Elliptic problems for some anisotropic operators

Ph.D. Thesis

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Contents

Introduction

This Ph.D. Thesis is devoted to the study of some classes of elliptic boundary value problems, associated with an anisotropic operator, in a bounded domain Ω of $\mathbb{R}^N$, $N \geq 3$. The interest in these problems relies on the fact that they are nonlinear. Indeed the anisotropic operator which we consider in our studies, weighs partial derivatives with different powers, $p_i > 1$, that is

$$-\sum_{i=1}^{N} \partial_i[|\partial_i u|^{p_i-2}\partial_i u], \tag{I.1}$$

where $\partial_i = \partial/\partial x_i$, for $i = 1, ..., N$. Moreover it is non homogeneous. Therefore in order to obtain existence, nonexistence and regularity results for both weak and distributional solutions, we need to perform some essential modifications of the classic methods developed by several authors in the study of partial differential equations with Dirichlet boundary conditions. We note that if $p_i = 2$ for all i, then (I.1) reduces to the well-known linear operator, the laplacian and if $p_i = p$ for all i we obtain the pseudo p-laplacian.

In the first chapter of the Thesis we study existence and regularity of the solutions for Dirichlet problems, such as,

$$\begin{cases} -\sum_{i=1}^{N} \partial_i[|\partial_i u|^{p_i-2}\partial_i u] = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{I.2}$$

where f is a given function belonging to either a Lebesgue space $L^m(\Omega)$ or a Marcienkiewicz space $M^m(\Omega)$, $m \geq 1$. Moreover, we consider the case of the datum in divergence form, that is $f = \sum_{i=1}^{N} \partial_i f_i$, with f_i in some Lebesgue spaces for any $i = 1, ..., N$.

We need to consider a different functional setting from the classical Sobolev spaces, in order to develop our theory for both weak and distributional solutions for (I.2), namely the anisotropic Sobolev spaces, to which the solutions for our problems naturally belong:

$$\begin{cases} W^{1,(p_i)}(\Omega) = \{v \in W^{1,1}(\Omega) : \partial_i v \in L^{p_i}(\Omega)\}, \\ W_0^{1,(p_i)}(\Omega) = W^{1,(p_i)}(\Omega) \cap W_0^{1,1}(\Omega). \end{cases} \tag{I.3}$$

We refer to [**54**], [**68**] and [**79**] for the theory of these spaces. Let us define

$$p_\infty = \max\{p_{\max}, \overline{p}^*\}, \quad p_{\max} = \max_i\{p_i\}, \quad \overline{p}^* = \frac{\overline{p}N}{N - \overline{p}}, \quad \overline{p} < N \tag{I.4}$$

and

$$\frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}, \tag{I.5}$$

where $\overline{p}^*$ is the "usual" critical exponent of the embedding theorems, related to the harmonic mean $\overline{p}$ of p_i's. The existence of weak solutions ((1.2.3) in Section 1.2) for problem (I.2), if $f \in L^m(\Omega)$, with $m \geq p'_\infty = p_\infty/(p_\infty - 1)$, is a consequence of the classic Leray-Lions theorem (see [**57**] and [**59**]) and suitable embedding theorems (Section 1.2). J. Leray and J.-L. Lions have showed that it is possible to extend the existence theory for monotone operators to (I.1) and to more general anisotropic operators (see (I.19) below). The same result is also obtained if $f = \sum_{i=1}^N \partial_i f_i$, with $f_i \in L^{m_i}(\Omega)$, $m_i \geq p'_i$, for every $i = 1, ..., N$ (Section 1.4). Since the following inclusion between Marcinkiewicz and Lebesgue spaces holds

$$M^m(\Omega) \subset L^{m-\varepsilon}(\Omega) \qquad \forall\ m > 1 \text{ and } 0 < \varepsilon \leq m - 1, \tag{I.6}$$

we also obtain the existence of at least a weak solution for (I.2) if $f \in M^m(\Omega)$ with $m > p'_\infty$, see Section 1.3.

The principal and more interesting subject of the first chapter of this Thesis concerns the regularity of solutions. If $m \geq p'_\infty$ (resp. $m > p'_\infty$) we show that the weak solutions, a priori only belonging to the energy space $W_0^{1,(p_i)}(\Omega)$, have an extra summability, that is $u \in L^s(\Omega)$ (resp. $M^s(\Omega)$) for some s depending on the summability of the datum f. If $f \in L^m(\Omega)$, $1 \leq m < p'_\infty$ (resp. $f \in M^m(\Omega)$, $1 < m \leq p'_\infty$), we prove that the distributional solutions belong to some anisotropic Sobolev spaces, not finite energy spaces but better than $W_0^{1,1}(\Omega)$, to which u a priori belongs, as well as its existence (using an approximation method, a priori estimates and compactness results in suitable spaces).

We recall that problems as (I.2) have been studied by several authors. In [**26**], the global L^∞-boundedness of solutions for some differential problems including (I.2) has been studied with f in a divergence form, that is $f = \sum_{i=1}^{N} \partial_i f_i$, $f_i \in L^{m_i}(\Omega)$, $m_i \geq p_i'$, for $i = 1, ..., N$ and

$$\frac{\overline{p}^*}{p_{\max}} \min_i \left\{ 1 - \frac{p_i'}{m_i} \right\} > 1, \tag{I.7}$$

with $\overline{p}^*$ and $p_{\max}$ as in (I.4). Moreover similar results for minimizers of some functionals of the Calculus of Variations are proved. Subsequently, the result of [**26**] has been improved in [**76**], replacing the assumption (I.7) with the more general condition

$$\frac{\overline{p}^*}{\overline{p}} \min_i \left\{ 1 - \frac{p_i'}{m_i} \right\} > 1. \tag{I.8}$$

The case

$$\frac{\overline{p}^*}{\overline{p}} \min_i \left\{ 1 - \frac{p_i'}{m_i} \right\} < 1, \tag{I.9}$$

has been studied in [**25**] for minimizers of some functionals, always with datum f in divergence form. To be complete we also report these results in Section 1.4 but we will give slightly different proofs (see Theorems 1.29 and 1.31 in Section 1.5). In particular to prove Theorem 1.31 we will use a new inequality: a weighted anisotropic Sobolev inequality (Lemma 1.2, proved in Section 1.1).

Finally in [**19**] (I.2) has been considered, on the right hand side, a bounded Radon measure. The existence of a solution is shown in the anisotropic Sobolev space $W_0^{1,(s_i)}(\Omega)$, as in (I.3), with s_i such that

$$1 \leq s_i < \frac{N(\overline{p}-1)}{\overline{p}(N-1)} p_i \qquad \forall\ i = 1, ..., N,$$

$\overline{p}$ as in (I.5) and with the additional assumption $2 - 1/N < \overline{p} < N$. It is well known that in the classic case, i.e. $p_i = p$ for all i, if $p \in (1, 2 - 1/N]$ one cannot expect solutions to belong to $W_0^{1,1}(\Omega)$ and hence, the notion of weak derivatives and distributional solutions breaks down. This problem is dealt with in literature using, for example, the notion of entropy solutions and of "approximated gradient", that has been introduced in [**11**] (see also [**21**]). So, for an anisotropic problem as (I.2), we cannot expect solutions to belong to $W_0^{1,1}(\Omega)$ as long as $\overline{p} \in (1, 2 - 1/N]$, but we can extend the notion of entropy solutions (see Section 1.6). This kind of solution is stronger than the distributional one. They do not belong to $W_0^{1,(p_i)}(\Omega)$ but their truncations to any level $k > 0$, $T_k(s) = \max\{-k, \min\{k, s\}\}$ (see also (1.1.7) in Section 1.1), are in the energy space. Moreover an entropy solution verifies the following inequality

$$\sum_{i=1}^{N}\int_{\Omega}|\partial_i u|^{p_i-2}\partial_i u\partial_i T_k(u-\varphi)\leq\int_{\Omega} fT_k(u-\varphi) \tag{I.10}$$

$$\forall\ k>0 \quad \text{and} \quad \forall\ \varphi\in W_0^{1,(p_i)}(\Omega)\cap L^{\infty}(\Omega).$$

This definition also allows us to obtain an existence result without further assumptions on $\overline{p}$, also for $\overline{p}\in(1,2-1/N]$ and to prove a uniqueness result. We show that the solution, obtained using approximation techniques (as in [**19**]) is an entropy one (see Theorem 1.14 and Theorem 1.17).

As already mentioned, the aim of the first chapter is to complete the framework of regularity, covering all the possible data f. We wish to call to the reader's attention the known results about the regularity of solutions for the problems

$$\begin{cases}-\Delta_p u=f & \text{in } \Omega,\\ \quad u=0 & \text{on } \partial\Omega.\end{cases} \tag{I.11}$$

It is well known, that, if $f\in L^m(\Omega)$, then

1a) $m>N/p$ implies $u\in W_0^{1,p}(\Omega)\cap L^{\infty}(\Omega)$;
2a) $m=N/p$ implies $u\in W_0^{1,p}(\Omega)$ and

$$\int_{\Omega} e^{\beta|u|}<+\infty,$$

for some constant $\beta>0$;
3a) $(\frac{pN}{N-p})'=(p^*)'\leq m<N/p$ implies $u\in W_0^{1,p}(\Omega)\cap L^s(\Omega)$, $s=\frac{mN(p-1)}{N-mp}$;
4a) $1<m<(p^*)'$ implies $u\in W_0^{1,s}(\Omega)$, $s=\frac{mN(p-1)}{N-m}$, with $p>1/m^*+1$;
5a) $m=1$ implies $u\in W_0^{1,s}(\Omega)$, for all $1<s<\frac{N(p-1)}{N-1}$ and $2-\frac{1}{N}<p<N$.

Let $f\in M^m(\Omega)$, then

1b) $m>N/p$ implies $u\in W_0^{1,p}(\Omega)\cap L^{\infty}(\Omega)$;
2b) $m=N/p$ implies $u\in W_0^{1,p}(\Omega)$ and

$$\int_{\Omega} e^{\beta|u|}<+\infty,$$

for some constants $\beta>0$;
3b) $(\frac{pN}{N-p})'=(p^*)'<m<N/p$ implies $u\in W_0^{1,p}(\Omega)\cap M^s(\Omega)$, $s=\frac{mN(p-1)}{N-mp}$;
4b) $1<m\leq(p^*)'$ implies $u\in W_0^{1,1}(\Omega)\cap M^s(\Omega)$, $s=\frac{mN(p-1)}{N-mp}$ and $\nabla u\in M^s(\Omega)$, $s=\frac{mN(p-1)}{N-m}$, always for $2-\frac{1}{N}<p<N$.

For linear operators, that is if $p = 2$ in (I.11), $1a)$, $2a)$, $3a)$, $1b)$, $2b)$, $3b)$ were proved for the first time by G. Stampacchia (see [**75**]). For the nonlinear operators the techniques, introduced by G. Stampacchia, can be adapted to prove $1b)$, $2b)$, $3b)$. Hence, also $1a)$ and $2a)$ hold, because of the following inclusion

$$L^m(\Omega) \subset M^m(\Omega), \qquad \forall\ m > 1. \tag{I.12}$$

A different proof is necessary for $3a)$, see [**23**] and [**24**]. If $f \in L^m(\Omega)$, with $1 \le m < (p^*)'$, the existence and the regularity of the distributional solutions for problem (I.11), (or more in general for a Leray-Lions type operator), have been proved in [**16**] and [**17**], and in the case f belonging in a Marcinkiewicz space in the recent paper [**14**].

For anisotropic problems such as (I.2), we need to distinguish two cases: $p_\infty = \overline{p}^*$ and $p_\infty = p_{\max}$, as in (I.4). It is well known that both the theory of existence and the regularity results for elliptic problems are strongly affected by the embedding results concerning the spaces in which we look for solutions. For anisotropic Sobolev spaces the critical embedding exponent depends on the type of anisotropy. Said exponent is $\overline{p}^*$, if the anisotropy is concentrated, while it is $p_{\max}$, if the anisotropy is not concentrated (see [**45**]). So if $p_\infty = \overline{p}^* \ge p_{\max}$, roughly speaking, we prove the same results of (I.11), by substituting p with the harmonic mean $\overline{p}$ of the p_i's, as in (I.5), and by substituting the "standard" Sobolev spaces with the anisotropic Sobolev spaces (I.3). Moreover, we obtain new regularity results for some choices of m and $p_{\max}$ close to $\overline{p}^*$, namely $p_{\max} \in ((N-1)\overline{p}^*/N, \overline{p}^*)$ (see Remark 1.16). If $p_\infty = p_{\max} > \overline{p}^*$ we obtain new results concerning both regularity and existence. As a matter of fact, if $p'_{\max} \le m < (\overline{p}^*)'$ we obtain weak solutions, not distributional ones and in particular finite energy solutions (see Theorems 1.7, 1.17, 1.20, 1.26 and 1.25).

The plan of Chapter 1 is the following: we start dealing with f belonging to a Lebesgue space $L^m(\Omega)$ (Section 1.2) and then we study the new case of f in a Marcinkiewicz space $M^m(\Omega)$ (Section 1.3). In Section 1.4 we consider a datum f in divergence form. In Section 1.5 we give the proofs of the main results and in the last section of Chapter 1 we deal with the uniqueness problem. For weak solutions, uniqueness is a direct consequence of the monotone property of the operator (I.1) (see Remark 1.12). In order to obtain uniqueness for distributional solutions, we introduce the notion of entropy solutions (Section 1.6). The main results presented in this chapter are contained in [**34**].

In Chapter 2 we consider nonlinear elliptic problems as (I.2) but with additional lower order terms, that play the role of perturbations terms. These have a so called natural growth with respect to the gradient, that means growths of the same order of the operator (I.1), because they naturally appear if we write the Euler-Lagrange equations of suitable functionals of Calculus of Variations. With no hope of being complete, in the isotropic case, i.e. $p_i = 2$ or $p_i = p$ for all i, we mention some papers regarding the study of these problems with natural growth terms, and the references therein [**12**], [**13**], [**18**], [**20**], [**22**], [**28**], [**29**], [**30**] and [**33**].

In Section 2.1 we study the following problem

$$\begin{cases} -\sum_{i=1}^N \partial_i[|\partial_i u|^{p_i-2}\partial_i u] + \mu_0 u = \sum_{i=1}^N b_i(x,u,\nabla u) + f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{I.13}$$

where $\mu_0 > 0$,

$$f \in L^m(\Omega), \qquad m > \frac{\overline{p}^*}{\overline{p}^* - p_{\max}} \quad \text{and} \quad \overline{p}^* > p_{\max},$$

$$b_i(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R},$$

is a Carathéodory function, for all $i = 1, ..., N$ and there exists $\gamma > 0$ such that the following inequality is true for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and a.e. $x \in \Omega$

$$|b_i(x,s,\xi)| \le \gamma |\xi_i|^{p_i}, \qquad \forall\ i = 1, ..., N. \tag{I.14}$$

In this problem we do not have any information about the sign of b_i, for all i, and so we must add the zero-th term $\mu_0 u$ to prove the existence of a weak solution. As a matter of fact, it is well known, as in the special case $p_i = 2$ or $p_i = p$ for all i, that the sign condition has a strong regularity effect that allows us to easily obtain a priori estimates from the equation. On the other hand, if the sign condition is not satisfied, the problem (I.13) may not even have solutions. In fact, also if $\mu_0 = 0$, a solution for problem (I.13) exists only if we assume that the norm of the datum f is small. We present an existence result following the techniques in [**30**] and [**13**].

In Section 2.2 we analyse the following problem

$$\begin{cases} -\sum_{i=1}^N \partial_i[|\partial_i u|^{p_i-2}\partial_i u] + \sum_{i=1}^N g_i(x,u,\nabla u) = f & \text{in } \Omega, \\ u \in W_0^{1,(p_i)}(\Omega) \qquad g_i(x,u,\nabla u) \in L^1(\Omega) & \forall\ i = 1, ..., N. \end{cases} \tag{I.15}$$

where $g_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ are Carathéodory functions such that for almost every $x \in \Omega$ and for all $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$

$$g_i(x,s,\xi)s \ge 0, \qquad \forall\ i = 1, ..., N, \tag{I.16}$$

$$|g_i(x,s,\xi)| \le b(|s|)|\xi_i|^{p_i}, \qquad \forall\ i=1,...,N, \tag{I.17}$$

where $b:\mathbb{R}\to\mathbb{R}$ is a continuous and nondecreasing function. Hence in this case we assume the sign condition (I.16). Moreover we also suppose, as well as (I.17), one of the following two assumptions: either f belongs to the dual space of $W_0^{1,(p_i)}(\Omega)$ or $f\in L^1(\Omega)$, but we also require a sort of coercivity, namely that $\sigma>0$ and $\gamma>0$ exist such that

$$|g_i(x,s,\xi)| \ge \gamma|\xi_i|^{p_i} \text{ when } |s|>\sigma,\ \ \forall\ i=1,...,N. \tag{I.18}$$

We prove the existence of weak solutions for problem (I.15) belonging to the finite energy space $W_0^{1,(p_i)}(\Omega)$, also in the case f only in $L^1(\Omega)$. This fact depends on the extra assumption on g_i, (I.18). So the presence of the terms g_i's turns out to provide more regular solutions. The role of (I.18) is to give an a priori estimate in energy space $W_0^{1,(p_i)}(\Omega)$ which allows us to deal with the lower order term.

Both (I.13) and (I.15) do not correspond to the Euler-Lagrange equation of any functional of Calculus of Variations. So we will use, as in Chapter 1, direct methods to deal with them. Namely we build approximating problems, we derive from the equations a priori estimates and then we use compact results for anisotropic Sobolev spaces.

We highlight that all the results in Chapter 1 and 2, are still valid if we substitute the operator (I.1) with a more general one, namely as

$$\begin{cases} -\sum_{i=1}^N \partial_i[a_i(x,\nabla u)] = f & \text{in } \Omega, \\ \qquad\qquad u=0 & \text{on } \partial\Omega, \end{cases} \tag{I.19}$$

with $a_i(x,\xi):\Omega\times\mathbb{R}^N\to\mathbb{R}^N$, Carathéodory functions that satisfy, for some constants α, $\beta>0$, for a.e. $x\in\Omega$, and for every ξ, $\eta\in\mathbb{R}^N$ with $\xi\neq\eta$,

$$\sum_{i=1}^N (a_i(x,\xi)-a_i(x,\eta))(\xi_i-\eta_i)>0, \tag{I.20}$$

$$\sum_{i=1}^N a_i(x,\xi)\xi_i \ge \alpha\sum_{i=1}^N |\xi_i|^{p_i} \tag{I.21}$$

and

$$|a_i(x,\xi)| \le \beta\left(\sum_{j=1}^N |\xi_j|^{p_j}\right)^{1-\frac{1}{p_i}}, \qquad \forall\ i=1,...,N. \tag{I.22}$$

Note that the last growth condition is satisfied (see [**61**]) for example if $a_i(x,\xi) = \partial_{\xi_i} A(x,\xi)$, with $A(x,\xi)$ a Carathéodory function, convex with respect to ξ and such that

$$|A(x,\xi)| \leq C \left(\sum_{i=1}^{N} |\xi_i|^{p_i} \right).$$

This kind of equation belongs to a more general class

$$-\sum_{i=1}^{N} \partial_i [a_i(x,\nabla u)] = f, \tag{I.23}$$

where $a_i(x,\xi)$ satisfies the so called (p,q)-growth conditions (in our case $q = p_{\max}$ and $p = p_{\min} = \min_i\{p_i\}$), that is for every $\xi \in \mathbb{R}^N$ and for a.e. $x \in \Omega$, for some α, $\beta > 0$ and $q \geq p > 1$,

$$\sum_{i=1}^{N} a_i(x,\xi)\xi_i \geq \alpha |\xi|^p \tag{I.24}$$

and

$$|a_i(x,\xi)| \leq \beta(1 + |\xi|^{q-1}), \qquad \forall \ i = 1, ..., N. \tag{I.25}$$

The interest in the study of this type of problem is rather recent and it has been increasing in the last few years. After sporadic papers (see for instance [**81**] and related references) a systematic study of regularity of solutions for these kinds of equations (or minima of functionals), with growth of (p,q)-type, was initiated by P. Marcellini, (see [**61**], [**62**], [**63**]). He pointed out that suitable smoothness assumptions assure existence and regularity of solutions for equations of type (I.23), and of minima of functionals, since a related notion can be introduced for functionals. We do not deal with them in the first part of this Thesis. We also recall some interesting papers about the regularity of minimizers of functionals of Calculus of Variations with non standard growth conditions: [**1**], [**2**], [**46**], [**47**], [**48**] and [**73**]. All of these papers focus principally on local and higher regularity, such as boundedness of ∇u, Hölder continuity, Lipschitz regularity, without assuming Dirichet boundary conditions. As we will see later, Dirichlet boundary conditions allow us to avoid restricting the interval covered by $p_i's$. In fact Marcellini's approach works if the ratio q/p does not differ too much from 1, depending on the dimension N; which, roughly speaking, means that the numbers q and p cannot be too far apart. If we take (I.20), (I.21) and (I.22) into consideration, we obtain better results than the ones obtained assuming the (p,q)-growth condition, since the functions a_i are bounded from above and from below by the same quantity. Another relevant class of anisotropic operators, for which a general and almost complete theory is now available, is without doubt one of the equations with the so called $p(x)$-growth, i.e. the $p(x)$-Laplacian equation, that is

$$-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = f, \tag{I.26}$$

with $p : \Omega \to (1, \infty)$ a bounded and continuous function. We recall some papers (and references therein), in which this theory is developed: [**36**], [**37**], [**38**], [**39**], [**43**], [**52**], [**55**], [**71**] and [**72**].

In Chapter 3, whose main results are contained in a joint work with Eugenio Montefusco (see [**35**]), we study the question of existence, nonexistence and multiplicity of positive solutions for a class of semilinear elliptic problems, associated to the operator (I.1),

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_i \left[|\partial_i u|^{p_i-2}\, \partial_i u \right] = \lambda |u|^{q-2} u & \text{in } \Omega \\ \qquad\qquad\qquad\qquad u = 0 & \text{on } \partial\Omega \end{cases} \tag{I.27}$$

where Ω is always a bounded subset of $\mathbb{R}^N$, but $N \geq 2$. Moreover $\lambda > 0$ is a real parameter. In particular we are interested in the case

$$p_{\min} = \min_i \{p_i\} < q < p_{\max} = \max_i \{p_i\}. \tag{I.28}$$

In this chapter we exploit the methods of Calculus of Variations, that is, we consider the functional associated to our equation

$$J_\lambda(v) = \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i v|^{p_i} - \frac{\lambda}{q} \int_\Omega |v^+|^q, \tag{I.29}$$

where $v^+ = \max\{v, 0\}$ is the positive part of the function v, and we try to determine critical levels for it. The study of semilinear elliptic equations has produced a large amount of functionals, topological and variational techniques (and results) about existence and nonexistence of solutions for the Dirichlet problem (see for example [**7**], [**53**], [**77**]). Much less is known about anisotropic elliptic problems like (I.27), let us recall some recent works only [**4**], [**43**], [**44**], [**58**], [**64**], [**65**], [**67**]. Many of these articles concern the $p(x)$-operators. Recently two interesting papers on anisotropic elliptic problems appeared, connected to the problem (I.27): [**45**] and [**78**]. In [**45**] many existence and nonexistence results of positive solutions are proved. In particular, using variational methods, the case $q > p_{\max}$ is thoroughly investigated. In [**78**] a more general problem than (I.27) is handled: with respect to our problem the authors prove the existence of a solution, also for q positive and subcritical, using approximation methods.

In both the previous papers the case $p_{\min} < q < p_{\max}$ is not investigated in order to study a new situation. Indeed if (I.28) holds then the reaction term (that is the right hand side in the equation) has an "intermediate" growth with respect to the differential operator and the problem (I.27) looks in some sense like an eigenvalue problem. More precisely we are able to prove the existence of at least two positive solutions for sufficiently large values of λ (Theorems 3.9 and 3.13) and nonexistence (see Proposition 3.7), for small positive λ.

We point out to the reader the papers [**15**], [**5**], [**6**], [**9**] (see also the references therein) where some isotropic elliptic problems with convex-concave nonlinearities are considered. The nonlinearity produces some multiplicity results of positive solutions for small values of the parameter $\lambda > 0$ and nonexistence results for large λ. These results are to a certain degree, contrary to ours. We think that the nonlinearity of the reaction term produces a "superlinear" (or convex) effect interacting with the small growths of the operator (I.1), and a "sublinear" (or concave) effect with the higher exponents which appear in the differential operator. The key point in all of our arguments is the presence of different homogeneities. Clearly, such a situation does not hold if the operator is of an isotropic type, but it appears if the reaction term is the sum of different nonlinearities. As we have already mentioned, all the results we obtain are due to the variational structure of the problem. We want to stress that any critical point of J_λ is a weak non-negative solution of (I.27) (see (3.0.3)).

The plan of Chapter 3 is the following: first, in Section 3.1, we recall some known results, and we give a more possibly complete image of the actual stage of research on this topic.

- $q > p_{\max}$: the Mountain-Pass theorem can be applied in order to show that for any $\lambda > 0$ a weak solution of problem (I.27) exists.
- $q < p_{\min}$: the Ekeland Variational Principle can be used in order to prove the existence of λ^* such that, for any $\lambda \in (0, \lambda^*)$ there is a nontrivial positive weak solution of (I.27). Moreover, the energy functional, J_λ, has a nontrivial minimum for any positive λ large enough. So for any $\lambda \in (0, \lambda^*) \cup (\lambda^{**}, +\infty)$ at least one weak solution of (I.27) exists.

In the first case the result is due to [**45**], (for more general case $p_i = p_i(x)$ for $i = 1, ..., N$, see [**64**] and [**65**]) while in the case $q < p_{\min}$ to [**64**] and [**65**]. In Section 3.2 we begin with the new results. We prove that a weak solution of problem (I.27) does not exist, different from zero, for λ small (Proposition 3.7) and that there exists a global minimum for J_λ, which is a weak non-negative solution of problem (I.27), (Theorem 3.9). In Section 3.3 we show (under suitable assumptions) that the functional J_λ also possesses a Mountain-Pass critical point, that is a second non-negative solution of (I.27) (Theorem 3.13). In Section 3.4 we prove a maximum principle for anisotropic operators which implies that the weak solutions found in the preceding sections are positive. Finally, in Section 3.5 we study some global properties of the branch of positive solutions of (I.27) and we present some open problems connected with our studies.

CHAPTER 1

Existence and regularity in the monotone case

1.1. Notations and basic tools

We briefly recall the functional analytic framework of the differential operator which we are going to study. We assume Ω to be an open, bounded domain of $\mathbb{R}^N$, $N \geq 3$. Let

$$p_i > 1 \qquad \text{for } i = 1, ..., N, \qquad p_{\max} = \max_i \{p_i\}, \qquad \text{and} \qquad p_{\min} = \min_i \{p_i\}. \tag{1.1.1}$$

Without loss of generality, we can assume

$$p_1 \leq p_2 \leq ... \leq p_N,$$

so that $p_{\max} = p_N$ and $p_{\min} = p_1$. We will see that the "natural" spaces in which we search for solutions to Dirichlet elliptic problems (I.2), are the anisotropic Sobolev spaces

$$\begin{cases} W^{1,(p_i)}(\Omega) = \{v \in W^{1,1}(\Omega) : \partial_i v \in L^{p_i}(\Omega)\}, \\ W_0^{1,(p_i)}(\Omega) = W^{1,(p_i)}(\Omega) \cap W_0^{1,1}(\Omega). \end{cases} \tag{1.1.2}$$

$W_0^{1,(p_i)}(\Omega)$ can also be defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|v\|_{W_0^{1,(p_i)}(\Omega)} = \sum_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)}.$$

In [**45**], [**54**], [**68**], [**79**], the theory of these spaces is developed and in particular the corresponding Sobolev embedding theorems are studied. Let

$$\overline{p}^* = \frac{N\overline{p}}{N - \overline{p}}, \quad \text{for} \quad \overline{p} < N, \qquad \frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i} \quad \text{and} \quad p_\infty = \max\{p_N, \overline{p}^*\}. \tag{1.1.3}$$

In [**79**] it is proved that if $\overline{p} < N$, then

$$W_0^{1,(p_i)}(\Omega) \hookrightarrow L^r(\Omega), \qquad \forall \ r \in [1, \overline{p}^*]. \tag{1.1.4}$$

This embedding is continuous and also compact if $r < \overline{p}^*$. The following Sobolev type inequality is also proved: there exists a positive constant C, depending only on Ω, such that

$$\|v\|_{L^r(\Omega)} \le C \prod_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)}^{\frac{1}{N}}, \qquad \forall\ r \in [1, \overline{p}^*], \tag{1.1.5}$$

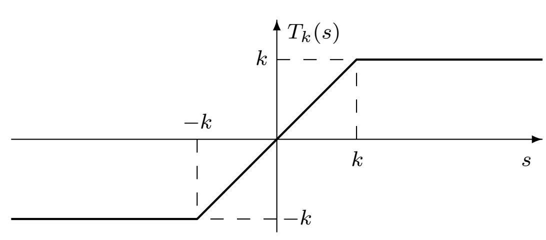
for any $v \in C_0^1(\Omega)$ where $p_i > 1$ for $i = 1, 2, ..., N$, and $\overline{p}^*$ as above. By density, (1.1.5) also holds for any $v \in W_0^{1,(p_i)}(\Omega)$. The inequality (1.1.5) then implies that

$$\|v\|_{L^r(\Omega)} \le C \sum_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)}, \qquad \forall\ r \in [1, \overline{p}^*]. \tag{1.1.6}$$

If $\overline{p} \ge N$, then (1.1.4) holds for any $r \ge 1$. Subsequently in [**45**] it is proved that the critical exponent depends on the kind of anisotropy. If the p_i's are not "too far apart" (i.e. the anisotropy is concentrated) the critical exponent is $\overline{p}^*$, like in [**79**], that is the "usual" critical exponent related to the harmonic mean $\overline{p}$ of the p_i's. While if the p_i's are "too spread out" it coincides with the maximum of the p_i's, i.e. p_N. Hence the effective critical exponent is p_∞, as in (1.1.3). This fact produces some technical difficulties as we will see later.

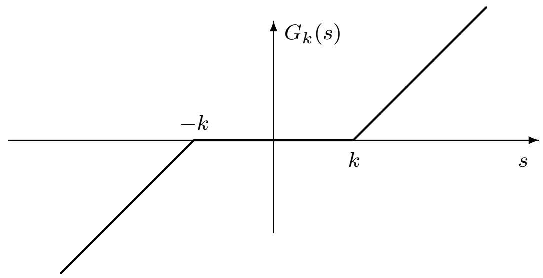
We consider the composition of functions in $W_0^{1,(p_i)}(\Omega)$ with some useful auxiliary functions of real variable. One of the most used, in the following, is the truncation function at level $k > 0$, T_k, that is

$$T_k(s) = \begin{cases} k\dfrac{s}{|s|} & \text{if} \quad |s| > k\,, \\ s & \text{if} \quad |s| \le k\,; \end{cases} \tag{1.1.7}$$



Moreover, let

$$G_k(s) = s - T_k(s), \quad \text{with } k \geq 0, \tag{1.1.8}$$



Now we recall some known lemmas we need in the following.

LEMMA 1.1. *Let p_1, p_i, p_N and $\overline{p}^*$ be as in* (1.1.1) *and* (1.1.3), *and let $B \geq 1$. Then the following inequality holds*

$$\|v\|^{p_N}_{L^{\overline{p}^*}(\Omega)} \leq C^{p_N} N^{p_N - 1} B^{p_N - p_1} \sum_{i=1}^{N} \|\partial_i v\|^{p_i}_{L^{p_i}(\Omega)}, \tag{1.1.9}$$

for all $v \in W_0^{1,1}(\Omega)$ such that

$$\|\partial_i v\|_{L^{p_i}(\Omega)} \leq B, \quad \forall\ i = 1, 2, ..., N.$$

PROOF. See [**26**]. ■

We remember also this Poincaré type inequality, valid for all $v \in W_0^{1,(p_i)}(\Omega)$,

$$\|v\|_{L^r(\Omega)} \leq \frac{ar}{2} \|\partial_i v\|_{L^r(\Omega)}, \qquad \forall\ r \geq 1, \tag{1.1.10}$$

where $a = \text{diam}(\Omega)$ (see [**45**]).

Now we prove a new technical Lemma that plays an important role in showing some results presented in the following sections, and in extending some techniques used in the isotropic case. It is a weighted Sobolev type inequality.

LEMMA 1.2. *Let $v \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)$, with the p_i's as above and suppose that $\sum_{i=1}^N 1/p_i > 1$, that is $\overline{p} < N$. Then it results*

$$\left(\int_\Omega |v|^r\right)^{\frac{N}{\overline{p}} - 1} \leq C \prod_{i=1}^{N} \left(\int_\Omega |\partial_i v|^{p_i} |v|^{t_i p_i}\right)^{\frac{1}{p_i}}, \tag{1.1.11}$$

for every r and $t_i \geq 0$ satisfying

$$\begin{cases} \dfrac{1}{r} = \dfrac{\gamma_i(N-1)-1+1/p_i}{t_i+1} > 0, & \forall\ i=1,...,N, \\ \\ \sum_{i=1}^N \gamma_i = 1, \qquad \gamma_i \geq 0, & \forall\ i=1,...,N. \end{cases} \tag{1.1.12}$$

The constant C depends only on p_i, $i=1,...,N$ and N. Moreover, (1.1.11) *holds also if $t_i<0$, γ_i and r as above, but we must already know that the integrals, which appear in the right side hand of* (1.1.11)*, are finite.*

Proof. Using the techniques introduced by Troisi (see Theorem 1.2 of [**79**]), we obtain

$$\left(\int_\Omega |v|^r\right)^{N-1} = \left(\int_\Omega |v|^{r\gamma_1+\ldots+r\gamma_N}\right)^{N-1} \leq \prod_{i=1}^N \int_{S_i} \sup_{x_i} |v|^{\gamma_i r(N-1)} dS_i,$$

where S_i, for every $i=1,...,N$, is the intersection between Ω and the hyperplane $x_i=0$ and $\gamma_i \geq 0$ for all i such that $\sum_{i=1}^N \gamma_i = 1$. On the other hand we have

$$|v(x)|^{\gamma_i r(N-1)} \leq \gamma_i r(N-1) \int_{-\infty}^{+\infty} |v|^{\gamma_i r(N-1)-1} |\partial_i v| dx_i.$$

Hence it results

$$\left(\int_\Omega |v|^r\right)^{N-1} \leq C \prod_{i=1}^N \int_\Omega |v|^{\gamma_i r(N-1)-1}|\partial_i v| = C\prod_{i=1}^N \int_\Omega |\partial_i v||v|^{t_i}|v|^{\beta_i}, \tag{1.1.13}$$

where $t_i \geq 0$ and $\beta_i > 0$ are chosen in such a way that

$$t_i + \beta_i = \gamma_i r(N-1) - 1, \qquad \forall\ i=1,...,N,$$

that is

$$\beta_i = -t_i + \gamma_i r(N-1) - 1,$$

and $C = r(N-1)$. By applying Hölder inequality with exponents p_i and p_i' to each term of the product which appears in (1.1.13), we obtain

$$\left(\int_\Omega |v|^r\right)^{N-1} \leq C \prod_{i=1}^N \left(\int_\Omega |\partial_i v|^{p_i}|v|^{t_i p_i}\right)^{\frac{1}{p_i}} \left(\int_\Omega |v|^{\beta_i p_i'}\right)^{\frac{1}{p_i'}}.$$

Now we define $r = \beta_i p_i'$, for every $i=1,...,N$. We get

$$\left(\int_\Omega |v|^r\right)^{N-1} \leq C \left(\int_\Omega |v|^r\right)^{\sum_{i=1}^N \frac{1}{p_i'}} \prod_{i=1}^N \left(\int_\Omega |\partial_i v|^{p_i}|v|^{t_i\, p_i}\right)^{\frac{1}{p_i}}.$$

Observing that

$$\sum_{i=1}^{N} \frac{1}{p_i'} < N-1 \quad \Leftrightarrow \quad \overline{p} < N,$$

(1.1.11) then follows. ∎

REMARK 1.3. We note that if $\overline{p} \geq N$, then (1.1.11) holds for every $r \geq 1$, $\gamma_i \geq 0$, such that $\sum_{i=1}^{N} \gamma_i = 1$, and $t_i \geq 0$, where $i = 1, ..., N$. Moreover in this case C also depends on the measure of the set Ω.

Finally, let us devote a few words to positive constants. We will write C to denote positive constants, possibly different depending on the data, that is they are fixed in the assumptions we make, as the dimension N, the bounded open set Ω, etc. During the proofs of our results, similar constants will also be indicated by C_i, $i = 0, 1, 2, ...$ to distinguish possibly different values. In any case the constants are always meant to not depend on n.

1.2. Data in Lebesgue spaces

In this and in the following sections, we present some results contained in [**34**]. We consider the following problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_i[|\partial_i u|^{p_i-2}\partial_i u] = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2.1}$$

We will give some results concerning existence and regularity of weak or distributional solutions of (1.2.1), where f is a given function belonging to a Lebesgue space.

Now and in the following, we assume that $\overline{p} < N$, otherwise the problem is more simple, because (1.1.4) holds for any $r \geq 1$.

We know, by a simple modification of the classic Leray-Lions theorem (see [**57**] and also [**26**]), thanks to the anisotropic Sobolev embeddings, that if $f \in L^m(\Omega)$, with $m \geq p_\infty'$ and

$$p_\infty = \max\{p_N, \ \overline{p}^*\}, \quad p_N = \max_i\{p_i\}, \quad \overline{p}^* = \frac{\overline{p}N}{N-\overline{p}} \quad \text{and} \quad \frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}, \tag{1.2.2}$$

there exists a weak solution to our problem, that is $u \in W_0^{1,(p_i)}(\Omega)$ such that

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i v = \int_\Omega f v, \qquad \forall \ v \in W_0^{1,(p_i)}(\Omega). \tag{1.2.3}$$

Since

$$M^m(\Omega) \subset L^{m-\varepsilon}(\Omega), \qquad \forall\ m > 1 \text{ and } 0 < \varepsilon \leq m-1, \tag{1.2.4}$$

we also obtain the existence of at least a weak solution of (1.2.1) when $f \in M^m(\Omega)$ with $m > p'_\infty$, defined in (1.2.2).

Now we consider $p_\infty = \overline{p}^*$, as in (1.2.2). We have the following result.

THEOREM 1.4. *Let $f \in L^m(\Omega)$.*

i) *If $m > \frac{N}{\overline{p}}$, then there exists a bounded weak solution u for the problem* (1.2.1), *as in* (1.2.3).

ii) *If $m = \frac{N}{\overline{p}}$, then there exists a weak solution u for the problem* (1.2.1) *and a constant $\beta > 0$ such that*

$$\int_\Omega e^{\beta|u|} < \infty. \tag{1.2.5}$$

iii) *If $(\overline{p}^*)' \leq m < \frac{N}{\overline{p}}$, then there exists a weak solution u for the problem* (1.2.1), *belonging to $L^s(\Omega)$, with*

$$s = \frac{m\overline{p}^*(\overline{p}-1)}{m\overline{p} + \overline{p}^* - m\overline{p}^*} = \frac{mN(\overline{p}-1)}{N - m\overline{p}}.$$

REMARK 1.5. i) and ii) are a direct consequence of i) and ii) of Theorem 1.19, in the following section, thanks to the following property of Lebesgue spaces

$$L^m(\Omega) \subset M^m(\Omega), \quad \forall\ m > 1. \tag{1.2.6}$$

REMARK 1.6. We note that the result ii) implies that the weak solution u of (1.2.1), that we obtain, belongs to $L^s(\Omega)$, $\forall\ 1 \leq s < +\infty$.

In the case $p_\infty = p_N = \max_i\{p_i\}$ we obtain this theorem.

THEOREM 1.7. *Let $f \in L^m(\Omega)$. i) and ii) of Theorem 1.4 hold true. Moreover*

iii) *if $\frac{N(p_N - \overline{p})}{\overline{p}(p_N - 1)} \leq m < \frac{N}{\overline{p}}$ then there exists a weak solution for the problem* (1.2.1), *belonging to $L^s(\Omega)$, with*

$$s = \frac{m\overline{p}^*(\overline{p}-1)}{m\overline{p} + \overline{p}^* - m\overline{p}^*} = \frac{mN(\overline{p}-1)}{N - m\overline{p}}.$$

iv) *If $p'_N \leq m < \frac{N(p_N - \overline{p})}{\overline{p}(p_N - 1)}$, then there exists a weak solution for the problem* (1.2.1), *belonging to $L^{\widetilde{s}}(\Omega)$, with $\widetilde{s} = m(p_N - 1)$.*

Remark 1.8. We note that, since $p_N > \overline{p}^*$,

$$\frac{N(p_N - \overline{p})}{\overline{p}(p_N - 1)} > (\overline{p}^*)' > p'_N,$$

and

$$\widetilde{s} = m(p_N - 1) > s = \frac{mN(\overline{p}-1)}{N - m\overline{p}} \quad \Leftrightarrow \quad m < \frac{N(p_N - \overline{p})}{\overline{p}(p_N - 1)}, \tag{1.2.7}$$

so that we have obtained a better summability of u, if $(\overline{p}^*)' < m < \frac{N(p_N - \overline{p})}{\overline{p}(p_N - 1)}$ and a new result if $p'_N < m < (\overline{p}^*)'$. While if $(\overline{p}^*)' < m < p'_N < N/\overline{p}$, $(p_N < \overline{p}^*)$, (1.2.7) never holds, so we do not improve the summability of u, obtained in Theorem 1.4.

Remark 1.9. We note that $\widetilde{s} = m(p_N - 1) > p_N$, since $m > p'_N$. Hence we improve the regularity, already known by the embeddings, of the weak solution of our problem.

Remark 1.10. We highlight that there is a continuity in our summability results. As a matter of fact, if $p_N = \overline{p}^*$, $m = (\overline{p}^*)'$, we have

$$\widetilde{s} = (\overline{p}^*)'(\overline{p}^* - 1) = \overline{p}^*$$

and this is the same exponent which appears in iii) of Theorem 1.4, with $m = (\overline{p}^*)'$.

Remark 1.11. Obviously the case iv) of Theorem 1.7 is new because our operator is anisotropic. If it is isotropic, $p_N = \overline{p} = p$ and $p < p^*$. Moreover

$$\frac{N(p_N - \overline{p})}{\overline{p}(p_N - 1)} = 0,$$

as we expect.

Remark 1.12. It is easy to prove that if $f \in L^m(\Omega)$, with $m \geq p'_\infty$, the weak solution of the problem (1.2.1) is unique due to the monotone property of anisotropic operators as (I.1). If we suppose that two solutions u_1 and u_2 exist, we have

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u_1|^{p_i - 2} \partial_i u_1 \partial_i v = \int_\Omega f v \quad \forall \ v \in W_0^{1,(p_i)}(\Omega)$$

and

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u_2|^{p_i - 2} \partial_i u_2 \partial_i v = \int_\Omega f v \quad \forall \ v \in W_0^{1,(p_i)}(\Omega).$$

Now we take as a test function $v = u_1 - u_2$ in both of them. We note that such a choice is possible since u_1, $u_2 \in W_0^{1,(p_i)}(\Omega)$. Then, subtracting the two expressions, we obtain

$$\sum_{i=1}^{N} \int_\Omega [|\partial_i u_1|^{p_i-2}\partial_i u_1 - |\partial_i u_2|^{p_i-2}\partial_i u_2]\partial_i(u_1 - u_2) = 0.$$

Then, if $p_i \geq 2$ for all $i = 1, ..., N$,

$$[|\partial_i u_1|^{p_i-2}\partial_i u_1 - |\partial_i u_2|^{p_i-2}\partial_i u_2]\partial_i(u_1 - u_2) \geq C_0|\partial_i(u_1 - u_2)|^{p_i}, \qquad \forall\ i.$$

Hence

$$\sum_{i=1}^{N} \int_\Omega |\partial_i(u_1 - u_2)|^{p_i} \leq 0,$$

that implies $u_1 = u_2$. A slight modification is needed to obtain the same result with $p_i < 2$.

Remark 1.13. We note that in our results we do not need to suppose that $\overline{p}^* > p_N$, thanks to the embeddings proved in [**45**]. This fact does not contradict the counterexample in [**60**] (see also [**49**]). As a matter of fact, in the cited paper, it is shown that (1.2.1) with $f = 0$ may have unbounded weak solutions, but the counterexample is not in the case of homogeneous Dirichlet boundary conditions.

Now we consider always the same Dirichlet problem (1.2.1) but if $m < p'_\infty$. We prove the existence of a distributional solution, that is $u \in W_0^{1,1}(\Omega)$, such that,

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i \phi = \int_\Omega f\phi, \qquad \forall\ \phi \in C_0^1(\Omega). \tag{1.2.8}$$

If $p_\infty = \overline{p}^*$, we have the following theorem.

Theorem 1.14. *Let $f \in L^m(\Omega)$.*

i) *If $m = 1$, then there exists a distributional solution u for* (1.2.1), *belonging to $W_0^{1,(s_i)}(\Omega)$, for all*

$$1 < s_i < p_i \frac{N(\overline{p} - 1)}{\overline{p}(N - 1)}, \quad \forall\ i = 1, ..., N.$$

ii) *If $1 < m < (\overline{p}^*)'$, then there exists a distributional solution u for* (1.2.1), *belonging to $W_0^{1,(s_i)}(\Omega)$, with*

$$1 < s_i = p_i \frac{mN(\overline{p} - 1)}{\overline{p}(N - m)},$$

for all $i = 1, ..., N$.

REMARK 1.15. Since $s_i > 1$, for any $i = 1, ..., N$, we deduce that

$$\overline{p} > 2 - \frac{1}{N}$$

in i), and in ii)

$$\overline{p} > 1 + \frac{1}{m^*},$$

as we expect recalling the classic case, i.e. $p_i = p$ for all i.

REMARK 1.16. Also in this case, we note that, since $\overline{p}^* \geq p_N$,

$$\frac{N(p_N - \overline{p})}{\overline{p}(p_N - 1)} < (\overline{p}^*)'$$

and

$$\frac{N(p_N - \overline{p})}{\overline{p}(p_N - 1)} > 1 \quad \Leftrightarrow \quad p_N > \frac{\overline{p}(N-1)}{N - \overline{p}} = \frac{N-1}{N}\,\overline{p}^*.$$

Hence we can improve the previous theorem if $\frac{\overline{p}(N-1)}{N-\overline{p}} < p_N \leq \overline{p}^*$ and $1 < m < \frac{N(p_N - \overline{p})}{\overline{p}(p_N-1)}$. As a matter of fact we have a distributional solution of (1.2.1) belonging to $W_0^{1,(\widetilde{s}_i)}(\Omega)$, with $1 < \widetilde{s}_i = p_i \frac{m}{p'_N}$ for all $i = 1, ..., N$ and by the restriction on m, it holds

$$p_i \frac{m}{p'_N} > p_i \frac{mN(\overline{p} - 1)}{\overline{p}(N - m)}, \qquad \forall\ i = 1, ..., N \quad \Leftrightarrow \quad m < \frac{N(p_N - \overline{p})}{\overline{p}(p_N - 1)}. \tag{1.2.9}$$

Also in this case, since $\widetilde{s}_i > 1$, we have the following condition on $\overline{p}$

$$\overline{p} > \frac{p'_N}{m}.$$

Moreover if $f \in L^1(\Omega)$, $u \in W_0^{1,(\widetilde{s}_i)}(\Omega)$, for all

$$\widetilde{s}_i < \frac{p_i}{p'_N} \quad \text{and} \quad \overline{p} > p'_N, \qquad \forall\ i = 1, ..., N.$$

In the case $p_\infty = p_N$, we have the theorem below.

THEOREM 1.17. *Let $f \in L^m(\Omega)$.*

i) *If $m = 1$, then there exists a distributional solution u for* (1.2.1)*, belonging to $W_0^{1,(\widetilde{s}_i)}(\Omega)$, for all*

$$\widetilde{s}_i < \frac{p_i}{p'_N} \quad \text{and} \quad \overline{p} > p'_N,$$

$\forall\ i = 1, ..., N$.

ii) *If $1 < m < (\overline{p}^*)'$, then there exists a distributional solution u for* (1.2.1)*, belonging to $W_0^{1,(\widetilde{s}_i)}(\Omega)$, with $\widetilde{s}_i = p_i \frac{m}{p'_N}$ for all $i = 1, ..., N$.*

REMARK 1.18. We note that, if $1 \leq m < p'_\infty$, the distributional solution may not be unique. In fact already in the isotropic and linear case the solution is not unique (see the counterexample presented in [**74**]). But it is possible to extend, in a natural way (see [**11**]), the definition of entropy solutions for the problem (1.2.1), see Section 1.6, to achieve an existence result without further assumptions on $\overline{p}$ and to have a unique solution.

1.3. Data in Marcinkiewicz spaces

In this section we present some results concerning the case of f belonging to a Marcinkiewicz space, $M^m(\Omega)$. As we have already mentioned, we know, by a simple modification of the classic Leray-Lions theorem, that if $f \in M^m(\Omega)$, with $m > p'_\infty$ there exists a weak solution, as in (1.2.3), of problem (1.2.1), due to (1.2.4). We begin to consider the case $p_\infty = \overline{p}^*$, where p_∞ and $\overline{p}^*$ are as in (1.2.2). We have the following result.

THEOREM 1.19. *Let $f \in M^m(\Omega)$.*

i) *If $m > \frac{N}{\overline{p}}$, then there exists a bounded weak solution u for the problem* (1.2.1).

ii) *If $m = \frac{N}{\overline{p}}$, then there exists a weak solution u for the problem* (1.2.1) *and a constant $\beta > 0$ such that*

$$\int_\Omega e^{\beta|u|} < \infty. \tag{1.3.1}$$

iii) *If $(\overline{p}^*)' < m < \frac{N}{\overline{p}}$, then there exists a weak solution u for the problem* (1.2.1), *belonging to $M^s(\Omega)$ with*

$$s = \frac{m\overline{p}^*(\overline{p}-1)}{m\overline{p} + \overline{p}^* - m\overline{p}^*} = \frac{mN(\overline{p}-1)}{N - m\overline{p}}.$$

If $p_\infty = p_N > \overline{p}^*$, we have the theorem below.

THEOREM 1.20. *Let $f \in M^m(\Omega)$. i) and ii) of Theorem 1.19 hold true. Moreover*

iii) *if $\frac{N(p_N - \overline{p})}{\overline{p}(p_N - 1)} \leq m < \frac{N}{\overline{p}}$ then there exists a weak solution u for the problem* (1.2.1), *belonging to $M^s(\Omega)$, with*

$$s = \frac{m\overline{p}^*(\overline{p}-1)}{m\overline{p} + \overline{p}^* - m\overline{p}^*} = \frac{mN(\overline{p}-1)}{N - m\overline{p}}.$$

iv) *If $p'_N < m < \frac{N(p_N - \overline{p})}{\overline{p}(p_N - 1)}$, then there exists a weak solution u for the problem* (1.2.1), *belonging to $M^{\widetilde{s}}(\Omega)$, with $\widetilde{s} = m(p_N - 1)$.*

Remarks 1.6, 1.8, 1.9, 1.10 and 1.11 hold true also in this case.

REMARK 1.21. We note that if, in the previous theorems, we let m tend to $N/\overline{p}$, we obtain $s \to +\infty$. Moreover the values of s and $\widetilde{s}$ obtained in Theorems 1.19 and 1.20 are the same of Theorems 1.4 and 1.7, as we expected.

REMARK 1.22. By (1.2.4), also if f belongs to $M^m(\Omega)$, with $m > p'_\infty$, the weak solution of (1.2.1) is unique.

For the case $1 < m \le p'_\infty$ we prove the existence of a distributional solution for the problem (1.2.1), as in (1.2.8).

As before, we distinguish between $p_\infty = \overline{p}^*$ and $p_\infty = p_N$. In the first case we have the following result.

THEOREM 1.23. *If f belongs to $M^m(\Omega)$, with $1 < m \le (\overline{p}^*)'$, then there exists a distributional solution u for* (1.2.1)*, belonging to $M^s(\Omega)$, with*

$$s = \frac{m\overline{p}^*(\overline{p}-1)}{m\overline{p} + \overline{p}^* - m\overline{p}^*} = \frac{mN(\overline{p}-1)}{N - m\overline{p}}$$

and $\partial_i u \in M^{s_i}(\Omega)$, with

$$1 < s_i = p_i \, \frac{mN(\overline{p}-1)}{\overline{p}(N-m)},$$

for all $i = 1, ..., N$.

REMARK 1.24. So $s_i > 1$, for every $i = 1, ..., N$, on the condition that

$$\overline{p} > 1 + \frac{1}{m^*}.$$

REMARK 1.25. We note that since $\overline{p}^* \ge p_N$,

$$\frac{N(p_N - \overline{p})}{\overline{p}(p_N - 1)} < (\overline{p}^*)'$$

and

$$\frac{N(p_N - \overline{p})}{\overline{p}(p_N - 1)} > 1 \quad \Leftrightarrow \quad p_N > \frac{\overline{p}(N-1)}{N - \overline{p}}.$$

So we can improve the previous theorem if $\frac{\overline{p}(N-1)}{N-\overline{p}} < p_N \le \overline{p}^*$ and $1 < m < \frac{N(p_N-\overline{p})}{\overline{p}(p_N-1)}$, by (1.2.7). As a matter of fact, we have a distributional solution for (1.2.1) belonging to $M^{\widetilde{s}}(\Omega)$, with $\widetilde{s} = m(p_N - 1)$. Moreover $\partial_i u \in M^{\widetilde{s}_i}(\Omega)$, with $1 < \widetilde{s}_i = p_i \frac{m}{p'_N}$ for all $i = 1, ..., N$, since it also holds (1.2.9). We note that, in order to have $\widetilde{s}_i > 1$, we assume

$$\overline{p} > \frac{p'_N}{m}.$$

In the second case, $p_\infty = p_N$, we get the following result.

THEOREM 1.26. *If f belongs to $M^m(\Omega)$, with $1 < m \le p_N'$, then there exists a distributional solution u for (1.2.1), belonging to $M^{\widetilde{s}}(\Omega)$, with $\widetilde{s} = m(p_N - 1)$ and $\partial_i u \in M^{\widetilde{s}_i}(\Omega)$, with $\widetilde{s}_i = p_i \frac{m}{p_N'}$, for all $i = 1, ..., N$. We also suppose that*

$$\overline{p} > \frac{p_N'}{m}.$$

REMARK 1.27. Under the assumption $p_N > \overline{p}^*$, it holds

$$\frac{N(p_N - \overline{p})}{\overline{p}(p_N - 1)} > (\overline{p}^*)' > p_N'$$

and if

$$m < \frac{N(p_N - \overline{p})}{\overline{p}(p_N - 1)},$$

(1.2.7) and (1.2.9) are true.

REMARK 1.28. We recall, as already mentioned in the introduction, that if we choose $p_i = 2$, for any $i = 1, ..., N$, (or equivalently $p_i = p$, for any $i = 1, ..., N$), we obtain the classic regularity results.

1.4. Data in divergence form

In this section we consider the problem (1.2.1), with datum f in divergence form, namely

$$\begin{cases} -\sum_{i=1}^N \partial_i[|\partial_i u|^{p_i-2}\partial_i u] = -\sum_{i=1}^N \partial_i f_i & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4.2}$$

with $f_i \in L^{m_i}(\Omega)$, $m_i \ge p_i'$, for all $i = 1, ..., N$.

This problem has already been studied by several authors, see [**19**], [**25**] and [**26**]. For due diligence we report these results, but we give slightly different proofs. In [**26**] it was proved that if

$$\frac{\overline{p}^*}{p_N} \min_i \left\{1 - \frac{p_i'}{m_i}\right\} > 1, \tag{1.4.3}$$

that is

$$\begin{cases} \overline{p}^* > p_N, \\ \\ m_i > \dfrac{\overline{p}^*}{\overline{p}^* - p_N}\, p_i', \qquad \forall\ i = 1, ..., N, \end{cases} \tag{1.4.4}$$

then any weak solution of (1.4.2) is bounded. This result can be proved without the assumption $\overline{p}^* > p_N$ and under the following weaker regularity assumption on the data

$$\min_i \left\{ 1 - \frac{p_i'}{m_i} \right\} \frac{\overline{p}^*}{\overline{p}} > 1, \tag{1.4.5}$$

(see [**76**]). We have the following theorem.

Theorem 1.29. *If $f_i \in L^{m_i}(\Omega)$, such that* (1.4.5) *holds, then there exists a bounded weak solution u for the problem* (1.4.2).

Remark 1.30. The assumption (1.4.5) is equivalent to require that

$$m_i > \frac{N}{\overline{p}}\, p_i', \qquad \forall\ i = 1, ..., N. \tag{1.4.6}$$

Moreover

$$\frac{\overline{p}^*}{\overline{p}^* - p_N}\, p_i' > \frac{N}{\overline{p}}\, p_i'.$$

Hence the boundedness of a weak solution u also holds true if f_i's are less regular and if $\overline{p}^* < p_N$.

For the case

$$\min_i \left\{ 1 - \frac{p_i'}{m_i} \right\} \frac{\overline{p}^*}{\overline{p}} < 1, \tag{1.4.7}$$

see [**25**], in which the following result is presented for minima of some functionals.

Theorem 1.31. *If $f_i \in L^{m_i}(\Omega)$, with*

$$\min_i \left\{ 1 - \frac{p_i'}{m_i} \right\} \frac{\overline{p}^*}{\overline{p}} = \min_i \left\{ 1 - \frac{p_i'}{m_i} \right\} \frac{N}{N - \overline{p}} < 1, \tag{1.4.8}$$

then there exists a weak solution u for the problem (1.4.2) and it is belongs to $L^s(\Omega)$, where

$$s = \frac{\overline{p} N \mu}{N - \overline{p}\mu} \quad \textit{and} \quad \mu = \min_i \left\{ \frac{m_i}{p_i'} \right\}.$$

REMARK 1.32. We note that if $\min_i \left\{1 - \frac{p_i'}{m_i}\right\} \frac{N}{N-\overline{p}}$ goes to 1, s goes to infinity. Moreover if $f \in L^m(\Omega)$ and $p_i = 2$, for all $i = 1, ..., N$ (or equivalently $p_i = p$, $\forall$ $i = 1, ..., N$) we obtain the known classic results, that is if

$$\frac{m-2}{m} \cdot \frac{2^*}{2} > 1 \qquad \Leftrightarrow \qquad m > N,$$

then there exists a bounded weak solution for the isotropic problem, corresponding to (1.4.2) and if $m < N$ a weak solution belongs to $L^s(\Omega)$ with $s = m^*$.

REMARK 1.33. All the results, except the uniqueness results, presented in these sections also hold if our anisotropic operator is exchanged by a more general one, i.e., A a non linear differential operator from $W_0^{1,(p_i)}(\Omega)$ into its dual of the form

$$A(u) = -\mathrm{div}(a(x, u, \nabla u)),$$

where $a(x, s, \xi) = (a_i(x, s, \xi))$ is a Carathéodory vector valued function on $\Omega \times \mathbb{R} \times \mathbb{R}^N$ such that, for some constant $\beta \geq \alpha > 0$

$$\sum_{i=1}^{N} a_i(x, s, \xi)\xi_i \geq \alpha \sum_{i=1}^{N} |\xi_i|^{p_i},$$

$$|a_i(x, s, \xi)| \leq \beta \left(\sum_{j=1}^{N} |\xi_j|^{p_j} \right)^{1-1/p_i}, \qquad \forall\ i = 1, .., N$$

and for a.e. $x \in \Omega$ and $\forall$ $s \in \mathbb{R}$, ξ, $\eta \in \mathbb{R}^N$, $\xi \neq \eta$

$$\sum_{i=1}^{N} (a_i(x, s, \xi) - a_i(x, s, \eta))(\xi_i - \eta_i) > 0.$$

1.5. Proofs of the results

Now we wish to prove the results presented in the previous sections.

1.5.1. Proofs of Theorem 1.4 iii) and Theorem 1.7. We only prove the part iii) of Theorem 1.4, because i) and ii) are a direct consequence of i) and ii) of Theorem 1.19, shown in the following, since

$$L^m(\Omega) \subset M^m(\Omega), \qquad \forall\ m \geq 1. \tag{1.5.9}$$

We take as a test function in (1.2.3) $v=|T_k(u)|^{t_j\,p_j}T_k(u)$, where $j=1,...,N$, T_k is as in (1.1.7) and t_j's are positive real numbers which we will fix later. We have

$$\sum_{i=1}^{N}\int_\Omega |\partial_i u|^{p_i-2}\partial_i u\partial_i(|T_k(u)|^{t_j\,p_j}T_k(u))=\int_\Omega f|T_k(u)|^{t_j\,p_j}T_k(u), \tag{1.5.10}$$

$\forall\ j=1,...,N$. For the first term in (1.5.10) we have

$$\sum_{i=1}^{N}\int_\Omega |\partial_i u|^{p_i-2}\partial_i u\partial_i(|T_k(u)|^{t_j\,p_j}T_k(u))\geq C_0\int_\Omega |\partial_j T_k(u)|^{p_j}|T_k(u)|^{t_j\,p_j},$$

$\forall\ j=1,...,N$. For the second term of (1.5.10), by applying Hölder inequality with exponents m and m', we obtain

$$\int_\Omega f\,|T_k(u)|^{t_j\,p_j}T_k(u)\leq\left(\int_\Omega |f|^m\right)^{\frac{1}{m}}\left(\int_\Omega |T_k(u)|^{(t_j\,p_j+1)m'}\right)^{\frac{1}{m'}},\ \ j=1,...,N.$$

From the previous inequalities, by multiplying on j, we deduce

$$\prod_{j=1}^{N}\left(\int_\Omega |\partial_j T_k(u)|^{p_j}|T_k(u)|^{t_j\,p_j}\right)^{\frac{1}{p_j}}\leq C_1\prod_{j=1}^{N}\left(\int_\Omega |T_k(u)|^{(t_j\,p_j+1)m'}\right)^{\frac{1}{m'\,p_j}}. \tag{1.5.11}$$

By (1.1.11) of Lemma 1.2, with $v=T_k(u)$ and $r=s$, and (1.5.11), we have

$$\left(\int_\Omega |T_k(u)|^s\right)^{\frac{N}{\overline{p}}-1}\leq C_2\prod_{j=1}^{N}\left(\int_\Omega |T_k(u)|^{(t_j\,p_j+1)m'}\right)^{\frac{1}{m'\,p_j}}. \tag{1.5.12}$$

Since we also want $s=(t_j\,p_j+1)m'$ in (1.5.12), for any $j=1,...,N$, we have to solve the following system

$$\begin{cases} s=\frac{1+t_j}{\gamma_j(N-1)-1+1/p_j}, & \forall\ j=1,...,N,\\[2em] s=(t_j\,p_j+1)m', & \forall\ j=1,...,N,\\[2em] \sum_{j=1}^{N}\gamma_j=1, & \gamma_j\geq 0\quad\text{and}\quad t_j\geq 0\quad\forall\ j=1,...,N.\end{cases}$$

From the first two equations in the previous system, after some lengthy but easy calculations, we have

$$t_j=\frac{2mp_j-m-p_j-m\,p_j\,\gamma_j(N-1)}{p_j(m\,p_j\,\gamma_j(N-1)-p_j\,m+1)},\qquad\forall\ j=1,...,N, \tag{1.5.13}$$

and

$$\gamma_j = \left(1 - \frac{1}{p_j}\right) \frac{N - m\overline{p}}{N(\overline{p}-1)m(N-1)} + \left(1 - \frac{1}{m\, p_j}\right) \frac{1}{N-1}, \qquad \forall\ j = 1, ..., N. \tag{1.5.14}$$

It is easy to prove that γ_j's satisfy the condition $\sum_{j=1}^{N} \gamma_j = 1$. Moreover $\gamma_j \geq 0$, since $m > 1$ and $p_j > 1$, for all $j = 1, ..., N$. Also by (1.5.13) and (1.5.14), we have

$$t_j = \frac{(p_j - 1)[N\overline{p}(m-1) - m(N - \overline{p})]}{p_j(p_j - 1)(N - m\overline{p})} = \frac{N\overline{p}(m-1) - m(N-\overline{p})}{p_j(N - m\overline{p})}.$$

Hence $t_j \geq 0$, for all $j = 1, ..., N$, by the assumptions on m. As a matter of fact

$$N - m\overline{p} > 0 \quad \Leftrightarrow \quad m < \frac{N}{\overline{p}},$$

and

$$N\overline{p}(m-1) - m(N - \overline{p}) \geq 0 \quad \Leftrightarrow \quad m \geq \frac{N\overline{p}}{N\overline{p} - N + \overline{p}} = (\overline{p}^*)'.$$

Moreover, by the choice of t_j and γ_j, we have

$$s = \frac{mN(\overline{p}-1)}{N - m\overline{p}},$$

as in iii) of Theorem 1.4. Therefore, thanks to (1.5.12), by noting that $N/\overline{p} - 1 > N/m'\,\overline{p}$, since $m < N/\overline{p}$,

$$\|T_k(u)\|_{L^s(\Omega)} \leq C_3, \qquad \forall\ k \in \mathbb{N}.$$

Using Fatou Lemma we can pass to the limit as $k \to +\infty$ to obtain

$$\|u\|_{L^s(\Omega)} \leq C_4,$$

as desired. ■

Remark 1.34. The proof of iv) of Theorem 1.7, is very similar to the previous one. For the sake of simplicity, we omit it, by remarking that we must only use Poincaré type inequality (1.1.10), with $r = p_N$ and $i = N$, instead of the Sobolev type. This is the same reason for which we do not prove Remark 1.16, Theorem 1.17, iv) of Theorem 1.20, Remark 1.25 and Theorem 1.26.

Now we prove the existence and regularity results if $f \in L^m(\Omega)$, with $1 \leq m < (\overline{p}^*)'$. We use the techniques of [**19**], introduced for the first time in [**16**] and [**17**].

1.5.2. Proof of Theorem 1.14. We begin with the case $m = 1$, i). We consider a sequence $\{f_n\} \subset L^\infty(\Omega)$, such that

$$f_n \to f \text{ in } L^1(\Omega) \text{ and } \|f_n\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)}$$

and let u_n be the solutions to the following problems

$$u_n \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega) \;:\; -\sum_{i=1}^N \partial_i[|\partial_i u_n|^{p_i-2}\partial_i u_n] = f_n, \tag{1.5.15}$$

which exist due to the previous results. We take as a test function in the weak formulation of problems (1.5.15), $v = T_1(G_k(u_n))$, where T_1 and G_k are defined in Section 1.1. We have

$$\sum_{i=1}^N \int_{B_{k,n}} |\partial_i u_n|^{p_i} \le \|f\|_{L^1(\Omega)},$$

with $B_{k,n} = \{x \in \Omega : k < |u_n(x)| \le k+1\}$. So,

$$\int_{B_{k,n}} |\partial_i u_n|^{p_i} \le \|f\|_{L^1(\Omega)}, \qquad \forall\; i = 1, ..., N. \tag{1.5.16}$$

Let $s_i = \theta p_i$, $\beta = \frac{(1-\theta)\overline{s}^*}{\theta}$ and $\theta \in \left(0, \frac{N(\overline{p}-1)}{\overline{p}(N-1)}\right)$. By using Hölder inequality with exponents p_i/s_i and $(p_i/s_i)'$, we obtain

$$\int_\Omega |\partial_i u_n|^{s_i} = \int_\Omega |\partial_i u_n|^{s_i}(1+|u_n|)^{-\beta\frac{s_i}{p_i}}(1+|u_n|)^{\beta\frac{s_i}{p_i}} \le$$

$$\le \left(\sum_{k=0}^\infty \int_{B_{k,n}} |\partial_i u_n|^{p_i}(1+|u_n|)^{-\beta}\right)^{\frac{s_i}{p_i}} \left(\int_\Omega (1+|u_n|)^{\frac{\beta s_i}{p_i-s_i}}\right)^{1-\frac{s_i}{p_i}}.$$

Since on $B_{k,n}$

$$\frac{1}{1+|u_n|} < \frac{1}{k+1},$$

we have

$$\int_\Omega |\partial_i u_n|^{s_i} \le \left(\sum_{k=0}^\infty \frac{1}{(1+k)^\beta}\int_{B_{k,n}} |\partial_i u_n|^{p_i}\right)^{\frac{s_i}{p_i}} \left(\int_\Omega (1+|u_n|)^{\frac{\beta s_i}{p_i-s_i}}\right)^{1-\frac{s_i}{p_i}}. \tag{1.5.17}$$

By (1.5.16) and since $\beta > 1$ we obtain

$$\sum_{k=0}^\infty \frac{1}{(1+k)^\beta}\int_{B_{k,n}} |\partial_i u_n|^{p_i} \le C_0,$$

hence

$$\left(\int_\Omega |\partial_i u_n|^{s_i}\right)^{\frac{1}{Ns_i}} \leq C_1 \left(\int_\Omega (1+|u_n|)^{\frac{\beta s_i}{p_i - s_i}}\right)^{\left(\frac{1}{s_i}-\frac{1}{p_i}\right)\frac{1}{N}}. \tag{1.5.18}$$

Now we apply the anisotropic Sobolev inequality (1.1.5), with $r = \overline{s}^*$, to obtain, thanks to the choice of β,

$$\|u_n\|_{L^{\overline{s}^*}} \leq C_2 \left(\int_\Omega (1+|u_n|)^{\overline{s}^*}\right)^{\left(\frac{1}{\theta}-1\right)\frac{1}{\overline{p}}}. \tag{1.5.19}$$

By (1.5.19) and (1.5.18), we get that $\partial_i u_n$ is bounded in $L^{s_i}(\Omega)$ uniformly in n. So we can assume that, for some u and for some subsequence, which we still denote by u_n, that

$$\partial_i u_n \to \partial_i u \ \text{ weakly in } \ L^{s_i}(\Omega), \qquad \forall\ i = 1, ..., N, \tag{1.5.20}$$

$$u_n \to u \ \text{ strongly in } \ L^{\overline{s}}(\Omega), \tag{1.5.21}$$

$\overline{s}$, the harmonic mean of the s_i's. It is not enough to pass to the limit, but we can claim that

$$\partial_i u_n \to \partial_i u \ \text{ strongly in } \ L^{r_i}(\Omega), \qquad \forall\ r_i < s_i, \tag{1.5.22}$$

$i = 1, ..., N$. In fact we have, for all $\eta > 0$ and for all $i = 1, ..., N$,

$$\int_{\{|u_n-u_h|\leq\eta\}} \left(|\partial_i u_n|^{p_i-2}\partial_i u_n - |\partial_i u_h|^{p_i-2}\partial_i u_h\right)\partial_i(u_n-u_h) \leq 2\eta\|f\|_{L^1(\Omega)}.$$

If we fix i and $p_i \geq 2$, we deduce that

$$\int_{\{|u_n-u_h|\leq\eta\}} |\partial_i(u_n-u_h)|^{p_i} \leq C_3\eta$$

and so by Hölder inequality with exponents p_i/r_i and $(p_i/r_i)'$ and by simple calculations, we have

$$\int_\Omega |\partial_i(u_n-u_h)|^{r_i} \leq C_4\eta^{\frac{r_i}{p_i}} + C_5\,\text{meas}(\{|u_n-u_h|>\eta\})^{1-\frac{r_i}{s_i}}.$$

We obtain the same result also for $p_i < 2$ by a slight modification. We recall that u_n converges to u in measure because $u_n \to u$ in $L^{\overline{s}}(\Omega)$. So, since the above inequality holds true for any $\eta > 0$, we obtain that $\partial_i u_n$ is a Cauchy sequence in $L^{r_i}(\Omega)$. So, by (1.5.22), we have

$$|\partial_i u_n|^{p_i-2}\partial_i u_n \to |\partial_i u|^{p_i-2}\partial_i u \ \text{ in } \ L^1(\Omega).$$

now we can pass to the limit for $n \to +\infty$ in the weak formulation of (1.5.15) and we obtain that u is a distributional solution for the equation (1.2.1).

Now we deal with the case $1 < m < (\overline{p}^*)'$, part ii) of Theorem 1.14. As above we consider a sequence $\{f_n\} \subset L^\infty(\Omega)$, such that

$$f_n \to f \quad \text{in} \quad L^m(\Omega) \qquad \text{and} \qquad \|f_n\|_{L^m(\Omega)} \leq \|f\|_{L^m(\Omega)}$$

and let u_n be the solutions of the problems (1.5.15). We use, as a test function in the weak formulation of (1.5.15), $v = T_1(G_k(u_n))$, to obtain

$$\sum_{i=1}^N \int_{B_{k,n}} |\partial_i u_n|^{p_i} \leq \int_{A_{k,n}} |f_n|,$$

with $A_{k,n} = \{|u_n| > k\}$. We get

$$\int_{B_{k,n}} |\partial_i u_n|^{p_i} \leq \int_{A_{k,n}} |f_n|, \qquad \forall\ i = 1, ..., N. \tag{1.5.23}$$

If we go on as in the case $m = 1$, we have

$$\int_\Omega |\partial_i u_n|^{s_i} \leq \left(\sum_{j=0}^\infty \int_{B_{j,n}} |f_n| \sum_{k=0}^j \frac{1}{(1+k)^\beta} \right)^{\frac{s_i}{p_i}} \left(\int_\Omega (1+|u_n|)^{\frac{\beta s_i}{p_i - s_i}} \right)^{1-\frac{s_i}{p_i}}.$$

Since

$$\sum_{k=0}^j \frac{1}{(1+k)^\beta} \leq C_0(1 + j^{1-\beta}),$$

we obtain, by some calculations and by using Hölder inequality with exponents m and m',

$$\int_\Omega |\partial_i u_n|^{s_i} \leq C_1 \left[\|f_n\|_{L^m(\Omega)} \left(\int_\Omega (1+|u_n|)^{(1-\beta)m'} \right)^{\frac{1}{m'}} \right]^{\frac{s_i}{p_i}} \left(\int_\Omega (1+|u_n|)^{\frac{\beta s_i}{p_i - s_i}} \right)^{1-\frac{s_i}{p_i}},$$

for all $i = 1, ..., N$. Now we take $s_i = \theta p_i$, with $\theta \in [0, 1)$, such that

$$\frac{\beta s_i}{p_i - s_i} = \frac{\beta\theta}{1-\theta}, \qquad \forall\ i = 1, ..., N$$

and we choose β, such that

$$\frac{\beta\theta}{1-\theta} = (1-\beta)m' \ \Leftrightarrow\ \beta = \frac{m(1-\theta)}{m-\theta}.$$

We obtain

$$(1-\beta)m' = \frac{\theta m}{m-\theta}$$

and therefore

$$\left(\int_\Omega |\partial_i u_n|^{s_i}\right)^{\frac{1}{s_iN}} \le C_2\left(\int_\Omega (1+|u_n|)^{\frac{\theta m}{m-\theta}}\right)^{\left(\frac{1}{\theta}-\frac{1}{m}\right)\frac{1}{p_iN}}, \qquad \forall\ i=1,...,N.$$

By Sobolev type inequality, we have

$$\|u_n\|_{L^{\overline{s}^*}} \le C_3\left(\int_\Omega (1+|u_n|)^{\frac{m\theta}{m-\theta}}\right)^{\left(\frac{1}{\theta}-\frac{1}{m}\right)\frac{1}{N\overline{p}}}, \tag{1.5.24}$$

with

$$\overline{s}^* = \frac{\theta\overline{p}N}{N-\theta\overline{p}}.$$

Now we take θ, such that

$$\frac{\theta\overline{p}N}{N-\theta\overline{p}} = \frac{\theta m}{m-\theta} \quad\Leftrightarrow\quad \theta = \frac{mN(\overline{p}-1)}{\overline{p}(N-m)},$$

and hence

$$s_i = \frac{mN(\overline{p}-1)}{\overline{p}(N-m)}p_i, \qquad \forall\ i=1,...,N.$$

We remark that the value of θ, which we obtain, is smaller than 1, since $m<(\overline{p}^*)'$. Besides

$$\frac{1}{\overline{s}^*} > \left(\frac{1}{\theta}-\frac{1}{m}\right)\frac{1}{\overline{p}} \quad\Leftrightarrow\quad m<\frac{N}{\overline{p}}.$$

Then, if we proceed as in the case $m=1$, we can pass to the limit in the approximating problems (1.5.15) to obtain the desired result. ∎

1.5.3. Proof of Theorem 1.19. We take $v=G_k(u)$ in (1.2.3). We have

$$\sum_{i=1}^N \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i G_k(u) = \int_\Omega fG_k(u),$$

it implies

$$\left(\int_\Omega |\partial_i G_k(u)|^{p_i}\right)^{\frac{1}{p_iN}} \le \left(\int_\Omega fG_k(u)\right)^{\frac{1}{p_iN}}.$$

Therefore, by (1.1.5), with $r=\overline{p}^*$, we get

$$\|G_k(u)\|_{L^{\overline{p}^*}(\Omega)} \le C_0\prod_{i=1}^N\left(\int_\Omega |fG_k(u)|\right)^{\frac{1}{p_iN}} = C_0\left(\int_\Omega |fG_k(u)|\right)^{\frac{1}{\overline{p}}}.$$

By applying Hölder inequality with exponents $\overline{p}^*$ and $(\overline{p}^*)'$, due to the fact that $f \in L^{(\overline{p}^*)'}(\Omega)$ by (1.2.4), we obtain

$$C_0 \left(\int_\Omega |f G_k(u)| \right)^{\frac{1}{\overline{p}}} \le C_0 \left(\int_\Omega |G_k(u)|^{\overline{p}^*} \right)^{\frac{1}{\overline{p}^* \overline{p}}} \left(\int_{A_k} |f|^{(\overline{p}^*)'} \right)^{\left(1 - \frac{1}{\overline{p}^*}\right)\frac{1}{\overline{p}}},$$

where $A_k = \{|u| > k\}$. Hence

$$\left(\int_\Omega |G_k(u)|^{\overline{p}^*} \right)^{\frac{1}{\overline{p}^*} - \frac{1}{\overline{p}^* \overline{p}}} \le C_0 \left(\int_{A_k} |f|^{(\overline{p}^*)'} \right)^{\frac{1}{\overline{p}} - \frac{1}{\overline{p}^* \overline{p}}}.$$

We note that

$$\frac{1}{\overline{p}^*} - \frac{1}{\overline{p}^* \overline{p}} > 0 \quad \Leftrightarrow \quad \overline{p} > 1,$$

which is true, since $p_i > 1$ for every $i = 1, ..., N$. Since $f \in M^m(\Omega)$ and $m > (\overline{p}^*)'$, we have

$$\int_{A_k} |f|^{(\overline{p}^*)'} \le C_f \operatorname{meas}(A_k)^{1 - \frac{(\overline{p}^*)'}{m}}.$$

Hence by applying Hölder inequality with exponents $\overline{p}^*$ and $(\overline{p}^*)'$ to $\int_\Omega |G_k(u)|$ and by simplifying, we obtain

$$\int_\Omega |G_k(u)| \le C_1 \operatorname{meas}(A_k)^{\left(1 - \frac{(\overline{p}^*)'}{m}\right)\left(1 - \frac{1}{\overline{p}^*}\right)\frac{1}{\overline{p}-1} + 1 - \frac{1}{\overline{p}^*}}. \tag{1.5.25}$$

We define $g(k) = \int_\Omega |G_k(u)|$ and we recall that $g'(k) = -\operatorname{meas}(A_k)$, for almost every k (see [**51**], [**56**]). We obtain, from (1.5.25), that

$$g(k)^{\frac{1}{\gamma}} \le -C_2\, g'(k),$$

with $\gamma = \left(1 - \frac{(\overline{p}^*)'}{m}\right)\left(1 - \frac{1}{\overline{p}^*}\right)\frac{1}{\overline{p}-1} + 1 - \frac{1}{\overline{p}^*}$. Therefore

$$1 \le -C_2\, g'(k) g(k)^{-\frac{1}{\gamma}} = -\frac{C_2}{1 - \frac{1}{\gamma}} (g(k)^{1 - \frac{1}{\gamma}})'. \tag{1.5.26}$$

If we are in case i) of Theorem 1.19, we note that

$$1 - \frac{1}{\gamma} > 0.$$

Therefore, by integrating (1.5.26) from 0 to k, we get

$$k \le -C_3 [g(k)^{1 - \frac{1}{\gamma}} - g(0)^{1 - \frac{1}{\gamma}}],$$

i.e.

$$C_3 g(k)^{1 - \frac{1}{\gamma}} \le -k + C_3 \|u\|_{L^1(\Omega)}^{1 - \frac{1}{\gamma}}.$$

Since $g(k)$ is a non-negative and decreasing function, from the latter inequality we deduce that there exists k_0, such that $g(k_0) = 0$, and so $u \in L^\infty(\Omega)$. In case ii) of Theorem 1.19, since $m = N/\overline{p}$, $\gamma = 1$, we have

$$1 \leq -C_2 \frac{g'(k)}{g(k)}.$$

By integrating from 0 to k, we have

$$\frac{k}{C_2} \leq \log \left[\frac{\|u\|_{L^1(\Omega)}}{g(k)} \right],$$

and since the function $t \to e^t$ increases, we obtain

$$e^{\frac{k}{C_2}} \leq \frac{\|u\|_{L^1(\Omega)}}{g(k)} \quad \Rightarrow \quad g(k) e^{\frac{k}{C_2}} \leq \|u\|_{L^1(\Omega)}.$$

So, recalling that

$$g(k) = \int_\Omega |G_k(u)| \geq \int_{A_{2k}} |G_k(u)| \geq k \,\text{meas}(A_{2k}), \tag{1.5.27}$$

if $k \geq 1$, we have

$$g(k) \geq \text{meas}(A_{2k}) \quad \Rightarrow \quad \text{meas}(A_{2k}) e^{\frac{k}{C_2}} \leq \|u\|_{L^1(\Omega)}.$$

Hence, if $k \geq 2$, we get

$$\text{meas}(A_k) e^{\frac{k}{2C_2}} \leq \|u\|_{L^1(\Omega)}. \tag{1.5.28}$$

We prove now that the previous inequality implies that

$$\sum_{k=0}^{+\infty} e^{k\beta} \text{meas}(A_k) < \infty,$$

with $0 < \beta < \frac{1}{2C_2}$. Indeed, by (1.5.28),

$$\sum_{k=0}^{+\infty} e^{k\beta} \text{meas}(A_k) \leq (1+e)\text{meas}(\Omega) + \sum_{k=2}^{+\infty} \frac{\|u\|_{L^1(\Omega)}}{e^{k(1/2C_2 - \beta)}} < \infty.$$

Since

$$\sum_{k=0}^{+\infty} e^{\beta k} \text{meas}(A_k) < +\infty \quad \Leftrightarrow \quad \int_\Omega e^{\beta|u|} < +\infty,$$

ii) is proved.

To conclude, we consider case iii). In this case we have

$$1 - \frac{1}{\gamma} < 0.$$

Therefore,

$$1 \le C_4 (g(k)^{1-\frac{1}{\gamma}})'.$$

By integration from 0 to k, we obtain

$$k \le C_4 [g(k)^{1-\frac{1}{\gamma}} - g(0)^{1-\frac{1}{\gamma}}] \le C_4 g(k)^{1-\frac{1}{\gamma}}$$

and so

$$g(k)^{-1+\frac{1}{\gamma}} \le \frac{C_4}{k} \quad \Rightarrow \quad g(k) \le \frac{C_5}{k^{\frac{\gamma}{1-\gamma}}}.$$

Therefore, by (1.5.27), it holds true that

$$\mathrm{meas}(A_{2k}) \le \frac{g(k)}{k} \le \frac{C_5}{k^{\frac{\gamma}{1-\gamma}} k} = \frac{C_5}{k^{\frac{1}{1-\gamma}}}.$$

By recalling the definition of γ, we obtain

$$\frac{1}{1-\gamma} = \frac{mN(\overline{p}-1)}{N - m\overline{p}} = s,$$

so that $u \in M^s(\Omega)$. ∎

Now we use an idea, presented in a recent paper (see [**14**]), to prove the Theorem 1.23, namely the case $f \in M^m(\Omega)$, with $1 < m \le p'_\infty$.

1.5.4. Proof of Theorem 1.23. We consider a sequence $\{f_n\} \subset L^\infty(\Omega)$, such that

$$f_n \to f \text{ in } M^m(\Omega) \text{ and } \|f_n\|_{M^m(\Omega)} \le \|f\|_{M^m(\Omega)},$$

with $1 < m \le (\overline{p}^*)'$ and let u_n be the solutions of the following problems, which exist by the previous results,

$$u_n \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega) \ : \ -\sum_{i=1}^{N} \partial_i[|\partial_i u_n|^{p_i-2}\partial_i u_n] = f_n. \tag{1.5.29}$$

If $\varepsilon > 0$, and if

$$\sigma_i = (p_i - 1) + \frac{N(r-1)(\overline{p}-1)}{N - r\overline{p}}, \qquad \forall \ i = 1, ..., N, \tag{1.5.30}$$

and $r < m$, we take as a test function in the weak formulation of the problems (1.5.29) $v_{\varepsilon,n} = [(\varepsilon+|G_k(u_n)|)^{\sigma_i-(p_i-1)} - \varepsilon^{\sigma_i-(p_i-1)}]\mathrm{sgn}(u_n)$, where $G_k(s)$ has been recalled in (1.1.8) and σ_i as in (1.5.30), to obtain

$$[\sigma_i - (p_i-1)] \int_\Omega |\partial_i G_k(u_n)|^{p_i} (\varepsilon + |G_k(u_n)|)^{\sigma_i - p_i} \le \int_\Omega |f_n||v_{\varepsilon,n}|, \qquad \forall \ i = 1, ..., N. \tag{1.5.31}$$

We know, by the assumptions on m, that

$$p_i - 1 < \sigma_i \le p_i\,, \qquad \forall\ i = 1, ..., N. \tag{1.5.32}$$

We note that if we pass to the limit for $\varepsilon \to 0$ in (1.5.31), we have, by Fatou Lemma, and for every $k, n \in \mathbb{N}$,

$$[\sigma_i - (p_i - 1)] \int_\Omega |\partial_i G_k(u_n)|^{p_i} |G_k(u_n)|^{\sigma_i - p_i} \le \|f_n\|_{L^\infty(\Omega)}\, \|u_n\|_{L^\infty(\Omega)} < \infty, \tag{1.5.33}$$

$\forall\ i = 1, ..., N$. So even if $\sigma_i - p_i < 0$, by (1.5.33), we can apply Lemma 1.2 with $v = G_k(u_n)$, $r = s$ and $t_i = \sigma_i/p_i - 1$, to obtain

$$\left(\int_\Omega |G_k(u_n)|^s\right)^{\frac{N}{\overline{p}} - 1} \le C_0 \prod_{i=1}^N \left(\int_\Omega |f_n||G_k(u_n)|^{\sigma_i - (p_i - 1)}\right)^{\frac{1}{p_i}}, \tag{1.5.34}$$

with

$$\begin{cases} s = \frac{\sigma_i}{p_i\,\gamma_i(N-1) - (p_i - 1)}, & \forall\ i = 1, ..., N, \\ \\ \sum_{i=1}^N \gamma_i = 1, \qquad \gamma_i \ge 0, & \forall\ i = 1, ..., N. \end{cases} \tag{1.5.35}$$

If we use Hölder inequality with exponents $r < m$ and r' in (1.5.34) and since $f \in M^m(\Omega)$, we have
(1.5.36)

$$\left(\int_\Omega |G_k(u_n)|^s\right)^{\frac{N}{\overline{p}} - 1} \le C_1\, C_f \mathrm{meas}(A_{k,n})^{\left(1 - \frac{q}{m}\right)\frac{N}{r\overline{p}}} \prod_{i=1}^N \left(\int_\Omega |G_k(u_n)|^{[\sigma_i - (p_i - 1)]r'}\right)^{\frac{1}{r'\,p_i}},$$

$A_{k,n} = \{|u_n| > k\}$. Hence we have to solve the following system

$$\begin{cases} s = \frac{\sigma_i}{p_i\,\gamma_i(N-1) - (p_i - 1)}, & \forall\ i = 1, ..., N, \\ \\ s = [\sigma_i - (p_i - 1)]r', & \forall\ i = 1, ..., N, \\ \\ \sum_{i=1}^N \gamma_i = 1, \quad \gamma_i \ge 0, & \forall\ i = 1, ..., N. \end{cases} \tag{1.5.37}$$

If σ_i is as in (1.5.30) and

$$\gamma_i = \left(1 - \frac{1}{p_i}\right) \frac{N - r\overline{p}}{N(\overline{p} - 1)r(N - 1)} + \left(1 - \frac{1}{p_i}\right) \frac{1}{N - 1} + \frac{1}{p_i}\,\frac{r - 1}{r(N - 1)}, \tag{1.5.38}$$

it is easy to prove that

$$\sum_{i=1}^{N} \gamma_i = 1 \qquad \text{and} \qquad \gamma_i \geq 0, \qquad \forall\ i = 1, ..., N.$$

Moreover

$$s = \frac{rN(\overline{p}-1)}{N - r\overline{p}}.$$

Therefore, we obtain

$$\left(\int_\Omega |G_k(u_n)|^s\right)^{\frac{1}{s}} \leq C_2 \operatorname{meas}(A_{k,n})^{\left(1-\frac{r}{m}\right)\frac{N}{N-r\overline{p}}\frac{1}{s}}. \tag{1.5.39}$$

By applying Hölder inequality, with exponents s and s' to $\int_\Omega |G_k(u_n)|$, from (1.5.39), we have

$$\int_\Omega |G_k(u_n)| \leq C_2 \operatorname{meas}(A_{k,n})^{1-\frac{N-m\overline{p}}{mN(\overline{p}-1)}}.$$

If we define $\gamma = 1 - \frac{N-m\overline{p}}{mN(\overline{p}-1)}$ and $g_n(k) = \int_\Omega |G_k(u_n)|$, and we proceed as in the proof of Theorem 1.19, we get

$$1 \leq -\frac{C_3}{1-\frac{1}{\gamma}}(g_n(k)^{1-\frac{1}{\gamma}})'. \tag{1.5.40}$$

By the assumptions on m, we have $1 - 1/\gamma < 0$. By integrating, from 0 to k, the inequality (1.5.40), we obtain

$$k \leq C_4[g_n(k)^{1-\frac{1}{\gamma}} - g_n(0)^{1-\frac{1}{\gamma}}] \leq \frac{C_4}{g_n(k)^{\frac{1}{\gamma}-1}},$$

it implies

$$g_n(k) \leq \frac{C_5}{k^{\frac{\gamma}{1-\gamma}}}.$$

Using (1.5.27) again, we have

$$\operatorname{meas}(A_{k,n}) \leq \frac{C_5}{k^{\frac{\gamma}{1-\gamma}+1}} \quad \text{and} \quad \frac{\gamma}{1-\gamma} + 1 = \frac{mN(\overline{p}-1)}{N - m\overline{p}} = s.$$

Hence the sequence $\{u_n\}$ is uniformly bounded in $M^s(\Omega)$. For $\partial_i u_n$, we proceed as follows. We use as a test function $v = T_1(G_k(u_n))$ in the weak formulation of problems (1.5.29), and we have

$$\sum_{i=1}^{N} \int_\Omega |\partial_i T_1(G_k(u_n))|^{p_i} \leq \int_\Omega |f_n||T_1(G_k(u_n))| \leq \int_{A_{k,n}} |f_n|.$$

Now we use the assumptions on f_n and the fact that u_n is uniformly bounded in $M^s(\Omega)$, and we obtain

$$\sum_{i=1}^{N} \int_{B_{k,n}} |\partial_i u_n|^{p_i} \le C_f \mathrm{meas}(A_{k,n})^{1-\frac{1}{m}} \le C_6 \frac{1}{k^{s\left(1-\frac{1}{m}\right)}}, \tag{1.5.41}$$

with $B_{k,n} = \{k < |u_n| \le k+1\}$. We note that

$$0 < s\left(1 - \frac{1}{m}\right) \le 1 \quad \Leftrightarrow \quad 1 < m \le (\overline{p}^*)',$$

and that for $0 < \theta < 1$, we have that

$$\frac{(k-1)^{1-\theta}}{1-\theta} > \sum_{j=1}^{k-1} \frac{1}{j^\theta}.$$

For $k \ge 1$, we have, by (1.5.41) and the definition of $B_{k,n}$,

$$\sum_{i=1}^{N} \int_\Omega |\partial_i T_k(u_n)|^{p_i} \le N \mathrm{meas}(\Omega) + C_7 \sum_{j=1}^{k-1} \frac{1}{j^{s\left(1-\frac{1}{m}\right)}} \le C_8\, k^{1-s\left(1-\frac{1}{m}\right)}.$$

It implies

$$\int_\Omega |\partial_i T_k(u_n)|^{p_i} \le C_8\, k^{1-s\left(1-\frac{1}{m}\right)}.$$

So we obtain

$$t^{p_i} \mathrm{meas}(\{|u_n| \le k\} \cap \{|\partial_i u_n| > t\}) \le$$
$$\le \int_{\{|u_n|\le k\}\cap\{|\partial_i u_n|>t\}} |\partial_i u_n|^{p_i} \le \int_{\{|u_n|\le k\}} |\partial_i u_n|^{p_i} \le C_8\, k^{1-s\left(1-\frac{1}{m}\right)}, \qquad \forall\ i = 1, ..., N.$$

Since we also have

$$\mathrm{meas}(\{|\partial_i u_n| > t\}) \le \mathrm{meas}(\{|\partial_i u_n| > t, |u_n| \le k\}) +$$
$$+ \mathrm{meas}(\{|u_n| > k\}) \le C_8\, k^{1-s\left(1-\frac{1}{m}\right)} \cdot \frac{1}{t^{p_i}} + C_9 \frac{1}{k^s}, \qquad \forall\ i = 1, ..., N.$$

If we minimize on k, we find $k = k^*(t)$, such that

$$\mathrm{meas}(\{|\partial_i u_n| > t\}) \le \frac{C_{10}}{t^{\frac{p_i m s}{s+m} = s_i}}, \qquad \forall\ i = 1, ..., N,$$

with

$$s_i = p_i \frac{mN(\overline{p}-1)}{\overline{p}(N-m)}, \qquad \forall\ i = 1, ..., N.$$

Hence $\{\partial_i u_n\}$ is uniformly bounded in $M^{s_i}(\Omega)$ for all $i = 1, ..., N$. To pass to the limit in (1.5.29), we proceed as in Theorem 1.14, by the property of $M^m(\Omega)$ and so we can conclude the proof of Theorem 1.23. ∎

Now we prove the results concerning the regularity of the weak solution of problem (1.4.2). This Theorem has been showed in [**76**]. In order to be complete we report the proof of this result but here we give a little different proof.

1.5.5. Proof of Theorem 1.29. We use as a test function in the weak formulation of the problem (1.4.2), $v = G_k(u)$, as in (1.1.8), we obtain

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i G_k(u) = \sum_{i=1}^{N} \int_\Omega f_i \partial_i G_k(u)$$

$$\Downarrow$$

$$\sum_{i=1}^{N} \int_\Omega |\partial_i G_k(u)|^{p_i} \leq \sum_{i=1}^{N} \int_\Omega |f_i \partial_i G_k(u)|.$$

We apply Hölder inequality, with exponents p_i and p_i', to each term of the sum which appears to the second member of the previous inequality, we get

$$\sum_{i=1}^{N} \int_\Omega |\partial_i G_k(u)|^{p_i} \leq \sum_{i=1}^{N} \left(\int_{A_k} |f_i|^{p_i'}\right)^{\frac{1}{p_i'}} \left(\int_\Omega |\partial_i G_k(u)|^{p_i}\right)^{\frac{1}{p_i}},$$

where $A_k = \{x \in \Omega : |u(x)| > k\}$. Now we use Young inequality, for any $i = 1, ..., N$, we obtain

$$\sum_{i=1}^{N} \int_\Omega |\partial_i G_k(u)|^{p_i} \leq C_\varepsilon \sum_{i=1}^{N} \int_{A_k} |f_i|^{p_i'} + \frac{\varepsilon}{p_1} \sum_{i=1}^{N} \int_\Omega |\partial_i G_k(u)|^{p_i}.$$

By simplifying, we obtain

$$\left(1 - \frac{\varepsilon}{p_1}\right) \sum_{i=1}^{N} \int_\Omega |\partial_i G_k(u)|^{p_i} \leq C_0 \sum_{i=1}^{N} \int_{A_k} |f_i|^{p_i'}.$$

By choosing $\varepsilon : 1 - \varepsilon/p_1 > 0$, we get

$$\int_\Omega |\partial_i G_k(u)|^{p_i} \leq C_1 \sum_{i=1}^{N} \int_{A_k} |f_i|^{p_i'}$$

$$\Downarrow$$

$$\left(\int_\Omega |\partial_i G_k(u)|^{p_i}\right)^{\frac{1}{p_i N}} \leq \left(C_1 \sum_{i=1}^{N} \int_{A_k} |f_i|^{p_i'}\right)^{\frac{1}{p_i N}}, \qquad \forall\ i = 1, ..., N.$$

Therefore, by Sobolev inequality (1.1.5) with $r = \overline{p}^*$, we get

$$\|G_k(u)\|_{L^{\overline{p}^*}(\Omega)} \leq C_2 \prod_{i=1}^{N} \left(\int_\Omega |\partial_i G_k(u)|^{p_i}\right)^{\frac{1}{p_i N}} \leq$$

$$\leq C_2 \prod_{i=1}^{N} \left(C_1 \sum_{i=1}^{N} \int_{A_k} |f_i|^{p_i'} \right)^{\frac{1}{p_i N}} = C_3 \left(\sum_{i=1}^{N} \int_{A_k} |f_i|^{p_i'} \right)^{\frac{1}{\overline{p}}}.$$

By Hölder inequality with exponents m_i/p_i' and $(m_i/p_i')'$, for any $i = 1, ..., N$, we obtain

$$\|G_k(u)\|_{L^{\overline{p}^*}(\Omega)} \leq C_3 \left(\sum_{i=1}^{N} \|f_i\|_{L^{m_i}(\Omega)}^{p_i'} \mathrm{meas}(A_k)^{1-p_i'/m_i} \right)^{\frac{1}{\overline{p}}}.$$

Finally, for

$$\gamma = \min_i \left\{ 1 - \frac{p_i'}{m_i} \right\}, \tag{1.5.42}$$

$k \geq k_0$ and k_0 such that $\mathrm{meas}(A_{k_0}) \leq 1$,

$$\|G_k(u)\|_{L^{\overline{p}^*}(\Omega)} \leq C_4 \, \mathrm{meas}(A_k)^{\frac{\gamma}{\overline{p}}}. \tag{1.5.43}$$

If we apply again Hölder inequality with exponent $\overline{p}^*$ and $(\overline{p}^*)'$ to $\int_\Omega |G_k(u)|$ we get

$$\int_\Omega |G_k(u)| \leq \left(\int_\Omega |G_k(u)|^{\overline{p}^*} \right)^{\frac{1}{\overline{p}^*}} \mathrm{meas}(A_k)^{1-\frac{1}{\overline{p}^*}}.$$

Therefore, by (1.5.43) we have

$$\int_\Omega |G_k(u)| \leq C_4 \, \mathrm{meas}(A_k)^{\frac{\gamma}{\overline{p}}+1-\frac{1}{\overline{p}^*}}.$$

We put $g(k) = \int_\Omega |G_k(u)|$, then $g'(k) = -\mathrm{meas}(A_k)$ (see [**51**], [**56**]). We obtain

$$g(k)^{\frac{\overline{p}\,\overline{p}^*}{\gamma\overline{p}^* + \overline{p}\,\overline{p}^* - \overline{p}}} \leq -C_5 \, g'(k).$$

We define

$$\alpha = \frac{\overline{p}\,\overline{p}^*}{\gamma\overline{p}^* + \overline{p}\,\overline{p}^* - \overline{p}},$$

then

$$1 \leq -C_5 \, g'(k) g(k)^{-\alpha} = -\frac{C_5}{1-\alpha} (g(k)^{1-\alpha})'. \tag{1.5.44}$$

We remark that

$$1 - \alpha > 0 \;\; \Leftrightarrow \;\; \min_i \left\{ 1 - \frac{p_i'}{m_i} \right\} \frac{N}{N - \overline{p}} > 1,$$

as in (1.4.3). By integration from 0 to k of (1.5.44), we have

$$k \leq -C_5 [g(k)^{1-\alpha} - g(0)^{1-\alpha}]$$

$$\Downarrow$$

$$C_5 \, g(k)^{1-\alpha} \leq -k + C_5 \|u\|_{L^1(\Omega)}^{1-\alpha}.$$

Since $g(k)$ is a non-negative and decreasing function there exists $\overline{k}$ such that $g(\overline{k}) = 0$ and so $u \in L^\infty(\Omega)$. ■

By using the same techniques of Theorem 1.4 iii), by Lemma 1.2, we prove also the Theorem 1.31. This problem has been already studied in [**25**], but we present a more simple proof using the new Lemma 1.2, presented in Section 1.1.

1.5.6. Proof of Theorem 1.31. Also in this case we use as a test function $v = |T_k(u)|^{t_j\, p_j} T_k(u)$, for all $j = 1, ..., N$, in the weak formulation of problem (1.4.2), where T_k is as in (1.1.7) and t_j's are positive real numbers that we will choose later. We obtain

$$C_0 \sum_{i=1}^{N} \int_\Omega |T_k(u)|^{p_j\, t_j} |\partial_i T_k(u)|^{p_i} \le C_1 \sum_{i=1}^{N} \int_\Omega |f_i| |T_k(u)|^{\frac{p_j\, t_j}{p_i'}} |T_k(u)|^{\frac{p_j\, t_j}{p_i}} |\partial_i T_k(u)|,$$

$\forall\ j = 1, ..., N$. By applying Young inequality, it holds true

$$\sum_{i=1}^{N} \left(C_0 - C_1 \frac{\varepsilon^{p_i}}{p_i} \right) \int_\Omega |T_k(u)|^{p_j\, t_j} |\partial_i T_k(u)|^{p_i} \le C_\varepsilon \sum_{i=1}^{N} \int_\Omega |f_i|^{p_i'} |T_k(u)|^{p_j\, t_j},$$

$\forall\ j = 1, ..., N$. Now we choose ε such that $C_0 - C_1 \frac{\varepsilon^{p_i}}{p_i} > 0$, for all $i = 1, ..., N$, we have

$$\int_\Omega |T_k(u)|^{p_j\, t_j} |\partial_j T_k(u)|^{p_j} \le C_2 \sum_{i=1}^{N} \int_\Omega |f_i|^{p_i'} |T_k(u)|^{p_j\, t_j}, \qquad \forall\ j = 1, ..., N.$$

By Hölder inequality, with exponents m_i/p_i' and $(m_i/p_i')'$, it holds

$$\left(\int_\Omega |T_k(u)|^{p_j\, t_j} |\partial_j T_k(u)|^{p_j} \right)^{\frac{1}{p_j}} \le C_3 \left(\sum_{i=1}^{N} \left(\int_\Omega |f_i|^{m_i} \right)^{\frac{p_i'}{m_i}} \left(\int_\Omega (1 + |T_k(u)|)^{p_j\, t_j (m_i/p_i')'} \right)^{1 - \frac{p_i'}{m_i}} \right)^{\frac{1}{p_j}},$$

$\forall\ j = 1, ..., N$. Recalling that $\mu = \min_i \{m_i/p_i'\}$, we arrive to the following inequality

$$\prod_{j=1}^{N} \left(\int_\Omega |T_k(u)|^{p_j\, t_j} |\partial_j T_k(u)|^{p_j} \right)^{\frac{1}{p_j}} \le C_4 \left(\sum_{i=1}^{N} \left(\int_\Omega (1 + |T_k(u)|)^{\frac{p_j\, t_j\, \mu}{\mu - 1}} \right)^{1 - \frac{p_i'}{m_i}} \right)^{\frac{N}{\overline{p}}}.$$

Now we can use Lemma 1.2, we obtain

$$\left(\int_\Omega |T_k(u)|^s \right)^{\frac{N}{\overline{p}} - 1} \le C_5 \left(\sum_{i=1}^{N} \left(\int_\Omega (1 + |T_k(u)|)^{\frac{t_j\, p_j \mu}{\mu - 1}} \right)^{1 - \frac{p_i'}{m_i}} \right)^{\frac{N}{\overline{p}}}, \tag{1.5.45}$$

with

$$\begin{cases} s = \frac{1+t_j}{\gamma_j(N-1)-1+1/p_j} & \text{for all } j=1,...,N, \\ \\ \sum_{j=1}^N \gamma_j = 1, \qquad \gamma_j \geq 0, & \text{for all } j=1,...,N. \end{cases} \tag{1.5.46}$$

We also want in (1.5.45) $s = \frac{t_j\, p_j \mu}{\mu - 1}$, for any $j = 1, ..., N$. So we must solve the following system

$$\begin{cases} s = \frac{1+t_j}{\gamma_j(N-1)-1+1/p_j}, & \forall \ j=1,...,N, \\ \\ s = \frac{t_j\, p_j \mu}{\mu-1}, & \forall \ j=1,...,N, \\ \\ \sum_{j=1}^N \gamma_j = 1, & \gamma_j \geq 0 \quad \text{and} \quad t_j \geq 0, \quad \forall \ j=1,...,N. \end{cases}$$

By the first two equations in the previous system, after some lengthy but easy calculations, we get

$$t_j = \frac{\mu - 1}{\mu[p_j\, \gamma_j(N-1) - (p_j - 1)] - (\mu - 1)}, \qquad \forall \ j = 1, ..., N. \tag{1.5.47}$$

and

$$\gamma_j = \frac{p_j - 1}{p_j(N-1)} + \frac{\mu - 1}{\mu p_j(N-1)} + \frac{N - \overline{p}\mu}{\overline{p}N\mu(N-1)}, \qquad \forall \ j = 1, ..., N. \tag{1.5.48}$$

It is simple to show that $\sum_{j=1}^N \gamma_j = 1$, $\gamma_j \geq 0$ and $t_j \geq 0$, by our assumptions of m_j's. Moreover, by the choice of t_j and γ_j, we obtain

$$s = \frac{N\overline{p}\mu}{N - \mu\overline{p}}.$$

Therefore

$$\|T_k(u)\|_{L^s(\Omega)} \leq C_6, \qquad \forall \ k \in \mathbb{N}.$$

By Fatou Lemma we can pass to the limit as $k \to +\infty$, we obtain

$$\|u\|_{L^s(\Omega)} \leq C_7.$$

∎

1.6. Entropic solutions

In this section we want to extend a definition of solution, introduced for the first time in [**11**], which allows us to prove a uniqueness result and a existence result for $1 < \overline{p} < 2 - 1/N$. In the previous pages we have seen that there exists a distributional solution for the problem (1.2.1), namely u belongs to $W_0^{1,1}(\Omega)$, such that

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i \phi = \int_\Omega f\phi, \qquad \forall\ \phi \in C_0^1(\Omega),$$

if $f \in L^m(\Omega)$ with $m < p'_\infty$. Before all it is necessary to introduce functional spaces on Ω.

Definition 1.35. $\mathcal{T}^{1,1}_{\mathrm{loc}}(\Omega)$ is the set of measurable functions $u : \Omega \to \mathbb{R}$ such that for any $k > 0$ the truncation function $T_k(u)$, as in (1.1.7), belongs to $W^{1,1}_{\mathrm{loc}}(\Omega)$.

Definition 1.36. Set $1 < p_i < \infty$, for all $i = 1, ..., N$. $\mathcal{T}_0^{1,(p_i)}(\Omega)$ is the set of measurable functions $u : \Omega \to \mathbb{R}$ such that for any $k > 0$ the truncation function $T_k(u)$ belongs to $W_0^{1,(p_i)}(\Omega)$.

Before giving the definition of the entropy solution we present the following result.

Lemma 1.37. *For all $u \in \mathcal{T}_0^{1,(p_i)}(\Omega)$, a unique, measurable function $v_i : \Omega \to \mathbb{R}$, for every $i = 1, ..., N$, exists such that*

$$\partial_i T_k(u) = v_i \chi_{\{x\in\Omega : |u(x)|<k\}} \quad a.e.$$

Moreover, if $u \in W_0^{1,(p_i)}(\Omega)$, $v_i = \partial_i u$.

We do not show this result because the proof is pretty much the same of which in [**11**]. Now we can define the entropy solution for the problem (1.2.1).

Definition 1.38. Set $f \in L^1(\Omega)$. A function $u \in \mathcal{T}_0^{1,(p_i)}(\Omega)$ is a entropy solution of (1.2.1) if

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i T_k(u-\varphi) \le \int_\Omega f T_k(u-\varphi) \tag{1.6.1}$$

$$\forall\ k > 0 \quad \text{and} \quad \forall\ \varphi \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega).$$

Remark 1.39. We note that the above inequality is well defined because $u \in \mathcal{T}_0^{1,(p_i)}(\Omega)$.

Now we want to prove that the solution, which we have found in the previous sections by approximation methods, is an entropy solution. We have already proved in Chapter 1 (see i) of Theorem 1.14 and its proof) that the solutions u_n of approximating problems (1.5.15) converge to u in $W_0^{1,(s_i)}(\Omega)$ with

$$s_i < \frac{N(\overline{p}-1)}{\overline{p}(N-1)}p_i, \qquad \forall\ i = 1, ..., N.$$

Moreover $T_k(u_n) \to T_k(u)$ weakly in $W_0^{1,(p_i)}(\Omega)$ and also $f_n \to f$ in $L^1(\Omega)$. So we have

$$\lim_{n\to+\infty} \int_\Omega f_n T_k(u_n - \varphi) = \int_\Omega f T_k(u - \varphi).$$

Now by Fatou Lemma, we have

$$\liminf_{n\to+\infty} \sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i-2}\partial_i u_n \partial_i T_k(u_n - \varphi) \geq$$

$$\geq \liminf_{n\to+\infty} \sum_{i=1}^N \int_\Omega [|\partial_i u_n|^{p_i-2}\partial_i u_n - |\partial_i \varphi|^{p_i-2}\partial_i\varphi]\partial_i T_k(u_n - \varphi) +$$

$$+ \liminf_{n\to+\infty} \sum_{i=1}^N \int_\Omega |\partial_i\varphi|^{p_i-2}\partial_i\varphi\partial_i T_k(u_n - \varphi) \geq$$

$$\geq \sum_{i=1}^N \int_\Omega [|\partial_i u|^{p_i-2}\partial_i u - |\partial_i\varphi|^{p_i-2}\partial_i\varphi]\partial_i T_k(u - \varphi) +$$

$$+ \sum_{i=1}^N \int_\Omega |\partial_i\varphi|^{p_i-2}\partial_i\varphi\partial_i T_k(u - \varphi) = \sum_{i=1}^N \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i T_k(u - \varphi).$$

Hence u satisfies (1.6.1) and so it is an entropy solution.

Now we prove the following result.

PROPOSITION 1.40. *Set u an entropy solution for the problem* (1.2.1). *Then u belongs to $M^{\frac{N(\overline{p}-1)}{N-\overline{p}}}(\Omega)$ and $\partial_i u$ belongs to $M^{p_i\frac{N(\overline{p}-1)}{\overline{p}(N-1)}}(\Omega)$, for all $i = 1, ..., N$.*

PROOF. We choose $\varphi = 0$ as a test function in (1.6.1), we have

$$\sum_{i=1}^N \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i T_k(u) \leq \int_\Omega f T_k(u) \leq k\|f\|_{L^1(\Omega)}$$

$$\Downarrow$$

$$\sum_{i=1}^N \int_\Omega |\partial_i T_k(u)|^{p_i} \leq k\|f\|_{L^1(\Omega)}.$$

It implies

$$\int_\Omega |\partial_i T_k(u)|^{p_i} \le k\|f\|_{L^1(\Omega)}, \qquad \forall\ i = 1, ..., N. \tag{1.6.2}$$

Hence we obtain

$$\left(\int_\Omega |\partial_i T_k(u)|^{p_i}\right)^{\frac{1}{p_i N}} \le k^{\frac{1}{p_i N}} \|f\|_{L^1(\Omega)}^{\frac{1}{p_i N}}, \qquad \forall\ i = 1, ..., N$$

$$\Downarrow$$

$$\prod_{i=1}^N \left(\int_\Omega |\partial_i T_k(u)|^{p_i}\right)^{\frac{1}{p_i N}} \le k^{\frac{1}{\overline{p}}} \|f\|_{L^1(\Omega)}^{\frac{1}{\overline{p}}}.$$

Now we use the anisotropic Sobolev inequality (1.1.5), with $r = \overline{p}^*$ and $v = T_k(u)$ and we obtain

$$\int_\Omega |T_k(u)|^{\overline{p}^*} \le C_0 \|f\|_{L^1(\Omega)}^{\frac{\overline{p}^*}{\overline{p}}} k^{\frac{\overline{p}^*}{\overline{p}}}.$$

So

$$k^{\overline{p}^*} \mathrm{meas}(A_k) \le \int_{A_k} |T_k(u)|^{\overline{p}^*} \le \int_\Omega |T_k(u)|^{\overline{p}^*} \le C_0 \|f\|_{L^1(\Omega)}^{\frac{\overline{p}^*}{\overline{p}}} k^{\frac{\overline{p}^*}{\overline{p}}},$$

that implies

$$\mathrm{meas}(A_k) \le \frac{C_0 \|f\|_{L^1(\Omega)}^{\frac{\overline{p}^*}{\overline{p}}}}{k^{\frac{\overline{p}^*(\overline{p}-1)}{\overline{p}}}}. \tag{1.6.3}$$

Hence

$$u \in M^{\frac{N(\overline{p}-1)}{N-\overline{p}}}(\Omega).$$

Now we consider (1.6.2), we have

$$\int_\Omega |\partial_i T_k(u)|^{p_i} = \int_{\{|u|\le k\}} |\partial_i u|^{p_i} \ge \int_{\{|u|\le k\}\cap\{|\partial_i u|>\beta\}} |\partial_i u|^{p_i} \ge$$

$$\ge \beta^{p_i} \mathrm{meas}(\{|u| \le k\} \cap \{|\partial_i u| > \beta\})$$

and so

$$\mathrm{meas}(\{|u| \le k, |\partial_i u| > \beta\}) \le \frac{k\|f\|_{L^1(\Omega)}}{\beta^{p_i}}, \tag{1.6.4}$$

for all $i = 1, ..., N$. Then, for all $k > 0$, we obtain

$$\mathrm{meas}(\{|\partial_i u| > \beta\}) = \mathrm{meas}(\{|\partial_i u| > \beta, |u| \le k\}) +$$

$$+\mathrm{meas}(\{|\partial_i u| > \beta, |u| > k\}) \le \mathrm{meas}(\{|\partial_i u| > \beta, |u| \le k\}) + \mathrm{meas}(\{|u| > k\}) \le$$

$$\leq \frac{k\|f\|_{L^1(\Omega)}}{\beta^{p_i}} + \frac{C_0\|f\|_{L^1(\Omega)}^{\frac{\overline{p}^*}{\overline{p}}}}{k^{\frac{\overline{p}^*(\overline{p}-1)}{\overline{p}}}}, \qquad \forall\ i=1,...,N,$$

by (1.6.3) and (1.6.4). We minimize on respect to k. We consider the function

$$\psi(k) = C_1\frac{k}{\beta^{p_i}} + C_2\frac{1}{k^{\frac{\overline{p}^*(\overline{p}-1)}{\overline{p}}}}.$$

We have

$$\psi'(k) = \frac{C_1}{\beta^{p_i}} - C_3 k^{\frac{-\overline{p}^*(\overline{p}-1)-\overline{p}}{\overline{p}}}.$$

So

$$\psi'(k)=0 \quad \Rightarrow \quad k_0 = C_4\beta^{\frac{p_i\overline{p}}{\overline{p}^*(\overline{p}-1)+\overline{p}^*}}.$$

Now we get

$$\psi(k_0) = C_5\beta^{\frac{-p_i\overline{p}^*(\overline{p}-1)}{\overline{p}^*(\overline{p}-1)+\overline{p}}} + C_6\beta^{\frac{-p_i\overline{p}^*(\overline{p}-1)}{\overline{p}^*(\overline{p}-1)+\overline{p}}} = C_7\beta^{\frac{-p_i\overline{p}^*(\overline{p}-1)}{\overline{p}^*(\overline{p}-1)+\overline{p}}}.$$

Hence

$$\text{meas}(\{|\partial_i u| > \beta\}) \leq C_7\beta^{\frac{-p_i\overline{p}^*(\overline{p}-1)}{\overline{p}^*(\overline{p}-1)+\overline{p}}},$$

for all $i=1,...,N$. So $\partial_i u \in M^{p_i\frac{N(\overline{p}-1)}{\overline{p}(N-1)}}(\Omega)$ for all $i=1,...,N$. ∎

REMARK 1.41. We observe that if $p_i = 2$ for all $i=1,...,N$ (or equivalently $p_i = p$ for all $i=1,...,N$), we obtain the classic result (see [**11**]), namely $u \in M^{\frac{N}{N-2}}(\Omega)$ and $\nabla u \in M^{\frac{N}{N-1}}(\Omega)$.

REMARK 1.42. We note that

$$p_i\frac{N(\overline{p}-1)}{\overline{p}(N-1)}$$

is the same exponent that we have found in Theorem 1.14, as we expect according to the isotropic classic case.

Now we can prove the following result.

THEOREM 1.43. *The entropy solution of problem* (1.2.1) *is unique.*

For the following proof we use techniques introduced in [**69**].

PROOF. Let u be the entropy solution satisfying the regularity properties stated in the previous propositions, obtained by approximation, as in Theorem 1.14. It is sufficient to show that every entropy solution of (1.2.1) coincides with u. Let z be a second entropy solution, so $z \in \mathcal{T}_0^{1,(p_i)}(\Omega)$ and

$$\sum_{i=1}^{N} \int_\Omega |\partial_i z|^{p_i-2} \partial_i T_k(z-\varphi) \leq \int_\Omega f T_k(z-\varphi)$$

for all $k \in \mathbb{R}^+ \setminus \{0\}$ and for all $\varphi \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)$. Taking $\varphi = u_n$, because we know that $u_n \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)$ (see the proof of Theorem 1.14), we get

$$\sum_{i=1}^{N} \int_\Omega |\partial_i z|^{p_i-2} \partial_i z \partial_i T_k(z-u_n) \leq \int_\Omega f T_k(z-u_n). \tag{1.6.5}$$

Now we choose $T_k(z-u_n)$ as test function in the weak formulation of problems (1.5.15). It is possible since $u_n \in L^\infty(\Omega)$ and $z \in \mathcal{T}_0^{1,(p_i)}(\Omega)$ and hence $T_k(z-u_n)$ belongs to $W_0^{1,(p_i)}(\Omega)$ for all $k>0$. We have

$$-\sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i-2} \partial_i u_n \partial_i T_k(z-u_n) = -\int_\Omega f_n T_k(z-u_n). \tag{1.6.6}$$

Then adding (1.6.5) to (1.6.6), we obtain

$$\sum_{i=1}^{N} \int_\Omega [|\partial_i z|^{p_i-2} \partial_i z - |\partial_i u_n|^{p_i-2} \partial_i u_n] \partial_i T_k(z-u_n) \leq \int_\Omega (f-f_n) T_k(z-u_n).$$

We note that the integral in the left hand side is non-negative, and it is bounded from above by a constant $C_1\, k$, independent on n. Moreover the integrand function, $[|\partial_i z|^{p_i-2} \partial_i z - |\partial_i u_n|^{p_i-2} \partial_i u_n] \partial_i T_k(z-u_n)$, goes to $[|\partial_i z|^{p_i-2} \partial_i z - |\partial_i u|^{p_i-2} \partial_i u] \partial_i T_k(z-u)$ a.e., for all $i = 1, ..., N$. Hence, by using Fatou Lemma, we can pass to the limit as $n \to +\infty$, we get

$$\sum_{i=1}^{N} \int_\Omega [|\partial_i z|^{p_i-2} \partial_i z - |\partial_i u|^{p_i-2} \partial_i u] \partial_i T_k(z-u) \leq 0,$$

and so, we obtain

$$\sum_{i=1}^{N} \int_\Omega |\partial_i T_k(z-u)|^{p_i} \leq 0,$$

if $p_i \geq 2$. By the arbitrary choice of k, we have $z = u$ a.e. in Ω. It is possible to obtain the same result, by a slight modification, also if $1 < p_i < 2$, but for simplicity we omit the proof. ∎

CHAPTER 2

Elliptic problems with natural growth terms

In this chapter we deal with some elliptic problems, as (1.2.1), studied in the first chapter, with the presence of lower order terms, with respect to the gradient of u, which play the role of perturbation terms. These terms are called natural growth terms because they have the same growths of the operator (I.1), and they naturally appear if we write the Euler-Lagrange equation, associated to some functionals of the Calculus of Variations. But the boundary value problems, which we study may not be the Euler-Lagrange equation of some functionals. So we will use direct methods to solve these kinds of problems. We see that a sign condition on lower order terms plays a crucial role in finding solutions of our problems, since it allows us to easily obtain a priori estimates from the equation. On the other hand if this assumption fails to hold true we may not even have solutions. Moreover we prove that an extra assumption allows us to show an extra regularity for the solutions of the problems, presented in this chapter, even if the datum only belongs to $L^1(\Omega)$. Also this additional condition is useful to obtain a priori estimate in the energy space $W_0^{1,(p_i)}(\Omega)$.

2.1. Lower order terms without sign assumptions

In this first section we prove the existence of a solution for the following problem

$$(2.1.1)\qquad \begin{cases} -\sum_{i=1}^N \partial_i[|\partial_i u|^{p_i-2}\partial_i u] + \mu_0 u = \sum_{i=1}^N b_i(x,u,\nabla u) + f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mu_0 > 0$,

$$f \in L^m(\Omega), \qquad m > \frac{\overline{p}^*}{\overline{p}^* - p_N}, \qquad \overline{p}^* > p_N,$$

$$b_i(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$$

is a Carathéodory function, for all $i = 1, ..., N$ and there exists $\gamma > 0$ such that the following inequality is true for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and a.e. $x \in \Omega$

$$(2.1.2)\qquad |b_i(x,s,\xi)| \le \gamma |\xi_i|^{p_i}, \qquad \forall\ i = 1, ..., N,$$

or more in general

$$\sum_{i=1}^{N} |b_i(x,s,\xi)| \leq \gamma \sum_{i=1}^{N} |\xi_i|^{p_i}. \tag{2.1.3}$$

First of all we note that it is impossible to use Leray-Lions Theorem (as in Chapter 1) because the b_i's terms are not bounded and we do not have any information concerning on their sign. To prove an existence result for (2.1.1) we use approximating techniques and some results presented in Chapter 1. We also observe that the term, which appear in left side hand of the equation in (2.1.1), allows us to achieve an existence results. In fact if $\mu_0 = 0$ already in the isotropic case (i.e. $p_i = 2$ for all $i = 1, ..., N$) we have a simple counterexample to the existence of solutions for the problem (2.1.1). In the following section we see that if we assume a sign condition on b_i, for $i = 1, .., N$, we can even choose $\mu_0 = 0$.

We highlight that it is also possible to take a unique function b instead of a sum of b_i's, if it satisfies, $\forall\ (s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, a.e. $x \in \Omega$, and $\gamma > 0$, the following condition

$$|b(x,s,\xi)| \leq \gamma \sum_{i=1}^{N} |\xi_i|^{p_i}, \qquad \forall\ i = 1, ..., N. \tag{2.1.4}$$

But the first possibility is more natural always by considering the relation between these problems and some functionals of the Calculus of Variation. As a matter of fact, if we consider the following functional

$$J(v) = \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega a(x,v)|\partial_i v|^{p_i} - \int_\Omega f v, \tag{2.1.5}$$

where a is a bounded, smooth function, we obtain the following Euler-Lagrange equation

$$-\sum_{i=1}^{N} \partial_i[a(x,u)|\partial_i u|^{p_i-2}\partial_i u] + \sum_{i=1}^{N} a'(x,u)|\partial_i u|^{p_i} = f,$$

and in the right hand side a sum appears. We also note that, obviously, the problem (2.1.1) does not correspond to the Euler-Lagrange equation of a functional, as (2.1.5). Indeed if $a(x,u) \equiv 1$ then $a'(x,u) \equiv 0$ and so we find the problem (1.2.1) not (2.1.1).

To fix the ideas, one can consider, as a special example of (2.1.1), the Dirichlet problem:

$$\begin{cases} -\sum_{i=1}^{N} \partial_i[|\partial_i u|^{p_i-2}\partial_i u] + \mu_0 u = \gamma \sum_{i=1}^{N} |\partial_i u|^{p_i} + f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1.6}$$

Moreover we remark that since $f \in L^m(\Omega)$, with $m > \overline{p}^*/(\overline{p}^* - p_N)$, we also must assume $\overline{p}^* > p_N$ and so $p_\infty = \overline{p}^*$. We think that this hypothesis is technical, like we will see better in the following.

We have the result below.

THEOREM 2.1. *Let $f \in L^m(\Omega)$, $m > \overline{p}^*/(\overline{p}^* - p_N)$, $\overline{p}^* > p_N$ and*

i) *$b_i(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, for all $i = 1, ..., N$;*

ii) *there exists $\gamma > 0$, such that the following inequality is true for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and a.e. $x \in \Omega$*

$$|b_i(x,s,\xi)| \le \gamma |\xi_i|^{p_i}, \quad \forall\ i = 1, ..., N. \tag{2.1.7}$$

Then there exists a function $u \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)$, weak solution for the problem (2.1.1)*, namely*

$$\sum_{i=1}^N \int_\Omega |\partial_i u|^{p_i-2} \partial_i u \partial_i \varphi + \mu_0 \int_\Omega u\varphi = \sum_{i=1}^N \int_\Omega b_i(x,u,\nabla u)\varphi + \int_\Omega f\varphi \tag{2.1.8}$$

for all $\varphi \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)$.

The idea of the proof is taken from [**29**] (see also [**13**]). We consider for any $n \in \mathbb{N}$ the approximating problems

$$\sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i-2} \partial_i u_n \partial_i \varphi + \mu_0 \int_\Omega u_n\varphi = \sum_{i=1}^N \int_\Omega b_i^n(x,u_n,\nabla u_n)\varphi + \int_\Omega f_n\varphi \tag{2.1.9}$$

$$\forall\ \varphi \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)$$

where

$$b_i^n(x,s,\xi) = \frac{b_i(x,s,\xi)}{1 + \frac{1}{n}|b_i(x,s,\xi)|}, \qquad \forall\ i = 1, ..., N$$

and

$$f_n(x) = \frac{f(x)}{1 + \frac{1}{n}|f(x)|},$$

so that

$$|b_i^n(x,s,\xi)| \le |b_i(x,s,\xi)| \quad \text{and} \quad |f_n(x)| \le |f(x)|, \qquad \forall\ i = 1, ..., N, \ \text{ and } \ n \in \mathbb{N} \tag{2.1.10}$$

and

$$|b_i^n(x,s,\xi)| \le n \quad \text{and} \quad |f_n(x)| \le n, \qquad \forall\ i = 1, ..., N, \ \text{ and } \ n \in \mathbb{N}. \tag{2.1.11}$$

Then there exists a solution $u_n \in W_0^{1,(p_i)}(\Omega)$ of (2.1.9) by a simple modification of classic Leray-Lions Theorem. This solution is also bounded because $f \in L^m(\Omega)$ with $m > \frac{\overline{p}^*}{\overline{p}^* - p_N} > \frac{N}{\overline{p}}$, (see Theorem 1.4 i) in Section 1.2). We divide the proof in several steps.

2.1.1. A priori estimates. We prove the following result.

LEMMA 2.2. *There exist $\beta > 1$ and a constant $C > 0$, not depending on n, such that*

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i} e^{\beta p_N |u_n|} \leq C. \tag{2.1.12}$$

PROOF. We use as a test function in (2.1.9), $\varphi = (e^{\beta p_N |u_n|} - 1)\mathrm{sgn}(u_n)$, where $\beta > 1$ is a real number, that we will fix later, because $u_n \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)$. We have, by (2.1.7) and (2.1.10)

$$\beta p_N \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i} e^{\beta p_N |u_n|} + \mu_0 \int_\Omega |u_n|(e^{\beta p_N |u_n|} - 1) \leq$$

$$\leq \gamma \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i} e^{\beta p_N |u_n|} + \int_\Omega |f|(e^{\beta p_N |u_n|} - 1).$$

Hence

$$(\beta p_N - \gamma) \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i} e^{\beta p_N |u_n|} + \mu_0 \int_\Omega |u_n|(e^{\beta p_N |u_n|} - 1) \leq \int_\Omega |f|(e^{\beta p_N |u_n|} - 1).$$

By recalling that $(e^{pt} - 1) \geq (e^t - 1)^p$ for every $p > 1$, and that for every $M > 1$ there exists t^* such that $(e^{pt} - 1) \leq M(e^t - 1)^p$ for $t > t^*$, we obtain

$$(\beta p_N - \gamma) \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i} e^{\beta p_N |u_n|} + \mu_0 \int_\Omega |u_n|(e^{\beta |u_n|} - 1)^{p_N} \leq$$

$$\leq \int_\Omega |f|(e^{\beta p_N |u_n|} - 1) = \int_{\{|u_n| > \frac{t^*}{\beta}\}} |f|(e^{\beta p_N |u_n|} - 1) + \int_{\{|u_n| \leq \frac{t^*}{\beta}\}} |f|(e^{\beta p_N |u_n|} - 1) \leq$$

$$\leq M \int_\Omega |f|(e^{\beta |u_n|} - 1)^{p_N} + C_0 \int_\Omega |f|.$$

Now we apply the Hölder inequality with exponents m and m', to obtain

$$(\beta p_N - \gamma) \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i} e^{\beta p_N |u_n|} + \mu_0 \int_\Omega |u_n|(e^{\beta |u_n|} - 1)^{p_N} \leq$$

$$\leq M\|f\|_{L^m(\Omega)}\left(\int_\Omega (e^{\beta|u_n|}-1)^{p_N m'}\right)^{\frac{1}{m'}}+C_0\|f\|_{L^1(\Omega)}.$$

Since $p_N < p_N m' < \overline{p}^*$, ($p_N m' < \overline{p}^* \quad\Leftrightarrow\quad m > \frac{\overline{p}^*}{\overline{p}^*-p_N}$ with $\overline{p}* > p_N$), we use the interpolation inequality to have

$$(\beta p_N-\gamma)\sum_{i=1}^N\int_\Omega |\partial_i u_n|^{p_i}e^{\beta p_N|u_n|}+\mu_0\int_\Omega |u_n|(e^{\beta|u_n|}-1)^{p_N}\leq$$

$$\leq M\|f\|_{L^m(\Omega)}\|e^{\beta|u_n|}-1\|^{p_N(1-\theta)}_{L^{p_N}(\Omega)}\|e^{\beta|u_n|}-1\|^{\theta p_N}_{L^{\overline{p}^*}(\Omega)}+C_0\|f\|_{L^1(\Omega)}.$$

Now we apply Young inequality with exponents $\frac{1}{\theta}$ and θ, to obtain

$$(\beta p_N-\gamma)\sum_{i=1}^N\int_\Omega |\partial_i u_n|^{p_i}e^{\beta p_N|u_n|}+\mu_0\int_\Omega |u_n|(e^{\beta|u_n|}-1)^{p_N}\leq$$

$$\leq \varepsilon(M\|f\|_{L^m(\Omega)})^{\frac{1}{\theta}}\left(\int_\Omega (e^{\beta|u_n|}-1)^{\overline{p}^*}\right)^{\frac{p_N}{\overline{p}^*}}+C_\varepsilon\int_\Omega (e^{\beta|u_n|}-1)^{p_N}+C_0\|f\|_{L^1(\Omega)}.$$

By Lemma 1.1, we obtain

$$(\beta p_N-\gamma)\sum_{i=1}^N\int_\Omega |\partial_i u_n|^{p_i}e^{\beta p_N|u_n|}+\mu_0\int_\Omega |u_n|(e^{\beta|u_n|}-1)^{p_N}\leq$$

$$\leq \varepsilon(M\|f\|_{L^m(\Omega)})^{\frac{1}{\theta}}C_1^{p_N}N^{p_N-1}B^{p_N-p_1}\sum_{i=1}^N\int_\Omega |\partial_i(e^{\beta|u_n|}-1)|^{p_i}+$$

$$+C_\varepsilon\int_\Omega (e^{\beta|u_n|}-1)^{p_N}+C_0\|f\|_{L^1(\Omega)}=$$

$$=\varepsilon(M\|f\|_{L^m(\Omega)})^{\frac{1}{\theta}}C_1^{p_N}N^{p_N-1}B^{p_N-p_1}\sum_{i=1}^N\int_\Omega \beta^{p_i}|\partial_i u_n|^{p_i}e^{\beta p_i|u_n|}+$$

$$+C_\varepsilon\int_\Omega (e^{\beta|u_n|}-1)^{p_N}+C_0\|f\|_{L^1(\Omega)}.$$

We choose $\beta>1$ such that $\beta p_N-\gamma>0$ and, since $e^t>1$ for any $t>0$ and $p_i\leq p_N$ for any $i=1,...,N$, we have

$$(\beta p_N-\gamma)\sum_{i=1}^N\int_\Omega |\partial_i u_n|^{p_i}e^{\beta p_N|u_n|}+\mu_0\int_\Omega |u_n|(e^{\beta|u_n|}-1)^{p_N}\leq$$

$$\leq \varepsilon(M\|f\|_{L^m(\Omega)})^{\frac{1}{\theta}}C_1^{p_N}N^{p_N-1}B^{p_N-p_1}\beta^{p_N}\sum_{i=1}^N\int_\Omega |\partial_i u_n|^{p_i}e^{\beta p_N|u_n|}+$$

$$+C_\varepsilon \int_\Omega (e^{\beta|u_n|}-1)^{p_N} + C_0\|f\|_{L^1(\Omega)}.$$

Setting $C_2 = (M\|f\|_{L^m(\Omega)})^{\frac{1}{\theta}} C_1^{p_N} N^{p_N-1} B^{p_N-p_1}\beta^{p_N}$, we get

$$[(\beta p_N-\gamma)-\varepsilon C_2]\sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i} e^{\beta p_N|u_n|} + \mu_0\int_\Omega |u_n|(e^{\beta|u_n|}-1)^{p_N} \leq$$

$$\leq C_\varepsilon \int_\Omega (e^{\beta|u_n|}-1)^{p_N} + C_0\|f\|_{L^1(\Omega)}.$$

We take ε such that $(\beta p_N-\gamma)-\varepsilon C_2>0$ and we note that

$$\mu_0\int_\Omega |u_n|(e^{\lambda|u_n|}-1)^{p_N} \geq \mu_0\int_{\{|u_n|>\frac{C_\varepsilon}{\mu_0}\}} |u_n|(e^{\beta|u_n|}-1)^{p_N} \geq C_\varepsilon\int_{\{|u_n|>\frac{C_\varepsilon}{\mu_0}\}} (e^{\beta|u_n|}-1)^{p_N}.$$

Therefore, since

$$C_\varepsilon\int_\Omega (e^{\beta|u_n|}-1)^{p_N} = C_\varepsilon\int_{\{|u_n|>\frac{C_\varepsilon}{\mu_0}\}} (e^{\beta|u_n|}-1)^{p_N} + C_\varepsilon\int_{\{|u_n|\leq\frac{C_\varepsilon}{\mu_0}\}} (e^{\beta|u_n|}-1)^{p_N},$$

we have

$$[(\beta p_N-\gamma)-\varepsilon C_2]\sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i} e^{\beta p_N|u_n|} \leq$$

$$\leq C_\varepsilon\int_{\{|u_n|\leq\frac{C_\varepsilon}{\mu_0}\}} (e^{\beta|u_n|}-1)^{p_N} + C_0\|f\|_{L^1(\Omega)} \leq C_\varepsilon\int_\Omega (e^{\beta\frac{C_\varepsilon}{\mu_0}}-1)^{p_N} + C_0\|f\|_{L^1(\Omega)}.$$

At the end we get (2.1.12). ∎

The consequence of this lemma is that

$$\sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i} \leq C. \tag{2.1.13}$$

Now we claim the following lemma.

LEMMA 2.3. *There exists a constant $C>0$, not depending on n such that $\|u_n\|_{L^\infty(\Omega)} \leq C$.*

PROOF. We proceed like in the previous Lemma. We put, as test function in (2.1.9), $\varphi = (e^{\beta p_N|G_k(u_n)|}-1)\mathrm{sgn}(u_n)$, where $\beta>1$ is a real number that we will choose later. We have

$$\beta p_N\sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i-2}\partial_i u_n \partial_i G_k(u_n) e^{\beta p_N|G_k(u_n)|} + \mu_0\int_\Omega |u_n|(e^{\beta p_N|G_k(u_n)|}-1) \leq$$

$$\leq \gamma \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i} (e^{\beta |G_k(u_n)|} - 1) + \int_\Omega |f|(e^{\beta |G_k(u_n)|} - 1).$$

Therefore

$$(\beta p_N - \gamma) \sum_{i=1}^{N} \int_\Omega |\partial_i G_k(u_n)|^{p_i} e^{\beta p_N |G_k(u_n)|} + \mu_0 \int_\Omega |u_n|(e^{\beta p_N |G_k(u_n)|} - 1) \leq$$

$$\leq \int_\Omega |f|(e^{\beta |G_k(u_n)|} - 1).$$

If we use that $(e^{pt} - 1) \geq (e^t - 1)^p$ for all $p > 1$ and that exists $M > 1$ such that $(e^{pt} - 1) \leq M(e^t - 1)^p$ for $t > t^*$, we obtain

$$(\beta p_N - \gamma) \sum_{i=1}^{N} \int_\Omega |\partial_i G_k(u_n)|^{p_i} e^{\beta p_N |G_k(u_n)|} + \mu_0 \int_\Omega |u_n|(e^{\beta |G_k(u_n)|} - 1)^{p_N} \leq$$

$$\leq C_0 \int_{\{|u_n| > k\}} |f| + M \int_\Omega |f|(e^{\lambda |G_k(u_n)|} - 1)^{p_N}.$$

We apply Hölder inequality with exponents $m > 1$ and m' to the second term of the above expression, we have

$$(\beta p_N - \gamma) \sum_{i=1}^{N} \int_\Omega |\partial_i G_k(u_n)|^{p_i} e^{\beta p_N |G_k(u_n)|} + \mu_0 \int_\Omega |u_n|(e^{\beta |G_k(u_n)|} - 1)^{p_N} \leq$$

$$\leq C_0 \int_{\{|u_n| > k\}} |f| + M \|f\|_{L^m(\Omega)} \left(\int_\Omega (e^{\beta |G_k(u_n)|} - 1)^{p_N m'} \right)^{\frac{1}{m'}}.$$

Since $p_N < p_N m' < \overline{p}^*$, by interpolation inequality, we get

$$(\beta p_N - \gamma) \sum_{i=1}^{N} \int_\Omega |\partial_i G_k(u_n)|^{p_i} e^{\beta p_N |G_k(u_n)|} + \mu_0 \int_\Omega |u_n|(e^{\beta |G_k(u_n)|} - 1)^{p_N} \leq$$

$$\leq C_0 \int_{\{|u_n| > k\}} |f| + M \|f\|_{L^m(\Omega)} \left(\int_\Omega (e^{\beta |G_k(u_n)|} - 1)^{p_N} \right)^{1-\theta} \left(\int_\Omega (e^{\beta |G_k(u_n)|} - 1)^{\overline{p}^*} \right)^{\frac{p_N \theta}{\overline{p}^*}}.$$

By Young inequality with exponents $\frac{1}{\theta}$ and θ, we obtain

$$(\beta p_N - \gamma) \sum_{i=1}^{N} \int_\Omega |\partial_i G_k(u_n)|^{p_i} e^{\beta p_N |G_k(u_n)|} + \mu_0 \int_\Omega |u_n|(e^{\beta |G_k(u_n)|} - 1)^{p_N} \leq$$

$$\leq C_0 \int_{\{|u_n| > k\}} |f| + \varepsilon (M \|f\|_{L^m(\Omega)})^{\frac{1}{\theta}} \left(\int_\Omega (e^{\beta |G_k(u_n)|} - 1)^{\overline{p}^*} \right)^{\frac{p_N}{\overline{p}^*}} + C_\varepsilon \int_\Omega (e^{\beta |G_k(u_n)|} - 1)^{p_N}.$$

We remark, since $p_N \geq p_i$ for all $i = 1, ..., N$ and $e^t > 1$ for all $t > 0$,

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i G_k(u_n)|^{p_i} e^{\beta p_N |G_k(u_n)|} \geq \sum_{i=1}^{N} \int_{\Omega} |\partial_i G_k(u_n)|^{p_i} e^{\beta p_i |G_k(u_n)|} \geq$$

$$\geq \sum_{i=1}^{N} \frac{1}{\beta^{p_i}} \int_{\Omega} |\partial_i (e^{\beta p_N |G_k(u_n)|} - 1)|^{p_i}.$$

We choose $\beta > 1$, such that $\beta p_N - \gamma > 0$ and we apply Lemma 1.1, we have

$$\sum_{i=1}^{N} \frac{1}{\beta^{p_i}} \int_{\Omega} |\partial_i (e^{\beta p_N |G_k(u_n)|} - 1)|^{p_i} \geq \frac{1}{\beta^{p_N} C_1^{p_N} N^{p_N - 1} B^{p_N - p_1}} \left(\int_{\Omega} (e^{\beta p_N |G_k(u_n)|} - 1)^{\overline{p}^*} \right)^{\frac{p_N}{\overline{p}^*}}.$$

We define $C_2 = \beta^{p_N} C_1^{p_N} N^{p_N - 1} B^{p_N - p_1}$, we get

$$\left[\frac{\beta p_N - \gamma}{C_2} - \varepsilon (M \|f\|_{L^m(\Omega)})^{\frac{1}{\theta}} \right] \left(\int_{\Omega} (e^{\beta |G_k(u_n)|} - 1)^{\overline{p}^*} \right)^{\frac{p_N}{\overline{p}^*}} +$$

$$+ (\mu_0 k - C_\varepsilon) \int_{\Omega} (e^{\beta |G_k(u_n)|} - 1)^{p_N} \leq C_0 \int_{\{|u_n| > k\}} |f|.$$

We take ε such that $\left[\frac{\beta p_N - \gamma}{C_2} - \varepsilon (M \|f\|_{L^m(\Omega)})^{\frac{1}{\theta}} \right] > 0$ and then we choose k such that $(\mu_0 k - C_\varepsilon) > 0$, we have

$$C_3 \|e^{\lambda |G_k(u_n)|} - 1\|^{p_N}_{L^{\overline{p}^*}(\Omega)} + C_4 \|e^{\lambda |G_k(u_n)|} - 1\|^{p_N}_{L^{p_N}(\Omega)} \leq C_0 \int_{\{|u_n| > k\}} |f|.$$

Since $e^t - 1 \geq t$ for any $t \geq 0$ and using Hölder inequality with exponents $\overline{p}^*$ and $(\overline{p}^*)'$, we obtain

$$\beta \int_{\Omega} |G_k(u_n)| \leq \int_{\Omega} (e^{\beta |G_k(u_n)|} - 1) \leq \text{meas}(A_{k,n})^{1 - \frac{1}{\overline{p}^*}} \left(\int_{\Omega} (e^{\beta |G_k(u_n)|} - 1)^{\overline{p}^*} \right)^{\frac{1}{\overline{p}^*}},$$

where $A_{k,n} = \{x \in \Omega : |u_n(x)| > k\}$. Hence we use the previous estimate and we obtain

$$\beta \int_{\Omega} |G_k(u_n)| \leq \text{meas}(A_{k,n})^{1 - \frac{1}{\overline{p}^*}} C_5 \left(\int_{\{|u_n| > k\}} |f| \right)^{\frac{1}{p_N}}.$$

Now we apply Hölder inequality with exponents m and m', we have

$$\beta \int_{\Omega} |G_k(u_n)| \leq \text{meas}(A_{k,n})^{1 - \frac{1}{\overline{p}^*}} C_5 \|f\|^{\frac{1}{p_N}}_{L^m(\Omega)} \text{meas}(A_{k,n})^{\frac{1}{p_N} - \frac{1}{p_N m}} =$$

$$= C_5 \|f\|^{\frac{1}{p_N}}_{L^m(\Omega)} \text{meas}(A_{k,n})^{1 - \frac{1}{\overline{p}^*} + \frac{1}{p_N} - \frac{1}{p_N m}}.$$

We remark that

$$1 - \frac{1}{\overline{p}^*} + \frac{1}{p_N} - \frac{1}{p_N m} > 1 \iff m > \frac{\overline{p}^*}{\overline{p}^* - p_N}.$$

We put

$$g_n(k) = \int_\Omega |G_k(u_n)|, \quad \text{we have} \quad g_n'(k) = -\text{meas}(A_{k,n}),$$

as in many proofs of the results presented in Chapter 1. Therefore

$$g_n(k)^{\frac{\overline{p}^* p_N m}{\overline{p}^* p_N m - p_N m + \overline{p}^* m - \overline{p}^*}} \leq -C_6\, g_n'(k)$$

$$1 \leq \frac{-C_6}{1 - \frac{\overline{p}^* p_N m}{\overline{p}^* p_N m - p_N m + \overline{p}^* m - \overline{p}^*}} (g_n(k)^{1 - \frac{\overline{p}^* p_N m}{\overline{p}^* p_N m - p_N m + \overline{p}^* m - \overline{p}^*}})',$$

where

$$1 - \frac{\overline{p}^* p_N m}{\overline{p}^* p_N m - p_N m + \overline{p}^* m - \overline{p}^*} > 0$$

always due to the assumptions on m. If we integrate the previous expression from 0 to k, we obtain

$$k \leq \frac{-C_6}{1 - \frac{\overline{p}^* p_N m}{\overline{p}^* p_N m - p_N m + \overline{p}^* m - \overline{p}^*}} [g_n(k)^{1 - \frac{\overline{p}^* p_N m}{\overline{p}^* p_N m - p_N m + \overline{p}^* m - \overline{p}^*}} - \|u_n\|_{L^1(\Omega)}^{1 - \frac{\overline{p}^* p_N m}{\overline{p}^* p_N m - p_N m + \overline{p}^* m - \overline{p}^*}}].$$

By (2.1.13) estimate we can say that a constant $C_7 > 0$ exists, which does not depend on n, such that

$$\|u_n\|_{L^1(\Omega)} \leq C_7.$$

Hence we have

$$C_8 g_n(k)^{1 - \frac{\overline{p}^* p_N m}{\overline{p}^* p_N m - p_N m + \overline{p}^* m - \overline{p}^*}} \leq -k + C_9.$$

If $k > C_9$, the function $g_n(k)$ becomes negative; but it is impossible because $g_n(k)$ is a decreasing and positive function. So there exists k_0 such that $g_n(k_0) = 0$. Hence

$$\|u_n\|_{L^\infty(\Omega)} \leq C, \tag{2.1.14}$$

where C does not depend on n. ∎

If we want a solution for the problem (2.1.1) we have to pass to the limit in the approximating problems. By (2.1.13) we have, up to subsequence, that $u \in W_0^{1,(p_i)}(\Omega)$ exists, such that

$$\nabla u_n \rightharpoonup \nabla u \text{ in } L^{p_1}(\Omega), \tag{2.1.15}$$

$$u_n \to u \text{ in } L^{p_1}(\Omega) \tag{2.1.16}$$

and

$$u_n \to u \text{ a.e. in } \Omega. \tag{2.1.17}$$

By estimate (2.1.14), we also have that $g \in L^\infty(\Omega)$ exists such that $u_n \rightharpoonup^* g$ in $L^\infty(\Omega)$ and, since Ω is a finite measure set, $u_n \rightharpoonup g$ in $L^{p_1}(\Omega)$. Hence, by (2.1.16), $g = u$ and $u_n \rightharpoonup^* u$.

2.1.2. The strong convergence of the approximate solutions. Now we want to prove that the sequence of the approximate solutions strong convergence in $W_0^{1,(p_i)}(\Omega)$. We take as a test function in (2.1.9), $\varphi_n = (e^{\beta p_N |u_n - u|} - 1)\mathrm{sgn}(u_n - u)$, where β is a real parameter that we will choose later. We get

$$\beta p_N \sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i-2} \partial_i u_n \partial_i (u_n - u) e^{\beta p_N |u_n - u|} + \mu_0 \int_\Omega u_n \varphi_n =$$

$$= \sum_{i=1}^N \int_\Omega b_i^n(x, u_n, \nabla u_n)\varphi_n + \int_\Omega f_n \varphi_n.$$

We add and subtract the term

$$\beta p_N \sum_{i=1}^N \int_\Omega |\partial_i u|^{p_i-2} \partial_i u \partial_i (u_n - u) e^{\beta p_N |u_n - u|},$$

we obtain

$$\beta p_N \sum_{i=1}^N \int_\Omega [|\partial_i u_n|^{p_i-2} \partial_i u_n - |\partial_i u|^{p_i-2} \partial_i u] \partial_i (u_n - u) e^{\beta p_N |u_n - u|} \le$$

$$\le -\beta p_N \sum_{i=1}^N \int_\Omega |\partial_i u|^{p_i-2} \partial_i u \partial_i (u_n - u) e^{\beta p_N |u_n - u|} - \mu_0 \int_\Omega u_n \varphi_n +$$

$$+ \gamma \sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i} \varphi_n + \int_\Omega |f \varphi_n|.$$

We note, since $p_i > 1$ and $p_N \ge p_i$ for any $i = 1, ..., N$, that

$$\sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i} \varphi_n \le 2^{p_N - 1} \sum_{i=1}^N \int_\Omega |\partial_i (u_n - u)|^{p_i} \varphi_n + 2^{p_N - 1} \sum_{i=1}^N \int_\Omega |\partial_i u|^{p_i} \varphi_n.$$

Moreover, for all i, the following inequality holds, if $p_i \ge 2$ (it is possible to obtain the same result also for $1 < p_i < 2$ for all i),

$$[|\partial_i u_n|^{p_i-2} \partial_i u_n - |\partial_i u|^{p_i-2} \partial_i u] \partial_i (u_n - u) \ge C_0 \, |\partial_i (u_n - u)|^{p_i}.$$

We have

$$(C_0 \beta p_N - \gamma 2^{p_N - 1}) \sum_{i=1}^N \int_\Omega |\partial_i (u_n - u)|^{p_i} e^{\beta p_N |u_n - u|} \le$$

$$\leq -\beta p_N \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2} \partial_i u \partial_i (u_n - u) e^{\beta p_N |u_n - u|} +$$

$$-\mu_0 \int_\Omega u_n \varphi_n + \gamma 2^{p_N-1} \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i} \varphi_n + \int_\Omega |f \varphi_n|.$$

We fix β such that $C_1 = C_0 \beta p_N - \gamma 2^{p_N-1} > 0$ we obtain

$$C_1 \sum_{i=1}^{N} \int_\Omega |\partial_i (u_n - u)|^{p_i} e^{\beta p_N |u_n - u|} \leq -\beta p_N \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2} \partial_i u \partial_i (u_n - u) e^{\beta p_N |u_n - u|} +$$

$$-\mu_0 \int_\Omega u_n \varphi_n + \gamma 2^{p_N-1} \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i} \varphi_n + \int_\Omega |f \varphi_n|. \tag{2.1.18}$$

Now we claim that all the terms of the right hand side of the previous inequality go to 0. By (2.1.17), we get that $\varphi_n \to 0$ a.e. on Ω, and we also obtain $|\varphi_n| \leq C_2$ by the continuity of exponential function and the uniform boundedness in $L^\infty(\Omega)$ of u_n and u. Hence

$$\varphi_n \to 0 \text{ in } L^{\overline{p}^*}(\Omega).$$

Therefore

$$\int_\Omega |\partial_i u|^{p_i} \varphi_n \to 0,$$

$$\int_\Omega |f \varphi_n| \to 0,$$

and

$$-\mu_0 \int_\Omega u_n \varphi_n \to 0.$$

We also have that the first term in the right side hand of (2.1.18), goes to zero. So the sequence strong converges in $W_0^{1,(p_i)}(\Omega)$.

2.1.3. Passing to the limit in the approximate problems. Now we can pass to the limit in the approximate problems (2.1.9). We have

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i-2} \partial_i u_n \partial_i \varphi \to \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2} \partial_i u \partial_i \varphi,$$

$$\int_\Omega f_n \varphi \to \int_\Omega f \varphi,$$

since $f_n \to f$ in $L^m(\Omega)$ and $\varphi \in L^\infty(\Omega)$. Moreover

$$\mu_0 \int_\Omega u_n \varphi \to \mu_0 \int_\Omega u\varphi,$$

by (2.1.16) and

$$\int_\Omega b_n^i(x, u_n, \nabla u_n)\varphi \to \int_\Omega b^i(x, u, \nabla u)\varphi, \qquad \forall\, i = 1, ..., N,$$

since $\partial_i u_n \to \partial_i u$ in measure, by the strong convergence in $W_0^{1,(p_i)}(\Omega)$. So we also have

$$b_n^i(x, u_n, \nabla u_n) \to b^i(x, u, \nabla u), \qquad \forall\; i = 1, ..., N$$

in measure, since b^i is a Caratheodory function for all $i = 1, ..., N$. Moreover, thanks to the assumption (2.1.7) it also strongly converges in $L^1(\Omega)$, for all $i = 1, ..., N$ and hence

$$b_n^i(x, u_n, \nabla u_n)\varphi \to b^i(x, u, \nabla u)\varphi, \quad \text{in} \quad L^1(\Omega) \qquad \forall\; i = 1, ..., N.$$

We get that, for all $\varphi \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)$,

$$\sum_{i=1}^N \int_\Omega |\partial_i u|^{p_i - 2}\partial_i u \partial_i \varphi + \mu_0 \int_\Omega u\varphi = \sum_{i=1}^N \int_\Omega b_i(x, u, \nabla u)\varphi + \int_\Omega f\varphi$$

and so we have a weak solution for the problem (2.1.9). ■

Remark 2.4. We note that we have used the assumptions $\overline{p}^* > p_N$ in order to apply the interpolation inequality. Moreover we know by Theorem 1.4 that u_n, solutions of approximating problems (2.1.9), belong to $L^\infty(\Omega)$ also if $f \in L^m(\Omega)$, with $N/\overline{p} < m \leq \overline{p}^*/(\overline{p}^* - p_N)$. So we think that also for these values of m, we should have the same result. Moreover if Theorem 2.1 holds for $m > \frac{N}{\overline{p}}$ we do not need to assume $\overline{p}^* > p_N$.

2.2. Lower order terms with sign conditions

In this section we prove the existence of a solution for the following problem

$$(2.2.1) \qquad \begin{cases} -\sum_{i=1}^N \partial_i[|\partial_i u|^{p_i-2}\partial_i u] + \sum_{i=1}^N g_i(x, u, \nabla u) = f & \text{in } \Omega, \\ u \in W_0^{1,(p_i)}(\Omega), \qquad g_i(x, u, \nabla u) \in L^1(\Omega), & \forall\; i = 1, ..., N. \end{cases}$$

$$\left(\sum_{i=1}^N g_i(x, u, \nabla u) \in L^1(\Omega)\right)$$

where $g_i(x, u, \nabla u)$ are nonlinearities with natural growths respect to the gradient of u, for all $i = 1, ..., N$, which satisfy the sign condition $g_i(x, s, \xi)s \geq 0$. We also assume that

f belongs either to $L^{(\overline{p}^*)'}(\Omega)$, (or to the dual space of $W_0^{1,(p_i)}(\Omega)$), or to $L^1(\Omega)$. In the second case we also suppose that $|g_i(x,s,\xi)| \geq \gamma|\xi_i|^{p_i}$, for all i, and for $|s|$ sufficiently large. Let $g_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ a Carathéodory function such that, for almost every $x \in \Omega$ and for all $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$,

$$g_i(x,s,\xi)s \geq 0, \qquad \forall\ i = 1,...,N, \tag{2.2.2}$$

$$|g_i(x,s,\xi)| \leq b(|s|)|\xi_i|^{p_i}, \qquad \forall\ i = 1,...,N, \tag{2.2.3}$$

or more in general

$$\sum_{i=1}^{N} |g_i(x,s,\xi)| \leq b(|s|) \sum_{i=1}^{N} |\xi_i|^{p_i}, \tag{2.2.4}$$

where $b : \mathbb{R} \to \mathbb{R}$ is a continuous and nondecreasing function. Finally we assume one of the following two assumptions:

$$f \in \left[W_0^{1,(p_i)}(\Omega)\right]^*, \tag{2.2.5}$$

with $*$ we denote the dual space of $W_0^{1,(p_i)}(\Omega)$, or

$$\begin{cases} f \in L^1(\Omega), \\ \text{and there exists } \sigma > 0 \text{ and } \gamma > 0 \text{ such that} \\ |g_i(x,s,\xi)| \geq \gamma|\xi_i|^{p_i} \text{ when } |s| > \sigma,\ \ \forall\ i = 1,...,N. \end{cases} \tag{2.2.6}$$

We have the following result.

THEOREM 2.5. *Under the assumptions* (2.2.2), (2.2.3) *and either* (2.2.5) *or* (2.2.6), *there exists at least a weak solution for* (2.2.1), *that is* $u \in W_0^{1,(p_i)}(\Omega)$ *such that*

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i \varphi + \sum_{i=1}^{N} \int_\Omega g_i(x,u,\nabla u)\varphi = \int_\Omega f\varphi, \tag{2.2.7}$$

for all $\varphi \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)$.

To fix the ideas we can take like a model problem the following

$$\begin{cases} -\sum_{i=1}^N \partial_i[|\partial_i u|^{p_i-2}\partial_i u] + u\sum_{i=1}^N |\partial_i u|^{p_i} = f \quad \text{on } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.2.8}$$

We observe that the solution of (2.2.1) is a solution of finite energy ($u \in W_0^{1,(p_i)}(\Omega)$) even if $f \in L^1(\Omega)$. It seems to be strange since for $f \in L^1(\Omega)$ the solution u of

$$\begin{cases} -\sum_{i=1}^N \partial_i[|\partial_i u|^{p_i-2}\partial_i u] = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

is known to only belong to $W_0^{1,(s_i)}(\Omega)$ for all $1 < s_i < \frac{N(\bar{p}-1)}{\bar{p}(N-1)} p_i$ (see Theorem (1.14) i), Section 1.2). But this better regularity of u is due to the second part of assumption (2.2.6). In other words the sense of the result that we prove is that the term with natural growth, satisfying (2.2.6), brings an extra regularity to the solutions for the problem (2.2.1) with L^1-data even implying the existence of solutions in $W_0^{1,(p_i)}(\Omega)$. The role of (2.2.6) is to give an a priori estimate in the energy space $W_0^{1,(p_i)}(\Omega)$, which allows us to deal with the lower order terms with natural growth. Under the assumption (2.2.5) it is also true that $ug_i(x,u,\nabla u) \in L^1(\Omega)$ for all i, which in contrast is in general false (cf Remark 3 of [**18**]) if we only assume the first condition in (2.2.6). The proof of this theorem is divided in two parts, depending on (2.2.5) or (2.2.6). Each one consists in the following steps. Before we define approximating equations. Then we prove an a priori estimate in $W_0^{1,(p_i)}(\Omega)$ for the sequence $\{u_n\}$ of the weak solutions of these approximating equations. At the end we prove that the truncations $T_k(u_n)$ are relatively compact in the strong topology of $W_0^{1,(p_i)}(\Omega)$ (see [**27**]). The last result allows us to pass to the limit in the approximate equations and to obtain the existence result.

Proof. of Theorem 2.5 with the assumption (2.2.5). We consider the sequence of approximate equations

$$\begin{cases} -\sum_{i=1}^N \partial_i[|\partial_i u_n|^{p_i-2}\partial_i u_n] + \sum_{i=1}^N g_i^n(x,u_n,\nabla u_n) = f_n & \text{in } \Omega, \\ u_n \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega) \qquad g_i^n(x,u_n,\nabla u_n) \in L^1(\Omega) & \forall\ i=1,...,N. \end{cases} \tag{2.2.9}$$

where

$$g_i^n(x,s,\xi) = \frac{g_i(x,s,\xi)}{1+\frac{1}{n}|g_i(x,s,\xi)|} \qquad \forall\ i=1,...,N \tag{2.2.10}$$

and f_n is a sequence of L^∞-functions such that $f_n \to f$ in $[W_0^{1,(p_i)}(\Omega)]^*$. We remark that

$$g_i^n(x,s,\xi)s \geq 0, \quad |g_i^n(x,s,\xi)| \leq |g_i(x,s,\xi)| \text{ and } |g_i^n(x,s,\xi)| \leq n \qquad \forall\ i=1,...,N.$$

Since g_i^n is bounded for all i, for any fixed $n > 0$, (2.2.9) has at least one weak solution u_n by a simple modification of the result of J. Leray and J.L. Lions. Moreover by assumption of f_n and Theorem 1.4, $u_n \in L^\infty(\Omega)$. As in the previous Section we divide the proof in three parts.

2.2.1. A priori estimate with the assumption (2.2.5)**.** We take u_n as test function in the weak formulation of (2.2.9), we get

$$\|u_n\|_{W_0^{1,(p_i)}(\Omega)} \leq C_0 \tag{2.2.11}$$

$$\int_\Omega u_n g_i^n(x, u_n, \nabla u_n) \leq C_1 \qquad \forall\ i = 1, ..., N. \tag{2.2.12}$$

Then there exists $u \in W_0^{1,(p_i)}(\Omega)$ and a subsequence (still denoted by $\{u_n\}$) such that

$$u_n \to u \text{ weakly in } W_0^{1,(p_i)}(\Omega) \tag{2.2.13}$$

and

$$u_n \to u \quad \text{ a.e.} \tag{2.2.14}$$

2.2.2. Strong convergence of $T_k(u_n)$ with the assumption (2.2.5)**.** We already know that, for any fixed $k \in \mathbb{R}^+$, $T_k(u_n)$ weakly converges to $T_k(u)$ in $W_0^{1,(p_i)}(\Omega)$. We want to prove that this convergence is also strong. We choose in the weak formulation of (2.2.9) as a test function $\varphi_n = \psi(T_k(u_n) - T_k(u))$ where $\psi(s) = se^{\lambda s^2}$. It is simple to see that if $\lambda \geq (b(k)/2)^2$ the following numerical inequality holds for all $s \in \mathbb{R}$

$$\psi'(s) - b(k)|\psi(s)| \geq \frac{1}{2}. \tag{2.2.15}$$

Thanks to the previous step we already know that $\varphi_n \to 0$ weakly in $W_0^{1,(p_i)}(\Omega)$ and weakly* in $L^\infty(\Omega)$, we have

$$\sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i-2}\partial_i u_n \partial_i \varphi_n + \sum_{i=1}^N \int_\Omega g_i^n(x, u_n, \nabla u_n)\varphi_n \to 0. \tag{2.2.16}$$

Since $g_i^n(x, u_n, \nabla u_n)\varphi_n \geq 0$ on the set $\{x \in \Omega : |u_n(x)| \geq k\}$, we obtain by (2.2.16) that

$$\sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i-2}\partial_i u_n \partial_i \varphi_n + \sum_{i=1}^N \int_{\{|u_n| \leq k\}} g_i^n(x, u_n, \nabla u_n)\varphi_n \leq \omega_1(n), \tag{2.2.17}$$

where $\omega_1(n)$ is a sequence of real numbers which converges to zero when n goes to infinity. Also in the following we will denote with $\omega_i(n)$, $i = 1, 2, ...$ this type of sequences. For the first term in the left hand side of (2.2.17), we have, since $\partial_i \varphi_n = \psi'(T_k(u_n) - T_k(u))\partial_i(T_k(u_n) - T_k(u))$ and by easy calculation,

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i-2}\partial_i u_n \partial \varphi_n \leq \tag{2.2.18}$$

$$\leq \sum_{i=1}^{N} \int_\Omega [|\partial_i T_k(u_n)|^{p_i-2}\partial_i T_k(u_n) - |\partial_i T_k(u)|^{p_i-2}\partial_i T_k(u)]\partial_i(T_k(u_n) - T_k(u))\psi'(T_k(u_n) - T_k(u)) +$$

$$+\omega_2(n).$$

On the other hand

$$|\sum_{i=1}^{N} \int_{\{|u_n|\leq k\}} g_i^n(x, u_n, \nabla u_n)\varphi_n| \leq \tag{2.2.19}$$

$$\leq \sum_{i=1}^{N} \int_{\{|u_n|\leq k\}} b(|u_n|)|\partial_i u_n|^{p_i}|\varphi_n| \leq$$

$$\leq b(k)\sum_{i=1}^{N} \int_\Omega [|\partial_i T_k(u_n)|^{p_i-2}\partial_i T_k(u_n) - |\partial_i T_k(u)|^{p_i-2}\partial_i T_k(u)]\partial_i(T_k(u_n) - T_k(u))|\varphi_n| + \omega_3(n).$$

Putting together (2.2.17), (2.2.18) and (2.2.19), we obtain

$$\sum_{i=1}^{N} \int_\Omega [|\partial_i T_k(u_n)|^{p_i-2}\partial_i T_k(u_n) - |\partial_i T_k(u)|^{p_i-2}\partial_i T_k(u)]\partial_i(T_k(u_n) - T_k(u))[\psi' - b(k)|\varphi_n|] \leq \omega_3(n).$$

Recalling (2.2.15) and that if $p_i \geq 2$ holds true

$$[|\partial_i T_k(u_n)|^{p_i-2}\partial_i T_k(u_n) - |\partial_i T_k(u)|^{p_i-2}\partial_i T_k(u)]\partial_i(T_k(u_n) - T_k(u)) \geq C_4|\partial_i(T_k(u_n) - T_k(u))|^{p_i},$$

for $i = 1, ..., N$, we obtain

$$\sum_{i=1}^{N} \int_\Omega |\partial_i(T_k(u_n) - T_k(u))|^{p_i} \leq 2C_4\,\omega_3(n)$$

that implies

$$T_k(u_n) \to T_k(u) \qquad \text{strongly in } W_0^{1,(p_i)}(\Omega). \tag{2.2.20}$$

A slight modification is needed to prove the case $1 < p_i < 2$.

2.2.3. Passing to the limit with the assumption (2.2.5)**.** The strong convergence of $T_k(u_n)$ implies that for some subsequence, that we still denote by u_n,

$$\partial_i u_n \to \partial_i u \quad \text{a.e.} \qquad \forall \ i = 1, ..., N \tag{2.2.21}$$

and so

$$\nabla u_n \to \nabla u \quad \text{a.e.}. \tag{2.2.22}$$

It yields, since g_i is a Carathéodory function for any i,

$$g_i^n(x, u_n, \nabla u_n) \to g_i(x, u, \nabla u) \quad \text{a.e.}. \tag{2.2.23}$$

Now we prove that $g_i^n(x, u_n, \nabla u_n)$ is uniformly equiintegrable for $i = 1, ..., N$. For any measurable E of Ω and for any $m \in \mathbb{R}^+$, we have

$$\int_E |g_i^n(x, u_n, \nabla u_n)| = \int_{E \cap \{|u_n| \le m\}} |g_i^n(x, u_n, \nabla u_n)| + \int_{E \cap \{|u_n| > m\}} |g_i^n(x, u_n, \nabla u_n)| \le$$

$$\le \int_{E \cap \{|u_n| \le m\}} b(m) |\partial_i u_n|^{p_i} + \int_{E \cap \{|u_n| > m\}} |g_i^n(x, u_n, \nabla u_n)| =$$

$$= \int_{E \cap \{|u_n| \le m\}} b(m) |\partial_i T_m(u_n)|^{p_i} + \int_{E \cap \{|u_n| > m\}} |g_i^n(x, u_n, \nabla u_n)|, \tag{2.2.24}$$

for fixed m and for $i = 1, ..., N$. For the first term we recall that $\partial_i T_m(u_n)$ strongly converges to $\partial_i T_m(u)$ in $L^{p_i}(\Omega)$ for all i. For the second term in the right hand side of (2.2.31), we have

$$\int_{E \cap \{|u_n| > m\}} |g_i^n(x, u_n, \nabla u_n)| \le \int_{\{|u_n| > m\}} |g_i^n(x, u_n, \nabla u_n)| \le$$

$$\le \int_{\{|u_n| > m\}} \frac{1}{|u_n|} u_n \, g_i^n(x, u_n, \nabla u_n) \le \frac{1}{m} \int_{\{|u_n| > m\}} u_n \, g_i^n(x, u_n, \nabla u_n) \le \frac{C_2}{m},$$

thanks to (2.2.12). This complete the uniform equintegrability of g_i^n for any i. So thanks to (2.2.23) we get

$$g_i^n(x, u_n, \nabla u_n) \to g_i(x, u, \nabla u), \quad \text{strongly in } L^1(\Omega), \qquad \forall \ i = 1, ..., N.$$

By strong L^1-convergence of g_i and the fact that

$$|\partial_i u_n|^{p_i - 2} \partial_i u_n \to |\partial_i u|^{p_i - 2} \partial_i u \quad \text{weakly in } L^{p_i'}(\Omega), \qquad \forall \ i = 1, ..., N,$$

it is easy to pass to the limit in (2.2.9). ∎

Proof. of Theorem 2.5 with the assumptions (2.2.6). We consider the sequence of the approximating problems

$$\begin{cases} -\sum_{i=1}^N \partial_i[|\partial_i u_n|^{p_i-2}\partial_i u_n] + \sum_{i=1}^N g_i(x,u_n,\nabla u_n) = f_n & \text{in } \Omega, \\ u_n \in W_0^{1,(p_i)}(\Omega) \qquad g_i(x,u_n,\nabla u_n) \in L^1(\Omega) & \forall\ i=1,...,N. \end{cases} \tag{2.2.25}$$

where $\{f_n\}$ is a sequence of L^∞-functions such that $f_n \to f$ in $L^1(\Omega)$. The solutions of these problems there exist by the previous part of the proof, if we suppose (2.2.5). We proceed as before.

2.2.4. A priori estimate with the assumption (2.2.6)**.** In this case, the use in the weak formulation of (2.2.25) of the test function $T_k(u_n)$ yields for any $k > 0$

$$\sum_{i=1}^N \int_\Omega |\partial_i T_k(u_n)|^{p_i} \le C_2\, k \tag{2.2.26}$$

$$k\int_{\{|u_n|>k\}} |g_i^n(x,u_n,\nabla u_n)| \le \int_\Omega |f_n||T_k(u_n)| \le C_3\, k \qquad \forall\ i=1,...,N. \tag{2.2.27}$$

The last inequality combined with (2.2.26) and the second part of assumption (2.2.6) gives (2.2.11) again, that is

$$\|u_n\|_{W_0^{1,(p_i)}(\Omega)} \le C_0.$$

Then there exist $u \in W_0^{1,(p_i)}(\Omega)$ and a subsequence (still denoted by $\{u_n\}$) such that u_n weakly converges to u in $W^{1,(p_i)}(\Omega)$ and a.e.

To prove the strong convergence of $T_k(u_n)$ in $W_0^{1,(p_i)}(\Omega)$ we proceed as in subsection 2.2.2, so we can try to pass to the limit in (2.2.25).

2.2.5. Passing to the limit with the assumption (2.2.6)**.** Obviously, as before, the strong convergence of $T_k(u_n)$ implies that for some sequence

$$\partial_i u_n \to \partial_i u \quad \text{a.e.} \qquad \forall\ i=1,...,N \tag{2.2.28}$$

and so

$$\nabla u_n \to \nabla u \quad \text{a.e.,} \tag{2.2.29}$$

hence, it is also true

$$g_i(x,u_n,\nabla u_n) \to g_i(x,u,\nabla u) \quad \text{a.e..} \tag{2.2.30}$$

Now we prove that $g_i(x, u_n, \nabla u_n)$ is uniformly equiintegrable for $i = 1, ..., N$. For any measurable E of Ω and for any $m \in \mathbb{R}^+$. As before, we have

$$\int_E |g_i(x, u_n, \nabla u_n)| = \int_{E\cap\{|u_n|\le m\}} |g_i(x, u_n, \nabla u_n)| + \int_{E\cap\{|u_n|>m\}} |g_i(x, u_n, \nabla u_n)| \le$$

$$\le \int_{E\cap\{|u_n|\le m\}} b(m)|\partial_i u_n|^{p_i} + \int_{E\cap\{|u_n|>m\}} |g_i(x, u_n, \nabla u_n)| =$$

$$= \int_{E\cap\{|u_n|\le m\}} b(m)|\partial_i T_m(u_n)|^{p_i} + \int_{E\cap\{|u_n|>m\}} |g_i(x, u_n, \nabla u_n)|, \tag{2.2.31}$$

for fixed m and for $i = 1, ..., N$. The first term of the expression above is small uniformly in n and in E, recalling that $\partial_i T_m(u_n)$ strongly converges to $\partial_i T_m(u)$ in $L^{p_i}(\Omega)$ for all i. For the second term in this case we use as test function in the weak formulation of the problem (2.2.25) $T_m(G_{m-1}(u_n))$ we obtain

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i-2}\partial_i u_n \partial_i T_m(G_{m-1}(u_n)) + \sum_{i=1}^{N} \int_\Omega g_i(x, u_n, \nabla u_n) T_m(G_{m-1}(u_n)) =$$

$$= \int_\Omega f_n T_m(G_{m-1}(u_n)),$$

it implies

$$\sum_{i=1}^{N} \int_{\{|u_n|>m\}} |g_i(x, u_n, \nabla u_n)| \le \int_{\{|u_n|\le m-1\}} |f_n|$$

and hence

$$\limsup_{n\to+\infty} \int_{\{|u_n|>m\}} |g_i(x, u_n, \nabla u_n)| \le \int_{\{|u_n|\le m-1\}} |f|, \qquad \forall\ i = 1, ..., N.$$

So also the second term, which appear in the right hand side of (2.2.31), is small uniformly in n and in E when m is sufficiently large. Hence by (2.2.30) we obtain

$$g_i(x, u_n, \nabla u_n) \to g_i(x, u, \nabla u), \quad \text{strongly in } L^1(\Omega), \qquad \forall\ i = 1, ..., N.$$

So it is simple to pass to the limit in (2.2.25). This fact concludes the proof. ∎

Remark 2.6. We note that in both of the previous sections the anisotropic operator which appears in the problems (2.1.1) and (2.2.1) can be substituted by more general one, that is

$$A(u) = -\mathrm{div}(a(x, u, \nabla u)),$$

where $a(x,s,\xi) = (a_i(x,s,\xi))$ is a Carathéodory vector valued function on $\Omega \times \mathbb{R} \times \mathbb{R}^N$ such that, for some constant $\beta \geq \alpha > 0$

$$\sum_{i=1}^{N} a_i(x,s,\xi)\xi_i \geq \alpha \sum_{i=1}^{N} |\xi_i|^{p_i},$$

$$|a_i(x,s,\xi)| \leq \beta \left(\sum_{j=1}^{N} |\xi_j|^{p_j} \right)^{1-1/p_i}, \qquad \forall\ i = 1,..,N$$

and for a.e. $x \in \Omega$ and $\forall\ s \in \mathbb{R}$, ξ, $\eta \in \mathbb{R}^N$, $\xi \neq \eta$

$$\sum_{i=1}^{N} (a_i(x,s,\xi) - a_i(x,s,\eta))(\xi_i - \eta_i) > 0.$$

CHAPTER 3

Multiplicity and existence results for a semilinear problem

In this chapter we principally talk about some results contained in [**35**]. We study the questions of existence, nonexistence and multiplicity of positive solutions for the following class of anisotropic semilinear elliptic problems

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left[|\partial_i u|^{p_i-2}\, \partial_i u \right] = \lambda |u|^{q-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.0.1}$$

where

$$p_1 < q < p_N. \tag{3.0.2}$$

For due diligence, we deal with also the cases

$$1 < q < p_1 \quad \text{and} \quad p_N < q < \overline{p}^*, \quad \text{with} \quad \overline{p}^* = \frac{\overline{p}N}{N - \overline{p}}.$$

The previous cases have already been studied in several recent papers. We recall some of these [**4**], [**41**], [**42**], [**45**], [**64**], [**65**], [**66**] and [**67**].

First of all we give the definition of a weak solution of (3.0.1), it is a function belonging to $W_0^{1,(p_i)}(\Omega)$, such that

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2}\, \partial_i u \partial_i \phi = \lambda \int_\Omega |u|^{q-2} u\phi, \tag{3.0.3}$$

for any $\phi \in C_0^\infty(\Omega)$.

Remark 3.1. Note that any weak solution u of (3.0.1) is, actually, a strong solution in the sense of [**45**], mainly u belongs to $W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)$. It follows from Theorem 2 in [**45**] and from assumption (3.0.2).

All the results, in the following, are due to the variational structure of the problem. Indeed, if we define the functional

$$J_\lambda(v) = \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i v|^{p_i} - \frac{\lambda}{q} \int_\Omega |v^+|^q, \tag{3.0.4}$$

where $v^+ = \max\{v, 0\}$, then any critical point of J_λ is a weak non-negative solution of (3.0.1).

3.1. Known results

In this first section we recall about the known results regarding our problem. We report some results presented in [**45**], if $q > p_N$. The authors of this paper obtain several existence, nonexistence and regularity results. To be complete we give also these results. The following theorem is valid.

THEOREM 3.2. *Let $q < p_\infty$, defined in* (1.2.2)*. Then for any $\gamma > 0$ there exist $\lambda_\gamma > 0$ and $u_\gamma \in W_0^{1,(p_i)}(\Omega)$ such that $\|u_\gamma\|_{W_0^{1,(p_i)}(\Omega)} = \gamma$ and u_γ is a bounded weak solution of problem* (3.0.1) *when $\lambda = \lambda_\gamma$.*

REMARK 3.3. We underline, as already said in [**45**], that this theorem cannot be use to have existence of a solution to problem (3.0.1) for a given λ. This fact is due to the lack of homogeneity of the differential operator.

REMARK 3.4. Theorem 3.2 gives the existence of a continuum of pairs $(\lambda_\gamma, u_\gamma) \in (0, \infty) \times W_0^{1,(p_i)}(\Omega)$ which solves (3.0.1), seen as eigenvalue problem. Moreover it is not clear which exponent q yields a resonance situation, i.e. eigenvalue problem. In Problem 2 proposed in [**45**] the authors do a conjecture. They think that the resonance situation occurs as soon as $q \leq p_N$ (see also Section 8.1 [**45**]), but maybe there are some "spectral gaps", namely some $q \in (p_1, p_N)$ such that (3.0.1) admits a weak bounded solution for all $\lambda > 0$. This conjecture is partial confirmed by our following results (see Proposition 3.7, Theorem 3.9 and Theorem 3.13). Obviously if $p_i = p$ for all $i = 1, ..., N$, the resonance problem corresponds to $q = p$, see for example [**10**].

To achieve an existence result for fixed $\lambda > 0$, in [**45**] it has been proved Theorem 3.5 below, we report its proof.

In [**45**] an nonexistence result is also presented. The main tool to prove this result is a Pohožaev identity. But also the weakest formulation requires solutions of class $C^1(\overline{\Omega})$ in order to have well defined boundary terms and it seems a challenging problem to obtain such regularity for weak solution of (3.0.1), see for example [**49**]. To manage this difficulty they build a sequence of "doubly approximating " problems, then they prove a strong regularity result for the solution of the approximating problems (Theorem 5 in [**45**]). At the end they present their main nonexistence result (Theorem 6 in [**45**]). It states that, in at least one critical case (3.0.1), does not admit weak solutions, belonging to $W_0^{1,(p_i)}(\Omega) \cap L^{(q-1)p_1'}(\Omega)$, other than $u \equiv 0$. This result needs two different assumptions. First, the domain Ω must have a particular geometrical feature, which modifies the classic notion of starshapedness, according to the anisotropy of operator. Second, the exponents p_i's must be sufficiently concentrated, that is

$$p_i \geq 2 \qquad \forall \ i = 1, ..., N \qquad \text{and} \qquad p_N < \frac{N+2}{N}\, p_1. \tag{3.1.5}$$

If (3.1.5) holds we necessarily have $N \geq 3$ and $\overline{p}^* > p_N$, so $p_\infty = \overline{p}^*$. In [**45**] it is also supposed $q \geq \overline{p}^*$ and this assumption is in according with our results (see Theorems 3.9 and 3.13). For the case $p_N < q < \overline{p}^*$ we follow [**45**], (see also [**64**] and [**65**] for the more general case $p_i = p_i(x)$, for all $i = 1, ..., N$). We have the theorem below.

Theorem 3.5. *Let q such that*

$$p_N < q < \overline{p}^*, \tag{3.1.6}$$

then, for all $\lambda > 0$, the problem (3.0.1) *possesses a nontrivial positive weak solution.*

Proof. We consider the functional (3.0.4). In this case it is not possible to apply Weiestrass Theorem, which we will use in the next Theorem, because the functional is not coercive, but it is possible to apply a Mountain-Pass Theorem in order to obtain a critical level for J_λ and so a weak solution for problem (3.0.1). First of all we prove that the functional (3.0.4) satisfies the geometrical assumptions required by Mountain-Pass Theorem.

i) Obviously $J_\lambda(0) = 0$.

ii) There exists $\rho \in (0, 1)$ and $\alpha > 0$ such that $J_\lambda \geq \alpha > 0$, for any $v \in W_0^{1,(p_i)}(\Omega)$, with $\|v\|_{W_0^{1,(p_i)}(\Omega)} = \rho$. If we apply Hölder inequality, with exponents $\overline{p}^*/q$ and $(\overline{p}^*/q)'$ (we recall that $q < \overline{p}^*$), to the second term of our functional, we have

$$J_\lambda(v) \geq \sum_{i=1}^N \frac{1}{p_i} \int_\Omega |\partial_i v|^{p_i} - \frac{\lambda\, C_0}{q} \left(\int_\Omega |v|^{\overline{p}^*} \right)^{\frac{q}{\overline{p}^*}}.$$

Now we apply the anisotropic Sobolev inequality (1.1.6) to the last term of above expression, we get

$$J_\lambda(v) \geq \sum_{i=1}^{N} \frac{1}{p_i} \|\partial_i v\|^{p_i}_{L^{p_i}(\Omega)} - \frac{\lambda\, C_1}{q} \|v\|^q_{W_0^{1,(p_i)}(\Omega)}.$$

We recall the following result, there exists $C > 0$, that is not depend on ρ, such that

$$\sigma_i > 0 \qquad \forall\ i, \qquad \sum_{i=1}^{N} \sigma_i = \rho \in (0,1) \quad \Rightarrow \quad \sum_{i=1}^{N} \frac{\sigma_i^{p_i}}{p_i} \geq C\, \rho^{p_N}.$$

We take $\sigma_i = \|\partial_i v\|_{L^{p_i}(\Omega)}$, we have

$$\sum_{i=1}^{N} \sigma_i = \|v\|_{W_0^{1,(p_i)}(\Omega)} = \rho \in (0,1)$$

and so

$$J_\lambda(v) \geq C_2\, \|v\|^{p_N}_{W_0^{1,(p_i)}(\Omega)} - \frac{\lambda\, C_1}{q} \|v\|^q_{W_0^{1,(p_i)}(\Omega)} = \rho^{p_N} \left(C_2 - \frac{\lambda\, C_1}{q} \rho^{q-p_N} \right).$$

Since $p_N < q$, we can find α, $\rho > 0$ such that

$$J_\lambda(v) \geq \alpha \qquad \forall\ \|v\|_{W_0^{1,(p_i)}(\Omega)} = \rho < \left(\frac{C_2\, q}{\lambda C_1} \right)^{\frac{1}{q-p_N}}.$$

iii) There exists $v \in W_0^{1,(p_i)}(\Omega)$ and $\beta \geq \rho > 0$ with $\|v\|_{W_0^{1,(p_i)}(\Omega)} > \beta$ such that $J_\lambda(v) < 0$. We consider $v = tz$ for some $z \in W_0^{1,(p_i)}(\Omega) \setminus \{0\}$ and $t > 1$, we obtain

$$J_\lambda(tz) = \sum_{i=1}^{N} \frac{t^{p_1}}{p_i} \int_\Omega |\partial_i z|^{p_i} - \frac{\lambda\, t^q}{q} \int_\Omega |z|^q \leq \sum_{i=1}^{N} \frac{t^{p_N}}{p_i} \int_\Omega |\partial_i z|^{p_i} - \frac{\lambda\, t^q}{q} \int_\Omega |vz^q.$$

It is clear, by (3.1.6), that $\lim_{t\to+\infty} J_\lambda(tz) = -\infty$. Then for $t > 1$ large enough we can take $v = tz$ such that $\|tz\|_{W_0^{1,(p_i)}(\Omega)} > \beta$ and $J_\lambda(tz) < 0$.

Now we prove the compactness hypothesis of the Mountain-Pass Theorem. Let $\{v_n\}$ a Palais-Smale sequence, namely is such that

1) $J_\lambda(v_n) \to c$,
2) $J'_\lambda(v_n) \to 0$,

where $c = \inf_{\gamma\in\Gamma} \max_{t\in(0,1)} J_\lambda(\gamma(t))$, with $\Gamma = \{\gamma \in C^0([0,1]; W_0^{1,(p_i)}(\Omega)) : \gamma(0) = 0 \text{ and } \gamma(1) = tz\}$, where tz has been chosen in $iii)$. Moreover J'_λ is a Fréchet derivative of J_λ,

$$\langle J'_\lambda(v), \phi\rangle = \sum_{i=1}^N \int_\Omega |\partial_i v|^{p_i-2}\partial v \partial_i \phi - \lambda \int_\Omega |v|^{q-2} v\phi.$$

These two conditions are equivalents to the following ones.

1′) There exists a numerical sequence $\{a_n\}$ which converges to zero, such that $J_\lambda(v_n) = a_n + c$, i.e.

$$\sum_{i=1}^N \frac{1}{p_i} \int_\Omega |\partial_i v_n|^{p_i} - \frac{\lambda}{q} \int_\Omega |v_n|^q = a_n + c.$$

2′) There exists $\{y_n\} \subset [W_0^{1,(p_i)}(\Omega)]^* : y_n \to 0$ in $[W_0^{1,(p_i)}(\Omega)]^*$, such that

$$\sum_{i=1}^N \int_\Omega |\partial_i v_n|^{p_i-2}\partial v_n \partial_i \phi = \lambda \int_\Omega |v_n|^{q-2} v_n \phi - \langle y_n, \phi\rangle, \qquad \forall\ \phi \in W_0^{1,(p_i)}(\Omega).$$

Now we choose $\phi = v_n$ in 2′)

$$\sum_{i=1}^N \int_\Omega |\partial_i v_n|^{p_i} = \lambda \int_\Omega |v_n|^q - \langle y_n, v_n\rangle.$$

We divide by $\frac{1}{q}$

$$\frac{1}{q}\sum_{i=1}^N \int_\Omega |\partial_i v_n|^{p_i} = \frac{\lambda}{q} \int_\Omega |v_n|^q - \frac{1}{q}\langle y_n, v_n\rangle.$$

Then if we subtract this expression from 1′), we obtain

$$\sum_{i=1}^N \frac{1}{p_i} \int_\Omega |\partial_i v_n|^{p_i} - \frac{1}{q}\sum_{i=1}^N \int_\Omega |\partial_i v_n|^{p_i} - \frac{\lambda}{q} \int_\Omega |v_n|^q =$$

$$= a_n + c - \frac{\lambda}{q} \int_\Omega |v_n|^q + \frac{1}{q}\langle y_n, v_n\rangle$$

and so

$$\sum_{i=1}^N \left(\frac{1}{p_i} - \frac{1}{q}\right) \int_\Omega |\partial_i v_n|^{p_i} = a_n + c + \frac{1}{q}\langle y_n, v_n\rangle.$$

If we use the assumptions on y_n, a_n and q, we have there exists $M \in \mathbb{R}^+$ independent on n such that

$$\|v_n\|_{W_0^{1,(p_i)}(\Omega)} \leq M.$$

Therefore we have, up a subsequence, we still denote with v_n,

$$v_n \to v, \quad \text{weakly in} \quad W_0^{1,(p_i)}(\Omega).$$

By the anisotropic Sobolev embedding (1.1.4), we get

$$v_n \to v \quad \text{strong in} \quad L^r(\Omega) \qquad \forall\ r < \overline{p}^*.$$

Now we choose in $2')$, $\phi = v_n - v$ we obtain

$$\sum_{i=1}^N \int_\Omega |\partial_i v_n|^{p_i-2} \partial_i v_n \partial_i (v_n - v) = \lambda \int_\Omega |v_n|^{q-2} v_n (v_n - v) - \langle y_n, v_n - v \rangle.$$

We subtract from the terms of the above expression

$$\sum_{i=1}^N \int_\Omega |\partial_i v|^{p_i-2} \partial_i v \partial_i (v_n - v),$$

we have

$$\sum_{i=1}^N \int_\Omega (|\partial_i v_n|^{p_i-2} \partial_i v_n - |\partial_i v|^{p_i-2} \partial_i v) \partial_i (v_n - v) =$$

$$= - \sum_{i=1}^N \int_\Omega |\partial_i v|^{p_i-2} \partial_i v \partial_i (v_n - v) + \lambda \int_\Omega |v_n|^{q-2} v_n (v_n - v) - \langle y_n, v_n - v \rangle.$$

We use that

$$\int_\Omega (|\partial_i v_n|^{p_i-2} \partial_i v_n - |\partial_i v|^{p_i-2} \partial_i v) \partial_i (v_n - v) \geq C_3 \int_\Omega |\partial_i (v_n - v)|^{p_i} \qquad \forall\ i = 1, ..., N,$$

and $p_i \geq 2$ for all i, we obtain

$$C_3 \sum_{i=1}^N \int_\Omega |\partial_i (v_n - v)|^{p_i} \leq - \sum_{i=1}^N \int_\Omega |\partial_i v|^{p_i-2} \partial_i v \partial_i (v_n - v) + \lambda \int_\Omega |v_n|^{q-2} v_n (v_n - v) - \langle y_n, v_n - v \rangle.$$

The same result holds also if $1 < p_i < 2$ by a slight modification. Now

$$\langle y_n, v_n - v \rangle \to 0,$$

by the strong convergence of y_n in the dual space of $W_0^{1,(p_i)}(\Omega)$ and the weak convergence of v_n to v in $W_0^{1,(p_i)}(\Omega)$. Moreover

$$\sum_{i=1}^N \int_\Omega |\partial_i v|^{p_i-2} \partial_i v \partial_i (v_n - v) \to 0,$$

since $v_n \to v$ weakly in $W_0^{1,(p_i)}(\Omega)$. For the term

$$\int_\Omega |v_n|^{q-2} v_n(v_n - v),$$

we apply Hölder inequality with exponents $\frac{\overline{p}^*}{q-1}$ and $\left(\frac{\overline{p}^*}{q-1}\right)'$, we obtain

$$\int_\Omega |v_n|^{q-2} v_n(v_n - v) \leq \left(\int_\Omega |v_n|^{\overline{p}^*}\right)^{\frac{q-1}{\overline{p}^*}} \left(\int_\Omega |v_n - v|^{(\frac{\overline{p}^*}{q-1})'}\right)^{1-\frac{q-1}{\overline{p}^*}}.$$

We note that

$$\left(\frac{\overline{p}^*}{q-1}\right)' = \frac{\overline{p}^*}{\overline{p}^* - q + 1} < \overline{p}^*,$$

since $q < \overline{p}^*$, and that $\|v_n\|_{L^{\overline{p}^*}(\Omega)} \leq M$, we have

$$\int_\Omega |v_n|^{q-2} v_n(v_n - v) \to 0.$$

So

$$\|v_n - v\|_{W_0^{1,(p_i)}(\Omega)} \to 0.$$

Also the Palais-Smale condition is true. Hence, as consequence of Mountain-Pass Theorem, we deduce that J_λ admits a nontrivial critical point and so we obtain a weak nontrivial solution of problem (3.0.1). ■

Now we deal with the case

$$1 < q < p_1. \tag{3.1.7}$$

This case has already been studied in [**64**] and [**65**] (see also the references therein). The authors study the more general case $p_i = p_i(x)$, for all $i = 1, ..., N$. To prove the next results we follow these two papers. The following result holds.

THEOREM 3.6. *Let $q : 1 < q < p_1$, then there exists $\lambda^{**} > 0$ and $\lambda^* > 0$ such that, for any $\lambda > \lambda^{**}$ and $\lambda \in (0, \lambda^*)$, problem (3.0.1) possesses a nontrivial positive weak solution.*

PROOF. We prove that the functional (3.0.4) is coercive and weak lower semicontinuous. We use Hölder inequality with exponents $\overline{p}^*/q$ and $(\overline{p}^*/q)'$, and it is possible because (3.1.7) is true. We have

$$J_\lambda(v) \geq \sum_{i=1}^N \frac{1}{p_i} \int_\Omega |\partial_i v|^{p_i} - \frac{\lambda C_0}{q} \|v\|^q_{L^{\overline{p}^*}(\Omega)}.$$

Now we apply Sobolev type inequality (1.1.6), with $r = \overline{p}^*$, we get

$$J_\lambda(v) \geq \sum_{i=1}^{N} \frac{1}{p_i} \|\partial_i v\|_{L^{p_i}(\Omega)}^{p_i} - \frac{\lambda\, C_1}{q} \left(\sum_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)} \right)^q .$$

For a numerical inequality, i. e.

$$\left(\sum_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)} \right)^q \leq C_2 \sum_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)}^q ,$$

we obtain

$$J_\lambda(v) \geq \sum_{i=1}^{N} \frac{1}{p_i} \|\partial_i v\|_{L^{p_i}(\Omega)}^{p_i} - \frac{\lambda\, C_3}{q} \sum_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)}^q .$$

Since $p_N \geq p_i$, for all $i = 1, ..., N$, we have

$$J_\lambda(v) \geq \frac{1}{p_N} \sum_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)}^{p_i} - \frac{\lambda\, C_3}{q} \sum_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)}^q .$$

Now we note that $q < p_i$ for all i, because for hypothesis $q < p_1$, and

$$\|v\|_{W_0^{1,(p_i)}(\Omega)} \to +\infty \quad \Rightarrow \quad \|\partial_i v\|_{L^{p_i}(\Omega)}^{p_i} \to +\infty \ \text{ for some } i.$$

Hence we obtain the coercivity of J_λ, that is

$$J_\lambda(v) \to +\infty \quad \text{when} \quad \|v\|_{W_0^{1,(p_i)}(\Omega)} \to +\infty.$$

For the weak lower semicontinuity we consider a sequence $\{v_n\} \subset W_0^{1,(p_i)}(\Omega)$ such that

$$v_n \to v \quad \text{weakly in} \quad W_0^{1,(p_i)}(\Omega).$$

Since the embedding (1.1.4) is compact for any $r \in [1, \overline{p}^*)$, we also have

$$v_n \to v \quad \text{in} \quad L^r(\Omega) \qquad \forall \ r < \overline{p}^*,$$

it implies

$$\int_\Omega |v_n|^q \to \int_\Omega |v|^q,$$

since $q < p_1 < \overline{p}^*$. So

$$\liminf_{n\to+\infty} J_\lambda(v_n) \geq \liminf_{n\to+\infty} \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i v_n|^{p_i} - \frac{\lambda}{q} \lim_{n\to+\infty} \int_\Omega |v_n|^q \geq$$

$$\geq \sum_{i=1}^{N} \frac{1}{p_i} \liminf_{n\to+\infty} \int_\Omega |\partial_i v_n|^{p_i} - \frac{\lambda}{q} \int_\Omega |v|^q \geq$$

$$\geq \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i v|^{p_i} - \frac{\lambda}{q} \int_\Omega |v|^q = J_\lambda(v),$$

by the weak semicontinuity of the norm, that is

$$\liminf_{n\to+\infty} \|\partial_i v_n\|_{L^{p_i}(\Omega)} \geq \|\partial_i v\|_{L^{p_i}(\Omega)}, \qquad \forall\ i = 1, ..., N.$$

Now we can use a Weierstrass Theorem (see for example [**77**]) in order to find a global minimizer of J_λ, u_λ, that is a weak solution of problem (3.0.1). Now we show that u_λ is not trivial for λ large enough. Let $t > 1$, a fixed real number and $\Omega_1 \subset \Omega$, open, with $\text{meas}(\Omega_1) > 0$, we take a function $v_0 \in C_0^\infty(\Omega) \subset W_0^{1,(p_i)}(\Omega)$ such that $v_0 = t_0$ in $\overline{\Omega}_1$ and $0 \leq v_0 \leq t_0$ in $\Omega \setminus \Omega_1$. We have

$$J_\lambda(v_0) = \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i v_0|^{p_i} - \frac{\lambda}{q} \int_\Omega |v_0|^q.$$

Since $v_0 \in C_0^\infty(\Omega)$ we have

$$\|v_0\|_{W_0^{1,(p_i)}(\Omega)} \leq C_4,$$

so we get, by the definition of v_0,

$$J_\lambda(v_0) \leq \frac{C_5}{p_1} - \frac{\lambda}{q} \int_{\Omega_1} |v_0|^q \leq \frac{C_5}{p_1} - \frac{\lambda}{q}\, t_0^q\, \text{meas}(\Omega_1).$$

Hence we can choose

$$\lambda \geq \frac{C_7\, q}{p_1\, t_0^q\, \text{meas}(\Omega_1)} = \lambda^{**},$$

such that $J_\lambda(v_0) < 0$ for every $\lambda \geq \lambda^{**}$. Since u_λ is a global minimum, it follows that $J_\lambda(u_\lambda) < 0$ for any $\lambda \geq \lambda^{**}$ and thus u_λ is a nontrivial weak solution of problem (3.0.1). So we conclude the proof of the first part of Theorem 3.6. For the second part we strictly follow [**65**]. At the beginning we show that there exist $\rho \in (0, 1)$ and $\alpha \geq 0$ such that $J_\lambda(v) \geq \alpha > 0$ for any $v \in W_0^{1,(p_i)}(\Omega)$ with $\|v\|_{W_0^{1,(p_i)}(\Omega)} = \rho$. Since $q < p_1$, the imbedding (1.1.4) holds. We have, for (1.1.6) with $r = q$,

$$\|v\|_{L^q(\Omega)} \leq C_7 \|v\|_{W_0^{1,(p_i)}(\Omega)}, \qquad \forall\ v \in W_0^{1,(p_i)}(\Omega).$$

So we get

$$J_\lambda(v) \geq \sum_{i=1}^{N} \frac{1}{p_i} \|\partial_i v\|^{p_i}_{L^{p_i}(\Omega)} - \frac{\lambda\, C_7}{q} \|v\|^q_{W_0^{1,(p_i)}(\Omega)}.$$

Now, since $\|v\|_{W_0^{1,(p_i)}(\Omega)} = \rho$ we get

$$\|\partial_i v\|_{L^{p_i}(\Omega)} \leq \|v\|_{W_0^{1,(p_i)}(\Omega)} = \rho, \qquad \forall\ i = 1, ..., N$$

and since $\rho \in (0,1)$ and $p_N \geq p_i$ for any $i = 1, ..., N$, it holds

$$\|\partial_i v\|^{p_N}_{L^{p_i}(\Omega)} \leq \|\partial_i v\|^{p_i}_{L^{p_i}(\Omega)}, \qquad \forall\ i = 1, ..., N.$$

Hence

$$J_\lambda(v) \geq C_8 \|v\|^{p_N}_{W_0^{1,(p_i)}(\Omega)} - \frac{\lambda\, C_7}{q}\|v\|^q_{W_0^{1,(p_i)}(\Omega)} = \rho^q \left(C_8\, \rho^{p_N - q} - \frac{\lambda\, C_7}{q} \right).$$

Then for any

$$0 < \lambda < \lambda^* < \frac{q\, C_8\, \rho^{p_N - q}}{C_7} \tag{3.1.8}$$

and $v \in W_0^{1,(p_i)}(\Omega)$ with $\|v\|_{W_0^{1,(p_i)}(\Omega)} = \rho$, we obtain the claim. Now we prove that there exists $z \geq 0$, $z \neq 0$ and $J_\lambda(tz) < 0$ for $t > 0$ small enough. Let $z \in C_0^\infty(\Omega)$ such that $\mathrm{supp}(z) \supset \overline{\Omega}_2$, $z \equiv 1$ in $\overline{\Omega}_2$ and $0 \leq z \leq 1$ in Ω. Then for any $0 < t < 1$ we have

$$\begin{aligned} J_\lambda(tz) &= \sum_{i=1}^N \int_\Omega \frac{t^{p_i}}{p_i} |\partial_i z|^{p_i} - \frac{\lambda\, t^q}{q} \int_\Omega |z|^q \leq \\ &\leq \frac{t^{p_1}}{p_1} \sum_{i=1}^N \int_\Omega |\partial_i z|^{p_i} - \frac{\lambda\, t^q}{q} \int_\Omega |z|^q = \\ &= t^q \left(\frac{t^{p_1 - q}}{p_1} \sum_{i=1}^N \int_\Omega |\partial_i z|^{p_i} - \frac{\lambda}{q} \int_\Omega |z|^q \right). \end{aligned}$$

Therefore, since $q < p_1$, $J_\lambda(tz) < 0$ if

$$t < \min \left\{ 1, \left(\frac{\lambda\, p_1\, \int_\Omega |z|^q}{q\, \sum_{i=1}^N \int_\Omega |\partial_i z|^{p_i}} \right)^{\frac{1}{p_1 - q}} \right\}.$$

Let $\lambda^* > 0$, as in (3.1.8), and $\lambda \in (0, \lambda^*)$. By the previous considerations there exists a ball centered at the origin and of radius ρ in $W_0^{1,(p_i)}(\Omega)$, $B_\rho(0)$ such that $\inf_{\partial B_\rho(0)} J_\lambda > 0$. Moreover there exists $z \in W_0^{1,(p_i)}(\Omega)$ such that $J_\lambda(tz) < 0$ for all t small enough and for any $v \in B_\rho(0)$,

$$J_\lambda(v) \geq \frac{1}{p_N} \sum_{i=1}^N \|\partial_i v\|^{p_N}_{L^{p_i}(\Omega)} - \frac{\lambda\, C_9}{q} \sum_{i=1}^N \|\partial_i v\|^q_{L^{p_i}(\Omega)}.$$

It follows that

$$-\infty < \underline{c} = \inf_{B_\rho(0)} J_\lambda < 0.$$

Let $0 < \varepsilon < \inf_{\partial B_\rho(0)} J_\lambda - \inf_{B_\rho(0)} J_\lambda$. By applying Ekeland variational principle (see [**40**] and also [**32**]) to the functional $J_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$. We find $v_\varepsilon \in \overline{B_\rho(0)}$ such that

- $J_\lambda(v_\varepsilon) < \inf_{\overline{B_\rho(0)}} J_\lambda + \varepsilon$,
- $J_\lambda(v_\varepsilon) < J_\lambda(v) + \varepsilon \|v - v_\varepsilon\|_{W_0^{1,(p_i)}(\Omega)}$, $v \neq v_\varepsilon$.

Since

$$J_\lambda(v_\varepsilon) \leq \inf_{\overline{B_\rho(0)}} J_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} J_\lambda,$$

$v_\varepsilon \in B_\rho(0)$. Now we define $I_\lambda(v) = J_\lambda(v) + \varepsilon \|v - v_\varepsilon\|_{W_0^{1,(p_i)}(\Omega)}$. It is clear that v_ε is a minimum of I_λ and thus

$$\frac{I_\lambda(v_\varepsilon + tz) - I_\lambda(v_\varepsilon)}{t} \geq 0,$$

for small $t > 0$ and $z \in B_1(0)$. We obtain for J_λ

$$\frac{J_\lambda(v_\varepsilon + tz) - J_\lambda(v_\varepsilon)}{t} + \varepsilon \|z\|_{W_0^{1,(p_i)}(\Omega)} \geq 0.$$

Letting $t \to 0$ it follows that $\langle J'_\lambda(v_\varepsilon), z\rangle + \varepsilon \|z\|_{W_0^{1,(p_i)}(\Omega)} > 0$ which implies $\|J'_\lambda(v_\varepsilon)\| \leq \varepsilon$. We deduce that there exists $\{w_n\} \subset B_\rho(0)$ such that

$$J_\lambda(w_n) \to \underline{c} \qquad \text{and} \qquad J'_\lambda(w_n) \to 0. \tag{3.1.9}$$

Obviously $\{w_n\}$ is bounded in $W_0^{1,(p_i)}(\Omega)$. If we proceed as in Theorem 3.5, we obtain the strong convergence of $\{w_n\}$ to w in $W_0^{1,(p_i)}(\Omega)$. Thus for (3.1.9) we have $J_\lambda(w) = \overline{c} < 0$ and $J'_\lambda(w) = 0$, that is w is a nontrivial positive weak solution for the problem (3.0.1). It completes the proof of Theorem 3.6. ∎

3.2. A global minimum

Now we begin to present the results obtained in [**35**]. This section is devoted to prove the first results regarding the geometry of J_λ. In particular we show that the functional is coercive (we prove that any level set is bounded, see Theorem 3.9 below), and it implies that the functional possesses a global minimum. Since the problem is a variational one, any critical point of J_λ is a weak solution, as in (3.0.3). So we have to prove that the functional has a nontrivial geometry, in order to obtain the existence of nontrivial critical levels, and it depends on the size of λ. In fact we also prove that the functional does not possess any nontrivial critical points, for small values of the parameter (see Proposition 3.7 below), because in this case J_λ behaves like the norm of $W_0^{1,(p_i)}(\Omega)$. We have the following result.

PROPOSITION 3.7. *There exists $\overline{\lambda} > 0$ such that if $\lambda < \overline{\lambda}$,* (3.0.1) *does not possess any nontrivial weak solutions.*

Proof. By contradiction, we assume that there exists a nontrivial weak solution of (3.0.1), u. By multiplying the equation by u and by integrating on Ω, we obtain

$$\sum_{i=1}^{N}\int_\Omega |\partial_i u|^{p_i} = \sum_{i=1}^{N}\|\partial_i u\|^{p_i}_{L^{p_i}(\Omega)} = \lambda\|u\|^q_{L^q(\Omega)} = \lambda\int_\Omega |u|^q.$$

By applying the estimate (1.1.10) with $r = p_1$ and $r = p_N$, it holds

$$\left(\frac{2}{ap_1}\right)^{p_1}\|u\|^{p_1}_{L^{p_1}(\Omega)} + \left(\frac{2}{ap_N}\right)^{p_N}\|u\|^{p_N}_{L^{p_N}(\Omega)} + \sum_{i=2}^{N-1}\|\partial_i u\|^{p_i}_{L^{p_i}(\Omega)} \le \lambda\|u\|^q_{L^q(\Omega)}. \tag{3.2.10}$$

Since $p_1 < q < p_N$, by interpolation inequality, we have

$$\|u\|^q_{L^q(\Omega)} \le \|u\|^{\theta p_1}_{L^{p_1}(\Omega)}\|u\|^{(1-\theta)p_N}_{L^{p_N}(\Omega)},$$

where $\theta \in (0,1)$ is such that $1/q = \theta/p_1 + (1-\theta)/p_N$, and so, using Young inequality, we get

$$\|u\|^q_{L^q(\Omega)} \le \theta\|u\|^{p_1}_{L^{p_1}(\Omega)} + (1-\theta)\|u\|^{p_N}_{L^{p_N}(\Omega)}.$$

By the previous inequality in (3.2.10), we obtain

$$\left(\frac{2}{ap_1}\right)^{p_1}\|u\|^{p_1}_{L^{p_1}(\Omega)} + \left(\frac{2}{ap_N}\right)^{p_N}\|u\|^{p_N}_{L^{p_N}(\Omega)} + \sum_{i=2}^{N-1}\|\partial_i u\|^{p_i}_{L^{p_i}(\Omega)} \le \lambda[\theta\|u\|^{p_1}_{L^{p_1}(\Omega)} + (1-\theta)\|u\|^{p_N}_{L^{p_N}(\Omega)}]$$

and it implies

$$\sum_{i=2}^{N-1}\|\partial_i u\|^{p_i}_{L^{p_i}(\Omega)} \le \left[\lambda\theta - \left(\frac{2}{ap_1}\right)^{p_1}\right]\|u\|^{p_1}_{L^{p_1}(\Omega)} + \left[\lambda(1-\theta) - \left(\frac{2}{ap_N}\right)^{p_N}\right]\|u\|^{p_N}_{L^{p_N}(\Omega)}. \tag{3.2.11}$$

Now we choose λ such that

$$\lambda \le \min\left\{\frac{1}{\theta}\left(\frac{2}{ap_1}\right)^{p_1}, \frac{1}{1-\theta}\left(\frac{2}{ap_N}\right)^{p_N}\right\}; \tag{3.2.12}$$

so by (1.1.10), with $r = p_j$ for some $1 < j < N$ and (3.2.11), we have

$$0 \le \|u\|^{p_j}_{L^{p_j}(\Omega)} \le \left(\frac{ap_j}{2}\right)^{p_j}\|\partial_j u\|^{p_j}_{L^{p_j}(\Omega)} \le \left(\frac{ap_j}{2}\right)^{p_j}\sum_{i=2}^{N-1}\|\partial_i u\|^{p_i}_{L^{p_i}(\Omega)} \le$$

$$\le \left(\frac{ap_j}{2}\right)^{p_j}\left\{\left[\lambda\theta - \left(\frac{2}{ap_1}\right)^{p_1}\right]\|u\|^{p_1}_{L^{p_1}(\Omega)} + \left[\lambda(1-\theta) - \left(\frac{2}{ap_N}\right)^{p_N}\right]\|u\|^{p_N}_{L^{p_N}(\Omega)}\right\}.$$

Since the right hand of above expression is negative, we obtain a contradiction. Then $u \equiv 0$ is the unique solution of (3.0.1). ∎

REMARK 3.8. Since J_λ is a coercive functional (see the next result), if λ is sufficiently small, it follows that J_λ is a small perturbation of the norm in the Sobolev space $W_0^{1,(p_i)}(\Omega)$. We want to emphasize that, in this case, $u \equiv 0$ is the unique solution of the equation (nonnegative or not) and 0 is the global minimum of the functional. Note that a similar result is proved in Section 8 of [**45**] (see *failure of the mountain-pass geometry*).

Now we show the first existence result.

THEOREM 3.9. *If λ is sufficiently large,* (3.0.1) *possesses at least one non-negative (and nontrivial) weak solution.*

PROOF. We begin showing that the level sets $J_\lambda^b = \{v \in W_0^{1,(p_i)}(\Omega) : J_\lambda(v) \le b\}$ are bounded. For any $v \in J_\lambda^b$, it holds

$$\sum_{i=1}^{N} \frac{1}{p_i} \|\partial_i v\|_{L^{p_i}(\Omega)}^{p_i} - \frac{\lambda}{q} \|v\|_{L^q(\Omega)}^q \le b.$$

By applying Hölder inequality with exponents $p_N/q > 1$ and $(p_N/q)'$ to the left hand side of the previous expression, we obtain

$$\sum_{i=1}^{N} \frac{1}{p_i} \|\partial_i v\|_{L^{p_i}(\Omega)}^{p_i} - \frac{\lambda}{q} \|v\|_{L^{p_N}(\Omega)}^q \operatorname{meas}(\Omega)^{\frac{p_N - q}{p_N}} \le b.$$

Now we use (1.1.10) with $r = p_N$, we have

$$\sum_{i=1}^{N} \frac{1}{p_i} \|\partial_i v\|_{L^{p_i}(\Omega)}^{p_i} - \frac{\lambda}{q} \|\partial_N v\|_{L^{p_N}(\Omega)}^q \operatorname{meas}(\Omega)^{\frac{p_N - q}{p_N}} \left(\frac{a p_N}{2}\right)^q \le b,$$

from which it follows

$$\sum_{i=1}^{N-1} \frac{1}{p_i} \|\partial_i v\|_{L^{p_i}(\Omega)}^{p_i} + \|\partial_N v\|_{L^{p_N}(\Omega)}^q \left[\frac{1}{p_N} \|\partial_N v\|_{L^{p_N}(\Omega)}^{p_N - q} - \frac{\lambda}{q} \operatorname{meas}(\Omega)^{\frac{p_N - q}{p_N}} \left(\frac{a p_N}{2}\right)^q\right] \le b.$$

If

$$\|\partial_N v\|_{L^{p_N}(\Omega)} \le 2 \left(\frac{p_N \lambda}{q}\right)^{\frac{1}{p_N - q}} \operatorname{meas}(\Omega)^{\frac{1}{p_N}} \left(\frac{a p_N}{2}\right)^{\frac{q}{p_N - q}},$$

then $\|\partial_N v\|_{L^{p_N}(\Omega)}$ is bounded and so we also obtain from the previous inequality that

$$\frac{1}{p_i} \|\partial_i v\|_{L^{p_i}(\Omega)}^{p_i} \le b + C_0,$$

for all $i \ne N$, where $C_0 = C(p_N, \lambda, q, |\Omega|, a)$.

Instead if

$$\|\partial_N v\|_{L^{p_N}(\Omega)} \geq 2 \left(\frac{p_N \lambda}{q}\right)^{\frac{1}{p_N - q}} \operatorname{meas}(\Omega)^{1/p_N} \left(\frac{a p_N}{2}\right)^{\frac{q}{p_N - q}},$$

we have proved that

$$\frac{1}{p_i}\|\partial_i v\|^{p_i}_{L^{p_i}(\Omega)} \leq b, \quad \|\partial_N v\|^{q}_{L^{p_N}(\Omega)} \left[\frac{1}{p_N}\|\partial_N v\|^{p_N - q}_{L^{p_N}(\Omega)} - \frac{\lambda}{q}\operatorname{meas}(\Omega)^{\frac{p_N - q}{p_N}} \left(\frac{a p_N}{2}\right)^{q}\right] \leq b,$$

where i is different from N. Since the last inequality proves the existence of a bound K (depending on b) also for $\|\partial_N v\|_{L^{p_N}(\Omega)}$ (this because $p_N > q$), we have

$$\|v\|_{W_0^{1,(p_i)}(\Omega)} \leq \sum_{i=1}^{N-1} (b p_i)^{\frac{1}{p_i}} + K(b).$$

The boundedness of level sets assures that the functional J_λ is coercive. Moreover J_λ is weakly lower semicontinuous. In fact we consider a sequence $\{v_n\} \subset W_0^{1,(p_i)}(\Omega)$ such that

$$v_n \longrightarrow v \text{ weakly in } W_0^{1,(p_i)}(\Omega).$$

Since the embedding $W_0^{1,(p_i)}(\Omega) \hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, p_\infty)$, it follows

$$v_n \longrightarrow v \text{ in } L^r(\Omega), \qquad \forall\ r < p_\infty,$$

and, since $q < p_N$, it follows

$$\int_\Omega |v_n|^q \longrightarrow \int_\Omega |v|^q.$$

So we obtain

$$\liminf_{n\to+\infty} J_\lambda(v_n) = \liminf_{n\to+\infty} \sum_{i=1}^{N} \frac{1}{p_i}\int_\Omega |\partial_i v_n|^{p_i} - \frac{\lambda}{q}\lim_{n\to+\infty}\int_\Omega |v_n|^q \geq$$

$$\geq \sum_{i=1}^{N} \frac{1}{p_i}\liminf_{n\to+\infty}\int_\Omega |\partial_i v_n|^{p_i} - \frac{1}{q}\int_\Omega |v|^q \geq$$

$$\geq \sum_{i=1}^{N} \frac{1}{p_i}\int_\Omega |\partial_i v|^{p_i} - \frac{1}{q}\int_\Omega |v|^q = J_\lambda(v),$$

recalling that, for any $i = 1, ..., N$, the norms $\|\partial_i v\|_{L^{p_i}(\Omega)}$ are weakly lower semicontinuous. So, by a Weierstrass Theorem, we obtain the existence of a global minimum.

Finally let $w \in W_0^{1,(p_i)}(\Omega)$, $w \neq 0$ be fixed, then, if λ is sufficiently large $J_\lambda(w) < 0$. This assures that our minimum is nontrivial. ∎

Remark 3.10. Note that it is possible to show the above result using a slightly different proof. Indeed J_λ satisfies the global Palais-Smale condition (in the following (PS)-condition), by the compact Sobolev embeddings, and, since any minimizing sequence is bounded, the existence of a global minimum follows.

We recall the definition of (PS)-condition: any sequence $\{v_n\} \subset W_0^{1,(p_i)}(\Omega)$ such that $|J_\lambda(v_n)| \leq c$ and $\|J'_\lambda(v_n)\| \to 0$, has a strongly convergent subsequence.

3.3. A Mountain-Pass solution

In this section we want to prove that the geometry of J_λ is rather involved, provided λ is sufficiently large. We show that the trivial solution 0 is always a local minimum of the functional, so we can apply the Mountain-Pass Theorem (see [**8**], [**70**], [**77**]) between the two minima found, in order to prove the existence of another solution of (3.0.1).

Proposition 3.11. *Assume* (3.0.2) *and suppose that one of the following conditions holds*

$$\overline{p} < N, \qquad p_1 < q < \min\{\overline{p}^*, p_N\} \qquad \text{and} \qquad p_1 < \frac{\overline{p}}{N - \overline{p}}, \tag{3.3.1}$$

$$\overline{p} \geq N, \tag{3.3.2}$$

then 0 *is a local minimum of the functional* J_λ.

Proof. (3.3.1): $\overline{p} < N$ implies that the exponent $\overline{p}^*$ is well defined and, by assumption, we have that q belongs to $(p_1, \overline{p}^*)$, so there exists $\theta \in (0,1)$ such that

$$\frac{1}{q} = \frac{\theta}{\overline{p}^*} + \frac{1-\theta}{p_1}. \tag{3.3.3}$$

Hence we can apply interpolation inequality, getting

$$\|v\|^q_{L^q(\Omega)} \leq \|v\|^{\theta q}_{L^{\overline{p}^*}(\Omega)} \|v\|^{(1-\theta)q}_{L^{p_1}(\Omega)}.$$

By the assumption $q < \overline{p}^*$, we can use the anisotropic Sobolev inequality ((1.1.5) Section 1.1) and we obtain

$$\|v\|^q_{L^q(\Omega)} \leq C_0 \prod_{i=1}^N \|\partial_i v\|^{\theta q/N}_{L^{p_i}(\Omega)} \|v\|^{(1-\theta)q}_{L^{p_1}(\Omega)}.$$

By Poincarè type inequality (1.1.10), with $i = 1$ and $r = p_1$, we have

$$\|v\|^q_{L^q(\Omega)} \leq C_0 \prod_{i=1}^N \|\partial_i v\|^{\theta q/N}_{L^{p_i}(\Omega)} \|\partial_1 v\|^{(1-\theta)q}_{L^{p_1}(\Omega)}.$$

Since we want to prove a local property of J_λ, we restrict to a suitable neighborhood of 0 and, without loss of generality, we suppose $\|\partial_i v\|_{L^{p_i}(\Omega)} \leq 1$, for all $i = 1, ..., N$, so we obtain

$$\|v\|^q_{L^q(\Omega)} \leq C_1 \|\partial_1 v\|^{\theta q/N+(1-\theta)q}_{L^{p_1}(\Omega)}.$$

Hence we get

$$J_\lambda(v) \geq \sum_{i=1}^N \frac{1}{p_i}\|\partial_i v\|^{p_i}_{L^{p_i}(\Omega)} - C_2\lambda \|\partial_1 v\|^{\theta q/N+(1-\theta)q}_{L^{p_1}(\Omega)}.$$

By (3.3.3) we get $\theta = \overline{p}^*(q-p_1)/[q(\overline{p}^*-p_1)]$. So we want to show that

$$p_1 < \frac{\theta q}{N} + (1-\theta)q = \frac{\overline{p}^*(q-p_1) + Np_1(\overline{p}^*-q)}{N(\overline{p}^*-p_1)}$$

and it holds because the previous inequality is equivalent to

$$Np_1^2 - (\overline{p}^* + Nq)\, p_1 + \overline{p}^* q > 0,$$

which follows by the assumption $p_1 < \overline{p}^*/N = \overline{p}/\,(N-\overline{p})$. So we have obtained

$$J_\lambda(v) \geq \left(\frac{1}{p_1} - \mathrm{o}(1)\right)\|\partial_1 v\|^{p_1}_{L^{p_1}(\Omega)} + \sum_{i=2}^N \frac{1}{p_i}\|\partial_i v\|^{p_i}_{L^{p_i}(\Omega)} \geq c > 0, \tag{3.3.4}$$

for $\|v\|_{W_0^{1,(p_i)}(\Omega)}$ suitably small (we recall that o(1) stands for a quantity which tends to 0 as $\|\partial_1 v\|_{L^{p_1}(\Omega)}$ tends to 0), and it implies the claim.

(3.3.2): when $\overline{p} \geq N$ the anisotropic Sobolev inequality (1.1.5) holds for any $r \geq 1$, as we had already said in Section 1.1, so, arguing as above, we have

$$\|u\|_{L^q(\Omega)} \leq \|u\|^{1-\theta}_{L^{p_1}(\Omega)}\|u\|^{\theta}_{L^r(\Omega)} \leq C_3\|\partial_1 u\|^{1-\theta(N-1)/N}_{L^{p_1}(\Omega)},$$

where the exponent $\theta = r(q-p_1)/[q(r-p_1)]$ depends continuously on r. In particular $\theta \to (q-p_1)/q$, as r tends to $+\infty$, and it implies that, choosing r suitably large, the condition (3.3.4) is satisfied and the claim follows as in case i). ∎

REMARK 3.12. Note that the assumption $p_1 < \overline{p}/(N-\overline{p})$ (contained in (3.3.1)) is equivalent to

$$\overline{p} > N\frac{p_1}{1+p_1}, \tag{3.3.5}$$

and this inequality is always true in case (3.3.2) of the proposition above.

Now we can prove the following result.

THEOREM 3.13. *Assume* (3.0.2). *Moreover suppose that* λ *is sufficiently large and that one of the following conditions holds*

$$\overline{p} < N, \qquad p_1 < q < \min\{\overline{p}^*, p_N\} \qquad \textit{and} \qquad p_1 < \frac{\overline{p}}{N - \overline{p}}, \tag{3.3.6}$$

$$\overline{p} \geq N. \tag{3.3.7}$$

Then the problem (3.0.1) *possesses at least two nontrivial non-negative weak solution.*

PROOF. Theorem 3.9 proves the existence of a nontrivial global minimum for λ large; since 0 is a local minimum by Proposition 3.11, we have the geometry required by the Mountain-Pass Theorem. The (PS)-condition follows by the compactness of the Sobolev embeddings, so the claim follows (see also [**70**]). ∎

REMARK 3.14. We note that the assumption (3.3.1) (or (3.3.5)) is a little bit unusual. We do not know if it is only a technical condition, moreover we want to point out that, since $\overline{p} \geq p_1$, the condition (3.3.5) (equivalent to (3.3.1)) is satisfied if $p_1 > N - 1$. Hence this condition seems to be equivalent to require that p_i's are sufficiently large in relation to the dimension N.

REMARK 3.15. Now we want to study some simple situations, looking for the validity of Proposition 3.11. Clearly it implies the existence of two solution, at least for λ sufficiently large.

We start considering the case in which

$$p_1 = \cdots = p_J = p < s = p_{J+1} = \cdots = p_N.$$

With some easy calculations, we find that

$$\overline{p} = \frac{Nps}{Js + (N-J)p},$$

so the conditions in (3.3.1) (see Theorem 3.13) become

$$\frac{J}{p} + \frac{N-J}{s} > 1 > \frac{J-1}{p} + \frac{N-J}{s}, \quad \text{and} \quad \overline{p}^* = \frac{Nps}{Js + (N-J)p - ps} \geq s \quad \text{iff} \quad 1 \geq \frac{J}{p} - \frac{J}{s},$$

otherwise assumption (3.3.2) reads

$$1 \geq \frac{J}{p} + \frac{N-J}{s}.$$

In particular, we want to emphasize that if $N = 2$ it follows $J = 1$ and, recalling that $p < s$, the preceding inequalities are always satisfied and Theorem 3.13 holds assuming only (3.0.2).

Now consider the case

$$p_1 = \cdots = p_J = p < p_{J+1} = \cdots = p_{J+L} = r < p_{J+L+1} = \cdots = p_N = s,$$

avoiding the calculations, it is easy to see that

$$\overline{p} = \frac{Nprs}{Jsr + Lps + (N - J - L)pr}$$

and that (3.3.1) is equivalent to suppose that

$$\frac{J}{p} + \frac{L}{r} + \frac{N - J - L}{s} > 1 > \frac{J - 1}{p} + \frac{L}{r} + \frac{N - J - L}{s},$$

with

$$\overline{p}^* = \frac{Nprs}{Jsr + Lps + (N - J - L)pr - prs} \geq s \quad \text{iff} \quad 1 \geq J\left(\frac{1}{p} - \frac{1}{s}\right) + L\left(\frac{1}{r} - \frac{1}{s}\right)$$

and (3.3.2) now it is

$$1 \geq \frac{J}{p} + \frac{L}{r} + \frac{N - J - L}{s}.$$

If $N = 3$ and $J = L = 1$ we have two non-negative solutions when either

$$\frac{1}{p} > 1 - \frac{1}{r} - \frac{1}{s} > 0 \quad \text{and} \quad p < q < \min\left\{\overline{p}^*, s\right\},$$

or

$$\frac{1}{p} + \frac{1}{r} + \frac{1}{s} \leq 1.$$

Remark 3.16. We note that our results do not contradict the results in Section 8 of [**45**], in which the authors show an example of a functional J_λ (such that (3.0.2) holds) which does not satisfy the Mountain-Pass geometry. Indeed they assume that the parameter λ is sufficiently small and it agrees with Proposition 3.7.

3.4. Strong maximum principle

In this Section we want prove a strong maximum principle for our problems. We will use the techniques of [**80**]. The crucial result is the following weak Harnack inequality for weak supersolutions. A function $u \geq 0$, $u \in W_0^{1,(p_i)}(\Omega)$ is a positive weak supersolution for (3.0.1), if it satisfies the following inequality

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i u|^{p_i - 2} \partial_i u \partial_i \phi \geq 0, \qquad \forall\ \phi \in C_0^{\infty}(\Omega), \qquad \phi \geq 0. \tag{3.4.8}$$

The estimate is of local nature. For convenience we shall work in cubes. We will denote, in the following pages, with $K_{x_0}(\rho)$, the cube in $\mathbb{R}^N$ of side ρ and center x_0 whose sides are parallel to the coordinate axes. We also write $K_{x_0}(\rho) = K(\rho)$.

To prove the Harnack inequality we need to the next lemma.

Lemma 3.17. *For all nonnegative measurable functions $w \in [W_0^{1,(p_i)}(\Omega)]^*$ and for all $u \in W_0^{1,(p_i)}(\Omega)$, with $p_i \geq 2$ for any $i = 1, ..., N$, we have*

$$\int_{\Omega} |\nabla u|^{p_1} w \leq N \int_{\Omega} w + \sum_{i=1}^{N} \int_{\Omega} w |\partial_i u|^{p_i}. \tag{3.4.9}$$

Proof. We use some arguments in [**50**]. If $p_i = p_1$ for any i we have

$$\int_{\Omega} |\nabla u|^{p_1} w = \int_{\Omega} \left(\sum_{i=1}^{N} (\partial_i u)^2 \right)^{\frac{p_1}{2}} w \leq \int_{\Omega} \sum_{i=1}^{N} |\partial_i u|^{p_1} f \leq N \int_{\Omega} w + \sum_{i=1}^{N} \int_{\Omega} |\partial_i u|^{p_1} w,$$

by the convexity of the real function $t \to t^{\frac{p_1}{2}}$ ($\frac{p_1}{2} \geq 1$) and the positivity of w. If $p_i \neq p_1$ for some $i = 1, ..., N$ we apply Young inequality with exponents $p_i/p_1 > 1$ and $(p_i/p_1)'$. So

$$\int_{\Omega} |\nabla u|^{p_1} w \leq \int_{\Omega} \sum_{i=1}^{N} |\partial_i u|^{p_1} w \leq \int_{\Omega} \sum_{i=1}^{N} (|\partial_i u|^{p_i} + 1) w = N \int_{\Omega} w + \sum_{i=1}^{N} \int_{\Omega} |\partial_i u|^{p_i} w.$$

∎

Theorem 3.18. *Let u be a weak non-negative supersolution of* (3.0.1)*, such that $u < M$ in Ω, and assume that $p_1 \geq 2$. Then*

$$\rho^{\frac{-N}{\gamma}} \|u\|_{L^{\gamma}(K(2\rho))} \leq C \min_{K(\rho)} u, \tag{3.4.10}$$

for $\gamma < \frac{N(p_1-1)}{N-p_1}$, if $p_1 \leq N$; for any γ, if $p_1 > N$.

Proof. Without loss of generality, we can suppose that $u \geq \varepsilon > 0$, otherwise we can replace u by $u + \varepsilon$ and let $\varepsilon \to 0$ in the final result.

We take as a test function in (3.4.8)

$$\phi = \overline{\eta} u^{\beta}, \tag{3.4.11}$$

where $\beta < \beta_0 < 0$ and $\overline{\eta}$ is a function in $C_0^1(\Omega)$ defined as follows,

$$\overline{\eta} = \prod_{j=1}^{N} \eta_j^{p_j},$$

where $\eta_j = \eta(x_j)$, for $j = 1, \dots, N$, and η is a nonnegative real function, $0 \le \eta \le 1$, which will be chosen later. We also define $\overline{\eta}_i = \displaystyle\prod_{j=1,\, j\neq i}^{N} \eta_j^{p_j}$.

From (3.4.11) we get $\partial_i \phi = p_i \eta_i^{p_i-1}\eta_i' \overline{\eta}_i u^\beta + \overline{\eta}\beta u^{\beta-1}\partial_i u$. By using its in (3.4.8), we obtain

$$\sum_{i=1}^{N} p_i \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \eta_i^{p_i-1}\eta_i' \overline{\eta}_i u^\beta + \beta \sum_{i=1}^{N}\int_\Omega |\partial_i u|^{p_i}\overline{\eta} u^{\beta-1} \ge 0.$$

This inequality gives nontrivial information only if $\beta < \beta_0 < 0$, so we obtain

$$|\beta_0| \sum_{i=1}^{N}\int_\Omega |\partial_i u|^{p_i}\overline{\eta} u^{\beta-1} \le \sum_{i=1}^{N} p_i \int_\Omega |\partial_i u|^{p_i-1}\eta_i^{p_i-1}|\eta_i'|\overline{\eta}_i u^\beta.$$

Now we apply Young inequality to $|\partial_i u|^{p_i-1}\eta_i^{p_i-1}|\eta_i'| u^{\beta-\frac{\beta+p_i-1}{p_i}} u^{\frac{\beta+p_i-1}{p_i}}$ for all $i = 1, ..., N$. Hence we get

$$|\beta_0| \sum_{i=1}^{N}\int_\Omega |\partial_i u|^{p_i}\overline{\eta} u^{\beta-1} \le \sum_{i=1}^{N}\left(\frac{1}{\varepsilon}\right)^{p_i-1}\int_\Omega |\eta_i'|^{p_i} u^{\beta+p_i-1}\overline{\eta}_i + \sum_{i=1}^{N}\varepsilon(p_i-1)\int_\Omega |\partial_i u|^{p_i} u^{\beta-1}\overline{\eta}.$$

So

$$\sum_{i=1}^{N}[|\beta_0| - \varepsilon(p_i-1)]\int_\Omega |\partial_i u|^{p_i}\overline{\eta} u^{\beta-1} \le \left(\frac{1}{\varepsilon}\right)^{p_N-1}\sum_{i=1}^{N}\int_\Omega |\eta_i'|^{p_i} u^{\beta+p_i-1}\overline{\eta}_i.$$

We choose ε such that $(|\beta_0| - (p_i-1)\varepsilon) > 0$ for any $i = 1, \dots, N$, so we take

$$\varepsilon = \min\left\{1, \frac{|\beta_0|}{2(p_N-1)}\right\}.$$

We obtain

$$\sum_{i=1}^{N}\int_\Omega |\partial_i u|^{p_i}\overline{\eta} u^{\beta-1} \le \left(\frac{2(p_N-1)}{|\beta_0|}+1\right)^{p_N-1}\frac{2}{|\beta_0|}\sum_{i=1}^{N}\int_\Omega |\eta_i'|^{p_i} u^{\beta+p_i-1}\overline{\eta}_i,$$

and taking $w = \overline{\eta} u^{\beta-1}$ in Lemma 3.17 we get

$$\int_\Omega |\nabla u|^{p_1}\overline{\eta} u^{\beta-1} \le C_0\left(\int_\Omega \overline{\eta} u^{\beta-1} + \sum_{i=1}^{N}\int_\Omega |\eta_i'|^{p_i} u^{\beta+p_i-1}\overline{\eta}_i\right). \tag{3.4.12}$$

Now we define

$$v=\begin{cases} u^s & \text{where } p_1 s = p_1+\beta-1 \quad \beta\neq 1-p_1\\ \log u & \text{for } \beta = 1-p_1.\end{cases} \tag{3.4.13}$$

First we analyze the case $\beta\neq 1-p_1$, we obtain, since $p_1>1$ and $\overline{\eta}\leq 1$,

$$\int_\Omega |\overline{\eta}\nabla v|^{p_1} \leq \int_\Omega \overline{\eta}|\nabla v|^{p_1} = |s|^{p_1}\int_\Omega \overline{\eta} u^{(s-1)p_1}|\nabla u|^{p_1} = \tag{3.4.14}$$

$$= |s|^{p_1}\int_\Omega \overline{\eta} u^{\beta-1}|\nabla u|^{p_1} \leq$$

$$\leq |s|^{p_1} C_1\left(\int_\Omega \overline{\eta} u^{\beta-1} + \sum_{i=1}^N \int_\Omega |\eta_i'|^{p_i} u^{\beta+p_i-1}\overline{\eta}_i\right),$$

by (3.4.12). Now we use the classic Sobolev inequality

$$\left(\int_\Omega |\overline{\eta} v|^{\chi p_1}\right)^{\frac{1}{\chi}} \leq C_2\int_\Omega |\nabla(\overline{\eta v})|^{p_1}, \tag{3.4.15}$$

where $\chi = N/(N-p_1)$ if $p_1<N$, χ is arbitrarily large if $p_1=N$, and $\chi=\infty$ if $p_1>N$.

Then

$$\int_\Omega |\nabla(\overline{\eta v})|^{p_1} \leq \int_\Omega |\nabla\overline{\eta}|^{p_1} v^{p_1} + \int_\Omega \overline{\eta}^{p_1}|\nabla v|^{p_1} \leq \tag{3.4.16}$$

$$\leq \int_\Omega |\nabla\overline{\eta}|^{p_1} v^{p_1} + C_1|s|^{p_1}\left(\int_\Omega \overline{\eta} u^{\beta-1} + \sum_{i=1}^N \int_\Omega |\eta_i'|^{p_i} u^{\beta+p_i-1}\overline{\eta}_i\right) \leq$$

$$\leq C_3|s|^{p_1}\left(\int_\Omega |\nabla\overline{\eta}|^{p_1} v^{p_1} + \int_\Omega \overline{\eta} u^{\beta-1} + \sum_{i=1}^N \int_\Omega |\eta_i'|^{p_i} u^{\beta+p_i-1}\overline{\eta}_i\right).$$

Let now $1\leq h'<h''\leq 2$ and take $\overline{\eta}$ as a cutoff function for $K(h')$ ($\overline{\eta}\in C_0^\infty(K(h''))$), such that $0\leq \eta_i\leq 1$ for all i, $\overline{\eta}=1$ in $K(h')$, $\overline{\eta}=0$ outside $K(h'')$ and

$$|\eta_i'| \leq \frac{2}{h''-h'} \qquad \forall\ i=1,...,N \quad \Rightarrow \quad |\nabla\overline{\eta}|\leq \frac{2C_4}{h''-h'}.$$

Now we want to estimate any integral in (3.4.16) with $\int_{K(h'')} v^{p_1}$. By the choice of $\overline{\eta}$, for the first integral we obtain

$$\int_\Omega |\nabla\overline{\eta}|^{p_1} v^{p_1} \leq C_5\left(\frac{2}{h''-h'}\right)^{p_1}\int_{K(h'')} v^{p_1}. \tag{3.4.17}$$

By the boundedness of u, the second integral becomes

$$\int_\Omega \overline{\eta} u^{\beta-1} \leq \int_{K(h'')} u^{p_1(s-1)} \leq C_6 \int_{K(h'')} v^{p_1}. \tag{3.4.18}$$

Finally, for the third integral we have, always by the assumptions on u and $\overline{\eta}$,

$$\sum_{i=1}^N \int_\Omega u^{\beta-1+p_i} |\eta_i'|^{p_i} \overline{\eta}_i \leq \sum_{i=1}^N \int_{K(h'')} u^{p_1 s - p_1 + p_i} |\eta_i'|^{p_i} \overline{\eta}_i \leq \tag{3.4.19}$$

$$\leq \sum_{i=1}^N \left(\frac{2}{h''-h'}\right)^{p_i} \int_{K(h'')} u^{p_1 s} u^{p_i - p_1} \leq \left(\frac{2}{h''-h'}\right)^{p_N} \int_{K(h'')} v^{p_1}.$$

Putting together (3.4.17), (3.4.18), (3.4.19), (3.4.16) and (3.4.15) we arrive to

$$\left(\int_{K(h')} v^{\chi p_1}\right)^{\frac{1}{\chi}} \leq C_7 |s|^{p_1} (h''-h')^{-p_N} \int_{K(h'')} v^{p_1}.$$

By taking the p_1-th root of each side of the previous inequality, we obtain

$$\|v\|_{L^{\chi p_1}(K(h'))} \leq C_8 |s| (h''-h')^{-\frac{p_N}{p_1}} \|v\|_{L^{p_1}(K(h'))}. \tag{3.4.20}$$

The above inequality allows us to conclude the proof proceeding in the same way of Theorem 1.2 of [**80**]. For the case $\beta = 1-p_1$ we proceed in a similar way. ∎

The previous result easily implies the following statement.

Corollary 3.19. *Let u be a weak non-negative solution for* (3.0.1)*, and assume that $p_1 \geq 2$. Then either u is the trivial solution or u is strictly positive in Ω.*

3.5. On the set of positive solutions

Let us define $\Lambda > 0$ as the infimum of the λ such that the functional J_λ has at least a nontrivial critical point (or, equivalently, that (3.0.1) has at least a positive solution). Obviously (3.2.12) gives a lower bound on Λ. Now we want to show some global properties of the set of the positive solutions of (3.0.1). Throughout all this section we assume that $p_1 \geq 2$ in order to have positive solutions of (3.0.1) and no restriction on the range of the exponent q.

Proposition 3.20. *Let u_λ be the minimum solution of* (3.0.1)*. Then*

$$\|u_\lambda\|_{W_0^{1,(p_i)}(\Omega)} \to +\infty, \quad \textit{as } \lambda \to +\infty.$$

PROOF. It is easy to see that $\inf J_\lambda \to -\infty$ as λ increases to $+\infty$, then, using the weak formulation of (3.0.1), we have that

$$J_\lambda(u_\lambda) = \sum_{i=1}^{N} \left(\frac{1}{p_i} - \frac{1}{q}\right) \|\partial_i u_\lambda\|_{L^{p_i}(\Omega)}^{p_i} \to -\infty.$$

It implies that $\|\partial_i u_\lambda\|_{L^{p_i}(\Omega)} \to +\infty$, at least for some $p_i > q$, so the statement is proved. ∎

PROPOSITION 3.21. *Let u_λ be a positive solution of* (3.0.1). *Then*

$$\|u_\lambda\|_{L^\infty(\Omega)} \leq C(p_i, q, \lambda) a^\beta \mathrm{meas}(\Omega)^\alpha,$$

with

$$\alpha = \frac{m(\overline{p}^* - \overline{p})}{m\overline{p}(\overline{p}^* - 1) - m\overline{p}^*(q-1) - \overline{p}^*},$$

$$\beta = \frac{p_N}{p_N - q} \left(\frac{\gamma - 1}{\gamma}\right) \frac{(\overline{p} - 1)\gamma}{(\overline{p} - 1)\gamma - (q-1)}$$

and $m > N/\overline{p}$ is suitably chosen.

PROOF. First of all, we want to recall that all the solutions of (3.0.1) are bounded (see Remark 3.1), now we want to give an estimate on this bound depending on the measure of Ω. Using the weak formulation of the problem (3.0.1) (with u_λ as a test function), we have

$$\sum_{i=1}^{N} \|\partial_i u_\lambda\|_{L^{p_i}(\Omega)}^{p_i} = \lambda \|u_\lambda\|_{L^q(\Omega)}^q.$$

Now we use Hölder inequality with exponents p_N/q and $p_N/(p_N - q)$. By the assumption $q < p_N$ and Poincaré type inequality (1.1.10), we obtain

$$\|\partial_N u_\lambda\|_{L^{p_N}(\Omega)}^{p_N} \leq \sum_{i=1}^{N} \|\partial_i u_\lambda\|_{L^{p_i}(\Omega)}^{p_i} \leq \lambda \left(\frac{a p_N}{2}\right)^q \|\partial_N u_\lambda\|_{L^{p_N}(\Omega)}^q \mathrm{meas}(\Omega)^{\frac{p_N - q}{p_N}}.$$

Simplifying and using again (1.1.10), we get

$$\|u_\lambda\|_{L^{p_N}(\Omega)}^{p_N - q} \leq \lambda \left(\frac{a p_N}{2}\right)^{p_N} \mathrm{meas}(\Omega)^{\frac{p_N - q}{p_N}} \quad \Rightarrow \quad \|u_\lambda\|_{L^{p_N}(\Omega)} \leq \lambda^{\frac{1}{p_N - q}} \left(\frac{a p_N}{2}\right)^{\frac{p_N}{p_N - q}} \mathrm{meas}(\Omega)^{\frac{1}{p_N}}.$$

It is easy to prove that

$$\|u_\lambda\|_{L^s(\Omega)} \leq \lambda^{\frac{1}{p_N - q}} \left(\frac{a p_N}{2}\right)^{\frac{p_N}{p_N - q}} \mathrm{meas}(\Omega)^{\frac{1}{s}}, \qquad \forall\ s < p_N. \tag{3.5.21}$$

Now by the result in [**45**], that is

$$u_\lambda \in L^s(\Omega) \qquad \forall \ s < \infty,$$

and using, as in the previous chapter, the Stampacchia techniques (see [**75**]), we obtain

$$\|u_\lambda\|_{L^\infty(\Omega)} \leq \frac{\|u_\lambda\|_{L^1(\Omega)}^{1-\frac{1}{\gamma}} C_0 \|f\|_{L^m(\Omega)}^{\frac{1}{(\overline{p}-1)\gamma}}}{1-\frac{1}{\gamma}},$$

with

$$f = \lambda u_\lambda^{q-1} \quad \text{and} \quad \gamma = \frac{m\overline{p}(\overline{p}^*-1)-\overline{p}^*}{m\overline{p}^*(\overline{p}-1)},$$

for all $m > N/\overline{p}$. So we obtain

$$\|u_\lambda\|_{L^\infty(\Omega)} \leq \frac{C_1 \|u_\lambda\|_{L^1(\Omega)}^{1-\frac{1}{\gamma}} \lambda^{\frac{1}{(\overline{p}-1)\gamma}} \|u_\lambda\|_{L^{(q-1)m}(\Omega)}^{\frac{q-1}{(\overline{p}-1)\gamma}}}{1-\frac{1}{\gamma}}.$$

Now we apply the inequality (3.5.21) with $s = 1$, and we get

$$\|u_\lambda\|_{L^\infty(\Omega)} \leq \frac{\gamma \lambda^{\frac{1}{p_N-q}(1-\frac{1}{\gamma})+\frac{1}{(\overline{p}-1)\gamma}} (ap_N)^{\frac{p_N}{p_N-q}(1-\frac{1}{\gamma})} C_2 \operatorname{meas}(\Omega)^{1-\frac{1}{\gamma}} \|u_\lambda\|_{L^{(q-1)m}(\Omega)}^{\frac{q-1}{(\overline{p}-1)\gamma}}}{(\gamma-1) 2^{\frac{p_N}{p_N-q}(1-\frac{1}{\gamma})}} \leq$$

$$\leq C_3 \|u_\lambda\|_{L^\infty(\Omega)}^{\frac{q-1}{(\overline{p}-1)\gamma}} a^{\frac{p_N}{p_N-q}(1-\frac{1}{\gamma})} \operatorname{meas}(\Omega)^{1-\frac{1}{\gamma}+\frac{1}{m\gamma(\overline{p}-1)}};$$

where the constant C_3 does not depend on $\operatorname{meas}(\Omega)$ and $a = \operatorname{diam}(\Omega)$. So, since $u_\lambda \in L^\infty(\Omega)$, we have

$$\|u_\lambda\|_{L^\infty(\Omega)}^{1-\frac{q-1}{(\overline{p}-1)\gamma}} \leq C_4 \, a^{\frac{p_N}{p_N-q}(1-\frac{1}{\gamma})} \operatorname{meas}(\Omega)^{1-\frac{1}{\gamma}+\frac{1}{m\gamma(\overline{p}-1)}}. \tag{3.5.22}$$

In order to conclude the proof we note that

$$1 - \frac{q-1}{(\overline{p}-1)\gamma} > 0,$$

if and only if

$$m > \frac{\overline{p}^*}{[\overline{p}(\overline{p}^*-1)-(q-1)\overline{p}^*]}.$$

If

$$\frac{\overline{p}^*}{[\overline{p}(\overline{p}^*-1)-(q-1)\overline{p}^*]} \leq \frac{N}{\overline{p}},$$

the proof is finished. If

$$\frac{\overline{p}^*}{[\overline{p}(\overline{p}^*-1)-(q-1)\overline{p}^*]} > \frac{N}{\overline{p}},$$

we choose $m = \frac{\overline{p}^*}{[\overline{p}(\overline{p}^*-1)-(q-1)\overline{p}^*]}$ and it is possible because the inequality holds true for all $m > \frac{N}{\overline{p}}$. ■

REMARK 3.22. The above proposition implies that a positive solution of

$$\begin{cases} -\sum_{i=1}^N \partial_i \left[|\partial_i u|^{p_i-2}\, \partial_i u\right] = \lambda |u|^{q-2}u & \text{in } \Omega' \\ u = 0 & \text{on } \partial\Omega' \end{cases}$$

where Ω' is a non empty set strictly contained in Ω, can be a subsolution of (3.0.1) provided the measure of Ω' is sufficiently small. Indeed, since any positive solution of (3.0.1) is strictly positive (see Corollary 3.19), it is possible to choose Ω' in such a way that the corresponding solution $u_{\lambda,\Omega'}$ (extended to zero in $\Omega \setminus \Omega'$) is very small in L^∞-norm. Moreover Proposition 3.21 shows also that it is possible to build a constant supersolution. So Lemma 8 in [**9**] proves the existence of a continuum of positive solutions.

PROPOSITION 3.23. *Let w_λ be the Mountain-Pass solution of* (3.0.1)*. Then*

$$J_\lambda(w_\lambda) \to 0, \quad \text{as } \lambda \to +\infty.$$

PROOF. We define

$$J_\lambda^0 = \{v \in W_0^{1,(p_i)}(\Omega) : J_\lambda(v) \le 0\}.$$

We note that, if λ is sufficiently large, J_λ^0 is nonempty. In fact u_λ, the minimum solution, belongs to J_λ^0. We also have that $J_\lambda^0 \supseteq J_{\lambda'}^0$ if $\lambda > \lambda'$, so if $v \in J_{\lambda_0}^0$ then $v \in J_\lambda^0$ for all $\lambda \ge \lambda_0$. Now we fix $\lambda_0 \in \mathbb{R}$, sufficiently large, such that

$$u_{\lambda_0} \in J_{\lambda_0}^0.$$

For any $t \in [0,1]$, we have

$$J_\lambda(tu_{\lambda_0}) \le \frac{1}{p_1}\sum_{i=1}^N \|\partial_i u_{\lambda_0}\|_{L^{p_i}(\Omega)}^{p_i}\, t^{p_1} - \frac{\lambda}{q}\|u_{\lambda_0}\|_{L^q(\Omega)}^q\, t^q.$$

It is easy to prove that the real function

$$\phi_\lambda(t) = J_\lambda(tu_{\lambda_0})$$

has a maximum for

$$t_0 = \left[\frac{\sum_{i=1}^N \|\partial_i u_{\lambda_0}\|_{L^{p_i}(\Omega)}^{p_i}}{\lambda \|u_{\lambda_0}\|_{L^q(\Omega)}^q}\right]^{\frac{1}{q-p_1}}.$$

Since u_{λ_0} is a minimum of J_{λ_0}, using the weak formulation of (3.0.1), we get

$$\phi_\lambda(t_0) = \frac{(q-p_1)\lambda_0^{\frac{q}{q-p_1}} \|u_{\lambda_0}\|_q^{L^q(\Omega)}}{qp_1\lambda^{\frac{p_1}{q-p_1}}} > 0.$$

Let

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t))$$

with

$$\Gamma = \left\{ \gamma : [0,1] \to W_0^{1,(p_i)}(\Omega) : \gamma \text{ is continuous, } \gamma(0) = 0 \text{ and } \gamma(1) = u_{\lambda_0} \right\}.$$

So if we choose $\gamma(t) = tu_{\lambda_0}$, we obtain the claim, since

$$0 < c_\lambda \leq \phi(t_0) \to 0 \quad \text{as } \lambda \to +\infty.$$

■

The above results seem to show that the continuum of positive solutions of (3.0.1) has a $\subset$-shape. This because we have no solution for λ small (Proposition 3.7) and at least two positive solution for λ large enough (Theorems 3.9,3.13 and Corollary 3.19). Finally, this branch of solutions cannot bifurcate from the set of trivial solutions, since there is no correct “linearized” problem for (3.0.1).

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