B\,f_{0,P_s}^N(\zeta(0)) \tag{9.39}$$

We first consider the set

$$\int d\Lambda_{P_{s-1}}\chi_{\lambda,E,\delta,q}\chi\left\{|\hat{V_{k_1}'}\wedge\hat{U}_{k_2}|\leq C\epsilon_0^{\frac{1}{5}}(P_s-1)\right\}e^{-\frac{\beta}{2}|\xi^\epsilon(0)|^2} \tag{9.40}$$

We change the integration variables in the following way

$$(\nu_{k_1},w_{k_1},\nu_{k_2},w_{k_2})\to\left(\nu_{k_1}',V_{k_1}',\nu_{k_2},V_{k_2}\right) \tag{9.41}$$

where $V_{k_1}'=w_{k_1}'-\xi_{i_{k_1}}'(\tau_{k_1})$ and $V_{k_2}=w_{k_1}-\xi_{i_{k_1}}(\tau_{k_1})$. From Lemma 2.1 it follows that (9.41) is a change of variables that preserve the measure. Thanks to this change of variables a simple calculation leads to

$$\int d\Lambda_{P_{s-1}}\chi_{\lambda,E,\delta,q}\chi\left\{|\hat{V_{k_1}'}\wedge\hat{U}_{k_2}|\leq C\epsilon_0^{\frac{1}{5}}(P_s-1)\right\}e^{-\frac{\beta}{2}|\xi^\epsilon(0)|^2}\leq E^5\epsilon_0^{\frac{2}{5}}2^{s+1} \tag{9.42}$$

This implies that

$$\sum_{j_1=0}^{1}\cdots\sum_{j_s=0}^{2^s-1}(\alpha)^{P_s-1}\sum_{\Gamma(P_s-1)}\sum_{\sigma_{P_s-1}}\boldsymbol{\sigma}_{P_s-1}\int d\Lambda_{P_{s-1}}\chi_{\lambda,E,\delta,q}\sum_{k_1=1}^{P_s-1}\sum_{r_2=k_1+1}^{P_s-1}\chi\left\{|\hat{V_{k_1}'}\wedge\hat{U}_{k_2}|\leq C\epsilon_0^{\frac{1}{5}}(P_s-1)\right\}$$

$$\prod_{k=1}^{P_s-1}B\,f_{0,P_s}^N(\zeta(0))\leq\epsilon^{-\lambda}\|g_0\|_\infty\,(C\alpha t)^{2^{s+1}}\,2^{(s+2)(s+1)}E^5\epsilon_0^{\frac{2}{5}} \tag{9.43}$$

Finally from these estimates we arrive to

$$\sum_{j_1=0}^{1}\cdots\sum_{j_s=0}^{2^s-1}(\alpha)^{P_s-1}\sum_{\Gamma(P_s-1)}\sum_{\sigma_{P_s-1}}\boldsymbol{\sigma}_{P_s-1}\int d\Lambda_{P_{s-1}}\chi_{\lambda,E,\delta,q}\chi\left\{N^{P_s}(\epsilon_0)\right\}\prod_{k=1}^{P_s-1}B\,f_{0,P_s}^N(\zeta(0))\leq$$

$$\epsilon^{-\lambda}\|g_0\|_\infty\,(C\alpha t)^{2^{s+1}}\,E^8 2^{(s+4)(s+1)}\left(\epsilon_0^{\frac{2}{5}}+\frac{\epsilon_0^{\frac{4}{5}}}{\delta^2}+\epsilon_0^{\frac{4}{5}}\right) \tag{9.44}$$

□

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