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UNIVERSITÀ DI ROMA

FACOLTÀ DI SCIENZE MATEMATICHE FISICHE E NATURALI
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Certain braided weak Hopf C*-algebras associated to modular categories.

Advisor:
Prof. Claudia Pinzari

Candidate:
Sergio Ciamprone

Introduction

Motivation

It is well-known that the representation theory of groups, algebras, quantum groups, etc. is most naturally developed in the language of tensor categories [50]. The problem of constructing one of the previous objects (that we will call in general *quantum groupoid*) describing a given tensor category via representation theory originates in the physics literature (see [29]), but soon became relevant in the theory of categories [55]. This branch of study is called *reconstruction theory*, and tries to answer to the following questions:
(a) given a tensor category $\mathcal{C}$, is it possible to build (possibly in a canonical way) a quantum groupoid A such that a category of representation of A is equivalent to $\mathcal{C}$?
(b) Is A unique, under some suitable conditions?

Of course these questions can be formulated in different shapes, depending on the type of category we have, and consequently the answers might change. For a brief introduction to the reconstruction theorems see the beginning of the Chapter 2. Here we would like to focus more on how these reconstruction problems are linked to certain physical models. In classical mechanics, symmetries are elements of the group of transformation which acts on the phase space. In quantum mechanics, we cannot talk about points of the phase space, but we have a non-commutative algebra of observables. The class of symmetries acting on this algebra is larger than a group (see [1], [26], [47], [48]), and we will call it quantum groupoid. Tensor categories play a prominent role in this setting, since they are the main tools in order to model some quantum systems. In this way we come back to the questions shown above: physics gives us a category with some properties, and we would like to bring out a quantum groupoid from it, possibly unique or at least canonical, whose physical interpretation is quite easy to understand at this point. As we can see in Chapter 2, some of the reconstruction problems arising in this setting were successfully solved [19], but only when we are dealing with symmetric categories. Unfortunately, many categories arising in conformal field theory are not symmetric, but only braided [29], [25]. On one hand, the weakening of this condition does not prevent us from proving the existence of a large class of quantum groupoids whose representation theory is equivalent to the given category. On the other hand, it makes hard to prove a sort of uniqueness of the reconstructed object. This can be seen for a quite large class of braided tensor categories in [61]. In our work, we focus on a special class of modular tensor C*-categories, which sometimes are called *Verlinde categories* [23]. These categories are of interest because their fusion rings, at least in some cases, are found to be the same as those which come from certain quantum and conformal field theories, as for example the Ising model and the Wess-Zumino-Witten model [14]. Moreover, they played a crucial in role in the subfactor theory [34] and the study of invariants of 3-manifolds (see [70], [82]). We briefly introduce them. Let $\mathfrak{g}$ be a complex simple Lie algebra of type different from E_8, q a primitive root of unity, and let $U_q(\mathfrak{g})$ be the Lusztig's restricted form of the Drinfeld-Jimbo quantum group. We consider as its representation category $\mathrm{Rep}(U_q(\mathfrak{g}))$, whose objects are tensorially generated by the Weyl modules V_λ, where λ is a dominant weight. It is well-known that $U_q(\mathfrak{g})$ is not semisimple. As a consequence, $\mathrm{Rep}(U_q(\mathfrak{g}))$ is quite far from being semisimple. When $q = e^{\frac{i\pi}{dl}}$, and d is the ratio of the square lengths of a long root to a short root, we can restrict to the category of the tilting $U_q(\mathfrak{g})$-modules, which is proved to be the tensor category generated by V_λ's, where $\lambda \in \Lambda_l$. Λ_l is the so-called *principal Weyl alcove* and $\overline{\Lambda_l}$ is its closure. This category is said to be the *tilting category*, and it is labelled by $\mathcal{T}_l$. Before going further, we add that a tilting module T is irreducible if and only if $T = V_\lambda$, with $\lambda \in \overline{\Lambda_l}$; moreover, for every l and every $\mathfrak{g}$ there always exists an irreducible representation called *fundamental* such that every irreducible representation is a submodule of a tensor power of V. Our procedure is not complete, since $\mathcal{T}_l$'s are still not semisimple, but now we are close to obtain the desired semisimple categories. In fact, every $\mathcal{T}_l$ contains the ideal of the so-called *negligible modules*. Quotienting by this ideal, we obtain a semisimple category that we indicate with $\mathcal{F}_l$. $\mathcal{F}_l$ can be seen as a semisimple tensor category endowed with a suitable *truncated* tensor product, which allows to drop out all the negligible modules. This theory is developed in depth in [2]–[7] and in [27]. Kirillov [37] introduced a *-involution and an associated hermitian form on the arrow spaces of $\mathcal{F}_l$, conjecturing positivity. Wenzl [81] proved the conjecture, putting on $\mathcal{F}_l$ a C* structure which makes the braiding unitary. Putting everything together, it is quite natural to ask if there exists a quantum groupoid whose representation category is equivalent to $\mathcal{F}_l$. To the best of our knowledge, the problem was faced in two different ways. On one side, $\mathcal{F}_l$ can be considered as part of a larger class of semisimple fusion categories. Every category of this type can be seen as the representation category of a weak Hopf algebra [31],[61],[76]. It is worth to notice now that in our work we will give a different definition of weak Hopf algebra in comparison to [10],[11],[12],[60], as we will also point out later. However, this approach is quite unsatisfactory, since the relation between the reconstructed object and the original Lie algebra is not clear. Moreover, the construction is highly non-unique and non-canonical. These facts rely on the use of a fiber functor which does not reproduce the truncation procedure of $\mathcal{F}_l$. On the other side, a different approach can be found in the work of Mack and Schomerus [49]. They considered only the case $\mathfrak{g} = \mathfrak{sl}_2$, showing that the truncation of the tensor product at the categorical level leads to the construction of a weak quasi Hopf algebra, as we intend in our work. The non-coassociativity of the coproduct on this algebra is the price we have to pay in order to obtain a more natural construction.

Aim of the work

At this point we are ready to introduce our work. These thesis contains part of the results of a more general research project about quantum groups at roots of unity which involves Claudia Pinzari and me [15],[16]. So, the most of the results exposed here are products of this collaboration. However, Chapter 2 and Chapter 5 contain my own contribution, and in particular results exposed in Chapter 5 will be published separately [17].

In [15] Claudia Pinzari and me partially followed the approach of Mack and Schomerus in order to extend their construction from $\mathfrak{g} = \mathfrak{sl}_2$ to every $\mathfrak{g}$ of Lie type A, using a procedure which is the most possible canonical. In the following lines we explain our approach. Let U and V be two tilting modules in $\mathcal{T}_l$. The truncated tensor product $U\underline{\otimes}V$ could be obtained decomposing $U \otimes V$ in indecomposable tilting modules, throwing away the negligible ones. Unfortunately this procedure is non-canonical. Anyway, if we restrict to truncated tensor products of the type $V_\lambda\underline{\otimes}V$, where V is the fundamental representation and $\lambda \in \Lambda_l$, we can choice a canonical truncation. This result is due to Wenzl. Every V_λ is endowed with a hermitian form, that Wenzl proved to be an inner product under the conditions $q = e^{\frac{i\pi}{dl}}$ and $\lambda \in \overline{\Lambda_l}$. The truncated tensor product $V_\lambda\underline{\otimes}V$ can be still endowed with an inner product, which is a deformation of the product form using the R-matrix and the ribbon structure of (a suitable extension) of $U_q(\mathfrak{g})$. The corresponding projection from $V_\lambda \otimes V$ to $V_\lambda\underline{\otimes}V$ is self-adjoint w.r.t. the modified form. Therefore, we can define a natural functor $W_l : \mathcal{F}_l \to$ Hilb, where Hilb is the category of Hilbert spaces. This functor will be called throughout this work as the *Wenzl's functor*. Our strategy is to apply a Tannakian reconstruction to W_l. Anyway, we need several adjustments in comparison to the Tannakian classical case, since W_l is really far from being a tensor functor. The structure of W_l seems to be too much poor, even considering reconstruction theorems which use functors with weaker assumptions on them (see [30]). This fact is essentially due to the lack of associativity of the projections $p_n : V^{\otimes n} \to V^{\underline{\otimes} n}$. We briefly report our procedure. We are able to construct the bialgebra $\mathcal{D}(V, l) = \bigoplus_n V^{\underline{\otimes} n*} \otimes V^{\underline{\otimes} n}$, which is coassociative but non-associative, thanks to some remarkable properties of p_n which partially replace the associativity failure. Using the coboundary structure on $\mathcal{D}$ we can put on it an involution. Next, we obtain the quotient algebra $\mathcal{C}(G, l)$ quotienting $\mathcal{D}(V, l)$ by a *-coideal which is also a right ideal (and not a left ideal again because of lack of associativity of p_n). This coideal is the translation of some identifications that we can do at the level of the fusion category. $\mathcal{C}$ is naturally a coalgebra with an involution. The problem is to endow $\mathcal{C}$ with a product, which can be solved supposing that $\mathcal{C}(G, l)$ is cosemisimple. In this way, we can put on it a non-associative product which is the pull-back of the product on $\mathcal{D}(V, l)$. Passing to the dual algebra $\widehat{\mathcal{C}(G, l)}$ we obtain the desired groupoid, that is a weak Hopf C*-algebra with a R-matrix and a representation category equivalent to $\mathcal{F}_l$. Moreover, it is endowed with a twisted involution $(\Omega, ^*)$ in a sense that we soon explain, where the twist Ω is induced by the R-matrix. At this point, the last thing to do is to understand when it is possible to put the cosemisimplicity condition on $\mathcal{C}(G, l)$. We focus only on Lie algebras $\mathfrak{g}$ of type A, proving that in this case $\mathcal{C}(G, l)$ is cosemisimple for every l. Crucial in order to prove this fact is the existence of a Haar functional, which in turn is induced by a suitable filtration on $\mathcal{C}(G, l)$. We avoid to treat the other cases since they require more Lie technicalities. Anyway, this result likely extends to every simple Lie algebra of a type different from E_8.

Now it is worth to say something more about the kind of object we built. As we said before, weak quasi Hopf algebras are not completely new in literature, but since their first appearance they have been quite ignored in literature. Therefore, in this thesis we also give a systematic approach to this kind of object. This will be part of a second paper, which is now in preparation [16]. A weak quasi Hopf algebra A is a quasi Hopf algebra in the sense of Drinfeld [20], with a non-unital coproduct. Following what Gould and Lekatsas did for quasi Hopf algebras [28], we consider as involution the pair $(\Omega,^*)$, where $\Omega \in A \otimes A$ is a self-adjoint element satisfying the following identity:

$$\Delta(a)^*\Omega = \Omega\Delta(a^*) \tag{0.0.1}$$

and a compatibility condition with the associator Φ. Moreover, it is possible to put a suitable definition of R-matrix on it. We prove that this class of algebras is closed under a quite large class of twists, and we stress some interesting properties of the antipode. More precisely, it is quite easy to see (passing to the dual algebra) that the antipode S on A is not anti-comultiplicative. We prove the existence of a weak version of the anti-comultiplicative relation. Unlike the quasi Hopf algebra case, if we restrict to the coassociative case we do not get the usual anti-comultiplicative relation, and this is a consequence of the non-unitality of the coproduct. We next develop a theory of representation in the semisimple case. If we consider *-representations on hermitian spaces, it is worth to notice that the representation category is a tensor category, but with a hermitian product form which is a deformation of the product form, using the involution element Ω and the relation (0.0.1). It is possible to prove the existence of conjugate objects when the antipode S commutes with *. Probably it is possible to remove this condition as Woronowicz did for compact quantum groups [83], but this will not be part of the thesis and will be treated in [16]. Moreover, we see that the semisimplicity of A is equivalent to the existence of a Haar measure. Considering a suitable definition of integral on the dual weak quasi Hopf algebra [13], we also notice that, unlike the Drinfeld case, we cannot prove the existence of such an integral in the dual case. A deeper look to the object built in the construction reported above bring us to focus on a subclass (not closed under twists) of these algebras, that we call *weak Hopf algebra*. These are weak quasi Hopf algebras with some additional conditions on the idempotents P_2, P_3, Q_3, P_4, Q_4 defined in the following way:

$$P_2 = \Delta(I), \quad P_3 = \Delta \otimes \mathrm{id}(\Delta(I)), \quad Q_3 = \mathrm{id} \otimes \Delta(\Delta(I))$$
$$P_4 = \Delta \otimes \mathrm{id} \otimes \mathrm{id}(\Delta \otimes \mathrm{id}(\Delta(I))), \quad Q_4 = \mathrm{id} \otimes \mathrm{id} \otimes \Delta(\mathrm{id} \otimes \Delta(\Delta(I)))$$

As a consequence, the associators Φ and Φ^{-1} can be chosen in the following peculiar way:

$$\Phi = Q_3 P_3, \quad \Phi^{-1} = P_3 Q_3$$

These algebras will be analyzed more in detail in [16]. In this work we notice that weak (quasi) Hopf algebras can be seen as the counterparts of weak (quasi) tensor functors. More precisely, given a semisimple weak (quasi) Hopf algebra A and its representation category, the natural embedding functor of $\mathrm{Rep}(A)$ in Vect (or Hilb in the C* case) is a weak (quasi) tensor funtor. Conversely, given a semisimple tensor category $\mathcal{C}$ with some other mild conditions, and a weak (quasi) tensor functor which embeds $\mathcal{C}$ into Vect, we can reconstruct a weak (quasi) Hopf algebra whose representation category is equivalent to $\mathcal{C}$. This result is reported in Chapter 2 and it is a generalization of the work of Häring-Oldenburg. Specifically, we do not assume that the natural transformation associated to the functor is a coisometry. Correspondingly, the reconstructed groupoid satisfies the generalized commutation relation (0.0.1). It could seem surprising that we did not apply this reconstruction theorem to our categories, but the explanation is that it was not obvious before that Wenzl's functor W_l was a weak (quasi) tensor functor. Indeed the construction of such a tensor structure can be interpreted as our main contribution. Finally, in order to give a more concrete look to the algebras reconstructed from $\mathcal{F}_l$, we present the groupoid $\mathcal{C}(SU(2), l)$ by generators and relations. We have chosen only the case $\mathfrak{g} = \mathfrak{sl}_2$ because it is the most workable and there is much information about its representation theory. As recalled above, all this will be the subject of the paper [17]. Before concluding the introduction, we would like to give a sketch of the future directions of our research. First of all, we would like to extend our construction to other simple Lie algebras of type different from A. Moreover, we would like to study the uniqueness of the reconstructed groupoid, of course under some suitable assumptions on it.

Plan of the work

In the first two sections of the Chapter 1, we reported some well-known definitions and results about tensor categories, including the pivotal definition of weak (quasi) tensor functor. In the other sections of the first chapter we introduce weak (quasi) Hopf algebras, reporting most of the result that will appear in [16]. In the Chapter 2 we made a brief introduction to the reconstruction theorems, and then we explain the reconstruction theorems made by Majid and Häring-Oldenburg. We also give a generalization of the Häring-Oldenburg's construction, as we said before. In Chapter 3 we report some well-known results about ribbon categories and ribbon algebra. Moreover, we introduce the quantum group at root of unity $U_q(\mathfrak{g})$ and the main results about its representation theory (see references therein). We focus on the tilting categories, showing the result of Kirillov and Wenzl about the existence of a C* structure on them. Finally, we introduce the Wenzl's functor and clarify some aspects of the rigidity of $\mathrm{Rep}(U_q(\mathfrak{g}))$. In the Chapter 4, we report the main results of [15], showing in detail the construction briefly explained in this Introduction. We also show that the reconstructed quantum groupoid fits with the algebras introduced in the first chapter. Finally, in Chapter 5 we focus on some results about Weyl $U_q(\mathfrak{sl}_2)$-modules, which allow to give a presentation of $\mathcal{C}(SU(2), l)$ by generators and relations, for every even root of unity q.

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Contents

Chapter 1

Tensor categories and weak quasi Hopf algebras

1.1 Generalities on tensor categories

In this section we introduce some general notions about tensor categories. Our main references are [23], [55]. A tensor category is a sextuple $(\mathcal{C}, \otimes, a, \mathbf{1}, l, r)$, where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor called *tensor product*, a is a natural isomorphism such that:

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)\ ,\ \ X, Y, Z \in \mathcal{C}$$

called the *associativity costraint*, $\mathbf{1} \in \mathcal{C}$ is the *unit object* of $\mathcal{C}$ and l, r are natural isomorphisms such that:

$$l_X : \mathbf{1} \otimes X \xrightarrow{\sim} X \text{ and } r_X : X \otimes \mathbf{1} \xrightarrow{\sim} X$$

called *unit costraints*. They are subject to the following two axioms.
(a) *The pentagon axiom* The diagram:

$$\begin{tikzcd}
 & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "a_{W,X,Y} \otimes \mathrm{id}_Z"'] \arrow[dr, "a_{W \otimes X, Y, Z}"] & \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W, X \otimes Y, Z}"'] & & (W \otimes Y) \otimes (Y \otimes Z) \arrow[d, "a_{W \otimes X, Y, Z}"] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\mathrm{id}_W \otimes a_{X,Y,Z}"] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd} \tag{1.1.1}$$

is commutative for all objects X, Y, Z, W in $\mathcal{C}$.
(b) *The triangle axiom* The diagram:

$$\begin{tikzcd}
(X \otimes \mathbf{1}) \otimes Y \arrow[rr, "a_{X,\mathbf{1},Y}"] \arrow[dr, "r_X \otimes \mathrm{id}_Y"'] & & X \otimes (\mathbf{1} \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes l_Y"] \\
 & X \otimes Y &
\end{tikzcd} \tag{1.1.2}$$

is commutative for all objects X, Y in $\mathcal{C}$.

Definition 1.1.1. A tensor category is *strict* if for all objects X, Y, Z in $\mathcal{C}$ one has equalities $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and $X \otimes \mathbf{1} = X = \mathbf{1} \otimes X$, and the associativity and unit costraints are the identity maps.

Theorem 1.1.2 (MacLane)**.** *Any tensor category is tensorially equivalent to a strict tensor category.*

In a tensor category, one can form n-fold tensor products of any ordered sequence of objects $X_1, \ldots, X_n$. Of course such products can be parenthesized in different ways, obtaining possibly distinct objects of $\mathcal{C}$. For $n = 3$ we have two different parenthesizings, $(X_1 \otimes X_2) \otimes X_3$ and $X_1 \otimes (X_2 \otimes X_3)$, which are canonically identified by the associativity isomorphism. It is easy to see that one can identify any two parenthesized products of $X_1, \ldots, X_n$, $n \geq 3$, using a chain of associativity isomorphisms. The problem is that, when $n \geq 4$, there may be two or more possible identification, which could be not the same. If $n = 4$ the pentagonal axiom avoids the occurrence of this unpleasant situation. What does it happen if $n > 4$? This problem is solved by the following theorem of MacLane:

Theorem 1.1.3 (Coherence Theorem)**.** *Let $X_1, \ldots, X_n \in \mathcal{C}$. Let P_1, P_2 two parenthesized products of $X_1, \ldots, X_n$. Let f, g be two isomorphisms between P_1 and P_2, obtained by composing associativity and unit costraints. Then $f = g$.*

The next definition is crucial, because from now on we will only deal with this kind of categories.

Definition 1.1.4. A tensor category is called *linear* if every morphism space (X, Y) is a $\mathbb{C}$-linear space. Moreover we require the existence of direct sums and subobjects. More precisely, if X, Y, Z are objects in $\mathcal{C}$, Z is the direct sum of X and Y if there are morphisms $u \in (X, Z)$, $u' \in (Z, X)$, $v \in (Y, Z)$, $v' \in (Z, Y)$ such that:

$$u \circ u' + v \circ v' = \mathrm{id}_Z$$

and:

$$u' \circ u = \mathrm{id}_X \quad , \quad v' \circ v = \mathrm{id}_Y \quad \text{and} \quad v' \circ u = 0 = u' \circ v$$

Y is a subobject of X if there exists a morphism $p \in (X, X)$ such that $p = p \circ p$ and morphisms $u \in (Y, X)$, $u' \in (X, Y)$ such that $u' \circ u = \mathrm{id}_Y$ and $u \circ u' = p$.

We now want to introduce categories endowed with more structure. Let $\mathcal{C}$ be a tensor category. A *commutativity costraint* c is a natural isomorphism such that:

$$c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$$

c must also satisfies the following conditions:
(a)

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{c_{X,Y}} & Y \otimes X \\ {\scriptstyle f\otimes g}\downarrow & & \downarrow{\scriptstyle g\otimes f} \\ X' \otimes Y' & \xrightarrow{c_{X',Y'}} & Y' \otimes X' \end{array} \tag{1.1.3}$$

commutes for all objects X, Y, X', Y' and all morphisms $f \in (X, X')$ and $g \in (Y, Y')$.

(b)

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "c_{X,Y\otimes Z}"] & (Y \otimes Z) \otimes X \arrow[d, "a_{Y,Z,X}"] \\
(X \otimes Y) \otimes Z \arrow[u, "a_{X,Y,Z}"] \arrow[d, "c_{X,Y}\otimes \mathrm{id}_Z"'] & Y \otimes (Z \otimes X) \\
(Y \otimes X) \otimes Z \arrow[r, "a_{Y,X,Z}"] & Y \otimes (X \otimes Z) \arrow[u, "\mathrm{id}_Y \otimes c_{X,Z}"']
\end{tikzcd} \tag{1.1.4}$$

commutes for all objects X, Y, Z.

(c)

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "c_{X\otimes Y,Z}"] & Z \otimes (X \otimes Y) \arrow[d, "a^{-1}_{Z,X,Y}"] \\
X \otimes (Y \otimes Z) \arrow[u, "a^{-1}_{X,Y,Z}"] \arrow[d, "\mathrm{id}_X \otimes c_{Y,Z}"'] & (Z \otimes X) \otimes Y \\
X \otimes (Z \otimes Y) \arrow[r, "a^{-1}_{X,Z,Y}"] & (X \otimes Z) \otimes Y \arrow[u, "c_{X,Z}\otimes \mathrm{id}_Y"']
\end{tikzcd} \tag{1.1.5}$$

commutes for all objects X, Y, Z.

The next definition is due to Joyal and Street:

Definition 1.1.5. Let $\mathcal{C}$ be a tensor category. A commutativity costraint c on $\mathcal{C}$ is also called *braiding*. A braided tensor category is a tensor category with a braiding.

When the tensor category $\mathcal{C}$ is strict, then the conditions (1.1.4) and (1.1.5) simply become:

$$\begin{aligned} c_{X,Y\otimes Z} &= (\mathrm{id}_Y \otimes c_{X,Z})(c_{X,Y} \otimes \mathrm{id}_Z) \\ c_{X\otimes Y,Z} &= (c_{X,Z} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes c_{Y,Z}) \end{aligned} \tag{1.1.6}$$

Braided tensor categories are generalizations of symmetric categories:

Definition 1.1.6. A tensor category $\mathcal{C}$ is symmetric if it is equipped with a braiding c such that:

$$c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X\otimes Y} \tag{1.1.7}$$

for all objects X, Y in the category. If (1.1.7) holds, the braiding c is the symmetry for the category. In that case, diagrams (1.1.4) and (1.1.5) are equivalent.

Definition 1.1.7. A category $\mathcal{C}$ is a *-*category* if there exists an antilinear contravariant functor $^* : \mathcal{C} \to \mathcal{C}$ which is the identity on the objects and such that:

$$f^{**} = f \ \text{ for every morphisms } f$$

If $\mathcal{C}$ is also a tensor category, the morphisms must satisfy the following additional compatibility condition:

$$(f \otimes g)^* = f^* \otimes g^*$$

Moreover, we require that the commutativity costraint and the associativity costraint are unitary natural isomorphisms. In other words:

$$a^*_{X,Y,Z} = a^{-1}_{X,Y,Z} \ \ \forall X, Y, Z \in \mathrm{Ob}(\mathcal{C}) \tag{1.1.8}$$

$$c^*_{X,Y} = c^{-1}_{X,Y} \ \ \forall X, Y \in \mathrm{Ob}(\mathcal{C}) \tag{1.1.9}$$

Definition 1.1.8. A *-tensor category $\mathcal{C}$ is rigid if there is, for any object X, an object $\overline{X}$ called the *conjugate object* of X and two morphisms $r_X \in (\mathbf{1}, \overline{X} \otimes X)$ and $\overline{r}_X \in (\mathbf{1}, X \otimes \overline{X})$ such that:

$$r_X^* \otimes \mathrm{id}_{\overline{X}} \circ a^{-1}_{\overline{X},X,\overline{X}} \circ \mathrm{id}_{\overline{X}} \otimes \overline{r}_X = \mathrm{id}_{\overline{X}} \tag{1.1.10}$$

$$\mathrm{id}_X \otimes r_X^* \circ a_{X,\overline{X},X} \circ \overline{r}_X \otimes \mathrm{id}_X = \mathrm{id}_X \tag{1.1.11}$$

Definition 1.1.9. Let $f \in (X, Y)$ be an arrow in $\mathcal{C}$. Then the *transpose* $f^\vee \in (\overline{Y}, \overline{X})$ of f is an arrow defined in the following way:

$$f^\vee = (r_Y^* \otimes \mathrm{id}_{\overline{X}})(\mathrm{id}_{\overline{Y}} \otimes f \otimes \mathrm{id}_{\overline{X}})(\mathrm{id}_{\overline{Y}} \otimes \overline{r}_X)$$

Remark 1.1.10. **(a)** If we consider a *-category $\mathcal{C}$, we can give an alternative (and equivalent) definition of linearity. In fact, for a *-category $\mathcal{C}$ it is possible to replace u' by u^* and v' by v^*. As a consequence, p is self-adjoint.
(b) The definition of rigidity can be given in a more general setting. In fact, it is possible to say that a tensor category $\mathcal{C}$ is rigid if there exists a conjugate object $\overline{X}$ for all $X \in \mathrm{Ob}(\mathcal{C})$ and two maps $\overline{r} \in (\mathbf{1}, X \otimes \overline{X})$ and $r^\dagger \in (\overline{X} \otimes X, \mathbf{1})$ playing respectively the role of $\overline{r}$ and r^*.

Definition 1.1.11. **(a)** An object X in a category $\mathcal{C}$ is *simple* or *irreducible* if $(X, X) = \mathbb{C}\,\mathrm{id}_X$;
(b) A category $\mathcal{C}$ is called *semisimple* if it is linear and admits a family of simple objects X_i indexed by $i \in I$, such that every object X is a finite direct sum of X_i's;
(c) $\nabla \subset \mathrm{Ob}(\mathcal{C})$ denote a set containing one object out of every equivalence class of irreducible objects;
(d) A category $\mathcal{C}$ is *rational* if there are finitely many isomorphism classes of simple objects;

The next result is the well-known categorical version of the *Schur's Lemma*:

Proposition 1.1.12. *Let X_1 and X_2 be two different simple objects in a semisimple linear tensor category $\mathcal{C}$. Then either $(X_1, X_2) = \{0\}$ or $\dim((X_1, X_2)) = 1$. In the latter case, $X_1 \cong X_2$.*

Definition 1.1.13. A *-category $\mathcal{C}$ is a C*-category if:
(a) (X, Y) is a Banach space for every pair of objects X, Y;
(b) if $f \in (X, Y)$ and $g \in (Y, Z)$, then $\|g \circ f\| \leq \|g\|\,\|f\|$;
(c) if $f \in (X, Y)$, $\|f^* f\| = \|f\|^2$;
(d) $f^* f$ is positive in the C*-algebra (X, X) for every $f \in (X, Y)$ and every Y object in $\mathcal{C}$.

At this point we introduce the notion of tensor functor between two tensor categories.

Definition 1.1.14. Let $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ and $(\mathcal{C}', \otimes', \mathbf{1}', a', l', r')$ be two tensor categories. A *tensor functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a triple (F, φ_0, e), where $F : \mathcal{C} \to \mathcal{C}'$ is a functor, φ_0 is an isomorphism from $\mathbf{1}'$ to $F(\mathbf{1})$, and:

$$e_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y) \tag{1.1.12}$$

is a natural isomorphism such that the diagrams:

$$\begin{tikzcd}
(F(X) \otimes' F(Y)) \otimes' F(Z) \arrow[rr, "{a'_{F(X),F(Y),F(Z)}}"] \arrow[d, "{e_{X,Y} \otimes' \mathrm{id}_{F(Z)}}"'] & & F(X) \otimes' (F(Y) \otimes' F(Z)) \arrow[d, "{\mathrm{id}_{F(X)} \otimes' e_{Y,Z}}"] \\
F(X \otimes Y) \otimes' F(Z) \arrow[d, "{e_{X \otimes Y, Z}}"'] & & F(X) \otimes' F(Y \otimes Z) \arrow[d, "{e_{X, Y \otimes Z}}"] \\
F((X \otimes Y) \otimes Z) \arrow[rr, "{F(a_{X,Y,Z})}"] & & F(X \otimes (Y \otimes Z))
\end{tikzcd} \tag{1.1.13}$$

and:

$$\begin{tikzcd}
\mathbf{1}' \otimes F(X) \arrow[r, "l_{F(X)}"] \arrow[d, "\varphi_0 \otimes \mathrm{id}_{F(X)}"'] & F(X) \\
F(\mathbf{1}) \otimes F(X) \arrow[r, "{e_{\mathbf{1},X}}"] & F(\mathbf{1} \otimes X) \arrow[u, "F(l_X)"']
\end{tikzcd} \tag{1.1.14}$$

and:

$$\begin{tikzcd}
F(X) \otimes \mathbf{1}' \arrow[r, "r_{F(X)}"] \arrow[d, "\mathrm{id}_{F(X)} \otimes \varphi_0"'] & F(X) \\
F(X) \otimes F(\mathbf{1}) \arrow[r, "{e_{X,\mathbf{1}}}"] & F(X \otimes \mathbf{1}) \arrow[u, "F(r_X)"']
\end{tikzcd} \tag{1.1.15}$$

are commutative for all $X, Y, Z \in \mathcal{C}$. This is called the *tensor structure axiom*. A tensor functor is said to be an equivalence of tensor categories if it is an equivalence of ordinary categories.

Remark 1.1.15. It is worth to notice that a tensor functor is an ordinary functor with an additional structure satisfying certain equations (the tensor structure axiom). These equations may have more than one solution or no solution at all, so the same functor can be equipped with different tensor structures or not admit any tensor structure.

Definition 1.1.16. A morphism $g \in (X, Y)$ in $\mathcal{C}$ is an epimorphism if it has a right inverse $h \in (Y, X)$. A natural transformation g is a natural epimorphism if there exists a natural transformation h such that $g \circ h = \mathrm{id}$.

Definition 1.1.17. (a) A *quasi tensor functor* between two tensor categories $\mathcal{C}$ and $\mathcal{C}'$ is a functor $F : \mathcal{C} \to \mathcal{C}'$ together with an isomorphism $\varphi_0 : \mathbf{1}' \to F(\mathbf{1})$ in $\mathcal{C}'$ and a natural isomorphism e:

$$e_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$$

(b) A *weak tensor functor* between two tensor categories $\mathcal{C}$ and $\mathcal{C}'$ is a functor $F : \mathcal{C} \to \mathcal{C}'$ together with an isomorphism $\varphi_0 : \mathbf{1}' \to F(\mathbf{1})$ in $\mathcal{C}'$ and a natural epimorphism e:

$$e_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$$

(but with $e_{\mathbf{1},X}$ and $e_{X,\mathbf{1}}$ isomorphisms) such that the diagrams (1.1.13), (1.1.14), (1.1.15) and the diagram obtained from (1.1.13) reversing all the arrows and replacing all the natural transformations with their inverses, commute [16];

(c) A *weak quasi tensor functor* between two tensor categories $\mathcal{C}$ and $\mathcal{C}'$ is a functor

$F : \mathcal{C} \to \mathcal{C}'$ together with an isomorphism $\varphi_0 : \mathbf{1}' \to F(\mathbf{1})$ in $\mathcal{C}'$ and a natural epimorphism e:

$$e_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$$

but with $e_{\mathbf{1},X}$ and $e_{X,\mathbf{1}}$ isomorphisms.

Remark 1.1.18. In (b) of the last Definition it is necessary to require that both the diagram (1.1.13) and its inverse commute, unlike the case of a tensor functor, where the commutativity of the inverse diagram is a consequence of the commutativity of the diagram (1.1.13). This is due to the fact that in the case (b) e has only a right inverse and not a left inverse. The kind of functor introduced in (b) is new and it is linked to the examples that we will show in the rest of the work, as we can see more in detail in [16].

Definition 1.1.19. **(a)** A functor F between two braided tensor categories $(\mathcal{C}, c)$ and $(\mathcal{C}', c')$ is *braided* if $\forall X, Y \in \mathrm{Ob}(\mathcal{C})$ the following diagram commutes:

$$\begin{array}{ccc} F(X) \otimes F(Y) & \xrightarrow{e_{X,Y}} & F(X \otimes Y) \\ {\scriptstyle c'_{X,Y}}\big\downarrow & & \big\downarrow{\scriptstyle F(c_{X,Y})} \\ F(Y) \otimes F(X) & \xrightarrow{e_{Y,X}} & F(Y \otimes X) \end{array} \tag{1.1.16}$$

(b) A functor F between two *-tensor categories $\mathcal{C}$ and $\mathcal{C}'$ is **-preserving* if $F(f^*) = F(f)^* \; \forall f$ morphism of $\mathcal{C}$.
(c) A functor F between two rigid tensor categories $\mathcal{C}$ and $\mathcal{C}'$ is *rigid* if there exists a natural isomorphism $d_X : \overline{F(X)} \to F(\overline{X})$. If $\mathcal{C}$ and $\mathcal{C}'$ are *-categories, we require that $d_X^* = d_X^{-1}$.

Remark 1.1.20. We point out that associativity costraint, commutativity costraint and e are all natural transformations in the following sense. If $f \in (X, X')$, $g \in (Y, Y')$ and $h \in (Z, Z')$, then:

$$a_{X',Y',Z'} \circ (f \otimes g) \otimes h = f \otimes (g \otimes h) \circ a_{X,Y,Z}$$
$$g \otimes f \circ c_{X,Y} = c_{X',Y'} \circ f \otimes g$$
$$e_{X',Y'} \circ F(f) \otimes F(g) = F(f \otimes g) \circ e_{X,Y}$$

Finally, let f be a morphism in (Y, X). d natural transformation means that:

$$F(f^\vee) \circ d_X = d_Y \circ F(f)^\vee$$

Definition 1.1.21. Let $(F, \varphi_0, e^{(F)})$ and $(G, \gamma_0, e^{(G)})$ be two weak quasi tensor functors between two tensor categories $\mathcal{C}$ and $\mathcal{C}'$. A natural isomorphism $\eta : F \to G$ is *tensorial* if the following diagrams commute:

$$\begin{array}{ccc} F(X) \otimes F(Y) & \xrightarrow{e^{(F)}_{X,Y}} & F(X \otimes Y) \\ {\scriptstyle \eta_X \otimes \eta_Y}\big\downarrow & & \big\downarrow{\scriptstyle \eta_{X \otimes Y}} \\ G(X) \otimes G(Y) & \xrightarrow{e^{(G)}_{X,Y}} & G(X \otimes Y) \end{array} \tag{1.1.17}$$

and

$$\begin{tikzcd} F(\mathbf{1}) \arrow[rr, "\eta_{\mathbf{1}}"] & & G(\mathbf{1}) \\ & \mathbf{1}' \arrow[ul, "\varphi_0"] \arrow[ur, "\gamma_0"'] & \end{tikzcd} \tag{1.1.18}$$

1.2 Definition of weak quasi bialgebra

The notion of quasi Hopf algebra raised in the late '80's thanks to the Drinfeld's work [20], where he proved the equivalence between the braided tensor category of the modules over the Drinfeld-Jimbo algebra and the braided tensor category of modules over $U(\mathfrak{g})[[h]]$, equipped with a non-trivial associativity costraint. This fact led to a reformulation of the Kohno's theorem and an explicit expression of the monodromy of the Knizhnik-Zamolodchikov equation. The first appearance of the weak quasi Hopf algebras is probably in the works of Mack and Schomerus [48], [49]. In fact, physical motivations brought them to develop a generalization of quasi Hopf algebras, allowing the non-unitality of the coproduct. This approach can be also found in the work of Haring-Oldenburg [30], as we will see in depth in the next chapter. In the next pages, we have two goals: to give a more systematic approach to these algebras in comparison to the works previously cited, and to introduce an involution on them, following what Gould and Lekatsas did for quasi Hopf algebras [28]. We will finally give the definition of weak Hopf algebra. The facts exposed in this section will be treated more in detail in [16]. Classical results about quasi Hopf algebras which inspired us for this work can be found in [20] and [35].

Let A be an algebra, and $a \in A$. Moreover, let p, q be two idempotents in A. We can define the linear space (p, q):

$$(p, q) := qAp = \{a \in A : qa = a = ap\}$$

$\boldsymbol{D}(a) = p$ will be called the domain of a, and $\boldsymbol{R}(a) = q$ the range. $a \in (p, q)$ is *partially invertible* if there exists an element $a^{-1} \in (q, p)$ such that:

$$a^{-1}a = p \ , \ aa^{-1} = q$$

Definition 1.2.1. A *weak quasi bialgebra* A is a unital, complex, associative algebra, with a possibly non-unital algebra homomorphism $\Delta : A \to A \otimes A$ called *coproduct* and an algebra homomorphism $\varepsilon : A \to \mathbb{C}$ called *counit* such that:

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta \tag{1.2.1}$$

Furthermore, there exists a partially invertible element $\Phi \in A \otimes A \otimes A$ such that:

$$\boldsymbol{D}(\Phi) = \Delta \otimes \mathrm{id}(\Delta(I)) \ , \ \boldsymbol{R}(\Phi) = \mathrm{id} \otimes \Delta(\Delta(I)) \tag{1.2.2}$$

$$\Phi(\Delta \otimes \mathrm{id}(\Delta(a))) = \mathrm{id} \otimes \Delta(\Delta(a))\Phi \ , \ \forall a \in A \tag{1.2.3}$$

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta(\Phi))(\Delta \otimes \mathrm{id} \otimes \mathrm{id}(\Phi)) = (I \otimes \Phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id}(\Phi))(\Phi \otimes I) \tag{1.2.4}$$

$$\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}(\Phi) = \Delta(I) \tag{1.2.5}$$

From now on, we will use the following notation:

$$\Phi = \sum_i x_i \otimes y_i \otimes z_i \ , \ \Phi^{-1} = \sum_i p_i \otimes q_i \otimes r_i \tag{1.2.6}$$

Remark 1.2.2. **(a)** The identity $\varepsilon \otimes \mathrm{id} \otimes \mathrm{id}(\Phi) = \Delta(I) = \mathrm{id} \otimes \mathrm{id} \otimes \varepsilon(\Phi)$ follows as in the quasi Hopf algebra case, applying $\varepsilon \otimes \varepsilon \otimes \mathrm{id} \otimes \mathrm{id}$ to both sides of (1.2.4) and then using the relations (1.2.1), (1.2.2) and (1.2.5).
(b) In the next pages we will often need to manipulate the identity (1.2.4), using it in

different shapes. Therefore, it is worth to notice the following fact. For example, if we multiplicate both side of (1.2.4) on the right by $\Phi^{-1} \otimes I$, we get:

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta(\Phi))(\Delta \otimes \mathrm{id} \otimes \mathrm{id}(\Phi))(\Phi^{-1} \otimes I) = (I \otimes \Phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id}(\Phi))(\mathrm{id} \otimes \Delta(\Delta(I)) \otimes I)$$

The idempotent appeared in the left hand side can be removed, since:

$$\begin{aligned}
&(\mathrm{id} \otimes \Delta \otimes \mathrm{id}(\Phi))(\mathrm{id} \otimes \Delta(\Delta(I)) \otimes I) = \\
&= (\mathrm{id} \otimes \Delta \otimes \mathrm{id}(\Phi))(\mathrm{id} \otimes \Delta \otimes \mathrm{id}(\Delta \otimes \mathrm{id}(\Delta(I))))(\mathrm{id} \otimes \Delta \otimes (\Delta \otimes \mathrm{id}(I \otimes I))) = \\
&= (\mathrm{id} \otimes \Delta \otimes \mathrm{id}(\Phi))(\mathrm{id} \otimes \Delta \otimes \mathrm{id}(\Delta \otimes \mathrm{id}(\Delta(I)))) = \\
&= \mathrm{id} \otimes \Delta \otimes \mathrm{id}(\Phi(\Delta \otimes \mathrm{id}(\Delta(I)))) = \mathrm{id} \otimes \Delta \otimes \mathrm{id}(\Phi)
\end{aligned}$$

Calculations like this allow to remove idempotent which come out from manipulations of this type.

A morphism of weak quasi bialgebras is a map:

$$\nu : (A, \Delta, \varepsilon, \Phi) \longrightarrow (A', \Delta', \varepsilon', \Phi')$$

which is a morphism of algebra such that:

$$\nu \otimes \nu \circ \Delta = \Delta' \circ \nu \text{ and } (\nu \otimes \nu \otimes \nu)(\Phi) = \Phi'$$

Of course, if Δ is unital the definition of weak quasi bialgebra reduces to that of quasi bialgebra. From now on we will indicate with m the multiplication map $m : A \otimes A \to A$ such that $m(a \otimes b) = ab$.

Definition 1.2.3. A *weak quasi Hopf algebra* is a weak quasi bialgebra with an anti-automorphism S together with two elements α and β in A such that:

$$m(I \otimes \alpha(S \otimes \mathrm{id} \circ \Delta(a))) = \varepsilon(a)\alpha \tag{1.2.7}$$

$$m(I \otimes \beta(\mathrm{id} \otimes S \circ \Delta(a))) = \varepsilon(a)\beta \tag{1.2.8}$$

$$\sum_i x_i \beta S(y_i) \alpha z_i = I = \sum_j S(p_j) \alpha q_j \beta S(r_j) \tag{1.2.9}$$

The antipode S is *strong* if $\alpha = \beta = I$.

Remark 1.2.4. The compatibility condition (1.2.9) between S and Φ is related to the rigidity of the representation category of A, as we can see in the section 1.4.

Definition 1.2.5. A weak quasi bialgebra A is said to be *involutive* (or a weak quasi *-bialgebra) if there exists an involutive map $^* : A \to A$ and a partially invertible element $\Omega \in A \otimes A$ such that:

$$\Omega = \Omega^* \tag{1.2.10}$$

$$\boldsymbol{D}(\Omega) = \Delta(I) \, , \boldsymbol{R}(\Omega) = \Delta(I)^* \tag{1.2.11}$$

$$\Delta(a)^* = \Omega \Delta(a^*) \Omega^{-1} \ \ \forall a \in A \tag{1.2.12}$$

$$\varepsilon \otimes \mathrm{id}(\Omega) = I = \mathrm{id} \otimes \varepsilon(\Omega) \tag{1.2.13}$$

$$(I \otimes \Omega)(\mathrm{id} \otimes \Delta(\Omega))\Phi(\Delta \otimes \mathrm{id}(\Omega^{-1}))(\Omega^{-1} \otimes I) = \Phi^{-1*} \tag{1.2.14}$$

We will use the following notation:

$$\Omega = \sum_i c_i \otimes d_i \ \text{ and } \ \Omega^{-1} = \sum_i \widetilde{c_i} \otimes \widetilde{d_i}$$

Definition 1.2.6. A weak quasi Hopf *-algebra A is a weak quasi Hopf algebra endowed with an involutive map * and a $\Omega \in A \otimes A$ satisfying (1.2.10) - (1.2.14)

Notation.

(1) Let $A = a_1 \otimes \ldots \otimes a_j$ be an element in $V^{\otimes j}$, where V is a linear space, and consider the pairwise different indices $i_1, \ldots, i_j \in \{1, \ldots, n\}$. We indicate with $A_{i_1 \ldots i_j} = a'_1 \otimes \ldots \otimes a'_n$ a tensor product in $V^{\otimes n}$, where $n \geq j$, defined in the following way: $a'_{i_l} = a_l$, and $a'_k = I$ if $k \neq i_h$ for all $h \in \{1, \ldots, j\}$. For example, if $A \in V^{\otimes 4}$, $A_{2145} \in V^{\otimes 5}$ satisfies the following identity:

$$A_{2145} = a_2 \otimes a_1 \otimes I \otimes a_3 \otimes a_4$$

(2) Let V be a linear space. $\Sigma : V^{\otimes 2} \to V^{\otimes 2}$ is the map which sends $v \otimes w$ to $w \otimes v$. $\Sigma_{i,j} : V^{\otimes n} \to V^{\otimes n}$ is the map which switches the first i elements with the second j elements, where $i + j = n$. More precisely:

$$\Sigma_{i,j}(v_1 \otimes \ldots \otimes v_i \otimes v_{i+1} \otimes \ldots \otimes v_n) = v_{i+1} \otimes \ldots \otimes v_n \otimes v_1 \otimes \ldots \otimes v_i$$

On the other hand,

$$\Sigma_{ij}(v_1 \otimes \ldots \otimes v_i \otimes \ldots \otimes v_j \otimes \ldots \otimes v_n) = v_1 \otimes \ldots \otimes v_j \otimes \ldots \otimes v_i \otimes \ldots \otimes v_n$$

Finally, $\Sigma_n : V^{\otimes n} \to V^{\otimes n}$ reverses the order of the n factors.

Definition 1.2.7. A weak quasi bialgebra A is said to be *braided* if there exists an R-matrix $R \in A \otimes A$ partially invertible such that:

$$\boldsymbol{D}(R) = \Delta(I)\,,\, \boldsymbol{R}(R) = \Delta^{\mathrm{op}}(I) \tag{1.2.15}$$

$$\Delta^{\mathrm{op}}(a)R = R\Delta(a) \tag{1.2.16}$$

$$\mathrm{id} \otimes \Delta(R) = \Phi_{312}^{-1} R_{13} \Phi_{213} R_{12} \Phi^{-1} \tag{1.2.17}$$

$$\Delta \otimes \mathrm{id}(R) = \Phi_{231} R_{13} \Phi_{132}^{-1} R_{23} \Phi \tag{1.2.18}$$

If A is a weak quasi *-bialgebra we require the following additional condition:

$$\Omega_{21} R = R^{-1^*} \Omega \tag{1.2.19}$$

The same definition of R-matrix holds if A is a weak quasi Hopf algebra. We will use the following notation:

$$R = \sum_i a_i \otimes b_i \ \text{ and } \ R^{-1} = \sum_i \widetilde{a_i} \otimes \widetilde{b_i}$$

Definition 1.2.8. **(a)** A morphism of weak quasi Hopf algebras $\nu : (A, S) \longrightarrow (A', S')$ is a morphism of weak quasi bialgebras such that:

$$\nu \circ S = S' \circ \nu$$

(b) A morphism of weak quasi *-bialgebras $\nu : (A, \Omega) \longrightarrow (A', \Omega')$ is a morphism of weak quasi bialgebras such that:

$$\nu \circ {}^* = {}^* \circ \nu \ \text{ and } \ \nu \otimes \nu(\Omega) = \Omega'$$

(c) A morphism of braided weak quasi bialgebras $\nu : (A, R) \longrightarrow (A', R')$ is a morphism of weak quasi bialgebras such that:

$$\nu \otimes \nu(R) = R'$$

Definition 1.2.9. Let A be a weak quasi bialgebra. A *twist* is a partially invertible element $F \in A \otimes A$ such that $\boldsymbol{D}(F) = \Delta(I)$ and:

$$\varepsilon \otimes \mathrm{id}(F) = I = \mathrm{id} \otimes \varepsilon(F) \tag{1.2.20}$$

The same definition of twist is valid for weak quasi Hopf algebras. We will use the following notation:

$$F = \sum_i e_i \otimes f_i \ \text{ and } \ F^{-1} = \sum_j \widetilde{e_j} \otimes \widetilde{f_j}$$

Using a twist F it is possible to build a new weak quasi bialgebra A_F, defining the coproduct $\Delta_F : A \to A \otimes A$ as follows:

$$\Delta_F(a) = F\Delta(a)F^{-1} \ \ \forall a \in A \tag{1.2.21}$$

and a new associator:

$$\Phi_F = (I \otimes F)(\mathrm{id} \otimes \Delta(F))\Phi(\Delta \otimes \mathrm{id}(F^{-1}))(F^{-1} \otimes I) \tag{1.2.22}$$

If A is a weak quasi Hopf algebra with an antipode (S, α, β), A_F is a weak quasi Hopf algebra with an antipode (S_F, α_F, β_F), such that:

$$S_F = S \ , \ \alpha_F = \sum_i S(\widetilde{e_i})\alpha\widetilde{f_i} \ \text{ and } \ \beta_F = \sum_i e_i \beta S(f_i)$$

We are going to prove these facts in the following:

Proposition 1.2.10. *For any weak quasi bialgebra $A = (A, \Delta, \varepsilon, \Phi)$ and any twist $F \in A \otimes A$, the algebra $A_F = (A, \Delta_F, \varepsilon, \Phi_F)$ is a weak quasi bialgebra. If A is a weak quasi Hopf algebra with an antipode (S, α, β), A_F is still a weak quasi Hopf algebra with antipode (S_F, α_F, β_F).*

Proof. We check the relation (1.2.1) - (1.2.5) involving Δ_F and Φ_F. Relations (1.2.1) and (1.2.5) are very easy to obtain using (1.2.20). We prove now that $\Phi_F(\Delta_F \otimes \mathrm{id}(\Delta_F(I))) = \Phi_F$:

$$\begin{aligned}
&\Phi_F(\Delta_F \otimes \mathrm{id}(\Delta_F(I))) = \\
&= (I \otimes F)(\mathrm{id} \otimes \Delta(F))\Phi(\Delta \otimes \mathrm{id}(F^{-1}))(F^{-1}F \otimes I)(\Delta \otimes \mathrm{id}(FF^{-1}))(F^{-1} \otimes I) = \\
&= (I \otimes F)(\mathrm{id} \otimes \Delta(F))\Phi(\Delta \otimes \mathrm{id}(F^{-1}))(\Delta \otimes \mathrm{id}(F))(\Delta \otimes \mathrm{id}(F^{-1}))(F^{-1} \otimes I) = \\
&= (I \otimes F)(\mathrm{id} \otimes \Delta(F))\Phi(\Delta \otimes \mathrm{id}(\Delta(I)))(\Delta \otimes \mathrm{id}(F^{-1}))(F^{-1} \otimes I) = \\
&= (I \otimes F)(\mathrm{id} \otimes \Delta(F))\Phi(\Delta \otimes \mathrm{id}(F^{-1}))(F^{-1} \otimes I) = \Phi_F
\end{aligned}$$

In a similar way we obtain that $(\mathrm{id}\otimes\Delta_F(\Delta_F(I)))\Phi_F = \Phi_F$. So we have proved that Δ_F and Φ_F satisfy (1.2.2). In the next calculation we prove that they satisfy (1.2.3):

$$\begin{aligned}
&(\mathrm{id}\otimes\Delta_F(\Delta_F(a)))\Phi_F =\\
&= (I\otimes F)(\mathrm{id}\otimes\Delta(F\Delta(a)F^{-1}))(I\otimes F^{-1}F)(\mathrm{id}\otimes\Delta(F))\Phi(\Delta\otimes\mathrm{id}(F^{-1}))(F^{-1}\otimes I) =\\
&= (I\otimes F)(\mathrm{id}\otimes\Delta(F))(\mathrm{id}\otimes\Delta(\Delta(a)))\Phi(\Delta\otimes\mathrm{id}(F^{-1}))(F^{-1}\otimes I) =\\
&= (I\otimes F)(\mathrm{id}\otimes\Delta(F))\Phi(\Delta\otimes\mathrm{id}(\Delta(a)))(\Delta\otimes\mathrm{id}(F^{-1}))(F^{-1}\otimes I) =\\
&= (I\otimes F)(\mathrm{id}\otimes\Delta(F))\Phi(\Delta\otimes\mathrm{id}(F^{-1}))(F^{-1}F\otimes I)(\Delta\otimes\mathrm{id}(F\Delta(a)F^{-1}))(F^{-1}\otimes I) =\\
&= \Phi_F(\Delta_F\otimes\mathrm{id}(\Delta_F(a)))
\end{aligned}$$

It remains to prove the relation (1.2.4), or, explicitly:

$$(\mathrm{id}\otimes\mathrm{id}\otimes\Delta_F(\Phi_F))(\Delta_F\otimes\mathrm{id}\otimes\mathrm{id}(\Phi_F)) = (I\otimes\Phi_F)(\mathrm{id}\otimes\Delta_F\otimes\mathrm{id}(\Phi_F))(\Phi_F\otimes I)$$

Hence:

$$\begin{aligned}
&(\mathrm{id}\otimes\mathrm{id}\otimes\Delta_F(\Phi_F))(\Delta_F\otimes\mathrm{id}\otimes\mathrm{id}(\Phi_F)) =\\
&= F_{34}(\mathrm{id}\otimes\mathrm{id}\otimes\Delta(F_{23}(\mathrm{id}\otimes\Delta(F))\Phi(\Delta\otimes\mathrm{id}(F^{-1}))F_{12}^{-1}))F_{34}^{-1}\cdot\\
&\cdot F_{12}(\Delta\otimes\mathrm{id}\otimes\mathrm{id}(F_{23}(\mathrm{id}\otimes\Delta(F))\Phi(\Delta\otimes\mathrm{id}(F^{-1}))F_{12}^{-1}))F_{12}^{-1} =\\
&= F_{34}(\mathrm{id}\otimes\mathrm{id}\otimes\Delta(F_{23}))(\mathrm{id}\otimes\mathrm{id}\otimes\Delta(\mathrm{id}\otimes\Delta(F)))(\mathrm{id}\otimes\mathrm{id}\otimes\Delta(\Phi))\cdot\\
&\cdot(\Delta\otimes\Delta(F^{-1}))F_{12}^{-1}F_{34}^{-1}F_{12}F_{34}(\Delta\otimes\Delta(F))(\Delta\otimes\mathrm{id}\otimes\mathrm{id}(\Phi))\cdot\\
&\cdot(\Delta\otimes\mathrm{id}\otimes\mathrm{id}(\Delta\otimes\mathrm{id}(F^{-1})))(\Delta\otimes\mathrm{id}\otimes\mathrm{id}(F_{12}^{-1}))F_{12}^{-1}
\end{aligned}$$

Since F_{12} e F_{34} commute and using (1.2.3) on Φ we get:

$$\begin{aligned}
&(I\otimes I\otimes F)(\mathrm{id}\otimes\mathrm{id}\otimes\Delta(I\otimes F))(I\otimes\Phi)(\mathrm{id}\otimes\Delta\otimes\mathrm{id}(\mathrm{id}\otimes\Delta(F)))\cdot\\
&\cdot(I\otimes\Phi^{-1})(\mathrm{id}\otimes\mathrm{id}\otimes\Delta(\Phi))(\Delta\otimes\mathrm{id}\otimes\mathrm{id}(\Phi))(\Phi^{-1}\otimes I)\cdot\\
&\cdot(\mathrm{id}\otimes\Delta\otimes\mathrm{id}(\Delta\otimes\mathrm{id}(F^{-1})))(\Phi\otimes I)(\Delta\otimes\mathrm{id}\otimes\mathrm{id}(F^{-1}\otimes I))(F^{-1}\otimes I\otimes I)
\end{aligned}$$

Using (1.2.4) on Φ we obtain:

$$\begin{aligned}
&(I\otimes I\otimes F)(\mathrm{id}\otimes\mathrm{id}\otimes\Delta(I\otimes F))(I\otimes\Phi)(\mathrm{id}\otimes\Delta\otimes\mathrm{id}(\mathrm{id}\otimes\Delta(F)))\cdot\\
&\cdot(\mathrm{id}\otimes\Delta\otimes\mathrm{id}(\Phi))(\mathrm{id}\otimes\Delta\otimes\mathrm{id}(\Delta\otimes\mathrm{id}(F^{-1})))\cdot\\
&\cdot(\Phi\otimes I)(\Delta\otimes\mathrm{id}\otimes\mathrm{id}(F^{-1}\otimes I))(F^{-1}\otimes I\otimes I) =\\
&= (I\otimes I\otimes F)(\mathrm{id}\otimes\mathrm{id}\otimes\Delta(I\otimes F))(I\otimes\Phi)(\mathrm{id}\otimes\Delta\otimes\mathrm{id}(I\otimes F^{-1}))(I\otimes F^{-1}\otimes I)\cdot\\
&\cdot(I\otimes F\otimes I)(\mathrm{id}\otimes\Delta\otimes\mathrm{id}((I\otimes F)(\mathrm{id}\otimes\Delta(F))\Phi(\Delta\otimes\mathrm{id}(F^{-1}))(F^{-1}\otimes I)))(I\otimes F^{-1}\otimes I)\cdot\\
&\cdot(I\otimes F\otimes I)(\mathrm{id}\otimes\Delta\otimes\mathrm{id}(F\otimes I))(\Phi\otimes I)(\Delta\otimes\mathrm{id}\otimes\mathrm{id}(F^{-1}\otimes I))(F^{-1}\otimes I\otimes I) =\\
&= (I\otimes\Phi_F)(\mathrm{id}\otimes\Delta_F\otimes\mathrm{id}(\Phi_F))(\Phi_F\otimes\mathrm{id})
\end{aligned}$$

Let's pass to the weak quasi Hopf algebra case. We start proving (1.2.7) on A_F:

$$\begin{aligned}
m(I\otimes\alpha_F(S_F\otimes\mathrm{id}\circ\Delta_F(a))) &= \sum_{i,j} S(e_i a_{(1)}\widetilde{e_j})\alpha_F f_i a_{(2)}\widetilde{f_j} =\\
&= \sum_{i,j,h} S(\widetilde{e_j})S(a_{(1)})S(e_i)S(\widetilde{e_h})\alpha\widetilde{f_h}f_i a_{(2)}\widetilde{f_j}
\end{aligned}$$

Since $F^{-1}F = \Delta(I)$:

$$\sum_{i,h} S(e_i)S(\widetilde{e_h})\alpha\widetilde{f_h}f_i = \alpha \qquad (1.2.23)$$

$$\sum_{i,h} \widetilde{e_h}e_i\beta S(f_i)S(\widetilde{f_h}) = \beta \qquad (1.2.24)$$

Hence:

$$m(I \otimes \alpha_F(S_F \otimes \mathrm{id} \circ \Delta_F(a))) = \sum_j S(\widetilde{e_j})S(a_{(1)})\alpha a_{(2)}\widetilde{f_j} =$$
$$= \varepsilon(a)\sum_j S(\widetilde{e_j})\alpha\widetilde{f_j} = \varepsilon(a)\alpha_F$$

The condition (1.2.8) can be proved in the same way. It remains to prove the relation (1.2.9). We will prove only the first of the two identities. Remembering the definition of Φ_F, we have:

$$\Phi_F = \sum_i x_i^{(F)} \otimes y_i^{(F)} \otimes z_i^{(F)} =$$
$$= \sum_{i,j,h,k,l} e_j x_h \widetilde{e}_{k(1)}\widetilde{e_l} \otimes e_i f_{j(1)} y_h \widetilde{e}_{k(2)}\widetilde{f_l} \otimes f_i f_{j(2)} z_h \widetilde{f_k}$$

We want to prove that:

$$\sum_i x_i^{(F)}\beta_F S_F(y_i^{(F)})\alpha_F z_i^{(F)} = I$$

In fact:

$$\sum_i x_i^{(F)}\beta_F S_F(y_i^{(F)})\alpha_F z_i^{(F)} = \sum_{j,h,k} e_j x_h \widetilde{e}_{k(1)}\left(\sum_{l,n}\widetilde{e_l} e_n \beta S(f_n)S(\widetilde{f_l})\right)\cdot$$
$$\cdot S(\widetilde{e}_{k(2)})S(y_h)S(f_{j(1)})\left(\sum_{i,m} S(e_i)S(\widetilde{e_m})\alpha\widetilde{f_m} f_i\right) f_{j(2)} z_h \widetilde{f_k}$$

Now, using (1.2.23) and (1.2.24), we obtain:

$$\sum_i x_i^{(F)}\beta_F S_F(y_i^{(F)})\alpha_F z_i^{(F)} =$$
$$= \sum_{j,h,k} e_j x_h \widetilde{e}_{k(1)}\beta S(\widetilde{e}_{k(2)})S(y_h)S(f_{j(1)})\alpha f_{j(2)} z_h \widetilde{f_k} =$$
$$= \left(\sum_j e_j \varepsilon(f_j)\right)\left(\sum_h x_h \beta S(y_h)\alpha z_h\right)\left(\sum_k \varepsilon(\widetilde{e_k})\widetilde{f_k}\right) = I$$

using (1.2.9) for A and (1.2.20). □

If we are dealing with weak quasi *-bialgebras A, the weak quasi bialgebra A_F is still involutive with:

$$\Omega_F = F^{-1^*}\Omega F^{-1}$$

This is what we are going to prove in the next:

Proposition 1.2.11. *Given a weak quasi *-bialgebra A and a twist F, the algebra A_F is a weak quasi *-bialgebra.*

Proof. We need to prove the relations (1.2.10) - (1.2.14) related to Ω_F. Identities (1.2.10) - (1.2.12) are very easy to prove. We prove the relation (1.2.13):

$$\Omega_F\Delta_F(a^*) = F^{-1^*}\Omega F^{-1}F\Delta(a^*)F^{-1} =$$
$$= F^{-1^*}\Omega\Delta(a^*)F^{-1} = F^{-1^*}\Delta(a)^*\Omega F^{-1} = \Delta_F(a)^*\Omega_F$$

It remains to prove (1.2.14):

$$(I \otimes \Omega_F)(\mathrm{id} \otimes \Delta_F(\Omega_F))\Phi_F(\Delta_F \otimes \mathrm{id}(\Omega_F^{-1}))(\Omega_F^{-1} \otimes I) =$$
$$= (I \otimes F^{-1^*})(I \otimes \Omega)(I \otimes F^{-1})(I \otimes F)(\mathrm{id} \otimes \Delta(F^{-1^*}))(\mathrm{id} \otimes \Delta(\Omega))(\mathrm{id} \otimes \Delta(F^{-1}))$$
$$(I \otimes F^{-1})(I \otimes F)(\mathrm{id} \otimes \Delta(F))\Phi(\Delta \otimes \mathrm{id}(F^{-1}))(F^{-1} \otimes I)(F \otimes I)$$
$$(\Delta \otimes \mathrm{id}(F))(\Delta \otimes \mathrm{id}(\Omega^{-1}))(\Delta \otimes \mathrm{id}(F^*))(F^{-1} \otimes I)(F \otimes I)(\Omega^{-1} \otimes I)(F^* \otimes I) =$$
$$= (I \otimes F^{-1^*})(\mathrm{id} \otimes \Delta(F^{-1}))^*(I \otimes \Omega)(\mathrm{id} \otimes \Delta(\Omega))\Phi$$
$$(\Delta \otimes \mathrm{id}(\Omega^{-1}))(\Omega^{-1} \otimes I)(\Delta \otimes \mathrm{id}(F))^*(F^* \otimes I) =$$
$$= (I \otimes F^{-1^*})(\mathrm{id} \otimes \Delta(F^{-1}))^*\Phi^{-1^*}(\Delta \otimes \mathrm{id}(F))^*(F^* \otimes I) = \Phi_F^{-1^*}$$

□

Now, suppose that $(A, \Delta, \varepsilon, \Phi, R)$ is a braided weak quasi bialgebra, and $F \in A \otimes A$ a twist. The twisted R-matrix R_F is:

$$R_F = F_{21} R F^{-1} \tag{1.2.25}$$

We have:

Proposition 1.2.12. *The algebra $A_F = (A, \Delta_F, \varepsilon, \Phi_F, \Omega_F, R_F)$ is a braided weak quasi bialgebra. If A is a braided weak quasi *-bialgebra, A_F is a braided weak quasi *-bialgebra too.*

Proof. We have to prove relations (1.2.15) - (1.2.18) for R_F. (1.2.15) is quite easy to prove. (1.2.16) is a consequence of the following calculation:

$$\begin{aligned} R_F \Delta_F(a) &= F_{21} R F^{-1} F \Delta(a) F^{-1} = F_{21} R \Delta(a) F^{-1} = \\ &= F_{21} \Delta^{\mathrm{op}}(a) R F^{-1} = (F \Delta(a) F^{-1})_{21} F_{21} R F^{-1} = \Delta_F^{\mathrm{op}}(a) R_F \end{aligned}$$

We pass to (1.2.18). The relation (1.2.17) can be proved in the same way.

$$\begin{aligned} \mathrm{id} \otimes \Delta_F(R_F) &= (\mathrm{id} \otimes \Delta_F(F_{21}))(\mathrm{id} \otimes \Delta_F(R))(\mathrm{id} \otimes \Delta_F(F^{-1})) = \\ &= (I \otimes F)(\mathrm{id} \otimes \Delta(F_{21}))(\mathrm{id} \otimes \Delta(R))(\mathrm{id} \otimes \Delta(F^{-1}))(I \otimes F^{-1}) = \\ &= (I \otimes F)(\mathrm{id} \otimes \Delta(F_{21}))\Phi_{312}^{-1} R_{13} \Phi_{213} R_{12} \Phi^{-1}(\mathrm{id} \otimes \Delta(F^{-1}))(I \otimes F^{-1}) = \\ &= (I \otimes F)(\mathrm{id} \otimes \Delta(F_{21}))\Phi_{312}^{-1} R_{13} \Phi_{213} R_{12}(\Delta \otimes \mathrm{id}(F^{-1}))(F^{-1} \otimes I)\Phi_F^{-1} = \\ &= (I \otimes F)(\mathrm{id} \otimes \Delta(F_{21}))\Phi_{312}^{-1} R_{13} \Phi_{213}(\Delta^{\mathrm{op}} \otimes \mathrm{id}(F^{-1}))(F_{21}^{-1} \otimes I)(R_F \otimes I)\Phi_F^{-1} = \\ &= (I \otimes F)(\mathrm{id} \otimes \Delta(F_{21}))\Phi_{312}^{-1} R_{13}(\mathrm{id} \otimes \Delta(F^{-1})_{213}) F_{13}^{-1} \Phi_{F_{213}} R_{F_{12}} \Phi_F^{-1} = \\ &= (I \otimes F)(\mathrm{id} \otimes \Delta(F_{21}))\Phi_{312}^{-1}(\mathrm{id} \otimes \Delta^{\mathrm{op}}(F^{-1})_{213}) F_{31}^{-1} R_{F_{13}} \Phi_{F_{213}} R_{F_{12}} \Phi_F^{-1} = \\ &= \Phi_{F_{312}}^{-1} F_{31}^{-1} R_{F_{13}} \Phi_{F_{213}} R_{F_{12}} \Phi_F^{-1} \end{aligned}$$

Now, suppose that A is a weak quasi *-bialgebra. It remains to prove the relation (1.2.19):

$$\begin{aligned} \Omega_{21\,F} R_F &= F_{21}^{-1^*} \Omega_{21} F_{21}^{-1} F_{21} R F^{-1} = F_{21}^{-1^*} \Omega_{21} R F^{-1} = \\ &= F_{21}^{-1^*} R^{*-1} \Omega F^{-1} = F_{21}^{-1^*} R^{-1^*} F^* F^{-1^*} \Omega F^{-1} = \\ &= F_{21}^{-1^*} R^{*-1} F^* \Omega_F = (F_{21} R F^{-1})^{-1^*} \Omega_F = R_F^{-1^*} \Omega_F \end{aligned}$$

□

When F is a twist on A, then so is F^{-1} and we have:

$$(A_F)_{F^{-1}} = A = (A_{F^{-1}})_F \tag{1.2.26}$$

If F' is another twist, then so is the product FF' and:

$$(A_F)_{F'} = A_{FF'} \tag{1.2.27}$$

Definition 1.2.13. Two weak quasi bialgebras $(A, \Delta, \varepsilon, \Phi)$ and $(A', \Delta', \varepsilon', \Phi')$ are equivalent if there exist a twist F on A' and an isomorphism $\nu : A \longrightarrow A'_F$ of weak quasi bialgebras.

All the results we have seen before involving weak quasi bialgebras are still valid for weak quasi Hopf algebras. Before ending this section, we introduce a special kind of these algebras, which is not closed under twists. Let A be a weak quasi Hopf algebra. In order to shorten the formulas in the next Definition, we set:

$$\begin{aligned} &P_2 = \Delta(I), \quad P_3 = \Delta \otimes \mathrm{id}(\Delta(I)), \quad Q_3 = \mathrm{id} \otimes \Delta(\Delta(I)) \\ &P_4 = \Delta \otimes \mathrm{id} \otimes \mathrm{id}(\Delta \otimes \mathrm{id}(\Delta(I))), \quad Q_4 = \mathrm{id} \otimes \mathrm{id} \otimes \Delta(\mathrm{id} \otimes \Delta(\Delta(I))) \end{aligned} \tag{1.2.28}$$

Definition 1.2.14. A *weak Hopf algebra* is a weak quasi Hopf algebra with P_2, P_3, Q_3, P_4, Q_4 satisfying the following:

$$Q_3 \Delta \otimes \mathrm{id}(\Delta(a)) = \mathrm{id} \otimes \Delta(\Delta(a)) P_3 \tag{1.2.29}$$

$$Q_3 P_3 Q_3 = Q_3, \quad P_3 Q_3 P_3 = P_3 \tag{1.2.30}$$

$$(I \otimes Q_3)(\mathrm{id} \otimes \Delta \otimes \mathrm{id}(Q_3 P_3)(P_3 \otimes I) = Q_4(\Delta \otimes \Delta(P_2)) P_4 \tag{1.2.31}$$

As a consequence of the previous identities, the associator Φ and its inverse Φ^{-1} can be chosen in the following way:

$$\Phi = Q_3 P_3 = \mathrm{id} \otimes \Delta(\Delta(I)) \Delta \otimes \mathrm{id}(\Delta(I)) \tag{1.2.32}$$

$$\Phi^{-1} = P_3 Q_3 = \Delta \otimes \mathrm{id}(\Delta(I)) \, \mathrm{id} \otimes \Delta(\Delta(I)) \tag{1.2.33}$$

Remark 1.2.15. **(a)** A deeper look to these algebras will be given in [16]. At this stage we can notice that our definition of weak Hopf algebra is quite different from the one introduced by Böhm, Nill and Szlachanyi [10], [60]. In fact, their algebras are coassociative, with a non-counital coproduct and a non-multiplicative counit. This is due to the fact these two kinds of algebras arise in different contexts and with different motivations.
(b) Suppose that A is a weak quasi Hopf algebra with associators Φ and Φ^{-1}. Hence, since we already know that Φ and Φ^{-1} are associators, the identities (1.2.32) and (1.2.33) are equivalent to conditions (1.2.29) - (1.2.31). This fact will be used afterwards to prove that a peculiar weak quasi Hopf algebra is actually a weak Hopf algebra.

1.3 Weak quasi Hopf algebras: properties of the antipode

In this section we focus on some properties of weak quasi Hopf algebras, specially those involving the antipode. The most of the following results are new but heavily inspired by the Drinfeld's work about quasi Hopf algebras [20].

Proposition 1.3.1. *The counit ε of a weak quasi Hopf algebra A is unique and satisfies $\varepsilon \circ S = \varepsilon$. If A is involutive, ε also satisfies $\varepsilon(a^*) = \overline{\varepsilon(a)}$, for every $a \in A$.*

Proof. The first two statements can be proved in the same way as for quasi Hopf algebras, namely the first follows from (1.2.1) while the second from applying the counit to one of the equations (1.2.7) and (1.2.8). For the last statement it suffices to show that $\widetilde{\varepsilon}(a) := \overline{\varepsilon(a^*)}$ is a counit. We prove just one of the two counit identities:

$$\begin{aligned} (\mathrm{id} \otimes \widetilde{\varepsilon})(\Delta(a)) = a_{(1)}\widetilde{\varepsilon}(a_{(2)}) = \\ = (a^*_{(1)}\varepsilon(a^*_{(2)}))^* = (\mathrm{id} \otimes \varepsilon(\Delta(a)^*))^* = \\ = (\mathrm{id} \otimes \varepsilon(\Omega\Delta(a^*)\Omega^{-1}))^* = (\mathrm{id} \otimes \varepsilon(\Delta(a^*)))^* = a \end{aligned}$$

using (1.2.12) and (1.2.13). □

Proposition 1.3.2. *Let A be a weak quasi Hopf algebra with antipode (S, α, β). Then for every invertible $u \in A$, the triple $(\overline{S}, \overline{\alpha}, \overline{\beta})$ defined by*

$$\overline{S}(a) = uS(a)u^{-1}, \quad u\alpha = \overline{\alpha}, \quad \beta u^{-1} = \overline{\beta}. \tag{1.3.1}$$

is another antipode of A. Conversely, if (S, α, β) and $(\overline{S}, \overline{\alpha}, \overline{\beta})$ are two antipodes, there exists a unique invertible element $u \in A$ satisfying (1.3.1)*. In particular, if a strong antipode exists, it is unique.*

Proof. Straightforward calculations allow us to prove that, if (S, α, β) is an antipode, $(\overline{S}, \overline{\alpha}, \overline{\beta})$ defined as (1.3.1) is another antipode. Conversely, let (S, α, β) and $(\overline{S}, \overline{\alpha}, \overline{\beta})$ be two antipodes, and:

$$u = \sum_i \overline{S}(p_i)\overline{\alpha}q_i\beta S(r_i) \tag{1.3.2}$$

In some of the following calculations we will avoid the summation symbol. Applying the map $V : A \otimes A \otimes A \to A$ such that:

$$V(b \otimes c \otimes d) = \overline{S}(b)\overline{\alpha}c\beta S(d)$$

to both side of the identity $\Delta \otimes \mathrm{id}(\Delta(a))\Phi^{-1} = \Phi^{-1}(\mathrm{id} \otimes \Delta)(\Delta(a))$, we get $uS(a) = \overline{S}(a)u$. In fact:

$$\begin{aligned} V(\Delta \otimes \mathrm{id}(\Delta(a))\Phi^{-1}) &= V(a_{(1)(1)}p_i \otimes a_{(1)(2)}q_i \otimes a_{(2)}r_i) = \\ &= \overline{S}(a_{(1)(1)}p_i)\overline{\alpha}a_{(1)(2)}q_i\beta S(a_{(2)}r_i) = \\ &= \overline{S}(p_i)\overline{S}(a_{(1)(1)})\overline{\alpha}a_{(1)(2)}q_i\beta S(a_{(2)}r_i) = \\ &= \overline{S}(p_i)\overline{\alpha}q_i\beta S(r_i)S(a) = uS(a) \end{aligned}$$

Applying V to the right hand side, we obtain:

$$\begin{aligned} V(\Phi^{-1}(\mathrm{id} \otimes \Delta)(\Delta(a))) &= V(p_i a_{(1)} \otimes q_i a_{(2)(1)} \otimes r_i a_{(2)(2)}) = \\ &= \overline{S}(p_i a_{(1)})\overline{\alpha}q_i a_{(2)(1)}\beta S(r_i a_{(2)(2)}) = \\ &= \overline{S}(a)\overline{S}(p_i)\overline{\alpha}q_i\beta S(r_i) = \overline{S}(a)u \end{aligned}$$

Now, from (1.2.4) and **(b)** of Remark 1.2.2 it follows that:

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta(\Phi))(\Delta \otimes \mathrm{id} \otimes \mathrm{id}(\Phi))(\Phi^{-1} \otimes I) = (I \otimes \Phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id}(\Phi))$$

We apply the map $W : A^{\otimes 4} \to A$, to both sides of the previous identity, where

$$W(b \otimes c \otimes d \otimes e) = \overline{S}(b)\overline{\alpha}c\beta S(d)\alpha e$$

obtaining in this way $u\alpha = \overline{\alpha}$. In fact the left hand side gives

$$\begin{aligned}
&\overline{S}(x_i x_{j_{(1)}} p_k)\overline{\alpha} y_i x_{j_{(2)}} q_k \beta S(z_{i_{(1)}} y_j r_k)\alpha z_{i_{(2)}} z_j = \\
&= \overline{S}(p_k)\overline{S}(x_{j_{(1)}})\overline{S}(x_i)\overline{\alpha} y_i x_{j_{(2)}} q_k \beta S(r_k) S(y_j)\varepsilon(z_i)\alpha z_j = \\
&= \overline{S}(p_k)\overline{S}(x_{j_{(1)}})[\overline{S}(x_i)\overline{\alpha} y_i \varepsilon(z_i)] x_{j_{(2)}} q_k \beta S(r_k) S(y_j)\alpha z_j = \\
&= \overline{S}(p_k)\overline{S}(x_{j_{(1)}})\overline{\alpha} x_{j_{(2)}} q_k \beta S(r_k) S(y_j)\alpha z_j = \\
&= \overline{S}(p_k)\overline{\alpha} q_k \beta S(r_k)[\varepsilon(x_j) S(y_j)\alpha z_j] = u\alpha
\end{aligned}$$

We have used the following identity in the third line:

$$\begin{aligned}
\overline{S}(x_i)\overline{\alpha} y_i \varepsilon(z_i) &= m \circ [I \otimes \overline{\alpha} \cdot (\overline{S} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)(\Phi)] = \\
&= m \circ [I \otimes \overline{\alpha} \cdot (\overline{S} \otimes \mathrm{id})(\Delta(I))] = \varepsilon(I)\overline{\alpha} = \overline{\alpha}
\end{aligned}$$

and similarly we have used the relation $\varepsilon(x_j)S(y_j)\alpha z_j = \alpha$ in the last line. Analogous computations show that the right hand side leads to $\overline{\alpha}$. The relation involving β can be proved in the same way. Now we prove the invertibility of u. We define:

$$v = \sum_i S(p_i)\alpha q_i \overline{\beta}\, \overline{S}(r_i)$$

v is the inverse of u. Indeed:

$$\begin{aligned}
uv &= \sum_i u S(p_i)\alpha q_i \overline{\beta}\, \overline{S}(r_i) = \sum_i \overline{S}(p_i) u\alpha q_i \overline{\beta}\, \overline{S}(r_i) = \\
&= \sum_i \overline{S}(p_i)\overline{\alpha} q_i \overline{\beta}\, \overline{S}(r_i) = I
\end{aligned}$$

A similar calculation allows to prove the uniqueness of u:

$$\begin{aligned}
u = uI &= \sum_i u S(p_j)\alpha q_j \beta S(r_j) = \sum_i \overline{S}(p_j) u\alpha q_j \beta S(r_j) = \\
&= \sum_i \overline{S}(p_j)\overline{\alpha} q_j \beta S(r_j)
\end{aligned}$$

□

We now ask if there exists a relation between the antipode and the coproduct, or, more precisely, if the antipode enjoys a sort of anticomultiplicativity. The answer is quite similar to the quasi Hopf algebras case, but we must take into account the non-unitality of the coproduct. We define:

$$\begin{aligned}
\gamma &= V((I \otimes \Phi^{-1})(\mathrm{id} \otimes \mathrm{id} \otimes \Delta(\Phi))) \\
\delta &= V'((\Delta \otimes \mathrm{id} \otimes \mathrm{id}(\Phi))(\Phi^{-1} \otimes I))
\end{aligned}$$

where $V, V' : A^{\otimes 4} \to A^{\otimes 2}$ are:

$$\begin{aligned}
V(c \otimes d \otimes e \otimes f) &= S(d)\alpha e \otimes S(c)\alpha f \\
V'(c \otimes d \otimes e \otimes f) &= c\beta S(f) \otimes d\beta S(e)
\end{aligned}$$

Our goal is to prove the following:

Theorem 1.3.3. *There exists a unique quasi invertible element F in $A \otimes A$ such that:*

$$\boldsymbol{D}(F) = \Delta(I) \ , \ \boldsymbol{R}(F) = S \otimes S(\Delta^{\mathrm{op}}(I)) \tag{1.3.3}$$
$$F\Delta(S(a))F^{-1} = (S \otimes S(\Delta^{\mathrm{op}}(a))) \ \forall a \in A \tag{1.3.4}$$
$$\gamma = F\Delta(\alpha) \tag{1.3.5}$$
$$\delta = \Delta(\beta)F^{-1} \tag{1.3.6}$$

Moreover, we have that:

$$F = \sum_i S \otimes S(\Delta^{\mathrm{op}}(p_i))\gamma\Delta(q_i\beta S(r_i))$$

and

$$F^{-1} = \sum_i \Delta(S(p_i)\alpha q_i)\delta(S \otimes S(\Delta'(r_i)))$$

We start proving the following:

Lemma 1.3.4. *We have:*
(a)

$$\gamma = V((\Phi \otimes I)(\Delta \otimes \mathrm{id} \otimes \mathrm{id}(\Phi^{-1})))$$
$$\delta = V'((\mathrm{id} \otimes \mathrm{id} \otimes \Delta(\Phi^{-1}))(I \otimes \Phi))$$

(b) *If $a \in A$, then:*

$$(S \otimes S(\Delta^{\mathrm{op}}(a_{(1)})))\gamma\Delta(a_{(2)}) = \varepsilon(a)\gamma$$
$$\Delta(a_{(1)})\delta(S \otimes S(\Delta^{\mathrm{op}}(a_{(2)}))) = \varepsilon(a)\delta$$

(c)

$$\sum_i \Delta(x_i)\delta(S \otimes S(\Delta^{\mathrm{op}}(y_i)))\gamma\Delta(z_i) = \Delta(I)$$
$$\sum_j (S \otimes S(\Delta^{\mathrm{op}}(p_j)))\gamma\Delta(q_j)\delta(S \otimes S(\Delta^{\mathrm{op}}(r_j))) = \Delta(I)$$

Proof. **(a)** Using (1.2.4) we have

$$\gamma = V((\mathrm{id} \otimes \Delta \otimes \mathrm{id}(\Phi))(\Phi \otimes I)(\Delta \otimes \mathrm{id} \otimes \mathrm{id}(\Phi^{-1})))$$

It is easy to see that:

$$V(\mathrm{id} \otimes \Delta \otimes \mathrm{id}(\Phi) \cdot T) = V(T) \tag{1.3.7}$$

after the following calculation:

$$\begin{aligned} V(\mathrm{id} \otimes \Delta \otimes \mathrm{id}(\Phi) \cdot T) &= \sum_i \varepsilon(y_i)V(x_i \otimes I \otimes I \otimes z_i \cdot T) = \\ &= \sum_i \varepsilon(y_i)S(T_2)\alpha T_3 \otimes S(T_1)S(x_i)\alpha z_i T_4 = \\ &= \sum_i S(T_2)\alpha T_3 \otimes S(T_1)[S(x_i)\varepsilon(y_i)\alpha z_i]T_4 = \\ &= \sum_i S(T_2)\alpha T_3 \otimes S(T_1)\alpha T_4 = V(T) \end{aligned}$$

where the expression in the square brackets is equal to α thanks to (1.2.5). Therefore:

$$\begin{aligned}\gamma &= V((\mathrm{id}\otimes\Delta\otimes\mathrm{id}(\Phi))(\Phi\otimes I)(\Delta\otimes\mathrm{id}\otimes\mathrm{id}(\Phi^{-1}))) = \\ &= V((\Phi\otimes I)(\Delta\otimes\mathrm{id}\otimes\mathrm{id}(\Phi^{-1})))\end{aligned}$$

Similarly for the identity involving δ.

(b) We prove the first identity. The second identity will follow in the same way. $(S\otimes S(\Delta^{\mathrm{op}}(a_{(1)})))\gamma\Delta(a_{(2)})$ is equal to

$$V((I\otimes\Phi^{-1})(\mathrm{id}\otimes\mathrm{id}\otimes\Delta(\Phi))(\Delta\otimes\Delta(\Delta(a))))$$

which in turn is equal to:

$$V((\mathrm{id}\otimes\Delta\otimes\mathrm{id}(\mathrm{id}\otimes\Delta(\Delta(a))))(I\otimes\Phi^{-1})(\mathrm{id}\otimes\mathrm{id}\otimes\Delta(\Phi))) \tag{1.3.8}$$

after the following calculation:

$$\begin{aligned}&(I\otimes\Phi^{-1})(\mathrm{id}\otimes\mathrm{id}\otimes\Delta(\Phi))(\Delta\otimes\Delta(\Delta(a))) = \\ &= (I\otimes\Phi^{-1})(\mathrm{id}\otimes\mathrm{id}\otimes\Delta(\Phi(\Delta\otimes\mathrm{id}(\Delta(a))))) = \\ &= (I\otimes\Phi^{-1})(\mathrm{id}\otimes\mathrm{id}\otimes\Delta((\mathrm{id}\otimes\Delta(\Delta(a)))\Phi)) = \\ &= (\mathrm{id}\otimes\Delta\otimes\mathrm{id}(\mathrm{id}\otimes\Delta(\Delta(a))))(I\otimes\Phi^{-1})(\mathrm{id}\otimes\mathrm{id}\otimes\Delta(\Phi))\end{aligned}$$

Finally, (1.3.8) is equal to $\varepsilon(a)\gamma$ since:

$$V(\mathrm{id}\otimes\Delta\otimes\mathrm{id}(\mathrm{id}\otimes\Delta(\Delta(a)))\cdot T_1\otimes T_2\otimes T_3\otimes T_4) = \varepsilon(a)V(T_1\otimes T_2\otimes T_3\otimes T_4)$$

(c) We prove the first identity. The second will follow similarly. Let r be the following element in $A^{\otimes 6}$:

$$\begin{aligned}&[I\otimes I\otimes((I\otimes\Phi^{-1})(\mathrm{id}\otimes\mathrm{id}\otimes\Delta(\Phi)))](\Delta\otimes\Delta\otimes\Delta(\Phi))\cdot \\ &\cdot[(\Delta\otimes\mathrm{id}\otimes\mathrm{id}(\Phi))(\Phi^{-1}\otimes I))\otimes I\otimes I]\end{aligned}$$

We introduce the map $\phi : A^{\otimes 6}\to A^{\otimes 2}$, with:

$$\phi(a\otimes b\otimes c\otimes d\otimes e\otimes f) = (a\otimes b)(\beta\otimes\beta)(S(d)\otimes S(c))(\alpha\otimes\alpha)(e\otimes f)$$

A simple calculation leads to the identity:

$$\phi(r) = \Delta(x_i)\delta(S\otimes S(\Delta^{\mathrm{op}}(y_i)))\gamma\Delta(z_i)$$

In this way we obtain the LHS of the identity that we are going to prove. We need much more work in order to get the RHS. It is possible to write r in a different way:

$$\begin{aligned}r &= (\Delta\otimes\mathrm{id}\otimes\Delta\otimes\mathrm{id}(\mathrm{id}\otimes\mathrm{id}\otimes\Delta(\Phi)))(I^{\otimes 3}\otimes\Phi^{-1})\cdot \\ &\cdot(\Phi^{-1}\otimes I^{\otimes 3})(\mathrm{id}\otimes\Delta\otimes\mathrm{id}\otimes\Delta((\Delta\otimes\mathrm{id}\otimes\mathrm{id}(\Phi))))\end{aligned} \tag{1.3.9}$$

using repeatedly (1.2.3). We report the explicit computation:

$$\begin{aligned}r &= [(I^{\otimes 3}\otimes\Phi^{-1})(\mathrm{id}^{\otimes 4}\otimes\Delta((I^{\otimes 2}\otimes\Phi)))](\Delta^{\otimes 3}(\Phi))[(\Delta\otimes\mathrm{id}^{\otimes 4}(\Phi\otimes I^{\otimes 2}))(\Phi^{-1}\otimes I^{\otimes 3})] = \\ &= (I^{\otimes 3}\otimes\Phi^{-1})(\Delta\otimes\mathrm{id}^{\otimes 2}\otimes\Delta((I\otimes\Phi)(\mathrm{id}\otimes\Delta\otimes\mathrm{id}(\Phi))(\Phi\otimes I)))(\Phi^{-1}\otimes I^{\otimes 3}) = \\ &= (I^{\otimes 3}\otimes\Phi^{-1})(\Delta\otimes\mathrm{id}^{\otimes 2}\otimes\Delta((\mathrm{id}\otimes\mathrm{id}\otimes\Delta(\Phi))(\Delta\otimes\mathrm{id}\otimes\mathrm{id}(\Phi))))(\Phi^{-1}\otimes I^{\otimes 3}) = \\ &= ((\Delta\otimes\mathrm{id})^{\otimes 2}((\mathrm{id}^{\otimes 2}\otimes\Delta(\Phi))))(I^{\otimes 3}\otimes\Phi^{-1})(\Phi^{-1}\otimes I^{\otimes 3})((\mathrm{id}\otimes\Delta)^{\otimes 2}(\Delta\otimes\mathrm{id}^{\otimes 2}(\Phi)))\end{aligned}$$

Now, we apply ϕ to r written as in (1.3.9). We obtain the following identity simply using the definition of ϕ:

$$\phi(r) = \phi((\Delta \otimes \mathrm{id}^{\otimes 4}(\Phi_{125}))(\Phi^{-1} \otimes I^{\otimes 3})(I^{\otimes 3} \otimes \Phi^{-1})(\mathrm{id}^{\otimes 4} \otimes \Delta(\Phi_{145})))$$

We rewrite the cocycle relation, putting I at the positions 4 and 5:

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta(\Phi))_{1236}(\Delta \otimes \mathrm{id}^{\otimes 4}(\Phi_{125})) = \Phi_{236}(\mathrm{id} \otimes \Delta \otimes \mathrm{id}^{\otimes 3}(\Phi_{125}))\Phi_{123}$$

Multiplying both side on the left by $(\mathrm{id} \otimes \mathrm{id} \otimes \Delta(\Phi^{-1}))_{1236}$ and on the right by Φ_{123}^{-1} we get:

$$\begin{aligned}(\Delta \otimes \Delta(\Delta(I)))_{1236}((\Delta \otimes \mathrm{id}^{\otimes 4}(\Phi_{125})))\Phi_{123}^{-1} =\\ = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta(\Phi))_{1236}^{-1}\Phi_{236}(\mathrm{id} \otimes \Delta \otimes \mathrm{id}^{\otimes 3}(\Phi_{125}))(\mathrm{id} \otimes \Delta(\Delta(I)) \otimes I^{\otimes 3})\end{aligned}$$

We can also rewrite the cocycle relation putting I at the positions 2 and 3:

$$\begin{aligned}(I^{\otimes 3} \otimes \Phi^{-1})((\mathrm{id}^{\otimes 4} \otimes \Delta(\Phi_{145})))(\Delta \otimes \Delta(\Delta(I)))_{1456} =\\ = (I^{\otimes 3} \otimes (\Delta \otimes \mathrm{id}(\Delta(I))))((\mathrm{id}^{\otimes 3} \otimes \Delta \otimes \mathrm{id}(\Phi_{145})))\Phi_{145}(\Delta \otimes \mathrm{id} \otimes \mathrm{id}(\Phi^{-1}))_{1456}\end{aligned}$$

Proceeding as in **(b)** of Remark 1.2.2 we have:

$$((\Delta \otimes \mathrm{id}^{\otimes 4}(\Phi_{125})))\Phi_{123}^{-1} = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta(\Phi))_{1236}^{-1}\Phi_{236}(\mathrm{id} \otimes \Delta \otimes \mathrm{id}^{\otimes 3}(\Phi_{125}))$$

and:

$$\begin{aligned}&(I^{\otimes 3} \otimes \Phi^{-1})((\mathrm{id}^{\otimes 4} \otimes \Delta(\Phi_{145}))) =\\ &= ((\mathrm{id}^{\otimes 3} \otimes \Delta \otimes \mathrm{id}(\Phi_{145})))\Phi_{145}(\Delta \otimes \mathrm{id} \otimes \mathrm{id}(\Phi^{-1}))_{1456}\end{aligned}$$

We use the previous two identities in the following calculation:

$$\begin{aligned}\phi(r) &= \phi((\Delta \otimes \mathrm{id}^{\otimes 4}(\Phi_{125}))(\Phi^{-1} \otimes I^{\otimes 3})(I^{\otimes 3} \otimes \Phi^{-1})(\mathrm{id}^{\otimes 4} \otimes \Delta(\Phi_{145}))) =\\ &= \phi((\mathrm{id} \otimes \mathrm{id} \otimes \Delta(\Phi))_{1236}\Phi_{236}(\mathrm{id} \otimes \Delta \otimes \mathrm{id}^{\otimes 3}(\Phi_{125}))\cdot\\ &\cdot (\mathrm{id}^{\otimes 3} \otimes \Delta \otimes \mathrm{id}(\Phi_{145}))\Phi_{145}(\Delta \otimes \mathrm{id} \otimes \mathrm{id}(\Phi^{-1}))_{1456})\end{aligned}$$

Using the definition of ϕ we get the desired result:

$$\begin{aligned}\phi(r) &= p_i x_j x_k x_l p_{j_{(1)}} \beta S(y_{k_{(1)}} y_l p_{j_{(2)}})\alpha y_{k_{(2)}} z_l q_j \otimes\\ &\otimes q_i x_i y_{j_{(1)}} \beta S(r_{i_{(1)}} y_i y_{j_{(2)}})\alpha r_{i_{(2)}} z_i z_j z_k r_j =\\ &= \varepsilon(p_j)\varepsilon(y_k) p_i x_j x_k q_j \otimes \varepsilon(r_i)\varepsilon(y_j) q_i z_j z_k r_j =\\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon(\Phi^{-1}))(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}(\Phi))(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}(\Phi))\cdot\\ &\cdot (\varepsilon \otimes \mathrm{id} \otimes \mathrm{id}(\Phi^{-1})) = \Delta(I)\end{aligned}$$

□

Lemma 1.3.5. *Let B be an algebra, p an idempotent in B, $f : A \to B$ a homomorphism and $g : A \to B$ an anti-homomorphism with $f(I) = g(I) = p$, and $\rho, \sigma \in (p, p)$ such that:*

$$\begin{aligned}g(a_{(1)})\rho f(a_{(2)}) = \varepsilon(a)\rho\\ f(a_{(1)})\sigma g(a_{(2)}) = \varepsilon(a)\sigma\end{aligned} \tag{1.3.10}$$

where $a \in A$. Moreover,

$$\sum_i f(x_i)\sigma g(y_i)\rho f(z_i) = p \tag{1.3.11}$$

$$\sum_j g(p_j)\rho f(q_j)\sigma g(r_j) = p \tag{1.3.12}$$

In addition, we have an idempotent $q \in B$, $\overline{\rho}, \overline{\sigma} \in qBp$ and an anti-homomorphism $\overline{g} : A \to B$ with $\overline{g}(I) = q$ also satisfying (1.3.10) - (1.3.12) *(but in* (1.3.12) *q replaces p).*

Then there exists exactly one element $F \in B$, partially invertible with $D(F) = p$, $R(F) = q$, such that:

$$\begin{aligned} F\rho &= \overline{\rho} \\ \overline{\sigma}F &= \sigma \\ \overline{g}(a) &= Fg(a)F^{-1} \end{aligned} \tag{1.3.13}$$

As a consequence, we have $F = \sum_i \overline{g}(p_i)\overline{\rho} f(q_i)\sigma g(r_i)$ and $F^{-1} = \sum_i g(p_i)\rho f(q_i)\overline{\sigma}\, \overline{g}(r_i)$.

Proof. Let F be $\sum_i \overline{g}(p_i)\overline{\rho} f(q_i)\sigma g(r_i)$. We apply the map $V : A^{\otimes 3} \to B$, $V(b \otimes c \otimes d) = \overline{g}(b)\overline{\rho} f(c)\sigma g(d)$, respectively to $(\Delta \otimes \mathrm{id}(\Delta(a)))\Phi^{-1}$ and $\Phi^{-1}(\mathrm{id} \otimes \Delta(\Delta(a)))$, obtaining $Fg(a) = \overline{g}(a)F$. In the same way, we apply the map $V : A^{\otimes 4} \to B$ such that:

$$V(b \otimes c \otimes d \otimes e) = \overline{g}(b)\overline{\rho} f(c)\sigma g(d)\rho f(e)$$

to both sides of the identity:

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta(\Phi))(\Delta \otimes \mathrm{id} \otimes \mathrm{id}(\Phi))(\Phi^{-1} \otimes I) = (I \otimes \Phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id}(\Phi))$$

From the LHS we obtain $F\rho$, and from the RHS $\overline{\rho}$. In the same way for σ. It remains to prove the uniqueness of F. Suppose that F satisfies the first three identities of (1.3.13). Then:

$$\begin{aligned} F = Fp = \sum_j Fg(p_j)\rho f(q_j)\sigma g(r_j) = \sum_j \overline{g}(p_j)F\rho f(q_j)\sigma g(r_j) = \\ = \sum_j \overline{g}(p_j)\overline{\rho} f(q_j)\sigma g(r_j) \end{aligned}$$

□

At this point we are able to prove the Theorem 1.3.3:

Proof. It is straightforward if we apply the previous Lemma to $B = A \otimes A$, $p = \Delta(I)$, $q = S \otimes S(\Delta^{\mathrm{op}}(I))$, $f = \Delta$, $g = \Delta \circ S$, $\rho = \Delta(\alpha)$, $\sigma = \Delta(\beta)$, $\overline{g} = S \otimes S \circ \Delta^{\mathrm{op}}$, $\overline{\rho} = \gamma$, $\overline{\sigma} = \delta$. □

We now ask what does it happen if A is coassociative, or in other terms $\Phi = I \otimes I \otimes I$. In this case we do not recover the classical anticomultiplicativity property of the antipode, as we can see in the next:

Proposition 1.3.6. *The following identities are equivalent:*

$$\text{(i) } \Delta(I) = I \otimes I$$
$$\text{(ii) } \Delta \circ S = (S \otimes S) \circ \Delta^{\mathrm{op}}$$

Proof. (i)⇒(ii) is the ordinary case. Let us prove (ii)⇒(i).

$$\begin{aligned}
\Delta(I) &= \Delta(I)(I\otimes I) = \\
&= \Delta(I)\varepsilon(I_{(1)})I_{(2)}\otimes I = \Delta(\varepsilon(I_{(1)})I)(I_{(2)}\otimes I) = \\
&= \Delta(I_{(1)(1)}S(I_{(1)(2)}))(I_{(2)}\otimes I) = \\
&= \Delta(I_{(1)})\Delta(S(I_{(2)(1)}))(I_{(2)(2)}\otimes I) = \\
&= (I_{(1)(1)}\otimes I_{(1)(2)})(S(I_{(2)(1)})_{(1)}\otimes S(I_{(2)(1)})_{(2)})(I_{(2)(2)}\otimes I) = \\
&= I_{(1)(1)}S(I_{(2)(1)(2)})I_{(2)(2)}\otimes I_{(1)(2)}S(I_{(2)(1)(1)}) = \\
&= I_{(1)(1)}S(I_{(2)(2)(1)})I_{(2)(2)(2)}\otimes I_{(1)(2)}S(I_{(2)(1)}) = \\
&= I_{(1)(1)}\varepsilon(I_{(2)(2)})\otimes I_{(1)(2)}S(I_{(2)(1)}) = \\
&= I_{(1)(1)}\otimes I_{(1)(2)}S(I_{(2)}) = I_{(1)}\otimes I_{(2)(1)}S(I_{(2)(2)}) = \\
&= I_{(1)}\otimes\varepsilon(I_{(2)})I = \varepsilon(I_{(2)})I_{(1)}\otimes I = I\otimes I
\end{aligned}$$

□

Finally, we want to shed light on the structure of the dual algebra $\widehat{A}$. We recall the definition of the structure maps on $\widehat{A}$:

$$\begin{aligned}
\langle\phi\psi|a\rangle &:= \langle\phi\otimes\psi|\Delta(a)\rangle \\
\left\langle\widehat{I}|a\right\rangle &:= \varepsilon(a) \\
\left\langle\widehat{\Delta}(\phi)|a\otimes b\right\rangle &:= \langle\phi|ab\rangle \\
\widehat{\varepsilon}(\phi) &:= \langle\phi|I\rangle \\
\left\langle\widehat{S}(\phi)|a\right\rangle &:= \langle\phi|S(a)\rangle \\
\langle\phi^*|a\rangle &:= \overline{\langle\phi|a^*\rangle}
\end{aligned} \tag{1.3.14}$$

where $\phi,\psi\in\widehat{A}$, $a,b\in A$ and $\langle\cdot|\cdot\rangle$ denotes the dual pairing $\widehat{A}\otimes A\to\mathbb{C}$. Using the structure maps it is easy to see that:

$$\widehat{\Delta}(\phi)^* = \widehat{\Delta}'(\phi^*)\ \ \forall\phi\in\widehat{A}$$

Moreover, $\widehat{A}$ is a non-associative algebra, with a coassociative and unital coproduct and a non-multiplicative counit. The involution * is not antimultiplicative.

1.4 Weak quasi Hopf algebras: representation theory

In this section we will deal with the representation theory of a semisimple weak quasi Hopf algebra. We restrict to the semisimple case since in this situation talking about representation categories does not cause any kind of categorical issues. Moreover, we will consider only finite-dimensional representations. A representation of a weak quasi Hopf algebra A is a pair (V,π_V) where V is a finite-dimensional complex linear space and $\pi_V : A\to\mathrm{End}(V)$ an algebra homomorphism. We will indicate with $\mathrm{Rep}\,A$ the category whose objects are the representations of A and the arrows between two objects (π_V,V) and $(\pi_{V'},V')$ the subspace (V,V') of $\mathcal{L}(V,V')$ whose elements are the A-linear maps. This linear category can be made into a (non-strict) tensor category following, for example, [10].

We define $\times$ on representations:

$$\begin{aligned} \pi_{V\times W} &:= (\pi_V \otimes \pi_W)\circ\Delta \\ V\times W &:= \pi_{V\times W}(I)(V\otimes W) \end{aligned} \tag{1.4.1}$$

and then on A-linear morphisms:

$$f\times g := (f\otimes g)\circ\pi_{V\times W}(I) \tag{1.4.2}$$

where $f\in(V,V')$, $g\in(W,W')$.

If A is a weak quasi Hopf *-algebra, we restrict to the representations (π_V, V) such that $\pi_V(a^*)=\pi_V(a)^*$ and V is endowed with a hermitian form. We will call them **-representations*. The category whose objects are these representations will be called $\mathrm{Rep}_h\, A$. Finally, if A is a weak quasi Hopf C*-algebra and $\Omega\in A\otimes A$ is a positive element, we restrict to the *-representations π_H on the Hilbert spaces H. The category whose objects are these representations will be called $\mathrm{Rep}_H\, A$.

Our goal is to prove the following:

Theorem 1.4.1. *Let A be a weak quasi Hopf algebra. Then* $\mathrm{Rep}\, A$ *is a tensor category. If A is a weak quasi Hopf *-algebra,* $\mathrm{Rep}_h\, A$ *is a *-tensor category. If A is a C*-algebra and Ω a positive element in $A\otimes A$, then* $\mathrm{Rep}_H\, A$ *is a C*-tensor category. If the antipode S commute with the involution * in A,* $\mathrm{Rep}_H\, A$ *is rigid. If A is braided, then* $\mathrm{Rep}\, A$ *is braided. If A is endowed with the involution *, then the braiding is unitary.*

It is quite easy to see that the unit object exists and it is the trivial representation $\mathbf{1}=(\mathbb{C},\varepsilon)$, where A acts on the complex numbers in the following way

$$a\cdot 1=\varepsilon(a)1$$

We define the associativity costraints $a_{U,V,W}:(U\times V)\times W\to U\times(V\times W)$ and $a^{-1}_{U,V,W}: U\times(V\times W)\to(U\times V)\times W$. We will take into account that:

$$\begin{aligned} (u\otimes v)\otimes w &= \Delta\otimes\mathrm{id}(\Delta(I))((u\otimes v)\otimes w)\in(U\times V)\times W \\ u\otimes(v\otimes w) &= \mathrm{id}\otimes\Delta(\Delta(I))(u\otimes(v\otimes w))\in U\times(V\times W) \end{aligned}$$

Hence:

$$\begin{aligned} a_{U,V,W}((u\otimes v)\otimes w) &= \Phi((u\otimes v)\otimes w) \\ a^{-1}_{U,V,W}(u\otimes(v\otimes w)) &= \Phi^{-1}(u\otimes(v\otimes w)) \end{aligned}$$

while the unit costraints are:

$$r_V(v\otimes 1)=v \text{ and } l_V(1\otimes v)=v$$

Using (1.2.2) it is quite easy to see that the associativity costraints a and a^{-1} are well-defined. Furthermore, pentagon axiom and triangle axiom are direct consequence of (1.2.2) - (1.2.5). So $\mathrm{Rep}\, A$ is a tensor category.

The next step is to prove that $\mathrm{Rep}_h\, A$ is a *-tensor category when A is a weak quasi Hopf *-algebra. In this setting, the main problem is to prove that the product of two *-representations is still a *-representation. It is crucial the existence of Ω.

In fact, let (π_V, V) and (π_W, W) be two *-representations. We put on $V \times W$ the following hermitian form:

$$(v_1 \otimes w_1, v_2 \otimes w_2) := (v_1 \otimes w_1, \Omega(v_2 \otimes w_2))_p$$

where the form on the right hand side is the product form. Then:

Proposition 1.4.2. *Endowing $V \times W$ with $(\cdot,\cdot)$ defined above, we get that $\pi_{V\times W}$ is a *-representation.*

Proof. First of all, the form $(\cdot,\cdot)$ is hermitian because of the self-adjointness of Ω. Next:

$$\begin{aligned}(v_1 \otimes w_1, \pi_{V\times W}(a)^*(v_2 \otimes w_2)) &= (\pi_{V\times W}(a)(v_1 \otimes w_1), v_2 \otimes w_2) = \\ &= (\Delta(a)(v_1 \otimes w_1), \Omega(v_2 \otimes w_2))_p = (v_1 \otimes w_1, \Delta(a)^*\Omega(v_2 \otimes w_2))_p = \\ &= (v_1 \otimes w_1, \Omega\Delta(a^*)(v_2 \otimes w_2))_p = (v_1 \otimes w_1, \pi_{V\times W}(a^*)(v_2 \otimes w_2))\end{aligned}$$

□

What does it happen if we have n *-representations $V_1, \dots, V_n$? Since the coproduct is not coassociative, the category is not strict. So, it will be necessary to use parentheses to indicate the order of the products. According to the order that occurs, we define

$$\Delta^{(n)} = (\mathrm{id} \otimes \dots \otimes \mathrm{id} \otimes \Delta \otimes \mathrm{id} \otimes \dots \otimes \mathrm{id}) \circ \dots \circ \Delta$$

with $\Delta^{(1)} = \mathrm{id}$ and $\Delta^{(2)} = \Delta$.

Unlike the coassociative case, $\Delta^{(n)}$ depend on the positions of Δ's into the composition. When it will be possible, we will avoid the use of parentheses and we will consider $\Delta^{(i)}$ fixed, with $i < n$. If $\Delta^{(n)} = \Delta^{(r)} \otimes \Delta^{(s)} \circ \Delta$, then:

$$\Omega^{(n)} := (\Omega^{(r)} \otimes \Omega^{(s)})(\Delta^{(r)} \otimes \Delta^{(s)})(\Omega) \in A^{\otimes n} \tag{1.4.3}$$

with $\Omega^{(1)} = I$ and $\Omega^{(2)} = \Omega$. In other words, the definition of $\Omega^{(n)}$ depends on the $\Delta^{(n)}$, which in turn is chosen according to the order of the products.

Now we can define the hermitian form on $V_1 \times \dots \times V_n$:

$$(v_1 \otimes \dots \otimes v_n, v_1' \otimes \dots \otimes v_n') := (v_1 \otimes \dots \otimes v_n, \Omega^{(n)}(v_1' \otimes \dots \otimes v_n'))_p \tag{1.4.4}$$

As before:

Proposition 1.4.3. *Endowing $V_1 \times \dots \times V_n$ with the hermitian form* (1.4.4)*, we get that $\pi_{V_1\times\dots\times V_n}$ is a *-representation.*

Remark 1.4.4. The relation $\Delta(a)^*\Omega = \Omega\Delta(a^*)$ can be extended to $\Delta^{(n)}$ and $\Omega^{(n)}$, or more explicitly:

$$\Delta^{(n)}(a)^*\Omega^{(n)} = \Omega^{(n)}\Delta^{(n)}(a^*)$$

$\forall a \in A$ and $\forall n \in \mathbb{N}$. We prove it by induction.

$$\begin{aligned}\Delta^{(n)}(a)^*\Omega^{(n)} &= (\Delta^{(r)} \otimes \Delta^{(s)}(\Delta(a)))^*(\Omega^{(r)} \otimes \Omega^{(s)})(\Delta^{(r)} \otimes \Delta^{(s)}(\Omega)) = \\ &= (\Delta^{(r)}(a_{(1)})^*\Omega^{(r)} \otimes \Delta^{(s)}(a_{(2)})^*\Omega^{(s)})(\Delta^{(r)} \otimes \Delta^{(s)}(\Omega)) = \\ &= (\Omega^{(r)}\Delta^{(r)}(a^*_{(1)}) \otimes \Omega^{(s)}\Delta^{(s)}(a^*_{(2)}))(\Delta^{(r)} \otimes \Delta^{(s)}(\Omega)) = \\ &= (\Omega^{(r)} \otimes \Omega^{(s)})(\Delta^{(r)} \otimes \Delta^{(s)}(\Delta(a)^*\Omega)) = \\ &= (\Omega^{(r)} \otimes \Omega^{(s)})(\Delta^{(r)} \otimes \Delta^{(s)}(\Omega))(\Delta^{(r)} \otimes \Delta^{(s)}(\Delta(a^*))) = \\ &= \Omega^{(n)}\Delta^{(n)}(a^*)\end{aligned}$$

The following result completes the proof that $\mathrm{Rep}_h A$ is a *-tensor category:

Proposition 1.4.5. *The associativity costraint $a_{U,V,W}$ satisfies the following identity:*

$$a^*_{U,V,W} = a^{-1}_{U,V,W}$$

for all U, V, W in $\mathrm{Rep}_h A$.

Proof. It is merely a consequence of (1.2.14):

$$\begin{aligned}
&((u_1 \otimes v_1) \otimes w_1, a^*_{U,V,W}(u_2 \otimes (v_2 \otimes w_2))) = \\
&= (a_{U,V,W}((u_1 \otimes v_1) \otimes w_1), \Omega^{(3)}(u_2 \otimes (v_2 \otimes w_2)))_p = \\
&= (\Phi((u_1 \otimes v_1) \otimes w_1), (I \otimes \Omega)(\mathrm{id} \otimes \Delta(\Omega))(u_2 \otimes (v_2 \otimes w_2)))_p = \\
&= ((u_1 \otimes v_1) \otimes w_1, \Phi^*(I \otimes \Omega)(\mathrm{id} \otimes \Delta(\Omega))(u_2 \otimes (v_2 \otimes w_2)))_p = \\
&= ((u_1 \otimes v_1) \otimes w_1, (\Omega \otimes I)(\Delta \otimes \mathrm{id}(\Omega))\Phi^{-1}(u_2 \otimes (v_2 \otimes w_2)))_p = \\
&= ((u_1 \otimes v_1) \otimes w_1, a^{-1}_{U,V,W}(u_2 \otimes (v_2 \otimes w_2)))
\end{aligned}$$

□

It remains to talk about the C* case. If A is a C*-algebra and Ω a positive element in $A \otimes A$, it is very easy to prove that $\mathrm{Rep}_H A$ is a C*-tensor category. We just need to check the following:

Proposition 1.4.6. *Suppose that Ω is a positive element in $A \otimes A$. Then $\Omega^{(n)}$ is positive $\forall n \in \mathbb{N}$ and for every $\Delta^{(n)}$.*

Proof. We suppose that $\Delta^{(n)}$ is equal to $\Delta^{(r)} \otimes \Delta^{(s)} \circ \Delta$, and we proceed by induction. Since Ω and $\Omega^{(r)}$ are positive for all $r < n$, we have $\Omega = T_1^2$, $\Omega^{(r)} = T_2^2$ and $\Omega^{(s)} = T_3^2$, where T_1, T_2 and T_3 are self-adjoint. We will obtain the result after the following calculation:

$$\begin{aligned}
\Omega^{(n)} &= (\Omega^{(r)} \otimes \Omega^{(s)})(\Delta^{(r)} \otimes \Delta^{(s)}(\Delta(T_1^2))) = \\
&= (\Delta^{(r)} \otimes \Delta^{(s)}(T_1)^*)(\Omega^{(r)} \otimes \Omega^{(s)})(\Delta^{(r)} \otimes \Delta^{(s)}(T_1)) = \\
&= (\Delta^{(r)} \otimes \Delta^{(s)}(T_1)^*)(T_2 \otimes T_3)(T_2 \otimes T_3)(\Delta^{(r)} \otimes \Delta^{(s)}(T_1)) = B^*B
\end{aligned}$$

where $B = (T_2 \otimes T_3)(\Delta^{(r)} \otimes \Delta^{(s)}(T_1))$. □

Next, we focus on the rigidity of $\mathrm{Rep}_H A$ under the assumption:

$$S \circ^* =^* \circ S$$

Let V be a Hilbert space and $\overline{V}$ its conjugate space. If (π_V, V) is a *-representation of A, then $(\pi_{\overline{V}}, \overline{V})$ is also a *-representation of A, with the following action of $a \in A$ on $\overline{v} \in \overline{V}$:

$$a\overline{v} = \overline{S(a)^* v} \tag{1.4.5}$$

The next calculation shows that $(\pi_{\overline{V}}, \overline{V})$ is a *-representation of A:

$$\begin{aligned}
(\overline{v}, \pi_{\overline{V}}(a)^*(\overline{w})) &= (\pi_{\overline{V}}(a)(\overline{v}), \overline{w}) = (\overline{S(a)^* v}, \overline{w}) = \\
&= (w, S(a)^* v) = (S(a)w, v) = \\
&= (\overline{v}, \overline{S(a)w}) = (\overline{v}, \pi_{\overline{V}}(a^*)\overline{w})
\end{aligned}$$

Now we prove that $\overline{V}$ is the conjugate object of V in $\mathrm{Rep}_H A$. In order to prove this fact, we introduce the conjugation maps $r = r_V^{(\alpha)} \in (\mathbf{1}, \overline{V} \times V)$ and $\overline{r} = \overline{r}_V^{(\beta)} \in (\mathbf{1}, V \times \overline{V})$:

$$r_V^{(\alpha)}(1) = \Omega^{-1} \sum_{i=1}^{n} \overline{e_i} \otimes \alpha^* e_i \tag{1.4.6}$$

$$\overline{r}_V^{(\beta)}(1) = \sum_{i=1}^{n} \beta e_i \otimes \overline{e_i} \tag{1.4.7}$$

We have:

Proposition 1.4.7. *The map $r^*(\overline{v} \otimes w) = (v, \alpha w)$ is the adjoint map of r. Moreover, r and $\overline{r}$ defined respectively as in* (1.4.6) *and* (1.4.7) *are morphisms in* $\mathrm{Rep}_H A$.

Proof. We start with r^*:

$$\begin{aligned} r^*(\overline{v} \otimes w) &= (r(1), \Omega(\overline{v} \otimes w))_p = (\Omega^{-1}(\overline{e_i} \otimes \alpha^* e_i), \Omega(\overline{v} \otimes w))_p = \\ &= (\overline{e_i}, \overline{v})(\alpha^* e_i, w) = (v, e_i)(e_i, \alpha w) = (v, \alpha w) \end{aligned}$$

Of course r is a morphism if and only if r^* is a morphism. Therefore it is sufficient to prove that $r^* \in (\overline{V} \times V, \mathbf{1})$:

$$\begin{aligned} r^*(\Delta(a)(\overline{v} \otimes w)) &= (S(a_{(1)})^* v, \alpha a_{(2)} w) = (v, S(a_{(1)}) \alpha a_{(2)} w) = \\ &= \varepsilon(a)(v, \alpha w) = \varepsilon(a) r^*(\overline{v} \otimes w) \end{aligned}$$

using (1.2.7). In order to prove the second statement, we use the well-known existence of the linear space isomorphism $\lambda : V \otimes \overline{V} \to \mathcal{B}(V)$, such that:

$$\lambda(v \otimes \overline{w})(u) = (w, u)v$$

So it will be sufficient to prove that $\lambda(\Delta(a)\overline{r}(1))(u) = \lambda(\varepsilon(a)\overline{r}(1))(u)\ \forall u \in V$.

$$\begin{aligned} \lambda(\Delta(a)\overline{r}(1))(u) &= \lambda\left(\sum_{i=1}^{n} a_{(1)}\beta e_i \otimes a_{(2)}\overline{e_i}\right)(u) = \\ = \sum_{i=1}^{n} (e_i, S(a_{(2)})u) a_{(1)}\beta e_i &= a_{(1)}\beta S(a_{(2)})u = \varepsilon(a)\beta u = \\ &= \lambda\left(\varepsilon(a)\sum_{i=1}^{n} \beta e_i \otimes \overline{e_i}\right)(u) = \lambda(\varepsilon(a)\overline{r}(1))(u) \end{aligned}$$

using (1.2.8). □

Proposition 1.4.8. *The morphisms r and $\overline{r}$ satisfy the conjugate equations* (1.1.10) *and* (1.1.11)*. Hence $\overline{V}$ is the conjugate object of V in* $\mathrm{Rep}_H A$.

Proof. First of all, it is easy to see that the conjugate identities can be proved putting the standard tensor product $\otimes$ in place of the truncated product $\times$. It is merely a consequence of (1.2.2). We start proving the equation (1.1.10):

$$r^* \otimes \mathrm{id} \circ a^{-1}_{\overline{V},V,\overline{V}} \circ \mathrm{id} \otimes \overline{r}(v^*) = r^* \otimes \mathrm{id}\left(a^{-1}_{\overline{V},V,\overline{V}}\left(\overline{v} \otimes \left(\sum_{i=1}^n \beta e_i \otimes \overline{e_i}\right)\right)\right) =$$

$$= \sum_{i=1}^n r^* \otimes \mathrm{id}(\Phi^{-1}(\overline{v} \otimes \beta e_i \otimes \overline{e_i})) = \sum_{i,j} r^* \otimes \mathrm{id}(p_j\overline{v} \otimes q_j\beta e_i \otimes r_j\overline{e_i}) =$$

$$= \sum_{i,j}(v, S(p_j)\alpha q_j \beta e_i) r_j \overline{e_i} = \overline{\sum_{i,j}(e_i, \beta^* q_j^* \alpha^* S(p_j)^* v) S(r_j)^* e_i} =$$

$$= \overline{\sum_j S(r_j)^* \beta^* q_j^* \alpha^* S(p_j)^* v} = \overline{\left(\sum_j S(p_j)\alpha q_j \beta S(r_j)\right)^* v} = \overline{v}$$

We pass to the second identity (1.1.11):

$$\mathrm{id} \otimes r^* \circ a_{V,\overline{V},V} \circ \overline{r} \otimes \mathrm{id}(v) = \sum_{i=1}^n \mathrm{id} \otimes r^*(\Phi(\beta e_i \otimes \overline{e_i} \otimes v)) =$$

$$= \sum_{i,j} \mathrm{id} \otimes r^*(x_j \beta e_i \otimes y_j \overline{e_i} \otimes z_j v) = \sum_{i,j}(e_i, S(y_j)\alpha z_j v) x_j \beta e_i =$$

$$= \left(\sum_j x_j \beta S(y_j)\alpha z_j\right) v = v$$

In both cases we have used (1.2.9). □

Remark 1.4.9. We point out that $\mathrm{Rep}_h\, A$ is also a rigid category, but in this case we can choose r_V and $\overline{r}_V$ as in (1.4.6) and (1.4.7) only when the hermitian form on V is an inner product. Otherwise, we have to introduce slight modifications on r and $\overline{r}$.

Focusing on the proofs of the last two results, we can see that all the properties of the antipode S are crucial in order to prove the rigidity of the category. We wonder if it is possible to reverse this statement in some sense. Suppose that A is a (semisimple) weak quasi bialgebra, and S is a *-antiautomorphism. We define the representation on $\overline{V}$ as in (1.4.5) using S. Then:

Proposition 1.4.10. *Let A be a quasi C*-bialgebra and* (1.4.5) *the representation of A on $\overline{V}$, where S is a *-antiautomorphism of A. Then (S, α, β) is an antipode if and only if $\mathrm{Rep}_H\, A$ is rigid with conjugate object $\overline{V}$ and conjugate morphisms $r_V^{(\alpha)}$ and $\overline{r}_V^{(\beta)}$ respectively as in* (1.4.6) *and* (1.4.7)*.*

Proof. We have just proved that if (S, α, β) is an antipode commuting with the involution, then $\mathrm{Rep}_H\, A$ is rigid, with $r_V^{(\alpha)}$ and $\overline{r}_V^{(\beta)}$ as in (1.4.6) and (1.4.7). Conversely, we must first notice that, since A is a semisimple algebra, then there exists a faithful representation V. Suppose now that the rigidity of the category $\mathrm{Rep}_H\, A$ is given by the conjugate morphisms $r = r_V^{(\alpha)}$ and $\overline{r} = \overline{r}_V^{(\beta)}$ defined resp. as in (1.4.6) and (1.4.7). Then, $\forall v, w \in V$:

$$(v, S(a_{(1)})\alpha a_{(2)} w) = r^*(\Delta(a)(\overline{v} \otimes w)) = r^*(\varepsilon(a)(v^* \otimes w)) = (v, \varepsilon(a)\alpha w)$$

which implies (1.2.7). We pass to $\overline{r}$, recalling that $\lambda : V \otimes \overline{V} \to \mathcal{B}(V)$ is an isomorphism of linear spaces such that:

$$\lambda(v \otimes \overline{w})(u) = (w, u)v$$

So:

$$\varepsilon(a)\beta v=\varepsilon(a)\sum_{i=1}^{n}(e_i,v)\beta e_i=\lambda\left(\varepsilon(a)\sum_{i=1}^{n}\beta e_i\otimes\overline{e_i}\right)(v)=$$
$$=\lambda(\varepsilon(a)\overline{r}(1))(v)=\lambda(\Delta(a)\overline{r}(1))(v)=\lambda\left(\sum_{i=1}^{n}a_{(1)}\beta e_i\otimes a_{(2)}\overline{e_i}\right)(v)=$$
$$=\sum_{i=1}^{n}(e_i,S(a_{(2)})v)a_{(1)}\beta e_i=(a_{(1)}\beta S(a_{(2)}))v$$

We obtain (1.2.8) using the faithfulness of V as representation of A. Next, developing the calculations as in the proof of the Prop. 1.4.8 and proceeding as for (1.2.7) and (1.2.8) we get that (S,α,β) satisfies (1.2.9). □

In order to conclude the proof of the Theorem 1.4.1, we prove that if A is endowed with a R-matrix R satisfying the identities (1.2.15) - (1.2.18), then $\mathrm{Rep}\,A$ is braided.

Proposition 1.4.11. *If (π_U,U) and (π_V,V) are two representations of A, then:*

$$c_{U,V}:=\Sigma(\pi_1\otimes\pi_2)(R)\in(U\times V,V\times U) \tag{1.4.8}$$

are morphisms such that, for all $f\in(U,U')$ and $g\in(V,V')$:

$$c_{U',V'}\circ f\times g=g\times f\circ c_{U,V} \tag{1.4.9}$$

and:

$$\begin{aligned}c_{U,V\times W}&=a^{-1}_{V,W,U}\circ\mathrm{id}_V\times c_{U,W}\circ a_{V,U,W}\circ c_{U,V}\times\mathrm{id}_W\circ a^{-1}_{U,V,W}\\ c_{U\times V,W}&=a_{W,U,V}\circ c_{U,W}\times\mathrm{id}_V\circ a^{-1}_{U,W,V}\circ\mathrm{id}_U\times c_{V,W}\circ a_{U,V,W}\end{aligned} \tag{1.4.10}$$

Finally, if A is a weak quasi Hopf*-algebra, $c_{U,V}$ are unitary morphisms $\forall U,V$.

Proof. It is straightforward to see that $c_{U,V}$ are morphisms in $\mathrm{Rep}\,A$ using (1.2.15) and (1.2.16). The identity (1.4.9) follows after an easy calculation. (1.4.10) are also easy to obtain using (1.2.17) and (1.2.18). Finally, we want to show that in the involutive case $c_{U,V}$ is unitary. It will be trivial after finding the explicit expression of $c^*_{U,V}$:

$$(u_1\otimes v_1,c^*_{U,V}(v_2\otimes u_2))=(c_{U,V}(u_1\otimes v_1),\Omega(v_2\otimes u_2))_p=$$
$$=(\Sigma R(u_1\otimes v_1),\Omega(v_2\otimes u_2))_p=(u_1\otimes v_1,R^*\Sigma\Omega(v_2\otimes u_2))_p=$$
$$=(u_1\otimes v_1,R^*\Omega_{21}(u_2\otimes v_2))_p=(u_1\otimes v_1,\Omega R^{-1}(u_2\otimes v_2))_p=$$
$$=(u_1\otimes v_1,\Omega R^{-1}\Sigma(v_2\otimes u_2))_p=(u_1\otimes v_1,R^{-1}\Sigma(v_2\otimes u_2))$$

using (1.2.19). So $c^*_{U,V}=R^{-1}\Sigma$. □

Collecting results together we obtain the proof of the Theorem 1.4.1.

We close this section giving some additional results about the representation theory of a weak quasi Hopf algebra. We remember that two weak quasi bialgebras A and A' are equivalent if there exists a twist F on A' and an isomorphism $\nu:A\to A'_F$ of weak quasi bialgebras. We are going to prove that equivalent weak quasi bialgebras have equivalent representation theory. We start proving the following result. Let A be a weak quasi bialgebra and $F\in A\otimes A$ a twist. Define:

$$e^{(F)}_{V,W}(v\otimes w)=F^{-1}(v\otimes w)$$

where v and w respectively belong to V and W.

Lemma 1.4.12. *Under the previous hypothesis, the triple* $(\mathrm{id}, \mathrm{id}, e^{(F)})$ *is a tensor equivalence from the tensor category* $\mathrm{Rep}\, A$ *to the tensor category* $\mathrm{Rep}\, A_F$.

Proof. We have to prove that $e^{(F)}$ satisfies the relations (1.1.13) - (1.1.15). Relations (1.1.14) and (1.1.15) are immediate to prove using (1.2.20). Relation (1.1.13) is consequence of the following identity:

$$(\mathrm{id} \otimes \Delta(F^{-1}))F_{23}^{-1}\Phi_F = \Phi(\Delta \otimes \mathrm{id}(F^{-1}))F_{12}^{-1}$$

Finally, it is a tensor equivalence since the tensor functor $(\mathrm{id}, \mathrm{id}, e^{(F^{-1})})$ is the inverse of $(\mathrm{id}, \mathrm{id}, e^{(F)})$. □

Now we can prove the result stated before. Let A and A' such that:

$$\nu : A \xrightarrow{\sim} A'_F$$

is an isomorphism of weak quasi bialgebras. Now, given a A'_F-representation V, we can equip V with a A-module structure:

$$a \cdot v := \nu(a)v \ \ \forall a \in A, \ \forall v \in V$$

In this way we can define a tensor functor $(\nu^*, \mathrm{id}, \mathrm{id})$ from $\mathrm{Rep}\, A'_F$ to $\mathrm{Rep}\, A$ which is the identity on objects and arrows. Composing it with $(\mathrm{id}, \mathrm{id}, e^{(F)})$ we obtain the tensor functor $(\nu^*, \mathrm{id}, e^{(F)})$ which is a tensor equivalence between $\mathrm{Rep}\, A'$ and $\mathrm{Rep}\, A$.

It is possible to give an extension to the braided case. As before, we have the following definition and the following theorem:

Definition 1.4.13. Two braided weak quasi bialgebras $(A, \Delta, \varepsilon, \Phi, R)$ and $(A', \Delta', \varepsilon', \Phi', R')$ are equivalent if there exist a twist F on A' and an isomorphism $\nu : A \longrightarrow A'_F$ of braided quasi bialgebras.

Theorem 1.4.14. *In the previous hypotheses, the tensor functor* $(\nu^*, \mathrm{id}, e^{(F)})$ *is a braided tensor equivalence between the braided tensor categories* $\mathrm{Rep}\, A'$ *and* $\mathrm{Rep}\, A$.

Next, we want to give the categorical counterpart of the anticomultiplicativity relation. It is a well-known result in the category theory, but we shall give an alternative and more concrete proof, making use of what we have seen in the previous section. We set:

$$F = \sum_i f_i \otimes g_i \ \text{ and } F^{-1} = \sum_i \widetilde{f}_i \otimes \widetilde{g}_i$$

Then:

Proposition 1.4.15. *Let A be a weak quasi Hopf*-algebra and V, W two *-representations. Then $\overline{V \times W}$ and $\overline{W} \times \overline{V}$ are isomorphic as *-representations of A.*

Proof. We set the maps $\gamma \in (\overline{W} \times \overline{V}, \overline{V \times W})$ and $\gamma^{-1} \in (\overline{V \times W}, \overline{W} \times \overline{V})$:

$$\gamma(\overline{w} \otimes \overline{v}) = \overline{\Omega^{-1}F^*(v \otimes w)} = \sum_{i,j} \overline{\widetilde{c}_i f_j^* v \otimes \widetilde{d}_i g_j^* w}$$

$$\gamma^{-1}(\overline{v \otimes w}) = \sum_{i,j} \overline{\widetilde{g}_i^{\,*} d_j w} \otimes \overline{\widetilde{f}_i^{\,*} c_j v}$$

We first prove that γ and γ^{-1} are A-linear:

$$\begin{aligned}\gamma(a(\overline{w}\otimes\overline{v})) = \gamma(\overline{S(a_{(1)})^*w}\otimes\overline{S(a_{(2)})^*v}) = \\ = \overline{\Omega^{-1}F^*(S\otimes S(\Delta^{\mathrm{op}}(a))^*)(v\otimes w)} = \overline{\Omega^{-1}\Delta(S(a))^*F^*(v\otimes w)} = \\ = \overline{\Delta(S(a)^*)\Omega^{-1}F^*(v\otimes w)} = a\overline{\Omega^{-1}F^*(v\otimes w)} = a\gamma(\overline{w}\otimes\overline{v})\end{aligned}$$

and

$$\begin{aligned}\gamma^{-1}(a(\overline{v\otimes w})) = \gamma^{-1}(\overline{\Delta(S(a))^*(v\otimes w)}) = \\ = \gamma^{-1}(\overline{(S(a)^*)_{(1)}v\otimes(S(a)^*)_{(2)}w}) = \\ = \sum_{i,j}\overline{\widetilde{g_i}^*d_j(S(a)^*)_{(2)}w}\otimes\overline{\widetilde{f_i}^*c_j(S(a)^*)_{(1)}v}\end{aligned}$$

Since:

$$F_{21}^{*\,-1}\Omega_{21}\Delta^{\mathrm{op}}(S(a)^*) = S\otimes S(\Delta(a))^*F_{21}^{*\,-1}\Omega_{21}$$

we get:

$$\gamma^{-1}(a(\overline{v\otimes w})) = \sum_{i,j}\overline{S(a_{(1)})^*\widetilde{g_i}^*d_jw}\otimes\overline{S(a_{(2)})^*\widetilde{f_i}c_jv} = a\gamma^{-1}(\overline{v\otimes w})$$

It remains to prove that γ^{-1} is the left and right inverse map of γ:

$$\begin{aligned}\gamma(\gamma^{-1}(\overline{v\otimes w})) = \sum_{i,j}\gamma(\overline{\widetilde{g_i}^*d_jw}\otimes\overline{\widetilde{f_i}^*c_jv}) = \\ = \overline{\Omega^{-1}F^*F^{*-1}\Omega(v\otimes w)} = \overline{\Omega^{-1}\Delta(I)^*\Omega(v\otimes w)} = \\ = \overline{\Delta(I)(v\otimes w)} = \overline{v\otimes w}\end{aligned}$$

On the other hand:

$$\begin{aligned}\gamma^{-1}(\gamma(\overline{w}\otimes\overline{v})) = \sum_{i,j}\gamma^{-1}(\overline{\widetilde{c}_if_j^*v\otimes\widetilde{d}_ig_j^*w}) = \\ = \sum_{i,j,h,k}\overline{\widetilde{g_h}^*d_k\widetilde{d}_ig_j^*w}\otimes\overline{\widetilde{f_h}^*c_k\widetilde{c}_if_j^*v} = \\ = \overline{S(I_{(1)})^*w}\otimes\overline{S(I_{(2)})^*v} = \Delta(I)(\overline{w}\otimes\overline{v}) = \overline{w}\otimes\overline{v}\end{aligned}$$

where we use the following calculation:

$$\begin{aligned}\sum_{i,j,h,k}\widetilde{g_h}^*d_k\widetilde{d}_ig_j^*\otimes\widetilde{f_h}^*c_k\widetilde{c}_if_j^*v = F_{21}^{*\,-1}\Omega_{21}\Omega_{21}^{-1}F_{21}^* = \\ = F_{21}^{*\,-1}\Delta^{\mathrm{op}}(I)^*F_{21}^* = F_{21}^{*\,-1}F_{21}^* = S\otimes S(\Delta(I))^*\end{aligned}$$

□

We conclude this section showing a relation between weak (quasi)Hopf algebras and weak (quasi)tensor functor. Before proving it, we need the following:

Definition 1.4.16. Let F be a tensor functor from $\mathrm{Rep}(A)$ to the tensor category Vect of the f.d. linear spaces, such that:

$$F((\pi_V, V)) = V$$

and acting as the identity on the arrows. F is called *forgetful functor*. More generally, if $\mathcal{C}$ is a tensor category and F a tensor functor from $\mathcal{C}$ to Vect, F is called *fiber functor*.

Theorem 1.4.17. *Let A be a weak quasi Hopf algebra. Then the forgetful functor $F : \mathrm{Rep}(A) \to \mathrm{Vect}$ is a weak quasi tensor functor. If A has an involution $(\Omega{\cdot}*)$, F is *-preserving, and if A has a R-matrix, F is braided. Moreover, A is a weak Hopf algebra if and only if F is a weak tensor functor.*

Proof. Proving that F is a weak quasi tensor functor is quite easy. We set:

$$e_{V,W}(v \otimes w) = \Delta(I)(v \otimes w)$$

It is not an isomorphism but only an epimorphism, and the right inverse is the map:

$$e^{-1}_{V,W}(\Delta(I)(v \otimes w)) = \Delta(I)(v \otimes w)$$

which is merely the inclusion:

$$e^{-1}_{V,W} : V \times W \hookrightarrow V \otimes W$$

recalling that $V \times W = \Delta(I)(V \otimes W)$. The assertions about involution and braiding are very easy to prove, so we focus on the last statement. Looking back at the diagram (1.1.13), we have that F is a weak tensor functor if and only if $e_{X,Y}$ and Φ satisfy the identities:

$$F(a_{X,Y,Z}) \circ e_{X\times Y,Z} \circ e_{X,Y} \times \mathrm{id}_{F(Z)} = e_{X,Y\times Z} \circ \mathrm{id}_{F(X)} \times e_{Y,Z} \circ a'_{F(X),F(Y),F(Z)}$$

and

$$e^{-1}_{X,Y} \times \mathrm{id}_{F(Z)} \circ e^{-1}_{X\times Y,Z} \circ F(a^{-1}_{X,Y,Z}) = {a'_{F(X),F(Y),F(Z)}}^{-1} \circ \mathrm{id}_{F(X)} \times e^{-1}_{Y,Z} \circ e^{-1}_{X,Y\times Z}$$

In our situation, $a'_{F(X),F(Y),F(Z)}$ is the trivial associativity costraint in Vect. Using the fact that e has right inverse, we have:

$$\begin{aligned} F(a_{X,Y,Z}) &= e_{X,Y\times Z} \circ \mathrm{id}_{F(Z)} \times e_{X,Y} \circ e^{-1}_{X,Y} \times \mathrm{id}_{F(Z)} \circ e^{-1}_{X\times Y,Z} \\ F(a^{-1}_{X,Y,Z}) &= e_{X\times Y,Z} \circ e_{X,Y} \times \mathrm{id}_{F(Z)} \circ \mathrm{id}_{F(X)} \times e^{-1}_{Y,Z} \circ e^{-1}_{X,Y\times Z} \end{aligned}$$

Using the definition of the natural transformations a and e, the previous identities become:

$$\begin{aligned} \Phi &= (\mathrm{id} \otimes \Delta(\Delta(I)))(I \otimes \Delta(I))(\Delta(I) \otimes I)(\Delta \otimes \mathrm{id}(\Delta(I))) = \\ &= (\mathrm{id} \otimes \Delta(I))(\Delta \otimes \mathrm{id}(\Delta(I))) \end{aligned}$$

and in the same way:

$$\Phi^{-1} = (\Delta \otimes \mathrm{id}(\Delta(I)))(\mathrm{id} \otimes \Delta(\Delta(I)))$$

As a consequence, (1.2.29) and (1.2.30) are immediately satisfied. In order to prove (1.2.31), it is sufficient to see that $a_{X,Y,Z}$ and $a^{-1}_{X,Y,Z}$ must enjoy the pentagon axiom. □

Remark 1.4.18. The notion of weak Hopf algebra and its relation with weak tensor functors will be studied in detail in [16].

1.5 Weak quasi Hopf algebras: Haar measure and semisimplicity

It is well-known [67] in the theory of Hopf algebras that if A is an arbitrary Hopf algebra, then the left integral space has dimension ≤ 1, and if A is a finite-dimensional Hopf algebra, then this dimension is exactly 1. Same statement is true for the right integral space. Moreover, if the left integral space has dimension 1, then the right one has dimension 1 too. The two spaces coincide if and only if A is semisimple. In this case, we deduce the existence of unique idempotent self-adjoint two-sided integral such that $\varepsilon(h) = 1$. We call this element *Haar measure*. The same result can be given for the dual algebra $\widehat{A}$. These facts can be generalized to quasi Hopf algebras and dual quasi Hopf algebras, giving in the last case a suitable definition of integral which could be the extension of the classical integral to a non-associative algebra [13]. In this section, we focus on an analogous result for finite-dimensional weak quasi Hopf algebras. More precisely, we want to prove that if A is a f.d. weak quasi Hopf algebra, then A is semisimple if and only if A has a Haar measure. If we consider the dual algebra $\widehat{A}$, we will prove that it is not possible to extend the result concerning the existence of the integrals available for dual quasi Hopf algebras. In fact, there exists a class of weak quasi Hopf algebras A such that $\widehat{A}$ cannot have integrals.

Definition 1.5.1. (i) An element $l \in A$, $l \neq 0$, is a left integral if $xl = \varepsilon(x)l \ \forall x \in A$.
(ii) An element $r \in A$, $r \neq 0$, is a right integral if $rx = \varepsilon(x)r \ \forall x \in A$.
(iii) An element $h \in A$, $h \neq 0$, is a Haar measure if it is a two-sided integral such that $\varepsilon(h) = 1$.

It is possible to prove the following central result:

Theorem 1.5.2. *A has a Haar measure h iff A is semisimple. It is unique, idempotent, self-adjoint and such that $S(h) = h$.*

In order to prove the theorem, we need the following lemmas:

Lemma 1.5.3. *Let A be a semisimple unital algebra. If J is a left ideal, $J = Ae$, where $e^2 = e$. If J is a right ideal, $J = fA$, where $f^2 = f$.*

Proof. Let J be a left ideal of A. For semisimplicity it exists a left ideal K such that $A = J \oplus K$. So, $I = e + f$ in a unique way, with $e \in J$ and $f \in K$. $e = eI = e^2 + ef$. Since $f \in K$, $ef \in K$. But $ef = e^2 - e \in J$, so $ef \in J \cap K = \{0\}$. So $e^2 = e$. Let's prove that $J = Ae$. $Ae \subseteq J$ is trivial. On the other hand, let j be an element of J. $j = jI = j(e + f) = je + jf$. So $jf \in J \cap K = \{0\}$, and $j = je \in Ae$.

Let's pass to the right case. J is a right ideal. It is possible to consider the action $a \cdot x := xS(a)$. Since the invertibility of S, K is a A-module with this action if and only if K is a right ideal. So, it exists a right ideal K such that $A = J \oplus K$. As before, $I = e + f$, with $e \in J$, $f \in K$. So:

$$S^{-1}(e)I = S^{-1}(e) \cdot e + S^{-1}(e) \cdot f \Rightarrow e = e^2 + fe \Rightarrow fe = 0 \Rightarrow e^2 = e$$

Let's prove that $J = eA$. $eA \subseteq J$ is obvious, since $ea = S^{-1}(a) \cdot e \in J$. On the other hand, $j \in J$. So:

$$\begin{aligned} j = Ij = (e + f)j = ej + fj = S^{-1}(j) \cdot e + S^{-1}(j) \cdot f \Rightarrow \\ \Rightarrow fj = 0 \Rightarrow j = ej \in eA \end{aligned}$$

□

Lemma 1.5.4. *The following statements are equivalent:*

$$\text{(i)}l \textit{ is a left integral;}$$
$$\text{(ii)} \ker\varepsilon \cdot l = \{0\};$$
$$\text{(iii)}S(l) \textit{ is a right integral.}$$

Proof. $(i) \Rightarrow (ii)$ is obvious.
$(ii) \Rightarrow (i)$ $x - \varepsilon(x)I \in \ker\varepsilon$. So $xl = (x - \varepsilon(x)I)l + \varepsilon(x)l = \varepsilon(x)l$.
$(i) \Rightarrow (iii)$ $S(xl) = S(\varepsilon(x)l) \Rightarrow S(l)S(x) = \varepsilon(x)S(l)$. We can conclude since S is bijective.
$(iii) \Rightarrow (i)$ It is obvious since S is invertible. □

Now we can prove the Theorem 1.5.2. The proof is inspired by [62]:

Proof. $\ker\varepsilon$ is a *-two-sided ideal of A. $\varepsilon \neq 0$, so $\ker\varepsilon \neq A$. Using the lemma, we have that $\ker\varepsilon = Ap$ and $\ker\varepsilon = qA$, where $p^2 = p$, $q^2 = q$ and $p, q \neq I$. So $\ker\varepsilon(I - p) = (0)$, and $(I - q)\ker\varepsilon = (0)$. So $I - p$ is a left integral, and $I - q$ is a right integral. It is obvious that $p, q \in \ker\varepsilon$, so $q(I - p) = 0$ and $(I - q)p = 0$. Therefore $q = qp = p$. In this way we can say that $I - p$ is a two-sided integral. $I - p \neq 0$, and $\varepsilon(I - p) = \varepsilon(I) - \varepsilon(p) = \varepsilon(I) = 1$. Let $h := I - p$. It is unique. In fact, let $h' \neq 0$ another two-sided integral, with $\varepsilon(h') = 1$. So $h = hh' = h'$. By uniqueness it is immediate to prove that $h = h^*$ and $S(h) = h$.

Let's prove the opposite implication. Let h be the Haar measure. We consider two left A-module M, N, where N is a submodule of M. Let $E : M \to N$ be a projection, and $P : M \to N$, with:

$$P(m) = \sum x_i h_{(1)}\beta E(S(h_{(2)})S(y_i)\alpha z_i m)$$

If $n \in N$, then $P(n) = \sum_i x_i h_{(1)}\beta S(h_{(2)})S(y_i)\alpha z_i n = n$. So, P is a projection. Now, we prove that P is A-linear.

$$\begin{aligned} aP(m) &= a_{(1)}x_i h_{(1)}\beta E(S(h_{(2)})S(y_i)\varepsilon(a_{(2)})\alpha z_i m) = \\ &= a_{(1)}x_i h_{(1)}\beta E(S(h_{(2)})S(y_i)S(a_{(2)(1)})\alpha a_{(2)(2)} z_i m) \end{aligned}$$

Since $\mathrm{id}\otimes\Delta(\Delta(a)) = \Phi(\Delta\otimes\mathrm{id}(\Delta(a)))\Phi^{-1}$, we have

$$a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)} = \sum_{i,j}(x_i a_{(1)(1)}p_j \otimes y_i a_{(1)(2)}q_j \otimes z_i a_{(2)}r_j)$$

So:

$$\begin{aligned} aP(m) &= x_k a_{(1)(1)}p_j x_i h_{(1)}\beta E(S(h_{(2)})S(y_i)S(y_k a_{(1)(2)}q_j)\alpha z_k a_{(2)}r_j z_i m) = \\ &= x_k a_{(1)(1)}p_j x_i h_{(1)}\beta E(S(h_{(2)})S(q_j y_i)S(y_k a_{(1)(2)})\alpha z_k a_{(2)}r_j z_i m) \end{aligned}$$

Since $\sum_{i,j} p_j x_i \otimes q_j y_i \otimes r_j z_i = I_{(1)(1)} \otimes I_{(1)(2)} \otimes I_{(2)}$ we have:

$$\begin{aligned} aP(m) &= x_k a_{(1)(1)}I_{(1)(1)}h_{(1)}\beta E(S(h_{(2)})S(a_{(1)(2)}I_{(1)(2)})S(y_k)\alpha z_k a_{(2)}I_{(2)}m) \\ &= x_k a_{(1)(1)}h_{(1)}\beta E(S(a_{(1)(2)}h_{(2)})S(y_k)\alpha z_k a_{(2)}m) \end{aligned}$$

Since h is a Haar measure, we have $a_{(1)(1)}h_{(1)} \otimes a_{(1)(2)}h_{(2)} = \Delta(a_{(1)}h) = \varepsilon(a_{(1)})h_{(1)} \otimes h_{(2)}$. Hence,

$$aP(m) = x_k a_{(1)(1)}h_{(1)}\beta E(S(a_{(1)(2)}h_{(2)})S(y_k)\alpha z_k a_{(2)}m) =$$
$$= x_k h_{(1)}\beta E(S(h_{(2)})S(y_k)\alpha z_k am) = P(am)$$

□

We can conclude that the existence of a Haar measure is equivalent to the semisimplicity of the algebra.

Let us pass to $\widehat{A}$. Here, the situation is different in comparison to the ordinary case. In fact, the cosemisimplicity of $\widehat{A}$ is not a sufficient condition to get the existence of a Haar measure. More precisely, suppose that A is coassociative ($\Phi = I \otimes I \otimes I$). We have the following:

Proposition 1.5.5. *There are no invariant functionals on A, or, in other words, there are no $\psi \in \widehat{A}$ such that $\phi\psi = \phi(I)\psi \ \forall \phi \in \widehat{A}$.*

Proof. Let M be a left $\widehat{A}$-module. We can see it as a right A-comodule, in this way: let $\{b_i\}_{i=1}^n$ be a basis of A, and $\{\beta_i\}_{i=1}^n$ be its dual basis. $\delta : M \to M \otimes A$ is the associated right coaction, with $\delta(m) := \sum_{i=1}^n \beta^i \cdot m \otimes b_i$, where $\phi \cdot m$ is the left action of $\widehat{A}$ on M. Conversely, if $\delta : M \to M \otimes A$, with $\delta(m) = m_0 \otimes m_1$, is a right coaction of A on M, we can define the associated left action of $\widehat{A}$ on M in this way:

$$\phi \cdot m := \phi(m_1)m_0$$

Let's check that δ is a coaction. First of all, we can write $\Delta(b_i) = \sum \lambda^i_{hk} b_h \otimes b_k$. Therefore:

$$((\delta \otimes \mathrm{id}) \circ \delta)(m) = \sum_{j=1}^n \delta(\beta^j \cdot m) \otimes b_j = \sum_{i,j=1}^n (\beta^i\beta^j) \cdot m \otimes b_i \otimes b_j$$
$$((\mathrm{id} \otimes \Delta) \circ \delta)(m) = \sum_{i=1}^n \beta^i \cdot m \otimes \Delta(b_i) = \sum_{i,h,k}^n \lambda^i_{hk}\beta^i \cdot m \otimes b_h \otimes b_k$$

It is straightforward to see that $\beta^i\beta^j(b_l) = \lambda^l_{ij}$, and $\beta^i(b_l) = \delta_{il}$. So, $\beta^i\beta^j = \sum_l \lambda^l_{ij}\beta^l$ and:

$$((\mathrm{id} \otimes \Delta) \circ \delta)(m) = \sum_{i,h,k} \lambda^i_{hk}\beta^i \cdot m \otimes b_h \otimes b_k = \sum_{h,k}\left(\sum_i \lambda^i_{hk}\beta^i\right) \cdot m \otimes b_h \otimes b_k =$$
$$= \sum_{h,k} \beta^h\beta^k \cdot m \otimes b_h \otimes b_k = ((\delta \otimes \mathrm{id}) \circ \delta)(m)$$

Now we have to check that $(\mathrm{id}_M \otimes \varepsilon) \circ \delta = \mathrm{id}_M$:

$$(\mathrm{id}_M \otimes \varepsilon)(\delta(m)) = \sum_{i=1}^n \varepsilon(b_i)\beta^i \cdot m = \sum_{i=1}^n \widehat{I}(b_i)\beta^i \cdot m = \widehat{I} \cdot m = m$$

So, δ is a coaction. Similarly it is possible to prove that, if δ is a right coaction, $\phi \cdot m = \phi(m_1)m_0$ is a left action.

The space of invariants of a left $\widehat{A}$-module M is defined to be the subspace:

$$\mathrm{Inv}({}_{\widehat{A}}M) := \{m \in M : \phi{\cdot}m = \widehat{\varepsilon}(\phi)m, \forall \phi \in \widehat{A}\} = \{m \in M : \phi{\cdot}m = \phi(I)m, \forall \phi \in \widehat{A}\}$$

By duality, we define the coinvariants of a right A-comodule M as:

$$\mathrm{Coinv}(M^A) := \{m \in M :\ m_0 \otimes m_1 = m \otimes I\}$$

We want to see that in our case $\mathrm{Inv}({}_{\widehat{A}}M) = \mathrm{Coinv}(M^A)$. In fact, let $m \in \mathrm{Inv}({}_{\widehat{A}}M)$. So, $\phi \cdot m = \phi(I)m\ \forall \phi \in \widehat{A}$, and in particular $\beta^i \cdot m = \beta^i(I)m$. Then:

$$\delta(m) = \sum_{i=1}^{n} \beta^i(I)m \otimes b_i = \sum_{i=1}^{n} m \otimes \beta^i(I)b_i =$$
$$= m \otimes \sum_{i=1}^{n} \beta^i(I)b_i = m \otimes I$$

So, we have just proved that $\mathrm{Inv}({}_{\widehat{A}}M) \subseteq \mathrm{Coinv}(M^A)$. Now, let m be an element of $\mathrm{Coinv}(M^A)$. It exists a linear map $\Phi : M \otimes A \to \mathrm{Hom}(\widehat{A}, M)$ which is an isomorphism of linear spaces, with $\Phi(m \otimes x) = (\phi \mapsto \phi(x)m)$. So, $\forall \phi \in \widehat{A}$:

$$\sum_{i=1}^{n} \beta^i \cdot m \otimes b_i = m \otimes I \Rightarrow \Phi\left(\sum_{i=1}^{n} \beta^i \cdot m \otimes b_i\right)(\phi) = \Phi(m \otimes I)(\phi) \Rightarrow$$
$$\Rightarrow \sum_{i=1}^{n} \phi(b_i)\beta^i \cdot m = \phi(I)m \Rightarrow \phi \cdot m = \phi(I)m$$

In this way, we have proved that $\mathrm{Inv}({}_{\widehat{A}}M) = \mathrm{Coinv}(M^A)$. The main consequence is that $\mathrm{Inv}({}_{\widehat{A}}M) = \{0\}\ \forall M\ \widehat{A}$-module. In fact, if $m \in \mathrm{Inv}({}_{\widehat{A}}M)$, $m \neq 0$, then $m \in \mathrm{Coinv}(M^A)$, so $\delta(m) = m \otimes I$. But δ is a coaction, so $(\delta \otimes \mathrm{id}_A) \circ \delta = (\mathrm{id}_M \otimes \Delta) \circ \delta$. The identity implies that $m \otimes I \otimes I = m \otimes I_{(1)} \otimes I_{(2)}$, which in turn implies $\Delta(I) = I \otimes I$.

Let $\mathcal{I}_L(\widehat{A})$ be the space of the left integrals of $\widehat{A}$. If we consider $\widehat{A}$ as a $\widehat{A}$-module under left multiplication, for what we said previously, we have $\mathcal{I}_L(\widehat{A}) = \mathrm{Inv}({}_{\widehat{A}}\widehat{A}) = \{0\}$. $\square$

Chapter 2

The reconstruction theory for tensor categories

2.1 Generalities on reconstruction theorems

The topic exposed in this section will be the motivation for what the reader will find afterwards. We can start from a very basic problem. Suppose that G is a locally compact abelian group. It is possible to build another locally compact abelian group $\widehat{G}$ whose elements are the continuous homomorphisms from G to $\mathbb{T}$. $\widehat{G}$ is locally compact with the topology of uniform convergence on compact sets. The product on $\widehat{G}$ is the pointwise multiplication. $\widehat{G}$ is called the *dual* of G. Repeating the procedure, we can build the bidual $\widehat{\widehat{G}}$. The classical result by Pontrjagin [66] states:

Theorem 2.1.1. *There is a canonical group isomorphism between* G *and* $\widehat{\widehat{G}}$*, which is also a homeomorphism.*

Of course, one can ask if it is possible to generalize this theorem to the non-abelian case. If we preserve for $\widehat{G}$ the same definition, we will not be able to reconstruct our group G, so we need to reformulate the definition of $\widehat{G}$. In the abelian case, $\widehat{G}$ is the set of the unitary irreducible representations of G. So, we can give this definition for $\widehat{G}$ also in the non-abelian case. But in this case $\widehat{G}$ is no more a group, since the irreducible representations might have dimensions higher than 1, and the tensor product of two irreducible representations might be reducible. So we must change our point of view. We consider $\widehat{G} = \mathrm{Rep}(G)$ the category of all the f.d. representations of G. It is possible to see that $\mathrm{Rep}(G)$ is a symmetric rigid tensor category. Is there a way to reconstruct G from $\mathrm{Rep}(G)$? We consider the forgetful symmetric tensor functor E from $\mathrm{Rep}(G)$ to the category Vect of the f.d. linear spaces, and we call $\mathrm{Nat}(E)$ the set of the natural isomorphisms between E and itself. Tannaka [77] proved that:

Theorem 2.1.2. *Let* G *be a compact group and* $\mathrm{Nat}(E)$ *the set of the natural automorphisms of* E*, where* $E : \mathrm{Rep}(G) \to \mathrm{Vect}$ *is the forgetful functor. Then* $\mathrm{Nat}(E)$ *is a compact group and* $G \cong \mathrm{Nat}(E)$.

Suppose now that we start from a given symmetric rigid tensor category $\mathcal{C}$, with a fiber functor $E : \mathcal{C} \to \mathrm{Vect}$. We ask if it is always possible to find a compact group

G such that $\mathcal{C}$ can be identified with $\mathrm{Rep}(G)$. The answer is positive and is due to Krein [41]:

Theorem 2.1.3. *Let $\mathcal{C}$ be a rigid symmetric linear category with $\mathrm{End}(\mathbf{1}) = \mathbb{C}$ and $E : \mathcal{C} \to \mathrm{Vect}$ a forgetful functor. Let $G = \mathrm{Nat}(E)$ be the group of the natural automorphisms of E. Then there exists a functor $F : \mathcal{C} \to \mathrm{Rep}(G)$ such that:*

$$F(X) = (E(X), \pi_X) \text{ and } \pi_X(g) = g_X \quad (g \in G)$$

which is an equivalence of symmetric tensor categories. If $\mathcal{C}$ is a *-category, G is compact.

It is interesting to give a sketch of the proof, following [55]:

Proof. Let $E_1, E_2 : \mathcal{C} \to \mathrm{Vect}$ be two fiber functors. We define a unital algebra $A_0(E_1, E_2)$ by:

$$A_0(E_1, E_2) = \bigoplus_{X \in \mathcal{C}} (E_2(X), E_1(X))_{\mathrm{Vect}}$$

spanned by elements $[X, s]$, $X \in \mathcal{C}$, $s \in (E_2(X), E_1(X))_{\mathrm{Vect}}$, with $[X, s] \cdot [Y, t] = [X \otimes Y, u]$, where:

$$u = e_{X,Y} \circ (s \otimes t) \circ e^{-1}_{X,Y}$$

This is a unital associative algebra, and $A(E_1, E_2)$ is defined as the quotient by the ideal generated by:

$$[X, a \circ E_2(s)] - [Y, E_1(s) \circ a]$$

where $s \in (X, Y)_{\mathcal{C}}$ and $a \in (E_2(Y), E_1(X))_{\mathrm{Vect}}$. At this point one can proves that: if E_1 and E_2 are symmetric tensor functors, $A(E_1, E_2)$ is commutative; if $\mathcal{C}$ is a *-category and E_1, E_2 are *-preserving, then $A(E_1, E_2)$ is a *-algebra and has a C*-completion; if $\mathcal{C}$ is finitely generated, then $A(E_1, E_2)$ is finitely generated. Finally, there exists a bijection between natural (unitary) isomorphisms $\alpha : E_1 \to E_2$ and (*-)characters on $A(E_1, E_2)$. Now, if E_1, E_2 are symmetric and either $\mathcal{C}$ is a *-category or is finitely generated, then $A(E_1, E_2)$ admits characters, using the Gelfand's theory or the Nullstellensatz, so $E_1 \cong E_2$. Moreover, $G = \mathrm{Nat}(E_1)$ is the group of the (*-)characters of $A(E_1) = A(E_1, E_1)$, and so $A(E_1)$ is the algebra of the representative (continuous) functions on G. This allows to prove that $\mathcal{C}$ and $\mathrm{Rep}(G)$ are equivalent. □

A good review of the Tannaka-Krein theory can be found in [38]. As we can point out from the sketch of the proof just seen, the fiber functor is unique up to isomorphism, and the symmetry of the category is crucial to state this fact. Of course, this implies that the group G is unique up to isomorphism. Now we want to go one step further. Is it possible to state the same theorem for an abstract category? Or, in other words, without giving a fiber functor? The answer is "no" in general, but it becomes "yes" if we add some weak assumptions on $\mathcal{C}$. This is the theorem due to Doplicher and Roberts [19]:

Theorem 2.1.4. *Let $\mathcal{C}$ be a linear rigid symmetric C*-tensor category, with $\mathrm{End}(\mathbf{1}) = \mathbb{C}$. Then there exists a unique (up to isomorphism) compact group G such that the category of the unitary representation of G $\mathrm{Rep}(G)$ is equivalent to $\mathcal{C}$ as linear rigid symmetric C*-tensor category.*

In this theorem we don't need to give a fiber functor thanks to the C*-structure. So in the symmetric case we can say that everything works well. We can now try to give a further generalization, supposing that $\mathcal{C}$ is not symmetric. In this case, it may happen that we don't have any fiber functor, and if it exists, it could be not unique up to natural isomorphism. Moreover, if E is a fiber functor, it is not symmetric, so $A(E)$ is not a commutative algebra and the reconstructed object cannot be a group. The suitable object is a Hopf algebra, with its several generalizations (multiplier Hopf algebras, quantum groups,...) [33]. For example, if $\mathcal{C}$ is a linear rigid semisimple and finitely generated tensor category and E is a fiber functor, we can reconstruct a finite-dimensional Hopf algebra H such that $\mathcal{C}$ is equivalent to $\mathrm{Rep}(H)$. If the category is braided, H is braided; if $\mathcal{C}$ is a *-category and the fiber functor is *-preserving, H is a *-Hopf algebra. We now wonder if we are able to give a reconstruction theorem also when the category $\mathcal{C}$ does not admit a fiber functor, or in other words, when the functor $E : \mathcal{C} \to \mathrm{Vect}$ is not a tensor functor. The answer is positive if E is a quasi tensor functor or a weak quasi tensor functor [30]. This is what we are going to see in a detailed way in the next section.

2.2 Reconstruction theorems of weak quasi Hopf algebras

Our main reference in this section is [30]. We will also give an extension of the results exposed there, in two different directions. On one hand, we will talk about the case when the category $\mathcal{C}$ is endowed with a weak tensor functor, getting as reconstructed object a weak Hopf algebra. On the other hand, suppose that the category $\mathcal{C}$ is endowed with an involution *. In this case, Häring-Oldenburg considered only (weak) quasi tensor functors whose natural transformations $e_{X,Y}$ satisfy the identity $e^*_{X,Y} = e^{-1}_{X,Y}\ \forall X, Y$. This request forced the reconstruction object to have an involution and a coproduct satisfying $\Delta \circ ^* = ^* \circ \Delta$. We drop out this condition, obtaining a more general class of reconstructed objects, which fits exactly with the definition of weak quasi Hopf *-algebra introduced before.

From now on in this section, we will deal with a linear semisimple rational rigid braided tensor category $\mathcal{C}$. As we said in the previous section, if we have a non-symmetric tensor category, it is not always possible to give an associated fiber functor. In any case, it is always possible to build a weak quasi tensor functor.

Definition 2.2.1. A *weak dimension function* is a function defined on the irreducible objects of a semisimple, rigid braided tensor category $D : \mathrm{Ob}(\mathcal{C})_{\mathrm{Irr}} \to \mathbb{N}$. It is constant on equivalence classes, and:

$$D(\mathbf{1}) = 1\,,\, D(X) = D(\overline{X})$$

and

$$D(X)D(Y) \geq \sum_{Z \in \nabla} D(Z) \dim((X \otimes Y, Z)) \tag{2.2.1}$$

If the equality holds, D is called *dimension function*

Starting from dimension functions we can build tensor functors:

Proposition 2.2.2. *Let $\mathcal{C}$ be a semisimple, rigid, braided tensor category and D : $\mathrm{Ob}(\mathcal{C}) \to \mathbb{N}$ a weak dimension function. Then there is a faithful weak quasi tensor functor $F : \mathcal{C} \to \mathrm{Vect}$.*

In order to prove the above proposition, we introduce the following lemma:

Lemma 2.2.3. *Set $X \in \mathrm{Ob}(\mathcal{C})_{\mathrm{Irr}}$. Then for all $Y \in \mathrm{Ob}(\mathcal{C})$, we have:*

$$(Y, X) \cong (X, Y)^*$$

where $(X,Y)^*$ is the dual space (as vector space) of (Y,X).

Proof. Let $\Phi : (Y, X) \longrightarrow (X, Y)^*$, where $\Phi(g) = \lambda_g$ and:

$$\lambda_g(f) = g \circ f \in (X, X) \cong \mathbb{C}$$

where $f \in (X, Y)$. We distinguish two cases. If X does not appear in the decomposition of Y, then $(Y, X) = \{0\} = (X, Y)$. We prove this statement. We have $Y \cong \bigoplus_i X_i$, where X_i are simple objects which are not isomorphic to X. There exist morphisms $v_i \in (X_i, Y)$ and $v_i' \in (Y, X_i)$ such that $v_i' \circ v_j = \delta_{i,j} \,\mathrm{id}_{X_i}$ and:

$$\mathrm{id}_Y = \sum_i v_i \circ v_i'$$

So, if $g \in (Y, X)$, then:

$$g = g \circ \mathrm{id}_Y = \sum_i g \circ (v_i \circ v_i') = \sum_i (g \circ v_i) \circ v_i' = 0$$

since $g \circ v_i \in (X_i, X) = \{0\}$. So $(Y, X) = \{0\}$. In the same way, let f be a morphism in (X, Y). Then:

$$f = \mathrm{id}_Y \circ f = \sum_i v_i \circ (v_i' \circ f) = 0$$

So $(X, Y) = \{0\}$. Now, we pass to the case where X is a subobject of Y. In this case:

$$Y \cong \bigoplus_{i=1}^{n_1} X_i \oplus \bigoplus_{j=1}^{n_2} X_{n_1+j}$$

where X_i and X are isomorphic if $i \leq n_1$ and not isomorphic if $i > n_1$. As before, there exist $v_i \in (X_i, Y)$ and $v_i' \in (Y, X_i)$ such that:

$$v_i' \circ v_j = \delta_{i,j} \,\mathrm{id}_{X_i} \quad \text{and} \quad \mathrm{id}_Y = \sum_i v_i \circ v_i'$$

Now we want to prove that Φ is injective. If $\lambda_g = 0$, then $g \circ f = 0 \; \forall f \in (X, Y)$. Hence:

$$g = g \circ \mathrm{id}_Y = \sum_{i=1}^{n_1} (g \circ v_i) \circ v_i' + \sum_{i=n_1+1}^{n_1+n_2} (g \circ v_i) \circ v_i'$$

We know that (X_i, X) has an invertible arrow ϕ_i if $i \leq n_1$, while $(X_i, X) = \{0\}$ if $i > n_1$. So:

$$g = \sum_{i=1}^{n_1} (g \circ (v_i \circ \phi_i^{-1})) \circ \phi_i \circ v_i'$$

But $\forall i \in \{1, \ldots, n_1\}$, $g \circ (v_i \circ \phi_i^{-1}) = 0$ since $v_i \circ \phi_i^{-1} \in (X, Y)$. Therefore $g = 0$. Since morphism spaces are finite-dimensional vector spaces, it is sufficient to prove that (X, Y) and (Y, X) have the same dimension. Recalling Prop. 1.1.12, we have that $\dim((X_i, X)) = 1$ if $i \leq n_1$, so each one is generated by ϕ_i. Now, we can prove that (X, Y) has $B_1 = \{v_i \circ \phi_i^{-1}\}_{i=1}^{n_1}$ as basis, while $B_2 = \{\phi_i \circ v_i'\}_{i=1}^{n_1}$ is a basis of (Y, X). Let f be an arrow in (X, Y). Then:

$$f = \mathrm{id}_Y \circ f = \sum_{i=1}^{n_1} v_i \circ (v_i' \circ f)$$

Since $v_i' \circ f \in (X, X_i)$, there exists $\lambda_i^{(f)} \in \mathbb{C}$ such that:

$$v_i' \circ f = \lambda_i^{(f)} \phi_i^{-1}$$

So B_1 generates (X, Y). It is quite easy to see that it is linearly independent. In fact,

$$\sum_{i=1}^{n_1} \lambda_i v_i \circ \phi_i^{-1} = 0$$

Composing on the left by v_j', we obtain $\lambda_j \phi_j^{-1} = 0$ which implies $\lambda_j = 0$. We conclude repeating this argument for every $j \in \{1, \ldots, n_1\}$. Now, B_2 is a basis and can be proved in the same way. Since $|B_1| = |B_2|$, we get the result. □

Now we can prove Prop.2.2.2:

Proof. Let X be a simple object and $F(X) := \mathbb{C}^{D(X)}$. F is extended on the other objects in the following way:

$$F(Y) := \bigoplus_{X \in \nabla} (X, Y) \otimes F(X)$$

F has to map morphisms $f \in (Y_1, Y_2)$ to morphisms $F(f) \in (F(Y_1), F(Y_2))$. Because of linearity, $F(f)$ needs only to be defined on simple tensor products in $(X, Y_1) \otimes F(X)$. We define:

$$F(f)(g \otimes x) = f \circ g \otimes x \tag{2.2.2}$$

where $g \in (X, Y_1)$ and $x \in F(X)$. Now we want to prove that F is faithful. Let $F(f_1)$ be equal to $F(f_2)$. This implies that $f_1 \circ g = f_2 \circ g$ for all X simple object and for all $g \in (X, Y_1)$. Since $Y_1 \cong \bigoplus_i X_i$, then:

$$\mathrm{id}_{Y_1} = \sum_i v_i \circ v_i'$$

as in the previous lemma, where $v_i \in (X_i, Y_1)$ and $v_i' \in (Y_1, X_i)$. So:

$$f_1 = f_1 \circ \mathrm{id}_{Y_1} = \sum_i (f_1 \circ v_i) \circ v_i' = \sum_i (f_2 \circ v_i) \circ v_i' = f_2$$

Next we prove that $F(\overline{Y}) \cong F(Y)^* = \overline{F(Y)}$. We need some preliminary facts. First of all, we have:

$$(\overline{X}, \overline{Y}) \cong (\mathbb{C}, \overline{Y} \otimes X) \cong (Y, X) \cong (X, Y)^*$$

using rigidity and the previous lemma. More precisely, $\Phi : (\overline{X}, \overline{Y}) \to (\mathbb{C}, \overline{Y} \otimes X)$ such that:

$$\Phi(f) = f \otimes \mathrm{id}_X \circ \overline{r}_{\overline{X}}$$

is an isomorphism, where $f \in (\overline{X}, \overline{Y})$. The inverse map of Φ is:

$$\Phi^{-1}(g) = \mathrm{id}_Y \otimes r^{\dagger}_{\overline{X}} \circ g \otimes \mathrm{id}_{\overline{X}}$$

As a consequence, $\overline{X}$ is also a simple object, since $(X, X) \cong (\overline{X}, \overline{X})$. Finally, if X is a simple object, $F(X)^* \cong F(X) \cong F(\overline{X})$ because X and $\overline{X}$ have the same dimension. So:

$$\begin{aligned} F(\overline{Y}) = \sum_X (X, \overline{Y}) \otimes F(X) &\cong \sum_X (\overline{X}, \overline{Y}) \otimes F(\overline{X}) \cong \\ \cong \sum_X (X, Y)^* \otimes F(X)^* &\cong F(Y)^* \end{aligned}$$

Now, let X_1 and X_2 be two simple objects. We choose an arbitrary epimorphism:

$$E_{X_1,X_2} : F(X_1) \otimes F(X_2) \to F(X_1 \otimes X_2) = \bigoplus_{X \in \nabla} (X, X_1 \otimes X_2) \otimes F(X)$$

It is always possible to find such morphism thanks to the property (2.2.1) of the weak dimension function D. Now, we want to extend E to every object of $\mathcal{C}$. Therefore we define $e_{Y_1,Y_2} : F(Y_1) \otimes F(Y_2) \longrightarrow F(Y_1 \otimes Y_2)$, or, in other terms:

$$\begin{aligned} e_{Y_1,Y_2} : \left(\bigoplus_{X_1 \in \nabla} (X_1, Y_1) \otimes F(X_1) \right) \otimes \left(\bigoplus_{X_2 \in \nabla} (X_2, Y_2) \otimes F(X_2) \right) = \\ = \bigoplus_{X_1, X_2 \in \nabla} (X_1, Y_1) \otimes F(X_1) \otimes (X_2, Y_2) \otimes F(X_2) \to \bigoplus_{X \in \nabla} (X, Y_1 \otimes Y_2) \otimes F(X) \end{aligned}$$

in the following way:

$$e_{Y_1,Y_2} : \bigoplus_{X_1, X_2} (\Gamma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes E_{X_1,X_2}) \circ \mathrm{id} \otimes \Sigma \otimes \mathrm{id} \tag{2.2.3}$$

where Γ is a map from $(X_1, Y_1) \otimes (X_2, Y_2) \otimes (X, X_1 \otimes X_2)$ to $(X, Y_1 \otimes Y_2)$ such that:

$$\Gamma(f_1 \otimes f_2 \otimes g) := (f_1 \otimes f_2) \circ g$$

We need to prove that, given $f_i \in (Y_i, \widetilde{Y_i})$, $i = 1, 2$, we have:

$$F(f_1 \otimes f_2) \circ e_{Y_1,Y_2} = e_{\widetilde{Y_1},\widetilde{Y_2}} \circ (F(f_1) \otimes F(f_2))$$

We introduce the following vectors $v_i \in F(Y_i)$, where $i = 1, 2$:

$$v_i = \bigoplus_{Z_i \in \nabla} g^{(Z_i)} \otimes x^{(Z_i)}$$

with $x^{(Z_i)} \in F(Z_i)$ and $g^{(Z_i)} \in (Z_i, Y_i)$. Using the notation:

$$E_{Z_1,Z_2}(x^{(Z_1)} \otimes x^{(Z_2)}) = \bigoplus_{B \in \nabla} q^{(B)}_{Z_1,Z_2} \otimes x^{(B)}_{Z_1,Z_2}$$

we get:

$$
\begin{aligned}
e_{\widetilde{Y_1},\widetilde{Y_2}} \circ (F(f_1) \otimes F(f_2))(v_1 \otimes v_2) = \\
= e_{\widetilde{Y_1},\widetilde{Y_2}}(f_1 \circ g^{(Z_1)} \otimes x^{(Z_1)} \otimes f_2 \circ g^{(Z_2)} \otimes x^{(Z_2)}) = \\
= \bigoplus_{Z_1,Z_2 \in \nabla} \Big(\Gamma \otimes \mathrm{id}(f_1 \circ g^{(Z_1)} \otimes f_2 \circ g^{(Z_2)} \otimes E_{Z_1,Z_2}(x^{(Z_1)} \otimes x^{(Z_2)})) \Big) = \\
= \bigoplus_{Z_1,Z_2,B \in \nabla} (f_1 \circ g^{(Z_1)} \otimes f_2 \circ g^{(Z_2)}) \circ q^B_{Z_1,Z_2} \otimes x^B_{Z_1,Z_2} = \\
= F(f_1 \otimes f_2) \circ \left(\bigoplus_{Z_1,Z_2,B \in \nabla} (g^{(Z_1)} \otimes g^{(Z_2)}) \circ q^B_{Z_1,Z_2} \otimes x^B_{Z_1,Z_2} \right) = \\
= F(f_1 \otimes f_2) \circ e_{Y_1,Y_2}(v_1 \otimes v_2)
\end{aligned}
$$

□

Remark 2.2.4. **(a)** If $\mathcal{C}$ is a C*-category, we look for a forgetful functor F in the category of the finite-dimensional Hilbert space Hilb. We wonder if the functor F built in the previous proof goes into Hilb and it is *-preserving. It straightforward to see that $F(X)$ is a Hilbert space if X is simple. Otherwise, we recall that:

$$F(Y) = \bigoplus_{X \in \nabla} (X, Y) \otimes F(X)$$

We can put on (X, Y) the following form:

$$\langle f, g \rangle = f^* \circ g \in (X, X) = \mathbb{C}$$

Using the properties of a C*-category it is quite easy to see that it is an inner product. If $(\cdot, \cdot)$ is the inner product on $F(X)$:

$$\langle \cdot, \cdot \rangle' := \langle \cdot, \cdot \rangle \cdot (\cdot, \cdot)$$

is the inner product on $F(Y)$. It remains to prove that $F(f^*) = F(f)^*$. Let f be a morphism in (Y_1, Y_2), $x, y \in F(X)$ and $g, h \in (X, Y_2)$:

$$
\begin{aligned}
\langle h \otimes y, F(f)^*(g \otimes x) \rangle' = \langle F(f)(h \otimes y), g \otimes x \rangle' = \\
= \langle f \circ h \otimes y, g \otimes x \rangle' = \langle f \circ h, g \rangle \, (y, x) = \\
= h^* \circ f^* \circ g(y, x) = \langle h \otimes y, f^* \circ g \otimes x \rangle' = \\
= \langle h \otimes y, F(f^*)(g \otimes x) \rangle'
\end{aligned}
$$

(b) The proof of the last Proposition allows us to say that finding a weak quasi tensor functor means finding a weak dimension functor. Moreover, it is clear that the functor constructed in the previous proof is not compatible in general with the associativity costraint as in (1.1.13).

Proposition 2.2.5. *If $\mathcal{C}$ is a rational, semisimple, rigid, braided tensor category, there exist weak dimension functions D such that:*

$$D(\mathbf{1}) = 1 \ , \ D(X) = \dim \bigoplus_{Y,Z \in \nabla} (Y \otimes X, Z) = \sum_{Y,Z \in \nabla} N^Z_{X,Y} \tag{2.2.4}$$

Proof.

$$D(X)D(Y) = \left(\sum_{s,r} N^r_{X,s}\right)\left(\sum_{S,R} N^R_{Y,S}\right) = \sum_{s,r,S,R} N^r_{X,s}N^R_{Y,S} \geq \sum_{K,N,M} N^K_{X,N}N^M_{Y,K}$$

Explicitly,

$$\begin{aligned}\sum_{K,N,M} N^K_{X,N}N^M_{Y,K} &= \sum_{K,N,M} \dim(N\otimes X, K)\dim(K\otimes Y, M) = \\ &= \sum_{N,M}\dim\left(\bigoplus_K \dim(N\otimes X,K)(K\otimes Y,M)\right) = \\ &= \sum_{N,M}\dim(N\otimes X\otimes Y, M) = D(X\otimes Y)\end{aligned}$$

□

So, putting together the last Proposition and the last Remark we can state that every category $\mathcal{C}$ of our type can be endowed with a weak quasi tensor functor $F : \mathcal{C} \to \mathrm{Vect}$.

Now we are ready to introduce the Majid's reconstruction theorem (see [51], [52], [53] and [54]) and the generalization made by Häring-Oldenburg. We will consider the case when $\mathcal{C}$ is a *-category. Let $\mathrm{Nat}(F,F)$ be the set of natural transformations of F. We define:

$$\begin{aligned}&H = H(\mathcal{C},F) = \mathrm{Nat}(F,F) = \{h : \mathrm{Ob}(\mathcal{C}) \to \mathrm{End}_{\mathrm{Vect}} | h_X \in \mathrm{End}(F(X)) \\ &\text{and } F(f)\circ h_X = h_Y \circ F(f)\ \forall X,Y \in \mathrm{Ob}(\mathcal{C}), \forall f \in (X,Y)\}\end{aligned}$$

Proposition 2.2.6. *H is a (braided) quasi Hopf algebra if F is a (braided) quasi tensor functor.*

Proof. H is a vector space by pointwise addition. The multiplication is also pointwise:

$$(hg)_X = h_X \circ g_X \text{ , where } X \in \mathrm{Ob}(\mathcal{C}) \text{ , } h,g \in H \tag{2.2.5}$$

The unit is the natural transformation which send X to id_X. The coproduct is defined in the following way:

$$\Delta(h)_{X,Y} = e^{-1}_{X,Y} \circ h_{X\otimes Y} \circ e_{X,Y} \tag{2.2.6}$$

Δ is compatible with multiplication:

$$\begin{aligned}(\Delta(h)\Delta(g))_{X,Y} &= \Delta(h)_{X,Y}\circ\Delta(g)_{X,Y} = \\ &= e^{-1}_{X,Y}\circ h_{X\otimes Y}\circ e_{X,Y}\circ e^{-1}_{X,Y}\circ g_{X\otimes Y}\circ e_{X,Y} = \\ &= e^{-1}_{X,Y}\circ h_{X\otimes Y}\circ g_{X\otimes Y}\circ e_{X,Y} = \Delta(hg)_{X,Y}\end{aligned}$$

The counit is $\varepsilon : H \to \mathbb{C}$, where $\varepsilon(h) = h_{\mathbf{1}}$:

$$\begin{aligned}((\mathrm{id}\otimes\varepsilon)\Delta(h))_X &= (h_{(1)}\otimes h_{(2)_{\mathbf{1}}})_X = \Delta(h)_{X,\mathbf{1}} = \\ &= e^{-1}_{X,\mathbf{1}}\circ h_{X\otimes\mathbf{1}}\circ e_{X,\mathbf{1}} = h_X\end{aligned}$$

The associator $\Phi \in H \otimes H \otimes H$ is defined in the following way:

$$\Phi_{X,Y,Z} = (\mathrm{id} \otimes e^{-1}_{Y,Z}) \circ e^{-1}_{X,Y\otimes Z} \circ F(a_{X,Y,Z}) \circ e_{X\otimes Y,Z} \circ (e_{X,Y} \otimes \mathrm{id}) \qquad (2.2.7)$$

If F is a tensor functor, $\Phi_{X,Y,Z}$ is trivial because of (1.1.13). If F is a quasi tensor functor, we have that $\Phi_{X,Y,Z}$ is invertible with inverse:

$$\Phi^{-1}_{X,Y,Z} = (e^{-1}_{X,Y} \otimes \mathrm{id}) \circ e^{-1}_{X\otimes Y,Z} \circ F(a^{-1}_{X,Y,Z}) \circ e_{X,Y\otimes Z} \circ (\mathrm{id} \otimes e_{Y,Z})$$

Moreover, we want to prove that:

$$(\Phi^{-1}(\mathrm{id} \otimes \Delta(\Delta(h))))_{X,Y,Z} = ((\Delta \otimes \mathrm{id}(\Delta(h)))\Phi^{-1})_{X,Y,Z}$$

In fact:

$$\begin{aligned}
&(\Phi^{-1}(\mathrm{id} \otimes \Delta(\Delta(h))))_{X,Y,Z} = \\
&= \Phi^{-1}_{X,Y,Z} \circ (e_{X,Y\otimes Z} \circ (\mathrm{id} \otimes e_{Y,Z}))^{-1} \circ h_{X\otimes(Y\otimes Z)} \circ e_{X,Y\otimes Z} \circ (\mathrm{id} \otimes e_{Y,Z}) = \\
&= (e^{-1}_{X,Y} \otimes \mathrm{id}) \circ e^{-1}_{X\otimes Y,Z} \circ F(a^{-1}_{X,Y,Z}) \circ h_{X\otimes(Y\otimes Z)} \circ e_{X,Y\otimes Z} \circ (\mathrm{id} \otimes e_{Y,Z})
\end{aligned}$$

On the other side:

$$\begin{aligned}
&((\Delta \otimes \mathrm{id}(\Delta(h)))\Phi^{-1})_{X,Y,Z} = \\
&= (e^{-1}_{X,Y} \otimes \mathrm{id}) \circ e^{-1}_{X\otimes Y,Z} \circ h_{(X\otimes Y)\otimes Z} \circ e_{X\otimes Y,Z} \circ (e_{X,Y} \otimes \mathrm{id}) \circ \Phi^{-1}_{X,Y,Z} = \\
&= (e^{-1}_{X,Y} \otimes \mathrm{id}) \circ e^{-1}_{X\otimes Y,Z} \circ h_{(X\otimes Y)\otimes Z} \circ F(a^{-1}_{X,Y,Z}) \circ e_{X,Y\otimes Z} \circ (\mathrm{id} \otimes e_{Y,Z})
\end{aligned}$$

We can conclude proving that $F(a^{-1}_{X,Y,Z}) \circ h_{X\otimes(Y\otimes Z)} = h_{(X\otimes Y)\otimes Z} \circ F(a^{-1}_{X,Y,Z})$. But it is immediate because of naturality of h. Now we want to prove the cocycle relation on Φ. The LHS is:

$$\begin{aligned}
\mathrm{LHS} = {} & \mathrm{id} \otimes \mathrm{id} \otimes e^{-1}_{Z,W} \circ \mathrm{id} \otimes e^{-1}_{Y,Z\otimes W} \circ e^{-1}_{X,Y\otimes(Z\otimes W)} \circ F(a_{X,Y,Z\otimes W}) \circ \\
& \circ e_{X\otimes Y,Z\otimes W} \circ e_{X,Y} \otimes \mathrm{id} \otimes \mathrm{id} \circ \mathrm{id} \otimes \mathrm{id} \otimes e_{Z,W} \circ e^{-1}_{X,Y} \otimes \mathrm{id} \otimes \mathrm{id} \circ \mathrm{id} \otimes \mathrm{id} \otimes e^{-1}_{Z,W} \circ \\
& \circ e^{-1}_{X\otimes Y,Z\otimes W} \circ F(a_{X\otimes Y,Z,W}) \circ e_{(X\otimes Y)\otimes Z,W} \circ e_{X\otimes Y,Z} \otimes \mathrm{id} \circ e_{X,Y} \otimes \mathrm{id} \otimes \mathrm{id}
\end{aligned}$$

while the RHS is:

$$\begin{aligned}
\mathrm{RHS} = {} & \mathrm{id} \otimes \mathrm{id} \otimes e^{-1}_{Z,W} \circ \mathrm{id} \otimes e^{-1}_{Y,Z\otimes W} \circ \mathrm{id} \otimes F(a_{Y,Z,W}) \circ \mathrm{id} \otimes e_{Y\otimes Z,W} \circ \\
& \circ \mathrm{id} \otimes e_{Y,Z} \otimes \mathrm{id} \circ \mathrm{id} \otimes e^{-1}_{Y,Z} \otimes \mathrm{id} \circ \mathrm{id} \otimes e^{-1}_{Y\otimes Z,W} \circ e^{-1}_{X,(Y\otimes Z)\otimes W} \circ \\
& \circ F(a_{X,Y\otimes Z,W}) \circ e_{X\otimes(Y\otimes Z),W} \circ e_{X,Y\otimes Z} \otimes \mathrm{id} \circ \mathrm{id} \otimes e_{Y,Z} \otimes \mathrm{id} \circ \mathrm{id} \otimes e^{-1}_{Y,Z} \otimes \mathrm{id} \circ \\
& \circ e^{-1}_{X,Y\otimes Z} \otimes \mathrm{id} \circ F(a_{X,Y,Z}) \otimes \mathrm{id} \circ e_{X\otimes Y,Z} \otimes \mathrm{id} \circ e_{X,Y} \otimes \mathrm{id} \otimes \mathrm{id}
\end{aligned}$$

Simplifying both expressions, we get:

$$\begin{aligned}
\mathrm{LHS} = {} & \mathrm{id} \otimes \mathrm{id} \otimes e^{-1}_{Z,W} \circ \mathrm{id} \otimes e^{-1}_{Y,Z\otimes W} \circ e^{-1}_{X,Y\otimes(Z\otimes W)} \circ F(a_{X,Y,Z\otimes W}) \circ \\
& \circ F(a_{X\otimes Y,Z,W}) \circ e_{(X\otimes Y)\otimes Z,W} \circ e_{X\otimes Y,Z} \otimes \mathrm{id} \circ e_{X,Y} \otimes \mathrm{id} \otimes \mathrm{id}
\end{aligned}$$

and:

$$\begin{aligned}
\mathrm{RHS} = {} & \mathrm{id} \otimes \mathrm{id} \otimes e^{-1}_{Z,W} \circ \mathrm{id} \otimes e^{-1}_{Y,Z\otimes W} \circ \mathrm{id} \otimes F(a_{Y,Z,W}) \circ e^{-1}_{X,(Y\otimes Z)\otimes W} \circ \\
& \circ F(a_{X,Y\otimes Z,W}) \circ e_{X\otimes(Y\otimes Z),W} \circ F(a_{X,Y,Z}) \otimes \mathrm{id} \circ e_{X\otimes Y,Z} \otimes \mathrm{id} \circ e_{X,Y} \otimes \mathrm{id} \otimes \mathrm{id}
\end{aligned}$$

Now, using the naturality of e in the RHS expression, we have:

$$\begin{aligned}\text{RHS} = & \operatorname{id} \otimes \operatorname{id} \otimes e^{-1}_{Z,W} \circ \operatorname{id} \otimes e^{-1}_{Y,Z\otimes W} \circ e^{-1}_{X,Y\otimes(Z\otimes W)} \circ \operatorname{id} \otimes F(a_{Y,Z,W}) \circ \\ & \circ F(a_{X,Y\otimes Z,W}) \circ F(a_{X,Y,Z}) \otimes \operatorname{id} \circ e_{(X\otimes Y)\otimes Z,W} \circ e_{X\otimes Y,Z} \otimes \operatorname{id} \circ e_{X,Y} \otimes \operatorname{id} \otimes \operatorname{id}\end{aligned}$$

At this stage, we can conclude using the pentagon relation on a. The question of the antipode will be treated afterwards.

Now we want to prove the existence of a R-matrix R in $H \otimes H$. We define:

$$R_{X,Y} := c^{\text{Vect}^{-1}}_{F(X),F(Y)} \circ e^{-1}_{Y,X} \circ F(c_{X,Y}) \circ e_{X,Y} \tag{2.2.8}$$

We prove that:

$$(R\Delta(h)R^{-1})_{X,Y} = \Delta^{\text{op}}(h)_{X,Y}$$

We have:

$$\begin{aligned}(R\Delta(h)R^{-1})_{X,Y} &= c^{\text{Vect}^{-1}}_{F(X),F(Y)} \circ e^{-1}_{Y,X} \circ F(c_{X,Y}) \circ e_{X,Y} \circ \\ &\circ e^{-1}_{X,Y} \circ h_{X\otimes Y} \circ e_{X,Y} \circ e^{-1}_{X,Y} \circ F(c_{X,Y})^{-1} \circ e_{Y,X} \circ c^{\text{Vect}}_{F(X),F(Y)} = \\ &= c^{\text{Vect}^{-1}}_{F(X),F(Y)} \circ e^{-1}_{Y,X} \circ F(c_{X,Y}) \circ h_{X\otimes Y} \circ F(c_{X,Y})^{-1} \circ e_{Y,X} \circ c^{\text{Vect}}_{F(X),F(Y)}\end{aligned}$$

Using naturality of h, we obtain:

$$(R\Delta(h)R^{-1})_{X,Y} = c^{\text{Vect}^{-1}}_{F(X),F(Y)} \circ e^{-1}_{Y,X} \circ h_{Y\otimes X} \circ e_{Y,X} \circ c^{\text{Vect}}_{F(X),F(Y)} = \Delta^{\text{op}}(h)_{X,Y}$$

Next we prove that R satisfies (1.2.17) and (1.2.18). We prove the first of the two identities. The other one will follow similarly. On one side we have:

$$(\Delta \otimes \operatorname{id}(R))_{X,Y} = e^{-1}_{X,Y} \otimes \operatorname{id} \circ \Sigma_{1,2} \circ e^{-1}_{Z,X\otimes Y} \circ F(c_{X\otimes Y,Z}) \circ e_{X\otimes Y,Z} \circ e_{X,Y} \otimes \operatorname{id}$$

and on the other side:

$$\begin{aligned}&(\Phi_{312} \circ R_{13} \circ (\Phi_{132})^{-1} \circ R_{23} \circ \Phi)_{X,Y,Z} = \\ &= \Sigma_{1,2} \circ \operatorname{id} \otimes e^{-1}_{X,Y} \circ e^{-1}_{Z,X\otimes Y} \circ F(a_{Z,X,Y}) \circ e_{Z\otimes X,Y} \circ e_{Z,X} \otimes \operatorname{id} \circ \\ &\circ e^{-1}_{Z,X} \otimes \operatorname{id} \circ F(c_{X,Z}) \otimes \operatorname{id} \circ e_{X,Z} \otimes \operatorname{id} \circ e^{-1}_{X,Z} \otimes \operatorname{id} \circ e^{-1}_{X\otimes Z,Y} \circ \\ &\circ F(a^{-1}_{X,Z,Y}) \circ e_{X,Z\otimes Y} \circ \operatorname{id} \otimes e_{Z,Y} \circ \operatorname{id} \otimes e^{-1}_{Z,Y} \circ \operatorname{id} \otimes F(c_{Y,Z}) \circ \operatorname{id} \otimes e_{Y,Z} \circ \\ &\circ \operatorname{id} \otimes e^{-1}_{Y,Z} \circ e^{-1}_{X,Y\otimes Z} \circ F(a_{X,Y,Z}) \circ e_{X\otimes Y,Z} \circ e_{X,Y} \otimes \operatorname{id}\end{aligned}$$

Using the right invertibility of e we obtain:

$$\begin{aligned}&(\Phi_{312} \circ R_{13} \circ (\Phi_{132})^{-1} \circ R_{23} \circ \Phi)_{X,Y,Z} = \\ &= e^{-1}_{X,Y} \otimes \operatorname{id} \circ \Sigma_{1,2} \circ e^{-1}_{Z,X\otimes Y} \circ F(a_{Z,X,Y}) \circ e_{Z\otimes X,Y} \circ \\ &\circ F(c_{X,Z}) \otimes \operatorname{id} \circ e^{-1}_{X\otimes Z,Y} \circ F(a^{-1}_{X,Z,Y}) \circ e_{X,Z\otimes Y} \circ \\ &\circ \operatorname{id} \otimes F(c_{Y,Z}) \circ e^{-1}_{X,Y\otimes Z} \circ F(a_{X,Y,Z}) \circ e_{X\otimes Y,Z} \circ e_{X,Y} \otimes \operatorname{id}\end{aligned}$$

At this point we can conclude using the naturality of e and the identity (1.1.4) involving a and c. □

Lemma 2.2.7. *If F is a (braided) weak quasi tensor functor then H is a (braided) weak quasi Hopf algebra.*

Proof. Let I be the natural transformation such that $I_X = \mathrm{id}_X\ \forall X \in \mathrm{Ob}(\mathcal{C})$. It is quite easy to see that:

$$\Delta(I)_{X,Y} = e^{-1}_{X,Y} \circ e_{X,Y}$$

Remember that $e_{X,Y} \circ e^{-1}_{X,Y} = \mathrm{id}_{F(X\otimes Y)}$ and $e^{-1}_{X,Y} \circ e_{X,Y} \neq \mathrm{id}_{F(X)\otimes F(Y)}$. Now we are ready to prove that H is a weak quasi Hopf algebra:

$$\begin{aligned}
&(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}(\Phi))_{X,Y} = \\
&= (\mathrm{id}_X \otimes e^{-1}_{\mathbf{1},Y}) \circ e^{-1}_{X,\mathbf{1}\otimes Y} \circ F(a_{X,\mathbf{1},Y}) \circ e_{X\otimes \mathbf{1},Y} \circ e_{X,\mathbf{1}} \otimes \mathrm{id}_Y = \\
&= e^{-1}_{X,Y} \circ e_{X,Y} = \Delta(I)_{X,Y}
\end{aligned}$$

Next:

$$\begin{aligned}
(R^{-1}R)_{X,Y} &= e^{-1}_{X,Y} \circ F(c^{-1}_{X,Y}) \circ e_{Y,X} \circ c^{\mathrm{Vect}}_{F(X),F(Y)} \circ c^{\mathrm{Vect}^{-1}}_{F(X),F(Y)} \circ e^{-1}_{Y,X} \circ F(c_{X,Y}) \circ e_{X,Y} = \\
&= e^{-1}_{X,Y} \circ e_{X,Y} = \Delta(I)_{X,Y}
\end{aligned}$$

Similarly we get:

$$\begin{aligned}
&\Phi^{-1}_{X,Y,Z} \circ \Phi_{X,Y,Z} = \\
&= (e^{-1}_{X,Y} \otimes \mathrm{id}_Z) \circ e^{-1}_{X\otimes Y,Z} \circ F(a^{-1}_{X,Y,Z}) \circ e_{X,Y\otimes Z} \circ (\mathrm{id}_X \otimes e_{Y,Z}) \circ \\
&\circ (\mathrm{id}_X \otimes e^{-1}_{Y,Z}) \circ e^{-1}_{X,Y\otimes Z} \circ F(a_{X,Y,Z}) \circ e_{X\otimes Y,Z} \circ (e_{X,Y} \otimes \mathrm{id}_Z) = \\
&= (e^{-1}_{X,Y} \otimes \mathrm{id}_Z) \circ e^{-1}_{X\otimes Y,Z} \circ e_{X\otimes Y,Z} \circ (e_{X,Y} \otimes \mathrm{id}_Z) = \\
&= (e^{-1}_{X,Y} \otimes \mathrm{id}_Z) \circ \Delta(I)_{X\otimes Y,Z} \circ (e_{X,Y} \otimes \mathrm{id}_Z) = \\
&= (\Delta \otimes \mathrm{id}(\Delta(I)))_{X,Y,Z}
\end{aligned}$$

In the same way we can prove that $\Phi^{-1}_{X,Y,Z} \circ \Phi_{X,Y,Z} = (\mathrm{id} \otimes \Delta(\Delta(I)))_{X,Y,Z}$. □

Lemma 2.2.8. *If F is a (braided) weak tensor functor, then H is a (braided) weak Hopf algebra.*

Proof. We defined the associator Φ in the following way:

$$\Phi_{X,Y,Z} = (\mathrm{id} \otimes e^{-1}_{Y,Z}) \circ e^{-1}_{X,Y\otimes Z} \circ F(a_{X,Y,Z}) \circ e_{X\otimes Y,Z} \circ (e_{X,Y} \otimes \mathrm{id})$$

Looking back at the Theorem 1.4.17, we have:

$$F(a_{X,Y,Z}) = e_{X,Y\otimes Z} \circ \mathrm{id}_{F(X)} \otimes e_{Y,Z} \circ e^{-1}_{X,Y} \otimes \mathrm{id}_{F(Z)} \circ e^{-1}_{X\otimes Y,Z}$$

Using this identity in the defining expression of Φ, we get:

$$\begin{aligned}
\Phi &= (\mathrm{id}_{F(X)} \otimes e^{-1}_{Y,Z}) \circ e^{-1}_{X,Y\otimes Z} \circ e_{X,Y\otimes Z} \circ (\mathrm{id}_{F(X)} \otimes e_{Y,Z}) \circ \\
&\circ (e^{-1}_{X,Y} \otimes \mathrm{id}_{F(Z)}) \circ e^{-1}_{X\otimes Y,Z} \circ e_{X\otimes Y,Z} \circ (e_{X,Y} \otimes \mathrm{id}_{F(Z)}) = \\
&= (\mathrm{id} \otimes \Delta(\Delta(I)))(\Delta \otimes \mathrm{id}(\Delta(I)))
\end{aligned}$$

In a similar way we can prove the assertion concerning Φ^{-1}. Since Φ and Φ^{-1} are defined as in 2.2.6, they automatically satisfy all the conditions in order to be an associator and its inverse, respectively. As a consequence, the idempotents P_2, P_3, Q_3, P_4, Q_4 defined as in (1.2.28) must enjoy the relations (1.2.29), (1.2.30) and (1.2.31). □

Lemma 2.2.9. *The vector spaces $F(X)$ are representation spaces of H. The functor $G : \mathcal{C} \to \mathrm{Rep}(H)$ is a braided tensor functor.*

Proof. Let $\rho_X : H \to \mathrm{End}(F(X))$ be the representation such that:

$$\rho_X(h) = h_X \quad h \in H$$

It is straightforward to prove that ρ_X is a representation of H. This induces a functor $G : \mathcal{C} \to \mathrm{Rep}(H)$ such that $G(X) = F(X)$ and $G(X) \otimes G(Y) = \Delta(I)_{X,Y}(F(X) \otimes F(Y))$. In addition, we put $G(f) = F(f)$, so $G(f)$ is an intertwiner:

$$G(f) \circ \rho_X(h) = F(f) \circ h_X = h_Y \circ F(f) = \rho_Y(h) \circ G(f)$$

We can see that $e_{X,Y}$ is an isomorphism between $G(X) \otimes G(Y)$ and $G(X \otimes Y)$. In fact, being $\Delta(I)_{X,Y}(v \otimes w)$ an element in $G(X) \otimes G(Y)$, and using the identity $\Delta(I)_{X,Y} = e^{-1}_{X,Y} \circ e_{X,Y}$, we obtain:

$$\begin{aligned} e^{-1}_{X,Y} \circ e_{X,Y}(\Delta(I)_{X,Y}(v \otimes w)) = e^{-1}_{X,Y} \circ e_{X,Y} \circ (e^{-1}_{X,Y} \circ e_{X,Y}(v \otimes w)) = \\ = (e^{-1}_{X,Y} \circ (e_{X,Y} \circ e^{-1}_{X,Y}) \circ e_{X,Y})(v \otimes w) = e^{-1}_{X,Y} \circ e_{X,Y}(v \otimes w) = \Delta(I)_{X,Y}(v \otimes w) \end{aligned}$$

Moreover, the definitions of Φ and R in H allow us to state that G is a braided tensor functor. □

Lemma 2.2.10. *Set $X, Y \in \mathrm{Ob}(\mathcal{C})$ and $h \in H$. If X and Y are isomorphic, then h_X is determined uniquely by h_Y. If $\mathcal{C}$ is semisimple, then $h \in H$ is determined by its values on ∇.*

Proof. The first statement is a direct consequence of the naturality of h. Next, we assume that $\mathcal{C}$ is semisimple, so every object is isomorphic to the direct sum of simple objects. Therefore, it is sufficient to define h on the direct sums of simple objects. It remains to prove that it is determined by its values on ∇. Consider $\bigoplus_i X_i$, where $X_i \in \nabla$. We have morphisms $p_j \in (\bigoplus_i X_i, X_j)$ and $q_j \in (X_j, \bigoplus_i X_i)$, such that:

$$\sum_j q_j \circ p_j = \mathrm{id}_{\bigoplus X_i}$$

Naturality implies that:

$$F(p_j) \circ h_{\bigoplus X_i} = h_{X_j} \circ F(p_j)$$

Hence:

$$\begin{aligned} h_{\bigoplus X_i} = F\left(\sum_j q_j \circ p_j\right) \circ h_{\bigoplus X_i} = \sum_j F(q_j) \circ F(p_j) \circ h_{\bigoplus X_i} = \\ = \sum_j F(q_j) \circ h_{X_j} \circ F(p_j) \end{aligned}$$

□

Lemma 2.2.11. *There exists an algebra isomorphism:*

$$\psi : H \longrightarrow \bigoplus_i \mathrm{End}(F(X_i)) \tag{2.2.9}$$

Proof. We define ψ in the following way:

$$\psi(h) = \bigoplus_i h_{X_i}$$

It is easy to see that it is an algebra isomorphism. We need to prove that it is injective and surjective. If $\psi(h) = 0$, then $h_{X_i} = 0\ \forall X_i \in \nabla$. Let X be an object in $\mathcal{C}$. If φ is an isomorphism in $(X, \bigoplus X_i)$ and p_j, q_j as in the proof of the previous lemma, we define:

$$s_j = p_j \circ \varphi \text{ and } t_j = \varphi^{-1} \circ q_j$$

with $s_j \in (X, X_j)$ and $t_j \in (X_j, X)$. It is an easy computation to see that:

$$s_j \circ t_j = \mathrm{id}_{X_j} \text{ and } \sum_i t_i \circ s_i = \mathrm{id}_X$$

Proceeding as in the proof of the last lemma, we have:

$$h_X = \sum_j F(t_j) \circ h_{X_j} \circ F(s_j) = 0$$

We need to prove surjectivity of ψ. Given $b_i \in \mathrm{End}(F(X_i))$ where $X_i \in \nabla$, and $X \in \mathrm{Ob}(\mathcal{C})$, we define $b_X \in \mathrm{End}(F(X))$ in the following way:

$$b_X = \sum_j F(t_j) \circ b_j \circ F(s_j)$$

where $s_j \in (X, X_j)$ and $t_j \in (X_j, X)$ are as before, and $X \cong \bigoplus_{i\in\nabla} X_i$. The definition of b_X does not depend on the choice of s_j, t_j. In fact, let s'_j, t'_j be with the same features of s_j and t_j. We define:

$$\widetilde{b_X} = \sum_j F(t'_j) \circ b_j \circ F(s'_j)$$

Moreover:

$$s_i \circ t'_j = \delta_{i,j}\lambda_j^{(1)}\, \mathrm{id}_{X_j} \text{ and } s'_i \circ t_j = \delta_{i,j}\lambda_j^{(2)}\, \mathrm{id}_{X_j}$$

where $\lambda_j^{(1)}, \lambda_j^{(2)} \in \mathbb{C}$. So:

$$F(s'_j) \circ \widetilde{b_X} \circ F(t'_j) = b_j = {\lambda_j^{(1)}}^{-1} {\lambda_j^{(2)}}^{-1} F(s'_j) \circ b_X \circ F(t'_j) = F(s'_j) \circ b_X \circ F(t'_j)$$

In the next calculation we explain why $\lambda_j^{(1)}\lambda_j^{(2)} = 1$:

$$\lambda_j^{(1)}\lambda_j^{(2)}\, \mathrm{id}_{X_j} = s_j \circ t'_j \circ s'_j \circ t_j =$$
$$= s_j \circ \left(\sum_h t'_h \circ s'_h \right) \circ t_j = s_j \circ t_j = \mathrm{id}_{X_j}$$

It is quite obvious that:

$$F(s'_j) \circ \widetilde{b_X} \circ F(t'_k) = 0 = F(s'_j) \circ b_X \circ F(t'_k)$$

Putting everything together we get:

$$\begin{aligned}
\widetilde{b_X} &= \left(\sum_j F(t'_j) \circ F(s'_j)\right) \circ \widetilde{b_X} \circ \left(\sum_k F(t'_k) \circ F(s'_k)\right) = \\
&= \sum_{j,k} F(t'_j) \circ \left(F(s'_j) \circ \widetilde{b_X} \circ F(t'_k)\right) \circ F(s'_k) = \\
&= \sum_{j,k} F(t'_j) \circ \left(F(s'_j) \circ b_X \circ F(t'_k)\right) \circ F(s'_k) = \\
&= \left(\sum_j F(t'_j) \circ F(s'_j)\right) \circ b_X \circ \left(\sum_k F(t'_k) \circ F(s'_k)\right) = b_X
\end{aligned}$$

It remains to show the naturality of b. Let f be an arrow in (X, Y), $\widetilde{s_j} \in (Y, X_j)$ and $\widetilde{t_j} \in (X_j, Y)$ with the same properties of s_j, t_j such that $b_Y = \sum_j F(\widetilde{t_j}) \circ b_i \circ F(\widetilde{s_j})$. Then:

$$F(f) \circ b_X = \sum_i F(f \circ t_i) \circ b_i \circ F(s_i) = \sum_{i,j} F(\widetilde{t_j} \circ \widetilde{s_j} \circ f \circ t_i) \circ b_i \circ F(s_i)$$

Since $\widetilde{s_j} \circ f \circ t_i \in (X_i, X_j)$, it is a scalar multiple of id_{X_i} if $X_i \cong X_j$, or 0 otherwise. So:

$$\begin{aligned}
F(f) \circ b_X &= \sum_{i,j} F(\widetilde{t_j}) \circ b_j \circ F(\widetilde{s_j} \circ f \circ t_i \circ s_i) = \\
&= \sum_j F(\widetilde{t_j}) \circ b_j \circ F(\widetilde{s_j} \circ f \circ (\sum_i t_i \circ s_i)) = \\
&= \sum_j F(\widetilde{t_j}) \circ b_j \circ F(\widetilde{s_j} \circ f) = b_Y \circ F(f)
\end{aligned}$$

□

Lemma 2.2.12. *F and G are essentially surjective and full.*

Proof. It is a well-known result that if A is an algebra isomorphic to the direct sum of $\mathrm{End}(V_i)$, where V_i are vector spaces, then all the irreducible representations are equivalent to $p_i : A \to \mathrm{End}(V_i)$, and p_i, p_j are equivalent iff $i = j$. So, using the previous lemma, we can state that every irreducible representation V is isomorphic to $F(X_i)$ for some i as object in $\mathrm{Rep}(H)$. Since $\mathrm{Rep}(H)$ is semisimple, W as representation of H is isomorphic to the direct sum of $F(X_i)$. Now we prove that F and G are full. Since $F(X_i)$ is an irreducible representation, we have:

$$\begin{gathered}
(F(X_i), F(X_j)) = \{0\} \\
\mathrm{End}(F(X_i)) = \mathbb{C}\,\mathrm{id}_{F(X_i)}
\end{gathered}$$

Now, let f be an arrow in $(F(X), F(Y))$. We use the map $s_i \in (X, X_i)$, $t_i \in (X_i, X)$, $s'_i \in (Y, X_i)$, $t'_i \in (X_i, Y)$, introduced in the proof of the previous lemma. We have $F(s'_i) \circ f \circ F(t_i) \in \mathrm{End}(F(X_i))$, where X_i is a simple object appearing in the decomposition of X. So there exists a $\lambda_i \in \mathbb{C}$ such that:

$$F(s'_i) \circ f \circ F(t_j) = \delta_{i,j} \lambda_i \,\mathrm{id}_{F(X_i)}$$

Composing on the left by $F(t_i')$, on the right by $F(s_i)$ and summing up on i and j we get:

$$\sum_{i,j} F(t_i') \circ F(s_i') \circ f \circ F(t_j) \circ F(s_j) = \sum_{i,j} \delta_{i,j} \lambda_i F(t_i') \circ F(s_j) \Rightarrow$$

$$\Rightarrow \left(\sum_i F(t_i') \circ F(s_i') \right) \circ f \circ \left(\sum_j F(t_j) \circ F(s_j) \right) = \sum_i \lambda_i F(t_i') \circ F(s_i) \Rightarrow$$

$$\Rightarrow f = F\left(\sum_i \lambda_i t_i' \circ s_i \right)$$

□

Lemma 2.2.13. *Faithfulness of F implies that inequivalent objects yield inequivalent representations.*

Proof. Assume X and Y to be inequivalent object and $F(X) \cong F(Y)$, and call $\varphi : F(X) \to F(Y)$ an isomorphism. Since F is full and $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, we have that $\varphi = F(f)$ and $\varphi^{-1} = F(g)$ such that:

$$F(f) \circ F(g) = \mathrm{id}_{F(Y)} \Rightarrow F(f \circ g) = F(\mathrm{id}_Y)$$

Using again the faithfulness of F we get that $f \circ g = \mathrm{id}_Y$. Similarly we obtain $g \circ f = \mathrm{id}_X$. □

Lemma 2.2.14. *If $\mathcal{C}$ is a *-category and F a rigid and *-preserving functor, then H is a weak quasi Hopf *-algebra. If $\mathcal{C}$ is a C*-category, H is a C*-algebra.*

Proof. We define the involution on H in the following way:

$$(h^*)_X = (h_X)^*$$

It is straightforward to prove that it is an involutive anti-linear map on H as an algebra. Moreover, $\varepsilon(h^*) = \overline{\varepsilon(h)}$ since:

$$\varepsilon(h^*) = (h^*)_{\mathbf{1}} = (h_{\mathbf{1}})^* = \overline{h_{\mathbf{1}}} = \overline{\varepsilon(h)}$$

We need to prove the existence of an invertible self-adjoint element $\Omega \in H \otimes H$ such that:

$$\Delta(h)^* = \Omega \Delta(h^*) \Omega^{-1} \tag{2.2.10}$$

$$(\Phi^{-1})^* = (I \otimes \Omega)(\mathrm{id} \otimes \Delta(\Omega)) \Phi (\Delta \otimes \mathrm{id}(\Omega^{-1}))(\Omega^{-1} \otimes I) \tag{2.2.11}$$

$$\varepsilon \otimes \mathrm{id}(\Omega) = I = \mathrm{id} \otimes \varepsilon(\Omega) \tag{2.2.12}$$

We set $\Omega_{X,Y} = e^*_{X,Y} \circ e_{X,Y}$. Self-adjointness is obvious. The inverse element of Ω is:

$$\Omega^{-1} = e^{-1}_{X,Y} \circ {e^*_{X,Y}}^{-1}$$

So:

$$\Omega\Omega^{-1} = e^*_{X,Y} \circ {e^*_{X,Y}}^{-1} = (e^{-1}_{X,Y} \circ e_{X,Y})^* = \Delta(I)^*_{X,Y}$$

$$\Omega^{-1}\Omega = e^{-1}_{X,Y} \circ e_{X,Y} = \Delta(I)_{X,Y}$$

Now we prove (2.2.10):

$$\begin{aligned}\Delta(h)^*{}_{X,Y} &= (\Delta(h)_{X,Y})^* = e^*_{X,Y} \circ h^*_{X\otimes Y} \circ {e^{-1}_{X,Y}}^* = \\ &= e^*_{X,Y} \circ e_{X,Y} \circ e^{-1}_{X,Y} \circ (h^*)_{X\otimes Y} \circ e_{X,Y} \circ e^{-1}_{X,Y} \circ {e^{-1}_{X,Y}}^* = \\ &= (e^*_{X,Y} \circ e_{X,Y}) \circ \Delta(h^*)_{X,Y} \circ (e^{-1}_{X,Y} \circ {e^{-1}_{X,Y}}^*)\end{aligned}$$

The next calculation allows us to prove (2.2.11):

$$\begin{aligned}\Phi^{-1^*} &= \mathrm{id}_X \otimes e^*_{Y,Z} \circ e^*_{X,Y\otimes Z} \circ F({a^{-1}_{X,Y,Z}}^*) \circ {e^{-1}_{X\otimes Y,Z}}^* \circ {e^{-1}_{X,Y}}^* \otimes \mathrm{id}_Z = \\ &= (I\otimes\Omega)(\mathrm{id}\otimes\Delta(\Omega)) \circ \mathrm{id}_X \otimes e^{-1}_{Y,Z} \circ e^{-1}_{X,Y\otimes Z} \circ F({a^{-1}_{X,Y,Z}}^*) \circ \\ &\circ e_{X\otimes Y,Z} \circ \mathrm{id}_X \otimes e_{Y,Z} \circ (\Delta\otimes\mathrm{id}(\Omega^{-1}))(\Omega^{-1}\otimes I) = \\ &= (I\otimes\Omega)(\mathrm{id}\otimes\Delta(\Omega)) \circ \mathrm{id}_X \otimes e^{-1}_{Y,Z} \circ e^{-1}_{X,Y\otimes Z} \circ F(a_{X,Y,Z}) \circ \\ &\circ e_{X\otimes Y,Z} \circ \mathrm{id}_X \otimes e_{Y,Z} \circ (\Delta\otimes\mathrm{id}(\Omega^{-1}))(\Omega^{-1}\otimes I) = \\ &= (I\otimes\Omega)(\mathrm{id}\otimes\Delta(\Omega))\Phi(\Delta\otimes\mathrm{id}(\Omega^{-1}))(\Omega^{-1}\otimes I)\end{aligned}$$

while (2.2.12) is immediate. Suppose now that $\mathcal{C}$ is a C*-category. We want to prove that H is a C*-algebra. We put the following norm on H:

$$\|h\|^2 = \sum_{X_i\in\nabla} \|h_{X_i}\|_i^2$$

The definition does not depend on the choice of the X_i in the isomorphism classes. Completeness is a consequence of the completeness of $\mathrm{End}(F(X_i))$ as C*-algebras. Subadditivity is a consequence of the Cauchy-Schwartz inequality, while submultiplicativity is straightforward to prove. If $\|h\| = 0$, then $h_{X_i} = 0\ \forall X_i \in \nabla$. So, for what we said in Lemma 2.2.10, $h = 0$. Finally, the C*-property is a direct consequence of the fact that $\|\cdot\|_i$ is a C*-norm on $\mathrm{End}(F(X_i))$. Finally, we must prove that Ω and R satisfy the following relation:

$$\Omega_{21}R = R^{-1^*}\Omega$$

On the left hand side, we have:

$$\begin{aligned}\Sigma \circ (e^*_{Y,X} \circ e_{Y,X}) \circ e^{-1}_{Y,X} \circ F(c_{X,Y}) \circ e_{X,Y} &= \\ = \Sigma \circ e^*_{Y,X} \circ (e_{Y,X} \circ e^{-1}_{Y,X}) \circ F(c_{X,Y}) \circ e_{X,Y} &= \\ = \Sigma \circ e^*_{Y,X} \circ F(c_{X,Y}) \circ e_{X,Y}&\end{aligned}$$

On the right hand side, we have:

$$\begin{aligned}\Sigma \circ e^*_{Y,X} \circ F(c_{X,Y}) \circ {e^{-1}_{X,Y}}^* \circ (e^*_{X,Y} \circ e_{X,Y}) &= \\ = \Sigma \circ e^*_{Y,X} \circ F(c_{X,Y}) \circ ({e^{-1}_{X,Y}}^* \circ e^*_{X,Y}) \circ e_{X,Y} &= \\ = \Sigma \circ e^*_{Y,X} \circ F(c_{X,Y}) \circ e_{X,Y}&\end{aligned}$$

□

Remark 2.2.15. We introduce the antipode on H. Since $\mathcal{C}$ and F are rigid, for every object X there exists a conjugate object $\overline{X}$, and a natural isomorphism:

$$d_X : \overline{F(X)} \longrightarrow F(\overline{X})$$

We use it in the definition of the antipode:

$$(S(h))_X = d_X^\vee \circ h_{\overline{X}}^\vee \circ d_X^{\vee\,-1} \tag{2.2.13}$$

where $^\vee$: Vect $\to$ Vect is a functor which sends V into $\overline{V} = V^*$, and $f \in (U, V)$ into the dual map $f^\vee \in (\overline{V}, \overline{U})$, transpose arrow in Vect of $f \in (U, V)$. It is an easy calculation to see that:

$$f^\vee(\varphi_V) = \varphi_V \circ f$$

In fact, Vect is a rigid tensor category with the following maps:

$$\delta_V^\dagger : V^* \otimes V \to \mathbb{C} \text{ such that } \delta_V^\dagger(\varphi \otimes v) = \varphi(v)$$
$$\overline{\delta}_V : \mathbb{C} \to V \otimes V^* \text{ such that } \overline{\delta}_V(1) = \sum_i e_i \otimes e_i^*$$

Therefore:

$$\begin{aligned} f^\vee(\varphi_V) &= \delta_V^\dagger \otimes \mathrm{id}_U(\mathrm{id}_{V^*} \otimes f \otimes \mathrm{id}_{U^*}(\mathrm{id}_{V^*} \otimes \overline{\delta}_V(\varphi_V))) = \\ &= \delta_V^\dagger \otimes \mathrm{id}_U(\mathrm{id}_{V^*} \otimes f \otimes \mathrm{id}_{U^*}(\varphi_V \otimes e_i \otimes e_i^*)) = \\ &= \delta_V^\dagger(\varphi_V \otimes f(e_i)) \otimes e_i^* = \varphi_V(f(e_i))e_i^* = \\ &= (\varphi_V \circ f(e_i))e_i^* = \varphi_V \circ f \end{aligned}$$

Since $\mathcal{C}$ is a *-category, it is possible to rewrite S. In fact, let $J_V : V \to \overline{V}$ be the map which sends v into $\overline{v} = (v, \cdot)$. If $f \in (U, V)$, $f^* \in (V, U)$ and:

$$f^* = J_U^{-1} \circ f^\vee \circ J_V \tag{2.2.14}$$

This is a consequence of the following calculation:

$$\begin{aligned} (J_U^{-1} \circ f^\vee \circ J_V)(v) &= J_U^{-1}(f^\vee(\overline{v})) = J_U^{-1}(f^\vee((v, \cdot))) = \\ = J_U^{-1}((v, f(\cdot))) &= J_U^{-1}((f^*(v), \cdot)) = J_U^{-1}(\overline{f^*(v)}) = f^*(v) \end{aligned}$$

Hence, if $\mathcal{C}$ is a *-category, the antipode becomes:

$$S(h)_X = J_{F(X)}^{-1} \circ d_X^{-1} \circ (h_{\overline{X}})^* \circ d_X \circ J_{F(X)} \tag{2.2.15}$$

since:

$$\begin{aligned} & d_X^\vee \circ h_{\overline{X}}^\vee \circ d_X^{\vee\,-1} = \\ &= (J_{F(X)}^{-1} \circ d_X^* \circ J_{\overline{F(X)}}^{-1}) \circ (J_{\overline{F(X)}} \circ h_{\overline{X}}^* \circ J_{\overline{F(X)}}^{-1}) \circ (J_{F(\overline{X})} \circ d_X^{-1\,*} \circ J_{F(X)}) = \\ &= J_{F(X)}^{-1} \circ d_X^{-1} \circ (h_{\overline{X}})^* \circ d_X \circ J_{F(X)} \end{aligned}$$

Finally, the dual representation $\overline{V}$ is given by:

$$\pi_{\overline{X}}(h) = \pi_X(S(h))^\vee$$

Using (2.2.14) the dual representation becomes:

$$\pi_{\overline{X}}(h) = J_{F(X)} \circ \pi_X(S(h))^* \circ J_{F(X)}^{-1}$$

and, more explicitly:

$$\begin{aligned} \pi_{\overline{X}}(h)(\overline{v}) &= J_{F(X)} \circ \pi_X(S(h))^* \circ J_{F(X)}^{-1}(\overline{v}) = \\ = J_{F(X)} \circ (J_{F(X)}^{-1} \circ d_X^{-1} \circ h_{\overline{X}} \circ d_X \circ J_{F(X)}) &\circ J_{F(X)}^{-1}(\overline{v}) = \\ &= d_X^{-1} \circ h_{\overline{X}} \circ d_X(\overline{v}) \end{aligned}$$

Finally it is easy to see that S commutes with *.

Lemma 2.2.16. $\mathrm{Rep}(H)$ *is rigid, and* S *defined in the Remark 2.2.15 is an antipode.*

Proof. Suppose that $r^\dagger$ and $\overline{r}$ are the conjugate maps in $\mathcal{C}$. We will prove that $\mathrm{Rep}(H)$ is rigid with the following conjugate maps:

$$\rho^\dagger = F(r^\dagger) \circ e_{\overline{X},X} \circ d_X \otimes \mathrm{id} \tag{2.2.16}$$
$$\overline{\rho} = \mathrm{id} \otimes d_X^{-1} \circ e^{-1}_{X,\overline{X}} \circ F(\overline{r}) \tag{2.2.17}$$

The first step is to prove that $\rho^\dagger$ and $\overline{\rho}$ are morphisms:

$$\begin{aligned} &\rho^\dagger \circ \Delta(h)_{\overline{X},X} = \\ &= F(r^\dagger) \circ e_{\overline{X},X} \circ d_X \otimes \mathrm{id} \circ d_X^{-1} \otimes \mathrm{id} \circ e^{-1}_{\overline{X},X} \circ h_{\overline{X}\otimes X} \circ e_{\overline{X},X} \circ d_X \otimes \mathrm{id}_X = \\ &= F(r^\dagger) \circ h_{\overline{X}\otimes X} \circ e_{\overline{X},X} \circ d_X \otimes \mathrm{id} = \\ &= h_1 \circ F(r^\dagger) \circ e_{\overline{X},X} \circ d_X \otimes \mathrm{id} = \varepsilon(h)\rho^\dagger \end{aligned}$$

and:

$$\begin{aligned} &\Delta(h)_{X,\overline{X}} \circ \overline{\rho} = \\ &= \mathrm{id} \otimes d_X^{-1} \circ \Delta(h)_{X,\overline{X}} \circ \mathrm{id} \otimes d_X \circ \mathrm{id} \otimes d_X^{-1} \circ e^{-1}_{X,\overline{X}} \circ F(\overline{r}) = \\ &= \mathrm{id} \otimes d_X^{-1} \circ \Delta(h)_{X,\overline{X}} \circ e^{-1}_{X,\overline{X}} \circ F(\overline{r}) = \\ &= \mathrm{id} \otimes d_X^{-1} \circ e^{-1}_{X,\overline{X}} \circ h_{X\otimes\overline{X}} \circ e_{X,\overline{X}} \circ e^{-1}_{X,\overline{X}} \circ F(\overline{r}) = \\ &= \mathrm{id} \otimes d_X^{-1} \circ e^{-1}_{X,\overline{X}} \circ h_{X\otimes\overline{X}} \circ F(\overline{r}) = \\ &= \mathrm{id} \otimes d_X^{-1} \circ e^{-1}_{X,\overline{X}} \circ F(\overline{r}) \circ h_1 = \overline{\rho}\varepsilon(h) \end{aligned}$$

Next, we want to prove that $\rho^\dagger$ and $\overline{\rho}$ satisfy the conjugate equations. We will prove only one of the two equations. The other will follow similarly.

$$\begin{aligned} &\rho^\dagger \otimes \mathrm{id} \circ \Phi^{-1} \circ \mathrm{id} \otimes \overline{\rho} = \\ &= F(r^\dagger) \otimes \mathrm{id} \circ e_{\overline{X},X} \otimes \mathrm{id} \circ d_X \otimes \mathrm{id} \otimes \mathrm{id} \circ d_X^{-1} \otimes \mathrm{id} \otimes d_X^{-1} \circ \\ &\circ e^{-1}_{\overline{X},X} \otimes \mathrm{id} \circ e^{-1}_{\overline{X}\otimes X,\overline{X}} \circ F(a^{-1}_{\overline{X},X,\overline{X}}) \circ e_{\overline{X},X\otimes\overline{X}} \circ \mathrm{id}_{\overline{X}} \otimes e_{X,\overline{X}} \circ \\ &\circ d_X \otimes \mathrm{id} \otimes d_X \circ \mathrm{id} \otimes \mathrm{id} \otimes d_X^{-1} \circ \mathrm{id} \otimes e^{-1}_{X,\overline{X}} \circ \mathrm{id} \otimes F(\overline{r}) = \\ &= F(r^\dagger) \otimes \mathrm{id} \circ \mathrm{id} \otimes \mathrm{id} \otimes d_X^{-1} \circ e^{-1}_{\overline{X}\otimes X,\overline{X}} \circ F(a^{-1}_{\overline{X},X,\overline{X}}) \circ \\ &\circ e_{\overline{X},X\otimes\overline{X}} \circ d_X \otimes \mathrm{id} \otimes \mathrm{id} \circ \mathrm{id} \otimes F(\overline{r}) = \\ &= d_X^{-1} \circ F(r^\dagger \otimes \mathrm{id} \circ a^{-1}_{\overline{X},X,\overline{X}} \circ \mathrm{id} \otimes \overline{r}) \circ d_X = \mathrm{id} \end{aligned}$$

In order to prove the properties of the antipode, it is sufficient to find α and β such that:

$$\rho^\dagger(\overline{x} \otimes y) = (x, \alpha y) \text{ and } \overline{\rho}(1) = \sum_{i=1}^{n} \beta e_i \otimes \overline{e_i}$$

where $\{e_i\}_{i=1}^n$ is a basis of $F(X)$. It is straightforward to see that:

$$\alpha = (\mathrm{id} \otimes \rho^\dagger) \circ (\overline{\delta} \otimes \mathrm{id}) \tag{2.2.18}$$
$$\beta = (\mathrm{id} \otimes \delta^\dagger) \circ (\overline{\rho} \otimes \mathrm{id}) \tag{2.2.19}$$

where $\delta^\dagger : \overline{F(X)} \otimes F(X) \to \mathbb{C}$ is such that:

$$\delta^\dagger(\overline{x} \otimes y) = (x, y)$$

and $\overline{\delta} : \mathbb{C} \to F(X) \otimes \overline{F(X)}$ is such that:

$$\overline{\delta}(1) = \sum_{i=1}^{n} e_i \otimes \overline{e_i}$$

At this stage, proving that S is an antipode is straightforward, and it is merely a consequence of Prop. 1.4.10. $\square$

Summing up all the results, we can state:

Theorem 2.2.17. *Let $\mathcal{C}$ be a rational semisimple braided rigid tensor category, and $F : \mathcal{C} \to \mathrm{Vect}$ a weak quasi tensor functor. Then:*

(a) *$H = \mathrm{Nat}(F)$ is a f.d. weak quasi Hopf algebra;*

(b) *there is a functor $G : \mathcal{C} \to \mathrm{Rep}(H)$ such that $F = V \circ G$, where $V : \mathrm{Rep}(H) \to \mathrm{Vect}$ is the forgetful functor. G is full and essentially surjective. If F is faithful, G is faithful and maps inequivalent objects to inequivalent objects;*

(c) *$\mathcal{C}$ and $\mathrm{Rep}(H)$ are equivalent braided tensor categories;*

(d) *if F is faithful, $\mathrm{Rep}(H)$ is rigid;*

(e) *if $\mathcal{C}$ is a *-category and F is rigid and *-preserving, then H is a weak quasi Hopf *-algebra. If $\mathcal{C}$ is a C^*-category, H is a C^*-algebra;*

(f) *if F is a tensor functor, H is a Hopf algebra;*

(g) *If F is a quasi tensor functor, H is a quasi Hopf algebra;*

(h) *If F is a weak tensor functor, H is a weak Hopf algebra.*

Before concluding the chapter, we say something about the uniqueness of the construction. The reconstruction of H we have presented is highly non-unique. In fact, changing the weak dimension function we obtain a different weak quasi tensor functor F. Moreover, we can have two different weak quasi tensor functors with the same weak dimension function, because they can differ in the choice of the epimorphisms.

Proposition 2.2.18. *Let $G, \widetilde{G} : \mathcal{C} \to \mathrm{Vect}$ be two different faithful weak quasi tensor functors constructed by the same weak dimension function. Then the reconstructed weak quasi Hopf algebras H and $\widetilde{H}$ are equal up to twist equivalence.*

Proof. Since G and $\widetilde{G}$ share the same weak dimension function, there exists a natural isomorphism φ such that:

$$\widetilde{e}_{X,Y} = \varphi_{X\otimes Y} \circ e_{X,Y}$$

We can see that $\widetilde{H} = H_F$, where $F \in H \otimes H$ is such that $F_{X,Y} = e^{-1}_{X,Y} \circ \varphi^{-1}_{X\otimes Y} \circ e_{X,Y}$. It is quite easy to see that $F \in H \otimes H$. We don't check every calculation. For example,

$$\begin{aligned}
\widetilde{\Delta}(h)_{X,Y} &= \widetilde{e}^{-1}_{X,Y} \circ h_{X\otimes Y} \circ \widetilde{e}_{X,Y} = \\
&= e^{-1}_{X,Y} \circ \varphi^{-1}_{X\otimes Y} \circ h_{X\otimes Y} \circ \varphi_{X\otimes Y} \circ e_{X,Y} = \\
&= F_{X,Y} \circ e^{-1}_{X,Y} \circ h_{X\otimes Y} \circ e_{X,Y} \circ F^{-1}_{X,Y} = \\
&= F_{X,Y} \circ \Delta(h)_{X,Y} \circ F^{-1}_{X,Y}
\end{aligned}$$

and:

$$\begin{aligned}
\widetilde{R}_{X,Y} &= \Sigma \circ \widetilde{e}^{-1}_{Y,X} \circ F(c_{X,Y}) \circ \widetilde{e}_{X,Y} = \\
&= \Sigma \circ e^{-1}_{Y,X} \circ \varphi^{-1}_{Y\otimes X} \circ F(c_{X,Y}) \circ \varphi_{X\otimes Y} \circ e_{X,Y} = \\
&= \Sigma \circ e^{-1}_{Y,X} \circ \varphi^{-1}_{Y\otimes X} \circ e_{Y,X} \circ \Sigma \circ \Sigma \circ \\
&\circ e^{-1}_{Y,X} \circ F(c_{X,Y}) \circ e_{X,Y} \circ e^{-1}_{X,Y} \circ \varphi_{X\otimes Y} \circ e_{X,Y} = \\
&= \Sigma \circ F_{Y,X} \circ \Sigma \circ R_{X,Y} \circ F^{-1}_{X,Y} = (F_{21} R F^{-1})_{X,Y}
\end{aligned}$$

$\square$

Chapter 3

Quantum groups at roots of unity and Wenzl's functor

3.1 Ribbon categories

In this section and in the next one we will give a quick review of the main results about ribbon categories and ribbon algebras. We refer to [35] for a deeper look. Let $\mathcal{C}$ be a strict rigid braided tensor category, and $c_{X,Y}$ its braiding.

Proposition 3.1.1. *$c_{X,Y}$ satisfies the following relations:*

$$c_{\overline{X},Y} = r_X^\dagger \otimes \mathrm{id}_{\overline{X}\otimes Y} \circ \mathrm{id}_{\overline{X}} \otimes c_{X,Y}^{-1} \otimes \mathrm{id}_X \circ \mathrm{id}_{\overline{X}\otimes Y} \otimes \overline{r}_X \tag{3.1.1}$$

$$c_{X,\overline{Y}} = \mathrm{id}_{\overline{Y}} \otimes \mathrm{id}_X \otimes r_{\overline{Y}}^\dagger \circ \mathrm{id}_{\overline{Y}} \otimes c_{X,Y}^{-1} \otimes \mathrm{id}_{\overline{Y}} \circ \overline{r}_{\overline{Y}} \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{Y}} \tag{3.1.2}$$

Proof. Using the rigidity of $\mathcal{C}$ and the naturality of c we have:

$$\begin{aligned} c_{\overline{X},Y} &= c_{\overline{X},Y} \circ r_X^\dagger \otimes \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_Y \circ \mathrm{id}_{\overline{X}} \otimes \overline{r}_X \otimes \mathrm{id}_Y \circ \mathrm{id}_{\overline{X}} \otimes c_{\mathbf{1},Y}^{-1} = \\ &= c_{\overline{X},Y} \circ r_X^\dagger \otimes \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_Y \circ \mathrm{id}_{\overline{X}} \otimes c_{X\otimes\overline{X},Y}^{-1} \circ \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_Y \otimes \overline{r}_X \end{aligned}$$

Since $c_{X\otimes Y,Z} = c_{X,Z} \otimes \mathrm{id}_Y \circ \mathrm{id}_X \otimes c_{Y,Z}$ then:

$$c_{X\otimes\overline{X},Y}^{-1} = \mathrm{id}_X \otimes c_{\overline{X},Y}^{-1} \circ c_{X,Y}^{-1} \otimes \mathrm{id}_{\overline{X}}$$

Therefore:

$$\begin{aligned} c_{\overline{X},Y} = r_X^\dagger \otimes \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_Y \circ \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \otimes c_{\overline{X},Y} \circ \\ \circ \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \otimes c_{\overline{X},Y}^{-1} \circ \mathrm{id}_{\overline{X}} \otimes c_{X,Y}^{-1} \otimes \mathrm{id}_X \circ \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_Y \otimes \overline{r}_X \end{aligned}$$

The second equality follows in the same way. □

We are ready to introduce the following:

Definition 3.1.2. **(a)** A *twist* on $\mathcal{C}$ is a natural isomorphism $\theta_X \in (X, X)$ such that:

$$\theta_{X\otimes Y} = (\theta_X \otimes \theta_Y) c_{Y,X} c_{X,Y} \ \text{ and } \ \theta_{\overline{X}} = \theta_X^\vee$$

(b) A *ribbon category* is a strict braided rigid tensor category with a twist.

We omit the proof of the next:

Lemma 3.1.3. **(a)** *Given objects X and Y of $\mathcal{C}$ we have:*

$$\theta_{X\otimes Y} = c_{Y,X}c_{X,Y}(\theta_X \otimes \theta_Y) = c_{Y,X}(\theta_Y \otimes \theta_X)c_{X,Y}$$

(b) *We also have $\theta_I = \mathrm{id}_I$.*

Using the braiding and the twist we define morphisms $\overline{s}_X : \mathbb{C} \to \overline{X} \otimes X$ and $s_X^\dagger : X \otimes \overline{X} \to \mathbb{C}$ for any object X of the ribbon category $\mathcal{C}$ by:

$$\overline{s}_X = (\mathrm{id}_{\overline{X}} \otimes \theta_X)c_{X,\overline{X}}r_X \tag{3.1.3}$$

$$s_X^\dagger = r_X^\dagger c_{X,\overline{X}}(\theta_X \otimes \mathrm{id}_{\overline{X}}) \tag{3.1.4}$$

Now we prove a technical lemma which will be useful afterwards:

Lemma 3.1.4. *For any object X of a ribbon category, we have:*

$$\begin{aligned}\theta_X^{-2} &= (r_X^\dagger \otimes \mathrm{id}_X)(\mathrm{id}_{\overline{X}} \otimes c_{X,X}^{-1})(c_{X,\overline{X}}\overline{r}_X \otimes \mathrm{id}_X) = \\ &= (r_X^\dagger c_{X,\overline{X}} \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes c_{X,\overline{X}}\overline{r}_X) = \\ &= (\mathrm{id}_X \otimes r_X^\dagger c_{X,\overline{X}})(c_{X,\overline{X}}^{-1} \otimes \mathrm{id}_{\overline{X}})(\mathrm{id}_X \otimes \overline{r}_X)\end{aligned}$$

Proof. It is sufficient to prove the first equality. The others follow using the naturality of the braiding. We indicate with f the RHS of the first equality. We want to prove that:

$$\overline{r}_X = (\theta_X^2 f \otimes \mathrm{id}_X)\overline{r}_X \tag{3.1.5}$$

In fact, using the above expression, we get:

$$\mathrm{id}_X = (\mathrm{id}_X \otimes r_X^\dagger)(\overline{r}_X \otimes \mathrm{id}_X) = (\mathrm{id}_X \otimes r_X^\dagger)((\theta_X^2 f \otimes \mathrm{id}_X)\overline{r}_X \otimes \mathrm{id}_X) = \theta_X^2 f$$

Therefore:

$$\theta_X^{-2} = f$$

So, it remains to prove (3.1.5):

$$\begin{aligned}\overline{r}_X &= \overline{r}_X\theta_I = \theta_{X\otimes\overline{X}}\overline{r}_X = c_{\overline{X},X}c_{X,\overline{X}}\theta_X \otimes \theta_{\overline{X}}\overline{r}_X = \\ &= c_{\overline{X},X} \circ c_{X,\overline{X}} \circ \theta_X \otimes \mathrm{id}_{\overline{X}} \circ \mathrm{id}_X \otimes r_X^\dagger \otimes \mathrm{id}_{\overline{X}} \circ \\ &\circ \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \otimes \theta_X \otimes \mathrm{id}_{\overline{X}} \circ \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \otimes \overline{r}_X \circ \overline{r}_X = \\ &= c_{\overline{X},X} \circ c_{X,\overline{X}} \circ \theta_X \otimes \mathrm{id}_{\overline{X}} \circ \mathrm{id}_X \otimes r_X^\dagger \otimes \mathrm{id}_{\overline{X}} \circ \\ &\circ \overline{r}_X \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \circ \theta_X \otimes \mathrm{id}_{\overline{X}} \circ \overline{r}_X = \\ &= c_{\overline{X},X} \circ c_{X,\overline{X}} \circ \theta_X^2 \otimes \mathrm{id}_{\overline{X}} \circ \overline{r}_X = \\ &= r_X^\dagger \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \circ \mathrm{id}_{\overline{X}} \otimes c_{X,X}^{-1} \otimes \mathrm{id}_{\overline{X}} \circ \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \otimes \overline{r}_X \circ \\ &\circ c_{X,\overline{X}} \circ \theta_X^2 \otimes \mathrm{id}_{\overline{X}} \circ \overline{r}_X = \\ &= r_X^\dagger \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \circ \mathrm{id}_{\overline{X}} \otimes c_{X,X}^{-1} \otimes \mathrm{id}_{\overline{X}} \circ \mathrm{id}_{\overline{X}} \otimes \theta_X^2 \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \circ \\ &\circ \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \otimes \overline{r}_X \circ c_{X,\overline{X}} \circ \overline{r}_X = \\ &= r_X^\dagger \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \circ \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \otimes \theta_X^2 \otimes \mathrm{id}_{\overline{X}} \circ \mathrm{id}_{\overline{X}} \otimes c_{X,X}^{-1} \otimes \mathrm{id}_{\overline{X}} \circ \\ &\circ \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \otimes \overline{r}_X \circ c_{X,\overline{X}} \circ \overline{r}_X = \\ &= \theta_X^2 \otimes \mathrm{id}_{\overline{X}} \circ r_X^\dagger \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \circ \mathrm{id}_X \otimes c_{X,X}^{-1} \otimes \mathrm{id}_{\overline{X}} \circ \\ &\circ c_{X,\overline{X}} \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \circ \overline{r}_X \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \circ \overline{r}_X = \\ &= (\theta_X^2 f \otimes \mathrm{id}_{\overline{X}})\overline{r}_X\end{aligned}$$

In the first equality we used $\theta_I = I$; in the second one the naturality of the twist; in the third one the definition of the twist; in the fourth one (1.1.11) and the naturality of the twist; in the fifth one the naturality of the tensor product; in the sixth one (1.1.11) again; in the seventh one we used the Prop. 3.1.1; in the eighth and ninth ones the naturality of the twist; in the tenth one the naturality of the tensor product; in the eleventh one the definition of f. $\square$

It is possible to introduce the notions of quantum trace and quantum dimension in a ribbon category.

Definition 3.1.5. Let $\mathcal{C}$ be a ribbon category. For any object X in $\mathcal{C}$ and any endomorphism f of X, we define the quantum trace $\mathrm{Tr}_q(f)$ of f as the element:

$$\mathrm{Tr}_q(f) = s_X^\dagger (f \otimes \mathrm{id}_{\overline{X}})\overline{r}_X = r_X^\dagger c_{X,\overline{X}} (\theta_X f \otimes \mathrm{id}_{\overline{X}})\overline{r}_X$$

of $(\mathbf{1}, \mathbf{1})$

It is quite easy to see that in Vect the above notion of trace coincides with the usual one. The quantum trace enjoys the usual properties of trace:

Proposition 3.1.6. *Given endomorphisms f and g in a ribbon category, we have:*
(a) $\mathrm{Tr}_q(fg) = \mathrm{Tr}_q(gf)$ *whenever f and g are composable;*
(b) $\mathrm{Tr}_q(f \otimes g) = \mathrm{Tr}_q(f) \circ \mathrm{Tr}_q(g)$*;*
(c) $\mathrm{Tr}_q(f) = \mathrm{Tr}_q(f^*)$.

Proof. We will prove the equalities **(a)** and **(c)**. The equality **(b)** follows using the same technicalities. Let us start with **(a)**:

$$\begin{aligned}
\mathrm{tr}_q(fg) &= r_X^\dagger \circ c_{X,\overline{X}} \circ \theta_X fg \otimes \mathrm{id}_{\overline{X}} \circ \overline{r}_X = \\
&= r_X^\dagger \circ c_{X,\overline{X}} \circ \theta_X f \otimes \mathrm{id}_{\overline{X}} \circ \mathrm{id}_X \otimes r_X^\dagger \otimes \mathrm{id}_{\overline{X}} \circ \overline{r}_X \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \circ g \otimes \mathrm{id}_{\overline{X}} \circ \overline{r}_X = \\
&= r_X^\dagger \circ c_{X,\overline{X}} \circ \mathrm{id}_X \otimes r_X^\dagger \otimes \mathrm{id}_{\overline{X}} \circ \theta_X f \otimes \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \circ \\
&\circ \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \otimes g \otimes \mathrm{id}_{\overline{X}} \circ \overline{r}_X \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \circ \overline{r}_X = \\
&= r_X^\dagger \circ c_{X,\overline{X}} \circ \mathrm{id}_X \otimes r_X^\dagger \otimes \mathrm{id}_{\overline{X}} \circ \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \otimes g \otimes \mathrm{id}_{\overline{X}} \circ \\
&\circ \theta_X f \otimes \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \circ \overline{r}_X \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \circ \overline{r}_X = \\
&= r_X^\dagger \circ r_X^\dagger \otimes \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \circ \mathrm{id}_{\overline{X}} \otimes g \otimes \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \circ c_{X,\overline{X}\otimes X\otimes \overline{X}} \circ \\
&\circ \theta_X f \otimes \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \circ \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \otimes \overline{r}_X \circ \overline{r}_X = \\
&= r_X^\dagger \circ \mathrm{id}_{\overline{X}} \otimes g \otimes \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \circ \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \otimes r_X^\dagger \circ \\
&\circ \mathrm{id}_{\overline{X}} \otimes \overline{r}_X \otimes \mathrm{id}_{\overline{X}} \circ c_{X,\overline{X}} \circ \theta_X f \otimes \mathrm{id}_{\overline{X}} \circ \overline{r}_X = \\
&= r_X^\dagger \circ \mathrm{id}_{\overline{X}} \otimes g \circ c_{X,\overline{X}} \circ \theta_X f \otimes \mathrm{id}_{\overline{X}} \circ \overline{r}_X = \\
&= r_X^\dagger \circ c_{X,\overline{X}} \circ g \otimes \mathrm{id}_{\overline{X}} \circ \theta_X f \otimes \mathrm{id}_X \circ \overline{r}_X = \mathrm{Tr}_q(gf)
\end{aligned}$$

In the first and last equalities we used the definition of the quantum trace; in the second one (1.1.11); in the third and fourth ones the naturality of the tensor product; in the fifth one the naturality of the braiding; in the sixth one the naturality of the tensor product and of the braiding; in the seventh one (1.1.11); in the eighth one the naturality of the braiding again.

We pass to **(c)**:

$$\begin{aligned}
\mathrm{Tr}_q(f^\vee) &= r^\dagger_{\overline{X}} \circ c_{\overline{X},X} \circ \theta_{\overline{X}} f^\vee \otimes \mathrm{id}_X \circ \overline{r}_{\overline{X}} = \\
&= r^\dagger_{\overline{X}} \circ c_{\overline{X},X} \circ \theta^\vee_X f^\vee \otimes \mathrm{id}_X \circ \overline{r}_{\overline{X}} = \\
&= r^\dagger_{\overline{X}} \circ c_{\overline{X},X} \circ (f\theta_X)^\vee \otimes \mathrm{id}_X \circ \overline{r}_{\overline{X}} = \\
&= r^\dagger_{\overline{X}} \circ c_{\overline{X},X} \circ r^\dagger_X \otimes \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \circ \mathrm{id}_{\overline{X}} \otimes \theta_X f \otimes \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \circ \mathrm{id}_{\overline{X}} \otimes \overline{r}_X \otimes \mathrm{id}_X \circ \overline{r}_{\overline{X}} = \\
&= r^\dagger_{\overline{X}} \circ r^\dagger_X \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \circ \mathrm{id}_{\overline{X}} \otimes \theta_X f \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \circ \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \otimes c_{\overline{X},X} \circ \\
&\circ \mathrm{id}_{\overline{X}} \otimes \overline{r}_X \otimes \mathrm{id}_X \circ \overline{r}_{\overline{X}} = \\
&= r^\dagger_{\overline{X}} \circ r^\dagger_X \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \circ \mathrm{id}_{\overline{X}} \otimes \theta_X f \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \circ \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \otimes r^\dagger_X \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \circ \\
&\circ \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \otimes c^{-1}_{X,X} \otimes \mathrm{id}_{\overline{X}} \circ \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \otimes \overline{r}_X \circ \mathrm{id}_{\overline{X}} \otimes \overline{r}_X \otimes \mathrm{id}_X \circ \overline{r}_{\overline{X}} = \\
&= r^\dagger_{\overline{X}} \circ r^\dagger_X \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \circ \mathrm{id}_{\overline{X}} \otimes \theta_X f \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \circ \\
&\circ \mathrm{id}_{\overline{X}} \otimes c^{-1}_{X,X} \otimes \mathrm{id}_{\overline{X}} \circ \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \otimes \overline{r}_X \circ \overline{r}_{\overline{X}} = \\
&= r^\dagger_X \circ \mathrm{id}_{\overline{X}} \otimes \theta_X f \circ \mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X \otimes r^\dagger_X \circ \mathrm{id}_{\overline{X}} \otimes c^{-1}_{X,X} \otimes \mathrm{id}_{\overline{X}} \circ \overline{r}_{\overline{X}} \otimes \mathrm{id}_X \otimes \mathrm{id}_{\overline{X}} \circ \overline{r}_X = \\
&= r^\dagger_X \circ \mathrm{id}_{\overline{X}} \otimes \theta_X f \circ c_{X,\overline{X}} \circ \overline{r}_X = \\
&= r^\dagger_X \circ c_{X,\overline{X}} \circ \theta_X f \otimes \mathrm{id}_{\overline{X}} \circ \overline{r}_X = \mathrm{Tr}_q(f)
\end{aligned}$$

in the first equality and last equalities we used the definition of the trace; in the second one the definition of the twist; in the third one the antimultiplicativity of the transpose; in the fourth one the definition of transpose map; in the fifth one the naturality of the braiding and of the tensor product; in the sixth one the first identity in the Prop. 3.1.1; in the seventh and eighth ones the naturality of the tensor product; in the ninth one the second identity of the Prop. 3.1.1; in the tenth one the naturality of the braiding. □

We can derive the notion of dimension from the trace:

Definition 3.1.7. Let $\mathcal{C}$ be a ribbon category. For any object X of $\mathcal{C}$ we define the quantum dimension $\dim_q(X)$ as the element:

$$\dim_q(X) = \mathrm{Tr}_q(\mathrm{id}_X) = s^\dagger_X \circ \overline{r}_X$$

in $(\mathbf{1}, \mathbf{1})$.

As a consequence of the Prop. 3.1.6, we have:

Corollary 3.1.8. *Let X, Y be objects of a ribbon category. Then:*

$$\dim_q(X \otimes Y) = \dim_q(X)\dim_q(Y) \;\; \textit{and} \;\; \dim_q(\overline{X}) = \dim_q(X)$$

3.2 Ribbon algebras

Let A be a braided Hopf algebra [67]. For our purpose, we can see it as a braided weak quasi Hopf algebra with a coassociative and counital coproduct. We use the following notation:

$$R = \sum_i a_i \otimes b_i \;\text{ and }\; R^{-1} = \sum_i \widetilde{a_i} \otimes \widetilde{b_i}$$

We consider the element $u \in A$ given by:

$$u = \sum_i S(b_i) a_i \tag{3.2.1}$$

Proposition 3.2.1. *The element u defined as in* (3.2.1) *is invertible with inverse given by:*

$$u^{-1} = \sum_i S^{-1}(\widetilde{b_i})\widetilde{a_i} \tag{3.2.2}$$

and for all $a \in A$ we have:

$$S^2(a) = uau^{-1} \tag{3.2.3}$$

Proof. We first show that $S^2(a)u = ua$ for all $a \in A$. If $y \in A \otimes A$, we have the identity:

$$(\Delta^{\mathrm{op}} \otimes \mathrm{id}(y))(R \otimes I) = (R \otimes I)(\Delta \otimes \mathrm{id}(y))$$

in $A \otimes A \otimes A$. When $y = \Delta(a)$ for some $a \in A$, we have:

$$\sum_i a_{(2)}a_i \otimes a_{(1)}b_i \otimes a_{(3)} = \sum_i a_i a_{(1)} \otimes b_i a_{(2)} \otimes a_{(3)}$$

Now, let V be the linear map $m \circ m \otimes \mathrm{id} \circ \mathrm{id} \otimes S \otimes S^2$ from $A \otimes A \otimes A$ to A, where S is the antipode on A. We apply V to the previous identity, obtaining:

$$\sum_i S^2(a_{(3)})S(b_i)S(a_{(1)})a_{(2)}a_i = \sum_i S^2(a_{(3)})S(a_{(2)})S(b_i)a_i a_{(1)}$$

Using the well-known properties of the antipode for a coassociative Hopf algebra, it is straightforward to get $S^2(a)u$ on the left hand side, and ua on the right hand side. It remains to show that u is invertible. We set $v = \sum_i S^{-1}(\widetilde{b_i})\widetilde{a_i}$. Then:

$$uv = \sum_i uS^{-1}(\widetilde{b_i})\widetilde{a_i} = \sum_i S(\widetilde{b_i})u\widetilde{a_i}$$

using that $S^2(a)u = ua$. As a consequence,

$$uv = \sum_i S(\widetilde{b_i})u\widetilde{a_i} = \sum_{i,j} S(\widetilde{b_i})S(b_i)a_i\widetilde{a_i} =$$
$$= m(S \otimes \mathrm{id}(R_{21}R_{21}^{-1})) = m(S \otimes \mathrm{id}(I \otimes I)) = I$$

In the same way, $vu = I$. □

Corollary 3.2.2. *We have that $S(u)u = uS(u)$. Moreover, this element is central in A.*

Proof. Using the last proposition, we have $uS^{-1}(a) = S(a)u$. Applying S to both sides of the last expression, we get:

$$aS(u) = S(u)S^2(a) = S(u)uau^{-1}$$

and therefore $aS(u)u = S(u)ua$. This proves that $S(u)u$ is central in A. Moreover, if we take $a = u$ in the equality:

$$aS(u) = S(u)uau^{-1}$$

we obtain $uS(u) = S(u)u$. □

It is possible to prove that u satisfies some additional relations.

Proposition 3.2.3. *The element u satisfies the following identities:*

$$\varepsilon(u) = 1 \ , \ \Delta(u) = (R_{21}R)^{-1}(u \otimes u) = (u \otimes u)(R_{21}R)^{-1}$$
$$\Delta(S(u)) = (R_{21}R)^{-1}(S(u) \otimes S(u)) = (S(u) \otimes S(u))(R_{21}R)^{-1}$$
$$\Delta(uS(u)) = (R_{21}R)^{-2}(uS(u) \otimes uS(u)) = (uS(u) \otimes uS(u))(R_{21}R)^{-2}$$

Before proving the Proposition we need to prove the following result about braided Hopf algebras:

Proposition 3.2.4. *Let A be a braided Hopf algebra:*
(a) *The R-matrix R satisfies the equation:*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{3.2.4}$$

and we have:

$$(\varepsilon \otimes \mathrm{id}(R)) = I = (\mathrm{id} \otimes \varepsilon(R)) \tag{3.2.5}$$

(b) *If the antipode S is invertible, then:*

$$(S \otimes \mathrm{id}(R)) = R^{-1} = (\mathrm{id} \otimes S(R)) \tag{3.2.6}$$
$$(S \otimes S(R)) = R \tag{3.2.7}$$

Proof. **(a)** Using (1.2.17) and (1.2.18) in the coassociative and counital case we get:

$$\begin{aligned} R_{12}R_{13}R_{23} &= R_{12}(\Delta \otimes \mathrm{id}(R)) = (\Delta^{\mathrm{op}} \otimes \mathrm{id}(R))R_{12} = \\ &= \Sigma \otimes \mathrm{id}(\Delta \otimes \mathrm{id}(R))\Sigma \otimes \mathrm{id}\, R_{12} = \\ &= \Sigma \otimes \mathrm{id}(R_{13}R_{23})\Sigma \otimes \mathrm{id}\, R_{12} = R_{23}R_{13}R_{12} \end{aligned}$$

Next,

$$R = (\varepsilon \otimes \mathrm{id} \otimes \mathrm{id}(\Delta \otimes \mathrm{id}(R))) = (\varepsilon \otimes \mathrm{id} \otimes \mathrm{id}(R_{13}R_{23})) = I \otimes (\varepsilon \otimes \mathrm{id}(R)) \cdot R$$

Since R is invertible, we obtain $\varepsilon \otimes \mathrm{id}(R) = I$. Similarly for the other side.
(b) Using (a) we obtain:

$$(m \otimes \mathrm{id}(S \otimes \mathrm{id} \otimes \mathrm{id}(\Delta \otimes \mathrm{id}(R)))) = (\varepsilon \otimes \mathrm{id}(R)) = I$$

As a consequence:

$$I = (m \otimes \mathrm{id}(S \otimes \mathrm{id} \otimes \mathrm{id}(R_{13}R_{23}))) = (S \otimes \mathrm{id}(R))R$$

using the unitality of S. Since R is invertible:

$$(S \otimes \mathrm{id}(R)) = R^{-1}$$

Proceeding similarly in the opposite braided Hopf algebra we get:

$$(\mathrm{id} \otimes S^{-1}(R)) = R^{-1}$$

Finally, we have:

$$\begin{aligned} (S \otimes S(R)) &= (\mathrm{id} \otimes S(S \otimes \mathrm{id}(R))) = (\mathrm{id} \otimes S(R^{-1})) = \\ &= (\mathrm{id} \otimes S(\mathrm{id} \otimes S^{-1}(R))) = (\mathrm{id} \otimes \mathrm{id}(R)) = R \end{aligned}$$

□

At this point we can prove the Prop. 3.2.3:

Proof.

$$\varepsilon(u) = \sum_i \varepsilon(S(b_i))\varepsilon(a_i) = \sum_i \varepsilon(b_i)\varepsilon(a_i) = \varepsilon\left(\varepsilon(a_i)b_i\right) = I$$

Next we compute $\Delta(u)$. It is quite easy to see that, for all $a \in A$:

$$\Delta(a)R_{21}R = R_{21}R\Delta(a)$$

Since $R_{21}R$ is invertible, it is enough to show that $\Delta(u)R_{21}R = u \otimes u$:

$$\begin{aligned}\Delta(u)R_{21}R &= \sum_i \Delta(S(b_i))\Delta(a_i)R_{21}R = \\ &= \sum_i (S \otimes S(\Delta^{\mathrm{op}}(b_i)))\Delta(a_i)R_{21}R = \\ &= \sum_i (S \otimes S(\Delta^{\mathrm{op}}(b_i)))R_{21}R\Delta(a_i)\end{aligned}$$

Now we define the action of the algebra $A^{\otimes 4}$ on $A \otimes A$ on the right by:

$$(q \otimes t) \cdot (Q \otimes T) = (S \otimes S(T))(q \otimes t)Q$$

where $q, t \in A$ and $Q, T \in A \otimes A$. We can rewrite the previous equalities as:

$$\Delta(u)R_{21}R = R_{21} \cdot R\Delta(a_i) \otimes \Delta^{\mathrm{op}}(b_i) = R_{21} \cdot ((R \otimes I \otimes I)(\Delta \otimes \Delta^{\mathrm{op}}(R)))$$

Since $\mathrm{id} \otimes \Delta(R) = R_{13}R_{12}$, we have $\mathrm{id} \otimes \Delta^{\mathrm{op}}(R) = R_{12}R_{13}$, so:

$$\Delta \otimes \Delta^{\mathrm{op}}(R) = \Delta \otimes \mathrm{id} \otimes \mathrm{id}(\mathrm{id} \otimes \Delta^{\mathrm{op}}(R)) = \Delta \otimes \mathrm{id} \otimes \mathrm{id}(R_{12}R_{13}) = R_{13}R_{23}R_{14}R_{24}$$

Hence:

$$\Delta(u)R_{21}R = R_{21} \cdot R_{12}R_{13}R_{23}R_{14}R_{24}$$

Using (a) of the last proposition we get:

$$\Delta(u)R_{21}R = R_{21} \cdot R_{23}R_{13}R_{12}R_{14}R_{24}$$

Now we calculate the expression above. Using (b) of the last proposition we get:

$$\begin{aligned}R_{21} \cdot R_{23} &= \sum_{i,j} S(b_j)b_i \otimes a_i a_j = (S \otimes \mathrm{id}(\sum_{i,j} S^{-1}(b_i)b_j \otimes a_i a_j)) = \\ &= (S \otimes \mathrm{id}(R_{21}^{-1}R_{21})) = S \otimes \mathrm{id}(I \otimes I) = I \otimes I\end{aligned}$$

Hence,

$$R_{21} \cdot R_{23}R_{13} = I \otimes I \cdot R_{13} = \sum_i S(b_i)a_i \otimes I = u \otimes I$$

Next,

$$R_{21} \cdot R_{23}R_{13}R_{12} = (u \otimes I) \cdot R_{12} = (u \otimes I)R$$

and:

$$\begin{aligned}R_{21} \cdot R_{23}R_{13}R_{12}R_{14} &= (u \otimes I)(\sum_{i,j} a_i a_j \otimes S(b_j)b_i) = \\ &= (u \otimes I)(\mathrm{id} \otimes S(\sum_{i,j} a_i a_j \otimes S^{-1}(b_i)b_j)) = \\ &= (u \otimes I)(\mathrm{id} \otimes S(R^{-1}R)) = u \otimes I\end{aligned}$$

Finally:

$$R_{21} \cdot R_{23}R_{13}R_{12}R_{14}R_{24} = (u \otimes I) \cdot R_{24} = (u \otimes I)(I \otimes u) = u \otimes u$$

The formula involving $\Delta(S(u))$ is a consequence of the formula for $\Delta(u)$ and the anticomultiplicativity relation. The formula for $\Delta(uS(u))$ can be obtained putting together the formulas for $\Delta(u)$ and $\Delta(S(u))$ and using the centrality of $uS(u)$. □

At this stage we are able to give the following:

Definition 3.2.5. A braided Hopf algebra A is a ribbon algebra if there exists a central element $\theta \in A$ satisfying the relations:

$$\Delta(\theta) = (R_{21}R)^{-1}(\theta \otimes \theta)\ , \quad \varepsilon(\theta) = 1\ , \quad S(\theta) = \theta \tag{3.2.8}$$

We wonder if ribbon algebras produce ribbon categories.

Proposition 3.2.6. *Let A be a ribbon algebra. Then the category of finite-dimensional representations* $\mathrm{Rep}(A)$ *is a ribbon category with twist θ_X given by the action by the inverse of the element θ introduced in the previous Definition.*

Proof. We define the twist θ_V on the vector space V by $\theta_V(v) = \theta^{-1}v$, where $v \in V$. θ_V is clearly an A-linear automorphism since θ^{-1} is central and invertible. We need to prove that θ_V is actually a twist. We have:

$$\begin{aligned}(\theta_V \otimes \theta_W)c_{W,V}c_{V,W}(v \otimes w) &= (\theta^{-1} \otimes \theta^{-1})\Sigma R \Sigma R(v \otimes w) = \\ &= (\theta^{-1} \otimes \theta^{-1})R_{21}R(v \otimes w) = \\ &= \Delta(\theta^{-1})(v \otimes w) = \theta_{V \otimes W}(v \otimes w)\end{aligned}$$

Moreover, let v, w be elements in V. Rigidity is given on $\mathrm{Rep}(A)$ by the following maps:

$$\overline{\delta}_V(1) = \sum_{i=1}^{n} e_i \otimes e_i^* \text{ and } \delta_V^\dagger(\alpha \otimes v) = \alpha(v)$$

where $\{e_i\}_{i=1}^n$ is a basis of V, $\{e_i^*\}_{i=1}^n$ its dual basis, α an element in V^*. Of course $\overline{V} = V^*$. Therefore, omitting the summation symbol:

$$\begin{aligned}(\theta_V)^\vee(\alpha) &= \delta^\vee \otimes \mathrm{id}_{V^*}(\mathrm{id}_{V^*} \otimes \theta_V \otimes \mathrm{id}_{V^*}(\mathrm{id}_{V^*} \otimes \overline{\delta}(\alpha))) = \\ &= \delta^\vee \otimes \mathrm{id}_{V^*}(\mathrm{id}_{V^*} \otimes \theta_V \otimes \mathrm{id}_{V^*}(\alpha \otimes e_i \otimes e_i^*)) = \\ &= \delta^\vee \otimes \mathrm{id}_{V^*}(\alpha \otimes \theta^{-1}e_i \otimes e_i^*) = \\ &= \alpha(\theta^{-1}e_i)e_i^* = (S(\theta^{-1})\alpha)(e_i)e_i^* = \theta^{-1}\alpha\end{aligned}$$

□

Corollary 3.2.7. *The central element θ^2 in a ribbon algebra acts as $uS(u)$ on any f.d. representation. As a consequence, $\theta^2 = uS(u)$ if A is finite-dimensional.*

Proof. Using the last proposition we have that θ^2 acts as θ_V^{-2} on V. Using the Lemma 3.1.4 we have:

$$\theta_V^{-2} = (\mathrm{id}_V \otimes \delta_V^\dagger c_{V,V^*})(c_{V,V}^{-1} \otimes \mathrm{id}_{V^*})(\mathrm{id}_V \otimes \overline{\delta}_V)$$

We compute the right hand side of the equality, omitting the summation symbol and using the identity $R^{-1} = (S \otimes \mathrm{id}(R))$:

$$\begin{aligned}
(\mathrm{id}_V \otimes \delta_V^\dagger c_{V,V^*})(c_{V,V}^{-1} \otimes \mathrm{id}_{V^*})(\mathrm{id}_V \otimes \overline{\delta}_V)(v) &= \\
= (\mathrm{id}_V \otimes \delta_V^\dagger c_{V,V^*})(c_{V,V}^{-1} \otimes \mathrm{id}_{V^*})(v \otimes e_i \otimes e_i^*) &= \\
= (\mathrm{id}_V \otimes \delta_V^\dagger c_{V,V^*})(\widetilde{a_j} e_i \otimes \widetilde{b_j} v \otimes e_i^*) &= \\
= (\mathrm{id}_V \otimes \delta_V^\dagger)(S(a_j) e_i \otimes b_k e_i^* \otimes a_k b_j v) &= \\
= e_i^*(S(b_k) a_k b_j v) S(a_j) e_i = S(a_j) u b_j v = S(a_j) S^2(b_j) uv &= \\
= S(S(b_j) a_j) uv = S(u) uv = u S(u) v
\end{aligned}$$

□

It is quite easy to compute the quantum trace and the quantum dimension on the representations of a ribbon algebra:

Proposition 3.2.8. *Let V be a representation of the ribbon algebra A. Then:*

$$\mathrm{Tr}_q(f) = \mathrm{Tr}(v \mapsto \theta^{-1} u f(v))$$

where $f \in \mathrm{End}_A(V)$. In particular, $\dim_q(V)$ is the trace of the action of $\theta^{-1} u$ on V.

Proof.

$$\begin{aligned}
\mathrm{Tr}_q(f) &= \delta^\dagger(c_{X,X^*}(\theta_X f \otimes \mathrm{id}_{X^*}(\overline{\delta}(1)))) = \\
&= \delta^\dagger(c_{X,X^*}(\theta_X f \otimes \mathrm{id}_{X^*}(e_i \otimes e_i^*))) = \\
&= \delta^\dagger(c_{X,X^*}(\theta^{-1} f(e_i) \otimes e_i^*)) = \\
&= \delta^\dagger(b_j e_i^* \otimes a_j \theta^{-1} f(e_i)) = \\
&= e_i^*(S(b_j) a_j \theta^{-1} f(e_i)) = e_i^*(\theta^{-1} u f(e_i))
\end{aligned}$$

which is the trace of the endomorphism $v \mapsto \theta^{-1} u f(v)$. □

3.3 Quantum groups: definition and R-matrix

In this section we introduce $U_q(\mathfrak{g})$. We will mainly follow the presentation of Lusztig [46], which can also be found in [14] and [81]. Let $\mathfrak{g}$ be a complex simple Lie algebra, let $\mathfrak{h}$ be a Cartan subalgebra and $\mathfrak{h}^*$ its dual space. Let $\Phi \subseteq \mathfrak{h}^*$ be the root system of $\mathfrak{g}$. Let $\langle \cdot, \cdot \rangle$ be the unique inner product on $\mathfrak{h}$ such that $\langle \alpha, \alpha \rangle = 2$ for every short root $\alpha \in \Phi$. Let $\check{\Phi} = \{\check{\alpha} = \frac{2\alpha}{\langle \alpha, \alpha \rangle} | \alpha \in \Phi\}$ be the dual root system of Φ. Let $\Lambda = \{\lambda \in \mathfrak{h}^* | \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}, \forall \alpha \in \Phi\}$ be the weight lattice, $\Lambda_r = \mathbb{Z}\Phi \subseteq \Lambda$ be the root lattice, and $\check{\Lambda}_r = \mathbb{Z}\check{\Phi} \subseteq \frac{1}{D}\Lambda$ be the dual root lattice. Let $\mathcal{W}$ (the *Weyl group*) be the group of isometries of $\mathfrak{h}^*$ generated by reflections σ_α such that:

$$\sigma_\alpha(\lambda) = \lambda - \langle \lambda, \check{\alpha} \rangle \alpha$$

We will most deal with the translated action of the Weyl group, which is defined by:

$$\sigma \cdot \lambda = \sigma(\lambda + \rho) - \rho$$

where $\rho = \sum_{\alpha > 0} \frac{\alpha}{2}$.

Let L be the least integer such that $L\langle\lambda,\gamma\rangle \in \mathbb{Z}\,\forall\lambda,\gamma\in\Lambda$. Let $\Delta=\{\alpha_1,\ldots,\alpha_N\}$ be a basis and $(a_{ij})=\langle\alpha_i,\check{\alpha}_j\rangle$ be the Cartan matrix. $\alpha>\beta$ means that $\alpha-\beta$ is a nonnegative linear combination of the elements of Δ. Let $\Lambda^+=\{\lambda\in\Lambda|\,\langle\lambda,\alpha_i\rangle\geq 0,\forall\alpha_i\in\Delta\}$ be the set of nonnegative integral weights. Let θ be the longest root, or, in other words, the unique long root in $\Phi\cap\Lambda^+$, and let ϕ be the unique short root in the same intersection.

We consider the complex *-algebra $\mathbb{C}[x,x^{-1}]$ of Laurent polynomials with involution $x^*=x^{-1}$, and let $\mathbb{C}(x)$ be the associated quotient field, endowed with the involution naturally induced from $\mathbb{C}[x,x^{-1}]$. We consider Drinfeld-Jimbo quantum group $U_x(\mathfrak{g})$, i.e. the algebra over $\mathbb{C}(x)$ defined by generators E_i, F_i, K_i, K_i^{-1}, $i=1,\ldots,r$, and relations

$$K_iK_j=K_jK_i,\quad K_iK_i^{-1}=K_i^{-1}K_i=1,$$

$$K_iE_jK_i^{-1}=x^{\langle\alpha_i,\alpha_j\rangle}E_j,\quad K_iF_jK_i^{-1}=x^{-\langle\alpha_i,\alpha_j\rangle}F_j,$$

$$E_iF_j-F_jE_i=\delta_{ij}\frac{K_i-K_i^{-1}}{x^{d_i}-x^{-d_i}},$$

$$\sum_0^{1-a_{ij}}(-1)^kE_i^{(1-a_{ij}-k)}E_jE_i^{(k)}=0,\quad \sum_0^{1-a_{ij}}(-1)^kF_i^{(1-a_{ij}-k)}F_jF_i^{(k)}=0,\quad i\neq j,$$

where $d_i=\langle\alpha_i,\alpha_i\rangle/2$, and, for $k\geq 0$, $E_i^{(k)}=E_i^k/[k]_{d_i}!$, $F_i^{(k)}=F_i^k/[k]_{d_i}!$. We notice that $d_i=1$ if α_i is a short root and $d_i=d$ if α_i is a long root, where d is the ratio of the square lengths of the long and short roots. $d=1$ except in the cases B, C, F_4, where it is equal to 2, and G_2 where $d=3$. Quantum integers and factorials are defined in the following way:

$$[k]_x=\frac{x^k-x^{-k}}{x-x^{-1}}$$

$$[k]_x!=[k]_x\ldots[2]_x$$

We will often use the notation: $[k]_{d_i}:=[k]_{x^{d_i}}$. There is a unique *–involution on $U_x(\mathfrak{g})$ making it into a *–algebra over $\mathbb{C}(x)$ such that

$$K_i^*=K_i^{-1},\quad E_i^*=F_i.$$

The introduction of an involution * is due to Wenzl [81]. This algebra becomes a Hopf algebra, i.e. a coassociative coalgebra with coproduct Δ, counit ε and antipode S. In fact:

$\Delta(E_i)=K_i\otimes E_i+E_i\otimes I$, $\Delta(F_i)=I\otimes F_i+F_i\otimes K_i^{-1}$, and $\Delta(K_i)=K_i\otimes K_i$
$S(K_i)=K_i^{-1}$, $S(E_i)-K_1^{-1}E_i$ and $S(F_i)=-F_iK_i$
$\varepsilon(E_i)=0=\varepsilon(F_i)$, $\varepsilon(K_i)=1$

Using the above identities it is possible to check that:

$$\Delta(a^*)=\Delta^{\mathrm{op}}(a)^* \tag{3.3.1}$$

$$\varepsilon(a^*)=\overline{\varepsilon(a)} \tag{3.3.2}$$

$$S(a^*)=S(a)^* \tag{3.3.3}$$

$$S^2(a)=K_{2\rho}^{-1}aK_{2\rho} \tag{3.3.4}$$

for all $a \in U_x(\mathfrak{g})$. If α is in the root lattice, $K_\alpha := K_1^{k_1} \dots K_r^{k_r}$, where $\alpha = \sum_{i=1}^r k_i\alpha_i$. Our goal is to construct a braided tensor category starting from $U_x(\mathfrak{g})$. In order to achieve this goal, we consider a larger polynomial ring $\mathcal{A} = \mathbb{C}[x^{\frac{1}{2L}}, x^{-\frac{1}{2L}}]$, with L the smallest positive integer such that $L\langle\lambda,\mu\rangle \in \mathbb{Z}$ for all dominant weights λ, μ. We give the explicit values of L for all Lie types: $L = n+1$ if $\mathfrak{g}$ is of type A_n; $L = 1$ in the cases B_{2n}, C_n, E_8, F_4 and G_2; $L = 2$ in the cases B_{2n+1}, D_{2n} and E_7; $L = 3$ in the case E_6; $L = 4$ in the case D_{2n+1}.

We define the integral form $\mathfrak{U}_\mathcal{A}$ as the $\mathcal{A}$–subalgebra generated by the elements $E_i^{(k)}$, $F_i^{(k)}$ and K_i. It is quite easy to prove that it is a *–invariant Hopf $\mathcal{A}$–algebra with the structure inherited from $U_x(\mathfrak{g})$. $\mathfrak{U}_\mathcal{A}$ has not a R-matrix, even topologically, so we will need to extend it in a suitable way. Before doing it, we introduce $U_q(\mathfrak{g})$.

We fix $q \in \mathbb{T}$, and consider the *–homomorphism $\mathcal{A} \to \mathbb{C}$ which evaluates every polynomial in q, and form the tensor product *–algebra,

$$U_q(\mathfrak{g}) := \mathfrak{U}_\mathcal{A} \otimes_\mathcal{A} \mathbb{C},$$

which becomes a complex Hopf algebra with a *–involution. Properties (3.3.1) - (3.3.4) still hold for $U_q(\mathfrak{g})$.

Given a dominant weight λ of $\mathfrak{g}$, we can associate different modules $V_\lambda(x)$, $V_\lambda(\mathcal{A})$, and $V_\lambda(q)$ to $U_x(\mathfrak{g})$, $\mathfrak{U}_\mathcal{A}$ and $U_q(\mathfrak{g})$ respectively, usually called Weyl modules, and thus form corresponding representation categories as follows. We shall mostly be interested in $V_\lambda(q)$ that we will usually denote by V_λ as well.

Let $V_\lambda(x)$ be the irreducible representation of $U_x(\mathfrak{g})$ with highest weight λ and let v_λ be the highest weight vector of $V_\lambda(x)$. We can form the cyclic module of $\mathfrak{U}_\mathcal{A}$ generated by v_λ:

$$V_\lambda(\mathcal{A}) = \mathfrak{U}_\mathcal{A} \cdot v_\lambda$$

It is possible to see that:

$$V_\lambda(\mathcal{A}) \otimes_\mathcal{A} \mathbb{C}(x) = V_\lambda(x)$$

We denote by $\mathrm{Rep}(\mathfrak{U}_\mathcal{A})$ the linear category over $\mathcal{A}$ with objects finite tensor products of modules $V_\lambda(\mathcal{A})$. It becomes a tensor category in the natural way.

Every module $V_\lambda(\mathcal{A})$ gives rise to the complex $U_q(\mathfrak{g})$-modules via the map which sends x to a complex number q:

$$V_\lambda(q) := V_\lambda(\mathcal{A}) \otimes_\mathcal{A} \mathbb{C}.$$

The representation category of $U_q(\mathfrak{g})$ whose objects are finite tensor products of the modules $V_\lambda(q)$ is a braided tensor category, even though $U_q(\mathfrak{g})$ is not braided. In fact we will produce a braided Hopf algebra $U_\mathcal{A}^\dagger(\mathfrak{g})$ which is the extension of $U_q(\mathfrak{g})$ (see [75]). As we said before, $\mathcal{A}$ will be the ring $\mathbb{C}[x^{\frac{1}{2L}}, x^{-\frac{1}{2L}}]$, and Λ the weight lattice. Now we consider the Hopf algebra of functions on Λ as additive group. The collection of all set-theoretic functions from Λ to $\mathcal{A}$ will be indicated by $\mathrm{Map}(\Lambda, \mathcal{A})$, and it is naturally an algebra over $\mathcal{A}$ with pointwise multiplication. It is a topological Hopf algebra with the following:

$$\Delta(f)(\mu,\mu') = f(\mu+\mu')$$
$$\varepsilon(f) = f(0) \text{ and } S(f)(\mu) = f(-\mu)$$

for $f \in \mathrm{Map}(\Lambda, \mathcal{A})$ and $\mu, \mu' \in \Lambda$. Here:

$$\Delta : \mathrm{Map}(\Lambda, \mathcal{A}) \to \mathrm{Map}(\Lambda \times \Lambda, \mathcal{A})$$

The latter space contains $\mathrm{Map}(\Lambda, \mathcal{A}) \otimes \mathrm{Map}(\Lambda, \mathcal{A})$ as a dense subspace in the topology of pointwise convergence, and thus may be viewed as the completion of the tensor product. A topological basis for this Hopf algebra is given by $\{\delta_\lambda\}_{\lambda \in \Lambda}$, where $\delta_\lambda(\gamma) = \delta_{\lambda,\gamma}$. By topological basis we mean that the elements are linearly independent and span a dense subspace of $\mathrm{Map}(\Lambda, \mathcal{A})$ in the topology of pointwise convergence. It is a classical result that given a homomorphism between an abelian group and its dual, we can associate to the homomorphism a R-matrix in the Hopf algebra of function on the group. If the homomorphism is $\lambda \mapsto x^{L\langle \lambda, \cdot \rangle}$, the R-matrix is:

$$\mathcal{R} = \sum_{\lambda,\gamma} x^{\langle \lambda, \gamma \rangle} \delta_\lambda \otimes \delta_\gamma$$

which is an element in the completion of $\mathrm{Map}(\Lambda, \mathcal{A}) \otimes \mathrm{Map}(\Lambda, \mathcal{A})$. We will indicate with $U_{\mathcal{A}}^\dagger(\mathfrak{h})$ the topological Hopf algebra $\mathrm{Map}(\Lambda, \mathcal{A})$, and with x^λ the homomorphism $\sum_{\gamma \in \Lambda} x^{\langle \lambda, \gamma \rangle} \delta_\gamma$. $U_q(\mathfrak{g})$ acts on $\mathfrak{U}_{\mathcal{A}}$ via the Λ-grading of $\mathfrak{U}_{\mathcal{A}}$. More precisely, we define the weight of a monomial in $\{E_i, F_i, K_i\}$ to be the sum of α_i for each factor of E_i and $-\alpha_i$ for each factor of F_i. Then $f \in U_{\mathcal{A}}^\dagger(\mathfrak{h})$ acts on a monomial X by $f[X] = f(\mathrm{weight}(X))X$ and then extends linearly. This action is a homomorphism of Hopf algebras so we can form the semidirect product $U_{\mathcal{A}}^\dagger(\mathfrak{h}) \ltimes \mathfrak{U}_{\mathcal{A}}$. This Hopf algebra is topologically generated by $\{E_i, F_i, K_i\} \cup \{\delta_\lambda\}_{\lambda \in \Lambda}$ with the usual quantum group relations together with:

$$\delta_\lambda \delta_\gamma = \delta_{\lambda,\gamma} \delta_\lambda \ , \ \sum_{\lambda \in \Lambda} \delta_\lambda = 1$$

$$\delta_\lambda K_i = K_i \delta_\lambda \ , \ \delta_\lambda E_i = E_i \delta_{\lambda - \alpha_i} \ , \ \delta_\lambda F_i = F_i \delta_{\lambda + \alpha_i}$$

If $U_{\mathcal{A}}^\dagger(\mathfrak{h}) \ltimes \mathfrak{U}_{\mathcal{A}}$ acts on an $\mathcal{A}$-module V, we say $v \in V$ is of weight $\lambda \in \Lambda$ if $K_i v = x^{\langle \lambda, \alpha_i \rangle}$ and $fv = f(\lambda)v$ for $f \in U_{\mathcal{A}}^\dagger(\mathfrak{h})$. V is a λ weight space if it consists entirely of weight λ vector. Let $\mathfrak{M}$ be the direct product of all $U_{\mathcal{A}}^\dagger(\mathfrak{h}) \ltimes \mathfrak{U}_{\mathcal{A}}$-modules which are a finite direct sum of λ weight spaces on $\mathcal{A}$. Of course $U_{\mathcal{A}}^\dagger(\mathfrak{h}) \ltimes \mathfrak{U}_{\mathcal{A}}$ acts on $\mathfrak{M}$. The kernel of this action is a two-sided ideal I (which is not $\{0\}$ for sure since it contains $K_i - x^{\alpha_i}$). Moreover, I is a Hopf ideal, so $U_{\mathcal{A}}^\dagger(\mathfrak{h}) \ltimes \mathfrak{U}_{\mathcal{A}}/I$ is a Hopf algebra which embeds into $\mathrm{End}(\mathfrak{M})$. $\mathrm{End}(\mathfrak{M})$ is endowed with a topology given by the product topology on $\mathfrak{M}$, where a sequence converges if and only if it converges on each f.d. submodules. The closure of $U_{\mathcal{A}}^\dagger(\mathfrak{h}) \ltimes \mathfrak{U}_{\mathcal{A}}/I$ with respect to this topology is called $U_{\mathcal{A}}^\dagger$, and it is a ribbon Hopf algebra. The R-matrix is:

$$R = \mathcal{R} \sum_{t_1, \dots, t_N = 1}^{\infty} \prod_{r=1}^{N} q_{\beta_r}^{\frac{t_r(t_r+1)}{2}} (1 - q_{\beta_r}^{-2})^{t_r} [t_r]_{q_{\beta_r}}! E_{\beta_r}^{(t_r)} \otimes F_{\beta_r}^{(t_r)}$$

with $R \in U_{\mathcal{A}}^\dagger \overline{\otimes} U_{\mathcal{A}}^\dagger$. By q_{β_r} we mean q^{d_i} when β_r is the same length of α_i. The ribbon element related to R is q^ρ.

It is possible to see that $V_\lambda(\mathcal{A})$ is a $U_{\mathcal{A}}^\dagger(\mathfrak{g})$-module by letting $f \in U_{\mathcal{A}}^\dagger(\mathfrak{h})$ act on a weight λ vector by multiplication by $f(\lambda)$. On the other side, every $U_{\mathcal{A}}^\dagger(\mathfrak{g})$ is clearly a $\mathfrak{U}_{\mathcal{A}}$-module. Therefore $\mathrm{Rep}(\mathfrak{U}_{\mathcal{A}})$ and $\mathrm{Rep}(U_{\mathcal{A}}^\dagger(\mathfrak{g}))$ are the same category and hence

$\mathrm{Rep}(\mathcal{U}_{\mathcal{A}})$ is ribbon. Now we restrict x to a root of unity, proceeding as before. Thus we can define:

$$U_q^{\dagger}(\mathfrak{g}) = U_{\mathcal{A}}^{\dagger} \otimes_{\mathcal{A}} \mathbb{C}$$

We indicate with $U_q^{\dagger}\overline{\otimes}U_q^{\dagger}$ the space $(U_{\mathcal{A}}^{\dagger} \otimes U_{\mathcal{A}}^{\dagger}) \otimes \mathbb{C}$. It is the completed tensor product of $U_q^{\dagger} \otimes U_q^{\dagger}$ in the topology inherited from $(U_{\mathcal{A}}^{\dagger}\overline{\otimes}U_{\mathcal{A}}^{\dagger}) \otimes \mathbb{C}$. Therefore $U_q^{\dagger}(\mathfrak{g})$ is a ribbon algebra, and $\mathrm{Rep}(U_q^{\dagger}(\mathfrak{g}))$ is a ribbon category. Proceeding as before, we have that $\mathrm{Rep}(U_q(\mathfrak{g}))$ is the same as $\mathrm{Rep}(U_q^{\dagger}(\mathfrak{g}))$, so it is ribbon.

3.4 Irreducibility of representations of $U_q(\mathfrak{g})$ and quotient category

In the previous section we introduced the category $\mathrm{Rep}(U_q(\mathfrak{g}))$ generated by the Weyl modules $V_\lambda = V_\lambda(q)$. We focus on this category, trying to understand when V_λ's are simple and if there is a way to restrict the category of representations in order to obtain a semisimple one. We will give here an overview of the results about this topic, following [2]–[7] and [27]. Other reviews on this argument can be found in [15] and [81]. In order to answer to the first question, we introduce the so-called *linkage principle*. We consider q as primitive root of unity of the type $q = e^{\frac{\pi i}{dl}}$. Moreover, we define the *affine* Weyl group W_l which acts on the real vector space E spanned by the roots in the following way:

$$w \cdot x = w(x + \rho) - \rho$$

where $w \in W_l$ and $x \in E$. Under the above restriction on q, W_l is generated by the ordinary Weyl group W and the translation by $l\theta$, where θ is the highest root. The action of W_l on E admits a fundamental domain[1], called the *principal Weyl alcove*, which is:

$$\overline{\Lambda}_l := \{\lambda \in \Lambda^+ : \langle \lambda + \rho, \theta \rangle \leq dl\}$$

The linkage principle states that V_λ is irreducible if $\lambda \in \overline{\Lambda}_l$. Moreover, V_λ and V_μ are pairwise inequivalent if $\lambda, \mu \in \overline{\Lambda}_l$. The proof of these facts can be found in [4].

Now, we restrict to a peculiar category of representations of $U_q(\mathfrak{g})$. In order to do that, we introduce the notion of *tilting module*.

Definition 3.4.1. A finite-dimensional $U_q(\mathfrak{g})$-module V has a *Weyl filtration* if there exists a sequence of submodules

$$\{0\} = V_0 \subset V_1 \subset \ldots \subset V_p = V$$

with $V_r/V_{r-1} \cong V_{\lambda_i}$ for some $\lambda_i \in \Lambda^+$. A finite-dimensional $U_q(\mathfrak{g})$-module V is a tilting module if both V and its dual V^* has a Weyl filtration.

We fix for each Lie type a representation V of $\mathfrak{g}$ taken from a specific list, that we call *fundamental*. For example, if $\mathfrak{g}$ is of type A, V is the vector representation. Each fundamental representation has the property that every irreducible representation of

[1] Given a topological space and a group acting on it, the images of a single point under the group action form an orbit of the action. A *fundamental domain* is a subset of the space which contains exactly one point from each of these orbits.

$\mathfrak{g}$ is contained in some tensor power of V. We can form the category $\mathcal{T}_l = \mathcal{T}(\mathfrak{g}, l)$ whose objects are finite tensor powers of V and arrows the intertwining operators, completed with subobjects and direct sums. $\mathcal{T}_l$ is called *tilting* category, because every object of $\mathcal{T}_l$ is a tilting module, and conversely for l large enough every tilting $U_q(\mathfrak{g})$-module is isomorphic to an object of $\mathcal{T}_l$. How much does l have to be large? We need an order l such that $\kappa \in \Lambda_l$, where κ is the weight associated to V and

$$\Lambda_l = \{\lambda \in \Lambda^+ : \langle \lambda + \rho, \theta \rangle < dl\}$$

In a tilting module, Weyl filtrations are not unique. Anyway, for all filtrations of a tilting module W the number of factors isomorphic to a given $V_\lambda(q)$ is unique, and it is in fact given by the multiplicity of $V_\lambda(x)$ in $W(x)$ if W is obtained from a specialization $x \mapsto q$ of a module $W(x)$ of $U_x(\mathfrak{g})$. In particular, suppose that W is the tensor product $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}$ with $\lambda_i \in \overline{\Lambda}_l$ for all $i \in \{1, \ldots, n\}$. Then the multiplicities of the factors in its Weyl filtrations are the same as those determined by the decomposition into irreducibles of the corresponding tensor product in the classical case. We now want to lists some interesting results about tilting modules, but without giving proofs.

Proposition 3.4.2. **(a)** *The dual of a tilting module is tilting;*
(b) *Any finite direct sum of tilting modules is tilting;*
(c) *Any direct summand of a tilting module is tilting;*
(d) *Any finite tensor product of tilting modules is tilting.*

The Prop. 3.4.2 allows us to restrict our attention to indecomposable tilting modules. They are parametrized by Λ^+:

Proposition 3.4.3. *For any $\lambda \in \Lambda^+$ there exists, up to isomorphism, a unique indecomposable tilting module $T_\lambda = T_\lambda(q)$. In particular, if $\lambda \in \overline{\Lambda}_l$, $T_\lambda = V_\lambda$, so it is irreducible. If T is a tilting module for $U_q(\mathfrak{g})$, then:*

$$T \cong \bigoplus_{\lambda \in \Lambda^+} T_\lambda^{n_\lambda(T)}$$

where the multiplicities $n_\lambda(T)$ are uniquely determined by T.

At this stage it is quite interesting to discuss the point of the quantum dimensions of tilting modules. First of all, we need to clarify the form of the ribbon element in the (extended) quantum group $U_q(\mathfrak{g})^\dagger$. We have:

Proposition 3.4.4. *$U_q(\mathfrak{g})^\dagger$ is ribbon with ribbon element $v = K_{2\rho} u$, where $u = m(S \otimes \mathrm{id}(R_{21}))$.*

Proof. First of all, we need to prove that v is central. Since $S^2(x) = uxu^{-1}$ and $S^2(x) = K_{2\rho}^{-1} x K_{2\rho}$, we have:

$$vx = K_{2\rho} ux = K_{2\rho} S^2(x) u = K_{2\rho} K_{2\rho}^{-1} x K_{2\rho} u = x K_{2\rho} u = xv$$

It is easy to see that $\varepsilon(v) = 1$. Moreover we prove that $\Delta(v) = (v \otimes v)(R_{21}R)^{-1}$:

$$\Delta(v) = \Delta(K_{2\rho})\Delta(u) = (K_{2\rho} \otimes K_{2\rho})(u \otimes u)(R_{21}R)^{-1} = (v \otimes v)(R_{21}R)^{-1}$$

It remains to prove that $S(v) = v$. It is quite easy to see that $K_{2\rho}^{-1} S(u)$ acts as $K_{2\rho} u$ on V_λ^*. Since $V_\lambda^* \cong V_\mu$ for some $\mu \in \Lambda^+$, $K_{2\rho}^{-1} S(u)$ acts like $K_{2\rho} u$. We can extend this fact to any module, obtaining that $v = K_{2\rho} u = K_{2\rho}^{-1} S(u)$. This identity leads to the conclusion. $\square$

Using the last proposition and the Prop. 3.2.8, we have that the quantum trace of an endomorphism f of a finite-dimensional $U_q(\mathfrak{g})$-module is:

$$\mathrm{Tr}_q(f) = \mathrm{Tr}(K_{2\rho}^{-1} f)$$

Taking $f = \mathrm{id}_V$, we obtain the quantum dimension:

$$\dim_q(V) = \mathrm{Tr}(K_{2\rho}^{-1})$$

The following results will be crucial. We report them without proofs.

Proposition 3.4.5. *Let $\lambda \in \Lambda^+$. Then $\dim_q(T_\lambda) \neq 0$ if and only if $\lambda \in \Lambda_l$.*

Corollary 3.4.6. **(a)** *Let $\lambda \in \Lambda^+ \setminus \Lambda_l$ and $f \in \mathrm{End}(T_\lambda)$. Then $\mathrm{Tr}_q(f) = 0$.*
(b) *Let $\lambda \in \Lambda^+ \setminus \Lambda_l$ and V be any $U_q(\mathfrak{g})$-module. Then, every direct summand of $T_\lambda \otimes V \cong V \otimes T_\lambda$ has quantum dimension zero.*

Tilting modules with zero quantum dimension will be called *negligible*.

At this point we have all the tools to build a semisimple quotient category of $\mathcal{T}_l$, which will be indicated by $\mathcal{F}_l$. Using Prop. 3.4.3 we can state that every object of $\mathcal{T}_l$ decomposes as a direct sum of indecomposable tilting submodules, and this decomposition is unique (up to isomorphism). We can form two full linear (non-tensorial) subcategories, $\mathcal{T}^0$ and $\mathcal{T}^\perp$ of $\mathcal{T}_l$, with objects respectively those representations which can be written as direct sums of V_λ, with $\lambda \in \Lambda_l$, and those which have no such V_λ as direct summand. Therefore every object in $\mathcal{T}^\perp$ is a negligible representation.

Definition 3.4.7. Let W and W' be two objects in $\mathcal{T}_l$. An arrow $T \in (W, W')$ is said to be *negligible* if there exist two arrows $S_1 \in (W, N)$ and $S_2 \in (N, W')$, where N is negligible, such that $T = S_2 \circ S_1$.

The category $\mathcal{T}^\perp$ satisfies the following properties, which can be inferred from Prop. 3.4.3 and 3.4.5, and from Cor. 3.4.6:

Proposition 3.4.8. **(a)** *Any object $W \in \mathcal{T}_l$ is isomorphic to a direct sum $W \cong W_0 \oplus N$, where $W_0 \in \mathcal{T}^0$ and $N \in \mathcal{T}^\perp$;*
(b) *For any pair of arrows $T \in (W_1, N)$ and $S \in (N, W_2)$ in $\mathcal{T}_l$, where $N \in \mathcal{T}^\perp$ and $W_1, W_2 \in \mathcal{T}^0$, then:*

$$ST = 0$$

(c) *For any pair of objects $W \in \mathcal{T}_l$, $N \in \mathcal{T}^\perp$, we have that $N \otimes W$ and $W \otimes N$ are negligible.*

Property (a) follows from the decomposition of tilting modules shown in Prop. 3.4.3. Property (b) means that non-negligible modules cannot be summands of a negligible one. This can be easily proved by a dimensional argument. Same argument can be used to prove property (c). These properties were first shown by Andersen [2] and then abstracted by Gelfand and Kazhdan [27].

Now, let $\mathrm{Neg}(W, W')$ be the subspace of the negligible arrows of (W, W'). Then the quotient category $\mathcal{F}_l$ is the category with the same objects as $\mathcal{T}_l$ and arrows between the objects W and W' the quotient space:

$$(W, W')_{\mathcal{F}_l} = (W, W')/\,\mathrm{Neg}(W, W')$$

Therefore it is possible to define a functor $F : \mathcal{T}_l \longrightarrow \mathcal{F}_l$ which is the identity on the objects and the natural projection on morphisms. It is quite easy to see that $\mathcal{F}_l$ is a linear rigid braided tensor category using as associativity and commutativity costraints the images through F of those in $\mathcal{T}_l$, and the same for the rigidity maps. If we restrict F to $\mathcal{T}^0$ we obtain the functor F^0 which is an equivalence between the categories $\mathcal{T}^0$ and $\mathcal{F}_l$. Indeed, to prove this, it is enough to show that every object in $\mathcal{F}_l$ is isomorphic to the image under F^0 of an object of $\mathcal{T}^0$, and that F^0 is a bijection on sets of morphisms. The first assertion is quite easy to prove, since if $W \in \mathcal{T}_l$ is non-negligible, then W is equivalent to $F(W)$ in $\mathcal{F}_l$. Conversely, if N is negligible, then N is equivalent to $F(0)$ in $\mathcal{F}_l$. The second assertion is a consequence of the property (b) seen before. Since $\mathcal{F}_l$ is a linear rigid braided tensor category, so is $\mathcal{T}^0$. We will usually denote the tensor product of objects and arrows in $\mathcal{F}_l$ by $W \underline{\otimes} W'$ and $S \underline{\otimes} T$ respectively, and we will call it the *truncated* product. Dropping out all the indecomposable but not irreducible tilting modules, we get that $\mathcal{F}_l$ is a semisimple category, with $\{V_\lambda, \lambda \in \Lambda_l\}$ as complete set of irreducible objects. $\mathcal{F}_l$ or, equivalently $\mathcal{T}^0$, can be seen as the categorification of the fusion ring with the truncated tensor product. If $\lambda, \mu \in \Lambda_l$, we can decompose $V_\lambda \otimes V_\mu$ in $\mathcal{T}_l$ in the following way:

$$V_\lambda \otimes V_\mu \cong \bigoplus_{\nu \in \Lambda_l} m^\nu_{\lambda,\mu} V_\nu \oplus N$$

where N is negligible. The decomposition is unique up to isomorphism. Passing to $\mathcal{F}_l$, we obtain:

$$V_\lambda \underline{\otimes} V_\mu \cong \bigoplus_{\nu \in \Lambda_l} m^\nu_{\lambda,\mu} V_\nu$$

It is worth to notice that the decomposition of $V_\lambda \otimes V_\mu$ in $\mathcal{T}_l$ is unique up to isomorphism but not canonical.

3.5 Kirillov-Wenzl theory

The goal of this section is to prove the existence of a hermitian form on the representation spaces of $\mathrm{Rep}(\mathcal{U}_\mathcal{A})$. So it is required a double effort: to put a hermitian form on $V_\lambda(\mathcal{A})$, and to find a suitable hermitian form on every tensor product space generated by V_λ's. These results can be found in [37]. Wenzl [81] proved the existence of a subclasses of such representations where this hermitian form is a scalar product, also when we consider the product form. We will strictly follow the review of these results made in [15].

It was proved in the section 2 of this chapter that $\mathrm{Rep}(\mathcal{U}_\mathcal{A})$ is a ribbon category, since $U^\dagger_\mathcal{A}$ is a ribbon algebra. It is possible to put on $U^\dagger_\mathcal{A}$ an involution which extends the involution on $\mathcal{U}_\mathcal{A}$, defining:

$$R^* = (R^{-1})_{21} \tag{3.5.1}$$

where R is the R-matrix. Recalling the definition of u in the first section of this chapter, we have:

Proposition 3.5.1. *u is a unitary element in $U^\dagger_\mathcal{A}$. As a consequence, the ribbon element v is unitary too.*

Proof. Since S commute with $*$, we have that $u^* = \sum_i a_i^* S(b_i^*)$. Moreover, from Prop. 3.2.4 (b), we have $S \otimes S(R) = R$. Therefore $S \otimes S(R^*) = R^*$ and hence $S^{-1} \otimes S^{-1}(R^*) = R^*$. As a consequence:

$$u^* = m(\mathrm{id} \otimes S(R^*)) = m(\mathrm{id} \otimes S(S^{-1} \otimes S^{-1}(R^*))) = m(S^{-1} \otimes \mathrm{id}(R^*))$$

Using (3.5.1), we get:

$$u^* = m(S^{-1} \otimes \mathrm{id}(R^*)) = m(S^{-1} \otimes \mathrm{id}(R_{21}^{-1})) = \sum_i S^{-1}(\widetilde{b_i})\widetilde{a_i} = u^{-1}$$

Since $K_{2\rho}$ is also unitary, we get that $v^* = v^{-1}$. □

It is possible to derive the explicit expression of v as operator on the highest weight module V_λ, using R and $K_{2\rho}$. It is given by the scalar multiplication by $x^{-\langle \lambda, \lambda + 2\rho \rangle}$ on V_λ. Therefore it is possible to construct a central square root w of v in the topological completion of $U_{\mathcal{A}}^\dagger$ which acts as $x^{-\frac{1}{2}\langle \lambda, \lambda + 2\rho \rangle}$ on every V_λ. Therefore:

$$w^2 = v \text{ and } w^* = w^{-1}$$

Since v is the ribbon element, we have:

$$R_{21}R = (v \otimes v)\Delta(v^{-1})$$

For this reason, we have $(R_{21}R)^{\frac{1}{2}} = (w \otimes w)\Delta(w^{-1})$. We set

$$\Theta = (w^{-1} \otimes w^{-1})\Delta(w) = (R_{21}R)^{-\frac{1}{2}}$$

and $\overline{R} = R\Theta$. We give the following:

Lemma 3.5.2. *The element Θ satisfies the following relations:*
(a) $\Theta^* = \Theta_{21}^{-1}$
(b) $\Theta_{21}R = R\Theta$

Proof. We start proving **(a)**:

$$\Theta^* = \Delta(w)^*(w \otimes w) = \Delta^{\mathrm{op}}(w^{-1})(w \otimes w) = (w \otimes w)\Delta^{\mathrm{op}}(w^{-1}) = \Theta_{21}^{-1}$$

Next:

$$\Theta_{21}R = (w^{-1} \otimes w^{-1})\Delta^{\mathrm{op}}(w)R = (w^{-1} \otimes w^{-1})R\Delta(w) = R(w^{-1} \otimes w^{-1})\Delta(w) = R\Theta$$

□

The following result is crucial:

Proposition 3.5.3. *$\overline{R}$ is self-adjoint. More precisely:*

$$\overline{R}^* = \overline{R}_{21}^{-1} = \overline{R}$$

Proof. Using the **(a)** of the previous lemma, we have:

$$\overline{R}^* = (R\Theta)^* = \Theta^* R^* = \Theta_{21}^{-1} R_{21}^{-1} = (R_{21}\Theta_{21})^{-1} = \overline{R}_{21}^{-1}$$

For the second part of the equality, we proceed in the following way:

$$\overline{R}_{21}\overline{R} = R_{21}\Theta_{21}R\Theta = R_{21}R\Theta^2 = I$$

since $\Theta = (R_{21}R)^{-\frac{1}{2}}$. Therefore $\overline{R}$ is the inverse element of $\overline{R}_{21}$, so it must coincide with $\overline{R}^*$. □

We proved in the section 2 of this chapter that R satisfies the relation (3.2.4), also called the *Yang-Baxter* equation. It can be expressed in an alternative way, which can be inferred from (3.2.4) itself and (1.2.17) and (1.2.18) restricted to the coassociative and counital case:

$$R_{12}(\Delta \otimes \mathrm{id}(R)) = R_{23}(\mathrm{id} \otimes \Delta(R)) \tag{3.5.2}$$

This identity allows us to define the following elements:

$$R^{(n+1)} = (R^{(n)} \otimes I)(\Delta^{(n-1)} \otimes \mathrm{id}(R)) = (I \otimes R^{(n)})(\mathrm{id} \otimes \Delta^{(n-1)}(R))$$

where $R^{(2)} = R$. Moreover, we set $\Theta^{(n)} = (w^{-1} \otimes \ldots \otimes w^{-1})\Delta^{(n-1)}(w)$, with $\Theta^{(2)} = \Theta$. We have the following identity:

$$\begin{aligned}
&(\Theta^{(n)} \otimes I)(\Delta^{(n-1)} \otimes \mathrm{id}(\Theta)) = \\
&= (w^{-1 \otimes n-1} \otimes I)(\Delta^{(n-1)}(w) \otimes I)(\Delta^{(n-1)}(w^{-1}) \otimes w^{-1})(\Delta^{(n-1)} \otimes \mathrm{id}(\Delta(w))) = \\
&= w^{-1 \otimes n}\Delta^{(n)}(w) = \Theta^{(n+1)}
\end{aligned}$$

In the same way it is possible to prove that $\Theta^{(n+1)} = (I \otimes \Theta^{(n)})(\mathrm{id} \otimes \Delta^{(n-1)}(\Theta))$. Hence we can define the element $\overline{R}^{(n)}$ in $U_{\mathcal{A}}^{\dagger\,\overline{\otimes} n}$ in the following natural way:

$$\overline{R}^{(n)} = R^{(n)}\Theta^{(n)}$$

where $\overline{R}^{(2)} = \overline{R}$. Our goal is to prove that $\overline{R}^{(n)}$ is self-adjoint for $n > 2$. We need the following:

Lemma 3.5.4. *We have the following recursive relation:*

$$\overline{R}^{(n+1)} = (\overline{R}^{(n)} \otimes I)(\Delta^{(n-1)} \otimes \mathrm{id}(\overline{R})) = (I \otimes \overline{R}^{(n)})(\mathrm{id} \otimes \Delta^{(n)}(\overline{R}))$$

Proof. The first part of the equality follows after the following computation:

$$\begin{aligned}
&(\overline{R}^{(n)} \otimes I)(\Delta^{(n-1)} \otimes \mathrm{id}(\overline{R})) = \\
&= (R^{(n)} \otimes I)(\Theta^{(n)} \otimes I)(\Delta^{(n-1)} \otimes \mathrm{id}(R))(\Delta^{(n-1)} \otimes \mathrm{id}(\Theta)) = \\
&= (R^{(n)} \otimes I)(w^{-1 \otimes n} \otimes I)(\Delta^{(n-1)}(w) \otimes I)(\Delta^{(n-1)} \otimes \mathrm{id}(R))(\Delta^{(n-1)} \otimes \mathrm{id}(\Theta)) = \\
&= (R^{(n)} \otimes I)(\Delta^{(n-1)} \otimes \mathrm{id}(R))(w^{-1 \otimes n} \otimes I)(\Delta^{(n-1)}(w) \otimes I)(\Delta^{(n-1)} \otimes \mathrm{id}(\Theta)) = \\
&= R^{(n+1)}(\Theta^{(n)} \otimes I)(\Delta^{(n-1)} \otimes \mathrm{id}(\Theta)) = R^{(n+1)}\Theta^{(n+1)} = \overline{R}^{(n+1)}
\end{aligned}$$

We get the second part of the equality in the same way. □

At this stage we are ready to prove the following:

Proposition 3.5.5. $\overline{R}^{(n)}$ *is self-adjoint for all* n.

Proof. It is straightforward to prove (for example, by induction) that for all n:

$$\Delta^{(n)}(a)^* = \Delta^{\mathrm{op}(n)}(a^*)$$

Moreover, it is quite easy to see that $\Delta^{\mathrm{op}}(a)\overline{R} = \overline{R}\Delta(a)$, since Θ commutes with $\Delta(a)$ for all a. We define now:

$$\overline{R}_{n+1} := (\Delta^{\mathrm{op}(n-1)} \otimes \mathrm{id}(\overline{R}))(\overline{R}_n \otimes I) = (\mathrm{id} \otimes \Delta^{\mathrm{op}(n-1)}(\overline{R}))(I \otimes \overline{R}_n)$$

and $\overline{R}_2 = \overline{R}$. We can prove the following identity by induction:

$$\Delta^{\mathrm{op}(n-1)}(a)\overline{R}_n = \overline{R}_n\Delta^{(n-1)}(a)$$

for all a. When $n = 2$ the above identity is true. Suppose it is true for n. We prove it for $n+1$:

$$\begin{aligned}
\Delta^{\mathrm{op}(n)}(a)\overline{R}_{n+1} &= (\Delta^{\mathrm{op}(n-1)} \otimes \mathrm{id}(\Delta^{\mathrm{op}}(a)))(\Delta^{\mathrm{op}(n-1)} \otimes \mathrm{id}(\overline{R}))(\overline{R}_n \otimes I) = \\
&= (\Delta^{\mathrm{op}(n-1)} \otimes \mathrm{id}(\Delta^{\mathrm{op}}(a)\overline{R}))(\overline{R}_n \otimes I) = \\
&= (\Delta^{\mathrm{op}(n-1)} \otimes \mathrm{id}(\overline{R}))(\Delta^{\mathrm{op}(n-1)} \otimes \mathrm{id}(\Delta(a)))(\overline{R}_n \otimes I) = \\
&= (\Delta^{\mathrm{op}(n-1)} \otimes \mathrm{id}(\overline{R}))(\overline{R}_n \otimes I)(\Delta^{(n-1)} \otimes \mathrm{id}(\Delta(a))) = \\
&= \overline{R}_{n+1}\Delta^{(n)}(a)
\end{aligned}$$

Next, by induction again we have $(\overline{R}^{(n+1)})^* = \overline{R}_{n+1}$. In fact, recalling that $\overline{R}$ is self-adjoint and $\overline{R}^{(n+1)} = (\overline{R}^{(n)} \otimes I)(\Delta^{(n-1)} \otimes \mathrm{id}(\overline{R}))$, we get:

$$\begin{aligned}
(\overline{R}^{(n+1)})^* &= (\Delta^{\mathrm{op}(n-1)} \otimes \mathrm{id}(\overline{R}^*))(\overline{R}^{(n)^*} \otimes I) = \\
&= (\Delta^{\mathrm{op}(n-1)} \otimes \mathrm{id}(\overline{R}))(\overline{R}_n \otimes I) = \overline{R}_n
\end{aligned}$$

At this point, proceeding by induction once more, we obtain that $\overline{R}^{(n+1)}$ is self-adjoint:

$$\begin{aligned}
\overline{R}^{(n+1)^*} &= (\Delta^{\mathrm{op}(n-1)} \otimes \mathrm{id}(\overline{R}^*))(\overline{R}^{(n)^*} \otimes I) = \\
&= (\Delta^{\mathrm{op}(n-1)} \otimes \mathrm{id}(\overline{R}))(\overline{R}_n \otimes I) = (\overline{R}_n \otimes I)(\Delta^{(n-1)} \otimes \mathrm{id}(\overline{R})) = \\
&= (\overline{R}^{(n)^*} \otimes I)(\Delta^{(n-1)} \otimes \mathrm{id}(\overline{R})) = (\overline{R}^{(n)} \otimes I)(\Delta^{(n-1)} \otimes \mathrm{id}(\overline{R})) = \overline{R}^{(n+1)}
\end{aligned}$$

□

The existence of the R-matrix in $U_{\mathcal{A}}^{\dagger}\overline{\otimes}U_{\mathcal{A}}^{\dagger}$ allows us to define, for any pair of objects (π_U, U) and (π_V, V) in $\mathrm{Rep}(\mathfrak{U}_{\mathcal{A}})$ the braiding operators:

$$c_{U,V} = \Sigma(\pi_U \otimes \pi_V(R)) \in (U \otimes V, V \otimes U)$$

It is quite natural to consider the associated modified form:

$$\sigma_{U,V} = \Sigma(\pi_U \otimes \pi_V(\overline{R})) \in (U \otimes V, V \otimes U)$$

The operators $\sigma_{U,V}$ are called *coboundary* operators. We have the following:

Proposition 3.5.6. *$\sigma_{U,V}$ is a natural isomorphism satisfying:*

$$\sigma_{U,V} \circ \sigma_{V,U} = \mathrm{id}_{V\otimes U} \tag{3.5.3}$$

$$\sigma_{V\otimes U,W} \circ \sigma_{U,V} \otimes \mathrm{id}_W = \sigma_{U,W\otimes V} \circ \mathrm{id}_U \otimes \sigma_{V,W} \tag{3.5.4}$$

Proof. The first equality is a consequence of the identity $\overline{R}_{21}\overline{R} = I$. The second one is a consequence of Lemma 3.5.4. □

The identity (3.5.4) defines an intertwiner of the category $\mathrm{Rep}(\mathcal{U}_{\mathcal{A}})$ which reverses the order in a triple tensor product:

$$\sigma_3 := \sigma_{V\otimes U,W} \circ \sigma_{U,V} \otimes \mathrm{id}_W \in (U \otimes V \otimes W, W \otimes V \otimes U)$$

More generally, we can consider the arrows:

$$\sigma_n \in (V_1 \otimes \ldots \otimes V_n, V_n \otimes \ldots \otimes V_1)$$

which can be inductively defined in the following way:

$$\sigma_n = \sigma_{V_{n-1}\otimes\ldots\otimes V_1,V_n} \circ \sigma_{n-1} \otimes \mathrm{id}_{V_n}$$

Proposition 3.5.7. *We have:*

$$\sigma_n = \Sigma_n \overline{R}_n \tag{3.5.5}$$
$$\sigma_n^2 = \mathrm{id} \tag{3.5.6}$$

and:

$$\begin{aligned}\sigma_n &= \sigma_{n-1} \otimes \mathrm{id}_{V_1} \circ \sigma_{V_1,V_2\otimes\ldots\otimes V_n} = \\ &= \sigma_{V_1,V_n\otimes\ldots\otimes V_2} \circ \mathrm{id}_{V_1} \otimes \sigma_{n-1} = \\ &= \mathrm{id}_{V_n} \otimes \sigma_{n-1} \circ \sigma_{V_1\otimes\ldots\otimes V_{n-1},V_n}\end{aligned} \tag{3.5.7}$$

Proof. We start proving (3.5.5) by induction on n:

$$\begin{aligned}\sigma_n &= \Sigma_{n-1,1}(\Delta^{(n-1)} \otimes \mathrm{id}(\overline{R}))(\Sigma_{n-1}\overline{R}_{n-1} \otimes \mathrm{id}) = \\ &= (\Sigma_{n-1,1} \circ \Sigma_{n-1} \otimes \mathrm{id})(\Delta^{\mathrm{op}(n-1)} \otimes \mathrm{id}(\overline{R}))(\overline{R}_{n-1} \otimes \mathrm{id}) = \\ &= \Sigma_n(\Delta^{\mathrm{op}(n-1)} \otimes \mathrm{id}(\overline{R}))(\overline{R}_{n-1} \otimes \mathrm{id}) = \Sigma_n \overline{R}_n\end{aligned}$$

(3.5.7) can be proved in a similar way. For example:

$$\begin{aligned}\sigma_{n-1} \otimes \mathrm{id}_{V_1} \circ \sigma_{V_1,V_2\otimes\ldots\otimes V_n} &= \\ = (\Sigma_{n-1}\overline{R}_{n-1} \otimes \mathrm{id})(\Sigma_{1,n-1} \circ \mathrm{id} \otimes \Delta^{(n-1)}(\overline{R})) &= \\ = (\Sigma_{n-1} \otimes \mathrm{id} \circ \Sigma_{1,n-1})(\mathrm{id} \otimes \overline{R}_{n-1})(\mathrm{id} \otimes \Delta^{(n-1)}(\overline{R})) &= \\ = \Sigma_n(\mathrm{id} \otimes \Delta^{\mathrm{op}(n-1)}(\overline{R}))(\mathrm{id} \otimes \overline{R}_{n-1}) = \Sigma_n \overline{R}_n &= \sigma_n\end{aligned}$$

It remains to prove (3.5.6), which can be proved by induction and using (3.5.7) and (3.5.3):

$$\begin{aligned}\sigma_n^2 &= \sigma_n \circ \sigma_n = \mathrm{id}_{V_1} \otimes \sigma_{n-1} \circ \sigma_{V_n\otimes\ldots\otimes V_2,V_1} \circ \sigma_{V_1,V_n\otimes\ldots\otimes V_2} \circ \mathrm{id}_{V_1} \otimes \sigma_{n-1} = \\ &= \mathrm{id}_{V_1} \otimes \sigma_{n-1} \circ \mathrm{id}_{V_1} \otimes \sigma_{n-1} = \mathrm{id}_{V_1\otimes\ldots\otimes V_n}\end{aligned}$$

□

Following now Kirillov [37] and Wenzl's [81] approach we are able to introduce an involution on $\mathrm{Rep}(\mathcal{U}_{\mathcal{A}})$ making it into a *-tensor category. Let V be a representation of $\mathcal{U}_{\mathcal{A}}$. V is a hermitian space if it is endowed with a non-degenerate and sesquilinear $\mathcal{A}$-valued form (ξ, η) on V such that:

$$(\xi, \eta)^* = (\eta, \xi)$$

If T is a map from V to V', we can define the adjoint map $T^* : V' \to V$. Therefore the category of the hermitian spaces on $\mathcal{A}$ is a *-category. (π_V, V) is a *-representation of $\mathcal{U}_\mathcal{A}$ if V is a hermitian space and:

$$\pi_V(a^*) = \pi_V(a)^*$$

Weyl modules can be made into *-representations. First of all, we prove that $V_\lambda(\mathcal{A})$ is endowed with a sesquilinear form:

Proposition 3.5.8. *Let $V_\lambda(\mathcal{A})$ be a Weyl module and v_λ be its highest weight vector. Then there exists a sesquilinear form $(\ ,\ )$ on $V_\lambda(\mathcal{A})$ uniquely determined by $(v_\lambda, v_\lambda) = 1$ and by:*

$$(av_\lambda, bv_\lambda) = (b^*av_\lambda, v_\lambda) = (v_\lambda, a^*bv_\lambda) \tag{3.5.8}$$

Proof. The uniqueness is clear. We pass to the existence of such form. We know that $V_\lambda(\mathcal{A})^*$ is still a $\mathcal{U}_\mathcal{A}$-representation, with the following action:

$$a \cdot \phi(\xi) = \phi(S^{-1}(a)\xi)$$

where $\phi \in V_\lambda(\mathcal{A})^*$. $V_\lambda(\mathcal{A})^*$ has highest weight $-w_0(\lambda)$, where w_0 is the longest element in the Weyl group. Define $\phi_\lambda \in V_\lambda(\mathcal{A})^*$ by $\phi_\lambda(v_\lambda) = 1$ and $\phi_\lambda(v) = 0$ for any v weight vector not belonging to the weight λ. Using a little bit of Lie theory it is possible to prove that ϕ_λ is the lowest weight vector in $V_\lambda(\mathcal{A})^*$. Now, we consider the conjugate space $\overline{V_\lambda(\mathcal{A})}$ with the action of $\mathcal{U}_\mathcal{A}$ defined in the following way:

$$a\overline{v} = \overline{S^{-1}(a^*)v}$$

The highest weight in $\overline{V_\lambda(\mathcal{A})}$ is $-w_0(\lambda)$ and the lowest weight vector is $\overline{v_\lambda}$. Therefore we have an antilinear isomorphism $\Phi : \overline{V_\lambda(\mathcal{A})} \to V_\lambda(\mathcal{A})^*$ such that $\Phi(\overline{v_\lambda}) = \phi_\lambda$ and $\Phi(a\overline{v}) = a\Phi(\overline{v})$. At this point we can define the sesquilinear form on $V_\lambda(\mathcal{A})$:

$$(\xi, \eta) = \Phi(\overline{\xi})(\eta)$$

It is easy to see that this form is well-defined and $(v_\lambda, v_\lambda) = 1$. It remains to prove (3.5.8):

$$(\xi, a\eta) = \Phi(\overline{\xi})(a\eta) = S(a)\Phi(\overline{\xi})(\eta) = \Phi(S(a)\overline{\xi})(\eta) = \Phi(\overline{a^*\xi})(\eta) = (a^*\xi, \eta)$$

□

Proposition 3.5.9. *The sesquilinear form defined in* (3.5.8) *for an irreducible highest weight module $V_\lambda(\mathcal{A})$ is hermitian over $\mathcal{A}$. In other words:*

$$(\xi, \eta) = (\eta, \xi)^*$$

Proof. Since $V_\lambda(\mathcal{A})$ is spanned by elements of the type av_λ, and using the sesquilinearity of the form, it is sufficient to prove that $(a_1v_\lambda, a_2v_\lambda)$ is self-adjoint in $\mathcal{A}$. We can restrict to the case that a_1, a_2 are products of F_i's, and we proceed by induction on the number of factors. Assume $a_1 = F_r a_1'$. Then $(a_1v_\lambda, a_2v_\lambda) = (a_1'v_\lambda, E_r a_2 v_\lambda)$. It is sufficient to prove that $E_r a_2 v_\lambda = \sum_j f_j b_j v_\lambda$, where each b_j is a product of F_i's with less factors than a_2 and f_j are self-adjoint and commute with E_i and F_i for all

i. We proceed by induction on the number of factors F_r in a_2. If a_2 does not contain F_r, $E_r a_2 = a_2 E_r = 0$. Otherwise, we can write $a_2 = a_2' F_r a_2''$, where a_2' is a product of elements of $\{F_i, i \neq r\}$. Then:

$$E_r a_2 v_\lambda = a_2'([\langle \alpha_r, \mu \rangle]_x + F_r E_r) a_2'' v_\lambda$$

where μ is the weight of the weight vector $a_2'' v_\lambda$. $[\langle \alpha_r, \mu \rangle]_x$ is clearly self-adjoint and commute with E_i and F_i, while $E_r a_2''$ can be written as a linear combination as stated using the induction. Iterating we obtain in the end:

$$\sum_i (v_\lambda, f_i b_i v_\lambda)$$

The j-th addend of the above summation is 0 if $b_j \neq I$. So we obtain that $(a_1 v_\lambda, a_2 v_\lambda)$ is self-adjoint for all a_1, a_2. □

As a consequence of the last proposition we have that $V_\lambda(q)$ is also a hermitian space in $\mathrm{Rep}(U_q(\mathfrak{g}))$. This makes that module into a *-representation. At this stage, there is a problem to overcome. In fact, if we consider the product of two *-representation U, V of $\mathcal{U}_\mathcal{A}$, with the action of $\mathcal{U}_\mathcal{A}$ defined in the usual way using the coproduct, and we put on it the natural product form:

$$(\xi \otimes \eta, \xi' \otimes \eta')_p = (\xi, \xi')(\eta, \eta')$$

then $U \otimes V$ is not a *-representation, since Δ is not *-preserving. However, the coboundary structure fixes the problem.

Proposition 3.5.10. *For any pair of *-representations (π_U, U) and (π_V, V) of $\mathcal{U}_\mathcal{A}$, the following form:*

$$(\xi \otimes \eta, \xi' \otimes \eta') = (\xi \otimes \eta, \pi_U \otimes \pi_V(\overline{R})(\xi' \otimes \eta'))_p \tag{3.5.9}$$

is hermitian. Furthermore, $(U \otimes V, \pi_U \otimes \pi_V)$ is a *-representation.

Proof. We proved that $\overline{R}$ is self-adjoint and invertible in the extended algebra $U_\mathcal{A}^\dagger$. Hence the right hand side of (3.5.9) define a non-degenerate and hermitian form. It remains to prove that $\pi_U \otimes \pi_V(a^*) = \pi_U \otimes \pi_V(a)^*$:

$$\begin{aligned}
(\xi \otimes \eta, \pi_U \otimes \pi_V(a)^*(\xi' \otimes \eta')) &= (\pi_U \otimes \pi_V(a)(\xi \otimes \eta), \xi' \otimes \eta') = \\
= (\Delta(a)(\xi \otimes \eta), \overline{R}(\xi' \otimes \eta'))_p &= (\xi \otimes \eta, \Delta^{\mathrm{op}}(a^*)\overline{R}(\xi' \otimes \eta'))_p = \\
= (\xi \otimes \eta, \overline{R}\Delta(a^*)(\xi' \otimes \eta'))_p &= (\xi \otimes \eta, \pi_U \otimes \pi_V(a^*)(\xi' \otimes \eta'))
\end{aligned}$$

□

In this way, all the objects in $\mathrm{Rep}(\mathcal{U}_\mathcal{A})$ can be made into *-representations. More precisely, we endow the tensor product $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}$ with the form defined by the action of the matrix $\overline{R}^{(n)}$:

$$(\xi, \eta) = (\xi, \overline{R}^{(n)} \eta)_p$$

which is hermitian since $\overline{R}^{(n)}$ is self-adjoint. We have the following:

Theorem 3.5.11. *The hermitian forms so defined on objects of* $\mathrm{Rep}(\mathfrak{U}_{\mathcal{A}})$ *allows us to state that* $\mathrm{Rep}(\mathfrak{U}_{\mathcal{A}})$ *is a strict tensor* **-category. Furthemore, both the braiding operators* $c_{U,V} = \Sigma R \in (U \otimes V, V \otimes U)$ *and the coboundary operators* $\sigma_{U,V} = \Sigma \overline{R} \in (U \otimes V, V \otimes U)$ *are unitary arrows of* $\mathrm{Rep}(\mathfrak{U}_{\mathcal{A}})$.

Proof. The associativity of the tensor product easily follows from Lemma 3.5.4. Moreover, $(S \otimes T)^* = S^* \otimes T^*$, where S and T are arrows in the category, and it is merely a consequence of the $\mathfrak{U}_{\mathcal{A}}$-linearity of S and T. We pass to the unitarity of c and σ:

$$\begin{aligned}(\xi \otimes \eta, c^*_{U,V}(\eta' \otimes \xi')) &= (\Sigma R(\xi \otimes \eta), \overline{R}(\eta' \otimes \xi')) = \\ = (R(\xi \otimes \eta), \overline{R}_{21}(\xi' \otimes \eta')) &= (\xi \otimes \eta, R^* R_{21}\Theta_{21}(\xi' \otimes \eta')) = \\ = (\xi \otimes \eta, \Theta_{21}(\xi' \otimes \eta')) &= (\xi \otimes \eta, \Theta_{21} R R^{-1}(\xi' \otimes \eta')) = \\ &= (\xi \otimes \eta, R\Theta R^{-1}\Sigma(\eta' \otimes \xi'))\end{aligned}$$

Therefore $c^*_{U,V} = R^{-1}\Sigma = c^{-1}_{U,V}$. In a similar way one can prove the unitarity of $\sigma_{U,V}$. □

We now focus on $\mathrm{Rep}(U_q(\mathfrak{g}))$. As we said before, if we specialise x to $q \in \mathbb{T}$, we obtain a complex sesquilinear and hermitian form on $V_\lambda(q)$. We can ask if there are some cases where the hermitian form on $V_\lambda(q)$ is an inner product, or, equivalently, positive definite.

Theorem 3.5.12. *Let* $q \in \mathbb{T}$. *The hermitian form of* $V_\lambda(\mathcal{A})$ *specialises to a positive definite form on* $V_\lambda(q)$ *in the following cases:*
(a) *for* $\lambda \in \Lambda^+$ *satisfying* $\langle \lambda + \rho, \theta \rangle < d + \frac{1}{|t|}$ *if q is not a root of unity;*
(b) *for* $\lambda \in \overline{\Lambda}_l$ *if* $q = e^{\frac{\pi i}{dl}}$.
Alternatively, we can say that the hermitian form on $V_\lambda(q)$ *is positive definite if* $\lambda \in \Lambda^+$ *and* $q = e^{\pi i t}$, *with:*

$$|t| < \frac{1}{\langle \lambda + \rho, \theta \rangle - d}$$

We call I the set of such q.

Proof. Since $V_\lambda(q)$ is simple when $q \in I$, the hermitian form on it must be non-degenerate, otherwise its nilspace would be a non-trivial submodule. Let $\{v_i\}_i$ be the canonical basis of $V_\lambda(q)$, and $C = ((v_i, v_j))_{i,j}$. Obviously the coefficients and the eigenvalues of C depend continuously on q. They are all real and positive if $q = 1$ from the classical Lie theory, and non-zero for any $q \in I$. Therefore they have to be positive for all $q \in I$, by continuity. □

Let κ be the dominant weight of the fundamental representation $V = V_\kappa$ of $\mathfrak{g}$. The order of the root of unity q is chosen large enough to make V_κ non-negligible. The following result is crucial to understand the fusion rules of the tensor product of an irreducible representation with the fundamental one.

Theorem 3.5.13. *Let V be the fundamental representation of* $\mathfrak{g}$, *where* $\mathfrak{g}$ *is of a Lie type different from* E_8. *Let* $\lambda \in \Lambda_l$. *Then:*
(a) *All irreducible submodules* V_μ *of* $V_\lambda \otimes V$ *in* $\mathrm{Rep}(U_q(\mathfrak{g}))$ *have weights* $\mu \in \overline{\Lambda}_l$;

(b) *The maximal negligible and non-negligible summands $V_\lambda \otimes V$ in $\mathcal{T}_l$ are unique and given by:*

$$N_\lambda = \bigoplus_{\mu \in \overline{\Lambda}_l \setminus \Lambda_l} m_\mu V_\mu \; , \; V_\lambda \underline{\otimes} V = \bigoplus_{\mu \in \Lambda_l} m_\mu V_\mu$$

with multiplicities as in the classical case. Specifically, both decompositions are multiplicity free for $\mathfrak{g} \neq F_4$;
(c) *If $\mathfrak{g}$ is of type A, B, C, D, then:*

$$m_\mu = 1 \Longleftrightarrow \mu \in E_\lambda = \{\lambda + \gamma : \gamma \text{ weight of } V\} \cap \Lambda^+$$

Proof. The proof of (a) and (b) is essentially a consequence of the classical Lie theory. A sketch of it can be found in Theorem 3.5 of [Wenzl]. We prove (c). Every summand V_μ of $V_\lambda \otimes V$ has weight of the form $\mu = \lambda + \gamma$, where γ is a weight of V. But in addition $\mathfrak{g}$ is of type $ABCD$, so the previous statement has a converse: for any weight γ of V such that $\lambda + \gamma$ is dominant, $V_{\lambda+\gamma}$ does appear in $V_\lambda \otimes V$ [LerherZhang]. □

We know that the coboundary matrices $\overline{R}^{(n)}$ are invertible and self-adjoint, so the corresponding hermitian form on $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}$ is non-degenerate when $\lambda_i \in \overline{\Lambda}_l$, for all i. However, it may happen that this form degenerates on some subspaces. Thanks to the properties of the fundamental representation exposed in the Theorem 3.5.13, Wenzl proved the next important result:

Theorem 3.5.14. *For $q = e^{\frac{\pi i}{dl}}$ and $\lambda \in \Lambda_l$ the hermitian form* (3.5.9) *on $V_\lambda \otimes V$ is positive definite on $V_\lambda \underline{\otimes} V$. Furthermore, for any $\gamma \in \Lambda_l$, the canonical projection:*

$$p_{\lambda,\gamma} : V_\lambda \otimes V \to m_\gamma V_\gamma$$

is self-adjoint under the same form and $p_{\lambda,\gamma} p_{\lambda,\mu} = 0$ for $\gamma \neq \mu$.

3.6 Rigidity of $\mathrm{Rep}(U_q(\mathfrak{g}))$

It is well-known that $\mathrm{Rep}(U_q(\mathfrak{g}))$ is a rigid category. So, here in this section we would like to focus more on the explicit expression of the conjugate maps. In Theorem 3.5.12 we have seen that if $q = e^{\frac{\pi i}{dl}}$ we can put an inner product on V_λ when $\lambda \in \overline{\Lambda_l}$. Proceeding as in the section 1.4 and in particular as in the Remark 1.4.9, we can explicitly express the conjugate maps of $\mathrm{Rep}(U_q(\mathfrak{g}))$ using an orthonormal basis $\{e_i\}$, but only when $r = r_V$ and $\overline{r}_V$ are referred to V endowed with an inner product. Hence:

$$r(1) = \overline{R}_{21} \sum_{i=1}^{n} \overline{e_i} \otimes e_i \quad \overline{r}(1) = \sum_{i=1}^{n} e_i \otimes \overline{e_i} \tag{3.6.1}$$

We notice that r and $\overline{r}$ are as in (1.4.6) and (1.4.7), taking into account that $U_q(\mathfrak{g})$ is a weak quasi Hopf *-algebra with $\Phi = I^{\otimes 3}$, $\Omega = \overline{R}$ and $\alpha = I = \beta$. Nevertheless, r and $\overline{r}$ could be written differently in this special case. This fact will be useful in the next chapter.

Proposition 3.6.1. *Let r and $\overline{r}$ be as in* (3.6.1). *Then:*

$$r(1) = \sum_i \overline{\psi_i} \otimes K_{2\rho} \psi_i \tag{3.6.2}$$

$$\overline{r}^*(\psi \otimes \overline{\phi}) = (\phi, K_{2\rho}^{-1} \psi) \tag{3.6.3}$$

where $\psi, \phi \in V_\lambda$ and $\{\psi_i\}_i$ is a basis of V_λ.

Proof. We start calculating $\overline{r}^*$:

$$\begin{aligned}
\overline{r}^*(\psi \otimes \overline{\phi}) &= (\overline{r}(1), \overline{R}(\psi \otimes \overline{\phi})) = (\psi_i \otimes \overline{\psi_i}, \overline{R}(\psi \otimes \overline{\phi})) = \\
&= (\psi_i, a_j w^{-1} w_{(1)} \psi)(\overline{\psi_i}, \overline{S(b_i)^* S(w^{-1})^* S(w_{(2)})^* \phi}) = \\
&= (\psi_i, a_j w^{-1} w_{(1)} \psi)(S(b_j)^* S(w) S(w_{(2)})^* \phi, \psi_i) = \\
&= (S(b_j)^* S(w) S(w_{(2)})^* \phi, a_j w^{-1} w_{(1)} \psi) = \\
&= (\phi, S(w_{(2)}) S(w^{-1}) S(b_j) a_j w^{-1} w_{(1)} \psi) = \\
&= (\phi, S(w_{(2)}) S(w^{-1}) u w^{-1} w_{(1)} \psi) = \\
&= (\phi, S(w_{(2)}) u S^{-1}(w^{-1}) w^{-1} w_{(1)} \psi)
\end{aligned}$$

Since $w^2 = v$ and $S(v) = v$, we have $S(w^{-1}) = w^{-1}$. So:

$$\begin{aligned}
&(\phi, S(w_{(2)}) u S^{-1}(w^{-1}) w^{-1} w_{(1)} \psi) = \\
&= (\phi, S(w_{(2)}) u v^{-1} w_{(1)} \psi) = (\phi, S(w_{(2)}) K_{2\rho}^{-1} w_{(1)} \psi) = \\
&= (\phi, K_{2\rho}^{-1} S^{-1}(w_{(2)}) w_{(1)} \psi) = (\phi, K_{2\rho}^{-1} \psi)
\end{aligned}$$

using that $v = K_{2\rho} u$ and $S^2(a) = K_{2\rho}^{-1} a K_{2\rho}$. We pass to prove the identity involving r. Let f be a map defined as the right hand side of (3.6.2). It will be sufficient to prove that $f^* = r^*$ in order to prove that $f = r$:

$$\begin{aligned}
f^*(\overline{\psi} \otimes \phi) &= (f(1), \overline{R}(\overline{\psi} \otimes \phi)) = (f(1), \overline{R}^*(\overline{\psi} \otimes \phi)) = \\
&= (\overline{\psi_i}, (w_{(1)})^* w a_j^* \overline{\psi})(K_{2\rho} \psi_i, (w_{(2)})^* w b_j^* \phi) = \\
&= (\overline{\psi_i}, \overline{S(w_{(1)}) S(w^{-1}) S(a_j) \psi})(K_{2\rho} \psi_i, (w_{(2)})^* w b_j^* \phi) = \\
&= (S(w_{(1)}) S(w^{-1}) S(a_j) \psi, \psi_i)(\psi_i, K_{2\rho}^{-1} (w_{(2)})^* w b_j^* \phi) = \\
&= (S(w_{(1)}) S(w^{-1}) S(a_j) \psi, K_{2\rho}^{-1} (w_{(2)})^* w b_j^* \phi) = \\
&= (\psi, S(a_j)^* S(w) S(w_{(1)})^* K_{2\rho}^{-1} (w_{(2)})^* w b_j^* \phi)
\end{aligned}$$

Since:

$$S(a_j)^* S(w) S(w_{(1)})^* K_{2\rho}^{-1} (w_{(2)})^* w b_j^* = (b_j w^{-1} w_{(2)} K_{2\rho} S(w_{(1)}) S(w^{-1}) S(a_j))^*$$

it is sufficient to prove that $b_j w^{-1} w_{(2)} K_{2\rho} S(w_{(1)}) S(w^{-1}) S(a_j) = I$ in order to conclude. Using the tools we used before for $\overline{r}$ and the centrality of w^{-1} we obtain:

$$\begin{aligned}
&b_j w^{-1} w_{(2)} K_{2\rho} S(w_{(1)}) S(w^{-1}) S(a_j) = \\
&= b_j w^{-1} w_{(2)} S^{-1}(w_{(1)}) K_{2\rho} S(w^{-1}) S(a_j) = \\
&= b_j u^{-1} S(a_j) = b_j S^{-1}(a_j) u^{-1} = S(b_j) a_j u^{-1} = I
\end{aligned}$$

In the last line of the previous calculation we used that $b_j S^{-1}(a_j) = S(b_j) a_j = u$, and it is worth to explain why. The Proposition 3.2.4 tells us that $S \otimes S(R) = R$. Therefore $\mathrm{id} \otimes S(R) = S^{-1} \otimes \mathrm{id}(R)$. Applying the opposite multiplication to both sides of the identity we obtain $b_j S^{-1}(a_j) = S(b_j) a_j$. □

3.7 Wenzl's functor

In this section we construct the projections p_n on the tensor powers $V^{\otimes n}$ onto suitable Hilbert subspaces, and describe their main properties. Let λ be a weight in Λ_l and p_λ be the self-adjoint projection from $V_\lambda \otimes V$ to $V_\lambda \underline{\otimes} V$, given by $p_\lambda = \sum p_{\lambda,\gamma}$, where the sum is made over all $\gamma \in \Lambda_l$ such that V_γ is a summand of $V_\lambda \otimes V$. We denote with p_0 the identity map in $(\mathbb{C}, \mathbb{C})$ and p_1 the identity map in (V, V). If $\mathfrak{g} \neq D_n$, we denote with p_2 the projection p_κ, where κ is the dominant weight of V. In the D_n case we denote with κ_1 and κ_2 the dominant weights of the two half-spin irreducible subrepresentations of V. In this case we set $p_2 = p_{\kappa_1} + p_{\kappa_2}$. In general, we denote with p_n the canonical projection onto the non-negligible part of a canonical decomposition of $V^{\otimes n}$ into submodules:

$$V^{\otimes n} = \bigoplus_{\iota \in I_n} V_{\iota,n} \bigoplus N_n$$

with N_n negligible. Using what we said in the previous section, we have unitaries $U_{\iota,\mu} : V_\mu \to V_{\iota,n}$, with $\mu \in \Lambda_l$. It is possible to define the projections p_n iteratively:

$$p_{n+1} = \sum_{\iota \in I_n} U_{\iota,n} \otimes \mathrm{id}_V \circ p_\mu \circ U_{\iota,\mu}^{-1} \otimes \mathrm{id}_V \circ p_n \otimes \mathrm{id}_V$$

Using the iterative description, we can prove the next useful lemma:

Lemma 3.7.1.

(1) $p_n \circ p_m \otimes \mathrm{id}_{V^{\otimes r}} = p_n = p_m \otimes \mathrm{id}_{V^{\otimes r}} \circ p_n, \quad n = m + r$

(2) $A \otimes \mathrm{id}_{V^{\otimes r}} \circ p_{m+r} = p_{n+r} \circ A \otimes \mathrm{id}_{V^{\otimes r}} = A \underline{\otimes}\, \mathrm{id}_{V^{\underline{\otimes} r}}, \quad A \in (V^{\underline{\otimes} m}, V^{\underline{\otimes} n})$

Proof. The first assertion is easy to prove. In fact, for $r = 1$ it is a simple consequence of the iterative definition of p_n; a simple iteration gives the result for the generic case. We pass to prove the second statement. We assume $r = 1$ and $A = U_{\iota_2,\mu} U_{\iota_1,\mu}^{-1}$. So:

$$A : V_{\iota_1,m} \to V_\mu \to V_{\iota_2,n}$$

Therefore:

$$A \otimes \mathrm{id}_V \circ p_{m+1} = U_{\iota_2,\mu} \otimes \mathrm{id}_V \circ p_\mu \circ U_{\iota_1,\mu}^{-1} \otimes \mathrm{id}_V \circ p_m \otimes \mathrm{id}_V = p_{n+1} \circ A \otimes \mathrm{id}_V$$

Since $V^{\underline{\otimes} m}$ and $V^{\underline{\otimes} n}$ are completely reducible, the equality is true for every $A \in (V^{\underline{\otimes} m}, V^{\underline{\otimes} n})$. Now we prove the same assertion for $r > 1$ by induction on r, and using (1):

$$\begin{aligned} A \otimes \mathrm{id}_{V^{\otimes r}} \circ p_{m+r} &= (A \otimes \mathrm{id}_{V^{\otimes r-1}} \circ p_{m+r-1}) \otimes \mathrm{id}_V \circ p_{m+r} = \\ &= (p_{n+r-1} \circ A \otimes \mathrm{id}_{V^{\otimes r-1}}) \otimes \mathrm{id}_V \circ p_{m+r} = \\ &= p_{n+r} \circ (p_{n+r-1} \circ A \otimes \mathrm{id}_{V^{\otimes r-1}}) \otimes \mathrm{id}_V = p_{n+r} \circ A \otimes \mathrm{id}_{V^{\otimes r}} \end{aligned}$$

□

Proposition 3.7.2. *The projection p_n are self-adjoint for all n w.r.t. the Wenzl's hermitian form.*

Proof. We prove it by induction. If $n = 0$ and $n = 1$ the projections are the identity maps so it is straightforward. Suppose it is true for n, and we prove it for $n + 1$. Using the identity $p_n \otimes \mathrm{id}_V \circ p_{n+1} = p_{n+1}$ and the iterative description of p_{n+1} we have:

$$p_{n+1} = \sum_{\iota \in I_n} p_n \otimes \mathrm{id}_V \circ U_{\iota,n} \otimes \mathrm{id}_V \circ p_\mu \circ U_{\iota,\mu}^{-1} \otimes \mathrm{id}_V \circ p_n \otimes \mathrm{id}_V$$

Since p_μ is self-adjoint and $U_{\iota,n}^* = U_{\iota,n}^{-1}$, we obtain that $p_{n+1}^* = p_{n+1}$. □

In the Lemma 3.7.1 we saw that the projections p_n satisfy a sort of left associativity (property (1)). Conversely, this property is not true on the right. In other words, if $n = m + r$:

$$p_n \circ \mathrm{id}_{V^{\otimes r}} \otimes p_m \neq p_n \text{ and } \mathrm{id}_{V^{\otimes r}} \otimes p_m \circ p_n \neq p_n$$

This fact can be deduced from the next example.

Example 3.7.3. Let $\mathfrak{g}$ be $\mathfrak{sl}_2$ and $l = 3$. α is the unique simple positive root, so $d = 1$, $\theta = \alpha$, $\rho = \frac{1}{2}\alpha$ and $\langle \alpha, \alpha \rangle = 2$. Therefore:

$$\Lambda_l = \{\lambda \in \Lambda^+ : \left\langle \lambda + \frac{1}{2}\alpha, \alpha \right\rangle < 3\}$$

So the only admissible λ's are $\lambda = 0$ and $\lambda = \frac{1}{2}\alpha$. Hence the irreducible non-negligible representations are $V_0 = \mathbb{C}$ and $V_1 = V$ fundamental representation. We know that $V_1 \otimes V_1 = V_0 \oplus V_2$, so $V_1 \underline{\otimes} V_1 = V_0$. Therefore:

$$V_0 \otimes V_1 \cong V_1 \cong V_0 \underline{\otimes} V_1$$

As a consequence, $p_3 = p_2 \otimes \mathrm{id}_V$. We now want to understand how p_2 acts on $V_1 \otimes V_1$. We have the following explicit expression of the decomposition of $V_1 \otimes V_1$:

$$V_0 = \langle \psi_1 \otimes \psi_2 - q\psi_2 \otimes \psi_1 \rangle$$
$$V_2 = \langle \psi_1 \otimes \psi_1, \psi_1 \otimes \psi_2 + q^{-1}\psi_2 \otimes \psi_1, \psi_2 \otimes \psi_2 \rangle$$

where $\{\psi_1, \psi_2\}$ is a basis of V_1 such that:

$$E\psi_1 = 0 \,,\, F\psi_1 = \psi_2 \,,\, K\psi_1 = q\psi_1$$
$$E\psi_2 = \psi_1 \,,\, F\psi_2 = 0 \,,\, K\psi_2 = q^{-1}\psi_2$$

So:

$$p_2(\psi_1 \otimes \psi_1) = 0 = p_2(\psi_2 \otimes \psi_2)$$
$$p_2(\psi_1 \otimes \psi_2) = q^{-1}(\psi_1 \otimes \psi_2 - q\psi_2 \otimes \psi_1)$$
$$p_2(\psi_2 \otimes \psi_1) = -(\psi_1 \otimes \psi_2 - q\psi_2 \otimes \psi_1)$$

For instance,

$$p_3(\psi_2 \otimes \psi_1 \otimes \psi_1) = p_2 \otimes \mathrm{id}_V(\psi_2 \otimes \psi_1 \otimes \psi_1) = -\psi_1 \otimes \psi_2 \otimes \psi_1 + q\psi_2 \otimes \psi_1 \otimes \psi_1$$

On the other hand:

$$p_3 \circ \mathrm{id}_V \otimes p_2(\psi_2 \otimes \psi_1 \otimes \psi_1) = p_2 \otimes \mathrm{id}_V \circ \mathrm{id}_V \otimes p_2(\psi_2 \otimes \psi_1 \otimes \psi_1) = 0$$

So $p_3 \circ \mathrm{id}_V \otimes p_2 \neq p_3$. In the same way it is easy to see that $\mathrm{id}_V \otimes p_2 \circ p_3 \neq p_3$. For a deeper treatment of this kind of examples one can see the last chapter of this work.

The next lemma will be very useful for calculations, and it is a sort of replacement of the right associativity failure of the projections:

Lemma 3.7.4. *For any pair of morphisms $S \in (V^{\otimes m}, V^{\otimes n})$ and $T \in (V^{\otimes n}, V^{\otimes r})$, we have:*

$$p_r \circ T \circ \mathrm{id}_{V^{\otimes s}} \otimes p_t \otimes \mathrm{id}_{V^{\otimes z}} \circ S \circ p_m = p_r \circ S \circ T \circ p_m$$

Proof. The range of $\mathrm{id}_{V^{\otimes s}} \otimes (\mathrm{id}_{V^{\otimes t}} - p_t) \otimes \mathrm{id}_{V^{\otimes z}}$ is a negligible module by the property (c) of Prop. 3.4.8, while the ranges of p_m and p_r are non-negligible modules. Hence using propery (b) of Prop. 3.4.8, we obtain:

$$p_r \circ T \circ \mathrm{id}_{V^{\otimes s}} \otimes (\mathrm{id}_{V^{\otimes t}} - p_t) \otimes \mathrm{id}_{V^{\otimes z}} \circ S \circ p_m = 0$$

□

At this point we are ready to introduce the category $\mathcal{G}_l$, whose objects are:

$$V^{\underline{\otimes} n} := p_n V^{\otimes n}$$

and morphisms:

$$(V^{\underline{\otimes} n}, V^{\underline{\otimes} m}) = \{S \in (V^{\otimes n}, V^{\otimes m}) : S p_n = p_m S = S\}$$

We introduce a tensor product in $\mathcal{G}_l$:

$$V^{\underline{\otimes} m} \underline{\otimes} V^{\underline{\otimes} n} := V^{\underline{\otimes} m+n}$$
$$S \underline{\otimes} T := p_{m'+n'} \circ S \otimes T \circ p_{m+n}$$

where $S \in (V^{\underline{\otimes} m}, V^{\underline{\otimes} m'})$ and $T \in (V^{\underline{\otimes} n}, V^{\underline{\otimes} n'})$. The following theorem is due to Wenzl:

Theorem 3.7.5. *$\mathcal{G}_l$ is a strict tensor C^*-category with a unitary braiding simmetry given by:*

$$\underline{c}_{U,V} := p_n c_{U,V} p_n$$

Furthermore the composition of the inclusion $\mathcal{G}_l \to \mathcal{T}_l$ with the quotient $\mathcal{T}_l \to \mathcal{F}_l$ is an equivalence of braided tensor *-categories.

Proof. First of all, we need to prove that $\mathrm{id}_{V^{\underline{\otimes} m}} \underline{\otimes} \mathrm{id}_{V^{\underline{\otimes} n}} = \mathrm{id}_{V^{\underline{\otimes} m+n}}$. Since $\mathrm{id}_{V^{\underline{\otimes} n}} = p_n$, it means to prove that $p_m \underline{\otimes} p_n = p_{m+n}$. It is straightforward using property (1) of Lemma 3.7.1 and Lemma 3.7.4. Next, we need to prove that $(S \underline{\otimes} T)^* = S^* \underline{\otimes} T^*$. We have:

$$(S \underline{\otimes} T)^* = p^*_{m+n} \circ (S \otimes T)^* \circ p^*_{m'+n'} = p_{m+n} \circ (S^* \otimes T^*) \circ p_{m'+n'} = S^* \underline{\otimes} T^*$$

Now we prove the associativity of the tensor product:

$$\begin{aligned}
(Q \underline{\otimes} S) \underline{\otimes} T &= p_{m'+n+r'} \circ p_{m'+n'} \otimes \mathrm{id}_{r'} \circ Q \otimes S \otimes T \circ p_{m+n} \otimes \mathrm{id}_r \circ p_{m+n+r} = \\
&= p_{m'+n'+r'} \circ Q \otimes S \otimes T \circ p_{m+n+r} = \\
&= p_{m'+n'+r'} \circ \mathrm{id}_{m'} \otimes p_{n'+r'} \circ Q \otimes S \otimes T \circ \mathrm{id}_m \otimes p_{n+r} \circ p_{m+n+r} = \\
&= Q \underline{\otimes} (S \underline{\otimes} T)
\end{aligned}$$

using again property (1) of Lemma 3.7.1 and Lemma 3.7.4. Finally we prove that:

$$(S \circ T)\underline{\otimes}(S' \circ T') = (S\underline{\otimes}S') \circ (T\underline{\otimes}T')$$

where $S \in (V^{\underline{\otimes} n}, V^{\underline{\otimes} r}), S' \in (V^{\underline{\otimes} n'}, V^{\underline{\otimes} r'}), T \in (V^{\underline{\otimes} m}, V^{\underline{\otimes} n}), T' \in (V^{\underline{\otimes} m'}, V^{\underline{\otimes} n'})$. In fact:

$$\begin{aligned}(S \circ T)\underline{\otimes}(S' \circ T') &= p_{r+r'} \circ (S \circ T) \otimes (S' \circ T') \circ p_{m+m'} = \\ &= p_{r+r'} \circ (S \otimes S') \circ (T \otimes T') \circ p_{m+m'} = \\ &= p_{r+r'} \circ (S \otimes S') \circ p_{n+n'} \circ p_{n+n'} \circ (T \otimes T') \circ p_{m+m'} = \\ &= (S\underline{\otimes}S') \circ (T\underline{\otimes}T')\end{aligned}$$

using Lemma 3.7.4. The assertions about the braiding can be proved using the same tools and with similar calculations. The unitarity of the braiding is a direct consequence of the self-adjointness of p_n and of the Theorem 3.5.11. The C*-structure is inherited from $\mathcal{T}_l$. It remains to prove that the functor $F : \mathcal{G}_l \to \mathcal{F}_l$ is an equivalence of tensor categories. It is sufficient to proceed as in the case of the equivalence between $\mathcal{T}^0$ and $\mathcal{F}_l$. □

This realisation of the quotient category $\mathcal{F}_l$ is very useful in order to obtain a sort of fiber functor $W_l : \mathcal{G}_l \to \mathrm{Hilb}$ which send $V^{\underline{\otimes} n}$ to its Hilbert space $p_n V^{\otimes n}$, and acts identically on the arrows. Now it is quite natural to wonder if we are able to apply the reconstruction theorem seen in the previous chapter to $(\mathcal{G}_l, W_l)$. The answer is 'no', since it is not possible to state that W_l is at least a weak quasi tensor functor. In fact, it is not obvious at all to prove the existence of epimorphisms:

$$e_{U,V} : W(U) \otimes W(V) \longrightarrow W(U \otimes V)$$

and this fact is essentially a consequence of $\mathrm{id}_n \otimes p_m \circ p_{n+m} \neq p_{n+m}$ and $p_{n+m} \circ \mathrm{id}_n \otimes p_m \neq p_{n+m}$. Anyway, it is possible to develop a reconstruction theorem for the Wenzl's functors W_l when $\mathfrak{g} = \mathfrak{sl}_n$, and this is what we will do in the next chapter. After this construction we will be able to say that the Wenzl's functor W_l is actually a weak tensor functor.

Chapter 4

A reconstruction theorem for $\mathcal{F}_l$

4.1 The generic q case

In this section we fix $q \in \mathbb{T}$ not a root of unity. We treat this case even if we don't need of any kind of truncation in $\mathcal{T}_l$. Anyway, this construction is interesting because it conveys a general strategy that will be extended to the root of unity case in the next sections. The reconstructed object will be a quantum group $\mathcal{C}$ which can be seen as the quantization of the algebra function over the compact group G corresponding to the compact real form of $\mathfrak{g}$. We denote with $\mathcal{V}_{\overline{R}}$ the category whose objects are the object of $\mathcal{T}_l$ which can be endowed with a hermitian form, and the arrows are the linear maps. The tensor product of two spaces in $\mathcal{V}_{\overline{R}}$ is still a hermitian space, with the well-known hermitian form:

$$(v \otimes w, v' \otimes w')' := (v \otimes w, \overline{R}(v' \otimes w'))$$

as in the truncated case. Therefore it is possible to define the adjoint on $\mathcal{V}_{\overline{R}}$. It satisfies all the axioms of a tensor *-category, except the rule of the adjoint of a tensor product arrow, which is replaced by:

$$(S \otimes T)^* = \overline{R}^{-1} \circ (S^* \otimes T^*) \circ \overline{R} \tag{4.1.1}$$

Of course it becomes the usual rule if S and T are morphisms of representations. We indicate with V^n the n-th tensor power of the fundamental representation of $U_q(\mathfrak{g})$, and with $\phi_1 \dots \phi_n$ a simple tensor in V^n. In an obvious way, elements of V^n can be regarded as arrows of $(\mathbf{1}, V^n)$, and elements of $(V^n)^*$ can be regarded as arrows in $(V^n, \mathbf{1})$. We introduce the tensor algebra associated to V:

$$\mathcal{D} := \bigoplus_{n \in \mathbb{N}} (V^n)^* \otimes V^n$$

where the multiplication is given by:

$$(\varphi \otimes \psi)(\varphi' \otimes \psi') = \varphi\varphi' \otimes \psi\psi' \tag{4.1.2}$$

with $\varphi \in (V^n)^*$, $\varphi' \in (V^m)^*$, $\psi \in V^n$, $\psi' \in V^m$. It is easy to see that the subset of simple tensors:

$$\{v_{\phi,\psi} = \varphi \otimes \psi, \psi \in V, \varphi \in V^*\}$$

generates $\mathcal{D}$ as an algebra. Now, let $\mathcal{I}$ be the linear subspace of $\mathcal{D}$ generated by:

$$\{\varphi \circ A \otimes \psi - \varphi \otimes A \circ \psi, \psi \in V^n, \varphi \in (V^*)^m, A \in (V^n, V^m)\}$$

We have the following:

Proposition 4.1.1. *$\mathcal{I}$ is a two-sided ideal of $\mathcal{D}$, so the quotient space $\mathcal{C} = \mathcal{D}/\mathcal{I}$ is an associative and unital algebra.*

Proof. It is sufficient to show that $\mathcal{I}$ is stable under left and right multiplication by the generators $v_{\phi,\psi}$. In fact:

$$v_{\phi,\gamma}(\varphi \circ A \otimes \psi - \varphi \otimes A \circ \psi) = (\phi\varphi \circ \mathrm{id} \otimes A) \otimes (\gamma\psi) - (\phi\varphi) \otimes (\mathrm{id} \otimes A \circ \gamma\psi)$$

Since $\mathrm{id} \otimes A \in (V^{n+1}, V^{m+1})$, we obtain the left stability of $\mathcal{I}$. The right one can be obtained in the same way. □

We now want to give $\mathcal{C}$ a structure of involutive Hopf algebra. Let ψ be an element of V, and ψ^* the linear functional on V defined by:

$$\psi^*(\psi') = (\psi, \psi')$$

It is possible to identify the dual space $(V^n)^*$ with $(V^*)^n$, denoting by $\psi_1^* \ldots \psi_n^*$ the tensor product functional $\psi_1^* \otimes \ldots \otimes \psi_n^*$. In fact, $(V^*)^n$ is a subspace of $(V^n)^*$. But they have the same dimension, so they must coincide. We can define the involution * on $\mathcal{D}$ as follows:

$$(\phi_1^* \ldots \phi_n^* \otimes \psi_1 \ldots \psi_n)^* = \psi_n^* \ldots \psi_1^* \otimes \phi_1 \ldots \phi_n \tag{4.1.3}$$

where $\phi_i, \psi_i \in V$. It is now interesting to express this adjoint in terms of the hermitian structure of V^n:

Lemma 4.1.2. *Let $\psi_1, \ldots, \psi_n$ be vectors in V, and $\sigma_n = \Sigma_n \overline{R}^{(n)} \in (V^n, V^n)$ the coboundary operator. We have:*

$$(\psi_1 \ldots \psi_n)^* = \psi_n^* \ldots \psi_1^* \circ \sigma_n$$

Proof. Using (4.1.1), we have:

$$(\psi_1 \ldots \psi_n)^* = \overline{R}^{(n)-1} \circ \psi_1^* \ldots \psi_n^* \circ \overline{R}^{(n)}$$

$\overline{R}^{(n)-1}$ can be removed since it acts trivially on the unit object. Therefore:

$$(\psi_1 \ldots \psi_n)^* = \psi_1^* \ldots \psi_n^* \circ \overline{R}^{(n)} = \psi_n^* \ldots \psi_1^* \sigma_n$$

□

Representing simple tensors in $\mathcal{D}$ in the form $\phi^* \sigma_n^{-1} \otimes \phi$, one can define the adjoint map on $\mathcal{D}$ by:

$$(\psi^* \sigma_n^{-1} \otimes \phi)^* = \phi^* \sigma_n^{-1} \otimes \psi$$

where $\phi, \psi \in V^n$.

Proposition 4.1.3. *This involution makes $\mathcal{D}$ into a $*$-algebra and $\mathcal{I}$ into a $*$-ideal. Hence $\mathcal{C}$ is a $*$-algebra.*

Proof. It is straightforward to see that $\mathcal{D}$ is a $*$-algebra. Let us prove that $\mathcal{I}$ is a $*$-ideal. Suppose that $A \in (V^m, V^n)$, $\psi \in V^n$, $\phi \in V^m$, and we compute the adjoint of $\psi^*\sigma_n^{-1}A \otimes \phi$:

$$(\psi^*\sigma_n^{-1}A \otimes \phi)^* = ((\sigma_n^{-1}A^*\sigma_n\psi)^*\sigma_n^{-1} \otimes \phi)^* = \phi^*\sigma_n^{-1} \otimes \sigma_n^{-1}A^*\sigma_n\psi$$

using the unitarity of σ_n. In the same way:

$$(\psi^*\sigma_n^{-1} \otimes A\phi)^* = (A\phi)^*\sigma_n^{-1} \otimes \psi = \phi^*\sigma_n^{-1}(\sigma_n A^*\sigma_n^{-1}) \otimes \psi$$

Since $\sigma_n = \sigma_n^{-1}$, we have that the adjoint of an element of the form $\psi^*\sigma_n^{-1}A \otimes \phi - \psi^*\sigma_n^{-1} \otimes A\phi$ is of the same form. $\square$

Remark 4.1.4. Let $[\psi^* \otimes \psi']$ be a class element in $\mathcal{C}$, where $\psi, \psi' \in V^n$. The adjoint is:

$$[\psi^* \otimes \psi']^* = [\psi'^* \otimes \psi^*]$$

where the adjoints of ψ, ψ' are referred to the hermitian form on V^n. If V_α is a submodule of V^n, with $\alpha \in \Lambda_l$, then the hermitian form on V^n restricts to the hermitian form on V_α. Hence if $\psi, \psi' \in V_\alpha$, ψ^* and ψ'^* are the adjoints relative to the hermitian form of V_α.

We can now introduce a Hopf algebra structure on $\mathcal{C}$ first endowing $\mathcal{D}$ with a coproduct defined by:

$$\Delta(\phi^* \otimes \psi) = \sum_r (\phi^* \otimes \eta_r) \otimes (\eta_r^* \otimes \psi)$$

where $\{\eta_r\}_r$ is a basis of V^n and η_r^* its dual basis in $(V^n)^*$. An easy computation shows that the definition of Δ does not depend on the choice of the basis. In fact if $\Delta'(\phi^* \otimes \psi) = \sum_s (\phi^* \otimes \xi_s) \otimes (\xi_s^* \otimes \psi)$, we obtain:

$$\sum_s (\phi^* \otimes \xi_s) \otimes (\xi_s^* \otimes \psi) = \sum_{r,s} |(\eta_r, \xi_s)|^2 (\phi^* \otimes \eta_r) \otimes (\eta_r^* \otimes \psi) =$$
$$= \sum_r \|\eta_r\|^2 (\phi^* \otimes \eta_r) \otimes (\eta_r^* \otimes \psi) = \sum_r (\phi^* \otimes \eta_r) \otimes (\eta_r^* \otimes \psi)$$

Proposition 4.1.5. *The coproduct Δ is unital and coassociative, and satisfies $\Delta(a)^* = \Delta^{\mathrm{op}}(a^*)$. Moreover $\mathcal{I}$ is a coideal, so Δ is a coproduct on $\mathcal{C}$ satisfying the same properties.*

Proof. Unitality is straightforward. Let us pass to multiplicativity. We take $\phi, \psi \in V^n$ and $\gamma, \delta \in V^m$. Moreover, $\{\eta_r\}$ is a basis of V^n and $\{\xi_s\}$ is a basis of V^m:

$$\Delta(\phi^* \otimes \psi)\Delta(\gamma^* \otimes \delta) = \sum_{r,s} [(\phi^* \otimes \eta_r) \otimes (\eta_r^* \otimes \psi)][(\gamma^* \otimes \xi_s \otimes \xi_s^* \otimes \delta)] =$$
$$= \sum_{r,s} (\phi^*\gamma^* \otimes \eta_r\xi_s) \otimes (\eta_r^*\xi_s^* \otimes \psi\delta) = \Delta(\phi^*\gamma^* \otimes \psi\delta)$$

Now we focus on the central terms $\sum_{r,s} \eta_r\xi_s \otimes \eta_r^*\xi_s^*$. Let $\{\zeta_h\}$ be a basis of V^{n+m}. We define:

$$\zeta^h = (\zeta_h, \eta_r\xi_s)\eta_r^*\xi_s^*$$

Therefore:

$$(\zeta^h)^* = \sum_{r,s} (\eta_r \xi_s, \zeta_h)(\eta_r^* \xi_s^*)^* = \sum_{r,s} (\eta_r \xi_s, \zeta_h) \overline{R}_{21} \eta_r \xi_s =$$
$$= \sum_{r,s} \overline{R}_{21} (\eta_r \xi_s, \overline{R} \zeta_h)_p \xi_r \eta_s = \overline{R}_{21} \overline{R} \zeta_h = \zeta_h$$

So $\{\zeta^h\}_h$ is the dual basis of $\{\zeta_h\}_h$. Multiplicativity follows after the next computation:

$$\sum_h \zeta_h \otimes \zeta^h = \sum_{h,r,s} \zeta_h \otimes (\zeta_h, \eta_r \xi_s) \eta_r^* \xi_s^* =$$
$$= \sum_{r,s} \left(\sum_h (\zeta_h, \eta_r \xi_s) \zeta_h \right) \otimes \eta_r^* \xi_s^* = \sum_{r,s} \eta_r \xi_s \otimes \eta_r^* \xi_s^*$$

Coassociativity is very easy to prove. We pass to the relation $\Delta(a)^* = \Delta^{\mathrm{op}}(a^*)$. Using multiplicativity of Δ, it is sufficient to prove it for $a = \phi^* \otimes \psi \in V^* \otimes V$. We have:

$$\Delta^{\mathrm{op}}(\phi^* \otimes \psi)^* = \left(\sum_r (\eta_r^* \otimes \psi) \otimes (\phi^* \otimes \eta_r) \right)^* =$$
$$= \sum_r (\psi^* \otimes \eta_r) \otimes (\eta_r^* \otimes \phi) = \Delta(\psi^* \otimes \phi) = \Delta((\phi^* \otimes \psi)^*)$$

Finally we prove that $\mathcal{I}$ is a coideal. We take $A \in (V^n, V^m)$, $\{\eta_i\}_i$ basis of V^n, $\{\xi_j\}_j$ basis of V^m, $\psi \in V^n$ and $\phi \in (V^m)^*$:

$$\Delta(\phi \circ A \otimes \psi) = \phi \circ A \otimes \eta_i \otimes \eta_i^* \otimes \psi$$
$$\Delta(\phi \otimes A \circ \psi) = \phi \otimes \xi_j \otimes \xi_j^* \otimes A \circ \psi$$

Therefore:

$$\Delta(\phi A \otimes \psi - \phi \otimes A\psi) = \sum_{i,j} ((\phi A \otimes \eta_i - \phi \otimes A\eta_i) \otimes \eta_i^* \otimes \psi +$$
$$+ \phi \otimes \xi_j \otimes (\xi_j^* A \otimes \psi - \xi_j^* \otimes A\psi) + \phi \otimes A\eta_i \otimes \eta_i^* \otimes \psi - \phi \otimes \xi_j \otimes \xi_j^* A \otimes \psi)$$

It remains to check that:

$$\phi \otimes A\eta_i \otimes \eta_i^* \otimes \psi = \phi \otimes \xi_j \otimes \xi_j^* A \otimes \psi$$

In fact, this implies $\Delta(\mathcal{I}) \subset \mathcal{I} \otimes \mathcal{D} + \mathcal{D} \otimes \mathcal{I}$. Omitting all the summation symbols, we have:

$$\begin{aligned} \phi \otimes A\eta_i \otimes \eta_i^* \otimes \psi &= \phi \otimes (\xi_j, A\eta_i) \xi_j \otimes \eta_i^* \otimes \psi = \\ &= \phi \otimes \xi_j \otimes ((A\eta_i, \xi_j)\eta_i)^* \otimes \psi = \\ &= \phi \otimes \xi_j \otimes ((\eta_i, A^* \xi_j)\eta_i)^* \otimes \psi = \\ &= \phi \otimes \xi_j \otimes (A^* \xi_j)^* \otimes \psi = \\ &= \phi \otimes \xi_j \otimes \xi_j^* A \otimes \psi \end{aligned}$$

□

We next introduce the functional $h : \mathcal{C} \to \mathbb{C}$. Let $e_n : V^n \to V^n$ be the projection onto the isotypical component of the trivial subrepresentation of V^n. So:

$$h(\phi \otimes \psi) = \phi(e_n \psi)$$

where $\psi \in V^n$ and $\phi \in (V^n)^*$. Calculating h on a generic element of $\mathcal{I}$ we obtain:

$$h(\phi A \otimes \psi - \phi \otimes A\psi) = \phi(e_m A\psi) - \phi(Ae_n\psi) = 0$$

since $e_m A = Ae_n$ for any $A \in (V^n, V^m)$. In fact A sends every isotypical component of V^n in the same isotypical component of V^m, since it is $U_q(\mathfrak{g})$-linear. In particular it happens for the trivial subrepresentation. Hence h annihilates on $\mathcal{I}$. As a consequence, it is a linear functional on $\mathcal{C}$. This functional will turn out useful later. Consider a complete set V_λ of irreducible representations, where $\lambda \in \Lambda^+$. The isometric intertwiner $S \in (V_\lambda, V^{\otimes n})$ induces a linear inclusion $V_\lambda^* \otimes V_\lambda \hookrightarrow \mathcal{C}$ which takes a simple tensor $\phi \otimes \psi$ to $[\phi \circ S^* \otimes S \circ \psi]$.

Proposition 4.1.6. *The above inclusion does not depend on n and $S \in (V_\lambda, V^{\otimes n})$. Moreover, the image is a subcoalgebra.*

Proof. Let T be another isometry in $(V_\lambda, V^{\otimes m})$. Then:

$$[\phi S^* \otimes S\psi] = [\phi S^* \otimes ST^*T\psi] = [\phi S^* ST^* \otimes T\psi] = [\phi T^* \otimes T\psi]$$

since $ST^* \in (V^{\otimes m}, V^{\otimes n})$. It remains to prove the last assertion. A similar computation allows to prove that $[\psi^* \otimes \psi'] = 0$ when ψ and ψ' lie in orthogonal invariant subspaces of some $V^{\otimes n}$. The next step is to choose a basis of $V^{\otimes n}$ collecting the basis of every single irreducible components of $V^{\otimes n}$. So, if ϕ and ψ stay in the same irreducible component of $V^{\otimes n}$, then $\Delta([\phi^* \otimes \psi])$ can be expressed using only the orthonormal basis of that submodule. □

Lemma 4.1.7. *Let V_λ and V_μ be two irreducible representations. Then $V_\lambda \otimes V_\mu$ contains the trivial representation if and only if $\overline{V_\lambda} \cong V_\mu$.*

Proof. Suppose that $\overline{V_\lambda}$ is isomorphic to V_μ. Then the trivial representation is contained in $V_\lambda \otimes \overline{V_\lambda}$ using the rigidity of the category. On the other hand, suppose that $\mathbb{C}$ is contained in $V_\lambda \otimes V_\mu$. Then there exists a $U_q(\mathfrak{g})$-linear map φ from $V_\lambda \otimes V_\mu$ onto $\mathbb{C}$, such that $\xi \otimes \gamma \mapsto c_{\xi,\gamma}$. This induces a map φ' from V_λ to $\overline{V_\mu}$ where $\xi \mapsto \xi^*$ such that $\xi^*(\gamma) = c_{\xi,\gamma}$. Using the properties of φ we can see that $\varphi' \neq 0$ is in $(V_\lambda, \overline{V_\mu})$. Since V_λ and $\overline{V_\mu}$ are both irreducible we have $V_\lambda \cong \overline{V_\mu}$. □

Theorem 4.1.8. *For any $\lambda \in \Lambda^+$, the natural inclusion $V_\lambda^* \otimes V_\lambda \hookrightarrow \mathcal{C}$ is faithful. Therefore:*

$$\mathcal{C} = \bigoplus_{\lambda \in \Lambda^+} V_\lambda^* \otimes V_\lambda$$

as a coalgebra and $\mathcal{C}$ is cosemisimple.

Proof. Consider $S_{\lambda,i} \in (V_\lambda, V^{\otimes n})$ such that $S_{\lambda,i}^* S_{\lambda,i} = \mathrm{id}_{V_\lambda}$ and:

$$\sum_i S_{\lambda,i} S_{\lambda,i}^* = \mathrm{id}_{V^{\otimes n}}$$

The class of the element $\phi \otimes \psi \in (V^{\otimes n})^* \otimes V^{\otimes n}$ in $\mathcal{C}$ can be written in the form:

$$[\phi \otimes \psi] = \left[\phi \otimes \sum_i S_{\lambda,i} S^*_{\lambda,i} \psi\right] = \left[\sum_i \phi S_{\lambda,i} \otimes S^*_{\lambda,i}\psi\right]$$

As a consequence, $\mathcal{C}$ is linearly generated by $V^*_\lambda \otimes V_\lambda$. We prove now that the inclusion of $V^*_\lambda \otimes V_\lambda$ in $\mathcal{C}$ is faithful. In order to do this, we use the functional h. Let $a = \psi^* \otimes \psi'$ be an element in $V^*_\lambda \otimes V_\lambda$ and $b = \phi^* \otimes \phi'$ be an element in $V^*_\mu \otimes V_\mu$. If V_λ is not conjugate to V_μ, $h(ab) = 0$, since $V_\lambda \otimes V_\mu$ does not contain the trivial representation. Suppose now that $V_\lambda \cong \overline{V_\mu}$. If $\{\psi_i\}_i$ is an orthonormal basis of V_λ, we consider the conjugate map $\overline{r}(1) = \sum_i \psi_i \otimes \overline{\psi_i}$ in $(\mathbb{C}, V_\lambda \otimes \overline{V_\lambda})$. Suppose that $a = \phi^* \otimes \psi$ is equal to 0 as an element of $\mathcal{C}$, and we want to prove that it is 0 also as an element of $V^*_\lambda \otimes V_\lambda$. Let $b = \overline{\xi}^* \otimes \overline{\eta}$ be an element in $\overline{V_\lambda}^* \otimes \overline{V_\lambda}$. We know that $\overline{r}^*\overline{r} = \sum_i (\psi_i, K^{-1}_{2\rho}\psi_i) = d(\lambda)$, where $d(\lambda)$ is the quantum dimension of V_λ. As a consequence, $\frac{1}{d(\lambda)}\overline{r}\overline{r}^*$ is the projection of $V_\lambda \otimes \overline{V_\lambda}$ onto $\mathbb{C}$. Therefore:

$$\begin{aligned} 0 = d(\lambda)h(ab) = (\phi \otimes \overline{\xi}, \overline{r}\overline{r}^*(\psi \otimes \overline{\eta})) = \\ = (\overline{r}^*(\phi \otimes \overline{\xi}), \overline{r}^*(\psi \otimes \overline{\eta})) = (\xi, K^{-1}_{2\rho}\phi)(\eta, K^{-1}_{2\rho}\psi) \end{aligned}$$

for all $\xi, \eta \in V_\lambda$. Hence $K^{-1}_{2\rho}\phi = 0 = K^{-1}_{2\rho}\psi$ which implies $\phi = 0 = \psi$. □

Let us fix a complete set of irreducible representations parametrised by Λ^+. For any $\mu \in \Lambda^+$, the conjugate of V_μ is V_λ, where $\lambda = -w_0\mu$. The composition of the complex conjugation $J_\mu : V_\mu \to \overline{V_\mu}$ with a unitary intertwiner $U_\mu : \overline{V_\mu} \to V_\lambda$ is an antiunitary map $j_\mu : V_\mu \to V_\lambda$, unique up to a scalar $z \in \mathbb{C}$ such that $|z| = 1$. We can thus define a linear map, the antipode, $S : \mathcal{C} \to \mathcal{C}$, by:

$$S(\phi^* \otimes \psi) = (j_\mu\psi)^* \otimes j_\mu\phi$$

where $\phi, \psi \in V_\mu$. It is quite easy to see that the definition of S does not depend on the choice of the unitary intertwiner. We can also define the counit $\varepsilon : \mathcal{C} \to \mathbb{C}$, such that:

$$\varepsilon(\phi^* \otimes \psi) = (\phi, \psi)$$

Proposition 4.1.9. *Antipode S and counit ε make $\mathcal{C}$ into a Hopf *-algebra. Furthermore, S is invertible and S and ε commute with the adjoint map. Finally:*

$$S^2(\phi^* \otimes \psi) = \phi^* K^{-1}_{2\rho} \otimes K_{2\rho}\psi$$

Proof. The relations $S(a^*) = S(a)^*$ is easy to check. S is invertible and its inverse is the map:

$$S^{-1}(\phi^* \otimes \psi) = (j^{-1}_\mu\psi)^* \otimes j^{-1}_\mu\phi$$

where $j^{-1}_\mu = J^{-1}_\mu \circ U^*_\mu$. Next, we prove that ε is a counit. We check the identity $\varepsilon \otimes \mathrm{id} \circ \Delta = \mathrm{id}$. The identity with ε on the right can be proved in the same way.

$$\begin{aligned} \varepsilon \otimes \mathrm{id}(\Delta(\phi^* \otimes \psi)) &= \sum_r \varepsilon \otimes \mathrm{id}(\phi^* \otimes \eta_r \otimes \eta^*_r \otimes \psi) = \\ &= \sum_r (\phi, \eta_r)\eta^*_r \otimes \psi = \phi^* \otimes \psi \end{aligned}$$

where $\phi, \psi \in V_\lambda$ and $\{\eta_r\}_r$ is an orthonormal basis in V_λ. We pass to prove that S is an antipode:

$$\begin{aligned} m \circ \mathrm{id} \otimes S \circ \Delta(\phi^* \otimes \psi) &= m \circ \mathrm{id} \otimes S(\phi^* \otimes \eta_r \otimes \eta_r^* \otimes \psi) = \\ = m(\phi^* \otimes \eta_r \otimes (j_\mu \psi)^* \otimes (j_\mu \eta_r)) &= \phi^*(j_\mu\psi)^* \otimes \eta_r(j_\mu \eta_r) = \\ = \phi^*(j_\mu\psi)^* \otimes (\mathrm{id} \otimes U_\mu \circ \overline{r}(1)) &= \phi^*(j_\mu\psi)^* \circ \mathrm{id} \otimes U_\mu \circ \overline{r} \otimes 1 \end{aligned}$$

Now we calculate the left factor of the tensor product in the last expression of the above identity:

$$\begin{aligned} \phi^* \otimes (j_\mu\psi)^* \circ \mathrm{id} \otimes U_\mu \circ \overline{r}(1) &= \sum_r (\phi, \eta_r)(j_\mu \psi, j_\mu \eta_r) = \\ &= \sum_r (\phi, \eta_r)(\eta_r, \psi) = (\phi, \psi) = \varepsilon(\phi^* \otimes \psi) \end{aligned}$$

It remains to prove the relation involving S^2 and $K_{2\rho}$. We need to talk about some preliminary facts. First of all, it is possible to identify V_μ and $\overline{\overline{V_\mu}}$. In fact, let ξ be an element in $\overline{\overline{V_\mu}}$, and $a \in U_q(\mathfrak{g})$. We have:

$$a \cdot \xi = S^2(a)\xi = K_{2\rho}^{-1} a K_{2\rho} \xi$$

So $K_{2\rho} \in (\overline{\overline{V_\mu}}, V_\mu)\ \forall \mu$, and of course $K_{2\rho}^{-1} \in (V_\mu, \overline{\overline{V_\mu}})$. Furthermore, let U_λ and U_μ be the unitary intertwiners introduced before. More precisely, $U_\mu \in (\overline{V_\mu}, V_\lambda)$ and $U_\lambda \in (\overline{V_\lambda}, V_\mu)$. It is possible to express U_λ in terms of U_μ:

$$U_\lambda = K_{2\rho} \circ U_\mu^\vee$$

where $U_\mu^\vee \in (\overline{V_\lambda}, \overline{\overline{V_\mu}})$ is the transpose arrow of U_μ. Repeating the same argument of the Remark 2.2.15 we obtain that $U_\mu^\vee(\varphi_{V_\lambda}) = \varphi_{V_\lambda} \circ U_\mu$ and $U_\mu^\vee \circ J_\lambda = J_{\overline{V_\mu}} \circ U_\mu^*$. If we want to calculate S^2, we have:

$$S^2(\phi^* \otimes \psi) = (j_\lambda j_\mu \phi)^* \otimes (j_\lambda j_\mu \psi)$$

where $\phi, \psi \in V_\mu$. We look for the explicit expression of $j_\lambda j_\mu$:

$$\begin{aligned} j_\lambda j_\mu = U_\lambda \circ J_\lambda \circ U_\mu \circ J_\mu &= K_{2\rho} \circ U_\mu^\vee \circ J_\lambda \circ U_\mu \circ J_\mu = \\ = K_{2\rho} \circ J_{\overline{V_\mu}} \circ U_\mu^* \circ U_\mu \circ J_\mu &= K_{2\rho} \circ J_{\overline{V_\mu}} \circ J_\mu \end{aligned}$$

and this gives $j_\lambda j_\mu = K_{2\rho}$. □

We finally pass to the dual space $\mathcal{C}'$ and we endow it with the usual dual algebra structure given by:

$$\omega\omega' := \omega \otimes \omega' \circ \Delta$$

Moreover, we can define a coproduct:

$$\Delta'(\omega)(a \otimes b) = \omega(ab)$$

and an involution:

$$\omega^*(a) = \overline{\omega(a^*)}$$

Notice that the involution on the dual space of $\mathcal{C}$ differs from the one of a ordinary Hopf *-algebra. The counit $\varepsilon' : \mathcal{C}' \to \mathbb{C}$ and the antipode $S' : \mathcal{C}' \to \mathcal{C}'$ are defined as usual by:

$$\varepsilon'(\omega) = \omega(I) \text{ and } S'(\omega) = \omega \circ S$$

As a consequence of the Theorem 4.1.8, we have:

Theorem 4.1.10. *$\mathcal{C}'$ is isomorphic as a *-algebra to the direct product of full matrix algebras:*

$$\mathcal{C}' \cong \prod M_{n_\lambda}(\mathbb{C})\,,\ n_\lambda = \dim(V_\lambda)$$

The coproduct Δ' is an algebra homomorphism satisfying $\Delta'(\omega^*) = (\Delta')^{\mathrm{op}}(\omega)^*$

4.2 The universal algebra $\mathcal{D}(V, l)$

In this section the deformation parameter is a fixed root of unity of the form $q = e^{\frac{i\pi}{dl}}$. Our aim is to construct non-associative bialgebra $\mathcal{D}(V, l)$ endowed with an involution playing a role similar to that of $\mathcal{D}(V)$ in the generic case. Let V be Wenzl's fundamental representation of $\mathfrak{g}$ (where $\mathfrak{g}$ is not of type E_8). We consider the infinite-dimensional linear space:

$$\mathcal{D}(V, l) = \bigoplus_{n=0}^{\infty} (V^{\otimes n})^* p_n \otimes p_n V^{\otimes n}$$

Notice that $\mathcal{D} = \mathcal{D}(V, l)$ depends not only on V but also on the root of unity. We define the multiplication on $\mathcal{D}$ by:

$$\alpha\beta := \phi\phi' p_{m+n} \otimes p_{m+n} \psi\psi'$$

for

$$\alpha = \phi \otimes \psi \in (V^m)^* p_m \otimes p_m V^m \ \text{ and } \ \beta = \phi' \otimes \psi' \in (V^n)^* p_n \otimes p_n V^n$$

$\mathcal{D}$ is a unital but not associative algebra, as, if we pick a third element $\gamma = \phi'' \otimes \psi'' \in (V^r)^* p_r \otimes p_r V^r$, and we take into account the relation $p_{m+n+r} \circ p_{m+n} \otimes \mathrm{id}_r = p_{m+n+r}$, we see that:

$$(\alpha\beta)\gamma = (\phi\phi'\phi'')p_{m+n+r} \otimes p_{m+n+r}(\psi\psi'\psi'')$$

but

$$\alpha(\beta\gamma) = (\phi\phi'\phi'')\,\mathrm{id}_m \otimes p_{n+r} \circ p_{m+n+r} \otimes p_{m+n+r} \circ \mathrm{id}_m \otimes p_{n+r}(\psi\psi'\psi'')$$

that differs from the previous one since in general $p_{m+n+r} \circ \mathrm{id}_m \otimes p_{n+r} \neq p_{m+n+r}$. The elements:

$$v_{\xi,\eta} = \xi^* \otimes \eta, \quad \xi, \eta \in V$$

still generate $\mathcal{D}$ as an algebra. We next introduce an involution on $\mathcal{D}$ as suggested by the generic case. Specifically, we replace the coboundary operators σ_n by their truncated version:

$$\tau_n := p_n \sigma_n p_n \in (V^{\underline{\otimes} n}, V^{\underline{\otimes} n})$$

Proposition 4.2.1. *τ_n satisfies the following relations:*

$$\tau_n^* = \tau_n \ \text{ and } \ \tau_n^2 = p_n$$

for all $n \in \mathbb{N}$.

Proof. $\tau_n^* = p_n \sigma_n^* p_n = p_n \sigma_n p_n$ since $\sigma_n = \Sigma_n \overline{R}_n$ is self-adjoint with respect to the hermitian form. Furthermore, using Lemma 3.7.4 we have $\tau_n^2 = p_n \sigma_n p_n \sigma_n p_n = p_n \sigma_n^2 p_n = p_n$ since $\sigma_n^2 = \mathrm{id}$. $\square$

We can write any element $\phi \in (V^{\otimes n})^* p_n$ uniquely in the form $\phi = \psi^*$ with $\psi \in V^{\underline{\otimes} n}$. The adjoint of ψ is computed with respect to the Wenzl's inner product of $V^{\underline{\otimes} n}$. We set for $\psi, \psi' \in V^{\underline{\otimes} n}$:

$$(\psi^* \otimes \psi')^* := (\tau_n \psi')^* \otimes \tau_n \psi$$

Proposition 4.2.2. *The involution on $\mathcal{D}$ enjoys the following properties. Let a be an element in $\mathcal{D}$ and $v_{\xi,\eta}$ in $V^* \otimes V$:*
(a) *$a \mapsto a^*$ is antilinear;*
(b) *$a^{**} = a$;*
(c) *$v^*_{\xi,\eta} = v_{\eta,\xi}$;*
(d) *$(v_{\xi,\eta} a)^* = a^* v^*_{\xi,\eta}$.*

Proof. (a) is obvious, (b) is a consequence of the involutivity of τ_n and (c) of $\tau_1 = \mathrm{id}_V$. More effort is required to prove (d). Let a be of the following form:

$$a = \phi^* \otimes \psi \in (V^{\otimes n})^* p_n \otimes p_n V^{\otimes n}$$

We have:

$$\begin{aligned} a^* v^*_{\xi,\eta} &= (\psi^* \tau_n^{-1} \otimes \tau_n^{-1} \phi)(\eta^* \otimes \xi) = \\ &= (\psi^* \eta^* \circ \tau_n^{-1} \otimes \mathrm{id}_V \circ p_{n+1}) \otimes (p_{n+1} \circ \tau_n^{-1} \otimes \mathrm{id}_V \circ \phi\xi) = \\ &= ((\eta\psi)^* \circ \sigma(V, V^{\underline{\otimes} n})^{-1} \circ \tau_n^{-1} \otimes \mathrm{id}_V \circ p_{n+1}) \otimes (p_{n+1} \circ \tau_n^{-1} \otimes \mathrm{id}_V \circ \phi\xi) = \\ &= ((\eta\psi)^* \circ \mathrm{id}_V \otimes p_n \circ \sigma(V, V^{\underline{\otimes} n})^{-1} \circ \tau_n^{-1} \otimes \mathrm{id}_V \circ p_{n+1}) \otimes (p_{n+1} \circ \tau_n^{-1} \otimes \mathrm{id}_V \circ \phi\xi) \end{aligned}$$

Using the adjoint identity of the next lemma, we obtain that:

$$\mathrm{id}_V \otimes p_n \circ (\mathrm{id}_{n+1} - p_{n+1}) \circ \sigma^{-1}_{V, V^{\underline{\otimes} n}} \circ \tau_n^{-1} \otimes \mathrm{id}_V \circ p_{n+1} = 0$$

So we can add the projection p_{n+1} after $(\eta\psi)^*$ in the above computation, removing $\mathrm{id}_V \otimes p_n$. We obtain:

$$((\eta\psi)^* \circ p_{n+1} \circ \sigma(V, V^{\otimes n})^{-1} \circ \sigma_n^{-1} \otimes \mathrm{id}_V \circ p_{n+1}) \otimes (p_{n+1} \circ \tau_n^{-1} \otimes \mathrm{id}_V \circ \phi\xi)$$

At this point, using the iterative definition of σ_n, we get:

$$((\eta\psi)^* \circ p_{n+1} \sigma_{n+1}^{-1} p_{n+1}) \otimes (p_{n+1} \circ \tau_n^{-1} \otimes \mathrm{id}_V \circ \phi\xi)$$

We calculate now $(v_{\xi,\eta} a)^*$. We have:

$$(v_{\xi,\eta} a)^* = (\xi^* \phi^* \circ p_{n+1} \otimes p_{n+1} \eta\psi)^* = (\eta\psi)^* \circ \tau_{n+1}^{-1} \otimes \tau_{n+1}^{-1} \circ \sigma(V^{\underline{\otimes} n}, V) \circ \phi\xi$$

We focus on $\tau_{n+1}^{-1} \circ \sigma(V^{\underline{\otimes} n}, V)$. We have:

$$\begin{aligned} \tau_{n+1}^{-1} \circ \sigma(V^{\underline{\otimes} n}, V) &= p_{n+1} \sigma_{n+1}^{-1} p_{n+1} \sigma(V^{\otimes n}, V) \circ p_n \otimes \mathrm{id}_V = \\ &= p_{n+1} \circ \sigma_n^{-1} \otimes \mathrm{id}_V \circ \sigma(V^{\otimes n}, V)^{-1} \circ p_{n+1} \circ \sigma(V^{\otimes n}, V) \circ p_n \otimes \mathrm{id}_V = \\ &= p_{n+1} \circ \sigma_n^{-1} \otimes \mathrm{id}_V \circ \sigma(V^{\otimes n}, V)^{-1} \circ \sigma(V^{\otimes n}, V) \circ p_n \otimes \mathrm{id}_V = p_{n+1} \circ \tau_n^{-1} \otimes \mathrm{id}_V \end{aligned}$$

using Lemma 3.7.4. □

Lemma 4.2.3. *Let T be a negligible arrow of the tilting category $\mathcal{T}_l$. Then:*

$$p_n \circ T \circ \mathrm{id}_V \otimes p_{m-1} = 0$$

Proof. Set $Y = p_n \circ T \circ \mathrm{id}_V \otimes p_{m-1}$. $V^{\underline{\otimes} m-1}$ is the direct sum of V_λ, where $\lambda \in \Lambda_l$. $V \otimes V_\lambda$ is completely reducible and the dominant weights μ of the irreducible components V'_μ all lie in $\overline{\Lambda_l}$. If $\mu \in \Lambda_l$, V_μ must be in the kernel of T since T is negligible (it is a straightforward recalling the definition of negligible arrow). On the other hand, if $\mu \in \overline{\Lambda_l} \setminus \Lambda_l$, then $YV'_\mu = \{0\}$, as otherwise it would be an irreducible submodule of $p_n V^{\otimes n}$ of weight μ. □

We introduce a coproduct:

$$\Delta : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$$

in a way similar to the generic case. More precisely, let ϕ, ψ be elements in $p_n V^n$ and $\{\eta_r\}_r$ be an orthonormal basis in $p_n V^n$, with $\{\eta_r^*\}$ its dual basis in $(p_n V^n)^*$. Then:

$$\Delta(\phi^* \otimes \psi) = \phi^* \otimes \eta_r \otimes \eta_r^* \otimes \psi$$

It is easy to prove that the definition does not depend on the choice of the basis.

Theorem 4.2.4. *The coproduct Δ is unital, coassociative, and satisfies for $a, b \in \mathcal{D}$:*
(a) $\Delta(a^*) = \Delta^{\mathrm{op}}(a)^*$*;*
(b) $\Delta(ab) = \Delta(a)\Delta(b)$.

Proof. (b) can be obtained with an argument very similar to the one used in the generic case. So we focus on (a). Set $a = \psi^* \otimes \psi'$, where $\psi, \psi' \in p_n V^n$, and let $\{\psi_r\}_r$ be an orthonormal basis in $p_n V^n$. Since τ_n is unitary, we have that $\{\tau_n \psi_r\}_r$ is still an orthonormal basis:

$$\begin{aligned}\Delta(a^*) = \Delta((\tau_n\psi')^* \otimes \tau_n\psi) &= \sum_r ((\tau_n\psi')^* \otimes \tau_n\psi_r) \otimes ((\tau_n\psi_r)^* \otimes \tau_n\psi) = \\ &= \sum_r (\psi_r^* \otimes \psi')^* \otimes (\psi^* \otimes \psi_r)^* = \Delta^{\mathrm{op}}(a)^*\end{aligned}$$

□

Proposition 4.2.5. *The linear map $\varepsilon : \mathcal{D} \to \mathbb{C}$ which send $\phi^* \otimes \psi$ in (ϕ, ψ) is a counit for the coproduct Δ making $\mathcal{D}(V, l)$ into a coalgebra. Moreover $\varepsilon(a^*) = \overline{\varepsilon(a)}$.*

Proof. The first statement can be proved as in the generic case. The second one is consequence of the following calculation:

$$\begin{aligned}\varepsilon((\phi^* \otimes \psi)^*) = \varepsilon((\tau_n\psi)^* \otimes \tau_n\phi) = (\tau_n\psi, \tau_n\phi) = \\ = (\psi, \phi) = \overline{(\phi, \psi)} = \overline{\varepsilon(\phi^* \otimes \psi)}\end{aligned}$$

using the unitarity of τ_n. □

Remark 4.2.6. It is important to notice that, unlike the generic case, ε is not multiplicative on $\mathcal{D}(V, l)$, since in general $p_n \neq \mathrm{id}$.

4.3 The quantum groupoid $\mathcal{C}(G,l)$ and a corresponding associative filtration

In analogy to the generic case, we follow a Tannakian reconstruction from the quotient category $\mathcal{F}_l$ for which $\mathcal{D} = \mathcal{D}(V,l)$ plays the role of a universal algebra, obtaining a quantum groupoid $\mathcal{C}(G,l)$. The two main differences with the generic case are the associativity failure of $\mathcal{D}(V,l)$ and the fact that the ideal of $\mathcal{D}$ defining $\mathcal{C}(G,l)$ is only a right ideal. $\mathcal{C}(G,l)$ is naturally only a *-coalgebra, so we need a new effort in order to give an algebra structure in $\mathcal{C}(G,l)$ which is compatible with the coalgebra structure. In this section we define $\mathcal{C} = \mathcal{C}(G,l)$ and establish the main properties. We then introduce a non-trivial (possibly nilpotent) associative structure on $\mathcal{C}$, described by a finite sequence $\widetilde{\mathcal{C}}_k$ of *-coalgebras, which we regard as a generalised algebra filtration. We will eventually be able to give an algebra structure in $\mathcal{C}(G,l)$ in the type A case by analysing this filtration.

We introduce identifications in $\mathcal{D}$ as follows. Consider the linear span $\mathcal{I}$ of $\mathcal{D}$ of elements of the form:

$$[\phi, A, \psi] := \phi^* \otimes A\psi - \phi^* \circ A \otimes \psi$$

where $A \in (V^{\underline{\otimes} m}, V^{\underline{\otimes} n})$, and set:

$$\mathcal{C}(G,l) = \mathcal{D}(V,l)/\mathcal{I}$$

We start summarizing the properties that $\mathcal{C}$ inherits from $\mathcal{D}$.

Proposition 4.3.1. *$\mathcal{C}(G,l)$ is a finite-dimensional, coassociative, counital coalgebra with involution. More precisely:*
(a) *$\mathcal{I}$ is a right coideal;*
(b) *$\mathcal{C}$ is linearly spanned by class tensors $v^\lambda_{\phi,\psi} = [\phi^* \otimes \psi], \phi, \psi \in V_\lambda, \lambda \in \Lambda_l$;*
(c) *$\dim(\mathcal{C}) < \infty$;*
(d) *$\mathcal{I}$ is a coideal annihilated by ε;*
(e) *$\mathcal{I}$ is *-invariant, hence the involution of $\mathcal{D}$ factors through $\mathcal{C}$ and satisfies:*

$$\Delta(a^*) = \Delta^{\mathrm{op}}(a)^*, \quad \varepsilon(a^*) = \overline{\varepsilon(a)}, \ a \in \mathcal{C} \quad (v^\lambda_{\phi,\psi})^* = v^\lambda_{\psi,\phi}$$

Finally, the coproduct acts on the class tensors in the following way:

$$\Delta(v^\lambda_{\phi,\psi}) = \sum_r v^\lambda_{\phi,\eta_r} \otimes v^\lambda_{\eta_r,\phi}$$

Proof. We start proving (a). Set $x = \phi \otimes A \circ \psi - \phi \circ A \otimes \psi \in \mathcal{I}$ and $\zeta = \xi \otimes \eta \in (V^h)^* p_h \otimes p_h V^h$. We have:

$$\begin{aligned} x\zeta = \phi\xi p_{n+h} \otimes p_{n+h}(A\psi)\eta - (\phi A)\xi p_{m+h} \otimes p_{m+h}\psi\eta = \\ = (\phi\xi p_{n+h}) \otimes (p_{n+h} \circ A \otimes \mathrm{id}_{V^h}(\psi\eta)) - ((\phi\xi) \circ A \otimes \mathrm{id}_{V^h} \circ p_{m+h}) \otimes (p_{m+h}\psi\eta) = \\ = (\phi\xi p_{n+h}) \otimes ((A \underline{\otimes} \mathrm{id}_{V^h}) p_{m+h}\psi\eta) - (\phi\xi p_{n+h}(A \underline{\otimes} \mathrm{id}_{V^h})) \otimes (p_{m+h}\psi\eta) \in \mathcal{I} \end{aligned}$$

To show (b) we may argue as in the generic case, but now with a choice of isometries $S_i \in (V_{\lambda_i}, V^{\underline{\otimes} n})$, with $\lambda_i \in \Lambda_l$, in the C*-category $\mathcal{F}_l$ satisfying $\sum_i S_i S_i^* = p_n$. (c) is a consequence of (b), since $|\Lambda_l| < \infty$ and $\dim(V_{\lambda_i}) < \infty$. (d) and (e) can be proved as in the generic case, taking into account the results of the previous section. □

We now focus on the filtration $\widetilde{\mathcal{C}}_k$. We filter $\mathcal{D}$ by the size of the tensor product. Set:

$$\mathcal{D}_k = \bigoplus_{n\leq k} (V^{\otimes n})^* p_n \otimes p_n V^{\otimes n}$$

Proposition 4.3.2. (a) $\mathcal{D}_k$ *is a filtration of* $\mathcal{D}$*, i.e. it is an increasing sequence of subspaces satisfying:*

$$\mathcal{D}_0 = \mathbb{C}\,,\ \mathcal{D}_h\mathcal{D}_k \subset \mathcal{D}_{h+k}\,,\ \bigcup_{k=0}^{\infty} \mathcal{D}_k = \mathcal{D}$$

(b) $\mathcal{D}_k$ *are* **-invariant subcoalgebras:* $(\mathcal{D}_k)^* = \mathcal{D}_k$ *and* $\Delta(\mathcal{D}_k) \subset \mathcal{D}_k \otimes \mathcal{D}_k$.

Proof. (a) is an immediate consequence of the definition of $\mathcal{D}_k$. (b) follows using the definition of * and Δ on $\mathcal{D}$. $\square$

Set:

$$\mathcal{J}_k := \text{span}\{[\phi, A, \psi], A \in (V^{\underline{\otimes} m}, V^{\underline{\otimes} n}), \psi \in V^{\underline{\otimes} m}, \phi \in V^{\underline{\otimes} n}, m, n \leq k\}$$

and notice that $\mathcal{J}_k \subset \mathcal{J}$ and $\mathcal{J}_k \subset \mathcal{J}_{k+1}$. Set $\mathcal{C}_k = \mathcal{D}_k/\mathcal{J}_k$. Hence there are maps $\mathcal{C}_k \to \mathcal{C}_{k+1}$ and $\mathcal{C}_k \to \mathcal{C}$. Focusing on the former one, it is defined in the following way:

$$d_k + \mathcal{J}_k \mapsto (d_k \oplus 0_{k+1}) + \mathcal{J}_{k+1}$$

where $d_k \in \mathcal{D}_k$. Since $\mathcal{J}_k \subset \mathcal{J}_{k+1}$, the map is well-defined. The latter can be defined in a similar way.

Lemma 4.3.3. $\mathcal{J} \cap \mathcal{D}_k = \mathcal{J}_k$ *and* $\mathcal{J}_{k+1} \cap \mathcal{D}_k = \mathcal{J}_k$.

Proof. We prove the first identity. The inclusion $\mathcal{J}_k \subset \mathcal{J} \cap \mathcal{D}_k$ is obvious. Let now $X \in \mathcal{J}\cap\mathcal{D}_k$ be written as a finite sum of $[\phi, A, \psi]$ elements of $\mathcal{J}$. Since X is an element of $\mathcal{D}_k$, we can assume that the indices m, n appearing in the elements $[\phi, A, \psi]$ of the finite sum satisfy $\min(m, n) \leq k$. Therefore $X = Y + Z$, where $Y \in \mathcal{J}_k$ and:

$$Z = \sum(\phi \otimes A \circ \psi - \phi \circ A \otimes \psi) + \sum(\xi \otimes A' \circ \eta - \xi \circ A' \otimes \eta)$$

where $A \in (V^{\underline{\otimes} m}, V^{\underline{\otimes} n})$, $A' \in (V^{\underline{\otimes} q}, V^{\underline{\otimes} r})$, $n, q \leq k$, $m, r > k$. We know that $V^{\underline{\otimes} m}$ can be written as the direct sum of some V_λ's. Therefore we can consider the maps $S_\lambda = S_\lambda^{(m)} \in (V^{\underline{\otimes} n_\lambda}, V^{\underline{\otimes} m})$, where $V^{\underline{\otimes} n_\lambda}$ is the lowest tensor power where V_λ appears as summand of the irreducible decomposition. S_λ restricted to V_λ is an isometry, while it annihilates the other summands. Since $\sum_\lambda S_\lambda S_\lambda^* = p_m$, the first sum in X can be written as follows:

$$\sum(\phi \otimes (AS_\lambda)S_\lambda^*\psi - \phi \circ (AS_\lambda) \otimes (S_\lambda^*\psi)) + \sum(\phi \circ (AS_\lambda) \otimes (S_\lambda^*\psi) - \phi \circ A \otimes \psi)$$

and similarly for the second sum. If $n_\lambda > k$, $A \circ S_\lambda = 0$. Hence:

$$\sum(\phi \otimes (AS_\lambda)S_\lambda^*\psi - \phi \circ (AS_\lambda) \otimes (S_\lambda^*\psi))$$

is an element of $\mathcal{J}_k$. It remains to show that:

$$W = \sum((\phi A)S_\lambda \otimes S_\lambda^*\psi - \phi A \otimes \psi) + \sum(\xi \otimes A'\eta - \xi S_\lambda \otimes S_\lambda^*(A'\eta))$$

vanishes. Since $W \in \mathcal{I} \cap \mathcal{D}_k$, and the domain of A and the range of A' have large indices, hence the sum of the terms in second and third position must vanish. It remains to prove that the sum of the terms in first and fourth position vanish. We are thus reduced to show the general statement that given elements ϕ_i and ψ_i of a Hilbert space H such that $\sum_i \phi_i^* \otimes \psi = 0$ and operators $Y_i : K_i \to H$, where K_i are other Hilbert spaces, then $\sum_i \phi_i^* Y_i \otimes Y_i^* \psi_i = 0$, and this can be now checked by means of orthonormal basis decomposition of ϕ_i, ψ_i. □

Proposition 4.3.4. *The natural maps $\mathcal{C}_k \to \mathcal{C}_{k+1}$ are faithful and form a finite increasing sequence of inclusions of *-invariant subcoalgebras:*

$$\mathbb{C} = \mathcal{C}_0 \subset \mathcal{C}_1 \subset \mathcal{C}_2 \subset \ldots \subset \mathcal{C}_m = \mathcal{C}$$

stabilizing to $\mathcal{C}$.

Proof. Using the previous lemma it is straightforward to prove that the natural maps defined above are faithful. Passing to the second statement, there is an integer m such that every irreducible V_λ with $\lambda \in \Lambda_l$ is contained in $V^{\otimes n}$ for some $n \leq m$. Therefore:

$$\mathcal{D}_k = \bigoplus (V^{\underline{\otimes} m})^* \otimes V^{\underline{\otimes} m} \cong \bigoplus_{\lambda \in \Lambda_l} (V_\lambda^* \otimes V_\lambda)^{\oplus m_\lambda}$$

Quotienting by $\mathcal{I}_k$ we identify the different copies of $V_\lambda^* \otimes V_\lambda$'s, so:

$$\mathcal{C}_k = \mathcal{D}_k / \mathcal{I}_k \cong \left(\bigoplus_{\lambda \in \Lambda_l} (V_\lambda^* \otimes V_\lambda)^{\oplus m_\lambda} \right) / \mathcal{I}_k \cong \bigoplus_{\lambda \in \Lambda_l} V_\lambda^* \otimes V_\lambda \cong \mathcal{C}$$

□

We next pass from $\mathcal{C}_k$ to a quotient $\widetilde{\mathcal{C}}_k$ and from the linear (faithful) maps $\mathcal{C}_k \to \mathcal{C}_{k+1}$ to the linear (possibly non-faithful) maps $\widetilde{\mathcal{C}}_k \to \widetilde{\mathcal{C}}_{k+1}$. The advantage will be the existence of natural and associative multiplication maps $\widetilde{\mathcal{C}}_h \otimes \widetilde{\mathcal{C}}_k \to \widetilde{\mathcal{C}}_{h+k}$. More precisely, we observe that the associativity failure of $\mathcal{D}$ can be described in terms of certain negligible intertwiners Z of the tilting category, as follows.

Lemma 4.3.5. *For any triple $\alpha = \phi \otimes \psi$, $\beta = \phi' \otimes \psi'$ and $\gamma = \phi'' \otimes \psi''$ of elements of $\mathcal{D}$, of grades m, n, r respectively, we have:*

$$(\alpha\beta)\gamma - \alpha(\beta\gamma) =$$
$$= ((\phi\phi'\phi'')p_{m+n+r} \otimes Z(\psi\psi'\psi'')) + ((\phi\phi'\phi'') \circ Z^* \otimes (p_{m+n+r} \circ \mathrm{id}_m \otimes p_{n+r} \circ (\psi\psi'\psi''))) \tag{4.3.1}$$

where $Z = p_{m+n+r} \circ \mathrm{id}_m \otimes (\mathrm{id}_{n+r} - p_{n+r})$.

Proof. It is sufficient to put the explicit expression of Z in the identity (4.3.1). □

Consider the following spaces of negligible arrows of the tilting category:

$$\mathcal{Z}^{(k)} := \{p_{q+j+r} \circ \mathrm{id}_q \otimes (\mathrm{id}_j - p_j) \otimes \mathrm{id}_r, q + j + r \leq k\}$$

and then define:

$$\widetilde{\mathcal{I}}_k := \mathrm{span}\{\mathcal{I}_k,\ \phi \otimes Z \circ \psi',\ \phi' \circ (Z')^* \otimes \psi\}$$

where Z, Z' vary in $\mathcal{Z}^{(k)}$, ψ, ϕ in the canonical truncated tensor powers of V and ψ', ϕ' in the full tensor powers.

Proposition 4.3.6. *We have the following inclusions:*

$$\mathcal{D}_j\widetilde{\mathcal{I}}_k \subset \widetilde{\mathcal{I}}_{j+k} \text{ and } \widetilde{\mathcal{I}}_k\mathcal{D}_j \subset \widetilde{\mathcal{I}}_{j+k}$$

Proof. Arguments similar to those of Prop. 4.3.1 (a), but keeping track of the grades of homogeneous elements, show that $\mathcal{I}_k\mathcal{D}_j \subset \mathcal{I}_{j+k}$, and hence $\mathcal{I}_k\mathcal{D}_j$ is a subspace of $\widetilde{\mathcal{I}}_{j+k}$. Similar considerations hold for products $y\zeta$ with y of the form $y = \phi \otimes Z\psi$ or $y = \phi \circ Z^* \otimes \psi$ and $\zeta = \xi \otimes \eta \in (V^h)p_h \otimes p_h V^h$, where $Z \in \mathcal{Z}^{(k)}$ and $h \leq j$. For example, if $y = \phi \otimes Z\psi$:

$$y\zeta = \phi\xi p_{n+h} \otimes p_{n+h} \circ Z \otimes \mathrm{id}_h(\psi\eta)$$

and $p_{n+h} \circ Z \otimes \mathrm{id}_h$ is an element of $\mathcal{Z}^{k+j}$ since $n \leq k$. $\mathcal{D}_j\widetilde{\mathcal{I}}_k \subset \widetilde{\mathcal{I}}_{j+k}$ is more difficult to prove, due to the lack of associativity of the projections p_n. Let y and ζ be as in the calculation above. We have:

$$\zeta y = \xi\phi p_{h+n} \otimes p_{h+n} \circ \mathrm{id}_h \otimes Z(\eta\psi)$$

Z has the explicit expression:

$$Z = p_n \circ \mathrm{id}_q \otimes (\mathrm{id}_j - p_j) \otimes \mathrm{id}_r$$

So:

$$p_{h+n} \circ \mathrm{id}_h \otimes Z = p_{h+n} \circ \mathrm{id}_h \otimes p_n \circ \mathrm{id}_{h+q} \otimes (\mathrm{id}_j - p_j) \otimes \mathrm{id}_r$$

Now, set $Z_1 = p_{h+n} \circ \mathrm{id}_{h+q} \otimes (\mathrm{id}_j - p_j) \otimes \mathrm{id}_r$ and $Z_2 = p_{h+n} \circ \mathrm{id}_h \otimes (\mathrm{id}_n - p_n)$. It is straightforward to see that:

$$p_{h+n} \circ \mathrm{id}_h \otimes Z = Z_1 - Z_2 \circ \mathrm{id}_{h+q} \otimes (\mathrm{id}_j - p_j) \otimes \mathrm{id}_r$$

Since $Z_1, Z_2 \in \mathcal{Z}^{j+k}$, we have $\zeta y \in \widetilde{\mathcal{I}}_{j+k}$. We are left to show that $\zeta x \in \widetilde{\mathcal{I}}_{j+k}$, for all $\zeta \in \mathcal{D}_j$ and $x = \phi \otimes A \circ \psi - \phi \circ A \otimes \psi \in \mathcal{I}_k$. It is sufficient to prove it when $j = 1$ and $\zeta = v_{\xi,\eta}$, where $v_{\xi,\eta} := \xi \otimes \eta \in V^* \otimes V$. Indeed, finite products of simple tensors v_{ξ_i,η_i} are total in $(V^{\otimes h})^* \otimes V^{\otimes h}$, and multiplication of $\mathcal{D}$ is associative up to summing elements of $\widetilde{\mathcal{I}}_{j+k}$ thanks to (4.3.1). We thus compute:

$$\begin{aligned} v_{\xi,\eta}x &= \xi\phi p_{1+n} \otimes (p_{1+n} \circ \mathrm{id}_V \otimes A(\eta\psi)) - ((\xi\phi) \circ \mathrm{id}_V \otimes A \circ p_{1+m}) \otimes p_{1+m}\eta\psi = \\ &= \xi\phi p_{1+n} \otimes (\mathrm{id}_V \otimes A) \circ p_{1+m}\eta\psi - (\xi\phi p_{1+n} \circ \mathrm{id} \otimes A) \otimes (p_{1+m}\eta\psi) + \\ &+ \xi\phi p_{1+n} \otimes Y_1(\eta\psi) - (\xi\phi)Y_2 \otimes p_{1+m}\eta\psi \end{aligned}$$

where:

$$Y_1 = p_{1+n} \circ \mathrm{id}_V \otimes A \circ (\mathrm{id}_V - p_{1+m}) \text{ and } Y_2 = (\mathrm{id}_V - p_{1+n}) \circ \mathrm{id}_V \otimes A \circ p_{1+m}$$

Using Lemma 4.2.3 we have that $Y_1 \circ \mathrm{id}_V \otimes p_m = 0 = \mathrm{id}_V \otimes p_n \circ Y_2$. So the last two terms in the sum vanish, and hence $v_{\xi,\eta}x \in \widetilde{\mathcal{I}}_{k+1}$. □

Proposition 4.3.7. *Each subspace $\widetilde{\mathcal{I}}_k$ is *-invariant.*

Proof. As already noticed in the proof of the Prop. 4.3.1, the proof of the $*$-invariance of $\mathcal{I}_k$ is similar to that of the generic case. Now, let Z be an arrow in $\mathcal{Z}^{(k)} \cap (V^n, V^{\otimes n})$. Then:

$$\begin{aligned} (\psi^* \otimes Z\phi)^* &= (\tau_n Z\phi)^* \otimes \tau_n\psi = \\ &= ((\tau_n Z\phi)^* \otimes \tau_n\psi - (\tau_n Z\phi)^*\tau_n^{-1} \otimes \psi) + \phi^* \circ Z^* \otimes \psi \end{aligned}$$

which lies in $\widetilde{\mathcal{I}}_k$, with $\phi \in V^n$ and $\psi \in V^{\underline{\otimes} n}$. In the same way it is possible to prove that $((\psi \circ Z')^* \otimes \phi)^* \in \widetilde{\mathcal{I}}_k$ as well for $Z' \in (\widetilde{\mathcal{Z}}^{(k)})^*$, concluding that $\widetilde{\mathcal{I}}_k$ is *-invariant. $\square$

Proposition 4.3.8. *$\widetilde{\mathcal{I}}_k$ is a coideal in $\mathcal{D}_k$, or in other words:*

$$\Delta(\widetilde{\mathcal{I}}_k) \subset \widetilde{\mathcal{I}}_k \otimes \mathcal{D}_k + \mathcal{D}_k \otimes \widetilde{\mathcal{I}}_k$$

Proof. This statement can be proved with the same calculations of the proof of the Prop. 4.1.5, keeping track of the grades of the homogeneous elements. $\square$

We set $\widetilde{\mathcal{C}}_k = \mathcal{D}_k/\widetilde{\mathcal{I}}_k$, for $k \in \mathbb{N}$. The composition of the natural linear inclusion $\iota_{h,k} : \mathcal{D}_h \hookrightarrow \mathcal{D}_k$, where $k > h$, with projection $\widetilde{\pi}_k : \mathcal{D}_k \to \widetilde{\mathcal{C}}_k$ factors through a linear map $i_{h,k} : \widetilde{\mathcal{C}}_h \to \widetilde{\mathcal{C}}_k$. In fact, let d_h be an element in $\mathcal{D}_h$. $\iota_{h,k}(d_h) = d_h \oplus (0^*_{h+1} \otimes 0_{h+1}) \oplus \ldots \oplus (0^*_k \otimes 0_k)$, which can be shortly written $\iota_{h,k}(d_h) = d_h \oplus 0^{\oplus k-h}$, while $\pi_k(d_k) = d_k + \widetilde{\mathcal{I}}_k$. So:

$$\pi_k \circ \iota_{h,k}(d_h) = d_h \oplus 0^{\oplus k-h} + \widetilde{\mathcal{I}}_k$$

Suppose that $d_h \in \widetilde{\mathcal{I}}_h$. Since $0^{\oplus k-h} = \sum_{l,m} 0^*_l \otimes A \circ 0_m - 0^*_l \circ A \otimes 0_m$ for some $A_l \in (V^{\underline{\otimes} m}, V^{\underline{\otimes} l})$, we can conclude that $d_h \oplus 0^{\oplus k-h} \in \widetilde{\mathcal{I}}_k$, so $\pi_k \circ \iota_{h,k}$ factors through $i_{h,k}$, which is an inductive system. Moreover, we have natural quotient maps:

$$\mathcal{C}_k \to \widetilde{\mathcal{C}}_k$$

and we denote by $e^\lambda_{i,j} \in \widetilde{\mathcal{C}}_k$ the image of the matrix coefficient $v^\lambda_{i,j} \in \mathcal{C}_k$ corresponding to an orthonormal basis of V_λ. Let Λ^k_l denote the set of $\lambda \in \Lambda_l$ for which V_λ is a summand of some $V^{\underline{\otimes} n}$, with $n \leq k$. In analogy with the properties of Prop. 4.3.1 for $\mathcal{C}$, we also summarise the results about $\widetilde{\mathcal{C}}_k$:

Theorem 4.3.9. *Assume that $\mathfrak{g} \neq E_8$ and let V be Wenzl's fundamental representation of $\mathfrak{g}$. Then:*
(a) *$\widetilde{\mathcal{C}}_k$ is a *-coalgebra linearly spanned by elements $e^\lambda_{i,j}$ labelling matrix units corresponding to V_λ, for $\lambda \in \Lambda^k_l$;*
(b) *coproduct and involution satisfy:*

$$\Delta(e^\lambda_{i,j}) = \sum_r e^\lambda_{i,r} \otimes e^\lambda_{r,j} \quad \textit{and} \quad (e^\lambda_{i,j})^* = e^\lambda_{j,i}$$

In particular the involution is anticomultiplicative.
(c) *There are associative multiplication maps $\widetilde{\mathcal{C}}_h \otimes \widetilde{\mathcal{C}}_k \to \widetilde{\mathcal{C}}_{h+k}$ and an element $I \in \widetilde{\mathcal{C}}_0$ acting as the identity. The involution is antimultiplicative and the coproduct is unital and multiplicative.*

4.4 Quasi coassociative dual C*-quantum groupoids $\widehat{\mathcal{C}(G,l)}$

The aim of the present and the next section is to show that if $G = SU(N)$, then the dual groupoid $\widehat{\mathcal{C}(G,l)}$ can be made into a C*-quantum groupoid, satisfying the axioms of a weak quasi Hopf C*-algebra introduced in the first chapter. Furthermore, the representation category $\mathrm{Rep}_V(\widehat{\mathcal{C}})$ of $\widehat{\mathcal{C}}$ generated by the fundamental representation turns out to be a tensor C*-category equivalent to the original fusion category $\mathcal{F}_l$.

We shall divide the proof in two parts. Throughout this section $\mathfrak{g}$ is general (but not E_8) and we assume to know that $\mathcal{C}(G,l)$ is cosemisimple with respect to the coalgebra structure introduced in the previous section. We then show that the above conclusions hold for $\widehat{\mathcal{C}(G,l)}$. More precisely, we will first show that $\mathcal{C}(G,l)$ is a non-associative bialgebra with antipode, associated to a fixed section of the quotient map $\mathcal{D}(V,l) \to \mathcal{C}(G,l)$; then we will pass to $\widehat{\mathcal{C}(G,l)}$, constructing a Drinfeld's associator in $\widehat{\mathcal{C}(G,l)}$ and discussing the main properties; moreover, we will explicitly write down quasi invertible R-matrices for $\widehat{\mathcal{C}(G,l)}$ and we will discuss the relation among the groupoid structures associated to different sections; finally, we will show that $\mathrm{Rep}_V(\widehat{\mathcal{C}})$ is a tensor C*-category equivalent to $\mathcal{F}_l$. In the next section we will verify cosemisimplicity in the type A case.

4.4.1 Algebra structure and antipode in $\mathcal{C}(G,l)$

Let V_λ be a copy of the irreducible representation of $U_q(\mathfrak{g})$ with highest weight $\lambda \in \Lambda_l$ and contained in some $V^{\underline{\otimes} n}$, and let M_λ denote the image of $V_\lambda^* \otimes V_\lambda$ in $\mathcal{C}$ under the quotient map $\mathcal{D} \to \mathcal{C}$. We already know that M_λ are subcoalgebras independent of the choice of V_λ and spanning $\mathcal{C}$. We shall say that $\mathcal{C}(G,l)$ is cosemisimple if it is the direct sum of M_λ, which are not only subcoalgebras but also matrix coalgebras of full dimension $\dim(V_\lambda)^2$. If we know that $\mathcal{C}(G,l)$ is cosemisimple, we can endow it both with an invertible antipode and with a non-associative algebra structure. Let's start with the antipode, which we introduce in a way similar to the generic case. Fix a complete set V_λ, where $\lambda \in \Lambda_l$, of irreducibles contained in the various $V^{\underline{\otimes} n}$, and set, for $\phi^* \otimes \psi \in V_\lambda^* \otimes V_\lambda$:

$$S(v^\lambda_{\phi,\psi}) = v^{\overline{\lambda}}_{j_\lambda \psi, j_\lambda \phi}$$

which does not depend on the choice of V_λ. It satisfies the relations:

$$S(a)^* = S(a^*), \quad S^2(v^\lambda_{\phi,\psi}) = v^\lambda_{K_{2\rho}\phi, K_{2\rho}\psi}, \quad \Delta \circ S = S \otimes S \circ \Delta$$

The above properties can be proved as in the general case. As regards the algebra structure, we pull back the product of $\mathcal{D}(V,l)$ via the choice of a section $s : \mathcal{C}(G,l) \to \mathcal{D}(V,l)$ of the quotient map $\mathcal{D}(V,l) \to \mathcal{C}(G,l)$. Correspondingly, we have a choice of irreducibles V_λ and s takes $v^\lambda_{\phi,\psi}$ to $\phi^* \otimes \psi$, where $\phi, \psi \in V_\lambda$. We thus set:

$$v^\lambda_{\phi,\psi} v^\mu_{\xi,\eta} = [s(v^\lambda_{\phi,\psi}) s(v^\mu_{\xi,\eta})]$$

We always choose $V_0 = \mathbb{C}$ and $V_\kappa = V$ for the trivial and the fundamental representation, respectively. In this way, denoting as before by $v_{\xi,\eta}$ the class tensors of V, products of the form $v^\lambda_{\phi,\psi} v_{\xi,\eta}$ encode fusion decomposition of $V_\lambda \otimes V$ in $\mathcal{F}_l$. However, the section is not unique, and the product of $\mathcal{C}(G,l)$ depends on s (but later we will see that this is not really a problem).

Proposition 4.4.1. *The product makes $\mathcal{C}(G,l)$ into a non-associative unital algebra and the coproduct of $\mathcal{C}(G,l)$ is a unital homomorphism. Furthermore, the following relation holds:*

$$m \circ \mathrm{id} \otimes S \circ \Delta = \varepsilon(\cdot) I = m \circ S \otimes \mathrm{id} \circ \Delta$$

Proof. The fact that the coproduct is an algebra homomorphism can be proved as the statement (b) of the Theorem 4.2.4. Next we prove that $m \circ \mathrm{id} \otimes S \circ \Delta = \varepsilon(\cdot)I$. The right identity can be proved in the same way. Developing the same calculation as in the proof of the Prop. 4.1.9, we obtain:

$$m(\mathrm{id} \otimes S(\Delta(\phi^* \otimes \psi))) = \phi^*(j_\lambda \eta_r)^* \circ p_{h+m} \otimes p_{h+m} \circ \mathrm{id} \otimes U_\lambda \circ \overline{r}(1)$$

where h and m are the powers of V containing V_λ and $V_{\overline{\lambda}}$ respectively. We have that the image of $\mathbb{C}$ through $\mathrm{id} \otimes U_\lambda \circ \overline{r}$ is a copy of V_0 in $\overline{V_\lambda} \otimes V_\lambda = V_{\overline{\lambda}} \otimes V_\lambda$, which in turn is embedded in $V^{\underline{\otimes} h+m}$. As a consequence:

$$p_{h+m} \circ \mathrm{id} \otimes U_\lambda \circ \overline{r} = \mathrm{id} \otimes U_\lambda \circ \overline{r} = \mathrm{id} \otimes U_\lambda \circ \overline{r} \circ p_1$$

Hence:

$$m(\mathrm{id} \otimes S(\Delta(\phi^* \otimes \psi))) = \phi^*(j_\lambda \eta_r)^* \circ p_{h+m} \otimes p_{h+m} \circ \mathrm{id} \otimes U_\lambda \circ \overline{r}(1) =$$
$$\phi^*(j_\lambda \eta_r)^* \circ \mathrm{id} \otimes U_\lambda \circ \overline{r} \otimes 1$$

At this stage it is easy to conclude proceeding again as in the proof of the Prop. 4.1.9. □

Hence $\mathcal{C}(G,l)$ satisfies all the axioms of a Hopf algebra except associativity of the product and multiplicativity of the counit. The antipode is not antimultiplicative.

4.4.2 The dual quantum groupoid $\widehat{\mathcal{C}(G,l)}$

Starting from the results of the previous subsection we can state that the structure of $\mathcal{C}(G,l)$ is quite unsatisfactory. In fact, $\mathcal{C}(G,l)$ is quite far from admitting an interpretation as a non-commutative space as compared, for instance, to the compact quantum groups of Woronowicz. It is far more rewarding to pass to the dual $\widehat{\mathcal{C}(G,l)}$, and correspondingly consider its $*$-representations. In this subsection we show that $\widehat{\mathcal{C}(G,l)}$ satisfies the properties of the weak quasi Hopf C*-algebras.

We identify elements of tensor power algebras $\widehat{\mathcal{C}}^{\otimes n}$ with functionals on $\mathcal{C}^{\otimes n}$. We shall need various elements of these algebras, and we start with $P \in \widehat{\mathcal{C}}^{\otimes 2}$, defined as follows:

$$P(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}) = (\phi \otimes \xi, p_{h+k}(\psi \otimes \eta))_p$$

where h and k are such that V_λ and V_μ are summands of $V^{\underline{\otimes} h}$ and $V^{\underline{\otimes} k}$, respectively. Notice that these integers depend on the section. Furthermore, the form defining P is understood with respect to the product form of $V^{\underline{\otimes} h} \otimes V^{\underline{\otimes} k}$, where each factor is in turn endowed with the Kirillov-Wenzl inner product.

Proposition 4.4.2. $\widehat{\mathcal{C}} = \widehat{\mathcal{C}(G,l)}$ *is a unital complex associative* **-algebra such that:*

$$\widehat{\mathcal{C}} \cong \bigoplus_{\lambda \in \Lambda_l} \mathrm{End}(V_\lambda)$$

Hence it is a C*-algebra.

Proof. It is mainly a consequence of the duality maps and of the corresponding properties of $\mathcal{C}(G,l)$. More precisely, the associative algebra structure of $\widehat{\mathcal{C}}$ follows from the coassociative coalgebra structure of $\mathcal{C}$. Antimultiplicativity of the involution of $\widehat{\mathcal{C}}$ follows from anticomultiplicativity of that of $\mathcal{C}$. The isomorphism is a consequence of the cosemisimplicity of $\mathcal{C}$. □

Proposition 4.4.3. *$\widehat{\mathcal{C}}$ is endowed with a coproduct Δ and a counit ε such that:*

$$\Delta(I) = P \tag{4.4.1}$$
$$\Delta(\omega\tau) = \Delta(\omega)\Delta(\tau) \tag{4.4.2}$$
$$\varepsilon(\omega\tau) = \varepsilon(\omega)\varepsilon(\tau) \tag{4.4.3}$$
$$\varepsilon \otimes \mathrm{id} \circ \Delta = \mathrm{id} = \mathrm{id} \otimes \varepsilon \circ \Delta \tag{4.4.4}$$

Proof. We prove the first assertion:

$$\begin{aligned}\Delta(I)(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}) &= I(v^\lambda_{\phi,\psi} v^\mu_{\xi,\eta}) = \varepsilon([\phi^*\xi^* \circ p_{h+k} \otimes p_{h+k} \circ \psi\eta]) = \\ &= \phi^*\xi^* p_{h+k}(p_{h+k}\psi\eta) = (\phi\xi, p_{h+k}\psi\eta)_p = P(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta})\end{aligned}$$

We pass to the second statement. We start calculating the left hand side:

$$\begin{aligned}\Delta(\omega\tau)(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}) &= \omega\tau([\phi^*\xi^* p_{h+k} \otimes p_{h+k}\psi\eta]) = \\ &= \omega \otimes \tau([\phi^*\xi^* p_{h+k} \otimes \zeta_j], [\zeta_j^* \otimes p_{h+k}\psi\eta]) = \\ &= \omega([\phi^*\xi^* p_{h+k} \otimes \zeta_j])\tau([\zeta_j^* \otimes p_{h+k}\psi\eta])\end{aligned}$$

where $\{\zeta_j\}_j$ is an orthormal basis of $V^{\underline{\otimes} h+k}$. On the other side:

$$\begin{aligned}\Delta(\omega)\Delta(\tau) &= (m \otimes m \circ \mathrm{id} \otimes \Sigma \otimes \mathrm{id} \circ \Delta \otimes \Delta(\omega \otimes \tau))(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}) = \\ = (\mathrm{id} \otimes \Sigma \otimes \mathrm{id}(\Delta \otimes \Delta(\omega \otimes \tau)))&([\phi^* \otimes \gamma_i], [\gamma_i^* \otimes \psi], [\xi^* \otimes \delta_j], [\delta_j^* \otimes \eta]) = \\ = (\Delta \otimes \Delta(\omega \otimes \tau))&([\phi^* \otimes \gamma_i], [\xi^* \otimes \delta_j], [\gamma_i^* \otimes \psi], [\delta_j^* \otimes \eta]) = \\ = \omega \otimes \tau&([\phi^*\xi^* p_{h+k} \otimes p_{h+k}\gamma_i\delta_j], [\gamma_i^*\delta_j^* p_{h+k} \otimes p_{h+k}\psi\eta]) = \\ = \omega&([\phi^*\xi^* p_{h+k} \otimes p_{h+k}\gamma_i\delta_j])\tau([\gamma_i^*\delta_j^* p_{h+k} \otimes p_{h+k}\psi\eta])\end{aligned}$$

where $\{\gamma_i\}_i$ is an orthonormal basis of $V^{\underline{\otimes} h}$ and $\{\delta_j\}_j$ is an orthonormal basis of $V^{\underline{\otimes} k}$. The multiplicativity of ε is an easy consequence of the unitality of the coproduct of $\mathcal{C}$. The last statement is also immediate to prove using the duality. □

The coproduct Δ in $\widehat{\mathcal{C}}$ is not coassociative, and this fact is due to the lack of associativity of the multiplication in $\mathcal{C}$. However, we are going to show that it is possible to endow $\widehat{\mathcal{C}}$ with a Drinfeld's associator Φ, making $\widehat{\mathcal{C}}$ a weak quasi Hopf C*-algebra.

For a given weight $\lambda \in \Lambda_l$, let h_λ denote the truncated powers of V containing V_λ, as prescribed by the choice of a section s. It will be useful for later computations to have a multiplication rule for elements of $\widehat{\mathcal{C}}^{\otimes n}$ of the following form. Let $T = (T_n)$ be a sequence of linear maps $T_n : V^{\otimes n} \to V^{\otimes n}$ and associate the element ω_T of $\widehat{\mathcal{C}}^{\otimes n}$ defined by:

$$\omega_T(v^{\lambda_1}_{\phi_1,\psi_1}, \dots, v^{\lambda_n}_{\phi_n,\psi_n}) = (\phi_1 \otimes \dots \otimes \phi_n, T_{h_{\lambda_1}+\dots+h_{\lambda_n}}\psi_1 \otimes \dots \otimes \psi_n)_p$$

where the form on the right is the product form with n factors. The following lemma is a convenient formulation of the product of two elements in $\widehat{\mathcal{C}}^{\otimes n}$ of the above form.

Lemma 4.4.4. *Given $S = (S_n)$ and $T = (T_n)$ as above, set $\omega = \omega_S\omega_T$. Then:*

$$\omega(v^{\lambda_1}_{\phi_1,\psi_1}, \dots, v^{\lambda_n}_{\phi_n,\psi_n}) = (\phi_1 \dots \phi_n, S_{h_{\lambda_1}+\dots+h_{\lambda_n}} \circ p_{h_{\lambda_1}} \otimes \dots \otimes p_{h_{\lambda_n}} \circ T_{h_{\lambda_1}+\dots+h_{\lambda_n}}\psi_1 \dots \psi_n)_p$$

Proof. We have:

$$\begin{aligned}
&\omega(v^{\lambda_1}_{\phi_1,\psi_1},\dots,v^{\lambda_n}_{\phi_n,\psi_n}) =\\
&= \omega_S([\phi_1^*\otimes\xi^{(1)}_{i_1}],\dots,[\phi_n^*\otimes\xi^{(n)}_{i_n}])\omega_T([\xi^{(1)*}_{i_1}\otimes\psi_1],\dots,[\xi^{(n)*}_{i_n}\otimes\psi_n]) =\\
&= (\phi_1\dots\phi_n, S_{h_{\lambda_1}+\ldots+h_{\lambda_n}}\xi^{(1)}_{i_1}\dots\xi^{(n)}_{i_n})_p(\xi^{(1)}_{i_1}\dots\xi^{(n)}_{i_n}, T_{h_{\lambda_1}+\ldots+h_{\lambda_n}}\psi_1\dots\psi_n)_p =\\
&= (\phi_1\dots\phi_n, S_{h_{\lambda_1}+\ldots+h_{\lambda_n}}(\xi^{(1)}_{i_1}\dots\xi^{(n)}_{i_n}, T_{h_{\lambda_1}+\ldots+h_{\lambda_n}}\psi_1\dots\psi_n)_p\xi^{(1)}_{i_1}\dots\xi^{(n)}_{i_n})_p =\\
&= (\phi_1\dots\phi_n, S_{h_{\lambda_1}+\ldots+h_{\lambda_n}}\circ p_{h_{\lambda_1}}\otimes\ldots\otimes p_{h_{\lambda_n}}\circ T_{h_{\lambda_1}+\ldots+h_{\lambda_n}}\psi_1\dots\psi_n)_p
\end{aligned}$$

□

Now we introduce the elements $\Phi,\Psi\in\widehat{\mathcal{C}}^{\otimes 3}$. Set:

$$q_{h_\lambda,h_\mu,h_\nu} = \sum_{\gamma,i}\mathrm{id}_{h_\lambda}\otimes S_{\gamma,i}\circ p_{h_\lambda+h_\gamma}\circ\mathrm{id}_{h_\lambda}\otimes S^*_{\gamma,i}$$

where $S_{\gamma,i}\in(V_\gamma, V^{\underline{\otimes}(h_\mu+h_\nu)})$ are isometries of the fusion category satisfying

$$\sum_{\gamma,i}S_{\gamma,i}S^*_{\gamma,i} = p_{h_\mu+h_\nu}$$

It is quite easy to prove that q does not depend on the choice of the isometries. In fact, suppose that $S'_{\gamma,i}\in(V_\gamma, V^{\underline{\otimes}(h_\mu+h_\nu)})$ are other isometries, with $S'^*_{\gamma,i}S_{\eta,j} = c_{\gamma,i}\delta_{\gamma,\eta}\delta_{i,j}\,\mathrm{id}_\gamma$, where $c_{\gamma,i}\in\mathbb{T}$ and δ is the Kronecker symbol. Of course we also have:

$$\sum_{\gamma,i}S'_{\gamma,i}S'^*_{\gamma,i} = p_{h_\mu+h_\nu},\quad S'^*_{\gamma,i}S'_{\eta,j} = \delta_{\gamma,\eta}\delta_{i,j}\,\mathrm{id}_\gamma$$

We set:

$$q'_{h_\lambda,h_\mu,h_\nu} = \sum_{\gamma,i}\mathrm{id}_{h_\lambda}\otimes S'_{\gamma,i}\circ p_{h_\lambda+h_\gamma}\circ\mathrm{id}_{h_\lambda}\otimes S'^*_{\gamma,i}$$

Therefore

$$\begin{aligned}
q_{h_\lambda,h_\mu,h_\nu} &= \sum_{\gamma,i}\mathrm{id}_{h_\lambda}\otimes S_{\gamma,i}\circ p_{h_\lambda+h_\gamma}\circ\mathrm{id}_{h_\lambda}\otimes S^*_{\gamma,i} =\\
&= \sum_{\gamma,i}\mathrm{id}_{h_\lambda}\otimes p_{h_\mu+h_\nu}\circ\mathrm{id}_{h_\lambda}\otimes S_{\gamma,i}\circ p_{h_\lambda+h_\gamma}\circ\mathrm{id}_{h_\lambda}\otimes S^*_{\gamma,i}\circ\mathrm{id}_{h_\lambda}\otimes p_{h_\mu+h_\nu} =\\
&= \sum_{\gamma,\eta,\iota,i,j,l}\mathrm{id}_{h_\lambda}\otimes S'_{\eta,j}S'^*_{\eta,j}S_{\gamma,i}\circ p_{h_\lambda+h_\gamma}\circ\mathrm{id}_{h_\lambda}\otimes S^*_{\gamma,i}S'_{\iota,l}S'^*_{\iota,l} =\\
&= \sum_{\gamma,i}c_{\gamma,i}\overline{c_{\gamma,i}}\,\mathrm{id}_{h_\lambda}\otimes S'_{\gamma,i}\circ p_{h_\lambda+h_\gamma}\circ\mathrm{id}_{h_\lambda}\otimes S'^*_{\gamma,i} =\\
&= \sum_{\gamma,i}\mathrm{id}_{h_\lambda}\otimes S'_{\gamma,i}\circ p_{h_\lambda+h_\gamma}\circ\mathrm{id}_{h_\lambda}\otimes S'^*_{\gamma,i} = q'_{h_\lambda,h_\mu,h_\nu}
\end{aligned}$$

It remains to explain why $c_{\gamma,i}\in\mathbb{T}$. In fact:

$$\begin{aligned}
c_{\gamma,i}\overline{c_{\gamma,i}} &= S'^*_{\gamma,i}S_{\gamma,i}S^*_{\gamma,i}S'_{\gamma,i} =\\
&= S'^*_{\gamma,i}\circ\left(\sum_{\eta,j}S_{\eta,j}S^*_{\eta,j}\right)\circ S'_{\gamma,i} =\\
&= S'^*_{\gamma,i}\circ p_{h_\mu+h_\nu}\circ S'_{\gamma,i} = \mathrm{id}_\gamma
\end{aligned}$$

Next we set:

$$\Phi(v^{\lambda}_{\phi,\psi}, v^{\mu}_{\xi,\eta}, v^{\nu}_{\chi,\zeta}) = (\phi \otimes \xi \otimes \chi, q_{h_\lambda,h_\mu,h_\nu} \circ p_{h_\lambda+h_\mu+h_\nu} \psi \otimes \eta \otimes \zeta)_p \tag{4.4.5}$$
$$\Psi(v^{\lambda}_{\phi,\psi}, v^{\mu}_{\xi,\eta}, v^{\nu}_{\chi,\zeta}) = (\phi \otimes \xi \otimes \chi, p_{h_\lambda+h_\mu+h_\nu} \circ q_{h_\lambda,h_\mu,h_\nu} \psi \otimes \eta \otimes \zeta)_p \tag{4.4.6}$$

In the next theorem we will see that Φ and Ψ play the roles of the Drinfeld's associator and its inverse in a weak quasi bialgebra, respectively.

Theorem 4.4.5. *Φ and Ψ satisfy the following relations:*

$$\Psi\Phi = \Delta \otimes \mathrm{id}(\Delta(I)), \ \ \Phi\Psi = \mathrm{id} \otimes \Delta(\Delta(I)) \tag{4.4.7}$$
$$\Phi\Delta \otimes \mathrm{id}(\Delta(I)) = \Phi = \mathrm{id} \otimes \Delta(\Delta(I))\Phi \tag{4.4.8}$$
$$\Phi\Delta \otimes \mathrm{id}(\Delta(\omega)) = \mathrm{id} \otimes \Delta(\Delta(\omega))\Phi \tag{4.4.9}$$
$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta(\Phi))(\Delta \otimes \mathrm{id} \otimes \mathrm{id}(\Phi)) = (I \otimes \Phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id}(\Phi))(\Phi \otimes I) \tag{4.4.10}$$
$$\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}(\Phi) = \Delta(I) \tag{4.4.11}$$

So Φ is partially invertible and Ψ is its inverse.

Proof. First of all, we see that:

$$\Delta \otimes \mathrm{id}(\Delta(I))(v^{\lambda}_{\phi,\psi}, v^{\mu}_{\xi,\eta}, v^{\nu}_{\chi,\zeta}) = (\phi \otimes \xi \otimes \chi, p_{h_\lambda+h_\mu+h_\nu} \circ \psi \otimes \eta \otimes \zeta) \tag{4.4.12}$$
$$\mathrm{id} \otimes \Delta(\Delta(I))(v^{\lambda}_{\phi,\psi}, v^{\mu}_{\xi,\eta}, v^{\nu}_{\chi,\zeta}) = (\phi \otimes \xi \otimes \chi, \mathrm{id} \otimes S_{\gamma,i} \circ p_{h_\lambda+h_\gamma} \circ \mathrm{id} \otimes S^*_{\gamma,i} \circ \psi \otimes \eta \otimes \zeta) \tag{4.4.13}$$

omitting in (4.4.13) the summation symbol with indices γ and i. We prove (4.4.13):

$$\begin{aligned} &\mathrm{id} \otimes \Delta(\Delta(I))([\phi^* \otimes \psi], [\xi^* \otimes \eta], [\chi^* \otimes \zeta]) = \\ &= \Delta(I)([\phi^* \otimes \psi], [\xi^*\chi^* \circ p_{h_\mu+h_\nu} \otimes p_{h_\mu+h_\nu} \circ \eta\zeta]) \end{aligned}$$

Since $p_{h_\mu+h_\nu} = \sum_{\gamma,i} S_{\gamma,i}S^*_{\gamma,i}$ and $S_{\gamma,i} \in (V_\gamma, V^{\underline{\otimes} h_\mu+h_\nu})$, we obtain:

$$\begin{aligned} &\Delta(I)([\phi^* \otimes \psi], [\xi^*\chi^* \circ p_{h_\mu+h_\nu} \otimes p_{h_\mu+h_\nu} \circ \eta\zeta]) = \\ &= \sum_{\gamma,i} \Delta(I)([\phi^* \otimes \psi], [\xi^*\chi^* \circ S_{\gamma,i} \otimes S^*_{\gamma,i} \circ \eta\zeta]) = \\ &= \sum_{\gamma,i} (\phi \otimes \xi \otimes \chi, \mathrm{id} \otimes S_{\gamma,i} \circ p_{h_\lambda+h_\gamma} \circ \mathrm{id} \otimes S^*_{\gamma,i} \circ \psi\eta\zeta)_p \end{aligned}$$

We are ready now to prove (4.4.7). We start with the identity on the left. We know that $\Psi = \omega_S$ and $\Phi = \omega_T$, where S and T can be easily deduced from (4.4.5) and (4.4.6). Therefore, using Lemma 4.4.4, $\Psi\Phi = \omega_U$, where:

$$\begin{aligned} U = &\sum_{\gamma,\iota,i,j} p_{h_\lambda+h_\mu+h_\nu} \circ \mathrm{id} \otimes S_{\gamma,i} \circ p_{h_\lambda+h_\gamma} \circ \mathrm{id} \otimes S^*_{\gamma,i} \circ \\ &\circ p_{h_\lambda} \otimes p_{h_\mu} \otimes p_{h_\nu} \circ \mathrm{id} \otimes S_{\iota,j} \circ p_{h_\lambda+h_\iota} \circ \mathrm{id} \otimes S^*_{\iota,j} \circ p_{h_\lambda+h_\mu+h_\nu} = \\ = &\sum_{\gamma,\iota,i,j} p_{h_\lambda+h_\mu+h_\nu} \circ \mathrm{id} \otimes S_{\gamma,i} \circ \mathrm{id} \otimes S^*_{\gamma,i} \circ \mathrm{id} \otimes S_{\iota,j} \circ \mathrm{id} \otimes S^*_{\iota,j} \circ p_{h_\lambda+h_\mu+h_\nu} = \\ = &\sum_{\gamma,i} p_{h_\lambda+h_\mu+h_\nu} \circ \mathrm{id} \otimes S_{\gamma,i}S^*_{\gamma,i} \circ p_{h_\lambda+h_\mu+h_\nu} = p_{h_\lambda+h_\mu+h_\nu} \end{aligned}$$

The identity on the right can be proved using the same approach. We have $\Phi\Psi = \omega_V$, where:

$$\begin{aligned}
V = &\sum_{\gamma,\iota,i,j} \mathrm{id} \otimes S_{\gamma,i} \circ p_{h_\lambda + h_\gamma} \circ \mathrm{id} \otimes S^*_{\gamma,i} \circ p_{h_\lambda+h_\mu+h_\nu} \circ \\
&\circ p_{h_\lambda} \otimes p_{h_\mu} \otimes p_{h_\nu} \circ p_{h_\lambda+h_\mu+h_\nu} \circ \mathrm{id} \otimes S_{\iota,j} \circ p_{h_\lambda+h_\iota} \circ \mathrm{id} \otimes S^*_{\iota,j} = \\
&= \sum_{\gamma,\iota,i,j} \mathrm{id} \otimes S_{\gamma,i} \circ p_{h_\lambda+h_\gamma} \circ \mathrm{id} \otimes S^*_{\gamma,i} \circ \mathrm{id} \otimes S_{\iota,j} \circ p_{h_\lambda+h_\iota} \circ \mathrm{id} \otimes S^*_{\iota,j} = \\
&= \sum_{\gamma,i} \mathrm{id} \otimes S_{\gamma,i} \circ p_{h_\lambda+h_\gamma} \circ \mathrm{id} \otimes S^*_{\gamma,i}
\end{aligned}$$

We pass to (4.4.8). The left identity is very easy to prove. The next calculation allows us to prove the right one, using again Lemma 4.4.4. Suppose that $\Phi = \omega_T$ as before, and $\mathrm{id} \otimes \Delta(\Delta(I)) = \omega_S$, where S can be deduced from (4.4.13). Therefore $\mathrm{id} \otimes \Delta(\Delta(I))\Phi = \omega_Q$, and it will be sufficient to prove that $Q = T$:

$$\begin{aligned}
Q = &\sum_{\gamma,\iota,i,j} \mathrm{id} \otimes S_{\gamma,i} \circ p_{h_\lambda+h_\gamma} \circ \mathrm{id} \otimes S^*_{\gamma,i} \circ p_{h_\lambda} \otimes p_{h_\mu} \otimes p_{h_\nu} \circ \mathrm{id} \otimes S_{\iota,j} \circ p_{h_\lambda+h_\iota} \circ \mathrm{id} \otimes S^*_{\iota,j} \circ p_{h_\lambda+h_\mu+h_\nu} = \\
= &\sum_{\gamma,\iota,i,j} \mathrm{id} \otimes S_{\gamma,i} \circ p_{h_\lambda+h_\gamma} \circ \mathrm{id} \otimes S^*_{\gamma,i} \circ \mathrm{id} \otimes S_{\iota,j} \circ \mathrm{id} \otimes S^*_{\iota,j} \circ p_{h_\lambda+h_\mu+h_\nu} = \\
= &\sum_{\gamma,i} \mathrm{id} \otimes S_{\gamma,i} \circ p_{h_\lambda+h_\gamma} \circ \mathrm{id} \otimes S^*_{\gamma,i} \circ p_{h_\lambda+h_\mu+h_\nu} = T
\end{aligned}$$

Next we prove (4.4.9). We first calculate the left hand side:

$$\begin{aligned}
&\Phi\Delta \otimes \mathrm{id}(\Delta(\omega))([\phi^* \otimes \psi], [\xi^* \otimes \eta], [\chi^* \otimes \zeta]) = \\
&= \Phi_1\omega_{(1)(1)} \otimes \Phi_2\omega_{(1)(2)} \otimes \Phi_3\omega_{(2)}([\phi^* \otimes \psi], [\xi^* \otimes \eta], [\chi^* \otimes \zeta]) = \\
&= \Phi_1 \otimes \omega_{(1)(1)} \otimes \Phi_2 \otimes \omega_{(1)(2)} \otimes \Phi_3 \otimes \omega_{(2)}([\phi^* \otimes \delta_i], [\delta_i^* \otimes \psi], [\xi^* \otimes \iota_j], [\iota_j^* \otimes \eta], [\chi^* \otimes \kappa_l], [\kappa_l^* \otimes \zeta]) = \\
&= \Phi([\phi^* \otimes \delta_i], [\xi^* \otimes \iota_j], [\chi^* \otimes \kappa_l])\Delta \otimes \mathrm{id}(\Delta(\omega))([\delta_i^* \otimes \psi], [\iota_j^* \otimes \eta], [\kappa_l^* \otimes \zeta]) = \\
&= (\phi\xi\zeta, T\delta_i\iota_j\kappa_l)_p \omega([\delta_i^*\iota_j^*\kappa_l^* \circ p_{h_\lambda+h_\mu+h_\nu} \otimes p_{h_\lambda+h_\mu+h_\nu}\psi\eta\zeta])
\end{aligned}$$

Using the symbol $\diamond$ to indicate the adjoint w.r.t. the product form, we have:

$$\begin{aligned}
(\phi\xi\zeta, T\delta_i\iota_j\kappa_l)_p \omega([\delta_i^*\iota_j^*\kappa_l^* \circ p_{h_\lambda+h_\mu+h_\nu} \otimes p_{h_\lambda+h_\mu+h_\nu}\psi\eta\zeta]) = \\
\omega([((\delta_i\iota_j\kappa_l, T^*\phi\xi\zeta)\delta_i\iota_j\kappa_l)^\diamond \circ p_{h_\lambda+h_\mu+h_\nu} \otimes p_{h_\lambda+h_\mu+h_\nu}\psi\eta\zeta]) = \\
= \omega([\phi^*\xi^*\zeta^* \circ T \circ p_{h_\lambda} \otimes p_{h_\mu} \otimes p_{h_\nu} \circ p_{h_\lambda+h_\mu+h_\nu} \otimes p_{h_\lambda+h_\mu+h_\nu}\psi\eta\zeta])
\end{aligned}$$

Using the same tools in the calculation of the right hand side we get:

$$\begin{aligned}
\mathrm{id} \otimes \Delta(\Delta(\omega))\Phi([\phi^* \otimes \psi], [\xi^* \otimes \eta], [\chi^* \otimes \zeta]) = \\
= \omega([\phi^*\xi^*\chi^* \circ \mathrm{id} \otimes S_{\gamma,i} \circ p_{h_\lambda+h_\gamma} \otimes p_{h_\lambda+h_\gamma} \circ \mathrm{id} \otimes S^*_{\gamma,i} \circ p_{h_\lambda} \otimes p_{h_\mu} \otimes p_{h_\nu} \circ T\psi\eta\zeta])
\end{aligned}$$

At this stage, using the explicit expression of T and the well-known properties of the projections we get:

$$\begin{aligned}
\omega([\phi^*\xi^*\chi^* \circ \mathrm{id} \otimes S_{\gamma,i} \circ p_{h_\lambda+h_\gamma} \otimes p_{h_\lambda+h_\gamma} \circ \mathrm{id} \otimes S^*_{\gamma,i} \circ p_{h_\lambda} \otimes p_{h_\mu} \otimes p_{h_\nu} \circ T\psi\eta\zeta]) = \\
= \omega([\phi^*\xi^*\chi^* \circ \mathrm{id} \otimes S_{\gamma,i} \circ p_{h_\lambda+h_\gamma} \otimes p_{h_\lambda+h_\gamma} \circ \circ \mathrm{id} \otimes S^*_{\gamma,i} \circ p_{h_\lambda+h_\mu+h_\nu} \circ \psi\eta\zeta])
\end{aligned}$$

Passing $\mathrm{id} \otimes S_{\gamma,i}$ from the left tensor factor to the right one and summing on γ and i we obtain (4.4.9). The relation (4.4.11) is very easy to prove using the same tools. Therefore it remains to prove the identity (4.4.10). We compute the elements $T_1, T_2, T_3 \in \widehat{\mathcal{C}}^{\otimes 4}$ corresponding to $\Delta \otimes \mathrm{id} \otimes \mathrm{id}(\Phi)$, $\mathrm{id} \otimes \Delta \otimes \mathrm{id}(\Phi)$, $\mathrm{id} \otimes \mathrm{id} \otimes \Delta(\Phi)$. We start with T_1:

$$\begin{aligned}
&\Delta \otimes \mathrm{id} \otimes \mathrm{id}(\Phi)(v^{\lambda}_{\phi,\psi}, v^{\tau}_{\alpha,\beta}, v^{\mu}_{\xi,\eta}, v^{\nu}_{\chi,\zeta}) = \Phi(v^{\lambda}_{\phi,\psi} v^{\tau}_{\alpha,\beta}, v^{\mu}_{\xi,\eta}, v^{\nu}_{\chi,\zeta}) = \\
&= (\phi\alpha\xi\chi, S_{\sigma,j} \otimes \mathrm{id}_{h_\mu+h_\nu} \circ \mathrm{id}_{h_\sigma} \otimes S_{\gamma,i} \circ p_{h_\sigma+h_\gamma} \circ \\
&\circ \mathrm{id}_{h_\sigma} \otimes S^*_{\gamma,i} \circ p_{h_\sigma+h_\mu+h_\nu} \circ S^*_{\sigma,j} \otimes \mathrm{id}_{h_\sigma+h_\nu}\, \psi\beta\eta\zeta)
\end{aligned}$$

where $S_{\gamma,i} \in (V_\gamma, V^{\underline{\otimes} h_\mu+h_\nu})$ and $S_{\sigma,j} \in (V_\sigma, V^{\underline{\otimes} h_\lambda+h_\tau})$. Therefore:

$$\begin{aligned}
T_1 &= S_{\sigma,j} \otimes \mathrm{id}_{h_\mu+h_\nu} \circ \mathrm{id}_{h_\sigma} \otimes S_{\gamma,i} \circ p_{h_\sigma+h_\gamma} \circ \mathrm{id}_{h_\sigma} \otimes S^*_{\gamma,i} \circ p_{h_\sigma+h_\mu+h_\nu} \circ S^*_{\sigma,j} \otimes \mathrm{id}_{h_\mu+h_\nu} = \\
&= \mathrm{id}_{h_\lambda+h_\tau} \otimes S_{\gamma,i} \circ S_{\sigma,j} \otimes \mathrm{id}_{h_\mu+h_\nu} \circ p_{h_\sigma+h_\gamma} \circ \mathrm{id}_{h_\sigma} \otimes S^*_{\gamma,i} \circ S^*_{\sigma,j} \otimes \mathrm{id}_{h_\mu+h_\nu} \circ p_{h_\lambda+h_\tau+h_\mu+h_\nu} = \\
&= \mathrm{id}_{h_\lambda+h_\tau} \otimes S_{\gamma,i} \circ p_{h_\lambda+h_\tau+h_\gamma} \circ S_{\sigma,j} \otimes \mathrm{id}_{h_\gamma} \circ S^*_{\sigma,j} \otimes \mathrm{id}_{h_\gamma} \circ \mathrm{id}_{h_\lambda+h_\tau} \otimes S^*_{\gamma,i} \circ p_{h_\lambda+h_\tau+h_\mu+h_\nu}
\end{aligned}$$

Summing on σ and j we obtain:

$$T_1 = \mathrm{id}_{h_\lambda+h_\tau} \otimes S_{\gamma,i} \circ p_{h_\lambda+h_\tau+h_\gamma} \circ \mathrm{id}_{h_\lambda+h_\tau} \otimes S^*_{\gamma,i} \circ p_{h_\lambda+h_\tau+h_\mu+h_\nu}$$

One also finds:

$$T_2 = \mathrm{id}_{h_\lambda} \otimes S_{\sigma',j} \otimes \mathrm{id}_{h_\nu} \circ \mathrm{id}_{h_\lambda} \otimes S_{\gamma',i} \circ p_{h_\lambda+h_{\gamma'}} \circ \mathrm{id}_{h_\lambda} \otimes S^*_{\gamma',i} \circ p_{h_\lambda+h_{\sigma'}+h_\nu} \circ \mathrm{id}_{h_\lambda} \otimes S^*_{\sigma',j} \otimes \mathrm{id}_{h_\nu}$$

where $S_{\gamma',i} \in (V_{\gamma'}, V^{\underline{\otimes}(h_{\sigma'}+h_\nu)})$ and $S_{\sigma',j} \in (V_{\sigma'}, V^{\underline{\otimes}(h_\tau+h_\mu)})$, and:

$$T_3 = \mathrm{id}_{h_\lambda+h_\tau} \otimes S_{\sigma'',j} \circ \mathrm{id}_{h_\lambda} \otimes S_{\gamma'',i} \circ p_{h_\lambda+h_{\gamma''}} \circ \mathrm{id}_{h_\lambda} \otimes S^*_{\gamma'',i} \circ p_{h_\lambda+h_\tau+h_{\sigma''}} \circ \mathrm{id}_{h_\lambda+h_\tau} \otimes S^*_{\sigma'',j}$$

where $S_{\gamma'',i} \in (V_{\gamma''}, V^{\underline{\otimes}(h_\tau+h_{\sigma''})})$ and $S_{\sigma'',j} \in (V_{\sigma''}, V^{\underline{\otimes}(h_\mu+h_\nu)})$. At this point we are ready to prove the cocycle relation. Since the formulas are rather long, we will drop out many indices. For example, we indicate with p_2 the projections $p_{h_\lambda+h_\gamma}$, and so on. We first calculate the left hand side:

$$\begin{aligned}
T_3 \circ p_1^{\otimes 4} \circ T_1 &= \mathrm{id} \otimes \mathrm{id} \otimes S_1 \circ \mathrm{id} \otimes S_2 \circ p_2 \circ \mathrm{id} \otimes S^*_2 \circ p_3 \circ \mathrm{id} \otimes \mathrm{id} \otimes S^*_1 \circ p_1^{\otimes 4} \circ \mathrm{id} \otimes \mathrm{id} \otimes S_3 \circ p_3 \circ \\
&\circ \mathrm{id} \otimes \mathrm{id} \otimes S^*_3 \circ p_4 = \mathrm{id} \otimes \mathrm{id} \otimes S_1 \circ \mathrm{id} \otimes S_2 \circ p_2 \circ \mathrm{id} \otimes S^*_2 \circ \mathrm{id} \otimes \mathrm{id} \otimes S^*_1 \circ p_4
\end{aligned}$$

On the other side, if T is the matrix defining Φ:

$$\begin{aligned}
&(I \otimes T) \circ p_1^{\otimes 4} \circ T_2 \circ p_1^{\otimes 4} \circ (T \otimes I) = \\
&= \mathrm{id} \otimes \mathrm{id} \otimes S_1 \circ \mathrm{id} \otimes p_2 \circ \mathrm{id} \otimes \mathrm{id} \otimes S^*_1 \circ \mathrm{id} \otimes p_3 \circ p_1^{\otimes 4} \circ \mathrm{id} \otimes S_2 \otimes \mathrm{id} \circ \mathrm{id} \otimes S_3 \circ p_2 \circ \\
&\circ \mathrm{id} \otimes S^*_3 \circ p_3 \circ \mathrm{id} \otimes S^*_2 \otimes \mathrm{id} \circ p_1^{\otimes 4} \circ \mathrm{id} \otimes S_4 \otimes \mathrm{id} \circ p_2 \otimes \mathrm{id} \circ \mathrm{id} \otimes S^*_4 \otimes \mathrm{id} \circ p_3 \otimes \mathrm{id}
\end{aligned}$$

Using the usual tools we can delete the idempotents $\mathrm{id} \otimes p_3$ and the first $p_1^{\otimes 4}$. Moreover, we can delete the first three factors of the second copy of $p_1^{\otimes 4}$, moving the fourth to the far right. This has also allowed to use $S^*_2 \circ S_4 = \delta_{2,4}$. So we obtain:

$$\begin{aligned}
&\mathrm{id} \otimes \mathrm{id} \otimes S_1 \circ \mathrm{id} \otimes p_2 \circ \mathrm{id} \otimes \mathrm{id} \otimes S^*_1 \circ \mathrm{id} \otimes (S_2 \otimes \mathrm{id} \circ S_3) \circ p_2 \circ \\
&\circ \mathrm{id} \otimes S^*_3 \circ [p_3 \circ (p_2 \circ \mathrm{id} \otimes S^*_2 \circ p_3) \otimes \mathrm{id}] \circ \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes p_1
\end{aligned}$$

Next, write the term in the square bracket as $(p_2 \circ \mathrm{id} \otimes S^*_2 \circ p_3) \otimes \mathrm{id} \circ p_4$, so we can delete p_2 and p_3. Therefore the above term becomes:

$$\begin{aligned}
&\mathrm{id} \otimes \mathrm{id} \otimes S_1 \circ \mathrm{id} \otimes (p_2 \circ \mathrm{id} \otimes S^*_1 \circ S_2 \otimes \mathrm{id} \circ S_3) \circ p_2 \circ \\
&\circ \mathrm{id} \otimes S^*_3 \circ \mathrm{id} \otimes S^*_2 \otimes \mathrm{id} \circ p_4 \circ \mathrm{id} \otimes \mathrm{id} \otimes \otimes \mathrm{id} \otimes p_1 = \\
&= \mathrm{id} \otimes \mathrm{id} \otimes S_1 \circ \mathrm{id} \otimes (p_2 \circ \mathrm{id} \otimes S^*_1 \circ S_2 \otimes \mathrm{id} \circ S_3) \circ p_2 \circ \\
&\circ \mathrm{id} \otimes (S^*_3 \circ S^*_2 \otimes \mathrm{id} \circ \mathrm{id} \otimes S_1 \circ p_2) \circ \mathrm{id} \otimes \mathrm{id} \otimes S^*_1 \circ p_4 \circ \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes p_1
\end{aligned}$$

adding in the last equality $\mathrm{id} \otimes p_2$, $\mathrm{id} \otimes \mathrm{id} \otimes S_1$ and $\mathrm{id} \otimes \mathrm{id} \otimes S_1^*$ in the suitable position. It is possible to do that thanks to the well-known properties of the projections. We set now $S' = p_2 \circ \mathrm{id} \otimes S_1^* \circ S_2 \otimes \mathrm{id} \circ S_3$ another orthonormal system of isometries. In this way we get:

$$\mathrm{id} \otimes \mathrm{id} \otimes S_1 \circ \mathrm{id} \otimes S' \circ p_2 \circ \mathrm{id} \otimes S'^* \circ \mathrm{id} \otimes \mathrm{id} \otimes S_1^* \circ p_4 \circ \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes p_1$$

which coincides with the matrix defining the left hand side of the identity (4.4.10). □

Remark 4.4.6. We have already noticed that $\widehat{\mathcal{C}}$ is endowed with a quasi coassociative structure but not coassociative. We can say that this structure is a little bit stronger than quasi coassociative. In fact, there is also a simple relation between the iterated coproducts:

$$\Delta_\ell^{(n)}(I) := \Delta \otimes \mathrm{id}^{\otimes n-1} \circ \ldots \circ \Delta \otimes \mathrm{id} \circ \Delta(I)$$

and arbitrary iterated coproducts $\Delta^{(n)}$ of order n. More precisely, iterated coproducts can be obtained as compositions:

$$\widehat{\mathcal{C}} \to \widehat{\mathcal{C}} \otimes \widehat{\mathcal{C}} \to \ldots \to \widehat{\mathcal{C}}^{\otimes n+1}$$

where the maps $\widehat{\mathcal{C}}^{\otimes j} \to \widehat{\mathcal{C}}^{\otimes j+1}$ can be an arbitrary translates $\mathrm{id}^{\otimes r} \otimes \Delta \otimes \mathrm{id}^{\otimes j-r-1}$ of Δ. Therefore, for all possible choice of $\Delta^{(n)}$ we have:

$$\Delta_\ell^{(n)}(\omega) = \Delta_\ell^{(n)}(I)\Delta^{(n)}(\omega)\Delta_\ell^{(n)}(I)$$

We derive the above identity. The left hand side is:

$$\Delta_\ell^{(n)}(\omega)(v^{\lambda_1}_{\phi_1,\psi_1}, \ldots, v^{\lambda_n}_{\phi_n,\psi_n}) =$$
$$= \omega(\phi_1^* \ldots \phi_n^* \circ p_{h_{\lambda_1}+\ldots+h_{\lambda_n}} \otimes p_{h_{\lambda_1}+\ldots+h_{\lambda_n}} \circ \psi_1 \ldots \psi_n)$$

The right hand side is:

$$\Delta_\ell^{(n)}(I)\Delta^{(n)}(\omega)\Delta_\ell^{(n)}(I)(v^{\lambda_1}_{\phi_1,\psi_1}, \ldots, v^{\lambda_n}_{\phi_n,\psi_n}) = I_1\omega_1 I'_n \otimes \ldots \otimes I_n\omega_n I'_n(v^{\lambda_1}_{\phi_1,\psi_1}, \ldots, v^{\lambda_n}_{\phi_n,\psi_n}) =$$
$$= I_1 \otimes \omega_1 \otimes I'_1 \otimes \ldots \otimes I_n \otimes \omega_n \otimes I'_n([\phi_1^* \otimes \xi_{(1)}],$$
$$[\xi^*_{(1)} \otimes \delta_{(1)}], [\delta_{(1)} \otimes \psi_1], \ldots, [\phi_n^* \otimes \xi_{(n)}], [\xi^*_{(1)} \otimes \delta_{(n)}], [\delta^*_{(n)} \otimes \psi_n]) =$$
$$= (\phi_1 \ldots \phi_n, p_{h_{\lambda_1}+\ldots+h_{\lambda_n}} \xi_{(1)} \ldots \xi_{(n)})(\delta_{(1)} \ldots \delta_{(n)}, p_{h_{\lambda_1}+\ldots+h_{\lambda_n}} \psi_1 \ldots \psi_n) \cdot$$
$$\cdot\, \omega(\xi^*_{(1)} \ldots \xi^*_{(n)} \circ T^* \otimes T \circ \delta_{(1)} \ldots \delta_{(n)})$$

where $\{\xi_{(i)}\}$ and $\{\delta_{(j)}\}$ are orthonormal basis of V_{λ_i}, and $T = p_k \circ T'$, where T' is a composition of isometries $S_{\gamma,i}$ and projections p_h (we have dropped out many indices). Proceeding as in the proof of the last Theorem and setting $p_n = p_{h_{\lambda_1}+\ldots+h_{\lambda_n}}$ we get:

$$\omega(\phi_1^* \ldots \phi_n^* \circ p_n \circ p_1^{\otimes n} \circ T'^* \circ p_k \otimes p_k \circ T' \circ p_1^{\otimes n} \circ p_n \circ \psi_1 \ldots \psi_n)$$

Now we can conclude using the well-known properties of the projections.

Proposition 4.4.7. *$\widehat{\mathcal{C}}$ is a weak quasi Hopf algebra. More precisely, it is endowed with an anti-automorphism S such that:*

$$m(\mathrm{id} \otimes S(\Delta(\omega))) = \varepsilon(a)I = m(S \otimes \mathrm{id}(\Delta(\omega))) \tag{4.4.14}$$
$$\sum_i x_i S(y_i) z_i = I = \sum_j S(p_j) q_j S(r_j) \tag{4.4.15}$$

where $\Phi = \sum_i x_i \otimes y_i \otimes z_i$ and $\Psi = \sum_i p_i \otimes q_i \otimes r_i$. Therefore S is an antipode with $\alpha = \beta = I$.

Proof. We define S by duality on $\widehat{C}$:

$$S(\omega)(v^{\lambda}_{\phi,\psi}) = \omega(S(v^{\lambda}_{\phi,\psi})) = \omega(v^{\overline{\lambda}}_{j\psi,j\phi})$$

We prove now that S is anti-multiplicative. Let $\{\eta_i\}_i$ be an orthonormal basis of V_λ and $\{\xi_i\}_i$ be an orthonormal basis of $\overline{V_\lambda}$. We have:

$$\begin{aligned} S(\omega\tau)([\phi^* \otimes \psi]) = \omega\tau([(j_\lambda\psi)^* \otimes (j_\lambda\phi)]) = \\ = \omega \otimes \tau([(j_\lambda\psi)^* \otimes \xi_i], [\xi_i^* \otimes (j_\lambda\phi)]) = \omega([(j_\lambda\psi)^* \otimes \xi_i])\tau([\xi_i^* \otimes (j_\lambda\phi)]) \end{aligned}$$

On the other side:

$$\begin{aligned} S(\tau)S(\omega)([\phi^* \otimes \psi]) = S(\tau) \otimes S(\omega)([\phi^* \otimes \eta_i], [\eta_i^* \otimes \psi]) = \\ = \tau([(j_\lambda\eta_i)^* \otimes (j_\lambda\phi)])\omega([(j_\lambda\psi)^* \otimes (j_\lambda\eta_i)]) \end{aligned}$$

Identifying ξ_i with $j_\lambda\eta_i$ we obtain $S(\omega\tau) = S(\tau)S(\omega)$. The relation (4.4.14) is also a consequence of the duality, since it is a self-dual relation and it has been proved on $\mathcal{C}$. It remains to check (4.4.15). We omit, as usual, the summation symbol on i:

$$\begin{aligned} x_i S(y_i) z_i(v^{\mu}_{\phi,\psi}) &= x_i S(y_i) \otimes z_i(v^{\mu}_{\phi,\eta_j}, v^{\mu}_{\eta_j,\psi}) = \\ &= x_i \otimes S(y_i) \otimes z_i(v^{\mu}_{\phi,\xi_k}, v^{\mu}_{\xi_k,\eta_j}, v^{\mu}_{\eta_j,\psi}) = \\ &= \Phi(v^{\mu}_{\phi,\xi_k}, v^{\overline{\mu}}_{j\eta_j,j\xi_k}, v^{\mu}_{\eta_j,\psi}) = \\ &= (\phi \otimes (j_\mu\eta_j) \otimes \eta_j, \mathrm{id}_{h_\mu} \otimes S_{\gamma,i} \circ p_{h_\mu+h_\gamma} \circ \mathrm{id}_{h_\mu} \otimes S^*_{\gamma,i} \circ p_{3h_\mu}\xi_k \otimes (j_\mu\xi_k) \otimes \psi) \end{aligned}$$

We now that $\overline{r}(1) = \sum_k \xi_k \otimes (j_\mu\xi_k)$ as arrow in $(\mathbb{C}, V_\mu \otimes \overline{V_\mu})$, so:

$$\begin{aligned} (\phi \otimes (j_\mu\eta_j) \otimes \eta_j, \mathrm{id}_{h_\mu} \otimes S_{\gamma,i} \circ p_{h_\mu+h_\gamma} \circ \mathrm{id}_{h_\mu} \otimes S^*_{\gamma,i} \circ p_{3h_\mu}\xi_k \otimes (j_\mu\xi_k) \otimes \psi) = \\ = (\phi \otimes (j_\mu\eta_j) \otimes \eta_j, \mathrm{id}_{h_\mu} \otimes S_{\gamma,i} \circ p_{h_\mu+h_\gamma} \circ \mathrm{id}_{h_\mu} \otimes S^*_{\gamma,i} \circ p_{3h_\mu}\overline{r}(1) \otimes \psi) \end{aligned}$$

We know that the range of $\overline{r}$ in $V_\mu \otimes \overline{V_\mu}$ is a copy of $\mathbb{C}$ in it. Hence $\overline{r}(1) \otimes \psi$ is an element in $V_{\mathbb{C}} \otimes V_\mu \cong V_\mu$, where $V_{\mathbb{C}} \cong \mathbb{C}$. Therefore the projections are trivial:

$$\begin{aligned} (\phi \otimes (j_\mu\eta_j) \otimes \eta_j, \mathrm{id}_{h_\mu} \otimes S_{\gamma,i} \circ p_{h_\mu+h_\gamma} \circ \mathrm{id}_{h_\mu} \otimes S^*_{\gamma,i} \circ p_{3h_\mu}\overline{r}(1) \otimes \psi) = \\ = (\phi \otimes (j_\mu\eta_j) \otimes \eta_j, \xi_k \otimes (j_\mu\xi_k) \otimes \psi) = (\phi, \xi_k)(j_\mu\eta_j, j_\mu\xi_k)(\eta_j, \psi) = \\ = (\phi, \psi) = I(v^{\mu}_{\phi,\psi}) \end{aligned}$$

The right part of the identity can be proved in the same way. □

Proposition 4.4.8. *$\widehat{\mathcal{C}}$ is a weak quasi Hopf *-algebra. In other terms, there exists an involution map $^*: \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ such that $\widehat{\mathcal{C}}$ is a *-algebra with respect to it and a partially invertible element Ω such that:*

$$\Omega = \Omega^* \tag{4.4.16}$$

$$\Omega\Delta(I) = \Omega = \Delta(I)^*\Omega \tag{4.4.17}$$

$$\Delta(\omega)^* = \Omega\Delta(\omega^*)\Omega^{-1} \quad \forall \omega \in \widehat{\mathcal{C}} \tag{4.4.18}$$

$$\varepsilon \otimes \mathrm{id}(\Omega) = I = \mathrm{id} \otimes \varepsilon(\Omega) \tag{4.4.19}$$

$$\Phi^{-1*} = (I \otimes \Omega)(\mathrm{id} \otimes \Delta(\Omega))\Phi(\Delta \otimes \mathrm{id}(\Omega^{-1}))(\Omega^{-1} \otimes I) \tag{4.4.20}$$

$$\tag{4.4.21}$$

More precisely, * is the dual involution of * on $\mathcal{C}$, and $\Omega = \omega_S$, where $S = \overline{R} \circ p_{h_\lambda + h_\mu}$

Proof. First of all, we know that * is defined on the simple tensor of $\mathcal{C}$ in the following way:

$$v^{\lambda *}_{\phi,\psi} = v^\lambda_{\psi,\phi}$$

Therefore we define * on $\widehat{\mathcal{C}}$ by duality:

$$\omega^*(v^\lambda_{\phi,\psi}) = \overline{\omega(v^\lambda_{\psi,\phi})}$$

We want to prove that * is anti-multiplicative. On one side:

$$\begin{aligned}(\omega\tau)^*([\phi^* \otimes \psi]) &= \overline{\omega\tau([\psi^* \otimes \phi])} = \overline{\omega \otimes \tau([\psi^* \otimes \eta_i], [\eta_i^* \otimes \phi])} = \\ &= \overline{\omega([\psi^* \otimes \eta_i])\tau([\eta_i * \otimes \phi])}\end{aligned}$$

On the other side:

$$\begin{aligned}\tau^*\omega^*([\phi^* \otimes \psi]) &= \tau^* \otimes \omega^*([\phi^* \otimes \eta_i], [\eta_i^* \otimes \psi]) = \\ &= \overline{\tau([\eta_i^* \otimes \phi])\omega([\psi^* \otimes \eta_i])}\end{aligned}$$

Next we check the relations involving Ω. We define Ω in the following way:

$$\Omega(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}) = (\phi \otimes \xi, \overline{R}p_{h_\lambda + h_\mu}\psi \otimes \eta)$$

where $\overline{R}$ is the coboundary matrix introduced in the Chapter 3. We start with the self-adjointness of Ω. Set $p = p_{h_\lambda + h_\mu}$:

$$\begin{aligned}\Omega^*(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}) &= \overline{\Omega(v^\lambda_{\psi,\phi}, v^\mu_{\eta,\psi})} = \overline{(\psi \otimes \eta, \overline{R}p\phi \otimes \xi)_p} = \\ &= (\overline{R}p\phi \otimes \xi, \psi \otimes \eta)_p = (\phi \otimes \xi, p^*\overline{R}\psi \otimes \eta)_p = \\ &= (\phi \otimes \xi, \overline{R}p\overline{R}_{21}\overline{R}\psi \otimes \eta)_p = (\phi \otimes \xi, \overline{R}p\psi \otimes \eta)_p = \Omega(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta})\end{aligned}$$

where we used the fact that p is self-adjoint w.r.t the modified hermitian form and hence $p^* = \overline{R}p\overline{R}_{21}$. Proceeding in the same way we have:

$$\Delta(I)^*(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}) = (\phi \otimes \xi, \overline{R}p_{h_\lambda + h_\mu}\overline{R}_{21}\psi \otimes \eta)$$

Using Lemma 4.4.4 we have (4.4.17). Next we prove (4.4.18) in the form $\Delta(\omega)^*\Omega = \Omega\Delta(\omega^*)$. The left hand side is:

$$\begin{aligned}\Delta(\omega)^*\Omega([\phi^* \otimes \psi], [\xi^* \otimes \eta]) &= (\omega_{(1)})^*\Omega_1 \otimes (\omega_{(2)})^*\Omega_2([\phi^* \otimes \psi], [\xi^* \otimes \eta]) = \\ &= (\omega_{(1)})^* \otimes \Omega_1 \otimes (\omega_{(2)})^* \otimes \Omega_2([\phi^* \otimes \chi_i], [\chi_i^* \otimes \psi], [\xi^* \otimes \delta_j], [\delta_j^* \otimes \eta]) = \\ &= \Delta(\omega)^*([\phi^* \otimes \chi_i], [\xi^* \otimes \delta_j])\Omega([\chi_i^* \otimes \psi], [\delta_j^* \otimes \eta]) = \\ &= \overline{\Delta(\omega)([\chi_i^* \otimes \phi], [\delta_j^* \otimes \xi])}(\chi_i \otimes \delta_j, \overline{R}p\psi \otimes \eta) = \\ &= \overline{\omega(\chi_i^*\delta_j^* \circ p \otimes p \circ \phi\xi)}(\chi_i\delta_j, \overline{R}p\psi\eta) = \overline{\omega(\psi^*\eta^* \circ \overline{R}p \otimes p \circ \phi\xi)}\end{aligned}$$

On the other side:

$$\begin{aligned}\Omega\Delta(\omega^*)([\phi^*\otimes\psi],[\xi^*\otimes\eta]) &= \Omega([\phi^*\otimes\chi_i],[\xi^*\otimes\delta_j])\Delta(\omega^*)([\chi_i^*\otimes\psi,\delta_j^*\otimes\eta]) = \\ &= (\phi\otimes\xi,\overline{R}p\chi_i\otimes\delta_j)\omega^*([\chi^*\delta_j^*\circ p\otimes p\circ\psi\eta]) = \\ &= \omega^*([\phi^*\xi^*\circ\overline{R}p\otimes p\circ\psi\eta]) = \\ &= \overline{\omega([\psi^*\eta^*\circ p^*\circ\overline{R}\otimes p\phi\xi])} = \overline{\omega([\psi^*\eta^*\circ\overline{R}p\otimes p\phi\xi])}\end{aligned}$$

(4.4.19) is merely a consequence of the identity $\varepsilon\otimes\mathrm{id}(\overline{R}) = I = \mathrm{id}\otimes\varepsilon(\overline{R})$. In fact:

$$\begin{aligned}\varepsilon\otimes\mathrm{id}(\Omega)(v^\lambda_{\phi,\psi}) &= \Omega(v^{\mathbb{C}}_{1,1},v^\lambda_{\phi,\psi}) = \\ = (1\otimes\phi,\overline{R}p_{h_\lambda}1\otimes\psi) &= (\phi,\varepsilon\otimes\mathrm{id}(\overline{R})\psi) = (\phi,\psi)\end{aligned}$$

The right part of (4.4.19) can be proved in the same way. It remains to prove (4.4.20). First of all, there is an equivalent formulation of the identity:

$$\Phi^{-1*} = (\mathrm{id}\otimes\Delta(\Omega)^*)(I\otimes\Omega)\Phi(\Omega^{-1}\otimes I)(\Delta\otimes\mathrm{id}(\Omega^{-1}))$$

This allows us to see that it is not restrictive to multiply on the right by $\Delta\otimes\mathrm{id}(\Delta(I))^*$. Now we turn back to the original formulation. The right hand side is ω_T, where:

$$\begin{aligned}T = \mathrm{id}\otimes\overline{R}\circ\mathrm{id}\otimes p_2\circ p_1\otimes p_1'\otimes p_1''\circ\mathrm{id}\otimes S_i\circ\overline{R}p_2\circ\mathrm{id}\otimes S_i^*\circ \\ \circ p_1\otimes p_1'\otimes p_1''\circ p_1\otimes p_1'\otimes p_1''\circ\mathrm{id}\otimes S_j\circ p_2\circ\mathrm{id}\otimes S_j^*\circ p_3\circ p_1\otimes p_1'\otimes p_1''\circ S_h\otimes\mathrm{id}\circ \\ \circ p_2\overline{R}_{21}\circ S_h^*\otimes\mathrm{id}\circ p_1\otimes p_1'\otimes p_1''\circ p_2\overline{R}_{21}\otimes\mathrm{id}\circ p_1\otimes p_1'\otimes p_1''\circ\overline{R}^{(3)}p_3\overline{R}^{(3)}_{21}\end{aligned}$$

Using the well-known properties of the projections we get:

$$\begin{aligned}T = p_1\otimes\mathrm{id}\otimes\mathrm{id}\circ\mathrm{id}\otimes\overline{R}\circ\mathrm{id}\otimes p_2\circ\mathrm{id}\otimes S_i\circ\overline{R}p_2\circ\mathrm{id}\otimes S_i^*\circ\mathrm{id}\otimes S_j\circ \\ \circ\mathrm{id}\otimes S_j^*\circ\mathrm{id}\otimes S_h\circ\overline{R}_{21}\circ\mathrm{id}\otimes S_h^*\circ\mathrm{id}\otimes\overline{R}_{21}\circ\overline{R}^{(3)}p_3\overline{R}^{(3)}_{21}\end{aligned}$$

Now, it is possible to use the properties of the isometries and omit the first projection. We have:

$$T = \mathrm{id}\otimes\overline{R}\circ\mathrm{id}\otimes p_2\circ\mathrm{id}\otimes S_i\circ\overline{R}p_2\overline{R}_{21}\circ\mathrm{id}\otimes S_i^*\circ\mathrm{id}\otimes\overline{R}_{21}\circ\overline{R}^{(3)}p_3\overline{R}^{(3)}_{21}$$

We know that $p_2 = \sum_l S_l S_l^*$, so using this expression in place of the first p_2 and the properties of the isometries again we obtain:

$$T = \mathrm{id}\otimes\overline{R}S_i\circ\overline{R}p_2\overline{R}_{21}\circ\mathrm{id}\otimes S_i^*\overline{R}_{21}\circ\overline{R}^{(3)}p_3\overline{R}^{(3)}_{21}$$

On the other side, we know that $\Phi^{-1*} = \Psi^* = \omega_{V^\diamond}$, where:

$$V^\diamond = \mathrm{id}\otimes\overline{R}S_i\overline{R}_{21}\circ\overline{R}p_2\overline{R}_{21}\circ\mathrm{id}\otimes\overline{R}S_i^*\overline{R}_{21}\circ\overline{R}^{(3)}p_3\overline{R}^{(3)}_{21}$$

We use $\diamond$ to indicate the adjoint with respect to the product form. In order to obtain $V^\diamond = T$, we can see that:

$$\overline{R}S_i\overline{R}_{21} = \overline{R}S_i \text{ and } \overline{R}S_i^*\overline{R}_{21} = S_i^*\overline{R}_{21}$$

In fact, if $S_i \in (V_\gamma, V^{\underline{\otimes}N})$ for some γ and some N, we have that $\overline{R}_{21}$ acts trivially on V_γ. For the same reason we can omit $\overline{R}$ in $\overline{R}S_i^*\overline{R}_{21}$, since $S_i^* \in (V^{\underline{\otimes}N}, V_\gamma)$. □

Proposition 4.4.9. *Φ and Ψ satisfy the following identities:*

$$\Phi = \mathrm{id} \otimes \Delta(\Delta(I))\Delta \otimes \mathrm{id}(\Delta(I)) \tag{4.4.22}$$
$$\Psi = \Delta \otimes \mathrm{id}(\Delta(I))\, \mathrm{id} \otimes \Delta(\Delta(I)) \tag{4.4.23}$$

Therefore, $\widehat{\mathcal{C}}$ is a weak Hopf *-algebra.

Proof. It is straightforward to prove using Lemma 4.4.4. □

We can see that $\widehat{\mathcal{C}}$ is endowed with a R-matrix $\mathcal{R} \in \widehat{\mathcal{C}}^{\otimes 2}$, defined in the following way:

$$\mathcal{R}(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}) = (\phi \otimes \xi, \Sigma_{k,h} p_{k+h} \Sigma_{h,k} R p_{h+k}(\psi \otimes \eta))_p$$

where $\phi, \psi \in V^{\underline{\otimes} h}$ and $\xi, \eta \in V^{\underline{\otimes} k}$.

Proposition 4.4.10. *$\mathcal{R}$ satisfies the following relations:*

$$\mathcal{R}\Delta(I) = \mathcal{R} = \Delta^{\mathrm{op}}(I)\mathcal{R} \tag{4.4.24}$$
$$\Delta^{\mathrm{op}}(\omega)\mathcal{R} = \mathcal{R}\Delta(\omega) \tag{4.4.25}$$
$$\mathrm{id} \otimes \Delta(\mathcal{R}) = \Phi^{-1}_{312}\mathcal{R}_{13}\Phi_{213}\mathcal{R}_{12}\Phi^{-1} \tag{4.4.26}$$
$$\Delta \otimes \mathrm{id}(\mathcal{R}) = \Phi_{231}\mathcal{R}_{13}\Phi^{-1}_{132}\mathcal{R}_{23}\Phi \tag{4.4.27}$$
$$\mathcal{R}^*\Omega_{21} = \Omega\mathcal{R}^{-1} \tag{4.4.28}$$

Finally, $\mathcal{R}$ is invertible and its inverse is:

$$\mathcal{R}^{-1}(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}) = (\phi \otimes \xi, p_{h+k} R^{-1} \Sigma_{k,h} p_{k+h} \Sigma_{h,k} \psi \otimes \eta)_p \tag{4.4.29}$$

Proof. Recalling that:

$$\Delta(I)(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}) = (\phi \otimes \xi, p_{h+k}\psi \otimes \eta)$$
$$\Delta^{\mathrm{op}}(I)(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}) = (\phi \otimes \xi, \Sigma_{k,h} p_{k+h} \Sigma_{h,k}\psi \otimes \eta)$$

and using Lemma 4.4.4 the first assertion is very easy to prove. We pass to the second one. Suppose that $\{\delta_i\}_i$ is an orthonormal basis of V_λ and $\{\chi_j\}_j$ is an orthonormal basis of V_μ. The left hand side is:

$$\begin{aligned}\Delta^{\mathrm{op}}(\omega)\mathcal{R}(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}) &= \omega_{(2)}\mathcal{R}_1 \otimes \omega_{(1)}\mathcal{R}_2(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}) = \\ &= \omega_{(2)}([\phi^* \otimes \delta_i])\mathcal{R}_1([\delta_i^* \otimes \psi])\omega_{(1)}([\xi^* \otimes \chi_j])\mathcal{R}_2([\chi_j^* \otimes \eta]) = \\ &= (\chi_j\delta_i, p_{k+h}\Sigma_{h,k}Rp_{h+k}\psi\eta)\omega([\xi^*\phi^* \circ p_{k+h} \otimes p_{k+h} \circ \chi_j\delta_i]) = \\ &= \omega([\xi^*\phi^* \circ p_{k+h} \otimes p_{k+h}\Sigma_{h,k}Rp_{h+k}\psi\eta])\end{aligned}$$

On the other side:

$$\begin{aligned}\mathcal{R}\Delta(\omega)(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}) &= (\xi\phi, p_{k+h}\Sigma_{h,k}Rp_{h+k}\delta_i\chi_j)\omega([\delta_i^*\chi_j^* \circ p_{k+h} \otimes p_{k+h} \circ \psi\eta]) = \\ &= \omega([\xi^*\phi^* \circ p_{k+h}\Sigma_{h,k}Rp_{h+k} \otimes p_{k+h} \circ \psi\eta])\end{aligned}$$

Since $p_{k+h}\Sigma_{h,k}Rp_{h+k} \in (V^{\underline{\otimes} h+k}, V^{\underline{\otimes} k+h})$ we can conclude. Next, we treat the identities (4.4.26) and (4.4.27). We focus on the first one, since the second can be proved in the same way. We will use the well-known computation rule shown in Lemma 4.4.4, but with a slight modification. In fact, we know that if $\omega = \omega_S\omega_T$, then $\omega = \omega_V$, where $V = S \circ p_1' \otimes \ldots \otimes p_1^{(n)} \circ T$. In the next calculation we omit the projections in the middle, since they will soon disappear thanks to the properties of the projections. So, the right hand side is ω_T, where T is:

$$\begin{aligned}T =&\Sigma_{2,1} \circ p_3 \circ 1 \otimes S_i \circ p_2 \circ 1 \otimes S_i^* \circ 1 \otimes p_2 \circ 1 \otimes \Sigma \circ 1 \otimes R \circ 1 \otimes p_2 \circ 1 \otimes S_j \circ\\ &\circ p_2 \circ 1 \otimes S_j^* \circ p_3 \circ p_2 \otimes 1 \circ \Sigma \otimes 1 \circ R \otimes 1 \circ p_2 \otimes 1 \circ p_3 \circ 1 \otimes S_k \circ p_2 \circ 1 \otimes S_k^*\end{aligned}$$

using the short notation. We can delete all the projections except the first and the last one, so:

$$T = \Sigma_{2,1} \circ p_3 \circ 1 \otimes \Sigma R \circ \Sigma R \otimes 1 \circ 1 \otimes S_k \circ p_2 \circ 1 \otimes S_k^*$$

using the relations involving the isometries. We know that the R-matrix R on the extended form of $U_q(\mathfrak{g})^{\otimes 2}$ satisfies the identity:

$$\mathrm{id} \otimes \Delta(R) = R_{13} R_{12}$$

which can be rewritten in the following way:

$$\Sigma_{1,2}\, \mathrm{id} \otimes \Delta(R) = (I \otimes \Sigma R)(\Sigma R \otimes I)$$

So:

$$T = \Sigma_{2,1} \circ p_3 \circ \Sigma_{1,2} \circ \mathrm{id} \otimes \Delta(R) \circ 1 \otimes S_k \circ p_2 \circ 1 \otimes S_k^*$$

Since $\mathrm{id} \otimes S_k$ is $U_q(\mathfrak{g})$-linear, we have:

$$\begin{aligned}T &= \Sigma_{2,1} \circ p_3 \circ \Sigma_{1,2} \circ \mathrm{id} \otimes S_k \circ R \circ p_2 \circ \mathrm{id} \otimes S_k^* =\\ &= \Sigma_{2,1} \circ p_3 \circ S_k \otimes 1 \circ \Sigma R p_2 \circ 1 \otimes S_k^* =\\ &= \Sigma_{2,1} \circ S_k \otimes 1 \circ p_2 \Sigma R p_2 \circ 1 \otimes S_k^* =\\ &= 1 \otimes S_k \circ \Sigma p_2 \Sigma R p_2 \circ 1 \otimes S_k^* = T'\end{aligned}$$

where T' is such that $\mathrm{id} \otimes \Delta(\mathcal{R}) = \omega_{T'}$. It remains to check that $\mathcal{R}^* \Omega_{21} = \Omega \mathcal{R}^{-1}$. If $\mathcal{R} = \omega_U$, then $\mathcal{R}^* = \omega_{U^\diamond}$, where $U^\diamond$ is the adjoint of U w.r.t. the product form. So:

$$U^\diamond = \overline{R} p_2 \overline{R}_{21} R^* \Sigma \overline{R} p_2 \overline{R}_{21} \Sigma$$

Therefore, using Lemma 4.4.4, we have that the left hand side is ω_S, where S is:

$$S = \overline{R} p_2 \overline{R}_{21} R^* \Sigma \overline{R} p_2 \overline{R}_{21} \circ p_1 \otimes p_1' \circ \Sigma \overline{R} p_2 \Sigma$$

Using the well-known relations $R_{21} R \Theta^2 = I \otimes I$, $R^* = R_{21}^{-1}$, $\Sigma \overline{R} = \overline{R}_{21} \Sigma$ and $\overline{R} = R\Theta$ we get:

$$\begin{aligned}S &= \overline{R} p_2 R_{21} \Theta_{21} R^* R_{21} \Theta_{21} \Sigma p_2 \overline{R}_{21} \Sigma \circ p_1 \otimes p_1' \circ \Sigma \overline{R} p_2 \Sigma =\\ &= \overline{R} p_2 R_{21} \Theta_{21}^2 \Sigma p_2 \overline{R}_{21} \Sigma \circ p_1 \otimes p_1' \circ \Sigma \overline{R} p_2 \Sigma =\\ &= \overline{R} p_2 (R\Theta^2)_{21} \Sigma p_2 \overline{R}_{21} \Sigma \circ p_1 \otimes p_1' \circ \Sigma \overline{R} p_2 \Sigma =\\ &= \overline{R} p_2 R^{-1} \Sigma p_2 \overline{R}_{21} \Sigma \circ p_1 \otimes p_1' \circ \Sigma \overline{R} p_2 \Sigma\end{aligned}$$

Using Lemma 3.7.4 we obtain:

$$S = \overline{R} p_2 R^{-1} \Sigma \overline{R}_{21} \Sigma \Sigma \overline{R} p_2 \Sigma = \overline{R} p_2 R^{-1} \Sigma p_2 \Sigma$$

On the other side, $\Omega \mathcal{R}^{-1} = \omega_T$, where:

$$T = \overline{R} p_2 \circ p_1 \otimes p_1' \circ p_2 R^{-1} \Sigma p_2 \Sigma = S$$

using again Lemma 3.7.4 and Lemma 4.4.4. Finally, similar calculations allow to prove that $\mathcal{R}^{-1}$ can be defined as in (4.4.29). □

We next briefly discuss a relation between the original quantum group $U_q(\mathfrak{g})$ and $\widehat{\mathcal{C}(G,l)}$. There is a natural map:

$$\pi : U_q(\mathfrak{g}) \to \widehat{\mathcal{C}(G,l)}$$

taking an element $a \in U_q(\mathfrak{g})$ to the functional:

$$\pi(a)(v^{\lambda}_{\phi,\psi}) = (\phi, a\psi)$$

where $\phi, \psi \in V_\lambda$ and $\lambda \in \Lambda_l$.

Proposition 4.4.11. *π is a surjective homomorphism of *-algebras satisfying:*

$$P\pi \otimes \pi(\Delta(a)) = \Delta(\pi(a)) = \pi \otimes \pi(\Delta(a))P \tag{4.4.30}$$

Proof. We first check that $\pi(a)^* = \pi(a^*)$. In fact:

$$\begin{aligned} \pi(a^*)(v^{\lambda}_{\phi,\psi}) &= (\phi, a^*\psi) = (a\phi, \psi) = \\ = \overline{(\psi, a\phi)} &= \overline{\pi(a)(v^{\lambda}_{\psi,\phi})} = \pi(a)^*(v^{\lambda}_{\phi,\psi}) \end{aligned}$$

Secondly we prove that $\pi(ab) = \pi(a)\pi(b)$. The left hand side is $\pi(ab)(v^{\lambda}_{\phi,\psi}) = (\phi, ab\psi)$. The right hand side is:

$$\begin{aligned} \pi(a)\pi(b)(v^{\lambda}_{\phi,\psi}) &= \pi(a) \otimes \pi(b)(v^{\lambda}_{\phi,\delta_i}, v^{\lambda}_{\delta_i,\psi}) = \\ &(\phi, a\delta_i)(\delta_i, b\psi) = (\phi, ab\psi) \end{aligned}$$

In the same way it is easy to prove (4.4.30). Finally, surjectivity is a consequence of the identification of $\widehat{\mathcal{C}}$ with $\bigoplus_{\lambda\in\Lambda_l} \mathcal{B}(V_\lambda)$ and of the irreducibility of the V_λ. □

4.4.3 Comparing the groupoids associated to different sections

Let $s : \mathcal{C}(G,l) \to \mathcal{D}(V,l)$ and $s' : \mathcal{C}(G,l) \to \mathcal{D}(V,l)$ be different sections of the quotient map $\mathcal{D}(V,l) \to \mathcal{C}(G,l)$. Correspondingly, we can put two different structures $(\Delta, \Phi, \mathcal{R})$ and $(\Delta', \Phi', \mathcal{R}')$ of weak quasi Hopf *-algebras on $\widehat{\mathcal{C}}$, noticing that the antipode S is the same because it does not depend on the choice of the section. We claim that these algebras are related by a twisting procedure as shown in Chapter 1, induced by special quasi invertible elements in $\widehat{\mathcal{C}} \otimes \widehat{\mathcal{C}}$.

Proposition 4.4.12. *Set $P = \Delta(I)$ and $P' = \Delta'(I)$. Then for $\omega \in \widehat{\mathcal{C}}$:*

$$P\Delta'(\omega)P = \Delta(\omega)\ ,\ P'\Delta(\omega)P' = \Delta'(\omega)$$

Proof. To show the first relation, we first write P in a different way, as follows. Let s and s' be defined by two choices of irreducible summands V_λ and V'_λ both of highest weight λ in $V^{\otimes h_\lambda}$ and $V^{\otimes h'_\lambda}$, respectively. For each λ there exists a unitary intertwiner $U_\lambda : V'_\lambda \to V_\lambda$, unique up to a scalar multiple by an element of $\mathbb{T}$. We associate the element $G \in \widehat{\mathcal{C}} \otimes \widehat{\mathcal{C}}$ defined by:

$$G(v^{\lambda}_{\phi',\psi'}, v^{\mu}_{\xi',\eta'}) = (U_\lambda\phi' \otimes U_\mu\psi', p_{h_\lambda+h_\mu} \circ U_\lambda\psi' \otimes U_\mu\eta')_p$$

where $\phi', \psi' \in V'_\lambda$, $\xi', \eta' \in V'_\mu$. On one hand we may use the defining identifications in $\mathcal{C}(G,l)$ and write, for $\phi, \psi \in V_\lambda$, $v^{\lambda}_{\phi,\psi} = v^{\lambda}_{U^*_\lambda\phi, U^*_\lambda\psi}$, and this shows that $G = P$. On the other hand, we can develop the usual calculations and derive the relation $G\Delta'(\omega)G = \Delta(\omega)$. We can proceed similarly in order to prove the second identity. □

Remark 4.4.13. $\widehat{\mathcal{C}}' = \widehat{\mathcal{C}}_F$, where $F = \Delta'(I)\Delta(I)$ and $F^{-1} = \Delta(I)\Delta'(I)$. We need to check that:

$$\varepsilon \otimes \mathrm{id}(F) = I = \mathrm{id} \otimes \varepsilon(F)$$

and $F^{-1}F = \Delta(I)$, $FF^{-1} = \Delta'(I)$. In order to prove the first assertion, it is sufficient to prove that:

$$\varepsilon \otimes \mathrm{id}(\Delta'(I)) = I = \mathrm{id} \otimes \varepsilon(\Delta'(I))$$

We have:

$$\begin{aligned} \varepsilon \otimes \mathrm{id}(\Delta'(I))(v^{\lambda}_{\phi,\psi}) &= \Delta'(I)([1^* \otimes 1], [\phi^* \otimes \psi]) = \\ &= (1 \otimes U_\lambda \phi, p_{h_\lambda} \circ 1 \otimes U_\lambda \psi) = (\phi, \psi) \end{aligned}$$

Next we check that $F^{-1}F = \Delta(I)$. It is equivalent to prove that $\Delta(I)\Delta'(I)\Delta(I) = \Delta(I)$. Using the well-known Lemma 4.4.4, we have $\Delta(I)\Delta'(I)\Delta(I) = \omega_T$, where:

$$T = p_{h_\lambda + h_\mu} \circ p_{h_\lambda} \otimes p_\mu \circ U^*_\lambda \otimes U^*_\mu \circ p_{h'_\lambda + h'_\mu} \circ U_\lambda \otimes U_\mu \circ p_{h_\lambda} \otimes p_{h_\mu} \circ p_{h_\lambda + h_\mu}$$

which is equal to $p_{h_\lambda + h_\mu}$. The identity $FF^{-1} = \Delta'(I)$ can be checked in the same way.

4.4.4 The *-tensor equivalence between $\mathcal{F}_l$ and $\mathrm{Rep}_V(\widehat{\mathcal{C}(G,l)})$.

In this subsection we show that if $\mathcal{C}(G,l)$ is cosemisimple then the smallest full tensor subcategory $\mathrm{Rep}_V(\widehat{\mathcal{C}(G,l)})$ of the representation category of $\widehat{\mathcal{C}}$ containing the fundamental representation is a tensor C*-category equivalent to the fusion category $\mathcal{F}_l$. Let $\mathrm{Rep}(\widehat{\mathcal{C}})$ be the category of unital representations of $\widehat{\mathcal{C}}$ on f.d. vector spaces. In the first chapter we showed that $\mathrm{Rep}(\widehat{\mathcal{C}})$ can be made a tensor category. Moreover, if we restrict to the f.d. vector spaces endowed with an inner product, we get that $\mathrm{Rep}_s(\widehat{\mathcal{C}})$ is a *-tensor category. Now if u and v are Hilbert space representations, then $u \otimes v$ is still a Hilbert space representation provided $u \otimes v(\Omega)$ is a positive operator with respect to the product form. We consider the map $\widehat{V} : \widehat{\mathcal{C}} \to \mathcal{B}(V)$ defined by:

$$(\xi, \widehat{V}(\omega)\eta) = \omega([\xi^* \otimes \eta])$$

where $\omega \in \widehat{\mathcal{C}}$ and $\xi, \eta \in V$, and $(\cdot,\cdot)$ is the inner product on V. We restrict to the category $\mathrm{Rep}_V(\widehat{\mathcal{C}})$, which is the smallest full tensor subcategory of $\mathrm{Rep}_s(\widehat{\mathcal{C}})$ containing $\widehat{V}$. The objects of $\mathrm{Rep}_V(\widehat{\mathcal{C}})$ are the trivial representation, $\widehat{V}$ and all the representations of the form $\omega \mapsto \widehat{V} \otimes \ldots \otimes \widehat{V} \circ \Delta^{(n)}(\omega)$, $n \geq 1$. We have:

Theorem 4.4.14. $\mathrm{Rep}_V(\widehat{\mathcal{C}})$ *is a tensor C*-category.*

Proof. It is sufficient to prove that $\widehat{V} \otimes \widehat{V}(\Omega)$ is a positive operator on $V \otimes V$. We use the notation $\Omega = \sum_i c_i \otimes d_i$. Set $\xi_1, \xi_2 \in V$. We have:

$$\begin{aligned} (\xi_1 \otimes \xi_2, \widehat{V} \otimes \widehat{V}(\Omega)\xi_1 \otimes \xi_2) &= \sum_i (\xi_1, \widehat{V}(c_i)\xi_1)(\xi_2, \widehat{V}(d_i)\xi_2) = \\ &= \sum_i c_i([\xi_1^* \otimes \xi_1]) d_i([\xi_2^* \otimes \xi_2]) = \Omega([\xi_1^* \otimes \xi_1], [\xi_2^* \otimes \xi_2]) = \\ &= (\xi_1 \otimes \xi_2, \overline{R}p_2 \circ \xi_1 \otimes \xi_2) = (p_2 \circ \xi_1 \otimes \xi_2, \overline{R}p_2 \circ \xi_1 \otimes \xi_2) \geq 0 \end{aligned}$$

using the positivity of the Wenzl's form. $\square$

We finally discuss the relation between the original (strict) fusion category $\mathcal{F}_l$ and $\mathrm{Rep}_V(\widehat{\mathcal{C}})$. Let $X = V^{\underline{\otimes} n}$ be regarded as an object of $\mathcal{F}_l$, meaning that X is the truncated $U_q(\mathfrak{g})$-submodule of $V^{\otimes n}$, endowed with a Hilbert space structure, as we showed in the Chapter 3 of this work. We associate to X the map:

$$\widehat{X} : \widehat{\mathcal{C}(G,l)} \to \mathcal{B}(X)\ ,\ (\phi, \widehat{X}(\omega)\psi) = \omega([\phi^* \otimes \psi])$$

for $\phi, \psi \in V^{\underline{\otimes} n}$, $\omega \in \widehat{\mathcal{C}(G,l)}$. This formula extends the previously introduced $\widehat{V}$ to all objects of $\mathcal{F}_l$, and, as before, it is easily seen that $\widehat{X}$ is a unital *-representation of $\widehat{\mathcal{C}(G,l)}$ on $V^{\underline{\otimes} n}$. One has:

$$\widehat{X} = \widehat{V} \otimes \ldots \otimes \widehat{V} \circ \Delta_\ell^{(n-1)}$$

Hence $\widehat{X}$ is an object of $\mathrm{Rep}_V(\widehat{\mathcal{C}})$.

Theorem 4.4.15. *The functor $\mathcal{E} : \mathcal{F}_l \to \mathrm{Rep}_V(\widehat{\mathcal{C}(G,l)})$ sending X to $\widehat{X}$ and acting identically on the arrows is a tensor *-equivalence.*

Proof. An arrow $T \in (X, X')$ in $\mathcal{F}_l$ is an intertwiner of the corresponding modules of $U_q(\mathfrak{g})$. But it also lies in the arrow space $(\widehat{X}, \widehat{X}')$ of $\mathrm{Rep}(\widehat{\mathcal{C}}(G,l))$, as:

$$\begin{aligned}(\phi, T\widehat{X}(\omega)\psi) &= (T^*\phi, \widehat{X}(\omega)\psi) = \omega([(T^*\phi)^* \otimes \psi]) = \\ &= \omega([\phi^* \circ T \otimes \psi]) = \omega([\phi^* \otimes T\psi]) = (\phi, \widehat{X}'(\omega)T\psi)\end{aligned}$$

hence $\mathcal{E}$ is a functor between the stated categories, and it is easy to see that it is actually a *-functor since in a *-category the involution acts trivially on the objects. $\mathcal{E}$ is obviously faithful, so we verify that it has full image. In fact, let $\widehat{X}$ be a *-representation of $\widehat{\mathcal{C}}$. It is a routine computation to see that $X = \widehat{X} \circ \pi$ is a *-representation of $U_q(\mathfrak{g})$, where $\pi : U_q(\mathfrak{g}) \to \widehat{\mathcal{C}}$ is the surjection introduced before. So if $T \in (\widehat{X}, \widehat{X}')$, then $T\widehat{X}(\pi(a)) = \widehat{X}'(\pi(a))T$, and therefore T is an intertwiner of the corresponding $U_q(\mathfrak{g})$-representations. Essential surjectivity is very easy to prove, since the $\widehat{\mathcal{C}}$-representation $\widehat{X} = \widehat{V} \otimes \ldots \otimes \widehat{V} \circ \Delta^{(n)}$ is equivalent to $\mathcal{E}(X)$, where $X = p_n V^{\otimes n}$. Finally, it remains to prove that $\mathcal{E}$ is a tensor functor. Explicitly, since $\mathcal{F}_l$ is strict while $\mathrm{Rep}_V(\widehat{\mathcal{C}})$ is not, we look for natural isomorphisms $\mathcal{E}_{X,Y} \in (\widehat{X} \otimes \widehat{Y}, \widehat{X \otimes Y})$ such that:

$$\mathcal{E}_{W, X\otimes Y} \circ \mathrm{id}_{\widehat{W}} \otimes \mathcal{E}_{X,Y} \circ \Phi_{\widehat{W},\widehat{X},\widehat{Y}} = \mathcal{E}_{W\otimes X, Y} \circ \mathcal{E}_{X,Y} \otimes \mathrm{id}_{\widehat{Y}}$$

For $W = V^{\underline{\otimes} m}$, $X = V^{\underline{\otimes} n}$, $Y = V^{\underline{\otimes} r}$, we write:

$$\mathcal{E}_{X,Y} = \widehat{V}^{\otimes(n+r)}(Q_{n,r})$$

and we are reduced to look for a quasi invertible $Q_{n,r} \in \widehat{\mathcal{C}}^{\otimes(n+r)}$ satisfying the intertwining relation:

$$Q_{n,r}\Delta_\ell^{(n-1)} \otimes \Delta_\ell^{(r-1)}(\Delta(\omega)) = \Delta_\ell^{(n+r-1)}(\omega)Q_{n,r}$$

It is solved by:

$$Q_{n,r} = \Delta_\ell^{(n+r-1)}(I)\Delta_\ell^{(n-1)} \otimes \Delta_\ell^{(r-1)}(\Delta(I))$$

which also allows to obtain naturality. So it remains to check that $Q_{n,r}$ satisfies the tensorial equation:

$$Q_{m,n+r}(I_m \otimes Q_{n,r})\Delta_\ell^{(m-1)} \otimes \Delta_\ell^{(n-1)} \otimes \Delta_\ell^{(r-1)}(\Phi) = Q_{m+n,r}(Q_{m,n} \otimes I_r) \quad (4.4.31)$$

Using now the short notation, we have that $Q_{n,r} = \omega_{T_{n,r}}$, where:

$$T_{n,r} = p_{n+r} \circ \mathrm{id}_n \otimes S_\gamma \circ p_{n+1} \circ \mathrm{id}_n \otimes S_\gamma^*$$

and this expression shows that $Q_{n,r}$ is quasi invertible, so $\mathcal{E}_{X,Y}$ are isomorphisms. Furthermore $\Delta_\ell^{(m-1)} \otimes \Delta_\ell^{(n-1)} \otimes \Delta_\ell^{(r-1)}(\Phi)$ corresponds to:

$$S_\sigma \otimes S_{\sigma'} \otimes S_{\sigma''} \circ \mathrm{id} \otimes S_\gamma \circ p_2 \circ \mathrm{id} \otimes S_\gamma^* \circ p_3 \circ S_\sigma \otimes S_{\sigma'} \otimes S_{\sigma''}$$

and we next check the validity of the relation (4.4.31). At the left hand side we obtain:

$$\begin{aligned} p_{m+n+r} \circ 1_m \otimes S_\gamma \circ p_{m+h_\gamma} \circ 1_m \otimes S_\gamma^* \circ p_m \otimes 1_{n+r} \circ 1_m \otimes p_{n+r} \circ \\ \circ 1_{m+n} \otimes S_\delta \circ 1_m \otimes p_{n+h_\delta} \circ 1_{m+n} \otimes S_\delta^* \circ S_\sigma \otimes S_{\sigma'} \otimes S_{\sigma''} \circ \\ \circ 1_m \otimes S_\alpha \circ p_{h_\sigma + h_\alpha} \circ 1 \otimes S_\alpha^* \circ p_{h_\sigma + h_{\sigma'} + h_{\sigma''}} \circ S_\sigma^* \otimes S_{\sigma'}^* \otimes S_{\sigma''}^* \end{aligned}$$

which equals

$$p_{m+n+r} \circ 1_m \otimes S_{\sigma'} \otimes S_{\sigma''} \circ p_{m+h_{\sigma'}+h_{\sigma''}} \circ 1_m \otimes S_{\sigma'}^* \otimes S_{\sigma''}^*$$

by repeated use of Lemma 3.7.1 and Lemma 3.7.4. The right hand side becomes:

$$\begin{aligned} p_{m+n+r} \circ 1_{m+n} \otimes S_{\sigma''} \circ p_{m+n+h_{\sigma''}} \circ 1_{m+n} \otimes S_{\sigma''}^* \circ p_{m+n} \otimes p_r \circ \\ \circ 1_m \otimes S_{\sigma'} \otimes 1_r \circ p_{m+h_{\sigma'}} \otimes 1_r \circ 1_m \otimes S_{\sigma'}^* \otimes 1_r \end{aligned}$$

which in turn equals:

$$p_{m+n+r} \circ 1_{m+n} \otimes S_{\sigma''} \circ [p_{m+n+h_{\sigma''}} \circ (p_{m+n} \circ 1_m \otimes S_{\sigma'} \circ p_{m+h_{\sigma'}}) \otimes 1_{h_{\sigma''}}] \circ 1_m \otimes S_{\sigma'}^* \otimes S_{\sigma''}^*$$

It is easy to see that the expression in the square brackets can be rewritten as:

$$(p_{m+n} \circ 1_m \otimes S_{\sigma'} \circ p_{m+h_{\sigma'}}) \otimes 1_{h_{\sigma''}} \circ p_{m+h_{\sigma'}+h_{\sigma''}}$$

Substituting it in the right hand side formula and using the usual properties of projections we get the desired identity. □

4.5 Cosemisimplicity in type A case.

Crucial in the construction we carried out in the previous pages was the cosemisimplicity of $\mathcal{C}(G,l)$. In this section we will verify it for $G = SU(N)$. Our condition appeals to the existence of a Haar functional on $\mathcal{C}(G,l)$, and of the associative filtration $\widetilde{\mathcal{C}}_k$ built in this chapter. We prove the cosemisimplicity just in type A case because it is the easiest and most common case, but maybe not so much work will be required to prove the same result for the other Lie types (but not E_8).

4.5.1 A sufficient condition for cosemisimplicity.

It turns out useful to face the cosemisimplicity problem at the level of the filtration $\widetilde{\mathcal{C}}_k$, as this is a better behaved structure, in that it is provided with a multiplication. Of course, this involves the question of non-triviality of this filtration, or, more precisely, whether the image of $\widetilde{M^k_\lambda}$ of M_λ in $\widetilde{\mathcal{C}}_k$ under the quotient map $\mathcal{C}_k \to \widetilde{\mathcal{C}}_k$ is a matrix coalgebra for some $k \geq n$ and sufficiently many $\lambda \in \Lambda_l$.

Linear indipendence can be easily settled.

Proposition 4.5.1. (a) *The subcoalgebras M_λ are linearly independent in $\mathcal{C}$ as λ varies in Λ_l;*
(b) *$\widetilde{M_\lambda}$ are linearly independent in $\widetilde{\mathcal{C}}_k$ as $\lambda \in \Lambda^k_l$ for all k.*

Proof. It is sufficient to prove (b). (a) is easier and can be done along similar lines. Let $V_{\lambda,n}$ be the isotypic submodule of $p_n V^{\otimes n}$ of type V_λ, with orthogonal complement $V^\perp_{\lambda,n}$. The subspaces $V^*_{\lambda,n} \otimes V^\perp_{\lambda,n}$, $(V^\perp_{\lambda,n})^* \otimes V_{\lambda,n}$, and $V^*_{\lambda,n} \otimes V_{\lambda,n}$, for $n = 0, \dots, k$, are linearly independent in $\mathcal{D}_k$. Let W_λ denote their span. Consider the projection $E_\lambda : \mathcal{D}_k \to W_\lambda$ with complement:

$$\bigoplus_{n=0}^{k} (V^\perp_{\lambda,n})^* \otimes V^\perp_{\lambda,n}$$

The main point is that $\widetilde{\mathcal{J}}_k$ is stable under E_λ. This can be seen noticing that $\widetilde{\mathcal{J}}_k$ is linearly spanned by $\phi^* \otimes A\psi - \phi^* A \otimes \psi$, with $\psi \in V_{\lambda,n}$ and $\phi \in V_{\lambda,m}$ or $\phi \in V^\perp_{\lambda,n}$ and $\psi \in V^\perp_{\lambda,m}$; and also by $\phi^* \otimes Z\psi$ and $\phi^* Z \otimes \psi$, where $\phi \in V_{\lambda,n}$ and $\psi \in V^\perp_{\lambda,n}$, or $\phi \in V^\perp_{\lambda,n}$ and $\psi \in V_{\lambda,n}$, or $\phi, \psi \in V^\perp_{\lambda,n}$. Therefore if E_λ acts on elements of these types, then it annihilates them or it acts as the identity. Hence $E_\lambda(\widetilde{\mathcal{J}}_k) \subseteq \widetilde{\mathcal{J}}_k$. □

Cosemisimplicity in the generic case was studied by means of the Haar functional. We look for a generalisation of that approach to the present setting. Notice that if one can establish that $\widetilde{M^k_\lambda}$ is a matrix coalgebra in $\widetilde{\mathcal{C}}_k$ then M_λ is a matrix coalgebra in $\mathcal{C}(G, \ell)$ as well, by dimension count. We shall verify cosemisimplicity in this stronger form.

Definition 4.5.2. A linear functional h on $\widetilde{\mathcal{C}}_k$ is said to be a Haar functional if $h(I) = 1$ and h annihilates the subcoalgebras $\widetilde{M^k_\lambda}$ for $\lambda \in \Lambda^k_l \setminus \{0\}$.

Obviously a Haar functional on $\widetilde{\mathcal{C}}_k$ is unique. Furthermore, if $\widetilde{\mathcal{C}}_k$ admits a Haar functional then so does $\widetilde{\mathcal{C}}_h$ for $h < k$. For a given $\lambda \in \Lambda_l$, let $\deg(\lambda)$ denote the smallest integer k such that $\lambda \in \Lambda^k_l$, or, in other words, such that V_λ is a summand of $V^{\otimes k}$. Furthermore, set:

$$m(\mathfrak{g}, l) := \max\{\deg(\lambda), \lambda \in \Lambda_l\}$$

Definition 4.5.3. We will say that the pair $(\mathfrak{g}, l)$ satisfies the cosemisimplicity condition if

(1) there is $\widetilde{m} \geq m(\mathfrak{g}, l)$ such that $\widetilde{\mathcal{C}}_{\widetilde{m}}$ admits a Haar functional,

(2) every $\lambda \in \Lambda_l$ has a conjugate $\overline{\lambda} \in \Lambda_l$ satisfying $\deg(\lambda) + \deg(\overline{\lambda}) \leq \widetilde{m}$.

Lemma 4.5.4. *Let $\lambda \in \Lambda_l^k$ have a conjugate $\overline{\lambda} \in \Lambda_l^h$ such that $\widetilde{\mathcal{C}}_{h+k}$ admits a Haar functional. Then $\widetilde{M}_\lambda^k$ is a matrix coalgebra in $\widetilde{\mathcal{C}}_k$.*

Proof. Let V_λ be a summand of $V^{\otimes n}$, $n \leq k$. Let $r : \mathbb{C} \to V_{\overline{\lambda}} \otimes V_\lambda$ be as in (3.6.2). The composed arrow $r^* r : \mathbb{C} \to V_{\overline{\lambda}} \otimes V_\lambda \to \mathbb{C}$ is nonzero since $\lambda \in \Lambda_l$. More precisely:

$$r^* r(1) = \sum_{i=1}^{N} (\xi_i, K_{2\rho}^{-1} \xi_i) = \mathrm{tr}(K_{2\rho^{-1}}) = \dim(V_\lambda) = N$$

In particular, the trivial submodule defined by r is a summand of $V_{\overline{\lambda}} \otimes V_\lambda$. But $(1 - p_{h+n}) V_{\overline{\lambda}} \otimes V_\lambda$ can not contain a trivial submodule, as otherwise it would be a summand, by multiplicity count. This shows that $r \in p_{h+n} V^{\otimes h+n}$. If a linear combination $x = \sum_{i,j} \mu_{i,j} e_{i,j}^\lambda = 0$ vanishes in $\widetilde{\mathcal{C}}_k$ then $h(ax) = 0$ for all $a \in \widetilde{\mathcal{C}}_h$, where h is a Haar functional for $\widetilde{\mathcal{C}}_{h+k}$. Now computations analogous to those at the end of the proof of the Theorem 4.1.8 show that $\mu_{i,j} = 0$. □

Theorem 4.5.5. *If $(\mathfrak{g}, l)$ satisfies the cosemisimplicity condition in Definition 4.5.3, then:*

(a) *$\widetilde{M}_\lambda^k$ is a matrix coalgebra in $\widetilde{\mathcal{C}}_k$ for $k = \deg(\lambda)$,*

(b) *M_λ is a matrix coalgebra in $\mathcal{C}(G, l)$, for all $\lambda \in \Lambda_l$, hence $\mathcal{C}(G, l)$ is cosemisimple:*

$$\mathcal{C}(G, l) = \bigoplus_{\lambda \in \Lambda_l} M_\lambda$$

Proof. The proof follows from Proposition 4.5.1 and Lemma 4.5.4 □

4.5.2 The case $G = SU(N)$.

The rest of the section is dedicated to the proof of the following theorem, which concludes the main result of the paper.

Theorem 4.5.6. *If $\mathfrak{g} = \mathfrak{sl}_N$ then $(\mathfrak{g}, l)$, satisfies the cosemisimplicity condition for all $N \geq 2$ and $l \geq N + 1$ with $m(\mathfrak{g}, l) = (N-1)(l-N)$ and $\widetilde{m} := m(\mathfrak{g}, l) + l - 1$.*

We start fixing notation of type A_{N-1} root systems [Humphreys]. Consider $\mathbb{R}^N$ with the usual euclidean inner product, and let $e_1, \ldots, e_N$ be the canonical orthonormal basis. Consider the subspace $E \subset \mathbb{R}^N$ of elements $\mu_1 e_1 + \cdots + \mu_N e_N$ such that $\mu_1 + \cdots + \mu_N = 0$. The A_{N-1} root system is $\Phi = \{e_i - e_j, i \neq j\}$, the simple roots are $\alpha_i = e_i - e_{i+1}$, and the fundamental weights are $\omega_i = e_1 + \cdots + e_i - \frac{i}{N} e$, where $e := e_1 + \cdots + e_N$ and $i = 1, \ldots, N-1$. The weight lattice and the dominant Weyl chamber of (E, Φ) are respectively:

$$\Lambda = \{\lambda = \lambda_1 e_1 + \cdots + \lambda_{N-1} e_{N-1} - \frac{\lambda_1 + \cdots + \lambda_{N-1}}{N} e, \quad \lambda_i \in \mathbb{Z}\}$$

and:

$$\Lambda^+ = \{\lambda \in \Lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N-1} \geq 0\}$$

The highest root is $\theta = e_1 - e_N$, and:

$$\rho = \frac{1}{2} \sum_{i>j} (e_i - e_j) = \frac{1}{2}((N-1)e_1 + \ldots + (N - 2i + 1)e_i + \ldots + (-N+1)e_N)$$

In general we know that:

$$\Lambda_l = \{\lambda \in \Lambda^+ : \langle \lambda + \rho, \theta \rangle < dl\}$$

Direct computations bring to:

$$\Lambda_l = \{\lambda \in \Lambda^+ : \lambda_1 < l - N + 1\}, \quad \overline{\Lambda_l} = \{\lambda \in \Lambda^+ : \lambda_1 \leq l - N + 1\}$$

We consider the vector representation $V = V_{\omega_1}$. Its weights are:

$$\gamma_1 = \omega_1 \ , \gamma_i = e_i - \frac{1}{N} e \ , i = 2, \ldots, N$$

In the next lemmas we shall make use of the decomposition into irreducibles:

$$V_\lambda \otimes V \simeq \bigoplus_i V_{\lambda + \gamma_i} \quad \lambda \in \Lambda_l$$

in the category $\mathcal{T}_l$ of tilting modules, where the sum is extended to all i such that $\lambda + \gamma_i \in \overline{\Lambda_l}$ (Theorem 3.5.13). We derive two simple consequences.

Lemma 4.5.7. *For any $\lambda \in \Lambda_l$, the negligible submodule N_λ of $V_\lambda \otimes V$ is non-zero if and only if $\lambda_1 = l - N$, and one has $N_\lambda \simeq V_{\lambda + \omega_1}$.*

Proof. We set $\lambda^{(i)} = \lambda + \gamma_i$ when $i \geq 2$. We have:

$$\lambda^{(i)} = \lambda_1 e_1 + \ldots + \lambda_{i-1} e_{i-1} + (\lambda_i + 1) e_i + \lambda_{i+1} e_{i+1} + \ldots + \lambda_{N-1} e_{N-1} - \frac{\lambda_1 + \ldots + \lambda_{N-1} + 1}{N} e$$

Since $\lambda_1 < l - N + 1$ and $\lambda^{(i)} \in \overline{\Lambda_l} \setminus \Lambda_l$ if and only if $\lambda_1 = l - N + 1$, we conclude that $\lambda^{(i)} \notin \overline{\Lambda_l} \setminus \Lambda_l$. We focus now on $\lambda^{(1)} = \lambda + \omega_1$. We have:

$$\lambda^{(1)} = (\lambda_1 + 1) e_1 + \lambda_2 e_2 + \ldots + \lambda_{N-1} e_{N-1} - \frac{\lambda_1 + \ldots + \lambda_{N-1} + 1}{N} e$$

Therefore $\lambda^{(1)} \in \overline{\Lambda_l} \setminus \Lambda_l$ if and only if $\lambda_1 = l - N$. $\square$

Lemma 4.5.8. $m(\mathfrak{sl}_N, l) = (N-1)(l-N)$.

Proof. Let $\lambda \in \Lambda_l$ be determined by non negative integers $\lambda_1, \ldots, \lambda_{N-1}$ as above, and let us identify λ with $(\lambda_1, \ldots, \lambda_{N-1})$. The dominant weight with coordinates all equal to $l - N$ lies in Λ_l, it is a summand of $V^{\otimes (N-1)(l-N)}$ and this is the smallest possible power. This fact is merely a consequence of the decomposition $V_\lambda \otimes V \simeq \bigoplus_i V_{\lambda + \gamma_i}$, with $\lambda \in \Lambda_l$. We need to show that every module V_λ with $\lambda \in \Lambda_l$ is a summand of some $p_n V^n$ with $n \leq (N-1)(l-N)$. Notice that $(1, 0, \ldots, 0), \ldots,$ $(\lambda_1, 0, \ldots, 0)$, $(\lambda_1, 1, 0, \ldots, 0), \ldots,$ $(\lambda_1, \lambda_2, \ldots, 0)$, $\ldots$, $(\lambda_1, \ldots, \lambda_{N-1})$ is a sequence of $\lambda_1 + \lambda_2 + \cdots + \lambda_{N-1}$ dominant weights of Λ_l starting with ω_1 and obtained from one another by adding a weight of V. The fusion rules then show that V_λ is a summand of $p_n V^n$, where $n = \lambda_1 + \lambda_2 + \cdots + \lambda_{N-1} \leq (N-1)(l-N)$. $\square$

We next derive information on the negligible summands of $V^{\otimes n}$, including the non-canonical ones, for the bounded values of n. The following Lemma plays a crucial role for the Haar functional.

Lemma 4.5.9. *No negligible summand of $V^{\otimes n}$, with n up to $\widetilde{m} = m(\mathfrak{sl}_N, l) + l - 1$, contains the trivial module among the successive factors of its Weyl filtrations.*

Proof. A negligible summand of $V^{\otimes n}$ is isomorphic to a summand of:

$$N_n = (\mathrm{id} - p_n)V^{\otimes n}$$

Furthermore the inductive procedure described in Sect. 3.7 shows that N_n is in turn spanned by the summands:

$$N(p_r V^{\otimes r} \otimes V) \otimes V^{\otimes(n-r-1)}, \quad r = 1, \dots, n-1 \tag{4.5.1}$$

where $N(p_r V^{\otimes r} \otimes V)$ is the canonical negligible summand of $p_r V^{\otimes r} \otimes V$, hence we are reduced to show the statement for these modules. Using Lemma 4.5.7, we know that $N(p_r V^{\otimes r} \otimes V)$ is completely reducible and the dominant weights of the irreducible components are of the form $\lambda + \omega_1 = (l - N + 1, \lambda_2, \dots, \lambda_{N-1})$. On the other hand, the dominant weights appearing in the Weyl filtrations of (4.5.1) are the same as those appearing in the irreducible decomposition of the corresponding module at the level of the semisimple category $\mathrm{Rep}(\mathfrak{g})$, see Prop. 3 and Remark 2 in [Sawin]. Hence we are reduced to show that the smallest integer t such that $V_{\lambda+\omega_1} \otimes V^{\otimes t}$ contains the trivial module in $\mathrm{Rep}(\mathfrak{g})$ satisfies:

$$t + r + 1 > (N-1)(l-N) + l - 1$$

where $\lambda + \omega_1$ has the form shown above. We compute t. For a general dominant weight $\mu = (\mu_1, \dots, \mu_{N-1})$, the shortest path to the trivial module is obtained as follows. If $\mu_{N-1} > 0$ we consider the path:

$$\mu + \gamma_N, \quad \mu + 2\gamma_N, \quad \dots, \quad \mu + \mu_{N-1}\gamma_N$$

which lowers μ to:

$$\mu' = (\mu_1 - \mu_{N-1}, \dots, \mu_{N-2} - \mu_{N-1}, 0)$$

and we have thus used μ_{N-1} powers of V. In fact, recall that $\gamma_N = e_N - \frac{1}{N}e$. Then:

$$\begin{aligned}\mu + \gamma_N &= \mu_1 e_1 + \ldots + \mu_{N-1}e_{N-1} - \frac{\mu_1 + \ldots + \mu_{N-1}}{N}e + e_N - \frac{1}{N}e = \\ &= (\mu_1 - 1)e_1 + \ldots + (\mu_{N-1} - 1)e_{N-1} - \frac{\mu_1 + \ldots + \mu_{N-1} - N + 1}{N}e\end{aligned}$$

So previous calculation explains the expression of μ'. At this stage we need no powers of V if $\mu_{N-1} = 0$. We proceed in the same way for the $N-2$ coordinate and the new weight μ', but we now need to follow a longer path, due to vanishing of the last coordinate, and the shortest is:

$$\mu' + \gamma_{N-1}, \quad \mu' + \gamma_{N-1} + \gamma_N, \quad \mu' + 2\gamma_{N-1} + \gamma_N, \quad \mu' + 2\gamma_{N-1} + 2\gamma_N, \dots$$

using $2(\mu_{N-2} - \mu_{N-1})$ more powers of V. Continuing in this way, we find:

$$t = \mu_{N-1} + 2(\mu_{N-2} - \mu_{N-1}) + 3(\mu_{N-3} - \mu_{N-2}) + \cdots + (N-1)(\mu_1 - \mu_2)$$

Now we know that $\mu = \lambda + \omega_1$ and appears as dominant weight in $V^{\otimes r+1}$. Therefore $\mu_1 + \ldots + \mu_{N-1} \leq r + 1$, so:

$$t + r + 1 \geq t + \mu_1 + \cdots + \mu_{N-1} = N\mu_1 = N(l - N + 1) = (N-1)(l-N) + l$$

which finally gives the desired estimate. $\square$

Remark 4.5.10. Using the last Lemma and Lemma 4.5.7 we have:

$$N(p_r V^{\otimes r} \otimes V) = 0 \text{ if } r < l - N$$

Corollary 4.5.11. *Let* $n \leq \widetilde{m}$.

(a) e_n *is a central element of* $(V^{\otimes n}, V^{\otimes n})$,

(b) $e_n \circ \mathrm{id}_{V^q} \otimes (1 - p_j) \otimes \mathrm{id}_{V^u} = 0$, $q + j + u = n$.

Proof. (a) The multiplicity of the trivial representation in $p_n V^n$ is the same as that of the classical case, by the proof of the previous lemma, hence e_n is the specialisation of a central intertwiner of the generic case. (b) We know that:

$$e_n \circ \mathrm{id}_{V^q} \otimes (1 - p_j) \otimes \mathrm{id}_{V^u} \circ p_n = p_0 \circ e_n \circ \mathrm{id}_{V^q} \otimes (1 - p_j) \otimes \mathrm{id}_{V^u} \circ p_n$$

Now, using Lemma 3.7.4, we have:

$$p_0 \circ e_n \circ \mathrm{id}_{V^q} \otimes (1 - p_j) \otimes \mathrm{id}_{V^u} \circ p_n = p_0 \circ e_n \circ p_n - p_0 \circ e_n \circ \mathrm{id}_{V^q} \otimes p_j \otimes \mathrm{id}_{V^u} \circ p_n =$$
$$= p_0 \circ e_n \circ p_n - p_0 \circ e_n \circ p_n = 0$$

so $e_n \circ \mathrm{id}_{V^q} \otimes (1 - p_j) \otimes \mathrm{id}_{V^u} \circ p_n = 0$. Putting together this fact with the centrality of e_n we get:

$$e_n \circ \mathrm{id}_{V^q} \otimes (\mathrm{id}_{V^j} - p_j) \otimes \mathrm{id}_{V^u} = e_n \circ \mathrm{id}_{V^q} \otimes (\mathrm{id}_{V^j} - p_j) \otimes \mathrm{id}_{V^u} \circ (\mathrm{id}_{V^n} - p_n) =$$
$$= \mathrm{id}_{V^q} \otimes (\mathrm{id}_{V^j} - p_j) \otimes \mathrm{id}_{V^u} \circ e_n \circ (\mathrm{id}_{V^n} - p_n) = 0$$

□

We next consider the $\widetilde{m}$-th term, $\widetilde{\mathcal{C}}_{\widetilde{m}}$, of the associative filtration $\widetilde{\mathcal{C}}_k$ corresponding to $\mathcal{C}(\mathrm{SU}(N), l)$. For convenience we recall that $\widetilde{\mathcal{C}}_{\widetilde{m}} = \mathcal{D}_{\widetilde{m}} / \widetilde{\mathcal{J}}_{\widetilde{m}}$, where $\widetilde{\mathcal{J}}_{\widetilde{m}}$ is spanned by elements of the form $\phi \otimes A \circ \psi - \phi \circ A \otimes \psi$, together with $\phi \otimes Z \circ \psi'$, and $\phi' \circ Z' \otimes \psi$, where:

$$A \in (V^{\underline{\otimes} m}, V^{\underline{\otimes} n}), \qquad Z,\, Z'^{*} = p_{q+j+u} \circ 1_{V^q} \otimes (1 - p_j) \otimes 1_{V^u}$$

with m, n, $q + j + u \leq \widetilde{m}$.

We define the linear functional:

$$h : \mathcal{D}_{\widetilde{m}} \to \mathbb{C}$$

setting:

$$h(\phi \otimes \psi) = \phi(e_n \psi), \quad \phi \otimes \psi \in (V^n)^* p_n \otimes p_n V^n, \quad n \leq \widetilde{m}$$

where $e_n \in (p_n V^n, p_n V^n)$ is the orthogonal projection onto the isotypical component of the trivial representation.

Theorem 4.5.12. *The functional* h *annihilates* $\widetilde{\mathcal{J}}_{\widetilde{m}}$. *Hence it gives rise to a Haar functional on* $\widetilde{\mathcal{C}}_{\widetilde{m}}$.

Proof. Using the centrality of e_n, it is clear that h annihilates elements $\phi \otimes A\psi - \phi A \otimes \psi$. Furthermore, it also annihilates elements of the form $\phi \otimes Z\psi'$, $\phi' Z' \otimes \psi \in \widetilde{\mathcal{J}}_{\widetilde{m}}$ by (b) of the last Corollary. □

We finally verify the needed upper bound for $\deg(\lambda) + \deg(\overline{\lambda})$ for all $\lambda \in \Lambda_l$. It will be given by (c) of the following:

Proposition 4.5.13. *If* $\lambda = (\lambda_1, \ldots, \lambda_{N-1}) \in \Lambda^+$ *then:*

(a) $\deg(\lambda) = \lambda_1 + \cdots + \lambda_{N-1}$,

(b) $\overline{\lambda} = (\lambda_1, \lambda_1 - \lambda_{N-1}, \lambda_1 - \lambda_{N-2}, \ldots, \lambda_1 - \lambda_2)$,

(c) $\deg(\lambda) + \deg(\overline{\lambda}) = N\lambda_1 \leq \widetilde{m}$ *for* $\lambda \in \Lambda_l$.

Proof. (a) is a classical result. Let us pass to (b). Let w_0 be longest element of the Weyl group. For $\mathfrak{sl}_N$, this is the permutation group $\mathbb{P}_N$ and w_0 is the permutation reversing the order of $(e_1, \ldots, e_N)$. Then:

$$\overline{\lambda} = -w_0\lambda = -(\lambda_1 e_N + \lambda_2 e_{N-1} + \cdots + \lambda_{N-1} e_2) + \frac{\lambda_1 + \cdots + \lambda_{N-1}}{N} e =$$
$$= (\lambda_1, \lambda_1 - \lambda_{N-1}, \lambda_1 - \lambda_{N-2}, \ldots, \lambda_1 - \lambda_2)$$

□

Chapter 5

The quantum groupoid $\mathcal{C}(SU(2), l)$: generators and relations.

5.1 Representation theory of $U_q(\mathfrak{sl}_2)$ for q root of unity

We shall write down $\mathcal{C}(G, l)$ introduced in previous chapter by generators and relations, in the case $\mathfrak{g} = \mathfrak{sl}_2$ and $l = N + 1$. The presentation of $\widehat{\mathcal{C}}(SU_2, N + 1)$ will be obtained simply passing to the dual, and we will do it in a peculiar example. The case $\mathfrak{sl}_2$ is the most workable, since the decompositions of the tensor products of irreducible representations are multiplicity free, and there is a canonical choice of truncated tensor products. Our main references in this chapter are the works of Andersen and his collaborators: [2]–[7]. For classical results about Lie theory we refer to [32], and [35] for the quantum case. Other very useful references for specific results about $\mathfrak{sl}_2$ case are [25] and [70]. First of all, we recall the definition of $U_q(\mathfrak{g})$ in the case $\mathfrak{g} = \mathfrak{sl}_2$.

Definition 5.1.1. We denote by $U_x = U_x(\mathfrak{sl}_2)$ the associative, unital $\mathbb{C}(x)$-algebra generated by E, F, K, K^{-1} subject to the following relations:

$$\begin{aligned} &KK^{-1} = K^{-1}K = 1 \\ &EF - FE = \frac{K - K^{-1}}{x - x^{-1}} \\ &KE = x^2 EK \, , KF = x^{-2} FK \end{aligned} \tag{5.1.1}$$

It is well-known that U_x has the following Hopf *-algebra structure:

$$\begin{aligned} &\Delta(E) = E \otimes 1 + K \otimes E \, , \Delta(F) = F \otimes K^{-1} + 1 \otimes F \, , \Delta(K) = K \otimes K \\ &S(E) = -K^{-1}E \, , S(F) = -FK \, , S(K) = K^{-1} \\ &\varepsilon(E) = \varepsilon(F) = 0 \, , \varepsilon(K) = 1 \\ &E^* = F \, , F^* = E \, , K^* = K^{-1} \end{aligned} \tag{5.1.2}$$

As we did in the general case, we want to specialize this definition in the case of q root of unity. We set $A = \mathbb{C}[x, x^{-1}]$ and define for all $j \in \mathbb{N}$ the j-th divided powers:

$$E^{(j)} = \frac{E^j}{[j]!} \, , F^{(j)} = \frac{F^j}{[j]!}$$

We call U_A the A-subalgebra of U_x generated by K, K^{-1}, $E^{(j)}$ and $F^{(j)}$. We fix now $q \in \mathbb{T}$, where q is a root of unity of order $2N+2$, with $N > 0$. This means that we will work only with even roots of unity of order ≥ 4.

Definition 5.1.2. Let q be as above, and consider $\mathbb{C}$ as an A-module by specializing x to q. Then we define:

$$U_q = U_A \otimes_A \mathbb{C}$$

So, the generators of U_q will be $E^{(j)} \otimes 1$ and analogously for the others, but we will abuse notation and keep on calling them $E^{(j)}$ and so on.

Remark 5.1.3. It is easy to see that U_q has a Hopf structure inherited from U_x. Moreover, since $q^{2N+2} = 1$, we have $K^{2N+2} = 1$ and $E^{N+1} = F^{N+1} = 0$. In fact, $E^{N+1} = [N+1]!E^{(N+1)} = 0$ because $[j] = 0$ iff $[N+1] = 0$. In this way we can see that this approach is equivalent to the Reshetikhin-Turaev's one, with the only remarkable difference that we call q what they call t and q^2 what they call q.

Definition 5.1.4. Let V be a U_q-module and λ be a scalar. An element $v \neq 0$ in V is a highest weight vector of weight λ if $Ev = 0$ and $Kv = \lambda v$.

Proposition 5.1.5. *A non-zero U_q-module V contains a highest weight vector.*

Proof. Since $\mathbb{C}$ is algebraically closed and V is finite-dimensional, there exists a non-zero vector v_0 and a scalar α such that $Kv_0 = \alpha v_0$. Let n be the lowest integer positive number such that $E^n v_0 = 0$. It is at most $N+1$, since $E^{N+1} = 0$. So, setting $v = E^{n-1}v_0$, we have that v is a highest weight vector. □

Lemma 5.1.6. *Let v be a highest weight vector of weight λ. Set $v_0 = v$ and $v_p = \frac{1}{[p]!}F^p v$ for $p > 0$. Then:*

$$Kv_p = \lambda q^{-2p} v_p\ ,\ Ev_p = \frac{q^{-(p-1)}\lambda - q^{p-1}\lambda^{-1}}{q - q^{-1}} v_{p-1}\ ,\ Fv_{p-1} = [p]v_p$$

Proof. It is a direct consequence of (5.1.1). □

Theorem 5.1.7. *Let V be a U_q-module generated by a highest weight vector v of weight λ, such that* $\dim(V) = n+1 < N+1$. *Then:*
(a) *The scalar λ is of the form $\lambda = \epsilon q^n$, where $\epsilon = \pm 1$.*
(b) *Setting $v_p = \frac{F^p}{[p]!}v$, we have $v_p = 0$ for $p > n$ and, in addition, the set $\{v_0, \ldots, v_n\}$ is a basis of V.*
(c) *K acting on V is diagonalizable with $n+1$ distinct eigenvalues $\{\epsilon q^n, \epsilon q^{n-2}, \ldots, \epsilon q^{-n}\}$.*
(d) *Any other highest weight vector in V is a scalar multiple of v and is of weight λ.*
(e) *The module V is simple.*
Any simple finite-dimensional U_q-module V is generated by a highest weight vector. Two finite-dimensional U_q-modules generated by highest weight vectors of the same weight are isomorphic.

Proof. (a),(b) Using (5.1.1), we have that the sequence $\{v_p\}_p$ is a sequence of eigenvectors for K with distinct eigenvalues, so they are linearly independent. Since V is finite-dimensional, there has to exist a m such that $v_m \neq 0$ and $v_{m+1} = 0$ (and $v_l = 0$ if $l > m$ using the last Lemma). Hence:

$$0 = Ev_{m+1} = \frac{q^{-m}\lambda - q^n\lambda^{-1}}{q - q^{-1}} v_m$$

which is equivalent to require that $\lambda = \epsilon q^m$. Since $\dim(V) = n+1$, $m \leq n$. But m must be n, since V is generated as a module by v_0, so any element in V must be a linear combination of v_i. (c) is straightforward to prove now, using the Lemma 5.1.6 and (5.1.1). (d) Let v' another highest weight vector. It is an eigenvector for the action of K; hence, it is a scalar multiple of some v_i because of (c). Using once again Lemma 5.1.6 we have that v_i is killed by E iff $i = 0$. (e) Let V' be a non-zero U_q-submodule of V and let v' be a highest weight vector of V'. Then v' is also a highest weight vector for V. By (d), v' has to be a scalar multiple of v. Therefore $v \in V'$. Since v generates V, we have $V \subseteq V'$, which proves that V is simple. Finally, we can prove the last statement. Let v be a highest weight vector of V; if V is simple, then the submodule generated by v is necessarily equal to V. Consequently, V is generated by a highest weight vector. If V and V' are generated by highest weight vectors v and v' with the same weight λ, then the linear map sending v_i to v'_i for all i is an isomorphism of U_q-modules. □

Remark 5.1.8. It is important to notice that there are two types of U_q-module V_n, of type 1 and type -1. In the first one, the highest weight vector has eigenvalue q^n, and in the second one it has eigenvalue $-q^n$. Kassel usual indicates them with $V_{1,n}$ and $V_{-1,n}$. In [R-T] there are four types: $1, -1, i, -i$, since they call q our q^2. Anyway, we can just deal with type 1 modules, since $V_{1,n}$ and $V_{-1,n}$ are unitarily equivalent as objects in a C*-tensor category.

Proposition 5.1.9. *Any simple non-zero U_q-module of dimension $< N+1$ is isomorphic to a module of the form V_n, where $0 \leq n \leq N-1$. Moreover, there is no simple finite-dimensional U_q-module of dimension $> N+1$.*

Proof. The first statement has been proved in the Theorem 5.1.7. More effort is required to prove the second statement. Let us assume that there exists a simple finite-dimensional module V of dimension $> N+1$. We shall prove that V has a non-zero submodule of dimension $\leq N+1$. It is possible to prove that there exists a non-zero eigenvector $v \in V$ for the action of K such that $Fv = 0$. In fact, it is well-known from the Lie theory that there exists a non-zero eigenvector v_0 for the action of K, since K is semisimple, V is finite-dimensional and $\mathbb{C}$ is algebraically close. If we consider $v = F^N v_0$, it is still a K-eigenvector, and $Fv = 0$. We claim that the subspace V' generated by $\{v, Ev, \ldots, E^N v\}$ is a submodule of dimension $\leq N$. It is enough that it is stable under the action of generators E, F and K. This is straightforward using (5.1.1). □

According to the terminology used in the Chapter 3, $V_n = V_n(q)$ will be called the n-th Weyl module. This is the linear space with basis $\xi_0, \ldots, \xi_n$ and the U_q-action defined by:

$$K\xi_k = q^{n-2k}\xi_k \ , \ E^{(j)}\xi_k = \frac{[n-k+j]!}{[j]![n-k]!}\xi_{k-j} \ , \ F^{(j)}\xi_k = \frac{[k+j]!}{[j]![k]!}\xi_{k+j}$$

For $\mathfrak{g} = \mathfrak{sl}_2$ we can explicitly give the Weyl modules which generate the quotient category. In this case we just have one simple root α. So $d = 1$, where d is the ratio of the square length of a long root to a short root. Moreover, $\Lambda^+ = \frac{\mathbb{Z}}{2}\alpha$, $\rho = \frac{\alpha}{2}$ and $\theta = \alpha$. Hence:

$$\lambda \in \overline{\Lambda}_{N+1} \Longleftrightarrow \left\langle \lambda + \frac{\alpha}{2}, \alpha \right\rangle \leq N+1$$

Since $\langle\cdot,\cdot\rangle$ is linear on the left, $\langle\alpha,\alpha\rangle = 2$ and $\lambda = \frac{k}{2}\alpha$ ($k \in \mathbb{N}$), we get:

$$\lambda \in \overline{\Lambda}_{N+1} \Longleftrightarrow k \leq N$$

and $\lambda \in \Lambda_{N+1}$ iff $k < N$. So, every object in the quotient category is the direct sum of some Weyl modules V_n, where $n < N$. The Weyl module V_1 is the vector representation and it is the fundamental one. V_N is negligible. Now we want to know how we can decompose $V_{j_1} \otimes V_{j_2}$, when $j_1 + j_2 \leq N$.

Theorem 5.1.10. *If $j_1 + j_2 \leq N$, then:*

$$V_{j_1} \otimes V_{j_2} = \bigoplus_{p=0}^{\min(j_1,j_2)} V_{j_1+j_2-2p}$$

Proof. It is possible to prove the statement as for a generic q. All the highest weight vectors in $V_{j_1} \otimes V_{j_2}$ are of the following form:

$$v_0^{(j_1+j_2-2p)} = \sum_{i=0}^{p} (-1)^i \frac{[j_1-i]![j_2-p+i]!}{[j_1]![j_2-p]!} q^{i(j_1-i+1)} v_i^{(j_1)} \otimes v_{p-i}^{(j_2)}$$

where p is an integer, $0 \leq p \leq \min(j_1, j_2)$. It easy to see that $v_0^{(j_1+j_2-2p)}$ has weight $q^{j_1+j_2-2p}$, since $v_i^{(j_1)} \otimes v_{p-i}^{(j_2)}$ has weight $q^{j_1-2i+j_2-2p+2i} = q^{j_1+j_2-2p}$. Let us prove now that $\Delta(E)v_0^{(j_1+j_2-2p)} = 0$. Recall that $\Delta(E) = E \otimes 1 + K \otimes E$. It follows that:

$$\begin{aligned}
\Delta(E)v_0^{(j_1+j_2-2p)} &= \sum_{i=0}^{p} (-1)^i \frac{[j_1-i]![j_2-p+i]!}{[j_1]![j_2-p]!} q^{i(j_1-i+1)} E v_i^{(j_1)} \otimes v_{p-i}^{(j_2)} + \\
&+ \sum_{i=0}^{p} (-1)^i \frac{[j_1-i]![j_2-p+i]!}{[j_1]![j_2-p]!} q^{i(j_1-i+1)} K v_i^{(j_1)} \otimes E v_{p-i}^{(j_2)} = \\
&= \sum_{i=0}^{p} (-1)^i \frac{[j_1-i+1]![j_2-p+i]!}{[j_1]![j_2-p]!} q^{i(j_1-i+1)} v_{i-1}^{(j_1)} \otimes v_{p-i}^{(j_2)} + \\
&+ \sum_{i=0}^{p} (-1)^i \frac{[j_1-i]![j_2-p+i+1]!}{[j_1]![j_2-p]!} q^{(i+1)(j_1-i)} v_i^{(j_1)} \otimes v_{p-i-1}^{(j_2)} = \\
&= \sum_{i=1}^{p+1} -(-1)^i \frac{[j_1-i]![j_2-p+i+1]!}{[j_1]![j_2-p]!} q^{(i+1)(j_1-i)} v_i^{(j_1)} \otimes v_{p-i-1}^{(j_2)} + \\
&+ \sum_{i=0}^{p} (-1)^i \frac{[j_1-i]![j_2-p+i+1]!}{[j_1]![j_2-p]!} q^{(i+1)(j_1-i)} v_i^{(j_1)} \otimes v_{p-i-1}^{(j_2)} = 0
\end{aligned}$$

So, for all p such that $0 \leq p \leq \min(j_1, j_2)$, there exists a non-zero morphism of modules from $V_{j_1+j_2-2p}$ into $V_{j_1} \otimes V_{j_2}$. Being $V_{j_1+j_2-2p}$ simple, the morphism must be an embedding into $V_{j_1} \otimes V_{j_2}$, since its kernel must be zero. The submodules $V_{j_1+j_2-2p}$ are simple and of distinct weights, so their sum in $V_{j_1} \otimes V_{j_2}$ is direct. Since their direct sum has the same dimension of $V_{j_1} \otimes V_{j_2}$, we can conclude. □

Now it is time to shed light about some facts that will be useful later. Considering the last theorem, it is clear that we have two interesting bases in $V_{j_1} \otimes V_{j_2}$. One is inherited from the tensor product structure:

$$\{v_i^{(j_1)} \otimes v_h^{(j_2)}\}_{0 \leq i \leq j_1, 0 \leq h \leq j_2}$$

and the other one is inherited from the decomposition:

$$v_k^{(j_1+j_2-2p)} = \frac{1}{[k]!} F^k v_0^{(j_1+j_2-2p)}$$

where $0 \leq p \leq \min(j_1, j_2)$ and $0 \leq k \leq j_1 + j_2 - 2p$. If we want to pass from one basis to another, we need to introduce the so-called *quantum Clebsch-Gordan coefficients*:

$$\begin{bmatrix} j_1 & j_2 & j_1+j_2-2p \\ i & h & k \end{bmatrix}$$

defined for $0 \leq p \leq \min(j_1, j_2)$ and $0 \leq k \leq j_1 + j_2 - 2p$ by:

$$v_k^{(j_1+j_2-2p)} = \sum_{0\leq i\leq j_1, 0\leq h\leq j_2} \begin{bmatrix} j_1 & j_2 & j_1+j_2-2p \\ i & h & k \end{bmatrix} v_i^{(j_1)} \otimes v_h^{(j_2)}$$

We discover now some properties of these coefficients, also called *quantum 3j-symbols*.

Lemma 5.1.11. *Fix* p *and* k. *The vector* $v_k^{(j_1+j_2-2p)}$ *is a linear combination of vectors of the form* $v_i^{(j_1)} \otimes v_{p-i+k}^{(j_2)}$. *Therefore, we have:*

$$\begin{bmatrix} j_1 & j_2 & j_1+j_2-2p \\ i & h & k \end{bmatrix} = 0$$

when $i + h \neq p + k$. *We also have the induction relation:*

$$\begin{bmatrix} j_1 & j_2 & j_1+j_2-2p \\ i & h+1 & k+1 \end{bmatrix} = \frac{[i]q^{-(j_2-2(h+1))} + [h+1]}{[k+1]} \begin{bmatrix} j_1 & j_2 & j_1+j_2-2p \\ i & h & k \end{bmatrix}$$

Proof. We prove this by induction on k. The assertion holds for $k = 0$ thanks to the Theorem 5.1.10. Supposing:

$$v_k^{(j_1+j_2-2p)} = \sum_i \alpha_i v_i^{(j_1)} \otimes v_{p-i+k}^{(j_2)}$$

We have:

$$\begin{aligned}
[k+1]v_{k+1}^{(j_1+j_2-2p)} &= F v_k^{(j_1+j_2-2p)} = \\
&= \sum_i \alpha_i \left(F v_i^{(j_1)} \otimes K^{-1} v_{p-i+k}^{(j_2)} + v_i^{(j_1)} \otimes F v_{p-i+k}^{(j_2)} \right) = \\
&= \sum_i \alpha_i ([i+1]q^{-(j_2-2(p-i+k))} v_{i+1}^{(j_1)} \otimes v_{p-i+k}^{(j_2)} + \\
&+ [p-i+k+1] v_i^{(j_1)} \otimes v_{p-i+k+1}^{(j_2)}) = \\
&= \sum_i \alpha_i \left([i]q^{-(j_2-2(p-i+k+1))} + [p-i+k+1] \right) v_i^{(j_1)} \otimes v_{p-i+k+1}^{(j_2)}
\end{aligned}$$

□

Remark 5.1.12. From the proof of the last theorem it is possible to get an explicit formula for quantum Clesbsch-Gordan coefficients when $k = 0$:

$$\begin{bmatrix} j_1 & j_2 & j_1+j_2-2p \\ i & p-i & 0 \end{bmatrix} = (-1)^i \frac{[j_1-i]![j_2-p+i]!}{[j_1]![j_2-p]!} q^{i(j_1-i+1)}$$

This fact together with the induction formula allows us to calculate all the possible quantum Clebsch-Gordan coefficients.

We now want to express the basis $\{v_i^{(j_1)} \otimes v_h^{(j_2)}\}_{i,h}$ in terms of the basis $\{v_k^{(j_1+j_2-2p)}\}_{p,k}$. We need to introduce an alternative scalar product on V_n when $n \leq N$, in order to do that. First of all, we need the following:

Proposition 5.1.13. *There exists a unique unital algebra *-antiautomorphism T of U_q such that $T(E) = KF$, $T(F) = EK^{-1}$ and $T(K) = K$. T is also a morphism of coalgebras.*

Proof. The following calculations will be sufficient to get the result:

$$\begin{aligned}
&\text{(i) } \Delta(T(E)) = \Delta(KF) = K \otimes KF + KF \otimes 1 = \\
&\quad = T \otimes T(K \otimes E + E \otimes 1) = T \otimes T(\Delta(E)) \\
&\text{(ii) } \Delta(T(F)) = \Delta(EK^{-1}) = 1 \otimes EK^{-1} + EK^{-1} \otimes K^{-1} = \\
&\quad = T \otimes T(1 \otimes F + F \otimes K^{-1}) = T \otimes T(\Delta(F)) \\
&\text{(iii) } \Delta(T(K)) = \Delta(K) = K \otimes K = T \otimes T(\Delta(K)) \\
&\text{(iv) } T(E^*) = T(F) = EK^{-1} = F^* K^* = (KF)^* = T(E)^* \\
&\text{(v) } T(F^*) = T(E) = KF = K^{-1*} E^* = (EK^{-1})^* = T(F)^* \\
&\text{(vi) } T(K^*) = T(K^{-1}) = K^{-1} = K^* = T(K)^*
\end{aligned}$$

□

Theorem 5.1.14. *On V_n with $n \leq N$ there exists a unique non-degenerate symmetric bilinear form such that $(v_0, v_0) = 1$ and:*

$$(xv, v') = (v, T(x)v') \tag{5.1.3}$$

If $v_i = \frac{F^i}{[i]!} v_0$, then:

$$(v_i, v_j) = \delta_{i,j} q^{-(n-i-1)i} \begin{bmatrix} n \\ i \end{bmatrix} \tag{5.1.4}$$

Proof. We first assume that there exists this scalar product on V_n, and we want to show that (v_i, v_j) is necessarily of the prescribed form. By definition we have:

$$(v_i, v_j) = \frac{1}{[i]!}(F^i v_0, v_j) = \frac{1}{[i]!}(v_0, T(F)^i v_j) = \frac{1}{[i]!}(v_0, (EK^{-1})^i v_j)$$

It is easy to prove that $(EK^{-1})^i = q^{i(i+1)} K^{-i} E^i$ for any $i > 0$. Consequently, the vector $T(F)^i v_j$ is a scalar multiple of $E^i v_j$ which vanishes as soon as $i > j$. By symmetry, we also have $(v_i, v_j) = 0$ if $i < j$. Now we compute (v_i, v_i). We need the formula:

$$E^i v_j = \frac{[n-j+i]!}{[n-j]!} v_{j-i}$$

to compute (v_i, v_i). We have:

$$\begin{aligned}
(v_i, v_i) &= \frac{1}{[i]!} q^{i(i+1)} (v_0, K^{-i} E^i v_i) = \\
&= q^{i(i+1)} \frac{[n]!}{[i]![n-i]!} (v_0, K^{-i} v_0) = \\
&= q^{i(i+1)-ni} \begin{bmatrix} n \\ i \end{bmatrix} (v_0, v_0)
\end{aligned}$$

So, we have the uniqueness of this inner product. We need to prove now the existence. In other words, we need to prove that the non-degenerate symmetric bilinear form such that:

$$(v_i, v_j) = \delta_{i,j} q^{-(n-i-1)i} \begin{bmatrix} n \\ i \end{bmatrix}$$

satisfies (5.1.3). It is enough to check it on the generators E, F and K. We will prove it for $x = E$, since the other two will follow similarly. On one hand:

$$(Ev_i, v_j) = [n-i+1](v_{i-1}, v_j) = \delta_{i-1,j} q^{-(n-i)(i-1)} \frac{[n]!}{[i-1]![n-i]!}$$

On the other hand:

$$\begin{aligned}(v_i, T(E)v_j) = (v_i, KFv_j) &= q^{n-2(j+1)}[j+1](v_i, v_{j+1}) = \\ &= \delta_{i,j+1} q^{-(n-i-1)i+n-2(j+1)}[j+1]\frac{[n]!}{[i]![n-i]!} = \\ &= \delta_{i,j+1} q^{-(n-i)(i-1)} \frac{[n]!}{[i-1]![n-i]!} = (Ev_i, v_j)\end{aligned}$$

□

Let us now equip V_{j_1} and V_{j_2} of the scalar product defined in the last Theorem. We put on $V_{j_1} \otimes V_{j_2}$ the following symmetric bilinear form:

$$(v_1 \otimes v_1', v_2 \otimes v_2') = (v_1, v_2)(v_1', v_2') \qquad (5.1.5)$$

where $v_1, v_2 \in V_{j_1}$ and $v_1', v_2' \in V_{j_2}$.

Lemma 5.1.15. *The symmetric bilinear form* (5.1.5) *is non-degenerate and the basis* $\{v_i^{(j_1)} \otimes v_h^{(j_2)}\}$ *is orthogonal. Moreover,* $\forall x \in U_q$ *and* $w_1, w_2 \in V_{j_1} \otimes V_{j_2}$*, we have:*

$$(xw_1, w_2) = (w_1, T(x)w_2)$$

Proof. All the assertions are very easy to prove. Regarding the last one, it is crucial the fact that T is a morphism of coalgebras. □

The next result is what we were looking for:

Proposition 5.1.16. **(a)** *The basis* $\{v_k^{(j_1+j_2-2p)}\}_{p,k}$ *is orthogonal.*
(b) *Fix integers* p, q, k, l*. We have the following relations:*

$$0 = \sum_{i,j} q^{-i(j_1-i-1)-h(j_2-h-1)} \begin{bmatrix} j_1 \\ i \end{bmatrix} \begin{bmatrix} j_2 \\ h \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j_1+j_2-2p \\ i & h & k \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j_1+j_2-2q \\ i & h & l \end{bmatrix}$$

when $p \neq q$ *and* $k \neq l$*, and:*

$$\sum_{i,j} q^{-i(j_1-i-1)-h(j_2-h-1)} \begin{bmatrix} j_1 \\ i \end{bmatrix} \begin{bmatrix} j_2 \\ h \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j_1+j_2-2p \\ i & h & k \end{bmatrix}^2 = q^{-k(j_1+j_2-2p-k-1)} \begin{bmatrix} j_1+j_2-2p \\ k \end{bmatrix}$$

(c) *Given i and h, we have:*

$$v_i^{(j_1)} \otimes v_h^{(j_2)} = q^{-i(n-i-1)-h(j_2-h-1)} \begin{bmatrix} j_1 \\ i \end{bmatrix} \begin{bmatrix} j_2 \\ h \end{bmatrix} \sum_{p=0}^{\min(j_1,j_2)} \sum_{k=0}^{j_1+j_2-2p} q^{k(j_1+j_2-2p-k-1)} \frac{\begin{bmatrix} j_1 & j_2 & j_1+j_2-2p \\ i & h & k \end{bmatrix}}{\begin{bmatrix} j_1+j_2-2p \\ k \end{bmatrix}} v_k^{(j_1+j_2-2p)} \tag{5.1.6}$$

Proof. (a) Arguing as in the proof of the Theorem 5.1.14 we have:

$$(v_k^{(j_1+j_2-2p)}, v_l^{(j_1+j_2-2p)}) = 0$$

when $k \neq l$. We pass to the case $p \neq q$. First of all, we need to show that the highest weight vectors $v_0^{(j_1+j_2-2p)}$ and $v_0^{(j_1+j_2-2q)}$ are orthogonal. In fact:

$$\begin{aligned}(v_0^{(j_1+j_2-2p)}, v_0^{(j_1+j_2-2q)}) &= \sum_{i,h} \alpha_i \beta_j (v_i^{(j_1)}, v_h^{(j_1)})(v_{p-i}^{(j_2)}, v_{q-i}^{(j_2)}) = \\ &= \sum_i \alpha_i \beta_i (v_i^{(j_1)}, v_i^{(j_1)})(v_{p-i}^{(j_2)}, v_{q-i}^{(j_2)}) = 0\end{aligned}$$

since $p-i \neq q-i$. It remains to prove that:

$$(v_k^{(j_1+j_2-2p)}, v_l^{(j_1+j_2-2q)}) = 0$$

when $k, l > 0$. Since this form is symmetric, it is sufficient to prove it when $k \geq l$. We have:

$$\begin{aligned}(v_k^{(j_1+j_2-2p)}, v_l^{(j_1+j_2-2q)}) &= \gamma(F^k v_0^{(j_1+j_2-2p)}, v_l^{(j_1+j_2-2q)}) = \\ &= \gamma'(v_0^{(j_1+j_2-2p)}, E^k v_l^{(j_1+j_2-2q)})\end{aligned}$$

for some scalars γ and γ'. If $k > l$, $E^k v_l^{(j_1+j_2-2q)} = 0$; if $k = l$, then $E^k v_k^{(j_1+j_2-2q)}$ is a scalar multiple of $v_0^{(j_1+j_2-2q)}$, so we are back to the previous case.
(b) We calculate $(v_k^{(j_1+j_2-2p)}, v_l^{(j_1+j_2-2q)})$. It is equal to:

$$\begin{aligned}&\sum_{i+h=p+k} \sum_{r+s=q+l} \begin{bmatrix} j_1 & j_2 & j_1+j_2-2p \\ i & h & k \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j_1+j_2-2q \\ r & s & l \end{bmatrix} (v_i^{(j_1)}, v_r^{(j_1)})(v_h^{(j_2)}, v_s^{(j_2)}) = \\ &= \sum_{i+h=p+k} \begin{bmatrix} j_1 & j_2 & j_1+j_2-2p \\ i & h & k \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j_1+j_2-2q \\ i & h & k \end{bmatrix} (v_i^{(j_1)}, v_i^{(j_1)})(v_h^{(j_2)}, v_h^{(j_2)}) = \\ &= \sum_{i+h=p+k} q^{-i(j_1-i-1)-h(j_2-h-1)} \begin{bmatrix} j_1 \\ i \end{bmatrix} \begin{bmatrix} j_2 \\ h \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j_1+j_2-2p \\ i & h & k \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j_1+j_2-2q \\ i & h & l \end{bmatrix}\end{aligned}$$

On the other hand:

$$(v_k^{(j_1+j_2-2p)}, v_l^{(j_1+j_2-2q)}) = \delta_{p.q} \delta_{k,l} q^{-k(j_1+j_2-2p-k-1)} \begin{bmatrix} j_1+j_2-2p \\ k \end{bmatrix}$$

(c) We have:

$$v_i^{(j_1)} \otimes v_h^{(j_2)} = \sum_{p=0}^{\min(j_1,j_2)} \sum_{k=0}^{j_1+j_2-2p} \gamma_{pk} v_k^{(j_1+j_2-2p)}$$

for some coefficients γ_{pk}. Therefore:

$$\begin{aligned}\gamma_{pk}(v_k^{(j_1+j_2-2p)}, v_k^{(j_1+j_2-2p)}) &= (v_i^{(j_1)} \otimes v_h^{(j_2)}, v_k^{(j_1+j_2-2p)}) = \\ &= \begin{bmatrix} j_1 & j_2 & j_1+j_2-2p \\ i & h & k \end{bmatrix} (v_i^{(j_1)}, v_i^{(j_1)})(v_h^{(j_2)}, v_h^{(j_2)})\end{aligned}$$

Applying (5.1.4) we get the explicit expression of γ_{pk}. □

Now we are ready to build the quantum groupoids $\mathcal{C}(N) = \mathcal{C}(SU_2, N+1)$.

5.2 General setting

In this section we set the generators and calculate how involution, counit, coproduct and antipode act on them. Let $\{\psi_1, \psi_2\}$ a basis of $V = V_1$, where ψ_1 is a highest weight vector and $\psi_2 = F\psi_1$. Applying Lemma 5.1.6, we have:

$$\begin{aligned} K\psi_1 = q\psi_1 \,,\, E\psi_1 = 0 \,,\, F\psi_1 = \psi_2 \\ K\psi_2 = q^{-1}\psi_1 \,,\, E\psi_2 = \psi_1 \,,\, F\psi_2 = 0 \end{aligned}$$

It is well-known that $\mathcal{C}(N)$ are the quotients of:

$$\mathcal{D}(N) = \bigoplus_{n=0}^{\infty} (V^{\otimes n})^* \otimes V^{\otimes n}$$

All the elements in $\mathcal{D}$ have the form $1 \otimes 1$ and $\psi_{i_1}^* \dots \psi_{i_n}^* \otimes \psi_{j_1} \dots \psi_{j_n}$, where $\psi_{i_k}, \psi_{j_h} \in \{\psi_1, \psi_2\}$. Therefore the generators of $\mathcal{C}(N)$ are:

$$\begin{aligned} I = 1 \otimes 1 \,,\, e_1 = \psi_1^* \otimes \psi_1 \\ e_2 = \psi_1^* \otimes \psi_2 \,,\, e_3 = \psi_2^* \otimes \psi_2 \end{aligned}$$

Let ϕ, ψ be in V_i. Then:
(a) *Involution* Since $(\phi^* \otimes \psi)^* = \psi^* \otimes \phi$, we have:

$$I^* = I \,,\, e_1^* = e_1 \,,\, e_3^* = e_3$$

and $e_2^* = \psi_2^* \otimes \psi_1$.

(b) *Counit* We know that $\varepsilon(\phi^* \otimes \psi) = (\phi, \psi)$. So:

$$\varepsilon(I) = 1 \,,\, \varepsilon(e_1) = 1 = \varepsilon(e_3) \,,\, \varepsilon(e_2) = 0$$

(c) *Coproduct* From the general theory we know that:

$$\Delta(\phi^* \otimes \psi) = \sum_{j=0}^{i} \phi^* \otimes \xi_j \otimes \xi_j^* \otimes \psi$$

where $\{\xi_j\}_{j=0}^{i}$ is a basis of V_i. So:

$$\Delta(I) = I \otimes I \, , \Delta(e_1) = e_1 \otimes e_1 + e_2 \otimes e_2^*$$
$$\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_3 \, , \Delta(e_3) = e_2^* \otimes e_2 + e_3 \otimes e_3$$

The previous relations can be easily proved:

$$\begin{aligned}
\Delta(I) &= \Delta(1^* \otimes 1) = 1^* \otimes 1 \otimes 1^* \otimes 1 = I \otimes I \\
\Delta(e_1) &= \Delta(\psi_1^* \otimes \psi_1) = \psi_1^* \otimes \psi_1 \otimes \psi_1^* \otimes \psi_1 + \psi_1^* \otimes \psi_2 \otimes \psi_2^* \otimes \psi_1 = \\
&= e_1 \otimes e_1 + e_2 \otimes e_2^* \\
\Delta(e_2) &= \Delta(\psi_1^* \otimes \psi_2) = \psi_1^* \otimes \psi_1 \otimes \psi_1^* \otimes \psi_2 + \psi_1^* \otimes \psi_2 \otimes \psi_2^* \otimes \psi_2 = \\
&= e_1 \otimes e_2 + e_2 \otimes e_3 \\
\Delta(e_3) &= \Delta(\psi_2^* \otimes \psi_2) = \psi_2^* \otimes \psi_1 \otimes \psi_1^* \otimes \psi_2 + \psi_2^* \otimes \psi_2 \otimes \psi_2^* \otimes \psi_2 = \\
&= e_2^* \otimes e_2 + e_3 \otimes e_3
\end{aligned}$$

(d) *Antipode* We will prove that:

$$S(e_1) = e_3 \, , S(e_3) = e_1 \, , S(e_2) = -q^{-1} e_2 \tag{5.2.1}$$

We recall how the antipode S acts on E, F and K in $U_q(\mathfrak{sl}_2)$:

$$S(K) = K^{-1} \, , S(E) = -K^{-1}E \, , S(F) = -FH$$

Moreover, we know that S acts on $\phi^* \otimes \psi$ in the following way:

$$S(\phi^* \otimes \psi) = (j_i \psi)^* \otimes j_i \phi$$

and $a \in U_q(\mathfrak{sl}_2)$ acts on $\overline{\gamma}$, where $\gamma \in V_i$, in the following way:

$$a \cdot \overline{\gamma} = \overline{S(a^*)\gamma}$$

adopting the Wenzl's notation. Let J_i be the complex conjugation map from V_i to $\overline{V_i}$, such that $J_i(\gamma) = \overline{\gamma} \; \forall \gamma \in V_i$. It is well-known that $\overline{V_i}$ is still a Weyl module, with $\dim(\overline{V_i}) = \dim(V_i)$. Therefore, by the Theorem 5.1.7, we have $\overline{V_i} \cong V_i$. j_i is an automorphism of V_i, and it is obtained composing J_i with U_i, where U_i is a unitary intertwiner between $\overline{V_i}$ and V_i. We explicitly calculate j_1. First of all, we write down how K, E and F act on $\overline{\psi_1}, \overline{\psi_2}$:

$$\begin{aligned}
K\overline{\psi_1} &= \overline{S(K^*)\psi_1} = \overline{K\psi_1} = q^{-1}\overline{\psi_1} \\
E\overline{\psi_1} &= \overline{S(E^*)\psi_1} = \overline{-FK\psi_1} = q^{-1}\overline{-F\psi_1} = -q^{-1}\overline{\psi_2} \\
F\overline{\psi_1} &= \overline{S(F^*)\psi_1} = \overline{S(E)\psi_1} = \overline{-K^{-1}E\psi_1} = 0 \\
K\overline{\psi_2} &= \overline{S(K^*)\psi_2} = \overline{K\psi_2} = q\overline{\psi_2} \\
E\overline{\psi_2} &= \overline{S(E^*)\psi_2} = \overline{-FK\psi_2} = q\overline{-F\psi_2} = 0 \\
F\overline{\psi_2} &= \overline{S(F^*)\psi_2} = \overline{-K^{-1}E\psi_2} = -\overline{K^{-1}\psi_1} = -q^{-1}\overline{\psi_1}
\end{aligned}$$

It is clear that the map $U_1 : \overline{V_1} \to V_1$ defined in the following way:

$$U_1(\overline{\psi_1}) = -q^{-1}\psi_2 \, , U_1(\overline{\psi_2}) = \psi_1$$

is a unitary intertwiner. So $j_1 = U_1 \circ J_1$ acts on the basis in the following way:

$$j_1\psi_1 = -q^{-1}\psi_2\,,\, j_1\psi_2 = \psi_1$$

Finally we obtain (5.2.1) after the following calculation:

$$\begin{aligned}
S(e_1) &= (j_1\psi_1)^* \otimes j_1\psi_1 = (-q\psi_2^*) \otimes (-q^{-1}\psi_2) = \psi_2^* \otimes \psi_2 = e_3 \\
S(e_3) &= (j_1\psi_2)^* \otimes j_1\psi_2 = \psi_1^* \otimes \psi_1 = e_1 \\
S(e_2) &= (j_1\psi_2)^* \otimes j_1\psi_1 = \psi_1^* \otimes (-q^{-1}\psi_2) = -q^{-1}e_2
\end{aligned}$$

It is important to notice that generators, involution, counit, coproduct and antipode do not depend on the order of the root of unity. Conversely, relations on the products will be heavily influenced by the order of the root. More precisely, let $2N+2$ be the order of the primitive root of unity q. In our treatment we will distinguish three cases:

(i) $N = 1$ or, in other words, q is a fourth primitive root of unity;

(ii) $N = 2$ or, in other words, q is a sixth primitive root of unity;

(iii) $N > 2$ or, in other words, q is a nth primitive root of unity, with $n > 6$

From now on we will use the following notation:

$$\alpha\gamma := \alpha\underline{\otimes}\gamma$$

where $\alpha, \gamma \in V_1$.

Moreover, we recall that in $\mathcal{C}(N)$ we have the following identification: if $\phi \in V_1^{\underline{\otimes} n}$, $\psi \in V_1^{\underline{\otimes} m}$ and $A \in (V_1^{\underline{\otimes} n}, V_1^{\underline{\otimes} m})$, then:

$$\psi^* \otimes A(\phi) = \psi^* \circ A \otimes \phi$$

We will repeatedly use this fact in the following sections.

5.3 Case (i) and (ii): q fourth and sixth root of unity

The case $N = 1$ is very simple to analyze. In this case, the only Weyl modules are $V_0 = \mathbb{C}$ and V_1, and V_1 is negligible. So, $e_1 = e_2 = e_3 = 0$, and $\mathcal{C}(1) = \mathbb{C}I$. Let us pass to the case $N = 2$. In this case, we have three Weyl modules: V_0, V_1 and V_2, and V_2 is negligible. Since $V_1 \otimes V_1 = V_0 \otimes V_2$, we have:

$$V_1\underline{\otimes}V_1 = V_0$$

We explicitly write down the decomposition of $V_1 \otimes V_1$:

$$\begin{aligned}
V_0 &= \langle \psi_1 \otimes \psi_2 - q\psi_2 \otimes \psi_1 \rangle \\
V_2 &= \langle \psi_1 \otimes \psi_1, \psi_1 \otimes \psi_2 + q\psi_2 \otimes \psi_1, \psi_2 \otimes \psi_2 \rangle
\end{aligned}$$

Since V_2 is negligible, we have that:

$$\psi_1\psi_1 = 0 = \psi_2\psi_2 \tag{5.3.1}$$

For the same reason:

$$\begin{aligned}
\psi_1^*\psi_1^* &= (\psi_1\psi_1)^* \circ \overline{R}^{-1} = 0 \\
\psi_2^*\psi_2^* &= (\psi_2\psi_2)^* \circ \overline{R}^{-1} = 0
\end{aligned} \tag{5.3.2}$$

Consequently:

Proposition 5.3.1. *We have the following relations:*

$$e_1^2 = e_2^2 = e_3^2 = e_1e_2 = e_2e_1 = e_2e_3 = e_3e_2 = 0$$

Proof. It is straightforward using (5.3.1) and (5.3.2) □

It remains to show the relations involving e_1e_3, $e_2e_2^*$ and $e_2^*e_2$. We introduce the morphism $A_0 : V_0 \to V_1 \otimes V_1$, such that:

$$A_0(1) = \psi_1 \otimes \psi_2 - q\psi_2 \otimes \psi_1$$

Hence:

Proposition 5.3.2. *We have the following relations:*

$$\begin{aligned} e_2^*e_2 &= e_2e_2^* = -I \\ e_1e_3 &= q^{-1}I \end{aligned}$$

Proof. Let us start with e_1e_3:

$$e_1e_3 = \psi_1^*\psi_2^* \otimes \psi_1\psi_2$$

We want to decompose $\psi_1 \otimes \psi_2 \in V_1 \otimes V_1$ as element in $V_0 \oplus V_2$. We have:

$$\psi_1 \otimes \psi_2 = \lambda(\psi_1 \otimes \psi_2 - q\psi_2 \otimes \psi_1) + \mu(\psi_1 \otimes \psi_2 + q^{-1}\psi_2 \otimes \psi_1)$$

Since q is a primitive sixth root of unity, we have $q + q^{-1} = 2\cos(\frac{\pi}{3}) = 1$. So $\lambda = q^{-1}$ and $\mu = q$. Therefore:

$$\psi_1\psi_2 = \psi_1 \underline{\otimes} \psi_2 = q^{-1}(\psi_1 \otimes \psi_2 - q\psi_2 \otimes \psi_1)$$

Using the morphism A_0, we have:

$$\psi_1^*\psi_2^* \otimes \psi_1\psi_2 = \psi_1^*\psi_2^* \otimes q^{-1}A_0(1) = q^{-1}\psi_1^*\psi_2^* \circ A_0 \otimes 1 = q^{-1}1 \otimes 1 = q^{-1}I$$

since $\psi_1^*\psi_2^* \circ A_0$ is a linear functional on V_0 which gives 1 on 1. We pass to $e_2^*e_2$:

$$e_2^*e_2 = \psi_2^*\psi_1^* \otimes \psi_1\psi_2$$

As we have just seen, $\psi_1\psi_2 = q^{-1}(\psi_1 \otimes \psi_2 - q\psi_2 \otimes \psi_1)$. Using A_0, we have:

$$\psi_2^*\psi_1^* \otimes \psi_1\psi_2 = \psi_2^*\psi_1^* \otimes q^{-1}A_0(1) = q^{-1}\psi_2^*\psi_1^* \circ A_0 \otimes 1 = -1 \otimes 1 = -I$$

since $\psi_2^*\psi_1^* \circ A_0$ is a linear functional on V_0 which gives $-q$ on 1. Similarly we can prove that $e_2e_2^* = -I$. □

The relations that are left are the adjoint relations. In fact, $*$ is anti-multiplicative on the products of simple tensor products, and it is enough. For instance:

$$e_3e_1 = (e_1e_3)^* = (q^{-1}I)^* = qI$$

Resuming, we have the following:

Theorem 5.3.3. $\mathcal{C}(2)$ *is generated by two self-adjoint elements* e_1 *and* e_3 *and one normal element* e_2. *Antipode, counit, coproduct and involution are as in the previous section. The relations are:*

$$e_1^2 = e_2^2 = e_3^2 = e_1e_2 = e_2e_1 = e_2e_3 = e_3e_2 = 0 \tag{5.3.3}$$

$$e_2^*e_2 = e_2e_2^* = -I \tag{5.3.4}$$

$$e_1e_3 = q^{-1}I \tag{5.3.5}$$

Remark 5.3.4. It is quite easy to notice that $\mathcal{C}(2)$ is not associative and cannot be a C*-algebra. Moreover, $\dim(\mathcal{C}(2)) = 5$, with $\{I, e_1, e_2, e_2^*, e_3\}$ as linear basis. The dimension of $\mathcal{C}(2)$ of course agrees with the general theory exposed in the previous chapter.

We can also present $\widehat{\mathcal{C}(2)}$ as generators and relations.

(a) *Generators* The generators will be:

$$\eta_0 = \widehat{I} \text{ and } \eta_i = \widehat{e_i},\, i \in \{1,2,3\}$$

where $\widehat{e_i}(e_j) = \delta_{i,j}$ and $\widehat{e_i}(I) = 0$.

(b) *Involution* We know that, if $f \in \widehat{\mathcal{C}(2)}$, we have:

$$f^*(a) = \overline{f(a^*)}$$

where $a \in \mathcal{C}(2)$. Applying the last formula to η_i, we get:

$$\eta_0^* = \eta_0\ ,\ \eta_1^* = \eta_1\ ,\ \eta_3^* = \eta_3$$

while $\eta_2^* = \widehat{e_2}^* = \widehat{e_2^*}$.

(c) *Counit* We know that $\widehat{\varepsilon}(f) = f(I)$. So:

$$\widehat{\varepsilon}(\eta_0) = \eta_0(I) = 1$$
$$\widehat{\varepsilon}(\eta_i) = \eta_i(I) = 0$$

(d) *Coproduct* The general formula is:

$$\widehat{\Delta}(f)(a \otimes b) = f(ab)$$

where $f \in \widehat{\mathcal{C}(2)}$ and $a, b \in \mathcal{C}(2)$. So:

$$\widehat{\Delta}(\eta_0) = \eta_0 \otimes \eta_0 + \eta_1 \otimes \eta_3 + \eta_3 \otimes \eta_1 + \eta_2^* \otimes \eta_2 + \eta_2 \otimes \eta_2^*$$
$$\widehat{\Delta}(\eta_i) = \eta_0 \otimes \eta_i + \eta_i \otimes \eta_0 \quad \forall i \in \{1,2,3\}$$

(e) *Antipode* From the general theory we know that:

$$(\widehat{S}(f))(a) = f(S(a))$$

where $f \in \widehat{\mathcal{C}(2)}$ and $a \in \mathcal{C}(2)$. So:

$$\widehat{S}(\eta_0) = \eta_0\ ,\ \widehat{S}(\eta_1) = \eta_3\ ,\ \widehat{S}(\eta_3) = \eta_1$$
$$\widehat{S}(\eta_2) = -q^{-1}\eta_2$$

(f) *Products* It is well-known that:

$$fg(a) = f \otimes g(\Delta(a))$$

where $f, g \in \widehat{\mathcal{C}(2)}$ and $a \in \mathcal{C}(2)$. So:

$$\begin{aligned}
&\eta_0^2 = \eta_0 \ , \ \eta_1^2 = \eta_1 \ , \ \eta_2^2 = 0 \ , \ \eta_3^2 = \eta_3 \\
&\eta_0\eta_i = 0 = \eta_i\eta_0 \ \ \forall i \in \{1, 2, 3\} \\
&\eta_1\eta_3 = \eta_3\eta_1 = \eta_2\eta_1 = \eta_3\eta_2 = 0 \\
&\eta_1\eta_2 = \eta_2 \ , \ \eta_2\eta_3 = \eta_2 \ , \ \eta_2^*\eta_2 = \eta_3 \ , \ \eta_2\eta_2^* = \eta_1
\end{aligned}$$

It is easy to see that the unit in $\widehat{\mathcal{C}(2)}$ is $\widetilde{I} = \eta_0 + \eta_1 + \eta_3$.

5.4 Case (iii): q nth primitive root of unity, with $n > 6$

Let n be $2N + 2$, with $N > 2$. As we saw before, the Weyl modules are $V_0, \ldots, V_N$, and V_N is negligible. Moreover:

$$\dim(\mathcal{C}(N)) = \sum_{i=1}^{N} i^2 = \frac{1}{6}N(N + 1)(2N + 1)$$

The next result give us the commutation relations on our algebra. They do not depend on $N > 2$:

Theorem 5.4.1. *We have the following relations for all $N \in \mathbb{N}$, with $N > 2$:*

$$\begin{aligned}
\text{(i) } & e_2e_1 = q^{-1}e_1e_2 \\
\text{(ii) } & e_3e_2 = q^{-1}e_2e_3 \\
\text{(iii) } & e_2^*e_2 = e_2e_2^* \\
\text{(iv) } & e_2^*e_1 = q^{-1}e_1e_2^* \\
\text{(v) } & e_3e_2^* = q^{-1}e_2^*e_3 \\
\text{(vi) } & e_2e_2^* = q^{-1}e_1e_3 - q^{-1}I \\
\text{(vii) } & e_3e_1 = q^{-2}e_1e_3 + (1 - q^{-2})I
\end{aligned}$$

Proof. Remember that $V_1 \otimes V_1 = V_0 \oplus V_2$, and we have the following decomposition:

$$\begin{aligned}
V_0 &= \langle \psi_1 \otimes \psi_2 - q\psi_2 \otimes \psi_1 \rangle \\
V_2 &= \langle \psi_1 \otimes \psi_1, \psi_1 \otimes \psi_2 + q\psi_2 \otimes \psi_1, \psi_2 \otimes \psi_2 \rangle
\end{aligned}$$

In comparison with the case (ii), we don't have any truncation here, so $V_1^{\underline{\otimes}2} = V_1^{\otimes 2}$. Let $A_0 \in (V_0, V_1^{\otimes 2})$ be a morphism, where $A_0(1) = \psi_1\psi_2 - q\psi_2\psi_1$. Therefore:

$$\psi_1^*\psi_1^* \otimes (\psi_1\psi_2 - q\psi_2\psi_1) = \psi_1^*\psi_1^* \otimes A_0(1) = \psi_1^*\psi_1^* \circ A_0 \otimes 1 = 0$$

Consequently we get $e_1e_2 - qe_2e_1 = 0$. Similarly:

$$\psi_2^*\psi_2^* \otimes (\psi_1\psi_2 - q\psi_2\psi_1) = \psi_2^*\psi_2^* \otimes A_0(1) = \psi_2^*\psi_2^* \circ A_0 \otimes 1 = 0$$

In this way we get $e_2^* e_3 - q e_3 e_2^* = 0$. So we have proved (i) and (v), and (ii) and (iv) immediately follow using the adjoint map. Going on with the same calculations, we have:

$$\psi_1^* \psi_2^* \otimes (\psi_1 \psi_2 - q\psi_2 \psi_1) = \psi_1^* \psi_2^* \otimes A_0(1) = \psi_1^* \psi_2^* \circ A_0 \otimes 1 = 1 \otimes 1$$

So $e_1 e_3 - q e_2 e_2^* = I$, getting (vi). In the same way, taking $\psi_2^* \psi_1^*$ in place of $\psi_1^* \psi_2^*$, we obtain $e_2^* e_2 - q e_3 e_1 = -qI$. Now, taking the adjoint of the relation (v), we have:

$$e_3 e_1 - q^{-1} e_2 e_2^* = I \Longrightarrow e_3 e_1 = q^{-1} e_2 e_2^* + I$$

Hence:

$$-qI = e_2^* e_2 - q e_3 e_1 = e_2^* e_2 - q(q^{-1} e_2 e_2^* + I) = e_2^* e_2 - e_2 e_2^* - qI$$

So $e_2^* e_2 = e_2 e_2^*$. It remains to prove the relation (vii). Using (vi) and its the adjoint relation we have:

$$e_3 e_1 = q^{-1} e_2 e_2^* + I = q^{-2} e_1 e_3 - q^{-2} I + I = q^{-2} e_1 e_3 + (1 - q^{-2}) I$$

□

Before going on, we need a very useful result about the representation theory of $U_q(\mathfrak{sl}_2)$. We need the following definition:

Definition 5.4.2. Let $\mathcal{I}_p^{(N)}$ the set whose elements are the maps $i : \{1, \ldots, N\} \to \{1, 2\}$, with $|i^{-1}(2)| = p$. $\sigma(i)$ is the minimum number of exchange we have to do in order to pass from the ordered set $\{1, \ldots, 1, 2, \ldots, 2\}$ to $\{i(1), \ldots, i(N)\}$.

Proposition 5.4.3. *V_N is the summand with the highest index in the decomposition of $V_1^{\otimes N}$ into the direct sum of irreducible representations, and its multiplicity is* 1. *$v_0^{(N)} = \psi_1^{\otimes N}$ is the highest weight vector, and:*

$$\frac{F^p}{[p]!} v_0^{(N)} = v_p^{(N)} = \sum_{i \in \mathcal{I}_p^{(N)}} q^{-\sigma(i)} \psi_{i(1)} \otimes \ldots \otimes \psi_{i(N)}$$

Proof. The first part of the Proposition can be proved by induction. If $N = 1$ the result is obvious. Suppose that the result is true for V_k, with $0 < k < N$, and we prove it for N:

$$V_1^{\otimes N} = V_1^{\otimes N-1} \otimes V_1 = \left(\bigoplus_{k \le N-2} V_k \oplus V_{N-1} \right) \otimes V_1 =$$

$$= \left(\bigoplus_{k \le N-2} V_k \otimes V_1 \right) \oplus (V_{N-1} \otimes V_1) = \left(\bigoplus_{k' \le N-1} V_{k'} \right) \oplus (V_{N-2} \oplus V_N) =$$

$$= \bigoplus_{k'' \le N-1} V_{k''} \oplus V_N$$

using the theorem we proved before. It is quite easy to see that $K v_0 = q^N v_0$, using that $\Delta^{(N)}(K) = K^{\otimes N}$. Now we prove that $E v_0 = 0$. We can proceed by induction.

If $N=1$, obviously $E\psi_1=0$, since ψ_1 is the highest weight vector of V_1. If it is true for $N-1$, let's prove it for N:

$$E\psi_1^{\otimes N}=E\psi_1^{\otimes N-1}\otimes\psi_1+K\psi_1^{\otimes N-1}\otimes E\psi_1=0$$

It remains to prove the last statement. Before, we need to prove the following identity. If $p\geq 1$:

$$v_p^{(N)}=v_{p-1}^{(N-1)}\otimes v_1^{(1)}+q^{-p}v_p^{(N-1)}\otimes v_0^{(1)} \tag{5.4.1}$$

We proceed by induction on p. If $p=1$:

$$\begin{aligned}v_1^{(N)}&=Fv_0^{(N)}=F(\psi_1^{\otimes N-1})\otimes K^{-1}\psi_1+\psi_1^{\otimes N-1}\otimes F\psi_1=\\&=q^{-1}F(\psi_1^{\otimes N-2})\otimes K^{-1}\psi_1\otimes\psi_1+q^{-1}\psi_1^{\otimes N-2}\otimes F\psi_1\otimes\psi_1+\psi_1^{\otimes N-1}\otimes\psi_2=\\&=q^{-2}F(\psi_1^{\otimes N-2})\otimes\psi_1^{\otimes 2}+q^{-1}\psi_1^{\otimes N-2}\otimes\psi_2\otimes\psi_1+\psi_1^{\otimes N-1}\otimes\psi_2=\\&=\ldots=\sum_{j=0}^{N-1}q^{-j}\psi_1^{\otimes N-j-1}\otimes\psi_2\otimes\psi_1^{\otimes j}=\sum_{i\in\mathcal{I}_1^{(N)}}q^{-\sigma(i)}\psi_{i(1)}\otimes\ldots\otimes\psi_{i(N)}\end{aligned}$$

On the other side:

$$\begin{aligned}&v_0^{(N-1)}\otimes v_1^{(1)}+q^{-1}v_1^{(N-1)}\otimes v_0^{(1)}=\\&=\psi_1^{\otimes N-1}\otimes\psi_2+q^{-1}\left(\sum_{j=0}^{N-2}q^{-j}\psi_1^{\otimes N-j-2}\otimes\psi_2\otimes\psi_1^{\otimes j}\right)\otimes\psi_1=\\&=\sum_{j=0}^{N-1}q^{-j}\psi_1^{\otimes N-j-1}\otimes\psi_2\otimes\psi_1^{\otimes j}\end{aligned}$$

Now, we need to prove the induction step. We need the following identity:

$$q[p-1]+q^{-(p-1)}=[p]$$

which can be easily proved:

$$\begin{aligned}q[p-1]+q^{-(p-1)}&=q\frac{q^{p-1}-q^{-(p-1)}}{q-q^{-1}}+q^{-(p-1)}=\\&=\frac{q^p-q^{-p+2}+q^{-p+2}-q^{-p}}{q-q^{-1}}=[p]\end{aligned}$$

At this stage it is easy to prove that, if (5.4.1) is true for $p-1$, then it is true for p:

$$\begin{aligned}v_p^{(N)}=\frac{F}{[p]}v_{p-1}^{(N)}=\frac{F}{[p]}(q^{-(p-1)}v_{p-1}^{(N-1)}\otimes v_0^{(1)}+v_{p-2}^{(N-1)}\otimes v_1^{(1)})=\\=\frac{1}{[p]}(q^{-p}[p]v_p^{(N-1)}\otimes v_0^{(1)}+q^{-(p-1)}v_{p-1}^{(N-1)}\otimes v_1^{(1)}+q[p-1]v_{p-1}^{(N-1)}\otimes v_1^{(1)})=\\=q^{-p}v_p^{(N-1)}\otimes v_0^{(1)}+v_{p-1}^{(N-1)}\otimes v_1^{(1)}\end{aligned}$$

Now we can conclude. We proceed again by induction on N, proving that:

$$v_p^{(N)}=\sum_{i\in\mathcal{I}_p^{(N)}}q^{-\sigma(i)}\psi_{i(1)}\otimes\ldots\otimes\psi_{i(N)}$$

If $N = 1$ it is obvious. We suppose that it is true for $N - 1$, and we prove it for N:

$$\begin{aligned} v_p^{(N)} &= v_{p-1}^{(N-1)} \otimes v_1^{(1)} + q^{-p} v_p^{(N-1)} \otimes v_0^{(1)} = \\ &= \left(\sum_{i \in \mathfrak{I}_{p-1}^{(N-1)}} q^{-\sigma(i)} \psi_{i(1)} \otimes \ldots \otimes \psi_{i(N-1)} \right) \otimes \psi_2 + \\ &+ q^{-p} \left(\sum_{i \in \mathfrak{I}_{p}^{(N-1)}} q^{-\sigma(i)} \psi_{i(1)} \otimes \ldots \otimes \psi_{i(N-1)} \right) \otimes \psi_1 \end{aligned}$$

and it is easy to see that this is exactly what we want to prove. □

Lemma 5.4.4.

$$v_l^{(n)*} = q^{l(n-l) - \frac{n(n-1)}{2}} \sum_{i \in \mathfrak{I}_l^{(n)}} q^{-\sigma(i)} \psi^*_{i(1)} \cdots \psi^*_{i(n)}$$

where $v_l^{(n)*} = (v_l^{(n)}, \cdot)$ is a functional on $V_1^{\otimes n}$.

Proof. We need to understand how $\overline{R}^{(n)}$ acts on:

$$w = v_l^{(n)} = \sum_{i \in \mathfrak{I}_l^{(n)}} q^{-\sigma(i)} \psi_{i(1)} \ldots \psi_{i(n)}$$

We follow what we did in Appendix [CP]. From there, it is well-known that $\overline{R}^{(n)} = R^{(n)} \Theta^{(n)}$, where $\Theta^{(n)}$ acts as scalar multiplication by:

$$q^{\frac{1}{2} n (\frac{\alpha}{2} + \alpha) - \frac{1}{2} (\frac{n}{2}\alpha + \alpha, \frac{n}{2}\alpha)} = q^{\frac{n(n-1)}{4}}$$

Moreover,

$$R^{(n)} = \Sigma_n \circ \varepsilon_{n-1} \circ (\varepsilon_{n-2}\varepsilon_{n-1}) \circ \ldots \circ (\varepsilon_1 \ldots \varepsilon_{n-1})$$

where $\varepsilon_i = q^{-\frac{1}{2}} g_i$ and $g_i = 1_{i-1} \otimes g \otimes 1_{n-i-1}$. $g \in (V_1^{\otimes 2}, V_1^{\otimes 2})$ acts in the following way:

$$\begin{aligned} g\psi_1 \otimes \psi_1 &= q\psi_1 \otimes \psi_1 \\ g\psi_2 \otimes \psi_2 &= q\psi_2 \otimes \psi_2 \\ g\psi_2 \otimes \psi_1 &= \psi_1 \otimes \psi_2 \\ g\psi_1 \otimes \psi_2 &= \psi_2 \otimes \psi_1 + (q - q^{-1})\psi_1 \otimes \psi_2 \end{aligned}$$

It is easy to prove that $g_i w = qw$, so putting everything together we get:

$$\begin{aligned} \overline{R}^{(n)} w &= q^{\frac{n(n-1)}{4}} R^{(n)} w = \\ &= \Sigma_n \circ g_{n-1} \circ \ldots \circ (g_1 \ldots g_{n-1}) w = \\ &= q^{\frac{n(n-1)}{2}} \Sigma_n w = q^{\frac{n(n-1)}{2}} \widetilde{w} \end{aligned}$$

where:

$$\widetilde{w} = \sum_{i \in \mathfrak{I}_l^{(n)}} q^{\sigma(i) - l(n-l)} \psi_{i(1)} \ldots \psi_{i(n)}$$

Now we want to write w^* in terms of $\psi^*_{i(1)}\dots\psi^*_{i(n)}$. Roughly speaking, we look for an explicit expression of λ_i, where:

$$w^* = \sum_{i\in\mathcal{I}^{(n)}_l} \lambda_i \psi^*_{i(1)}\dots\psi^*_{i(n)}$$

We obtain it after the following calculation:

$$\begin{aligned}
\lambda_{i_0} &= \sum_i \lambda_i \psi^*_{i(1)}\dots\psi^*_{i(n)}(\psi_{i_0(1)}\dots\psi_{i_0(n)}) = \\
&= w^*(\psi_{i_0(1)}\dots\psi_{i_0(n)}) = \left(\sum_i q^{-\sigma(i)}\psi_{i(1)}\dots\psi_{i(n)}, \overline{R}^{(n)}(\psi_{i_0(1)}\dots\psi_{i_0(n)})\right)_{p,n} = \\
&= \left(\overline{R}^{(n)}\sum_i q^{-\sigma(i)}\psi_{i(1)}\dots\psi_{i(n)}, \psi_{i_0(1)}\dots\psi_{i_0(n)}\right)_{p,n} = \\
&= \left(\sum_i q^{\frac{n(n-1)}{2}+\sigma(i)-l(n-l)}\psi_{i(1)}\dots\psi_{i(n)}, \psi_{i_0(1)}\dots\psi_{i_0(n)}\right)_{p,n} = \\
&= q^{l(n-l)-\frac{n(n-1)}{2}-\sigma(i_0)}
\end{aligned}$$

□

Proposition 5.4.5. *We have the following relations on* $\mathcal{C}(N)$*:*

(i) $e_i^N = 0 \ \ \forall i \in \{1,2,3\}$
(ii) $e_1^k e_2^{N-k} = e_1^k e_2^{*N-k} = e_2^k e_3^{N-k} = e_2^{*k} e_3^{N-k} = 0 \ \ \forall k \in \{0,\dots,N-1\}$

Proof. Using the Prop. 5.4.3, $v_0^{(N)} = \psi_1^{\otimes N}$ and $v_N^{(N)} = \psi_2^{\otimes N}$. Since V_N is negligible, $\psi_1^N = \psi_2^N = 0$ and:

$$\psi_1^*\dots\psi_1^* = (\psi_1\dots\psi_1)^* \circ \overline{R}^{(n)^{-1}} = 0 = (\psi_2\dots\psi_2)^* \circ \overline{R}^{(n)^{-1}} = \psi_2^*\dots\psi_2^*$$

Using this facts, it is straightforward to prove the above identities. □

It remains to discover the relations involving $e_1^h e_2^j e_3^k$ and $e_1^h e_2^{*j} e_3^k$ when $h+j+k=N$ and $hk \neq 0$:

Proposition 5.4.6. *We have the following relations on* $e_1^h e_2^j e_3^k$ *and* $e_1^h e_2^{*j} e_3^k$ *when* $h+j+k=N$ *and* $hk\neq 0$*:*

$$e_1^h e_2^j e_3^k = \sum_{l=1}^{\min(h,k)} C_l e_1^{h-l} e_2^j e_3^{k-l}$$

where $C_l \in \mathbb{C}\ \forall l \in \{1,\dots,\min(h,k)\}$. *We have analogous relations replacing* e_2 *by* e_2^*.

Proof. It is well-known that:

$$e_1^h e_2^j e_3^k = \psi_1^{*h+j}\psi_2^{*k} \otimes \psi_1^h \psi_2^{j+k}$$

We focus now on $\psi_1^{\otimes h} \otimes \psi_2^{\otimes j+k}$. It is an element in $V_1^{\otimes N}$, but more precisely we can see it as an element in $V_1^{\otimes h} \otimes V_1^{\otimes j+k}$, where $\psi_1^{\otimes h} \in V_1^{\otimes h}$ and $\psi_2^{\otimes j+k} \in V_1^{\otimes j+k}$.

Using the previous proposition, we have that $\psi_1^{\otimes h}$ is the highest weight vector in $V_h \hookrightarrow V_1^{\otimes h}$ and $\psi_2^{\otimes j+k}$ is the lowest weight vector in $V_{j+k} \hookrightarrow V_1^{\otimes j+k}$. So:

$$\psi_1^{\otimes h} \otimes \psi_2^{j+k} = v_0^{(h)} \otimes v_{j+k}^{(j+k)} \in V_h \otimes V_{j+k} \hookrightarrow V^{\otimes N}$$

Using the Theorem 5.1.10, we know that:

$$V_h \otimes V_{j+k} = \bigoplus_{p=0}^{\min(h,j+k)} V_{N-2p}$$

Using (5.1.6), we have:

$$v_0^{(h)} \otimes v_{j+k}^{(j+k)} = \sum_{p=0}^{\min(h,j+k)} d_p v_{j+k-p}^{(N-2p)} = d_0 v_{j+k}^{(N)} + \ldots + d_{j+k-\min(h,j+k)} v_0^{(|h-j-k|)}$$

where:

$$d_p = q^{j+k+(j+k-p)(h-p-1)} \frac{\begin{bmatrix} h & j+k & N-2p \\ 0 & j+k & j+k-p \end{bmatrix}}{\begin{bmatrix} N-2p \\ j+k-p \end{bmatrix}}$$

So, if we consider the truncated product $\psi_1^h \psi_2^{j+k}$, we have:

$$\psi_1^h \psi_2^{j+k} = \sum_{p=1}^{\min(h,j+k)} d_p v_{j+k-p}^{(N-2p)} = d_1 v_{j+k-1}^{(N-2)} + \ldots + d_{j+k-\min(h,j+k)} v_0^{(|h-j-k|)}$$

We can now use the map:

$$A_{N-2p}^{h,j+k} : V_1^{\otimes N-2p} \to V_{N-2p} \hookrightarrow V_h \otimes V_{j+k} \hookrightarrow V_1^{\otimes N}$$

where

$$A_{N-2p}^{h,j+k}|_{V_{N-2p}} = \mathrm{id}$$

and 0 elsewhere. Using the Prop. 5.4.3, we know that:

$$v_{j+k-p}^{(N-2p)} = \sum_{h \in \mathcal{I}_{j+k-p}^{(N-2p)}} q^{-\sigma(i)} \psi_{i(1)} \cdots \psi_{i(N-2p)}$$

In particular, $|i^{-1}(1)| = h - p$ and $|i^{-1}(2)| = j + k - p$. It is easy to see now that:

$$\psi_1^{*h+j} \psi_2^{*k} \circ A_{N-2p} = (v_{k-p}^{(N-2p)})^*$$

if $k \geq p$, and 0 otherwise. After an easy calculation we have:

$$\min(\min(h, j + k), k) = \min(h, k)$$

So:

$$\psi_1^{*h+j} \psi_2^{*k} \otimes \psi_1^h \psi_2^{j+k} = \sum_{p=1}^{\min(h,k)} d_p (v_{k-p}^{(N-2p)})^* \otimes v_{j+k-p}^{(N-2p)} =$$

$$= \sum_{p=1}^{\min(h,k)} d_p \sum_{i \in \mathcal{I}_{k-p}^{(N-2p)}, j \in \mathcal{I}_{j+k-p}^{(N-2p)}} c_{i,j} \psi_{i(1)}^* \cdots \psi_{i(N-2p)}^* \otimes \psi_{j(1)}^* \cdots \psi_{j(N-2p)}^*$$

where $c_{i,j} = q^{(k-p)(h+j-p)-\sigma(i)-\sigma(j)-\frac{(N-2p)(N-2p-1)}{2}}$. Using the commutation rules we can reorder the above summation, obtaining:

$$\psi_1^{*h+j}\psi_2^{*k} \otimes \psi_1^h\psi_2^{j+k} = \sum_{p=1}^{\min(h,k)} C_p\psi_1^{*h+j-p}\psi_2^{*k-p} \otimes \psi_1^{h-p}\psi_2^{j+k-p}$$

□

Remark 5.4.7. We are not able to give a concrete and explicit formula for C_l, but following the proof of the previous theorem it is easy to understand what is the procedure in order to obtain them.

Corollary 5.4.8. *The relations showed above together with the commutation rules are all the possible relations on $\mathcal{C}(N)$, and the following set is a linear basis of $\mathcal{C}(N)$:*

$$\begin{aligned} B =&\{e_1^i e_2^j e_3^k | i,j,k \in \{0,\dots,N-1\}, i+j+k < N\} \cup \\ &\cup \{e_1^i e_2^{*j} e_3^k | i,k \in \{0,\dots,N-2\}, j \in \{1,\dots,N-1\}, i+j+k < N\} \end{aligned}$$

Proof. Using the relations we found, it is easy to see that all the (non-commutative) monomials in e_1, e_2, e_2^*, e_3 are linear combinations of the monomials $e_1^i e_2^j e_3^k$ and $e_1^i e_2^{*j} e_3^k$. If $i+j+k = N$, then $e_1^i e_2^j e_3^k$ and $e_1^i e_2^{*j} e_3^k$ are equals to a linear combinations of monomials of degrees lower than N, because of truncation relations. If:

$$|B| = \sum_{i=1}^{N} i^2$$

then we can conclude for a dimensional argument. It is sufficient to count how many elements of the type $e_1^i e_2^j e_3^k$ and $e_1^i e_2^{*j} e_3^k$ we have for fixed $j > 0$ and $n = i+j+k$. Elements of the first type are $n-j+1$ since $i \in \{0,\dots,n-j\}$ and k is completely determined by i and j, and the same for the element of the second type. So, if we sum on $j = 1,\dots,n$, we have:

$$\sum_{j=1}^{n} 2(n-j+1) = 2[n^2 - (\sum_{i=1}^{n} j) + n] = 2[n^2 - \frac{n(n+1)}{2} + n] = n(n+1)$$

It remains the case when $j = 0$. In this case we only have elements of the first type, and they are $n+1$. Therefore, the elements of B of degree n are $n(n+1)+(n+1) = (n+1)^2$. Now, summing up on n we have:

$$|B| = \sum_{n=0}^{N-1} (n+1)^2 = \sum_{i=1}^{N} i^2$$

putting $i = n+1$. In this way we obtain the desired result. □

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