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Optimization and optimal control in economics: the resource curse and oil extraction model cases

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Introduction

This PhD thesis contains some applications of optimization and optimal control theories in two different economic model cases, the first one is a model of *resource curse* phenomenon and the second concerns the production of oil from a large number of producers.

The term resource curse briefly refers to countries with an abundance of natural resources that tend to have less economic growth than countries with fewer resources. There are many studies that try to explain this phenomenon, and one thinks that it happens for several reasons, among other for example decline in the competitiveness of other economic sectors, volatility of revenues from the natural resource sector, particular forms of government, geographical position of nations, government mismanagement of resources, or weak, ineffectual, unstable or corrupt institutions.

We start our work from a model, which is halfway between economy and politics, developed by Robinson, Torvik and Verdier [49, 50] in 2006 and in a simplified version in 2008, in which there is an incumbent politician who wants to be reelected, and a competitor. There are two periods with an election in the middle. The incumbent has to choice an economic policy, namely how many resources to extract and how many workers to employ in the public sector, in order to maximize a profit criterion, and the peculiarity of this model is that the politician can influence his reelection probability by hiring workers in the public sector.

We first formalize rigorously the results achieved by the authors, concerning the variations of the optimal resource extraction rate and the rate of people having public jobs with respect to the prices, using optimization tools like the Karush-Kuhn-Tucker conditions and the generalized implicit function theorem.

Subsequently we introduce some improvements to the model and at the same time some generalizations. We consider the case when one constraint is saturated, that is the point representing the optimal strategy belongs to the boundary of the considered domain, and we study also this problem in presence of multiple resources.

Then we introduce a completely new discrete time optimal control version of the model to describe the same framework by considering several time steps before the election. In this case the incumbent has to decide which policy to choose at every time.

To study this model we define a value function for this problem. Then we prove a Dynamical Programming Principle and derive the Bellman equation for the value function, proving also some properties of the solution. We numerically solve the equation and we present some graphs that show the situation with different choices of parameters. Lastly a table summarizes the different results in presence or absence of the ability of the incumbent to influence his reelection.

The last part of the thesis is devoted to an application of optimal control and mean field games theories in a model case of oil production in presence of a lot of producers. Mean field games theory was developed starting from 2006 by Lasry and Lions as a set of tools to model games with infinitely many players which can be thought as a continuum of agents.

Typically the model is described by a two differential equations mutually coupled, a backward Hamilton-Jacobi-Bellman equation which involves the determination of an optimal strategy, and a forward Kolmogorov-Fokker-Planck equation which describes the evolution of the population.

The model under consideration was proposed by Guéant, Lasry and Lions in 2011. They consider a large number (a continuum) of oil producers which an initial amount of reserves distributed among them according to a density function. The reserve of a single agent evolves according to a stochastic differential equation and productive choices are made to optimize a profit criterion, the same for all agents.

Our goal is to rigorously derive the two differential equations that model the situation. We prove a Dynamic Programming Principle and derive the Hamilton-Jacobi-Bellman equation that the value function satisfies in the viscosity sense. For the second equation, satisfied by the density function of the reserves, we use a procedure to correctly derive it.

At the very end of the work we formulate two verification theorems, for classic and viscosity solutions, that illustrate under which assumptions a solution of the Hamilton-Jacobi-Bellman equation is also a solution of the underlying optimization problem. Nevertheless there are still open problems, especially about uniqueness of equilibria in this model.

The thesis is organized as follows.

Chapter 1 is devoted to the theoretical background regarding optimization problems and optimal control theory. We recall the Karush-Kuhn-Tucker conditions for constrained optimization and we state the general implicit function theorem. Then, starting from a model problem, we introduce the tools of optimal control theory, such as Dynamic Programming Principle, Hamilton-Jacobi-Bellman equation and the synthesis of optimal controls. We consider also the case of discrete time dynamics and we give an introduction on viscosity approximation and stochastic control.

In Chapter 2 we consider the nonevolutive model on resource curse phenomenon. Recalling the original results we consider some improvements and particular cases. We give also several graphics which support the achieved results and show counter-intuitive situations.

Chapter 3 is devoted to the discrete time optimal control version of a similar problem. We solve it using the techniques of optimal control theory stated in Chapter 1, proving some regularity results and studying a numerical method based on Dynamic Programming Principle. We also give error estimates.

Finally in Chapter 4 we consider the mean field game model of oil production by a continuum of producers. We rigorously derive partial differential equations that describe the model. We give two verification theorems and some possible improvements to the considered model.

Chapter 1

Preliminaries

1.1 Mathematical optimization

In an optimization problem, one tries to minimize or maximize a quantity associated with a decision process, such as elapsed time or cost, by exploiting available degrees of freedom under a set of restrictions or constraints. Optimization problems arise, for example, in almost all branches of industry or in financial problems like strategic planning.

There are several methods and algorithms to deal with optimization problems (for further information see for example [52, 42]). In this section we present some very famous conditions that will be applied later in the elaborate. Moreover we recall a well known theorem, which is also used in the following.

1.1.1 Karush-Kuhn-Tucker conditions

The Karush-Kuhn-Tucker conditions[1] [34, 37] are first order necessary conditions for a solution in nonlinear programming to be optimal, provided that some regularity conditions are satisfied. Allowing inequality constraints, the KKT approach to nonlinear programming generalizes the method of Lagrange multipliers, which allows only equality constraints.

Theorem 1.1 (Karush-Kuhn-Tucker conditions for inequality constraints only)**.** *Suppose that $x \in \mathbb{R}^N$ is a maximum (or minimum) point of $f(x)$ in the region*

$$E = \{x \in \mathbb{R}^N \,|\varphi_j(x) \le 0,\ j = 1, \dots, m\}$$

where $f, \varphi_1, \dots, \varphi_m \colon \mathbb{R}^N \to \mathbb{R}$ are continuously differentiable functions. If the constraints φ_j satisfy some regularity conditions in x then there exists $\lambda \in \mathbb{R}^m$ such that

[1] For historical information (in Italian) see `http://www.unibg.it/static_content/presentazioneateneo/lhkuhn.htm`.

(x, λ) is a solution of the system

$$\begin{cases} \nabla f(x) - \sum_{j=1}^{m} \lambda_j \nabla \varphi_j(x) = 0 \\ \varphi_j(x) \leq 0, \ j = 1, \ldots, m \\ \lambda_j \geq 0, \ j = 1, \ldots, m \\ \lambda_j \varphi_j(x) = 0, \ j = 1, \ldots, m \end{cases}$$

There are several regularity conditions, also called constraints qualifications, that the constraints must alternatively satisfy. The constraints are qualified if, for example, they are affine functions (and no other conditions are required) or, in the general case, the gradients of the active inequality constraints and the gradients of the equality constraints are linearly independent at x (for a detailed discussione see for instance [1]).

Remark 1.2. *The well known first order condition, that says that in an internal maximum (or minimum) point x it results $\nabla f(x) = 0$, are a particular case of KKT conditions when we consider a solution (x, λ) of the system such that $\varphi_j(x) < 0$, i.e. x is an internal point of E, and consequently $\lambda = 0$.*

We want to point out that if some of the functions are nondifferentiable, subdifferential versions of KKT conditions are available (see for example [51]).

1.1.2 General implicit function theorem

The well known implicit function theorem is one of the most important, and one of the oldest, paradigms in modern mathematics.

The form of the implicit theorem has evolved. The theorem first was formulated in terms of complex analysis, then was formulated for functions of two real variables, and the assumption corresponding to the Jacobian matrix being nonsingular was simply the one partial derivative is nonzero. Finally Dini generalized the real variable version to the context of vector valued functions of any number of variables, and the statement is the following.

Theorem 1.3 (General implicit function theorem)**.** *Suppose that we are given a set of equations*

$$f_i(x_1, \ldots, x_l, y_1, \ldots, y_n) = 0, \ i = 1, \ldots, n$$

in which all the functions f_i are continuously differentiable. Assume that $(p, q) = (p_1, \ldots, p_l, q_1, \ldots, q_n)$ is a point such that all the equations hold and at which we have

$$\det\left(\frac{\partial(f_1, \ldots, f_n)}{\partial(y_1, \ldots, y_n)}\right) = \det\begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_n} \end{pmatrix} \neq 0$$

Then there exists a neighborhood $U \subset \mathbb{R}^l$ of p and a continuously differentiable function $\phi\colon U \to \mathbb{R}^n$ such that $\phi(p) = q$ and

$$f_i(x, \phi(x)) = 0, \ i = 1, \ldots, n$$

holds for $x \in U$ *and we have*

$$\frac{\partial(\phi_1,\ldots,\phi_n)}{\partial(x_1,\ldots,x_l)}(x) = -\left(\frac{\partial(f_1,\ldots,f_n)}{\partial(y_1,\ldots,y_n)}(x,\phi(x))\right)^{-1}\frac{\partial(f_1,\ldots,f_n)}{\partial(x_1,\ldots,x_l)}(x,\phi(x)) \tag{1.1}$$

A proof of this result, along with further generalizations and examples, can be found for example in [36] and for the last part in [31].

Remark 1.4. *The case* $n = 1$ *is of course the classic implicit function theorem for one dependent variable and one equation.*

1.2 Optimal control theory

Optimal control deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved. A control problem includes a cost functional that is a function of state and control variables. An optimal control is a set of differential equations describing the paths of the control variables that minimize the cost functional.

In this chapter we will present a model problem to focus on the main concept of the optimal control theory. Most of the contents can be founded for example in [47, 4, 23].

1.2.1 A model problem: the infinite horizon discounted regulator

Consider for example a control system governed by the state equation

$$\begin{cases} y'(t) = f(y(t),\alpha(t)) & t > 0 \\ y(0) = x \end{cases} \tag{1.2}$$

Here the control α is any measurable function $\alpha\colon [0,+\infty) \to A$ where A is the control space, typically a closed and bounded subset of $\mathbb{R}^M$.

Assume that the dynamics $f\colon \mathbb{R}^N \times A \to \mathbb{R}^N$ is such that, for every choice of the control α and of the initial position $x \in \mathbb{R}^N$ the state equation (1.2) has a unique solution denoted by $y_x(t,\alpha)$.

The model also includes a running cost, associated with this controlled evolution, described by a given function $\ell\colon \mathbb{R}^N \times A \to \mathbb{R}$.

The cost functional to be minimized is, by definition,

$$J(x,\alpha) = \int_0^{+\infty} \ell(y_x(t,\alpha),\alpha(t))e^{-\lambda t}dt \tag{1.3}$$

where $\lambda > 0$ represents a fixed discount factor.

1.2.2 Dynamic Programming Principle

A very powerful approach to this optimal control problem is to use the dynamic programming, which was first introduced in 1954 by Bellman with his principle of optimality [5] (see also [6]) which says that

> *"an optimal policy has the property that whatever the initial state and initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decisions".*

and which decomposes the problem into a sequence of simpler minimization (or maximization) problems. The Bellman equation was first applied in engineering, then it became an important tool in economic theory (see for example [44] for a first and celebrated economic application of a Bellman equation, other examples can be found for instance in [7, 41]).

To apply the dynamic programming approach to this model problem we introduce one of the most important concept of the theory. We define the value function

$$v(x) = \inf_{\alpha \in \mathcal{A}} J(x, \alpha) \tag{1.4}$$

where $\mathcal{A}$ denotes the set of control functions.

The fundamental idea of dynamic programming is that the value function v satisfies a functional equations, often called the Dynamic Programming Principle, and a partial differential equation, the so called Hamilton-Jacobi-Bellman equation.

To illustrate the Dynamic Programming Principle, let us assume that an optimal control α_x^* exists for every starting point x so that, by definition of v,

$$v(x) = J(x, \alpha_x^*) = \int_0^{+\infty} \ell(y_x(t, \alpha_x^*), \alpha_x^*) e^{-\lambda t} dt$$

Observe now the obvious fact that, for every $T > 0$, we have

$$J(x, \alpha_x^*) = \int_0^{T} \ell(y_x(t, \alpha_x^*), \alpha_x^*) e^{-\lambda t} dt + \int_T^{+\infty} \ell(y_x(t, \alpha_x^*), \alpha_x^*) e^{-\lambda t} dt$$

Exploiting the semigroup property of the optimal trajectory $y_x(t, \alpha_x^*)$, that is[2]

$$y_x(t + s, \alpha_x^*) = y_{y_x(t, \alpha_x^*)}(s, \alpha_x^*)$$

and performing a suitable change of variable we get

$$v(x) = \int_0^{T} \ell(y_x(t, \alpha_x^*), \alpha_x^*) e^{-\lambda t} dt + v(y_x(T, \alpha_x^*)) e^{-\lambda T} \tag{1.5}$$

which holds for all $T > 0$ and $x \in \mathbb{R}^N$.

If we consider now the general case in which the optimal control is unknown, the previous identity is replaced by

$$v(x) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^{T} \ell(y_x(t, \alpha), \alpha) e^{-\lambda t} dt + v(y_x(T, \alpha)) e^{-\lambda T} \right\} \tag{1.6}$$

which is the statement of the Dynamic Programming Principle for this model problem.

Remark 1.5. *When ℓ and, consequently, v is bounded* (1.6) *characterizes the value function v in the sense that if u is a bounded function satisfying* (1.6) *for all $x \in \mathbb{R}^N$ and $T > 0$ then $u \equiv v$.*

[2] With the control on the right-hand side shifted in time by t.

1.2.3 Viscosity solutions

In this section we want to show the procedure to derive the Hamilton-Jacobi-Bellman equation associated with this control problem exploiting the Dynamic Programming Principle stated before.

The standard derivation of the equation requires, of course, some smoothness of v. To perform this calculation in the general case it is useful to introduce the notion of viscosity solution of a general Hamilton-Jacobi equation

$$F(x, u(x), Du(x)) = 0 \qquad x \in \mathbb{R} \tag{1.7}$$

where Ω is an open domain of $\mathbb{R}^N$ and the Hamiltonian $F\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is continuous. The term viscosity solutions first appear in a work of Crandall and Lions (see [18, 19] and, for further details, [17, 33]) in 1981, but the definition of solution had actually been given earlier by Evans [21] in 1980.

Definition 1.6. *A function $u \in C$ is a viscosity subsolution of* (1.7) *if, for any $\varphi \in C^1$, one has*

$$F(x_0, u(x_0), D\varphi(x_0)) \le 0 \tag{1.8}$$

at any local maximum point $x_0 \in \mathbb{R}$ of $u - \varphi$.

Similarly, $u \in C$ is a viscosity supersolution of (1.7) *if, for any $\varphi \in C^1$, one has*

$$F(x_0, u(x_0), D\varphi(x_0)) \ge 0$$

at any local minimum point $x_0 \in \mathbb{R}$ of $u - \varphi$.

Finally, u is a viscosity solution of (1.7) *if it is simultaneously a viscosity sub- and supersolution.*

Remark 1.7. *In the definition of subsolution we can always assume that x_0 is a local strict maximum point for $u - \varphi$. Moreover, since* (1.8) *depends only on the value of $D\varphi$ at x_0, it is not restrictive to assume that $u(x_0) = \varphi(x_0)$. Similar remarks apply of course to the definition of supersolution. Geometrically this means that the validity of the subsolution condition* (1.8) *for u is tested on smooth function touching from above the graph of u at x_0.*

To give an equivalent definition of viscosity solution let us introduce the concept of superdifferential and subdifferential of a function at a point x.

Definition 1.8. *The superdifferential of v at x is the set*

$$D^+v(x) = \left\{ p \in \mathbb{R}^N \mid \limsup_{y\to x} \frac{v(y) - v(x) - p(y-x)}{|y-x|} \le 0 \right\}$$

The subdifferential of v at x is the set

$$D^-v(x) = \left\{ p \in \mathbb{R}^N \mid \liminf_{y\to x} \frac{v(y) - v(x) - p(y-x)}{|y-x|} \ge 0 \right\}$$

Remark 1.9. *Observe that if both $D^+v(x)$ and $D^-v(x)$ are nonempty at some x then $D^+v(x) = D^-v(x) = \{Dv(x)\}$ and v is differentiable at x.*

Definition 1.10 (Alternative definition of viscosity solution)**.** *A function $u \in C$ is a viscosity subsolution of* (1.7) *if*

$$F(x, u(x), p) \leq 0$$

for all $x \in \mathbb{R}$ and $p \in D^+u(x)$.

Similarly, $u \in C$ is a viscosity supersolution of (1.7) *if*

$$F(x, u(x), p) \geq 0$$

for all $x \in \mathbb{R}$ and $p \in D^-u(x)$.

Finally, u is a viscosity solution of (1.7) *if it is simultaneously a viscosity sub- and supersolution.*

1.2.4 Hamilton-Jacobi-Bellman equation

In this section we want to show how to derive the Hamilton-Jacobi-Bellman equation

$$\lambda v(x) + \sup_{a \in A}\{-f(x,a)Dv(x) - \ell(x,a)\} = 0 \qquad x \in \mathbb{R}^N \tag{1.9}$$

satisfied in the viscosity sense by the value function v. For brevity we only show that v is a subsolution since the other part requires some technicalities.

Let $\varphi \in C$ and x be a local maximum point of $v - \varphi$ that is, for some $r > 0$,

$$v(x) - v(z) \geq \varphi(x) - \varphi(z) \qquad \forall z \in B_r(x)$$

Fix an arbitrary $a \in A$ and let $y_x(t, a)$ be the solution corresponding to the constant control $\alpha(t) \equiv a$. For t small enough one can prove that $y_x(t, a) \in B_r(x)$ and then

$$\varphi(x) - \varphi(y_x(t,a)) \leq v(x) - v(y_x(t,a)) \qquad \forall t \in [0, t_0]$$

Now by using the inequality $\leq$ in the Dynamic Programming Principle (1.6) we get

$$\varphi(x) - \varphi(y_x(t,a)) \leq \int_0^t \ell(y_x(s,a), a)e^{-\lambda s} ds + v(y_x(t,a))(e^{-\lambda t} - 1)$$

Dividing by $t > 0$ and letting t goes to 0 we obtain, under suitable assumptions,

$$-D\varphi(x)y_x'(0) = -D\varphi(x)f(x,a) \leq \ell(x,a) - \lambda v(x)$$

Since $a \in A$ is arbitrary we have proved that

$$\lambda v(x) + \sup_{a \in A}\{-f(x,a)D\varphi(x) - \ell(x,a)\} \leq 0$$

that is v is a viscosity subsolution of (1.9).

Moreover, we want to point out that the value function v is characterized by the Hamilton-Jacobi-Bellman equation (1.9) under general assumptions on the data. The next proposition, of which we omit the rather technical proof, contains an uniqueness result.

Proposition 1.11. *If u is any bounded continuous function in R^N satisfying* (1.9) *in the viscosity sense then, under suitable assumptions,*

$$u \equiv v$$

The proof of the preceding proposition show that the following comparison principle holds.

Proposition 1.12. *If u_1 and u_2 are, respectively, viscosity sub- and supersolution of* (1.9)*, then*

$$u_1 \leq u_2$$

1.2.5 Synthesis of optimal controls

This section is devoted to the recall of the classical synthesis procedure, that allows to design an optimal feedback map for the model problem exploiting again the Dynamic Programming Principle, when the value function v is smooth.

In the derivation of (1.6) we observed from identity (1.5) that the function

$$h(t) := v(y_x^*(t))e^{-\lambda t} + \int_0^t \ell(y_x^*(s), \alpha_x^*)e^{-\lambda s} ds$$

is constant for all $t > 0$ if and only if (α_x^*, y_x^*) is a pair of optimal control and trajectory for a fixed initial position x. Therefore if v is smooth the optimality condition is $h'(t) \equiv 0$, that is

$$e^{-\lambda t}[\lambda v(y_x^*(t)) - f(y_x^*(t), \alpha_x^*)Dv(y_x^*(t)) - \ell(y_x^*(t), \alpha_x^*)] \equiv 0$$

Since is this case v is a classical solution of (1.9) we conclude that the control α_x^* is optimal for the initial state x if and only if

$$\alpha_x^*(t) = S(y_x^*(t))$$

for any choice of $S(z)$ such that

$$S(z) \in \arg\max_{a \in A}\{-f(z,a)Dv(z) - \ell(z,a)\} \tag{1.10}$$

This characterization of optimal open loop control provides a method for construction a pair of optimal control and trajectory for every initial condition x.

The first step is to find a map $S\colon \mathbb{R}^N \to A$ with the property (1.10), but if v is known it is a static, finite dimensional, mathematical programming problem. Such a map S is called an optimal feedback map. The second step is solving (1.2) with $\alpha(t) = S(y_x^*(t))$ and a solution $y_x^*(t)$ generates a control $\alpha_x^*(t) := S(y_x^*(t))$ which is optimal for the initial state x.

1.2.6 Discrete time dynamic programming

In this section we show briefly how to apply the Dynamic Programming Principle when we deal with a discrete time optimal control problem.

In order to illustrate the approach assume that the evolution in given by the control system

$$\begin{cases} y_{n+1} = y_n + f(y_n, a_n) \\ y_0 = x \in \mathbb{R}^N \end{cases}$$

for $n \in \mathbb{N}$ and $a_n \in A$. The infinite horizon cost functional to be minimized is, in this case, the series

$$J(x, \alpha) = \sum_{n=0}^{+\infty} \ell(y_n, a_n)\beta^n$$

with $\beta \in (0,1)$. Consider now the value function

$$v(x) = \inf_{\alpha \in \mathcal{A}} J(x, \alpha)$$

and, under suitable assumptions, it can be proved that the value function satisfies the discrete time Dynamic Programming Principle

$$v(x) = \inf_{a \in A} \{\beta v(x + f(x,a)) + \ell(x,a)\} \qquad x \in \mathbb{R}^N \tag{1.11}$$

This procedure still allows the computing of the optimal control via the synthesis method stated before.

We want to conclude this section observing that the discrete equation (1.11) is deeply related with the continuous one (1.9). We can in fact lead to a discrete problem considering a discretization of the control system (1.2) assuming that it is observed only at a sequence of times $t_j = jh$ for $j \in \mathbb{N}$ and where h, the discretization step, is a fixed positive real number and that the dynamics f and the running cost ℓ remain constant between two subsequent observation. One can prove that the discrete value function converges, as $h \to 0^+$ and under suitable assumptions, to a function which is the (unique) value function of the continuous problem.

1.2.7 Numerical approximation

Let us recall that the value function v defined in (1.4) is the unique viscosity solution of the Hamilton-Jacobi-Bellman equation (1.9) that we restate for convenience

$$\lambda u(x) + \sup_{a \in A} \{-f(x,a) Du(x) - \ell(x,a)\} = 0 \qquad x \in \mathbb{R}^N$$

Starting from the continuous time problem, we can lead us to a discrete time problem, as described in the previous subsection, by making a discretization in time of the original control problem. It consists in replacing the dynamics (1.2) by a one-step scheme (for example by the Euler method) and the cost functional (1.3) by its discretization by a quadrature formula (for example the rectangle rule), one can get a new control problem in discrete time.

The value function v_h for this problem satisfies a more general (namely with a generic time step h) discrete Dynamic Programming Principle which gives the following approximation scheme

$$u_h(x) = \min_{a \in A} \{(1 - \lambda h) u_h(x + h f(x,a)) + h\ell(x,a)\} \qquad x \in \mathbb{R}^N$$

Remark 1.13. *If $\lambda < 1$, choosing $h = 1$ the previous equation becomes equation* (1.11) *with $\beta = 1 - \lambda$.*

In order to compute an approximate value function we have to make also a discretization in space. To do this we build a grid in the state space and, to simplify our presentation, we will assume that there exists a bounded polyhedron $\Omega \subset \mathbb{R}^N$ such that, for h sufficiently small, we have

$$x + h f(x,a) \in \overline{\Omega} \qquad \forall (x,a) \in \overline{\Omega} \times A$$

so that the discrete controlled trajectory never leaves the domain Ω. We construct a regular triangulation of Ω made by a family of simplices[3] S_j such that $\Omega = \bigcup_j S_j$, denoting by x_i, $i = 1, \ldots, L$, the nodes of the triangulation and by

$$k = \max_j \operatorname{diam}(S_j)$$

the size of the mesh. Finally, the discretized in space-time Hamilton-Jacobi-Bellman equation becomes

$$u_h(x_i) = \min_{a \in A} \{(1 - \lambda h) u_h(x_i + h f(x_i, a)) + h \ell(x_i, a)\} \qquad i = 1, \ldots, L$$

and we look for a solution in the space of piecewise linear functions on Ω

$$W^k := \{w \colon \Omega \to \mathbb{R} \text{ s.t. } w \in C(\Omega) \text{ and } Dw(x) \equiv c_j \ \ \forall x \in S_j, \forall j\} \tag{1.12}$$

Proposition 1.14. *Under suitable hypotheses there exists a unique solution v_h^k of* (1.12) *in W^k and the following estimate holds*

$$\|v - v_h^k\|_\infty \leq C_1 h^{1/2} + C_2 \frac{k}{h}$$

where $C_2 = C_2(\lambda, f, \ell)$ and v is the unique solution of (1.9).

1.2.8 Viscosity approximation and stochastic control

A different insight to viscosity solutions is provided by the so-called viscosity approximation of equation (1.9), namely

$$-\varepsilon \Delta u^\varepsilon + \lambda u^\varepsilon + \sup_{a \in A} \{-f(x, a) D u^\varepsilon - \ell(x, a)\} = 0 \qquad x \in \mathbb{R}^N \tag{1.13}$$

where ε is a positive parameter. This equation can be interpreted in terms of stochastic optimal control theory. Consider the stochastic differential equation

$$\begin{cases} dy(t) = f(y(t), \alpha(t)) dt + \sqrt{2\varepsilon} d\omega(t) \\ y(0) = x \end{cases} \tag{1.14}$$

where ω is a N-dimensional standard Brownian motion. Also in this case we can define the value function for this stochastic version of the infinite horizon problem where one tries to control in an optimal way the trajectories of (1.14), a random perturbation of system (1.2), which takes the form

$$v^\varepsilon(x) = \inf_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_0^{+\infty} \ell(y_x^\varepsilon(t), \alpha) e^{-\lambda t} dt \right]$$

Under suitable conditions of the data, v^ε happens to be a C^2 solution of (1.13) and this follows from the Dynamic Programming Principle and Itō's stochastic calculus (see [27, 45] for more details).

Moreover it can be proved, as was natural to expect, that when the randomness parameter ε goes to 0 the value function v^ε converges to the value function v, viscosity solution of (1.9). This limiting procedure to deal with the singular perturbation problem can be thought as a way to define weak (viscosity) solutions of (1.9) and this is actually the motivation for the terminology viscosity solutions in the original paper of Crandall and Lions [18].

[3] Or, for simplicity, we can think of building a lattice.

Chapter 2

Resource curse: a nonevolutive model

2.1 An overview on resource curse

Scholars and economic historians traditionally emphasized the great benefits that natural resources give to a country (see for example [56] on the British case). However it seems that in many cases an abundance of natural resources leads to a poor economic development. In this regard in 1993 Auty [3] coined the term *resource curse thesis* to characterize the phenomenon whereby some countries rich in natural resources had lower economic growth than nation without this huge availability[4].

A crucial point is to understand the linking between natural resource availability and their prices, form and quality of institutions and development of countries. Empirical literature on the resource curse emphasizes that resource dependent economies and resource booms seem to lead to highly dysfunctional state behavior, particularly large public sectors and unsustainable budgetary policies. Such a literature suggests that many different reasons, for example government mismanagement of resources or weak, ineffectual, unstable or corrupt institutions can lead to an incorrect exploitation of abundant natural resources.

In this context there is an ample literature about this topic, for example it was studied in [2] if the resource curse phenomenon is due to presidentialism or parliamentarism or in [43] if it is caused by grabber friendly or producer friendly institutions, or more in [16] it was analyzed the causal relationship between economic growth, human development and sustainability. Moreover (see for example [20] for a very brief introduction) it has been suggested that the poor development of countries has geographical origins also, such as location or climate, as well as political ones.

2.2 The considered framework

In 2006 Robinson, Torvik and Verdier [49] published the first explicitly political model of resource extraction and public employment. Within the framework of a nation with natural resources that are publicly owned such as oil, gas, diamond and other

[4]For a general framework about the resource curse phenomenon see [54].

minerals[5] and whose revenue from sales goes to the government, this model features an incumbent politician wishing to be reelected.

There are two periods with the election occurring at the end of the first one and an amount of natural resources than can be extracted and sold in the two periods at different prices. The income from natural resources accrues directly to the government and the incumbent must decide how much of the resources to extract in the first period and consequently how much will left for the future.

Finally, there are a population of voters/workers which gives to the incumbent a certain "probability" of reelection and an equilibrium policy to choose that maximizes a given income.

We consider an incumbent that distribute its rent as patronage to influence the outcome of the election. Patronage is to be understood in the definition given by Weingrod in [55] that is the way in which party politicians distribute public jobs or special favors in exchange for electoral support. It is widely believed that public employing is, politically speaking, a very profitable way to distribute rents (see for example [15], about the phenomenon of political recommendation in Palermo).

So we choose to model patronage as offering job in the public sector to take the favors of voters. An alterative form of patronage can be the investment in white elephants, as in [48], namely valuable but burdensome possessions or investment projects whose costs are out of proportion to its usefulness or worths[6].

In this framework the model under consideration concerns basically an optimization problem, that will be tackled with the mathematical tools described in Section 1.1.

2.3 Model description

Following and generalizing the simplified version [50] of the original work we deal with a model where there are two politicians, an incumbent politician wishing to be reelected and a competitor. The mass of voters is normalized to 1. There are two periods with an election occurring at the end of the first one in which the incumbent is challenged by the alternative politician.

There are also d stocks of different nonrenewable natural resources $E = (E_1, \dots, E_d)$ and all the income from selling E accrues directly to the government. The selling prices of the natural resources in the two periods are $p_1 = (p_1^1, \dots, p_1^d)$ in the first period and $p_2 = (p_2^1, \dots, p_2^d)$ in the second one and we assume they are determined on world market[7] and taken as given by the country under consideration.

The incumbent must decide how much of the resource to extract in the first period, denoted $e = (e_1, \dots, e_d)$, and consequently how much to left for the second one. We denote $R(e) = (R_1(e_1), \dots, R_d(e_d))$ the remaining resources available in the second period. We assume that every R_i for $i = 1, \dots, d$ is continuous and $R_i', R_i'' < 0$. These assumptions on the derivatives mean that every R_i is a strictly decreasing and concave function of e_i respectively.

[5] See [8] for a discussion on the effects of the interaction between the type of resources that a country has and the quality of its institutions on the economic development.

[6] An example of white elephant comes from the activities of the Industrial Development Corporation of Zambia, that was subject to a series of political and most importantly uneconomic directives on specific operational issues, including type and location of investments (see [53] for further details). Other examples can be founded in [35].

[7] We consider an open market, in which all economic actors have an equal opportunity of entry in that market.

This models the fact that, obviously, more resources are extracted less remain and that the total amount of resources that can be extracted depends in turn on the extraction rate in a way such that if too much is taken in the first period the total stock over the two periods falls down. Moreover, the sign of the second derivative implies that for every i there exists a value $\overline{e_i} < E_i$ such that $R_i(\overline{e_i}) = 0$.

We make the further technical assumption that $R_i'(0) \leq -1$, and therefore $R_i'(e_i) < -1$ for $e_i > 0$ since $R_i'' < 0$, to ensure that $R_i(e_i) \leq E_i - e_i$, that is $e_i + R_i(e_i) \leq E_i$ and the equality holds if and only if $e_i = 0$. In conclusion for every different resource i the incumbent can extract a quantity $e_i \in [0, \overline{e_i}]$ and we have $R_i(0) = E_i$ and $R_i(\overline{e_i}) = 0$.

To influence the outcome of the election the incumbent politician engages in clientelism and offers to employ voters in the public sector. We denote the function relating the number of workers employed to the reelection "probability" $\Pi = \Pi(G) \in (0, 1)$ where $G \in [0, 1]$ is the number of voters employed in the public sector in the first period[8].

We assume Π continuous, $\Pi' > 0$ and $\Pi'' < 0$. These assumptions imply that Π is strictly increasing and concave function and models a situation in which the reelection probability increases with respect to the number of voters employed but in a way such that if the number of workers employed is too high the reelection probability increases less if the incumbent hires other workers. We also assume $\Pi(0) = \frac{1}{2}$ so that if the incumbent does not employ any worker he has a fifty-fifty chance to be reelected.

We make the assumption that for a worker is better if he is offered a job in the public sector. On the other hand employing people in the public sector will be socially and economically inefficient because their productivity is lower than productivity of private sector workers.

Resource income can be spent by the incumbent politician or can be redistributed as patronage to increase reelection probability and to influence the outcome of voting. So the incumbent chooses its economic policy, namely $e \in [0, \overline{e_1}] \times \ldots \times [0, \overline{e_d}]$ and $G \in [0, 1]$, in order to maximize his own expected income[9]

$$I(e, G; p_1, p_2) := p_1 \cdot e - WG + \Pi(G)(p_2 \cdot R(e) - WG) \tag{2.1}$$

The first term $p_1 e - WG$ in the expression above is the difference between the income from the resource extraction and the outcome to employ workers while the second term $\Pi(G)(p_2 R(e) - WG)$ is the same for the second period yet discounted by a factor that is the reelection probability.

2.4 Case $d = 1$: a single natural resource

The case $d = 1$ models the simplest situation in which there is only one natural resource to extract, so we can omit the index associated with the variable e. The results presented in the first part of this section are the same of the original work, but here they are obtained in a rigorous way. In the second part there are some new results and considerations on several particular cases.

[8]We use the term "probability" only by analogy with probability density function of a continuous random variable since $\Pi \in (0, 1)$, but we do not make any assumption on its integral over the considered interval and in the rest of the work we will omit the quotes for ease of reading.

[9]In the rest of the work we will omit the scalar product symbol for simplicity of notation.

We recall that the goal of the incumbent is to maximize his expected income, in this case in the set

$$\Omega = \{(e, G) \in \mathbb{R}^2 \,|\varphi_j(e, G) \leq 0,\ j = 1, \dots, 4\}$$

where

$$\begin{aligned}
&\varphi_1(e, G) = -e && \nabla\varphi_1 = (-1, 0)\\
&\varphi_2(e, G) = e - \overline{e} && \nabla\varphi_2 = (1, 0)\\
&\varphi_3(e, G) = -G && \nabla\varphi_3 = (0, -1)\\
&\varphi_4(e, G) = G - 1 && \nabla\varphi_4 = (0, 1)
\end{aligned}$$

Theorem 1.1 says that if (e, G) is a maximum point for (2.1) in the set Ω and if the constraints are qualified in this point[10] then there exists $\lambda = (\lambda_1, \dots, \lambda_4) \in \mathbb{R}^4$ such that (e, G, λ) is a solution of

$$\begin{cases}
p_1 + \Pi(G)p_2R'(e) + \lambda_1 - \lambda_2 = 0\\
-(1 + \Pi(G))W + \Pi'(G)(p_2R(e) - WG) + \lambda_3 - \lambda_4 = 0\\
0 \leq e \leq \overline{e},\ 0 \leq G \leq 1 \qquad \text{(2.2a)}\\
\lambda_j \geq 0, \quad j = 1, \dots, 4\\
-\lambda_1 e = \lambda_2(e - \overline{e}) = -\lambda_3 G = \lambda_4(G - 1) = 0 \qquad \text{(2.2b)}
\end{cases}$$

To solve this system we can distinguish several cases by studying more deeply the last line and this is the goal of the next two subsections.

2.4.1 Maximum point in the interior

If we suppose that the maximum point is internal, this means that inequalities (2.2a) become $0 < e < \overline{e}$ and $0 < G < 1$, then equations (2.2b) become $\lambda_j = 0$ for all j. By virtue of Remark 1.2 the first two equations of system (2.2) are simply the two first order necessary conditions for this maximization problem

$$\begin{cases}
F^1(e, G; p_1, p_2) := I_e = p_1 + \Pi(G)p_2R'(e) = 0 & \text{(2.3a)}\\
F^2(e, G; p_1, p_2) := I_G = -(1 + \Pi(G))W + \Pi'(G)(p_2R(e) - WG) = 0 & \text{(2.3b)}
\end{cases}$$

For simplicity of notation we define

$$D_1 := F^1_eF^2_G - F^1_GF^2_e = p_2(-2W\Pi\Pi'R'' + \Pi\Pi''R''(p_2R - WG) - p_2(\Pi')^2(R')^2)$$

and we suppose that it is strictly positive[11].

Starting from the two first order conditions (2.3) we can prove immediately an important result whose proof is identical to that in [50].

Proposition 2.1. *Let e_0 be the socially optimal extraction rate in the first period, namely*

$$e_0 := \arg\max_{e \in [0, \overline{e}]}\{p_1e + p_2R(e)\}$$

Then $e > e_0$, that is the resources are inefficiently over-extracted.

[10] This is the case.

[11] This hypothesis is related to the second order sufficient condition for this maximization problem.

Proof. We recall that e is an internal point, and suppose that e_0 is internal too. We observe that e_0 is simply the value that maximizes the total income from selling the resource over the two periods and that satisfies

$$p_1 + p_2 R'(e_0) = 0$$

Now comparing (2.3a) with the last equality one has, since $\Pi < 1$ and $R' < 0$,

$$R'(e_0) = \Pi(G) R'(e) > R'(e)$$

which implies $e > e_0$ because R' is decreasing since $R'' < 0$.

□

The main tool to see how prices of the resource influence extraction and public sector employment, in other words how the maximum point (e, G) changes with respect to the parameters of the model p_1 and p_2, is the general implicit functions theorem applied to equations (2.3a) and (2.3b). We state the first result on this.

Proposition 2.2. *The resource extraction rate is an increasing function with respect to p_1, decreasing with respect to p_2 and decreasing with respect to both p_1 and p_2 also if they vary simultaneously but proportionally.*

Proof. We apply the result in Theorem 1.3 but in a constructive way. To do this we consider

$$\frac{\partial F^2}{\partial G} = -2\Pi' W + \Pi''(p_2 R - WG)$$

which is negative, and in particular nonzero, if we require in addition the quite natural hypothesis that $D_2 := p_2 R - WG > 0$ if $R \neq 0$. Equation (2.3b) implicitly defines a function $G = G(e; p_1, p_2)$. We substitute in (2.3a) and define

$$H(e; p_1, p_2) := F^1(e, G(e; p_1, p_2); p_1, p_2) = 0$$

We consider now

$$\frac{\partial H}{\partial e} = \frac{\partial F^1}{\partial e} + \frac{\partial F^1}{\partial G}\frac{\partial G}{\partial e} = F^1_e + F^1_G\left(-\frac{F^2_e}{F^2_G}\right) = \Pi p_2 R'' + \Pi' p_2 R' \left(-\frac{\Pi' p_2 R'}{-2\Pi' W + \Pi'' D_2}\right)$$

We suppose $H_e \neq 0$ and then $H = 0$ implicitly defines a function $e = e(p_1, p_2)$. Denoting $h(p_1, p_2) := G(e(p_1, p_2); p_1, p_2)$, the starting system is now

$$\begin{cases} F^1(e(p_1,p_2), h(p_1,p_2); p_1, p_2) = 0 \\ F^2(e(p_1,p_2), h(p_1,p_2); p_1, p_2) = 0 \end{cases}$$

Differentiating both equations by p_1 one has

$$\begin{cases} F^1_e \dfrac{\partial e}{\partial p_1} + F^1_G \dfrac{\partial h}{\partial p_1} = -F^1_{p_1} \\ F^2_e \dfrac{\partial e}{\partial p_1} + F^2_G \dfrac{\partial h}{\partial p_1} = -F^2_{p_1} \end{cases}$$

and by Cramer's rule we get

$$\frac{\partial e}{\partial p_1} = \frac{\begin{vmatrix} -F^1_{p_1} & F^1_G \\ -F^2_{p_1} & F^2_G \end{vmatrix}}{\begin{vmatrix} F^1_e & F^1_G \\ F^2_e & F^2_G \end{vmatrix}} = \frac{-F^1_{p_1} F^2_G}{D_1} = \frac{2\Pi' W - \Pi'' D_2}{D_1} > 0$$

Differentiating now both equations by p_2 one has

$$\begin{cases} F_e^1 \dfrac{\partial e}{\partial p_2} + F_G^1 \dfrac{\partial h}{\partial p_2} = -F_{p_2}^1 \\ F_e^2 \dfrac{\partial e}{\partial p_2} + F_G^2 \dfrac{\partial h}{\partial p_2} = -F_{p_2}^2 \end{cases}$$

and again by Cramer's rule we get

$$\frac{\partial e}{\partial p_2} = \frac{\begin{vmatrix} -F_{p_2}^1 & F_G^1 \\ -F_{p_2}^2 & F_G^2 \end{vmatrix}}{D_1} = \frac{2W\Pi\Pi' R' - \Pi\Pi'' R' D_2 + (\Pi')^2 R R' p_2}{D_1} < 0$$

To consider a simultaneous but proportional variation of p_1 and p_2 we compute the directional derivative of function $e(p_1, p_2)$ along the direction $\overrightarrow{u} = c(p_1, p_2)$ with c a normalizing constant[12]. After some calculation and using (2.3a) twice one has

$$\frac{de}{d\overrightarrow{u}} = \langle \nabla e, \overrightarrow{u} \rangle = c\left(\frac{\partial e}{\partial p_1} p_1 + \frac{\partial e}{\partial p_2} p_2\right) = c\frac{(\Pi')^2 R R'}{D_1} p_2^2 < 0 \tag{2.4}$$

and this concludes the proof.

□

This result leads us to draw some reflections. The extraction rate is an increasing function of p_1 because if, for example, price in the first period increases the resources become more valuable in the present than in the future, so the optimal response is to increase the extraction.

Vice versa the extraction rate is decreasing with respect to p_2. In fact if p_2 increases resources become more valuable in the future than now, so the optimal response is to decrease the extraction to leave more resources available for the second period.

In the last case the situation is a little bit different. If both p_1 and p_2, for example, increase the optimal response is not to leave the extraction rate unaltered (as in Propositions 2.5 and 2.6) but to decrease the extraction (and at the same time to increase the number of voters employed in the public sector, see Proposition 2.3) because this situation in any case makes more valuable to be in power in the future.

In order to present some graphical results which best show the correct behavior of optimal extraction and employment, even if the constraints are not observed due to the parameters chosen, we can choose for example the quadratic function $R(e) = -\frac{5}{16}e^2 - e + 1$ that satisfies all the hypothesis and models a situation in which $E = 1$ and $\overline{e} = \frac{4}{5}$, so the incumbent can extract only 80% of the total in the first period, in this case leaving nothing for the second one.

Similarly we can choose for example the function $\Pi(G) = -\frac{3}{10}G^2 + \frac{3}{4}G + \frac{1}{2}$ that satisfies all the hypothesis and at point $G = 1$ it is close to 1.

Now Figure 2.3 on the following page shows an example of function $e(p_1, p_2)$ in the region $1 \leq p_1 \leq 4$ and $6 \leq p_2 \leq 9$ when we choose $W = 1$ and functions R and Π as in Figures 2.1 and 2.2.

[12] In this case $c = \frac{1}{\sqrt{p_1^2 + p_2^2}} > 0$.

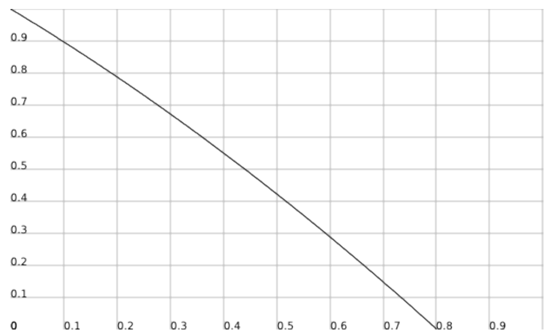

Figure 2.1: The function $R(e) = -\frac{5}{16}e^2 - e + 1$.

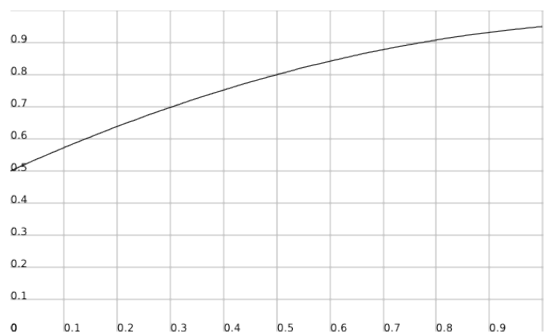

Figure 2.2: The function $\Pi(G) = -\frac{3}{10}G^2 + \frac{3}{4}G + \frac{1}{2}$.

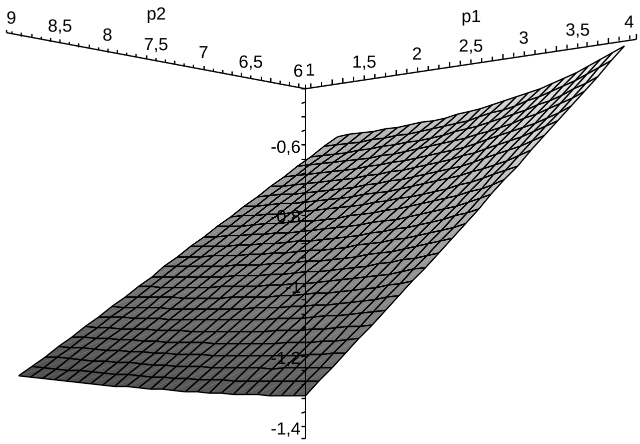

Figure 2.3: The function $e(p_1, p_2)$ when we choose $W = 1$ and functions $R(e)$ and $\Pi(G)$ as in Figures 2.1 and 2.2.

The second result concerns how the public sector employment is affected by a variation of prices of resource.

Proposition 2.3. *The rate of voters employed in the public sector is a decreasing function with respect to p_1, increasing with respect to p_2 and increasing with respect to both p_1 and p_2 also if they vary simultaneously but proportionally.*

Proof. We apply again the result in Theorem 1.3 in the same way. To do this we consider

$$\frac{\partial F^1}{\partial e} = \Pi p_2 R''$$

which is negative, and in particular nonzero. Equation (2.3a) implicitly defines a function $e = e(G; p_1, p_2)$. We substitute in (2.3b) and define

$$H(G; p_1, p_2) := F^2(e(G; p_1, p_2), G; p_1, p_2) = 0$$

We now consider

$$\frac{\partial H}{\partial G} = \frac{\partial F^2}{\partial e}\frac{\partial e}{\partial G} + \frac{\partial F^2}{\partial G}$$

We suppose $H_G \neq 0$ and then $H = 0$ implicitly defines a function $G = G(p_1, p_2)$. Denoting $h(p_1, p_2) := e(G(p_1, p_2); p_1, p_2)$, the starting system is now

$$\begin{cases} F^1(h(p_1, p_2), G(p_1, p_2); p_1, p_2) = 0 \\ F^2(h(p_1, p_2), G(p_1, p_2); p_1, p_2) = 0 \end{cases}$$

Differentiating both equations by p_1 one has

$$\begin{cases} F^1_e \dfrac{\partial h}{\partial p_1} + F^1_G \dfrac{\partial G}{\partial p_1} = -F^1_{p_1} \\ F^2_e \dfrac{\partial h}{\partial p_1} + F^2_G \dfrac{\partial G}{\partial p_1} = -F^2_{p_1} \end{cases}$$

and by Cramer's rule we get

$$\frac{\partial G}{\partial p_1} = \frac{\begin{vmatrix} F^1_e & -F^1_{p_1} \\ F^2_e & -F^2_{p_1} \end{vmatrix}}{D_1} = \frac{\Pi' p_2 R'}{D_1} < 0$$

Differentiating now both equations by p_2 one has

$$\begin{cases} F^1_e \dfrac{\partial h}{\partial p_2} + F^1_G \dfrac{\partial G}{\partial p_2} = -F^1_{p_2} \\ F^2_e \dfrac{\partial h}{\partial p_2} + F^2_G \dfrac{\partial G}{\partial p_2} = -F^2_{p_2} \end{cases}$$

and again by Cramer's rule we get

$$\frac{\partial G}{\partial p_2} = \frac{\begin{vmatrix} F^1_e & -F^1_{p_2} \\ F^2_e & -F^2_{p_2} \end{vmatrix}}{D_1} = \frac{-\Pi\Pi' p_2 R R'' + \Pi\Pi' p_2 (R')^2}{D_1} = \frac{\Pi\Pi' p_2((R')^2 - RR'')}{D_1} > 0$$

Considering a simultaneous but proportional variation of p_1 and p_2, after some calculation and using (2.3a) again we have

$$\frac{dG}{d\vec{u}} = \langle \nabla G, \vec{u} \rangle = c\left(\frac{\partial G}{\partial p_1}p_1 + \frac{\partial G}{\partial p_2}p_2\right) = c\frac{\Pi\Pi' R R''}{D_1}p_2^2 > 0 \tag{2.5}$$

and the proof is complete.

□

Regarding this case, the situation is exactly the opposite. The rate of voters employed in the public sector in a decreasing function of p_1. In fact if, for example, p_1 increases more resources are extracted in the first period. Since there are less resources remaining to exploit the incumbent has less incentive to be in power in the second period.

Vice versa the rate of voters is increasing with respect to p_2 because if, for example, the price in the second period increases then it is more valuable to be in power in this period so the incumbent is forced to increase the number of voters employed to increase his reelection probability.

Even in the latter case the optimal response is, if for example both p_1 and p_2 increase, to increase the number of voters employed in the public sector to increase the reelection probability because it is more valuable (as in Proposition 2.2) to be in power in the period after the elections.

In Figure 2.4 there is an example of function $G(p_1, p_2)$ in the same region and under the same choices made for the function $e(p_1, p_2)$.

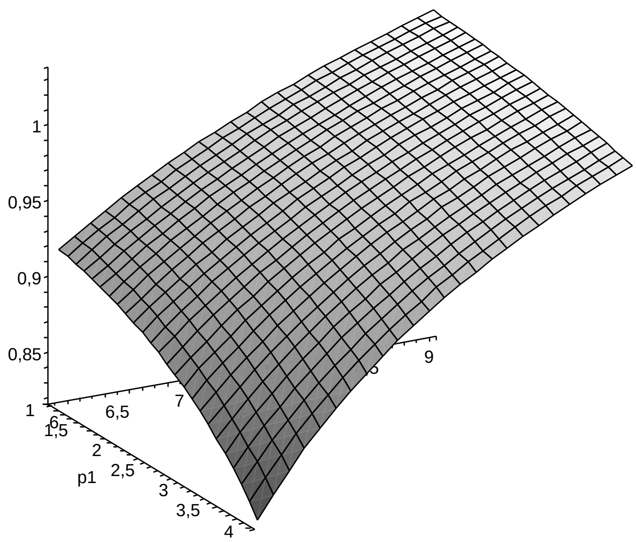

Figure 2.4: The function $G(p_1, p_2)$ when we choose $W = 1$ and functions $R(e)$ and $\Pi(G)$ as in Figures 2.1 and 2.2.

Lastly we present the most important result of this subsection that shows how prices, strength of institutions and total income in the economy of a nation are mutually related.

Proposition 2.4. *The behavior of the total income as a function of resource prices is strongly linked to the quality of institutions. In particular it is increasing if the latter are sufficiently strong, conversely it is decreasing if they are not.*

Proof. We consider for simplicity only the dependence with respect to both prices simultaneously, namely we compute the directional derivative with respect to $\overrightarrow{u}$, but the reasoning and the conclusions are the same even if we consider the dependence with respect to a single price. To quantify the total income Y we can use the well known Gross Domestic Product, that measures the market value of all final goods and services produced within a country in a given period of time, that is in this case

$$Y := p_1 e + p_2 R(e) + 2(1-G)H$$

We recall that we can see e and G as functions of p_1 and p_2, therefore one can prove that the derivative along the direction $\overrightarrow{u}$ in given by

$$\frac{dY}{d\overrightarrow{u}} = c(p_1 e + p_2 R) + \frac{\partial e}{\partial \overrightarrow{u}}(p_1 + p_2 R') - 2H\frac{\partial G}{\partial \overrightarrow{u}}$$

Now replacing expressions (2.4) and (2.5) in the previous formula, recalling the definition of D_1 and making a lot of computation one can show that it results

$$\begin{aligned}\operatorname{sgn}\frac{dY}{d\overrightarrow{u}} = &\operatorname{sgn}\left[2R''\left(-eW - \frac{p_2}{p_1}(W-H)R\right) + R''\frac{\Pi''}{\Pi'}(p_2 R - WG)\left(e + \frac{p_2}{p_1}R\right) - \right.\\ &\left. - \frac{\Pi'}{\Pi}(ep_2(R')^2 - p_2 R R')\right]\end{aligned}$$

Here the first and the last addendum, which are respectively positive and negative, are the same contained in the original paper while the second, which is positive, comes from considering a nonzero second derivative of the reelection probability. It is in general not possible to say that this derivative is monotone but if we look at as a function of Π', the argument of sign function is of the form

$$a + b\frac{1}{\Pi'} - c\Pi'$$

with $a, b, c > 0$. The largest zero of this function is

$$\widetilde{\Pi}' = \frac{-a - \sqrt{a^2 + 4bc}}{-2c}$$

which is positive[13]. Moreover it is easy to check that this function is decreasing and continuous in the semiaxis $\Pi' > 0$ (remember that it is an hypothesis of the model).

So $\widetilde{\Pi}'$ is a sort of critical value for the derivative of Y because we have $\frac{dY}{d\overrightarrow{u}} > 0$ if $\Pi' < \widetilde{\Pi}'$, that means that if prices increase then the economy of the nation increases and, on the contrary, $\frac{dY}{d\overrightarrow{u}} < 0$ if $\Pi' > \widetilde{\Pi}'$, so to an increase in prices follows a decrease of the total income.

[13] The other zero is negative.

The function Π' is in some sense related to the robustness of institutions. A small value means that the incumbent, for different reasons, has less chance to influence his reelection probability by employing people in the public sector, consequently the institutions are less sensitive to the phenomenon of clientelism. Exactly the contrary happens if Π' is sufficiently big so the statement is proved.

□

2.4.2 Maximum point on the boundary

In this completely new subsection we want to analyze the case in which the maximum point is on the boundary of the set Ω. This set is a rectangle of $\mathbb{R}^2$ and so its boundary Γ is composed by four subsets which are[14]

$$\begin{aligned}\Gamma_1 &:= [0, \bar{e}] \times \{0\}\\ \Gamma_2 &:= \{\bar{e}\} \times [0,1]\\ \Gamma_3 &:= [0, \bar{e}] \times \{1\}\\ \Gamma_4 &:= \{0\} \times [0,1]\end{aligned}$$

In the next propositions we show how the maximum point on the boundary changes with respect to the parameters and also the differences between every piece of the boundary Γ. In this case we exhibit some graphics that show the evolution of the maximum point in the region $1 \le p_1, p_2 \le 2$. We start assuming the maximum point is in the interior of Γ_1.

Proposition 2.5. *If we assume that the maximum point is in the interior of Γ_1 so $G = 0$ and $0 < e < \bar{e}$, then the resource extraction rate is an increasing function with respect to p_1, decreasing with respect to p_2 and constant with respect to both p_1 and p_2 if they vary simultaneously but proportionally.*

Proof. By looking at system (2.2), since $G = 0$ and $0 < e < \bar{e}$, equations (2.2b) become $\lambda_1 = \lambda_2 = \lambda_4 = 0$ so the system reduces to

$$(2.6) \qquad \begin{cases} F^1(e, \lambda_3; p_1, p_2) := p_1 + \Pi(0) p_2 R'(e) = 0 \\ F^2(e, \lambda_3; p_1, p_2) := -(1 - \Pi(0))W + \Pi'(0) p_2 R(e) + \lambda_3 = 0 \end{cases}$$

We apply again Theorem 1.3 and consider

$$\frac{\partial F^2}{\partial \lambda_3}$$

which is identically 1, and in particular nonzero. The second equation implicitly defines a function $\lambda_3 = \lambda_3(e; p_1, p_2)$. We substitute in the first one and define

$$H(e; p_1, p_2) := F^1(e, \lambda_3(e; p_1, p_2); p_1, p_2) = 0$$

We consider now

$$\frac{\partial H}{\partial e} = \frac{\partial F^1}{\partial e} + \frac{\partial F^1}{\partial \lambda_3} \frac{\partial \lambda_3}{\partial e}$$

[14] We chose to enumerate the subsets anticlockwise starting from the bottom rather then in analogy with the φ's functions.

and we suppose $H_e \neq 0$. So $H = 0$ implicitly defines a function $e = e(p_1, p_2)$. Denoting $h(p_1, p_2) := \lambda_3(e(p_1, p_2); p_1, p_2)$ the starting system is now

$$\begin{cases} F^1(e(p_1, p_2), h(p_1, p_2); p_1, p_2) = 0 \\ F^2(e(p_1, p_2), h(p_1, p_2); p_1, p_2) = 0 \end{cases}$$

Differentiating both equations by p_1 one has

$$\begin{cases} F^1_e \dfrac{\partial e}{\partial p_1} + F^1_{\lambda_3} \dfrac{\partial h}{\partial p_1} = -F^1_{p_1} \\ F^2_e \dfrac{\partial e}{\partial p_1} + F^2_{\lambda_3} \dfrac{\partial h}{\partial p_1} = -F^2_{p_1} \end{cases}$$

and by Cramer's rule we get

$$\frac{\partial e}{\partial p_1} = \frac{\begin{vmatrix} -F^1_{p_1} & F^1_{\lambda_3} \\ -F^2_{p_1} & F^2_{\lambda_3} \end{vmatrix}}{\begin{vmatrix} F^1_e & F^1_{\lambda_3} \\ F^2_e & F^2_{\lambda_3} \end{vmatrix}} = \frac{\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} \Pi(0)p_2R'' & 0 \\ F^2_e & 1 \end{vmatrix}} = -\frac{1}{\Pi(0)p_2R''} > 0$$

Differentiating now both equations by p_2 one has

$$\begin{cases} F^1_e \dfrac{\partial e}{\partial p_2} + F^1_{\lambda_3} \dfrac{\partial h}{\partial p_2} = -F^1_{p_2} \\ F^2_e \dfrac{\partial e}{\partial p_2} + F^2_{\lambda_3} \dfrac{\partial h}{\partial p_2} = -F^2_{p_2} \end{cases}$$

and again by Cramer's rule we get

$$\frac{\partial e}{\partial p_2} = \frac{\begin{vmatrix} -F^1_{p_2} & F^1_{\lambda_3} \\ -F^2_{p_2} & F^2_{\lambda_3} \end{vmatrix}}{\begin{vmatrix} F^1_e & F^1_{\lambda_3} \\ F^2_e & F^2_{\lambda_3} \end{vmatrix}} = \frac{\begin{vmatrix} -\Pi(0)R' & 0 \\ -F^2_{p_2} & 1 \end{vmatrix}}{\begin{vmatrix} \Pi(0)p_2R'' & 0 \\ F^2_e & 1 \end{vmatrix}} = -\frac{R'}{p_2R''} < 0$$

Lastly considering a simultaneous but proportional variation of p_1 and p_2 we have

$$\frac{de}{d\overrightarrow{u}} = \langle \nabla e, \overrightarrow{u} \rangle = -c\frac{p_1 + \Pi(0)p_2R'}{\Pi(0)p_2R''} \equiv 0$$

where the last equality turns out from (2.6).

□

Observing that since Ω is a rectangle its boundary is easily parameterizable we can provide a more direct proof of this and the next results by reducing to the case of maximization of function of only one variable.

Alternative proof of Proposition 2.5. The boundary $\Gamma_1 = [0, \overline{e}] \times \{0\}$ is parameterizable by $e(s) = s$ with $s \in [0, \overline{e}]$ and $G(s) \equiv 0$ so the income to maximize is, with an abuse of notation,

$$I^1(e; p_1, p_2) := I(e, 0; p_1, p_2) = p_1 e + \Pi(0)p_2R(e)$$

If we look for an internal maximum point the first order condition says that

$$\frac{\partial I^1}{\partial e} = p_1 + \Pi(0)p_2R'(e) = 0 \tag{2.7}$$

We have

$$I^1_{ee} = \Pi(0)p_2R''(e) < 0$$

which is in particular nonzero[15] then by the implicit function theorem equation (2.7) implicitly defines a function $e = e(p_1, p_2)$ which derivatives are

$$\frac{\partial e}{\partial p_1} = -\frac{I^1_{ep_1}}{I^1_{ee}} = -\frac{1}{\Pi(0)p_2R''} > 0$$
$$\frac{\partial e}{\partial p_2} = -\frac{I^1_{ep_2}}{I^1_{ee}} = -\frac{R'}{p_2R''} < 0$$
$$\frac{de}{d\overrightarrow{u}} = \langle \nabla e, \overrightarrow{u} \rangle = -c\frac{p_1 + \Pi(0)p_2R'}{\Pi(0)p_2R''} \equiv 0$$

where the last equality turns out again from (2.6).

□

The next case deals with the maximum point in the interior of Γ_3.

Proposition 2.6. *If we assume that the maximum point is in the interior of* Γ_3 *so* $G = 1$ *and* $0 < e < \overline{e}$*, then the resource extraction rate is an increasing function with respect to* p_1*, decreasing with respect to* p_2 *and constant with respect to both* p_1 *and* p_2 *if they vary simultaneously but proportionally.*

Proof. The proof in essentially the same of the last proposition. The boundary $\Gamma_3 = [0, \overline{e}] \times \{1\}$ is parameterizable by $e(s) = s$ with $s \in [0, \overline{e}]$ and $G(s) \equiv 1$ so the income to maximize is

$$I^3(e; p_1, p_2) := I(e, 1; p_1, p_2) = p_1e - W + \Pi(1)(p_2R(e) - W)$$

If we look for an internal maximum point the first order condition says that

$$\frac{\partial I^3}{\partial e} = p_1 + \Pi(1)p_2R'(e) = 0 \tag{2.8}$$

We have

$$I^3_{ee} = \Pi(1)p_2R''(e) < 0$$

which is in particular nonzero then by the implicit function theorem equation (2.8) implicitly defines a function $e = e(p_1, p_2)$ which derivatives are

$$\frac{\partial e}{\partial p_1} = -\frac{I^3_{ep_1}}{I^3_{ee}} = -\frac{1}{\Pi(1)p_2R''} > 0$$
$$\frac{\partial e}{\partial p_2} = -\frac{I^3_{ep_2}}{I^3_{ee}} = -\frac{R'}{p_2R''} < 0$$
$$\frac{de}{d\overrightarrow{u}} = \langle \nabla e, \overrightarrow{u} \rangle = -c\frac{p_1 + \Pi(1)p_2R'}{\Pi(1)p_2R''} \equiv 0$$

where the last equality turns out from (2.8).

□

[15] It is negative so the second order condition for functions of only one variable is fulfilled. The same for all the next two cases.

The previous two propositions show the same results. If the employment rate is fixed the incumbent can not influence his reelection probability by employing voters in the public sector.

This fact is crucial to explain in particular the third result. In fact while the first two results, and related consideration, are identical to those of Proposition 2.2, the latter is different and says that the optimal response to a simultaneous changing of prices is to leave unaltered the extraction rate. This because since the incumbent can not influence his reelection probability the policy is, in some sense, cut off. Consequently, since a proportional increase in both prices keeps unchanged the ratio $\frac{p_1}{p_2}$, the optimal response is what one would obtain by reasoning from a merely economic perspective.

The difference with respect to the case in which the incumbent maximizes also over G variable is that an increase in p_2 makes more valuable to be in power after the elections so the politician is forced to employ voters to increase his chances of success.

Figures 2.5 and 2.6 on the next page show the function $e(p_1, p_2)$ when G is fixed and takes value on the boundaries.

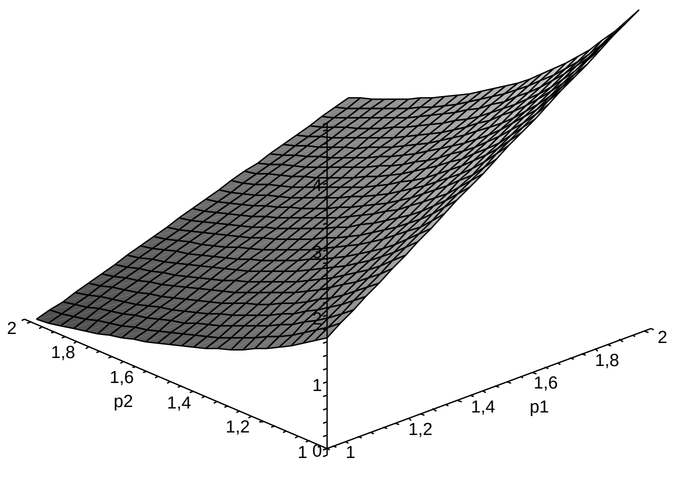

Figure 2.5: The function $e(p_1, p_2)$ in the case $G = 0$ when we choose $W = 1$ and functions $R(e)$ and $\Pi(G)$ as in Figures 2.1 and 2.2.

Now in the next two propositions we consider a maximum point on the sets in which e is fixed and G is variable, starting from the interior of Γ_4.

Proposition 2.7. *If we assume that the maximum point is in the interior of Γ_4 so $e = 0$, and consequently $R(e) = E$, and $0 < G < 1$, then the rate of voters employed in the public sector is a constant function with respect to p_1, increasing with respect*

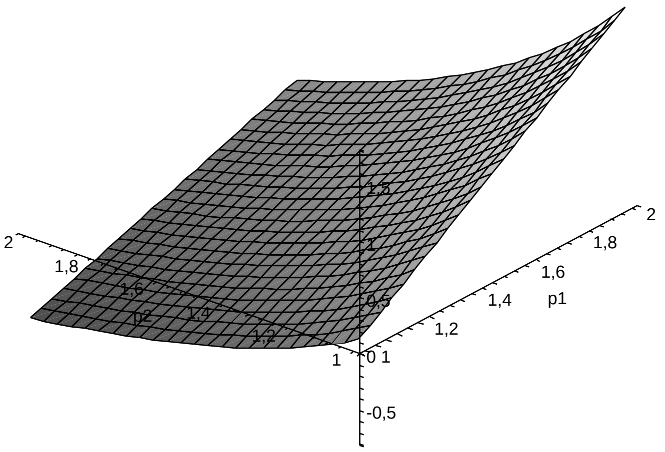

Figure 2.6: The function $e(p_1, p_2)$ in the case $G = 1$ when we choose $W = 1$ and functions $R(e)$ and $\Pi(G)$ as in Figures 2.1 and 2.2.

to p_2 and increasing with respect to both p_1 and p_2 also if they vary simultaneously but proportionally.

Proof. In this case the boundary $\Gamma_4 = \{0\} \times [0, 1]$ is parameterizable by $e(s) \equiv 0$ and $G(s) = s$ with $s \in [0, 1]$ so the income to maximize is

$$I^4(G; p_1, p_2) := I(0, G; p_1, p_2) = -WG + \Pi(G)(p_2 E - WG)$$

If we look for an internal maximum point the first order condition says that

$$\frac{\partial I^4}{\partial G} = -(1 + \Pi)W + \Pi'(p_2 E - WG) = 0 \tag{2.9}$$

We have

$$I^4_{GG} = -2\Pi' W + \Pi''(p_2 E - WG) < 0$$

which is in particular nonzero then by the implicit function theorem equation (2.9) implicitly defines a function $G = G(p_1, p_2)$ which derivatives are

$$\begin{aligned}
\frac{\partial G}{\partial p_1} &= -\frac{I^4_{Gp_1}}{I^4_{GG}} \equiv 0 \\
\frac{\partial G}{\partial p_2} &= -\frac{I^4_{Gp_2}}{I^4_{GG}} = -\frac{\Pi' E}{-2\Pi' W + \Pi''(p_2 E - WG)} > 0 \\
\frac{dG}{d\overrightarrow{u}} &= \langle \nabla G, \overrightarrow{u} \rangle = c \frac{\partial G}{\partial p_2} p_2 > 0
\end{aligned}$$

□

If we prescribe that in the first period we have no resource extraction, a variation in price p_1 is obviously meaningless. Conversely the number of voters employed by the incumbent is increasing in p_2 and in p_1 and p_2 simultaneously because if, for example, p_2 increases the politician is forced to employ voters in the public sector to guaranteed his victory.

Figure 2.7 shows function $G(p_1, p_2)$ when there is no extraction in the first period.

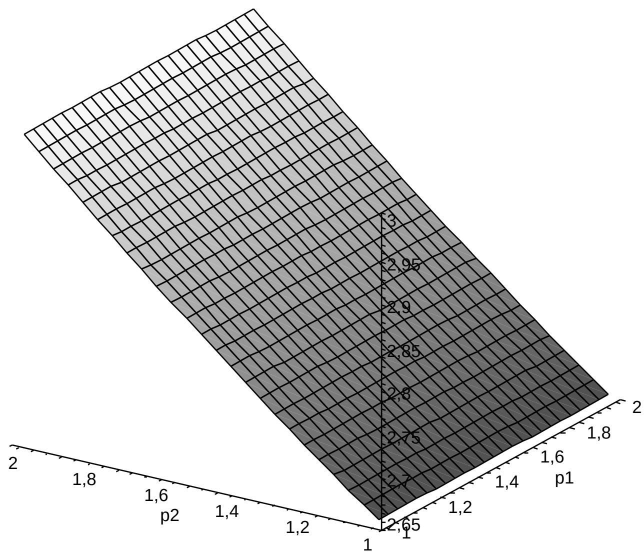

Figure 2.7: The function $G(p_1, p_2)$ in the case $e = 0$ when we choose $W = 1$ and functions $R(e)$ and $\Pi(G)$ as in Figures 2.1 and 2.2.

The last case deals with the maximum point in the interior of Γ_2.

Proposition 2.8. *If we assume that the maximum point is the interior of* Γ_2 *so* $e = \bar{e}$ *and consequently* $R(e) = 0$*, and* $0 < G < 1$*, then the rate of voters employed in the public sector is a constant function with respect to* p_1*, constant with respect to* p_2 *and constant with respect to both* p_1 *and* p_2 *also if they vary simultaneously but proportionally.*

Proof. In this case the boundary $\Gamma_2 = \{\bar{e}\} \times [0, 1]$ is parameterizable by $e(s) \equiv \bar{e}$ and $G(s) = s$ with $s \in [0, 1]$ so the income to maximize is

$$I^2(G; p_1, p_2) := I(\bar{e}, G; p_1, p_2) = p_1\bar{e} - (1 + \Pi(G))WG$$

It is easy to see that the maximum with respect to G of I^2 is obtained at $G = 0$ independently of p_1 and p_2, then the optimal rate of voters employed in the public sector does not change.

□

We can explain the last result considering that if all the resources are extracted in the first period the incumbent has obviously no interest to be reelected. Therefore the optimal response is to employ nobody in order to cancel hiring costs and this choice does not change if prices p_1 and p_2 vary.

Figure 2.8 shows the function $G(p_1, p_2)$ when the incumbent exploits all the extractable resource.

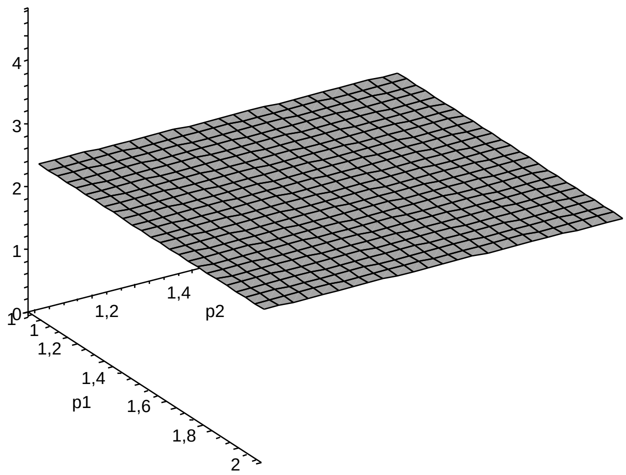

Figure 2.8: The function $G(p_1, p_2)$ in the case $e = \frac{4}{5}$ when we choose $W = 1$ and functions $R(e)$ and $\Pi(G)$ as in Figures 2.1 and 2.2.

Remark 2.9. *In the proof of the last result we can not follow the previous strategy because the term*

$$I^2_{GG} = -2\Pi' W - \Pi'' W G$$

is not surely different from 0*. Nevertheless if we suppose* $I^2_{GG} \neq 0$ *we can calculate explicitly that all the derivatives are identically* 0*.*

2.5 Case $d = 2$: two different natural resources

In this section we generalize the model by considering the case inn which the incumbent has two different resources to exploit and the maximum point is located in the interior of the region under consideration. We will just present the situation from a merely graphical point of view because the explicit expression of the derivatives of functions e_1, e_2 and G with respect to prices can be obtained directly from (1.1).

We decided to take the same choices made for the case of one single resource, so we set $W = 1$ and for the first resource we choose the same function, obviously now indexed with index 1, $R_1(e_1) = -\frac{5}{16}e_1^2 - e_1 + 1$ which describes the remaining. For the second resource we choose the very similar function $R_2(e_2) = -\frac{10}{9}e_2^2 - e_2 + 1$ that models a situation in which again $E = 1$ but $\overline{e} = \frac{3}{5}$, so the incumbent can extract only 60% of the total in the first period leaving nothing for the second one (see Figure 2.9).

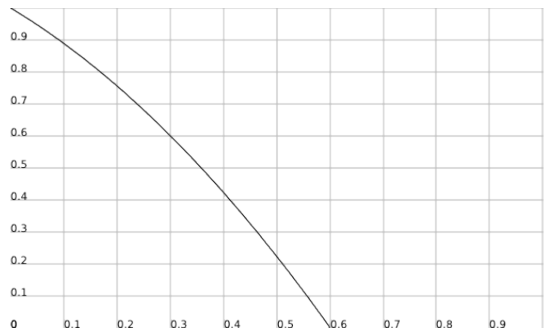

Figure 2.9: The function $R_2(e_2) = -\frac{10}{9}e_2^2 - e_2 + 1$.

In this situation, with two different natural resources, there are two rates of extraction and four selling prices, two for each period. We are going now to show six graphics that illustrate the trend of the rates of extraction as a function of selling prices. Obviously every rate will be plot as a function of only two prices, ranging from 20 to 24 except in a case that will be indicated, so we set the other two prices to an arbitrary value, in this case 28.

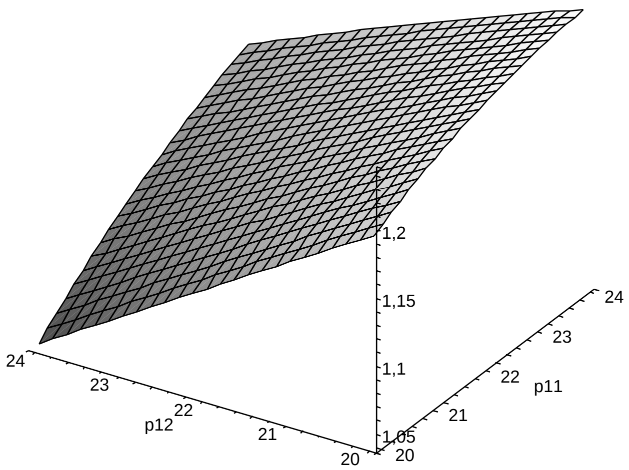

Figure 2.10: The function $e_1(p_1^1, p_2^1)$ when we choose $W = 1$ and functions $R_1(e_1)$, $R_2(e_2)$ and $\Pi(G)$ as in Figures 2.1, 2.9 and 2.2.

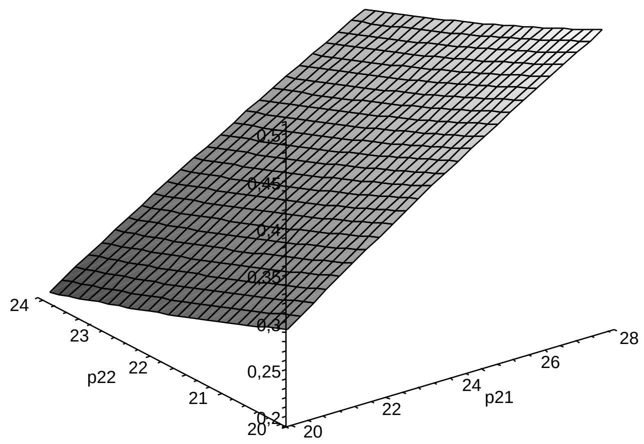

Figure 2.11: The function $e_2(p_1^2, p_2^2)$ when we choose $W = 1$ and functions $R_1(e_1)$, $R_2(e_2)$ and $\Pi(G)$ as in Figures 2.1, 2.9 and 2.2. In this case we have $20 \leq p_1^2 \leq 28$.

Figure 2.10 on the previous page shows that the rate of extraction of the first resource e_1 is increasing with respect to its selling price in the first period and decreasing with respect to its price in the second one. The same occurs to the rate of extraction of the second resource e_2 with respect to its selling prices, as shown in Figure 2.11. The explanation of this behavior is the same provided for Proposition 2.2. We show now the trend of extraction of a resource with respect to the prices of the other one.

Figure 2.12 on the next page shows that the rate of extraction of the first resource as a function of prices of the other has the opposite behavior compared to Figure 2.11. A possible explanation is that if, for example, the price p_1^2 of the second resource in the first period increases then the incumbent can afford to extract less resource of the first type without compromising the overall gain. Vice versa if price p_2^2 increases then the incumbent has to increase the extraction of the first resource to balance the lower extraction of the second one. The extraction rate of the second resource with respect to the selling prices of the first one has the same behavior, as shown in Figure 2.13 on the following page.

We focus at last on the rate of voters employed in the public sector as a function of the selling prices. In this case there is an unexpected phenomenon.

Figure 2.14 on page 34 shows that the rate of voters G is increasing with respect to p_1^1 and decreasing with respect to p_2^1, and the same happens if we consider the prices of the second resource, as shown in Figure 2.15 on page 34. A possible explanation

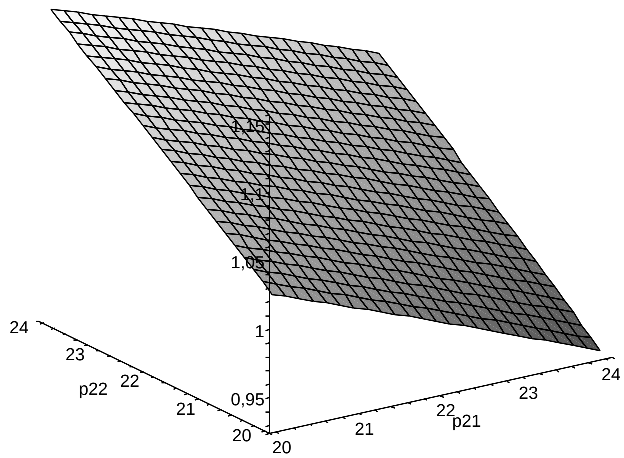

Figure 2.12: The function $e_1(p_1^2, p_2^2)$ when we choose $W = 1$ and functions $R_1(e_1)$, $R_2(e_2)$ and $\Pi(G)$ as in Figures 2.1, 2.9 and 2.2.

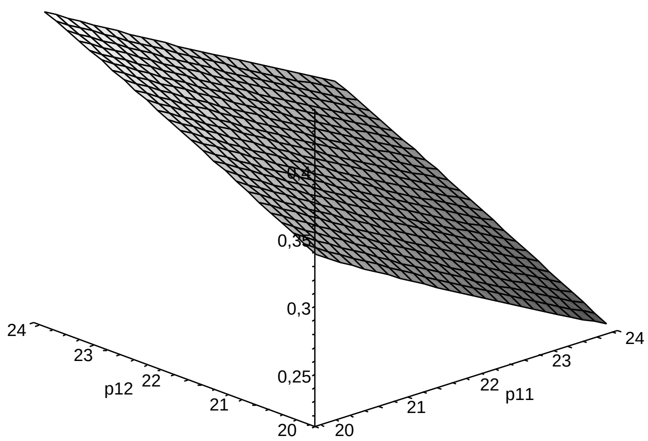

Figure 2.13: The function $e_2(p_1^1, p_2^1)$ when we choose $W = 1$ and functions $R_1(e_1)$, $R_2(e_2)$ and $\Pi(G)$ as in Figures 2.1, 2.9 and 2.2.

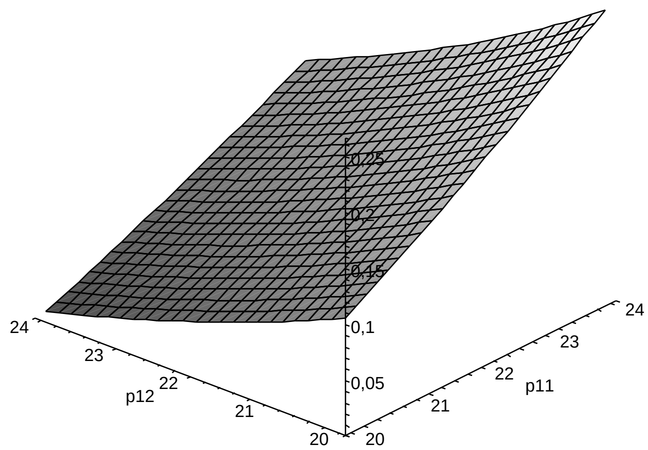

Figure 2.14: The function $G(p_1^1, p_2^1)$ when we choose $W = 1$ and functions $R_1(e_1)$, $R_2(e_2)$ and $\Pi(G)$ as in Figures 2.1, 2.9 and 2.2.

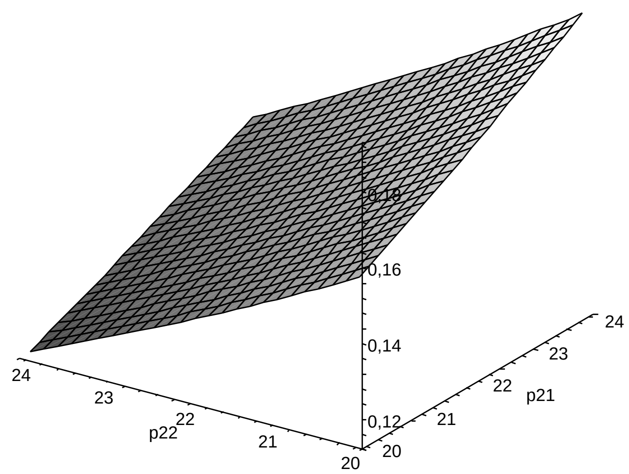

Figure 2.15: The function $G(p_1^2, p_2^2)$ when we choose $W = 1$ and functions $R_1(e_1)$, $R_2(e_2)$ and $\Pi(G)$ as in Figures 2.1, 2.9 and 2.2.

is that since in any case there is a resource whose extraction rate decreases, the incumbent has interest to be reelected to extract the remaining in the second period, so the optimal response is always to increase the rate of voters employed to guarantee his success.

Chapter 3

Resource curse: a discrete time model

In this chapter we present a further generalization of the previous model, considering a politician in charge up to a finite time which choices at every time step the extraction (for simplicity of only one resource) and recruitment rates to maximize his own income at the end of his mandate, while the rest of the framework remains the same.

In this case the way chosen to approaching this model is using the dynamic programming described extensively in Section 1.2. We defined a value function for this model, namely the maximum income that the incumbent can obtain starting with a certain amount of resources available and workers employed and remaining in power for a specified period, which satisfies a Dynamic Programming Principle which leads to a Bellman equation. We provided a regular result and we determined an optimal policy that the incumbent must follow to maximize his income.

Lastly, we also studied the problem from a numerical point of view. Bellman equation can in fact be solved by induction numerically on a computer. There is a large literature about numerical approximations of the equation, in particular in [11] Capuzzo Dolcetta proposed an uniformly convergent approximation of the Hamilton-Jacobi-Bellman equation related to an infinite horizon optimal control problem while in [25] Falcone and Giorgi deal with a finite horizon deterministic problem (for further works on this topic see also [13, 22, 12, 24]).

3.1 Model description

We state briefly the new context of study and the new assumptions we make. Anything not explicitly mentioned are subject to the assumptions made in the previous chapter.

In this case we deal with a model where there is a politician in charge up to time $t \in \mathcal{T} := \{1, \ldots, T\}$, with $T \in \mathbb{N}$, which represents for example the duration of his term in months or years, wishing to be reelected for a second mandate, and of course a competitor. There is a normalized amount of a single nonrenewable natural resource and for simplicity of notation we denote $r \in [0, 1]$ instead of E the extractable part. This quantity declines at the percentage rate the incumbent extracts it $e_n \in [0, 1]$, where 0 means no extraction and 1 means that the incumbent extracts all the available resource, and all the income from its sale accrues again directly to the government.

We denote also with $G_n \in [0,1]$ the numbers of employed at time n, that evolves according to an employing rate $h_n \in [0,1]$, $G \in [0,1]$ the starting number of workers and $\Pi_n := \Pi(G_n)$ the reelection probability as function of the latter. To give finally a justification to the choice $h_n \geq 0$ we can think to a situation in which the incumbent can not fire his employees during his mandate, for example due a particular form of contract.

Considering the whole system resource/workers, it is governed by the discrete-time state equation

$$\begin{cases} y_{n+1} = y_n + f(y_n, a_n) & n = 0, \dots, t-1 \\ y_0 = x \end{cases} \tag{3.1}$$

where $y_n = (r_n, G_n)$, $a_n = (e_n, h_n) \in A = [0,1] \times [0,1]$, $x = (r, G) \in Q = [0,1] \times [0,1]$ and $f\colon \mathbb{R}^2 \times A \to \mathbb{R}^2$ is a given mapping such that for every $x \in Q$ and $a \in A$ one has

$$x + f(x, a) \in Q \tag{3.2}$$

A possible choice for f is for example

$$f(x, a) = f((x_1, x_2), (a_1, a_2)) := (-a_1 x_1, a_2(1 - x_2)) = (f_1(x, a), f_2(x, a))$$

so the dynamics is forced to go to point $(0,1)$ regardless of the starting point $x \in Q$ (see Figure 3.1).

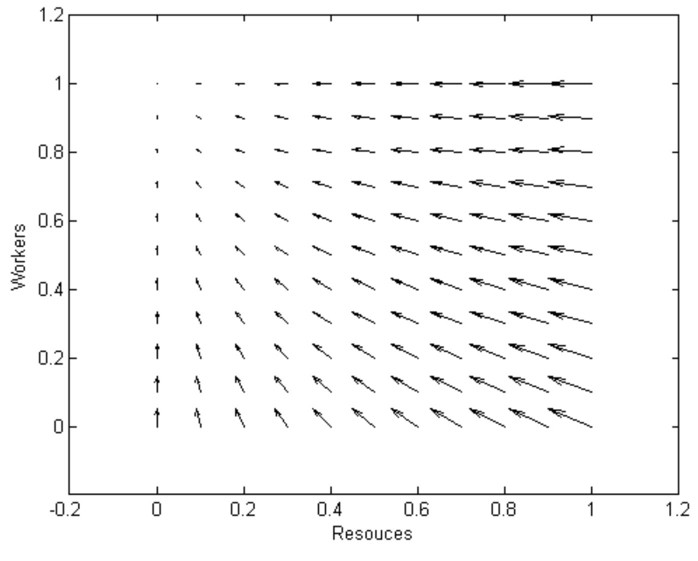

Figure 3.1: The vector field $f((x_1, x_2), (0.1, 0.05))$.

We denote by $\mathcal{A}$ the set of all the sequences $\alpha = \{a_n\} = \{(e_n, h_n)\} \subseteq A$ and by $y_n = y_n(x, \alpha)$ the corresponding trajectory of (3.1).

Resource income can again be spent by the incumbent politician or can be redistributed as patronage to increase reelection probability and to influence the outcome of voting. So the incumbent choose its economic policy $\alpha \in \mathcal{A}$ in order to maximize his own income

$$I(x,t,\alpha) := \sum_{n=0}^{t-1} \left(p_1 e_n r_n G_n - \frac{e_n^2}{r_n + \eta} - W G_n \right) \beta^n + \beta^t \Pi(G_t)(p_2 r_t - W G_t) \tag{3.3}$$

in which $\beta \in (0,1)$ represents a discount rate and $\eta \ll 1$ is a technical parameter which ensures that the denominator is never zero. The generic term of the sum in the expression above is the difference between the income from the resource extraction, where the presence of G_n means that only a part of the resource extracted can be sold, in particular the incumbent can sell all the resource extracted if and only if he employs the whole population, and the outcome resulting from extraction cost and from employing workers while the last term represents in some sense a scrap value, in this case the value of being in power for a second mandate. Note that this value is again discounted by a factor that is the reelection probability.

For sake of simplicity we define

$$\ell(x,a) = \ell((x_1,x_2),(a_1,a_2)) := -\left(p_1 a_1 x_1 x_2 - \frac{a_1^2}{x_1 + \eta} - W x_2 \right)$$
$$g(x) = g((x_1,x_2)) := -\Pi(x_2)(p_2 x_1 - W x_2)$$

so the income (3.3) becomes

$$I(x,t,\alpha) = \sum_{n=0}^{t-1} -\ell(y_n,a_n)\beta^n - \beta^t g(y_t)$$

3.2 Dynamic Programming Principle and corresponding Bellman equation

We start by defining what will be for us the value function of this optimization problem

$$u(x,t) := \inf_{\alpha \in \mathcal{A}} -I(x,t,\alpha) = \inf_{\{a_n\} \subseteq A} \left\{ \sum_{n=0}^{t-1} \ell(y_n,a_n)\beta^n + \beta^t g(y_t) \right\} \tag{3.4}$$

Remark 3.1. *To define this value function, in order to develop a theory analogue to the classical one, we considered the problem of minimizing the opposite of the total income instead of maximizing the latter. Obviously the "actual" value function is simply $-u(x,t)$.*

Again for simplicity of notation we define

$$J(x,t,\alpha) := -I(x,t,\alpha)$$

so the value function defined in (3.4) becomes

$$u(x,t) = \inf_{\alpha \in \mathcal{A}} J(x,t,\alpha)$$

The next results we are going to show, with appropriate changes, follows directly from [4] and are comparable to those contained in [25], to which we refer for further details. The first result is that the function u satisfies the following Dynamic Programming optimality Principle.

Proposition 3.2 (Dynamic Programming Principle). *The function $u(x,t)$ verifies*

$$u(x,t) = \inf_{\alpha \in \mathcal{A}} \left\{ \sum_{n=0}^{\tau-1} \ell(y_n, a_n)\beta^n + \beta^\tau u(y_\tau, t-\tau) \right\} \tag{3.5}$$

for all $x \in (0,1)\times(0,1)$ and $\tau = 0,\dots,t$. In particular for $\tau = 1$ we have, by swapping the addends,

$$u(x,t) = \inf_{a\in A}\{\beta u(y_1, t-1) + \ell(y_0, a)\} = \inf_{a\in A}\{\beta u(x + f(x,a), t-1) + \ell(x,a)\} \tag{3.6}$$

Proof. Since by definition $u(x,0) = g(x)$ then (3.5) reduces to the definition of u when $\tau = t$. For $\tau = 0,\dots,t-1$ and $\alpha = \{a_n\} = \{a_0,\dots,a_{t-1}\} \in \mathcal{A}$ fixed we define $\tilde{\alpha} = \{\tilde{a}_n\} := \{a_{n+\tau}\}$ and we denote $\tilde{y}_n$ the trajectory of (3.1) starting from y_τ and corresponding to $\tilde{\alpha}$. One has

$$\begin{aligned}
J(x,t,\alpha) &= \sum_{n=0}^{\tau-1} \ell(y_n,a_n)\beta^n + \sum_{n=\tau}^{t-1} \ell(y_n,a_n)\beta^n + \beta^t g(y_t) = \\
&= \sum_{n=0}^{\tau-1} \ell(y_n,a_n)\beta^n + \sum_{n=0}^{t-\tau-1} \ell(y_{n+\tau},\tilde{a}_n)\beta^{n+\tau} + \beta^\tau\beta^{t-\tau} g(\tilde{y}_{t-\tau}) = \\
&= \sum_{n=0}^{\tau-1} \ell(y_n,a_n)\beta^n + \beta^\tau J(y_\tau, t-\tau, \tilde{\alpha}) \geq \\
&\geq \sum_{n=0}^{\tau-1} \ell(y_n,a_n)\beta^n + \beta^\tau u(y_\tau, t-\tau)
\end{aligned}$$

By taking the infimum over $\mathcal{A}$ we get the inequality $\geq$ in (3.5). To prove the opposite inequality we fix $\alpha = \{a_n\} \in \mathcal{A}$ and we observe that, by definition of $u(x,t)$, for every $\delta > 0$ there exists $\alpha^\delta = \{a_n^\delta\}$ such that

$$u(y_\tau, t-\tau) \geq J(y_\tau, t-\tau, \alpha^\delta) - \delta$$

We define now the control $\overline{\alpha}$ such that its generic term is

$$\bar{a}_n := \begin{cases} a_n & 0 \leq n \leq \tau - 1 \\ a^\delta_{n-\tau} & \tau \leq n \leq t-1 \end{cases}$$

that is more compactly $\{\bar{a}_n\} = \{a_0,\dots,a_{\tau-1},a_0^\delta,\dots,a^\delta_{t-\tau-1}\}$, and let $\bar{y}_n$ and y_n^δ be the trajectories of (3.1) respectively starting from x and y_τ and corresponding to $\bar{\alpha}$ and α^δ. Then we have

$$\begin{aligned}
J(x,t,\bar{\alpha}) &= \sum_{n=0}^{\tau-1} \ell(\bar{y}_n,\bar{a}_n)\beta^n + \sum_{n=\tau}^{t-1} \ell(\bar{y}_n,\bar{a}_n)\beta^n + \beta^t g(\bar{y}_t) = \\
&= \sum_{n=0}^{\tau-1} \ell(y_n,a_n)\beta^n + \sum_{n=0}^{t-\tau-1} \ell(y^\delta_{n+\tau},a^\delta_n)\beta^{n+\tau} + \beta^\tau\beta^{t-\tau} g(y^\delta_{t-\tau}) = \\
&= \sum_{n=0}^{\tau-1} \ell(y_n,a_n)\beta^n + \beta^\tau J(y_\tau, t-\tau, \alpha^\delta) \leq \\
&\leq \sum_{n=0}^{\tau-1} \ell(y_n,a_n)\beta^n + \beta^\tau u(y_\tau, t-\tau) + \beta^\tau\delta
\end{aligned}$$

Since δ is arbitrary, by taking the infimum over $\mathcal{A}$ we get the inequality $\leq$ in (3.5) so the proof is complete.

$\square$

We observe that equation (3.6), if written as

$$u(x,t) + \sup_{a\in A}\{-\beta u(x + f(x,a), t-1) - \ell(x,a)\} = 0 \tag{3.7}$$

it can be seen as the discrete version of the classic Hamilton-Jacobi-Bellman equation

$$u_t(x,t) + \lambda u(x,t) + \sup_{a\in A}\{-f(x,a)\cdot Du(x,t) - \ell(x,a)\} = 0$$

for the finite horizon problem (analogous to (1.9) which comes from considering an infinite horizon) with the same dynamics f, cost function ℓ and the discount rate λ somehow related to β.

Before continuing with the study of the properties of the function u, we try to guess for a solution of (3.6). In particular, following an example in [26], we guess that the value function has the form

$$u(x,t) = u((x_1,x_2),t) = \varphi(t)\left(Ax_1x_2 + B\frac{1}{x_1+\eta} + Cx_2\right) \tag{3.8}$$

For this tentative solution the expression within the brackets of (3.6) is

$$\begin{aligned}\beta\varphi(t-1)\Bigg(A(x_1 - a_1x_1)(x_2 + a_2(1-x_2)) + \frac{B}{x_1 - a_1x_1 + \eta} + \\ + C(x_2 + a_2(1-x_2))\Bigg) - p_1a_1x_1x_2 + \frac{a_1^2}{x_1+\eta} + Wx_2\end{aligned}$$

Taking the gradient with respect to (a_1, a_2) of this expression and setting it equal to zero, we have that the minimum is achieved when

$$a_1 = \frac{Ax_1 + C}{Ax_1}$$

while for a_2 we have a much more complicated expression. To simplify the exposure we set now $A = B = C = 1$, so the value of the minimum is given by

$$\frac{\beta\varphi(t-1)}{\eta-1} - p_1(x_1+1)x_2 + \frac{(x_1+1)^2}{x_1^2(x_1+\eta)} + Wx_2$$

A sufficient condition for this quantity to be equal to $u(x,t)$ defined in (3.8) is that $\varphi(t)$ satisfy the recurrence equation

$$\varphi(t+1) = \left(x_1x_2 + \frac{1}{x_1+\eta} + x_2\right)^{-1}\left(\frac{\beta\varphi(t)}{\eta-1} - p_1(x_1+1)x_2 + \frac{(x_1+1)^2}{x_1^2(x_1+\eta)} + Wx_2\right)$$

with the initial point (x_1, x_2) fixed. If for example the initial point is $(x_1, x_2) = (1, 0)$ the recurrence equation becomes

$$\varphi(t+1) = \beta\frac{\eta+1}{\eta-1}\varphi(t) + 4$$

We couple now equation (3.6) with a natural initial condition so we have the discrete initial value problem

$$\begin{cases} u(x,t) = \inf_{a\in A}\{\beta u(x+f(x,a),t-1)+\ell(x,a)\} & \text{in } Q\times\mathcal{T} \\ u(x,0)=g(x) & \text{in } Q \end{cases} \tag{3.9}$$

We want to show now that the value function u is the unique solution of the previous problem and that a comparison principle between subsolution and supersolution of (3.9) holds. To do this we notice first of all that functions $f(x,a)$, $\ell(x,a)$ and $g(x)$ have some relevant properties, which are interesting in a more general theory.

Remark 3.3. *Functions $f(x,a)$, $\ell(x,a)$ and $g(x)$ are bounded respectively in $Q\times A$ by two constants M_1 and M_2 and in Q by a constant M_3 since they are continuous. Moreover $f(x,a)$ and $\ell(x,a)$ are Lipschitz continuous in x uniformly in a with constants respectively L_1 and L_2 since the are also C^1. We can finally observe that the last property of f implies*

$$|y_n(x,\alpha)-y_n(z,\alpha)| \le (1+L_1)^n|x-z| \tag{3.10}$$

We state now the first result concerning the function u.

Proposition 3.4. *The value function $u(x,t)$ is bounded in $Q\times\mathcal{T}_0$, in which $\mathcal{T}_0 := \{0,\dots,T\}$, and it is the complete solution, namely the maximal subsolution and the minimal supersolution, and therefore the unique solution of* (3.9) *in $B(Q\times\mathcal{T}_0)$. Moreover, if $u_1,u_2\in B(Q\times\mathcal{T}_0)$ are respectively a subsolution and a supersolution of* (3.9) *then*

$$\sup_{Q\times\mathcal{T}_0}(u_1-u_2) \le \sup_{Q\times\{0\}}(u_1-u_2) \tag{3.11}$$

Proof. From the previous remark and the definition of u we have

$$|u(x,t)| \le \sum_{n=0}^{t-1}\beta^n|\ell(y_n,a_n)|+\beta^t|g(y_t)| \le M_1\sum_{n=0}^{t-1}\beta^n+\beta^t M_2 \le M_1\frac{1-\beta^T}{1-\beta}+M_2$$

so that $u\in B(Q\times\mathcal{T}_0)$. For any subsolution u_1 of (3.9) we have by definition $u_1-u\le 0$ in $Q\times\{0\}$ so by (3.11) one has $u_1-u\le 0$ in $Q\times\mathcal{T}_0$. Similarly we can show that u is below any supersolution. Now we prove the comparison principle (3.11). By definition of supersolution, for any $\varepsilon>0$ and $(x,t)\in Q\times\mathcal{T}$ there exists $a_\varepsilon\in A$ such that

$$u_2(x,t) \ge \beta u_2(x+f(x,a_\varepsilon),t-1)+\ell(x,a_\varepsilon)-\varepsilon$$

On the other hand we have, by definition of subsolution,

$$u_1(x,t) \le \beta u_1(x+f(x,a_\varepsilon),t-1)+\ell(x,a_\varepsilon)$$

Hence, by subtracting u_2 from u_1 we get

$$u_1(x,t)-u_2(x,t) \le \beta u_1(x+f(x,a_\varepsilon),t-1)-\beta u_2(x+f(x,a_\varepsilon),t-1)+\varepsilon$$

By taking the supremum over $Q\times\{1\}$ we have

$$\sup_{Q\times\{1\}}(u_1-u_2) \le \beta\sup_{Q\times\{0\}}(u_1-u_2)+\varepsilon$$

Since $\beta \in (0,1)$ and ε is arbitrary, by repeating the procedure one has

$$\sup_{Q\times\{T\}}(u_1-u_2) \leq \sup_{Q\times\{T-1\}}(u_1-u_2) \leq \ldots \leq \sup_{Q\times\{1\}}(u_1-u_2) \leq \sup_{Q\times\{0\}}(u_1-u_2)$$

and finally we have the comparison principle (3.11) by taking the supremum over the union of all sets.

□

Next we show a regularity result for u.

Proposition 3.5. *For ever fixed $t \in \mathcal{T}$ function u is uniformly continuous in x. Moreover, u is Lipschitz continuous with Lipschitz constant*

$$L_2\frac{1-\beta^t(1+L_1)^t}{1-\beta(1+L_1)} + \beta^t(1+L_1)^t$$

Proof. For fixed $z \in Q$ and for an arbitrary $\varepsilon > 0$ there exists $\alpha^\varepsilon = \{a_n^\varepsilon\} \in \mathcal{A}$ such that

$$u(z,t) \geq J(z,t,\alpha^\varepsilon) - \varepsilon$$

Then

$$u(x,t) - u(z,t) \leq J(x,t,\alpha^\varepsilon) - J(z,t,\alpha^\varepsilon) + \varepsilon$$

for every $x, z \in Q$ and $t \in \mathcal{T}_0$. Consequently, by Remark 3.3, and in particular by property (3.10), we have

$$\begin{aligned} u(x,t) - u(z,t) &\leq \sum_{n=0}^{t-1} |\ell(y_n(x,\alpha^\varepsilon), a_n^\varepsilon) - \ell(y_n(z,\alpha^\varepsilon), a_n^\varepsilon)|\beta^n + \\ &+ \beta^t|y_t(x,\alpha^\varepsilon) - y_t(z,\alpha^\varepsilon)| + \varepsilon \leq \\ &\leq \sum_{n=0}^{t-1} L_2(1+L_1)^n\beta^n|x-z| + \beta^t(1+L_1)^t|x-z| + \varepsilon \end{aligned}$$

It is easy to check that the first two summands can be made smaller than ε for $|x-z|$ small enough, therefore u is uniformly continuous. Moreover, using the formula for the sum of the first t terms of a geometric series, it results

$$u(x,t) - u(z,t) \leq \left(L_2\frac{1-\beta^t(1+L_1)^t}{1-\beta(1+L_1)} + \beta^t(1+L_1)^t\right)|x-z| + \varepsilon$$

and since ε is arbitrary u has the desired regularity.

□

In the next result we state that there exists an optimal control for this problem and we show the procedure for its construction from equation (3.9).

Proposition 3.6. *The control $\alpha^* = \{a_n^*\} \in \mathcal{A}$, which depends on x and t, defined by*

$$a_n^* \in \arg\min_{a\in A}\{\beta u(y_n^* + f(y_n^*, a), t_n - 1) + \ell(y_n^*, a)\}$$

for $n = 0, \ldots, t-1$, where $y_0^* = x$, $\{y_n^*\}$ is the trajectory of (3.1) corresponding to α^* and $t_n = t - n$, is optimal for (x,t).

Proof. We recall that by the previous two propositions u is the unique solution of (3.9) in $BUC(Q \times \mathcal{T}_0)$. By Remark 3.3 the function

$$a \longmapsto \beta u(x + f(x,a), t-1) + \ell(x,a)$$

is continuous for every fixed $x \in Q$ and $t \in \mathcal{T}$. Since $A = [0,1] \times [0,1]$ is compact there exist $\bar{a} = \bar{a}(x,t) \in A$ such that

$$\inf_{a \in A} \beta u(x + f(x,a), t-1) + \ell(x,a) = \beta u(x + f(x,\bar{a}), t-1) + \ell(x,\bar{a})$$

On the account of (3.9) this means that the set

$$F(x,t) = \{a \in A \mid u(x,t) = \beta u(x + f(x,a), t-1) + \ell(x,a)\}$$

is nonempty for all $(x,t) \in Q \times \mathcal{T}$. For fixed (x,t) set $y_0^* = x$ and define recursively the sequences $\{y_n^*\} \subset Q$ and $\alpha^* = \{a_n^*\} \subset A$ for $n = 0, \ldots, t-1$ by setting

$$\begin{cases} a_n^* \in F(y_n^*, t_n) \\ y_{n+1}^* = y_n^* + f(y_n^*, a_n^*) \end{cases}$$

By definition of F we can show that y_{n+1}^*, y_n^* and a_n^* is related by

$$\beta^n u(y_n^*, t_n) - \beta^{n+1} u(y_{n+1}^*, t_{n+1}) = \beta^n \ell(y_n^*, a_n^*)$$

and therefore

$$\sum_{n=0}^{t-1} \beta^n (u(y_n^*, t_n) - \beta u(y_{n+1}^*, t_{n+1}) = \sum_{n=0}^{t-1} \beta^n \ell(y_n^*, a_n^*)$$

Since the left-hand side of the previous equality is the partial sum of a telescoping series, it is easy to check that it results

$$\sum_{n=0}^{t-1} \beta^n (u(y_n^*, t_n) - \beta u(y_{n+1}^*, t_{n+1}) = u(x,t) - \beta^t g(y_t^*)$$

by using the initial condition of (3.9). Hence

$$u(x,t) = \sum_{n=0}^{t-1} \ell(y_n^*, a_n^*) \beta^n + \beta^t g(y_t^*) = J(x,t,\alpha^*)$$

so α^* achieves the infimum and the proof of its optimality is complete.

□

Let the value function u be known, then the procedure to construct the optimal control α^* and the optimal trajectory $\{y_n^*\}$ is the following. Fix the final time t. Starting from the initial point $x = y_0^*$, compute

$$a_0^* = \arg\min_{a \in A} \{\beta u(y_0^* + f(y_0^*, a), t-1) + \ell(y_0^*, a)\}$$

Then the initial point moves, according to the dynamics (3.1), to the next point

$$y_1^* = y_0^* + f(y_0^*, a_0^*)$$

Compute now the optimal control at the next time

$$a_1^* = \arg\min_{a\in A}\{\beta u(y_1^* + f(y_1^*, a), t-2) + \ell(y_1^*, a)\}$$

and the next point

$$y_2^* = y_1^* + f(y_1^*, a_1^*)$$

Iterating the procedure, we arrive to the computation of the last optimal control

$$a_{t-1}^* = \arg\min_{a\in A}\{\beta u(y_{t-1}^* + f(y_{t-1}^*, a), 0) + \ell(y_{t-1}^*, a)\}$$

and the end point is given by

$$y_t^* = y_{t-1}^* + f(y_{t-1}^*, a_{t-1}^*)$$

3.3 Numerical approximation

In this section we want to present a numerical approximation for the model proposed. Theoretical results are in analogy to those contained in [23]. First of all we observe that dynamics (3.1) and Bellman equation (3.6) are already discretized in time, so we only need a discretization of the spaces of states.

We start building a grid in the state space Q. We choose a discretization step $h > 0$ such that $\frac{1}{h}$ is an integer N_h and build for simplicity a grid of N_h^2 squares of side h for a total amount of nodes of $L := (N_h + 1)^2$. Each node is denoted by x_i for $i = 1, \ldots, L$.

For simplicity of notation we set $u_i^t = u(x_i, t)$. Then the fully discretized Bellman equation is

$$u_i^t = \inf_{a\in A}\{\beta u^{t-1}(x_i + f(x_i, a)) + \ell(x_i, a)\} \tag{3.12}$$

with initial data $u_i^0 = g(x_i)$ for $i = 1, \ldots, L$.

Before tackling some aspects of the numerical calculation we want to give an estimate of the approximation error in the $L^\infty(Q)$ norm.

Proposition 3.7. *Let u^t and u_h^t be respectively solutions of* (3.6) *and* (3.12). *Then one has*

$$\|u^t - u_h^t\|_\infty \le tL_3 h \tag{3.13}$$

where

$$L_3 = L_2 \frac{1-\beta^t(1+L_1)^t}{1-\beta(1+L_1)} + \beta^t(1+L_1)^t$$

Proof. For any $(x, t) \in Q \times \mathcal{T}$ we can write

$$|u(x,t) - u_h(x,t)| \le \left|\sum_{i=1}^{L} \lambda_i(u(x,t) - u(x_i,t))\right| + \left|\sum_{i=1}^{L} \lambda_i(u(x_i,t) - u_h(x_i,t))\right|$$

where the λ_i's are the coefficients of a convex combination such that

$$x = \sum_{i=1}^{L} \lambda_i x_i$$

Since by Proposition 3.5 u is Lipschitz continuous with Lipschitz constant

$$L_3 := L_2 \frac{1-\beta^t(1+L_1)^t}{1-\beta(1+L_1)} + \beta^t(1+L_1)^t$$

we obtain

$$|u(x,t) - u(x_i,t)| \leq L_3|x - x_i| \leq L_3 h \tag{3.14}$$

By equations (3.6) and (3.12) we have

$$\begin{aligned} u(x_i,t) - u_h(x_i,t) &\leq \beta[u(x_i + f(x_i,\bar{a}), t-1) - u_h(x_i + f(x_i,\bar{a}), t-1)] \\ &\leq \beta\|u^{t-1} - u_h^{t-1}\|_\infty \end{aligned}$$

where $\bar{a}$ is a control giving the minimum in (3.12), and this implies

$$|u(x_i,t) - u_h(x_i,t)| \leq \beta\|u^{t-1} - u_h^{t-1}\|_\infty \tag{3.15}$$

By (3.14) and (3.15) we have, iterating the procedure,

$$\|u^t - u_h^t\|_\infty \leq L_3 h + \beta\|u^{t-1} - u_h^{t-1}\|_\infty \leq \ldots \leq tL_3 h + \beta\|u^0 - u_h^0\|_\infty = tL_3 h$$

since $\beta < 1$ and $\|u^0 - u_h^0\|_\infty = 0$ because initial data coincides on the nodes of the grid.

□

Following the procedure described at the end of the previous section we can reconstruct an approximate optimal feedback control α_h^* by means of our numerical approximation u_h at every time of the value function u obtained using equation (3.12). We want to compare now the optimal control α^* with the approximated optimal control α_h^*.

Proposition 3.8. *For any $(x,t) \in Q \times \mathcal{T}$ it results*

$$J(x,t,\alpha_h^*) \xrightarrow[h\to 0^+]{} J(x,t,\alpha^*)$$

Proof. Let the functions $L_h \colon Q \times \mathcal{T} \times A \to \mathbb{R}$ be defined by

$$L_h(x,t,a) := \beta u_h^{t-1}(x + f(x,a)) + \ell(x,a)$$

and $L \colon Q \times \mathcal{T} \times A \to \mathbb{R}$ be defined by

$$L(x,t,a) := \beta u(x + f(x,a), t-1) + \ell(x,a)$$

For any (x,t) fixed, we consider the control a_h^*, which depends on x and t, that by definition realizes

$$L_h(x,t,a_h^*) = \inf_{a\in A} L_h(x,t,a) = u_h$$

and the control a^*, which also depends on x and t, that by definition verifies

$$L(x,t,a^*) = \inf_{a\in A} L(x,t,a) = u$$

The estimate (3.13) implies that L_h converges uniformly on $Q \times \mathcal{T} \times A$ to L as h goes to 0. Moreover, since $A = [0,1] \times [0,1]$ is compact, for any sequence $\{a_h^*\}_{h\geq 0}$ of controls giving the minimum for L_h we can extract a subsequence $\{a^*_{h_k}\}$ converging to a limit $\bar{a}$. Then we have

$$L_{h_k}(x,t,a^*_{h_k}) \xrightarrow[h_k\to 0^+]{} L(x,t,\bar{a})$$

and, by definition of $\bar{a}$,

$$L(x,t,\bar{a}) \leq L(x,t,a^*)$$

On the other hand, by definition of a^*, one has the opposite inequality, so that

$$L(x,t,\bar{a}) = L(x,t,a^*)$$

which implies $\bar{a} = a^*$. For any subsequence we can repeat the same reasoning so we can conclude that the whole sequence $\{a^*_h\}$ converges, and it converges to the limit a^*. By definition of L_h we have

$$\ell(x,a) = L_h(x,t,a) - \beta u_h^{t-1}(x + f(x,a)) \tag{3.16}$$

We take now $x = y_n$, $t = t_n$ and the control[16] $a = a^*_h(y_n,t_n)$, so by definition of a^*_h we have

$$L_h(y_n,t_n,a^*_h(y_n,t_n)) = u_h^{t_n}(y_n)$$

Multiplying (3.16) by β^n one has

$$\ell(y_n,a^*_h(y_n,t_n))\beta^n = u_h^{t_n}(y_n)\beta^n - \beta^{n+1}u_h^{t_n-1}(y_n + f(y_n,a^*_h(y_n,t_n)))$$

We sum on n from 0 to $t-1$ and we get

$$\sum_{n=0}^{t-1}\ell(y_n,a^*_h(y_n,t_n))\beta^n = \sum_{n=0}^{t-1}u_h^{t_n}(y_n)\beta^n - \sum_{n=0}^{t-1}\beta^{n+1}u_h^{t_n-1}(y_n + f(y_n,a^*_h(y_n,t_n)))$$

Sending h to 0 we have that the left-hand side of the previous equality goes to[17]

$$\sum_{n=0}^{t-1}u(y_n,t_n)\beta^n - \sum_{n=0}^{t-1}\beta^{n+1}u(y_n + f(y_n,a^*(y_n,t_n)))$$

which is equal, expanding the sums and recalling that $y_0 = x$ and $t_0 = t$, to

$$u(x,t) + \beta u(y_1,t-1) + \ldots + \beta^{t-1}u(y_{t-1},1) - \beta u(y_1,t-1) - \ldots - \beta^t u(y_t,0)$$

Recalling now that $u(y_t,0) = g(y_t)$ we have finally

$$\sum_{n=0}^{t-1}\ell(y_n,a^*_h(y_n,t_n))\beta^n \xrightarrow[h\to 0^+]{} u(x,t) - \beta^t g(y_t)$$

which is the desired result.

□

[16]This is in practice the n-th component of α^*_h.
[17]Similarly $a^*(y_n,t_n)$ is the n-th component of α^*.

We show now two aspects related only on the implementation of the algorithm. The first is that since in general the point $x_i + f(x_i, a)$ is not a node of the grid but, by hypothesis (3.2), it belongs to one and only one square of the discretization[18] we must use an interpolation operator that involves the values of the function u at nodes of that square to reconstruct it in the interior.

The second concerns the determination of the minimum. In this case the simplest solution is to replace the set A by a finite set. To do this we discretize the set by choosing a step $m > 0$ such that $\frac{1}{m}$ is an integer N_m and build again for simplicity a grid of N_m^2 squares of side m for a total amount of discrete controls $M := (N_m + 1)^2$. Each control is denoted by a_s for $s = 1, \ldots, M$ and the discrete set of controls by A_M. So Bellman equation (3.12) is implemented in the further discretized form

$$u_i^t = \inf_{a_s \in A_M} \{\beta u^{t-1}(x_i + f(x_i, a_s)) + \ell(x_i, a_s)\}$$

and since we discretized the control space, the computation of the infimum for every i and t fixed is simply a search for the minimum in a set of M real numbers.

3.4 Numerical simulations

We want now to present some numerical tests performed on this model, and to compare them to the situation in which there is no discount, which we recall is the reelection probability of the incumbent.

Before starting with the numerical tests we fix some parameters of the approximation and the model also, while the other parameters of the model will wary at every test. So from now we set $h = \frac{1}{50}$, $m = \frac{1}{20}$, $\beta = .99$ and $\eta = 10^{-5}$.

Moreover, to describe the reelection probability as function of the number of workers we considered the same function presented in Figure 2.2 on page 20.

We set the initial point $(r, G) = (1, 0)$, unless otherwise indicated, so the incumbent starts his period of government with all resources available and without any worker hired.

Lastly, we set $u \equiv +\infty$ on the exit boundary of the domain, namely $r = 0$ and $G = 1$, to ensure that the dynamics will not ever reach it.

We summarize in the next table the parameters used for the different tests.

Test	t	p_1	p_2	W
1	6	100	60	1
2	6	300	60	1
3	6	100	100	1
4	6	100	200	10
5	6	100	200	50
6	6	100	95	20
7	24	500	50	20
8	6	2000	200	10

Table 3.1: Summary table of the tests performed.

[18] After choosing for every square a node representing and only two sides which refer to that node.

Tests 1 and 2

In Figure 3.2, by looking at the graph of u, we can see that when we have a lot of resources available the income is almost constant with respect to the number of workers, and this for the extremely low labor cost. On the contrary, when we have very few resources the income is decreasing with respect to the workers. In fact, more workers leads to higher cost which are not compensated by revenue due to the increasingly high costs of extraction that must be paid as the resources run out and because of the lack of the available resources themselves. Moreover, since $p_1 > p_2$, we can see that the best strategy is to employ the whole population immediately and then to extract all the available resource in the first period to maximize the income. Prize p_2 is lover than p_1 so it is not economically profitable to leave resources for the second period. If we consider the parameters of test 2, in which price p_1 increase, we can see (Figure 3.3) that this increasing leads to a higher extraction because the resource becomes more valuable in the present than in the future.

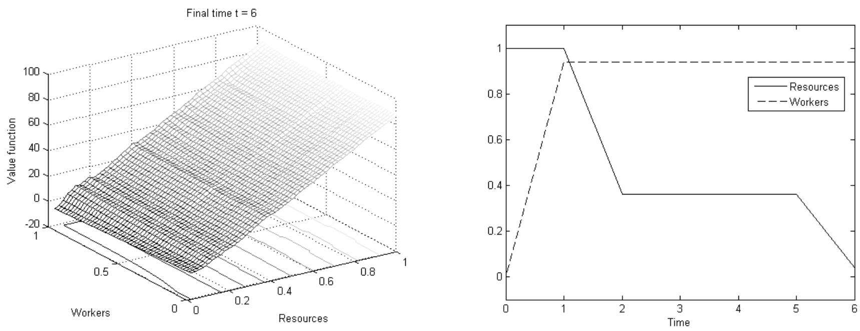

Figure 3.2: Test 1: The function $-u$ and the graphs of e_n and G_n.

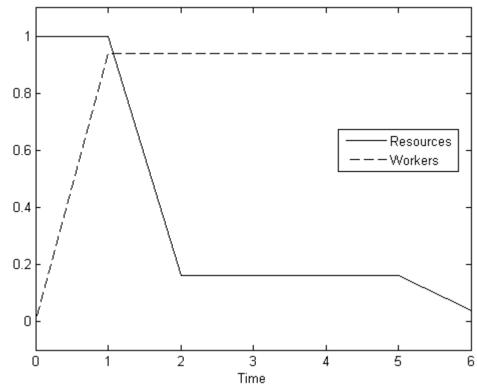

Figure 3.3: Test 2: The graphs of e_n and G_n.

Test 3

A different result is achieved with parameters of test 3, in which prices in the two periods coincide. In this case, as we can see in Figure 3.4, it is more profitable to extract some resource in the first period but leaving a part for the second one.

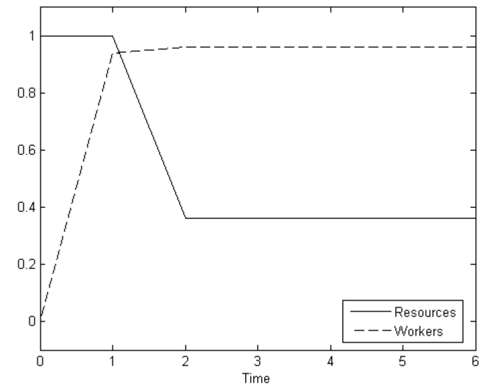

Figure 3.4: Test 3: The graphs of e_n and G_n.

Tests 4 and 5

In test 4 we note that, unlike the previous two tests, we have $p_1 < p_2$. In this case, as we can see in Figure 3.5, the best strategy is to leave all the resources for the second period and at the same time to employ all the workers to guarantee the maximum reelection probability, but at the final time to minimize expenditures related to the wages since the model does not penalize this "opportunistic" strategy.

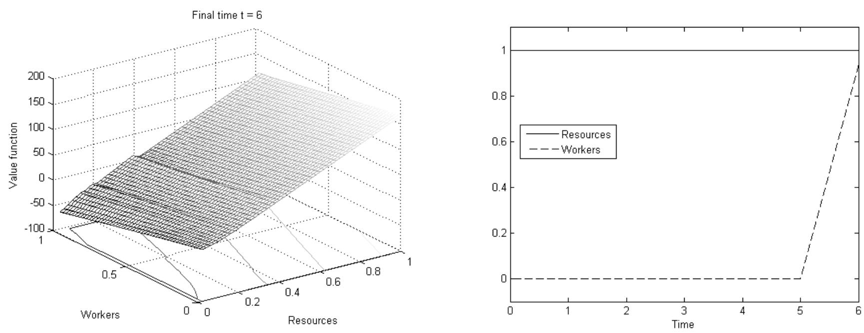

Figure 3.5: Test 4: The function $-u$ and the graphs of e_n and G_n.

In test 5 (see Figure 3.6 on the next page) we have the same situation as in test 4, except for the cost of the work W that is higher. In this case the optimal response is to hire a certain number of workers to guarantee the reelection, but not the whole population. Lastly, in both tests we note that the value function u is decreasing with respect to G, more decreasing in test 5 in which W in higher than in test 4.

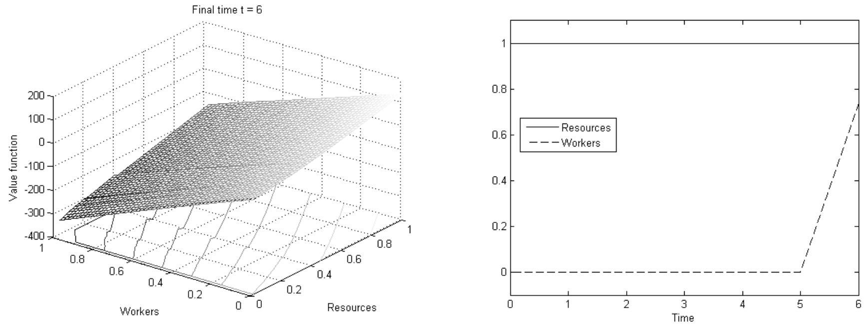

Figure 3.6: Test 5: The function $-u$ and the graphs of e_n and G_n.

Test 6

In this case we are again, as in test 1, in the situation $p_1 < p_2$, but now the two prices are very close and W is higher. So, as we can see in Figure 3.7, the best strategy is not to extract resources in the first period to exploit the slightly higher selling price, but is to leave them for the second period and hire worker at the end of the first one in order to save salary costs.

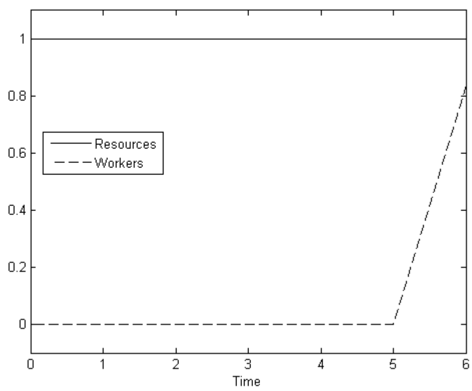

Figure 3.7: Test 6: The graphs of e_n and G_n.

Tests 7 and 8

This case is quite different from the previous. The situation is $p_1 \gg p_2$ and high labor cost. As we can see in Figure 3.8 on the next page, the value function is decreasing with respect to the workers G in all the domain except around the region

$0.8 \leq e, G \leq 1$ in which it is even slightly increasing and the optimal strategy is very different.

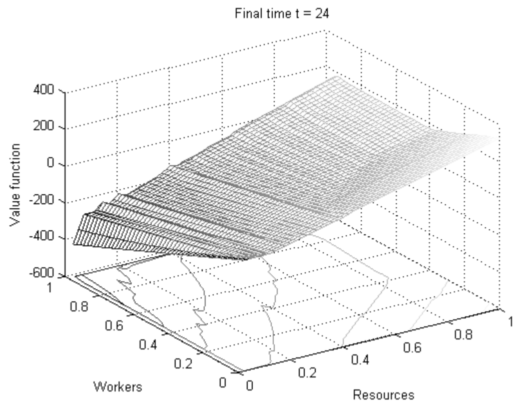

Figure 3.8: Test 7: The function $-u$.

In fact, as we can see in Figure 3.9, the optimal response when we start from the default starting point is to employ the whole population and extract all the resources at the penultimate and last time respectively, to minimize wages expenditures and to exploit the very higher selling price on the first period. But if we start from a point in that region, for example the point $(1, 0.8)$, the best strategy is, in addition to maximize the workforce, to extract part of the resource at the beginning of the first period, when the extraction cost is lower and the discount factor is higher and then to extract the remaining resource at the last time when the increased cost of extraction is amortized by the devaluation.

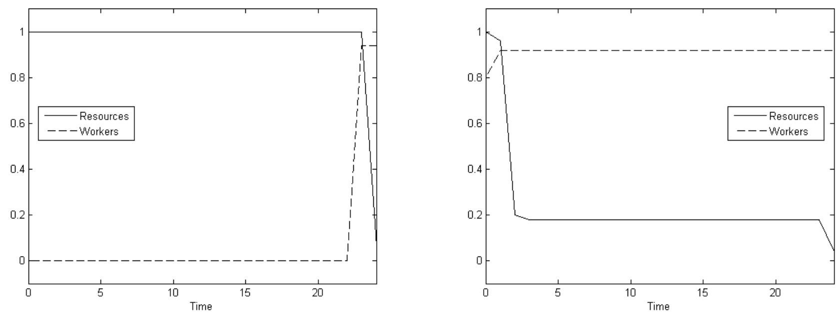

Figure 3.9: Test 7: The graphs of e_n and G_n when we start from $(1, 0)$ on the left and $(1, 0.8)$ on the right.

Anyway, we want to remark that in the second case, even it seems that in the period from 3 to 22 there is no extraction, the optimal strategy is instead to extract a very small amount of resource, as explained in Figure 3.10 on the next page, but

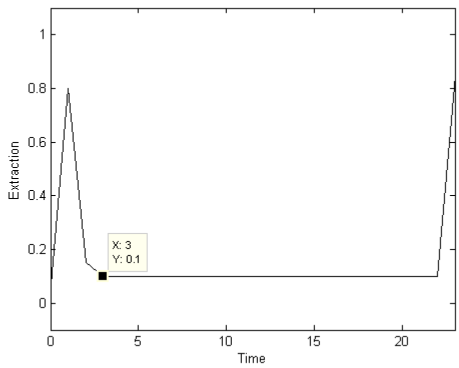

Figure 3.10: Test 7: The graph of a_n^* when we start from $(1, 0.8)$.

we can not see this due to numerical approximation of the calculator.

In test 8 we have the same situation since the parameters are comparable. As we can see in Figure 3.11, the optimal strategy, even if we start with no workers, is to hire the whole population then to extract resource during the entire period which, in this case, is shorter, only 6 time steps.

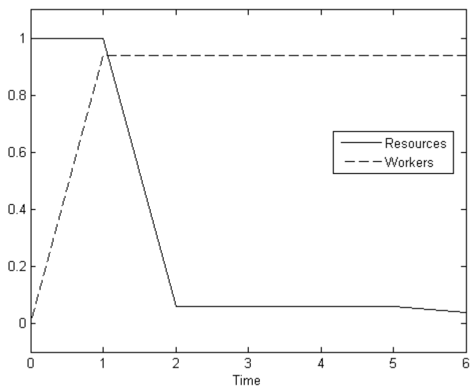

Figure 3.11: Test 8: The graphs of e_n and G_n.

3.4.1 Tests without probability factor

We consider, in this last subsection, a situation in which there is no probability factor in the income of second period, namely by setting $\Pi(G) \equiv 1$. This means that the incumbent makes decisions that maximize the income without taking in account his own interests, in particular he does not consider his probability to be in power in the second period. Roughly speaking, we deal with a government in which personal political interests and common economic interests are completely separated.

If we consider the parameters of test 3, then the optimal strategy is different than before. In fact, as we can see in Figure 3.12, the incumbent has no interest to employ workers to guarantee his reelection, so he prefers to leave all the resource available for the second period, since the prices are equal, to save salary costs. The same behavior appears on tests 4, 5 and 6 with the same motivation, while in the other tests there are no modification on the optimal strategy.

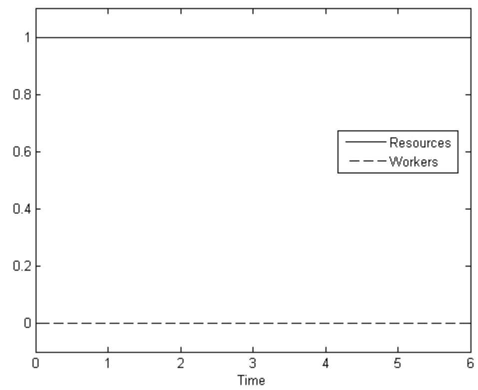

Figure 3.12: Test 3 without reelection probability factor: The graphs of e_n and G_n.

3.5 Some comments and remarks

To give some conclusion that can be made from the numerical tests, we point out that in most cases when there is not probability factor we can see an increasing in the total expected income, intended as the maximum of function $-u$, as we summarized in Table 3.2 on the following page. Typically, this maximum increase if there is no probability factor in the income of second period, while decrease if the incumbent politician discounts the future stock of resources by the probability he wins power.

In cases 7 and 8, on the contrary, we deal with a combination of prices and salary cost that leads to a situation in which being in power for a second mandate is unprofitable. Therefore replace the probability factor with the greater value 1, on an equal extraction and employment rates as in this case, worsens the situation.

To conclude, we can see that the condition in which there is not probability factor typically leads to optimal strategies which amount to extract less resources, avoiding in a certain sense wastes, and to employ less workers for the benefit of more productive private sector. Moreover in most cases, except in the "pathological" ones, the maximum of the value function $-u$ is strictly greater than in the case with a generic probability factor. This means that, generally, the total income increases with respect to the strength of institutions, measured by the personal interests of the incumbent to remain in power.

Test	With probability factor	Without probability factor	% difference
1	89.4021	89.6312	0.2563
2	281.1765	281.2812	0.0372
3	94.7955	96.5512	1.8521
4	167.1259	188.2960	12.6672
5	135.0464	188.2960	39.4306
6	66.7895	89.4406	33.9142
7	326.3555	325.6142	-0.2271
8	1.9271e+03	1.9271e+03	0

Table 3.2: Summary table of maximum of function $-u$ in presence or not of the probability reelection factor in the tests made.

3.6 Generic dynamics and state constraints implementation

We recall that assumption (3.2) forces us to choose only maps f that keep the dynamics inside the square Q. In this section we want to drop this assumption in order to consider a more general f and to do this we can use two different approaches.

The first approach is a penalty argument. In optimization, two types of penalty function are commonly used: interior and exterior penalty functions. Considering an exterior penalty function, as for example in [14], we can take a function $\tilde{\rho} \in BUC(\mathbb{R}^2)$ such that $\tilde{\rho} \equiv 0$ in Q and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\tilde{\rho}(x) \geq \delta$ if $\mathrm{dist}(x, Q) \geq \varepsilon$. Then we can define the running cost

$$\tilde{\ell}(x,a) := \ell(x,a) + \frac{1}{\varepsilon}\tilde{\rho}(x)$$

and, in analogy with what has been done previously, the cost functional

$$\tilde{J}(x,t,\alpha) := \sum_{n=0}^{t-1} \tilde{\ell}(y_n, a_n)\beta^n + \beta^t g(y_t)$$

and the value function

$$\tilde{u}(x,t) = \inf_{\alpha \in \mathcal{A}} \tilde{J}(x,t,\alpha)$$

It is quite simple to check that this new value function satisfies the same Dynamic Programming Principle (3.5) as before, obviously with running cost $\tilde{\ell}$, and the discrete Bellman equation

$$\tilde{u}(x,t) = \inf_{a \in A} \left\{ \beta\tilde{u}(x + f(x,a), t-1) + \ell(x,a) + \frac{1}{\varepsilon}\tilde{\rho}(x) \right\}$$

for $x \in \mathbb{R}^2$ and $t \in \mathcal{T}$ or, using the supremum as it is common in the literature,

$$\tilde{u}(x,t) + \sup_{a \in A}\{-\beta\tilde{u}(x + f(x,a), t-1) - \ell(x,a)\} = \frac{1}{\varepsilon}\tilde{\rho}(x)$$

which is almost identical to (3.7) except for the term $\frac{1}{\varepsilon}\tilde{\rho}(x)$.

Conversely, using an interior penalty function, we can take a function $\bar{\rho} \in BUC(Q)$ which is nonnegative and increasing as the constraint boundary ∂Q is approached (see for example [46] for further details). Then we define the running cost

$$\bar{\ell}(x,a) := \ell(x,a) + \varepsilon\bar{\rho}(x)$$

and cost functional $\bar{J}$ and value function $\bar{u}$ in the usual way. It is easy to show that the value function satisfies the discrete Bellman equation

$$\bar{u}(x,t) = \inf_{a\in A} \{\beta\bar{u}(x + f(x,a), t-1) + \ell(x,a) + \varepsilon\bar{\rho}(x)\}$$

for $x \in Q$ and $t \in \mathcal{T}$.

The second approach is to consider, in order to guarantee that the trajectory never leaves the domain Q, only the control functions that achieve that requirement. In particular, as in [23], for every $x \in Q$ we define the set

$$\mathcal{A}(x) := \{\alpha \in \mathcal{A} : y_n(x,\alpha) \in Q \ \forall n \in \mathcal{T}_0\}$$

and we assume that

$$\mathcal{A}(x) \neq \emptyset \tag{3.17}$$

for every $x \in Q$. So the value function for the constrained problem is

$$u(x,t) := \inf_{\alpha\in\mathcal{A}(x)} J(x,t,\alpha)$$

One has that the value function u satisfies the Dynamic Programming Principle (3.5) and it is the unique constrained solution of

$$u(x,t) = \inf_{a\in A(x)} \{\beta u(x + f(x,a), t-1) + \ell(x,a)\}$$

for $x \in Q$ and $t \in \mathcal{T}$ in which $A(x) := \{a \in A : x + f(x,a) \in Q\}$.

Chapter 4

Mean field games and oil production

4.1 An overview on mean field games

Mean field games theory was developed in 2006 by Lasry and Lions [38, 39, 40], and independently in the engineering community by Caines, Huang and Malhamé [9, 10], and it is a branch of game theory. It is therefore a set of concept and mathematical tools which help to model situations of infinitely many agents who take decisions in a context of strategic interactions.

In this framework we consider a huge number of agents, that can be viewed as a continuum of economic agents, whose are not able to influence singularly the strategies of the other.

Under some assumptions, mean field games can be expressed as a coupled system of two equations, a Kolmogorov-Fokker-Planck equation, evolving forward in time, that governs the evolution of the density function of the agents, and a Hamilton-Jacobi-Bellman equation, evolving backward in time, that governs the computation of the optimal path for each agent. Let us now show how this two equations arise.

In any (quantitative) model there must always be an equation to express the optimization problem of each agent. Usually this involves one equation for each agent. If agents are grouped together by similar agent classes, there is one equation for every class of agents. This equation is generally a Bellman equation, since a large proportion of optimization problems falls within the framework of dynamic programming. Hence Hamilton-Jacobi-Bellman equations will be used to compute optimal behaviors.

An equation is also needed to express the group's behavior. When agents are atomized, the group is represented in the model by a distribution on the state space. The dynamics of the distribution is governed by a transport equation that can be called a Kolmogorov-Fokker-Plank equation.

Lastly, this two equations are mutually coupled. In fact in the transport equation the optimal behaviors of agents occur as data, since it is the infinite collection (the continuum) of individual behaviors that it is aggregated and which constitutes collective behavior. Vice versa in the optimization equation the density of the agents occur as data, since they choose its own optimal strategy by considering the group's configuration.

4.2 The oil production framework

A fairly typical example of mean field game is that of the production of an exhaustible resource by a continuum of producers. We know from a seminal article by Hotelling [32], published in 1931, that there is a rent involved in the production of an exhaustible resource, but it is interesting to examine this in greater depth in a competitive situation and to understand the dynamics of exhaustion of a scarce resource.

In 2010 Guéant, Larsy and Lions [29] published the first model of mean field games applied to this very old and studied question of exploitation of natural resources in which they consider a large number of oil producers, that can be viewed as companies, with the only assumption that there is a sufficiently large number of them that one can think as a continuum.

Each oil producer initially has a reserve, and all the reserves are distributed according to an initial density function. The reserve of the generic agent evolves deterministically according to an extraction rate and its target is to maximize an assigned income functional.

In 2011 the same authors proposed a slightly different model [30] in which the reserve evolves according to a stochastic differential equation. In the following we present a procedure to rigorously derive the partial differential equation that describe the second model, along with some new results and simple generalizations.

4.3 Model description

We consider a large number of oil producers, which can be viewed as macro standpoints or oil companies. The only assumptions we make, that characterize the mean field games approach, is that there is a sufficiently large number of agents such that we can think as a continuum of them, and perfect competition.

Each of these oil producers initially has a certain reserve and we assume that there reserves are distributed among producers according to an initial distribution $m(0, R) = m_0(R)$. Moreover the reserves will contribute to production q for every specific agent according to the dynamics

$$dR(t) = -q(t)dt + \nu R(t)dB_t \tag{4.1}$$

where the Brownian motion is specific to the considered agent and ν is a parameter representing the uncertainty, due for example to an incorrect measurement of resources or losses during the extraction process.

Production choices will be made in order to optimize a profit criterion (the same for every agent) of the form

$$\max_{q(s)} \mathbb{E}\left[\int_0^{+\infty} (p(s)q(s) - C(q(s)))e^{-rs}ds\right]$$

such that $q(t), R(t) \geq 0$, where che cost function C is quadratic, for example $C(q) = \alpha q + \frac{1}{2}\beta q^2$ with $\alpha, \beta > 0$, and the selling price $p(t)$ is determined according to the supply/demand equilibrium on the market at each time.

In this regard we model the demand by a function $D(t, p) = We^{\rho t}p^{-\sigma}$ where $We^{\rho t}$ denotes the total wealth, affecter by a constant growth rate which models the economic growth, and σ is a general indicator of the elasticity of substitution between oil and any other good. The supply is given instead by the total oil production of all the agents.

This model can be dealt with in the deterministic case or in the stochastic case, depending on the value of ν, namely $\nu = 0$ is the deterministic case while $\nu \neq 0$ is the stochastic one. In the first case a solution can be found without the mean field methods, so we focus only on the stochastic case and in the next section we rigorously derive the mean field games equations.

4.4 Mean field games equations

The tool to perform a rigorous derivation of equations describing the model is also in this case the dynamic programming. The first step is to define the value function and to do this we rewrite the problem using the formulation most common in literature.

We consider in fact agents whose aim is to optimize the following profit criterion of the form

$$\min_{q(s)} \mathbb{E}\left[\int_0^{+\infty} (C(q(s)) - p(s)q(s))e^{-rs}ds\right]$$

which is simply to minimize the opposite of the quantity considered before.

Indicated with

$$J(t,R,q) := \mathbb{E}\left[\int_t^{+\infty} (C(q(s)) - p(s)q(s))e^{-r(s-t)}ds\right]$$

the functional to minimize then we can define the value function

$$u(t,R) := \inf_{q(s), s\geq t} J(t,R,q)$$

and illustrate the Dynamic Programming Principle that it satisfies.

Proposition 4.1 (Dynamic Programming Principle)**.** *The function $u(t,R)$ verifies*

$$u(t,R) = \inf_{q(s),s\geq t} \mathbb{E}\left[\int_t^{\tau} (C(q(s)) - p(s)q(s))e^{-r(s-t)}ds + \right. \\ \left. + e^{-r(\tau-t)}u(\tau, R(\tau))\right] \tag{4.2}$$

for all $\tau \geq t$.

Proof. We show separately that inequalities $\leq$ and $\geq$ hold, starting from the second one. Consider the equality

$$\underbrace{\int_t^{+\infty} (C(q(s)) - p(s)q(s))e^{-r(s-t)}ds}_{I} = \underbrace{\int_t^{\tau} (C(q(s)) - p(s)q(s))e^{-r(s-t)}ds}_{I_1} + \\ + \int_\tau^{+\infty} (C(q(s)) - p(s)q(s))e^{-r(s-t)}ds$$

and perform the linear changing of variable $z = s - (\tau - t) = s - \tau + t$ that is $s = z + \tau - t$. Then we get

$$I = I_1 + \int_t^{+\infty} (C(q(z+\tau-t)) - p(z+\tau-t)q(z+\tau-t))e^{-r(z+\tau-2t)}dz \tag{4.3}$$

By exploiting a well known property of powers we have

$$e^{-r(z+\tau-2t)} = e^{-r(\tau-t)}e^{-r(z-t)}$$

and then (4.3) becomes

$$I = I_1 + e^{-r(\tau-t)} \int_t^{+\infty} (C(q(z+\tau-t)) - p(z+\tau-t)q(z+\tau-t))e^{-r(z-t)}dz$$

Performing now the inverse change of variable $s = z + (\tau - t)$ so $z = s - \tau + t$ we get

$$I = I_1 + e^{-r(\tau-t)} \int_\tau^{+\infty} (C(q(s)) - p(s)q(s))e^{-r(s-\tau)}ds$$

Taking now the expected value in both sides of last equality and using the linearity we have

$$J(t, R, q) = \mathbb{E}[I_1] + e^{-r(\tau-t)}J(\tau, R(\tau), q) \geq \mathbb{E}[I_1] + e^{-r(\tau-t)}u(\tau, R(\tau))$$

Finally taking the infimum over $q(s)$ it results

$$\begin{aligned} u(t, R) &\geq \inf_{q(s), s\geq t} \left\{ \mathbb{E}[I_1] + e^{-r(\tau-t)}u(\tau, R(\tau)) \right\} = \\ &= \inf_{q(s), s\geq t} \mathbb{E} \left[\underbrace{\int_t^\tau (C(q(s)) - p(s)q(s))e^{-r(s-t)}ds}_{I_1} + e^{-r(\tau-t)}u(\tau, R(\tau)) \right] \end{aligned}$$

Regarding the other inequality we fix $\varepsilon > 0$ and a control q_1 such that

$$u(\tau, R(\tau)) + \varepsilon \geq J(\tau, R(\tau), q_1) \tag{4.4}$$

and define the control

$$\overline{q}(s) = \begin{cases} q(s) & t \leq s \leq \tau \\ q_1(s) & s > \tau \end{cases}$$

We have

$$\begin{aligned} u(t, R) \leq J(t, R, \overline{q}) &= \mathbb{E} \left[\underbrace{\int_t^{+\infty} (C(\overline{q}(s)) - p(s)\overline{q}(s))e^{-r(s-t)}ds}_{I_2} \right] = \\ &= \mathbb{E} \left[\int_t^\tau (C(q(s)) - p(s)q(s)))e^{-r(s-t)}ds \right] + \\ &\quad + \mathbb{E} \left[\int_\tau^{+\infty} (C(q_1(s)) - p(s)q_1(s))e^{-r(s-t)}ds \right] \end{aligned}$$

We operate the same change of variable used previously $z = s - \tau + t$ to obtain

$$\begin{aligned}
\mathbb{E}[I_2] = \mathbb{E}[I_1] + \mathbb{E}\Bigg[\int_t^{+\infty} &(C(q_1(z+\tau-t)) - \\
&- p(z+\tau-t)q_1(z+\tau-t))e^{-r(z+\tau-2t)}dz\Bigg] = \\
= \mathbb{E}[I_1] + \mathbb{E}\Bigg[e^{-r(\tau-t)}\int_t^{+\infty} &(C(q_1(z+\tau-t)) - \\
&- p(z+\tau-t)q_1(z+\tau-t))e^{-r(z-t)}dz\Bigg]
\end{aligned}$$

Performing the inverse change of variable $s = z + \tau - t$ we get now

$$\begin{aligned}
\mathbb{E}[I_2] &= \mathbb{E}[I_1] + \mathbb{E}\left[e^{-r(\tau-t)}\int_\tau^{+\infty}(C(q_1(s)) - p(s)q_1(s))e^{-r(s-\tau)}ds\right] = \\
&= \mathbb{E}[I_1] + e^{-r(\tau-t)}J(\tau, R(\tau), q_1) \le \\
&\le \mathbb{E}[I_1] + e^{-r(\tau-t)}(u(\tau, R(\tau)) + \varepsilon)
\end{aligned}$$

where the last inequality was obtained thanks to (4.4). Since q and ε are arbitrary and the expected value is linear we get the desired inequality by taking the infimum in the last one.

□

By exploiting the dynamic programming approach we can now show in a rigorous way that the value function satisfies a certain Hamilton-Jacobi-Bellman, noting that due to the presence of a stochastic term it is a second order equation.

Proposition 4.2. *The value function $u(t, R)$ is a viscosity solution of the Hamilton-Jacobi-Bellman equation*

$$-u_t - \frac{\nu^2}{2}R^2 u_{RR} + ru + \sup_{q\ge 0}\{pq - C(q) + qu_R\} = 0 \tag{4.5}$$

Proof. We show that u is simultaneously viscosity subsolution and supersolution, starting from the first one. Let $\varphi \in C^{1,2}$ and (t, R) be a local maximum point of $u - \varphi$, namely for some $\rho > 0$ such that $|t-s|, |R-Z| < \rho$ one has

$$u(t, R) - \varphi(t, R) \ge u(s, Z) - \varphi(s, Z)$$

and so

$$\varphi(t, R) - \varphi(s, Z) \le v(t, R) - v(s, Z)) \tag{4.6}$$

Fix a constant control $q(t) \equiv q$ and let $R(t)$ be the solution of (4.1) with that control, which has the property that for τ small enough one has $|R(\tau) - R| < \rho$. By taking $(s, Z) = (\tau, R(\tau))$, we have

$$\varphi(t, R) - \varphi(\tau, R(\tau)) \le v(t, R) - v(\tau, R(\tau))$$

By using the inequality $\leq$ in the Dynamic Programming Principle (4.2) we have

$$\varphi(t,R)-\varphi(\tau,R(\tau))\leq \mathbb{E}\left[\int_t^\tau (C(q)-p(s)q)e^{-r(s-t)}ds+\right.$$
$$\left.+(e^{-r(\tau-t)}-1)u(\tau,R(\tau))\right]$$

We divide both sides by $\tau-t$ and pass to the limit as $\tau\to t$ to obtain

$$-\frac{d}{ds}\varphi(s,R(s))\bigg|_{s=t}\leq C(q)-p(t)q-ru(t,R)$$

Expanding the derivative on the left-hand side by using Itō's formula one has

$$-\varphi_t+q\varphi_R-\frac{\nu^2}{2}R^2\varphi_{RR}-\nu R\varphi_R dB_t\leq C(q)-pq-ru$$

We take now the expected value and since dB_t has zero mean we get

$$-\varphi_t-\frac{\nu^2}{2}R^2\varphi_{RR}+ru+pq-C(q)+q\varphi_R\leq 0$$

and since q is arbitrary we finally have

$$-\varphi_t-\frac{\nu^2}{2}R^2\varphi_{RR}+ru+\sup_{q\geq 0}\{pq-C(q)+q\varphi_R\}\leq 0$$

which means that u is a viscosity subsolution. For the second part of the proof let (t,R) be a local minimum point of $u-\varphi$ then (4.6) holds with the opposite inequality. For all $\varepsilon>0$ and $\tau>t$, by the inequality $\geq$ in the Dynamic Programming Principle (4.2) there exists a control $\widetilde{q}$ depending on ε and $\tau-t$ such that

$$u(t,R)\geq \mathbb{E}\left[\int_t^\tau (C(\widetilde{q}(s))-p(s)\widetilde{q}(s))e^{-r(s-t)}ds+\right.$$
$$\left.+\,e^{-r(\tau-t)}u(\tau,\widetilde{R}(\tau))-(\tau-t)\varepsilon\right]$$

in which $\widetilde{R}(s)$ is the solution of the stochastic differential equation (4.1) with corresponding control $\widetilde{q}(s)$. Taking $(s,Z)=(\tau,\widetilde{R}(\tau))$ and using the linearity of the expected value one has

$$\varphi(t,R)-\varphi(\tau,\widetilde{R}(\tau))\geq \mathbb{E}\left[\int_t^\tau (C(\widetilde{q}(s))-p(s)\widetilde{q}(s))e^{-r(s-t)}ds\right]+$$
$$+\,(e^{-r(\tau-t)}-1)u(\tau,\widetilde{R}(\tau))-(\tau-t)\varepsilon$$

We now expand the left-hand side of the last inequality. We have

$$
\begin{aligned}
\varphi(t,R)-\varphi(\tau,\widetilde{R}(\tau)) &= -\int_t^\tau \frac{d}{ds}\varphi(s,\widetilde{R}(s))ds = \\
&= \int_t^\tau \Big(-\varphi_t(s,\widetilde{R}(s)) + \widetilde{q}(s)\varphi_R(s,\widetilde{R}(s)) - \\
&\quad - \frac{\nu^2}{2}\widetilde{R}^2(s)\varphi_{RR}(s,\widetilde{R}(s))\Big)\,ds - \int_t^\tau \nu\widetilde{R}(s)\varphi_R(s,\widetilde{R}(s))dB_s = \\
&= \int_t^\tau \Big(-\varphi_t(s,\widetilde{R}(s)) + \widetilde{q}(s)\varphi_R(s,R) - \\
&\quad - \frac{\nu^2}{2}\widetilde{R}^2(s)\varphi_{RR}(s,\widetilde{R}(s))\Big)\,ds - \int_t^\tau \nu\widetilde{R}(s)\varphi_R(s,\widetilde{R}(s))dB_s + \\
&\quad + o(\tau-t)
\end{aligned}
$$

where in the last step we used that $\varphi \in C^{1,2}$ and the property that the difference $\widetilde{R}(s)-R(t)$ with $s\in[t,\tau]$ goes to zero as $\tau-t\to 0$. Therefore we have

$$
\begin{aligned}
&\int_t^\tau \Big(-\varphi_t(s,\widetilde{R}(s)) + \widetilde{q}(s)\varphi_R(s,R) - \frac{\nu^2}{2}\widetilde{R}^2(s)\varphi_{RR}(s,\widetilde{R}(s))\Big)\,ds - \\
&\qquad - \int_t^\tau \nu\widetilde{R}(s)\varphi_R(s,\widetilde{R}(s))dB_s - \\
&\qquad - \mathbb{E}\left[\int_t^\tau (C(\widetilde{q}(s)) - p(s)\widetilde{q}(s))e^{-r(s-t)}ds\right] + \\
&\qquad + (1-e^{-r(\tau-t)})u(\tau,\widetilde{R}(\tau)) \geq \\
&\qquad\qquad \geq (\tau-t)\varepsilon + o(\tau-t)
\end{aligned}
$$

We take now the expected value in both sides and add and subtract the term

$$
\mathbb{E}\left[\int_t^\tau (p(s)\widetilde{q}(s) - C(\widetilde{q}(s)))ds\right]
$$

to obtain

$$
\begin{aligned}
&\mathbb{E}\left[\int_t^\tau (p(s)\widetilde{q}(s) - C(\widetilde{q}(s)) + \widetilde{q}(s)\varphi_R(s,R))ds\right] + \\
&\qquad + \mathbb{E}\left[\int_t^\tau \Big(-\varphi_t(s,\widetilde{R}(s)) - \frac{\nu^2}{2}\widetilde{R}^2(s)\varphi_{RR}(s,\widetilde{R}(s))\Big)\,ds\right] - \\
&\qquad - \mathbb{E}\left[\int_t^\tau \nu\widetilde{R}(s)\varphi_R(s,\widetilde{R}(s))dB_s\right] + \\
&\qquad + \mathbb{E}\left[\int_t^\tau (p(s)\widetilde{q}(s) - C(\widetilde{q}(s)))(e^{-r(s-t)}-1)ds\right] + \\
&\qquad + (1-e^{-r(\tau-t)})u(\tau,\widetilde{R}(\tau)) \geq \\
&\qquad\qquad \geq (\tau-t)\varepsilon + o(\tau-t)
\end{aligned}
$$

Again we would like to divide by $\tau - t$ and let τ to go to t and to do this we note that the fist integrand is estimated from above by

$$
\sup_{q\geq 0}\{pq - C(q) + q\varphi_R\}
$$

The second integral becomes a deterministic quantity by taking the expected value, the third one is zero thanks to a property of Itō's integral since the integrand function verifies some suitable hypotheses and the last one is $o(\tau - t)$ because $pq - C(q)$ is bounded if q belongs to a compact interval. Performing the desired operation we get

$$-\varphi_t - \frac{\nu^2}{2}R^2\varphi_{RR} + ru + \sup_{q\geq 0}\{pq - C(q) + q\varphi_R\} \geq \varepsilon$$

and since ε is arbitrary the proof that u is a viscosity supersolution is complete.

□

In the next proposition we show, again in a rigorous way, that the density function $m(t,R)$ of the reserves is transported by the optimal extraction rate according to a Kolmogorov-Fokker-Plank equation. The proof basically follows the ideas contained in [28].

Proposition 4.3. *The density function $m(t,R) \in C^{1,2}$ satisfies the Kolmogorov-Fokker-Planck equation*

$$\partial_t m + \partial_R(-qm) = \frac{\nu^2}{2}\partial^2_{RR}(R^2 m)$$

Proof. Let $f \in C^2$ and $R(t)$ be the solution of the stochastic differential equation (4.1) with a generic control $q(t,R)$ fixed. Using the Itō's formula one has

$$df(R(t)) = \left(-q(t,R(t))f'(R(t)) + \frac{\nu^2}{2}R^2(t)f''(R(t))\right)dt + \nu R(t)f'(R(t))dB_t$$

Taking the expected value with respect to the density m, so the last addend vanishes because Brownian motion has zero mean, and using the fact that the expected value of f' equals the derivative of the expected value of f we have

$$\frac{\partial}{\partial t}\int_{\mathbb{R}} f(R)m(t,R)dR = \int_{\mathbb{R}} -q(t,R)f'(R)m(t,R)dR + \frac{\nu^2}{2}\int_{\mathbb{R}} R^2 f''(R)m(t,R)dR$$

Performing some integrations by parts at the right-hand side and interchanging the order of differentiation at the left-hand side we obtain

$$\int_{\mathbb{R}} f(R)m_t(t,R)dR = \int_{\mathbb{R}} f(R)\partial_R(q(t,R)m(t,R))dR + \frac{\nu^2}{2}\int_{\mathbb{R}} f(R)\partial_{RR}(R^2 m(t,R))dR$$

and since the equality has to be satisfied for all f we finally have that

$$\partial_t m = \partial_R(qm) + \frac{\nu^2}{2}\partial^2_{RR}(R^2 m)$$

which is the thesis.

□

4.5 Interdependence of equations

Unlike the typical cases in which the interdependence of equations is explicit, namely in the transport equation the optimal behavior occurs as a data and vice versa, in this case we deal with a more complicated interdependence.

First of all m depends on u through the optimal extraction rate. Since the Hamiltonian is given by

$$\sup_{q\geq 0}\{pq - C(q) + qu_R\}$$

and since the verification theorems of the next section say that a solution of the Hamilton-Jacobi-Bellman equation (4.5) is also a solution of the optimization problem, if we keep considering the quadratic cost introduced before, then the optimal control is given by

$$q^*(t,R) = \left(\frac{p(t) - \alpha + \partial_R u(t,R)}{\beta}\right)^+$$

where $q^*(t,R)$ represents the optimal instantaneous extraction at time t of a producer with reserve R at this time. So equation (4.5) can be rewritten as

$$-u_t - \frac{\nu^2}{2}R^2 u_{RR} + ru + \frac{1}{2\beta}\left((p - \alpha + u_R)^+\right)^2 = 0$$

Moreover there is also a coupling through the price. In fact equation we recall that price $p(t)$ is determined by a global equilibrium between supply and demand. Since supply depends on the global production of the agents and then on the distribution of reserves, it can be seen as the time derivative, with correct sign, of the total reserve, which is

$$-\frac{d}{dt}\int_{\mathbb{R}} Rm(t,R)dR$$

So mathematically the equilibrium is achieved when the price is expressed by

$$p(t) = D(t,\cdot)^{-1}\left(-\frac{d}{dt}\int_{\mathbb{R}} Rm(t,R)dR\right)$$

To conclude this aspect of the model, we can see that the two equations are coupled thought optimal production and price in a way such that Hamilton-Jacobi-Bellman equation becomes

$$-u_t - \frac{\nu^2}{2}R^2 u_{RR} + ru + \frac{1}{2\beta}\left(\left(D(t,\cdot)^{-1}\left(-\frac{d}{dt}\int_{\mathbb{R}} Rm(t,R)dR\right) - \alpha + u_R\right)^+\right)^2 = 0$$

while the Kolmogorov-Fokker-Plank equation becomes

$$\partial_t m + \partial_R\left(-\left(\frac{D(t,\cdot)^{-1}\left(-\frac{d}{dt}\int_{\mathbb{R}} Rm(t,R)dR\right) - \alpha + \partial_R u(t,R)}{\beta}\right)^+ m\right) = \\ = \frac{\nu^2}{2}\partial^2_{RR}(R^2 m)$$

4.6 Verification theorems

In this section we present two verification theorems for this model, obtained following the ideas contained in [57]. This theorems say that a solution of the Hamilton-Jacobi-Bellman equation (4.5) is actually the required value function, and also give a condition to establish when a control is optimal. The first theorem deals with classical solutions.

Theorem 4.4 (Verification theorem for classical solutions). *Let $u \in C^{1,2}$ be a solution of the Hamilton-Jacobi-Bellman equation* (4.5) *Then one has*

$$u(t,R) \leq J(t,R,q)$$

Moreover $(q^*(\cdot), R^*(\cdot))$ is a pair of optimal control and trajectory if and only if

$$\begin{aligned}(4.7)\quad -u_t(\tau,R^*(\tau)) - \frac{\nu^2}{2}(R^*)^2(\tau)u_{RR}(\tau,R^*(\tau)) + ru(\tau,R^*(\tau)) + \\ + p(\tau)q^*(\tau) - C(q^*(\tau)) + q^*(\tau)u_R(\tau,R^*(\tau)) = 0\end{aligned}$$

for all $\tau \geq t$.

Proof. For all $q(\cdot)$ with corresponding trajectory $R(\cdot)$ one has by Itō's formula

$$\begin{aligned}-e^{r(s-t)}\frac{d}{ds}\left(e^{-r(s-t)}u(s,R(s))\right) &= -e^{r(s-t)}\Bigg(- re^{-r(s-t)}u(s,R(s)) + \\ &+ e^{-r(s-t)}\Bigg(u_t(s,R(s)) - q(s)u_R(s,R(s)) + \\ &+ \frac{\nu^2}{2}R^2(s)u_{RR}(s,R(s)) + \\ &+ \nu R(s)u_R(s,R(s))dB_s\Bigg)\Bigg) = \\ &= ru(s,R(s)) - u_t(s,R(s)) + q(s)u_R(s,R(s)) - \\ &- \frac{\nu^2}{2}R^2(s)u_{RR}(s,R(s)) - \nu R(s)u_R(s,R(s))dB_s\end{aligned}$$

Since u is a solution the inequality

$$-u_t - \frac{\nu^2}{2}R^2u_RR + ru + pq - C(q) + qu_R \leq 0$$

holds for all $q > 0$ and therefore, by the previous formula, we have

$$-e^{r(s-t)}\frac{d}{ds}\left(e^{-r(s-t)}u(s,R(s))\right) \leq -p(s)q(s) + C(q(s)) - \nu R(s)u_R(s,R(s))dB_s$$

We multiply by $-e^{-r(s-t)}$, take the expectation, integrate in ds from t to $+\infty$ both sides getting

$$\int_t^{+\infty}\frac{d}{ds}\left(e^{-r(s-t)}v(s,R(s))\right)ds \geq \int_t^{+\infty}(p(s)q(s) - C(q(s)))e^{-r(s-t)}ds$$

Taking again the expected value and changing sign one has

$$u(t,R) \leq J(t,R,q(\cdot)) \quad \forall q(\cdot)$$

Now let $(q^*(\cdot), R^*(\cdot))$ be such that (4.7) holds. One has

$$\begin{aligned} u(t,R) = &-\int_t^{+\infty} \frac{d}{ds}\left(e^{-r(s-t)}u(s,R^*(s))\right) ds = \\ = &\int_t^{+\infty}\Big(-u_t(s,R^*(s)) - \frac{\nu^2}{2}(R^*)^2(s)u_{RR}(s,R^*(s)) + \\ &+ ru(s,R^*(s)) + q^*(s)u_R(s,R^*(s))\Big)\, e^{-r(s-t)}ds + \\ &+ \int_t^{+\infty} -\nu R^*(s)u_R(s,R^*(s))e^{-r(s-t)}dB_s \end{aligned}$$

On the right-hand side we add and subtract the quantity

$$\int_t^{+\infty} (p(s)q^*(s) - C(q^*(s)))e^{-r(s-t)}ds$$

so the first integrand becomes exactly the expression in (4.7) that vanishes for assumption. Then it follows the equality

$$\begin{aligned} u(t,R) = &-\int_t^{+\infty} (p(s)q^*(s) - C(q^*(s)))e^{-r(s-t)}ds + \\ &+ \int_t^{+\infty} -\nu R^*(s)u_R(s,R^*(s))e^{-r(s-t)}dB_s \end{aligned}$$

Taking the expected value we have

$$u(t,R) = J(t,R,q^*(\cdot))$$

and therefore the control $q^*(\cdot)$ is the optimal one because it realizes the infimum.

□

The second one is instead a verification theorem for viscosity solutions.

Theorem 4.5 (Verification theorem for viscosity solutions)**.** *Let $u \in C$ be a viscosity solution of Hamilton-Jacobi-Bellman equation* (4.5)*. Therefore one has*

$$u(t,R) \leq J(t,R,q) \tag{4.8}$$

Moreover, fix (t,R) and let $(q^*(\cdot), R^*(\cdot))$ be a pair of admissible control and trajectory such that there exists $\varphi \in C^{1,2}$ that verifies

$$\begin{aligned} -\varphi_t(\tau,R^*(\tau)) - \frac{\nu^2}{2}(R^*)^2(\tau)\varphi_{RR}(\tau,R^*(\tau)) + ru(\tau,R^*(\tau)) + \\ + p(\tau)q^*(\tau) - C(q^*(\tau)) + q^*(\tau)\varphi_R(\tau,R^*(\tau)) \geq 0 \end{aligned} \tag{4.9}$$

and such that $(\tau, R^*(\tau))$ is a maximum point of $u - \varphi$ both for every $\tau \geq t$. Then $(q^*(\cdot), R^*(\cdot))$ is a pair of optimal control and trajectory.

Remark 4.6. *Since* (4.9) *does not depend on the value of φ at $(\tau, R^*(\tau))$ it is not restrictive to assume that u and φ take the same value at the maximum points $(\tau, R^*(\tau))$.*

Proof of Theorem 4.5. The first part follows from the uniqueness of solution to equation (4.5) that holds under standard assumptions. We now prove the second part. By definition of maximum point one has

$$u(\tau+h, R^*(\tau+h)) - \varphi(\tau+h, R^*(\tau+h)) \leq u(\tau, R^*(\tau)) - \varphi(\tau, R^*(\tau))$$

that is

$$u(\tau+h, R^*(\tau+h)) - u(\tau, R^*(\tau)) \leq \varphi(\tau+h, R^*(\tau+h)) - \varphi(\tau, R^*(\tau))$$

We add both sides the expression

$$\left(e^{-r(\tau+h-t)} - 1\right)\varphi(\tau+h, R^*(\tau+h)) + \left(1 - e^{-r(\tau-t)}\right)\varphi(\tau, R^*(\tau))$$

We now perform the calculations separately, starting from the right-hand side. We have

$$\begin{aligned}
&\varphi(\tau+h, R^*(\tau+h)) - \varphi(\tau, R^*(\tau)) + \\
&\quad + \left(e^{-r(\tau+h-t)} - 1\right)\varphi(\tau+h, R^*(\tau+h)) + \left(1 - e^{-r(\tau-t)}\right)\varphi(\tau, R^*(\tau)) = \\
&\quad = e^{-r(\tau+h-t)}\varphi(\tau+h, R^*(\tau+h)) - e^{-r(\tau-t)}\varphi(\tau, R^*(\tau)) = \\
&\quad = \int_\tau^{\tau+h} \frac{d}{ds}\left(e^{-r(s-t)}\varphi(s, R^*(s))\right) ds = \\
&= \int_\tau^{\tau+h} e^{-r(s-t)} \Big(- r\varphi(s, R^*(s)) + \varphi_t(s, R^*(s)) - q^*(s)\varphi_R(s, R^*(s)) + \\
&+ \frac{\nu^2}{2}(R^*)^2(s)\varphi_{RR}(s, R^*(s))\Big) ds + \int_\tau^{\tau+h} \nu R^*(s)\varphi_R(s, R^*(s)) e^{-r(s-t)} dB_s = \\
&= h e^{-r(\tau-t)} \Big(- r\varphi(\tau, R^*(\tau)) + \varphi_t(\tau, R^*(\tau)) - q^*(\tau)\varphi_R(\tau, R^*(\tau)) \\
&+ \frac{\nu^2}{2}(R^*)^2(\tau)\varphi_{RR}(\tau, R^*(\tau))\Big) + \int_\tau^{\tau+h} \nu R^*(s)\varphi_R(s, R^*(s)) e^{-r(s-t)} dB_s + o(h)
\end{aligned}$$

where the last equality holds if τ is a Lebesgue point of the integrand. Take the expected value, divide by h, pass to the limit inferior as $h \to 0$, add and subtract the term

$$(p(\tau)q^*(\tau) - C(q^*(\tau)))e^{-r(\tau-t)}$$

and integrate from t to $+\infty$ in $d\tau$, so the right-hand side is equal to

$$\begin{aligned}
(4.10) \quad &\int_t^{+\infty} e^{-r(\tau-t)} \Big(\varphi_t(\tau, R^*(\tau)) + \frac{\nu^2}{2}(R^*)^2(\tau)\varphi_{RR}(\tau, R^*(\tau)) - r\varphi(\tau, R^*(\tau)) - \\
&\quad - p(\tau)q^*(\tau) + C(q^*(\tau)) - q^*(\tau)\varphi_R(\tau, R^*(\tau))\Big) d\tau + \\
&\quad + \int_t^{+\infty} (p(\tau)q^*(\tau) - C(q^*(\tau)))e^{-r(\tau-t)} d\tau \leq -J(t, R, q^*)
\end{aligned}$$

where the last inequality follows, after taking again the expectation, from Remark 4.6 and inequality (4.9). Consider now the left-hand side and first of all note that it consists by only deterministic quantities so taking the expected value has no practical effect. One has, again from Remark 4.6,

$$\begin{aligned} & u(\tau+h, R^*(\tau+h)) - u(\tau, R^*(\tau)) + \\ & \quad + \left(e^{-r(\tau+h-t)} - 1\right) \varphi(\tau+h, R^*(\tau+h)) + \left(1 - e^{-r(\tau-t)}\right) \varphi(\tau, R^*(\tau)) = \\ & \qquad = e^{-r(\tau+h-t)} u(\tau+h, R^*(\tau+h)) - e^{-r(\tau-t)} u(\tau, R^*(\tau)) \end{aligned}$$

We divide by h, pass to the superior limit as $h \to 0$ and integrate from t and $+\infty$ in $d\tau$ obtaining that the right-hand side is equal to

$$\begin{aligned} & \int_t^{+\infty} \limsup_{h\to 0} \frac{e^{-r(\tau+h-t)} u(\tau+h, R^*(\tau+h)) - e^{-r(\tau-t)} u(\tau, R^*(\tau))}{h} = \\ & = e^{rt} \int_t^{+\infty} \limsup_{h\to 0} \frac{e^{-r(\tau+h)} u(\tau+h, R^*(\tau+h)) - e^{-r\tau} u(\tau, R^*(\tau))}{h} \geq \\ & \quad \geq e^{rt} \left[e^{-r\tau} u(\tau, R^*(\tau))\right]_{\tau=t}^{\tau=+\infty} = -u(t,R) \end{aligned} \tag{4.11}$$

where the inequality follow from Fatou's lemma. Then, combining (4.10) and (4.11) and changing sign we get

$$u(t,R) \geq J(t,R,q^*)$$

and so, from (4.8), one has

$$u(t,R) = J(t,R,q^*)$$

and therefore q^* is optimal because it realizes the infimum.

□

4.7 Possible variations to the model

We conclude the work on this model by presenting some changes that can be made.

4.7.1 Depletion time of resources

In this variation we consider

$$t_0 = t_0(t,R,q) := \inf\{s \geq t \text{ s.t. } R(s) = 0\}$$

where $R(\cdot)$ solves the stochastic differential equation (4.1). Roughly speaking, t_0 is the first time the trajectory $R(\cdot; q(\cdot))$ hits the given target $\mathcal{T} = \{0\}$, which corresponds to the depletion of reserves. The cost to minimize is then

$$J(t,R,q) := \mathbb{E} \int_t^{t_0} (C(q(s)) - p(s)q(s)) e^{-r(s-t)} ds + e^{-r(t_0-t)} g(t_0, R(t_0))$$

with some exit cost g if $t_0 < +\infty$, or the same as before if $t_0 = +\infty$. The value function is also in this case

$$u(t,R) = \inf_{q(s), s\geq t} J(t,R,q(s))$$

We now show the Dynamic Programming Principle and the Hamilton-Jacobi-Bellman equation satisfied by this value function.

Proposition 4.7 (Dynamic Programming Principle)**.** *The value function $u(t,R)$ verifies*

$$u(t,R) = \inf_{q(s), s\geq t} \mathbb{E}\left[\int_t^{\tau\wedge t_0} (C(q(s)) - p(s)q(s))e^{-r(s-t)}ds + + e^{-r((\tau\wedge t_0)-t)} u(\tau\wedge t_0, R(\tau\wedge t_0))\right] \tag{4.12}$$

for all $\tau \geq t$.

The proof is essentially the same as the proof of Proposition 4.1 and we omit it.

Remark 4.8. *Note that* (4.12) *can be rewritten as*

$$u(t,R) = \inf_{q(s), s\geq t} \mathbb{E}\left[\int_t^{\tau\wedge t_0} (C(q(s)) - p(s)q(s))e^{-r(s-t)}ds + + e^{-r(\tau-t)}\chi_{\{\tau<t_0\}} u(\tau, R(\tau)) + e^{-r(t_0-t)}\chi_{\{\tau\geq t_0\}} g(t_0, R(t_0))\right]$$

Proposition 4.9. *The value function $u(t,R)$ is a viscosity solution of the Hamilton-Jacobi-Bellman equation*

$$-u_t - \frac{\nu^2}{2}R^2 u_{RR} + ru + \sup_q\{pq - C(q) + qu_R\} = 0 \text{ in } [0,+\infty)\times\mathcal{T}^c$$

The proof is the same as that of Proposition 4.2 and we omit it.

4.7.2 Introduction of state constraints

The aim of this variation is to force the trajectory to stay in a given set for all time. To do this we consider two different approaches, in the first one we deal with a restriction over control $q(\cdot)$ while in the second one we introduce a sort of penalization term in the cost functional.

More precisely, in addition to the obvious restriction $0 \leq q(\cdot) \leq q_0$, i.e. the quantity of oil extracted is a nonnegative number and it is less or equal to a limit q_0 depending on the extraction technology used, we want to consider the additional restriction $q(s) \leq R(s)$, i.e. the quantity extracted can not be greater than the available reserves.

Following the first approach we introduce the set of admissible controls

$$\mathcal{Q}_{t,R} := \{q(s) \in \mathcal{Q} \text{ s.t. } R(s,q) \geq 0 \ \forall s \geq t\}$$

where $R(s,q)$ is the solution of (4.1). We assume that for every (t,R) we have

$$\mathcal{Q}_{t,R} \neq \emptyset$$

The cost functional to minimize is the same but the value function becomes

$$u(t,R) := \inf_{q(\cdot)\in\mathcal{Q}_{t,R}} \mathbb{E}\int_t^{+\infty} (C(q(s)) - p(s)q(s))e^{-r(s-t)}ds$$

By standard arguments we can deduce that the value function satisfies the same Dynamic Programming Principle and Hamilton-Jacobi-Bellman equation as before.

We derive now a differential boundary condition satisfied by the value function when a state constraint is imposed. To derive de boundary condition formally assume that $u \in C^{1,2}$ and $q^* \in \mathcal{Q}_{t,R}$ is an optimal control. Then u satisfies

$$-u_t - \frac{\nu^2}{2}R^2 u_{RR} + ru + \sup_q\{pq - C(q) + qu_R\} = 0 \tag{4.13}$$

in the classical sense and

$$\sup_q\{pq - C(q) + qu_R\} = p(t)q^*(t,R) - C(q^*(t,R)) + q^*(t,R)u_R(t,R) \tag{4.14}$$

We observe now that, for every control, the state constraint imposes

$$-q(t,x)\cdot n(x) \leq 0 \tag{4.15}$$

for $x = 0$, where $n(x)$ is the exterior normal to $(0,+\infty)$ at x.

Remark 4.10. *In this case, since $n(0) = -1$ we simply have $0 \leq q(t,0) \leq 0$ at $x = 0$, which obviously implies $q(t,0) = 0$ in according to the fact that we are dealing with a nonrenewable resource.*

For any $\beta \geq 0$ from (4.15) one has

$$\begin{aligned} p(t)q^*(t,x) - C(q^*(t,x)) + q^*(t,x)v_R(t,x) \leq \\ \leq p(t)q^*(t,x) - C(q^*(t,x)) + q^*(t,x)(v_R(t,x) + \beta n(x)) \end{aligned} \tag{4.16}$$

for $x = 0$. Note now that for all $\varphi \in C^\infty([0,+\infty)\times[0,+\infty))$ such that $v - \varphi$ has a minimum at a boundary point $(t,x) \in [0,+\infty)\times\{0\}$, by the Lagrange multiplier rule we have $\varphi_t(t,x) \leq v_t(t,x)$, $\varphi_R(t,x) = v_R(t,x) + \beta n(x)$ for some $\beta \geq 0$ and $\varphi_{RR}(t,x) \leq v_{RR}(t,x)$. Combining (4.13), (4.14) and (4.16) one has

$$-\varphi_t(t,x) - \frac{\nu^2}{2}x^2\varphi_{RR}(t,x) + ru(t,x) + \sup_q\{p(t)q - C(q) + q\varphi_R(t,x)\} \geq 0$$

We can now introduce a new definition of viscosity solution in presence of state constraints

Definition 4.11. *$u \in C$ is a viscosity supersolution of (4.13) in $[0,+\infty)\times[0,+\infty)$ if the last inequality holds for any $\varphi \in C^{1,2}(\mathbb{R}\times\mathbb{R})$ such that $u - \varphi$ has a local minimum point at $(t,R) \in [0,+\infty)\times[0,+\infty)$*

Definition 4.12. *$u \in C$ is a constrained viscosity solution of (4.13) in $[0,+\infty)\times[0,+\infty)$ if it is a subsolution in $(0,+\infty)\times(0,+\infty)$ and a supersolution in $[0,+\infty)\times[0,+\infty)$*

To follow the second approach we consider a penalty term h with the property that $h \equiv 0$ on $[0,+\infty)$ and

$$\forall\varepsilon > 0\ \exists\delta > 0 \text{ such that } h(x) \geq \delta \text{ if } d(x,[0,+\infty)) \geq \varepsilon$$

In other words h is null as soon as the trajectory remains inside $[0,+\infty)$ otherwise it assumes positive values. We now define the cost functional

$$J(t,R,q) := \mathbb{E}\int_t^{+\infty}(C(q(s)) - p(s)q(s) + \frac{1}{\varepsilon}h(R(s)))e^{-r(s-t)}ds$$

and with few adjustments we can prove that the value function u satisfies the same dynamic programming principle and the equation

$$-u_t - \frac{\nu^2}{2} R^2 u_{RR} + ru + \sup_q \{pq - C(q) + qu_R\} = \frac{1}{\varepsilon} h$$

in the viscosity sense.

To model the fact that reserves can not be a negative quantity we can use, for example, the penalization

$$h(x) := -d^2(x, [0, +\infty)) = \begin{cases} -x^2 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

4.7.3 Extraction cost depending on reserves

In this last variation we consider an extraction cost $C(q, R)$ which depends also on the available reserves. The cost to minimize is then

$$J(t, R, q) := \mathbb{E} \int_t^{+\infty} (C(q(t), R(t)) - p(t)q(t)) e^{-r(s-t)} ds$$

We have the same value function and Dynamic Programming Principle and also essentially the same equation.

Proposition 4.13. *The value function $u(t, R)$ is a viscosity solution of the Hamilton-Jacobi-Bellman equation*

$$-u_t - \frac{\nu^2}{2} R^2 u_{RR} + ru + \sup_q \{pq - C(q, R) + qu_R\} = 0$$

The proof is essentially the same as the proof of Proposition 4.5 and we omit it. A way to model the dependence of the extraction cost from the available reserves is to consider, for example, a function like

$$C(q, R) = (\alpha q + \frac{1}{2} \beta q^2) \frac{1}{R^\gamma}$$

with $\gamma > 1$ which is increasing in R and such that

$$\lim_{R \to +\infty} C(q, R) = 0$$

for all fixed $q > 0$ and such that $C(0, R) \equiv 0$.

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