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Ph.D. Thesis

Bruhat order on the involutions of classical Weyl groups

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Contents

Introduction

It is known that a Coxeter group W, partially ordered by the Bruhat order, is a graded poset, with rank function given by the length, and that it is also EL-shellable, hence Cohen-Macaulay, and Eulerian.

The aim of this work is to investigate whether a particular subposet of W, namely that induced by the set of involutions of W, which we denote by $Invol(W)$, is endowed with similar properties.

The problem arises from a geometric question. It is known that the symmetric group S_n, partially ordered by the Bruhat order, encodes the cell decomposition of Schubert varieties. In [RS1], [RS2] Richardson and Springer introduce a vast generalization of this partial order, in relation to the cell decomposition of certain symmetric varieties. In a particular case they obtain the subposet of S_n induced by the involutions.

In this work the problem is completely solved for an important class of Coxeter groups, namely that of classical Weyl groups. Our main result is that, if W is a classical Weyl group, then the poset $Invol(W)$ is graded, with rank function given by the average between the length and the absolute length, and that it is EL-shellable, hence Cohen-Macaulay, and Eulerian.

The proofs are combinatorial and use the descriptions of classical Weyl groups in terms of permutation groups: the symmetric group S_n, the hyperoctahedral group B_n and the even-signed permutation group D_n.

In particular we obtain, as new results, a combinatorial description of the absolute length of the involutions in classical Weyl groups, and a combinatorial description of the covering relation in the Bruhat order of the hyperoctahedral group and of the even-signed permutation group.

It is also conjectured that the result proved for classical Weyl groups actually holds for every Coxeter group, and it is shown that for the class of dihedral groups, which are Coxeter groups but not Weyl groups, the result is valid.

This work collects the results obtained in [Inc1], [Inc2], [Inc3]. In [Inc1] we study the poset $Invol(S_n)$, showing that this is a graded poset, with rank function given by the average between the number of inversions and the number of excedances, and that it is EL-shellable and Eulerian. In [Inc2] we extend these results to the poset $Invol(B_n)$, finding an explicit formula for the rank function. Finally, in [Inc3], we give a unified description of the results obtained in [Inc1] and [Inc2], extending them to the even-signed permutation group.

The organization of this work is as follows. In Chapter 1 we give the basic definitions, notation and results that are needed in the sequel. In particular we focus on posets, Coxeter groups, Bruhat order, Weyl groups and classical Weyl groups, with their combinatorial descriptions. In Chapter 2 we introduce the main problem, saying something about the motivations, stating the main result about classical Weyl groups and giving the conjecture on Coxeter groups. In Chapter 3 we expose some general methods to prove that a poset is graded and EL-shellable: we use these methods in the proofs of the following chapters. In Chapter 4 some preliminary results are illustrated, about the absolute length of the involutions and about the covering relation in the Bruhat order of classical Weyl groups. In Chapters 5, 6 and 7 the main results are stated and proved, about, respectively, the symmetric group, the hyperoctahedral group and the even-signed permutation group.

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Chapter 1

Notation and preliminaries

1.1 Notation

We let $\mathbf{N} = \{1, 2, 3, \ldots\}$ and $\mathbf{Z}$ be the set of integers.

For $n, m \in \mathbf{Z}$, with $n \leq m$, we let

$$[n, m] = \{n, n + 1, \ldots, m\}.$$

For $n \in \mathbf{N}$, we let

$$\begin{aligned} [n] &= [1, n]; \\ [-n] &= [-n, -1]; \\ [\pm n] &= [-n, n] \setminus \{0\} = [-n] \cup [n]. \end{aligned}$$

We denote by $\equiv$ the congruency modulo 2: $n \equiv m$, with $n, m \in \mathbf{Z}$, means that $n - m$ is even.

Finally, we denote simply by $<$ the lexicographic order:

$$(a_1, a_2, ..., a_n) < (b_1, b_2, ..., b_n)$$

means that $a_k < b_k$, where $k = \min\{i \in [n] : a_i \neq b_i\}$.

1.2 Posets

We follow [Sta1, Chapter 3] for poset notation and terminology.

The word *poset* is an abbreviation of *partially ordered set.* Thus, a poset is a pair $(P, \leq)$ consisting of a set P together with a partial order relation $\leq$. The relation is suppressed from the notation when it is clear from context.

Let P be a poset. If $Q \subseteq P$, then the *subposet* of P *induced* by Q is the poset $(Q, \leq)$ whose order relation is the restriction of the order relation of P.

Let $x, y \in P$. We write $x < y$ to mean that $x \leq y$ and $x \neq y$. We denote by $\lhd$ the *covering relation*: $x \lhd y$ means that $x < y$ and there is no z such that $x < z < y$. The *Hasse diagram* of a finite poset P is the graph whose vertices are the elements of P, whose edges are the covering relations, and such that if $x < y$, then y is drawn "above" x.

If $x, y \in P$, with $x \leq y$, we let $[x, y] = \{z \in P : x \leq z \leq y\}$, and we call it an *interval* of P. The set $(x, y) = \{z \in P : x < z < y\}$ is an *open interval.* If $x, y \in P$, with $x < y$, a *chain* from x to y of *length* k is a $(k+1)$-tuple $(x_0, x_1, ..., x_k)$ such that $x = x_0 < x_1 < ... < x_k = y$, denoted simply by "$x_0 < x_1 < ... < x_k$". A chain $x_0 < x_1 < ... < x_k$ is said to be *saturated* if all the relations in it are covering relations: $x_0 \lhd x_1 \lhd ... \lhd x_k$.

A poset is said to be *bounded* if it has a minimum and a maximum, denoted by $\hat{0}$ and $\hat{1}$ respectively.

A poset P is said to be *graded* of *rank* n if it is finite, bounded and if all maximal chains of P have the same length n. If P is a graded poset of rank n, then there is a unique *rank function* $\rho : P \to [0, n]$ such that $\rho(\hat{0}) = 0$, $\rho(\hat{1}) = n$ and $\rho(y) = \rho(x) + 1$ whenever y covers x in P. Conversely, if P is finite and bounded, and if such a function exists, then P is graded of rank n.

If P is a graded poset and Q is a totally ordered set, an *edge-labelling* of P with values in Q is a function $\lambda : \{(x, y) \in P^2 : x \lhd y\} \to Q$. If λ is an edge-labelling of P, for every saturated chain $x_0 \lhd x_1 \lhd ... \lhd x_k$ we set

$$\lambda(x_0, x_1, ..., x_k) = (\lambda(x_0, x_1), \lambda(x_1, x_2), ..., \lambda(x_{k-1}, x_k)).$$

An edge-labelling λ of P is said to be an *EL-labelling* if for every $x, y \in P$, with $x < y$, the following properties hold:

1. there is exactly one saturated chain from x to y, say

$$x = x_0 \lhd x_1 \lhd ... \lhd x_k = y,$$

such that $\lambda(x_0, x_1, ..., x_k)$ is a non-decreasing sequence, that is

$$\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq ... \leq \lambda(x_{k-1}, x_k);$$

2. this chain has the lexicographically minimal labelling: if

$$x = y_0 \lhd y_1 \lhd ... \lhd y_k = y,$$

is a saturated chain from x to y, different from the previous one, we have

$$\lambda(x_0, x_1, ..., x_k) < \lambda(y_0, y_1, ..., y_k).$$

A graded poset P is said to be *lexicographically shellable*, or *EL-shellable*, if it has an EL-labelling.

Finally, a graded poset P with rank function ρ is said to be *Eulerian* if

$$|\{z \in [x, y] : \rho(z) \text{ is even}\}| = |\{z \in [x, y] : \rho(z) \text{ is odd}\}|,$$

for every $x, y \in P$ such that $x < y$.

In an EL-shellable poset there is a necessary and sufficient condition for the poset to be Eulerian. We state it in the following form (see [Bjö, Theorem 2.7] and [Sta3, Theorem 1.2] for proofs of more general results).

Theorem 1.2.1 *Let P be a graded EL-shellable poset and let λ be an EL-labelling of P. Then P is Eulerian if and only if for every $x, y \in P$, with $x < y$, there is exactly one saturated chain from x to y with decreasing labels.*

Connections between EL-shellable posets and shellable complexes, Cohen-Macaulay rings and Gorenstein rings can be found, for example, in [Bac], [BGS], [Bjö], [Gar], [Hoc], [Rei] and [Sta2]. Here we only recall some basic facts.

A finite simplicial complex Δ is said to be *pure d-dimensional* if all maximal faces are of dimension d. A pure d-dimensional complex Δ is said to be *shellable* if its maximal faces can be arranged in sequence $\sigma_1, \sigma_2, \ldots, \sigma_t$ in such a way that

$$\sigma_i \cap \bigcup_{j=1}^{i-1} \overline{\sigma}_j$$

is a pure $(d-1)$-dimensional complex for $i \in [2, t]$, where

$$\overline{\sigma} = \{\tau : \tau \subseteq \sigma_i\}.$$

Such an ordering of the maximal faces is called a *shelling* of Δ.

The *order complex* $\Delta(P)$ of a poset P is the simplicial complex of all chains of P. A poset P is said to be *shellable* if $\Delta(P)$ is shellable.

Let $P = \{x_1, x_2, \ldots, x_f\}$ be a finite poset. Let K be a field or $K = \mathbf{Z}$. The *Stanley-Reisner ring* associated with P is

$$R_P = \frac{K[x_1, x_2, \ldots, x_f]}{I_P},$$

where I_P is the ideal in the polynomial ring $K[x_1, x_2, \ldots, x_f]$ generated by all monomials $x_i x_j$ for which x_i and x_j are uncomparable in P.

A poset P is said to be *Cohen-Macaulay* if R_P is a Cohen-Macaulay ring. Hochster and Stanley have shown that if the order complex $\Delta(P)$ is shellable then P is Cohen-Macaulay (see [Hoc] and [Sta2]). Finally Björner has proved the following (see [Bjö, Theorem 2.3]).

Theorem 1.2.2 *Let P be a graded poset. If P is EL-shellable then P is shellable, hence Cohen-Macaulay.*

A poset P is said to be *Gorenstein* if R_P is a Gorenstein ring. Hochster and Stanley have proved the following (see [Hoc, p. 211] or [Sta2, p. 57]).

Theorem 1.2.3 *Let P be a graded poset and suppose that $\Delta(P)$ triangulates a sphere, or a multiple cone over a sphere. Then P is Gorenstein.*

1.3 Coxeter groups

A *Coxeter matrix* of order n is a matrix $m : [n] \times [n] \to \mathbf{N} \cup \{\infty\}$ such that

1. $m(i,i) = 1, \quad$ for every $i \in [n]$;
2. $m(i,j) = m(j,i) \geq 2, \quad$ for every $i, j \in [n]$, with $i \neq j$.

A *Coxeter system* is a pair (W, S), consisting of a group W and a subset $S = \{s_1, \ldots, s_r\}$ of generators of W, subject only to the relations

$$(s_i s_j)^{m(i,j)} = 1, \quad \text{for every } i, j \in [r],$$

where m is a Coxeter matrix of order r. In particular $m(i,j) = \infty$ means that no relation occurs for the pair (s_i, s_j). Formally, W is the quotient F/N, where F is a free group on the set S and N is the normal subgroup generated by all elements $(s_i s_j)^{m(i,j)}$. The group W is called *Coxeter group.*

The *Coxeter graph* (or *diagram*) of a Coxeter system (W, S) is the graph whose node set is S and whose edges are the unordered pairs $\{s_i, s_j\}$ such that $m(i,j) \geq 3$. The edges $\{s_i, s_j\}$ such that $m(i,j) \geq 4$ are labeled by the number $m(i,j)$.

Since $m(i,i) = 1$, for $i \in [r]$, every generator s_i is an involution. Moreover, the relation $(s_i s_j)^{m(i,j)} = 1$ is equivalent to

$$\underbrace{s_i s_j s_i s_j \ldots}_{m(i,j)} = \underbrace{s_j s_i s_j s_i \ldots}_{m(i,j)}.$$

In particular $m(i,j) = 2$ (that is, s_i and s_j are not neighbours in the Coxeter graph) if and only if s_i and s_j commute in W.

The *length* of an element $w \in W$, denoted by $l(w)$, is the minimal k such that w can be written as a product of k generators. If $w = s_{i_1} s_{i_2} \ldots s_{i_k}$, with $k = l(w)$, then the word $s_{i_1} s_{i_2} \ldots s_{i_k}$ is called a *reduced expression* of w.

A *reflection* in a Coxeter group W is a conjugate of some element in S. The elements of S are also called *simple reflections*. The set of all reflections is usually denoted by T:

$$T = \{wsw^{-1} : s \in S,\ w \in W\}.$$

The *absolute length* of an element $w \in W$, denoted by $al(w)$, is the minimal k such that w can be written as a product of k reflections.

Since the generators are particular reflections, for every $w \in W$ we have

$$al(w) \leq l(w).$$

If a Coxeter system (W, S) has Coxeter graph G, then we say that the *type* of (W, S) is G. A Coxeter system is *irreducible* if its type is connected.

The finite irreducible Coxeter systems have been completely classified, as we state in the following (see, e.g., [Hum1]).

Theorem 1.3.1 *If (W, S) is a finite irreducible Coxeter system, then its type is necessarily one of those in Table 1.1.*

Note that the groups (whose types are) represented in Table 1.1, are pairwise non-isomorphic, except that $\mathbf{I}_2(3) = \mathbf{A}_2$, $\mathbf{I}_2(4) = \mathbf{B}_2$ and $\mathbf{I}_2(6) = \mathbf{G}_2$.

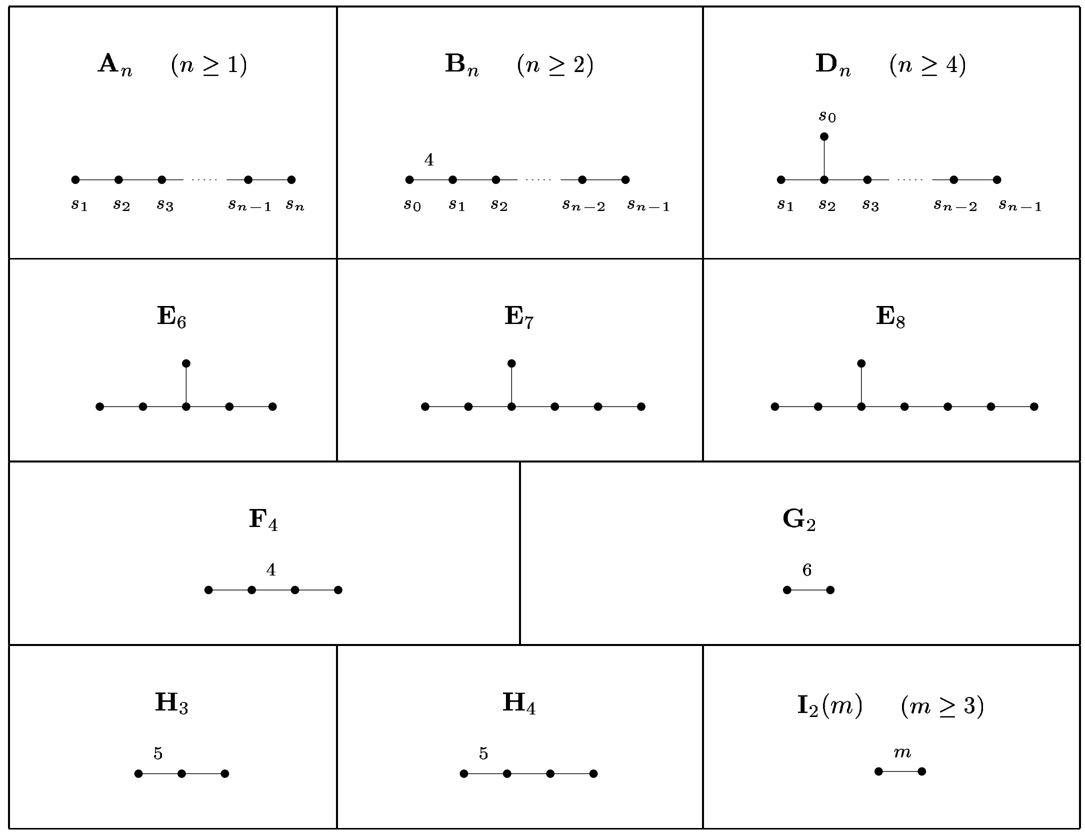


Table 1.1: The finite irreducible Coxeter systems.

1.4 Bruhat order

Let (W, S) be a Coxeter system.

The *Bruhat graph* of W is the directed graph whose vertex set is W and such that an ordered pair (u, v) of elements of W is an edge if and only if $v = ut$, for some reflection $t \in T$, and $l(u) < l(v)$. If (u, v) is an edge of the Bruhat graph, then we write $u \to v$.

The *Bruhat order* of W is the partial order relation which is the transitive closure of the relation $\to$ defined above: given $u, v \in W$, then $u \leq v$ in the Bruhat order if there exists a chain

$$u = u_0 \to u_1 \to u_2 \to \ldots \to u_k = v.$$

A *subword* of a word $a_1 a_2 \ldots a_k$ is a word $a_{i_1} a_{i_2} \ldots a_{i_q}$, with

$$1 \leq i_1 \leq i_2 \leq \ldots \leq i_q \leq k.$$

The following result, known as "subword property", gives a characterization of the Bruhat order relation (see, e.g., [BB]).

Theorem 1.4.1 (Subword property) *Let $u, v \in W$ and let $s_{i_1} s_{i_2} \ldots s_{i_k}$ be a reduced expression of v. Then $u \leq v$ if and only if there exists a reduced expression of u which is a subword of $s_{i_1} s_{i_2} \ldots s_{i_k}$.*

It is known that the map which associates with every element $w \in W$ its inverse w^{-1} is an automorphism of the Bruhat order, as we state in the following.

Proposition 1.4.2 *Let W be a Coxeter group and let $u, v \in W$. Then the following are equivalent:*

1. $u \leq v$;
2. $u^{-1} \leq v^{-1}$.

If W is finite it is known that W has a maximum, which is usually denoted by w_0. This element is an involution: $w_0^2 = 1$. Moreover, translation and conjugacy by w_0 induce (anti)automorphisms of the Bruhat order.

Proposition 1.4.3 *Let W be a finite Coxeter group, with maximum w_0, and let $u, v \in W$. Then the following are equivalent:*

1. $u \leq v$;
2. $w_0 v \leq w_0 u$;
3. $v w_0 \leq u w_0$;
4. $w_0 u w_0 \leq w_0 v w_0$.

The Bruhat order of Coxeter groups has been studied extensively (see, e.g., [BW], [Deo], [Ede], [Pro], [Rea] and [Ver]). In particular it is known that it gives to W the structure of a graded poset, whose rank function is the length. It has been also proved that this poset is always EL-shellable, hence Cohen-Macaulay (see [Ede], [Pro] and [BW]), and Eulerian (see [Ver]).

The aim of this work is to investigate whether a particular subposet of W, namely that induced by the set of involutions of W, is endowed with similar properties. The problem will be completely solved for an important class of Coxeter groups, namely that of classical Weyl groups.

1.5 Weyl groups

In this section we give some basic notions about Weyl groups. For preliminaries about this part we refer to [Hum2].

Let G be a connected and reductive linear algebraic group. Let T be a maximal torus of G and let N be the normalizer of T in G. The quotient $W = N/T$ is the corresponding *Weyl group*.

Now let B be a Borel subgroup of G containing T. Then B, acting by left translations, has a finite number of orbits on the flag variety $X = G/B$. The set V of these orbits has a natural partial order: if $u, v \in V$, then $u \leq v$ if u is contained in the Zariski closure of v.

The Borel subgroup B defines a set S of involutorial generators of W such that (W, S) is a Coxeter system. It is known that there is a bijection of the set of orbits V onto the Weyl group W. Moreover, the order on V corresponds to the combinatorially defined Bruhat order on W.

1.6 Classical Weyl groups

Among the finite irreducible Coxeter systems (see Table 1.1) we find all the irreducible Weyl groups: the *classical* Weyl groups, whose types are $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{D}_n$, and the *exceptional* Weyl groups, whose types are $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$, $\mathbf{F}_4$, $\mathbf{G}_2$.

The classical Weyl groups have nice combinatorial descriptions in terms of permutation groups: the symmetric group is a representative for type $\mathbf{A}_n$, the hyperoctahedral group for type $\mathbf{B}_n$ and the even-signed permutation group for type $\mathbf{D}_n$. In this section we describe in detail such groups.

1.6.1 The symmetric group

We denote by S_n the *symmetric group*, defined by

$$S_n = \{\sigma : [n] \rightarrow [n] : \sigma \text{ is a bijection}\}$$

and we call its elements *permutations*.

To denote a permutation $\sigma \in S_n$ we often use the *one-line notation*: we write $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$, to mean that $\sigma(i) = \sigma_i$ for every $i \in [n]$. We also write σ in *disjoint cycle form*, omitting to write the 1-cycles of σ: for example, if $\sigma = 364152$, then we also write $\sigma = (1,3,4)(2,6)$. Given $\sigma, \tau \in S_n$, we let $\sigma\tau = \sigma \circ \tau$ (composition of functions) so that, for example, $(1,2)(2,3) = (1,2,3)$.

Given $\sigma \in S_n$, the *diagram* of σ is a square of $n \times n$ cells, with the cell (i, j) (that is, the cell in column i and row j, with the convention that the first column is the leftmost one and the first row is the lowest one) filled with a dot if and only if $\sigma(i) = j$. For example, in Figure 1.1 the diagram of $\sigma = 35124 \in S_5$ is represented.

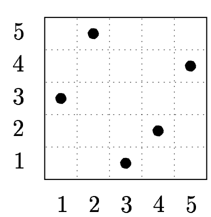


Figure 1.1: Diagram of $\sigma = 35124 \in S_5$.

The *diagonal* of the diagram is the set of cells $\{(i, i) : i \in [n]\}$.

As a set of generators for S_n we take

$$S = \{s_1, s_2, \dots, s_{n-1}\},$$

where $s_i = (i, i+1)$ for every $i \in [n-1]$.

As we have already mentioned, the symmetric group is a representative for the Coxeter groups of type $\mathbf{A}_n$ (see, e.g., [BB]).

Theorem 1.6.1 *(S_n, S) is a Coxeter system of type $\mathbf{A}_{n-1}$.*

The concepts introduced in general for Coxeter groups (length, reflections, Bruhat order) have all a simple combinatorial description in the symmetric group. For example the length of a permutation $\sigma \in S_n$ is given by

$$l(\sigma) = inv(\sigma),$$

where

$$inv(\sigma) = |\{(i,j) \in [n]^2 : i < j,\ \sigma(i) > \sigma(j)\}|$$

is the number of *inversions* of σ.

In the symmetric group the reflections are the transpositions:

$$T = \{(i,j) \in [n]^2 : i < j\}.$$

In order to give a characterization of the covering relation in the Bruhat order of the symmetric group, we introduce the following definition.

Definition 1.6.2 *Let $\sigma \in S_n$. A* rise *of σ is a pair $(i,j) \in [n]^2$ such that*

1. $i < j$,
2. $\sigma(i) < \sigma(j)$.

A rise (i,j) is said to be free *if there is no $k \in [n]$ such that*

1. $i < k < j$,
2. $\sigma(i) < \sigma(k) < \sigma(j)$.

For example, the rises of $\sigma = 35124 \in S_5$ are $(1,2)$, $(1,5)$, $(3,4)$, $(3,5)$ and $(4,5)$. They are all free except $(3,5)$. The following is a well-known result.

Proposition 1.6.3 *Let $\sigma, \tau \in S_n$, with $\sigma < \tau$. Then $\sigma \lhd \tau$ if and only if*

$$\tau = \sigma(i, j),$$

where (i, j) is a free rise of σ.

In order to give a characterization of the Bruhat order relation in S_n, we introduce the following notation: for $\sigma \in S_n$ and for $(h, k) \in [n]^2$, we set

$$\sigma[h, k] = |\{i \in [h] : \sigma(i) \in [k, n]\}|.$$

The characterization is the following (see, e.g., [Pro]).

Theorem 1.6.4 *Let $\sigma, \tau \in S_n$. Then $\sigma \leq \tau$ if and only if*

$$\sigma[h, k] \leq \tau[h, k],$$

for every $(h, k) \in [n]^2$.

Finally, the maximum of S_n is

$$w_0 = n\,(n-1)\,(n-2)\,\ldots\,3\,2\,1.$$

Note that, given $\sigma \in S_n$, the diagrams of the permutations σ^{-1}, $w_0\sigma$, σw_0 and $w_0\sigma w_0$ are obtained from the diagram of σ by, respectively, interchanging rows and columns (transposing), reversing the rows, reversing the columns and reversing rows and columns. So the effects of these operations on the Bruhat order are described in Propositions 1.4.2 and 1.4.3.

1.6.2 The hyperoctahedral group

We denote by $S_{\pm n}$ the symmetric group on the set $[\pm n]$:

$$S_{\pm n} = \{\sigma : [\pm n] \to [\pm n] : \sigma \text{ is a bijection}\},$$

which is clearly isomorphic to S_{2n}.

We denote by B_n the *hyperoctahedral group*, defined by

$$B_n = \{\sigma \in S_{\pm n} : \sigma(-i) = -\sigma(i) \text{ for every } i \in [n]\}$$

and we call its elements *signed permutations*.

To denote a signed permutation $\sigma \in B_n$ we use the *window notation*: we write $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n]$, to mean that $\sigma(i) = \sigma_i$ for every $i \in [n]$ (the images of the negative entries are then uniquely determined). We also denote σ by the sequence $|\sigma_1|\,|\sigma_2|\,\ldots\,|\sigma_n|$, with the negative entries underlined. For example, $4\,\underline{2}\,1\,\underline{5}\,\underline{3}$ denotes the signed permutation $[4, -2, 1, -5, -3]$. We also write σ in disjoint cycle form.

Signed permutations are particular permutations of the set $[\pm n]$, so they inherit the notion of diagram. Note that the diagram of a signed permutation is symmetric with respect to the center. In Figure 1.2, the diagram of $\sigma = \underline{3}\,\underline{2}\,1 \in B_3$ is represented.

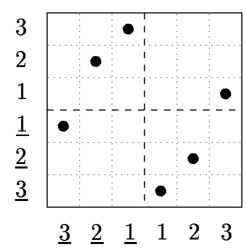


Figure 1.2: Diagram of $\sigma = \underline{3}\,\underline{2}\,1 \in B_3$.

The (*main*) *diagonal* of the diagram is the set of cells $\{(i,i) : i \in [\pm n]\}$, and the *antidiagonal* is the set of cells $\{(i,-i) : i \in [\pm n]\}$.

As a set of generators for B_n we take

$$S = \{s_0, s_1, \ldots, s_{n-1}\},$$

where $s_0 = (1,-1)$ and $s_i = (i, i+1)(-i, -i-1)$ for every $i \in [n-1]$.

The hyperoctahedral group is a representative for Coxeter groups of type $\mathbf{B}_n$ (see, e.g., [BB]).

Theorem 1.6.5 *(B_n, S) is a Coxeter system of type $\mathbf{B}_n$.*

There are various known formulas for computing the length in B_n (see, e.g., [Bre, Proposition 3.1]). Here we present a new one: the length of $\sigma \in B_n$ is given by

$$l(\sigma) = \frac{inv(\sigma) + neg(\sigma)}{2}, \tag{1.1}$$

where

$$inv(\sigma) = |\{(i,j) \in [\pm n]^2 : i < j,\ \sigma(i) > \sigma(j)\}|,$$

which is also the length of σ in the symmetric group $S_{\pm n}$, and

$$neg(\sigma) = |\{i \in [n] : \sigma(i) < 0\}|.$$

It is known (see, e.g., [BB]) that the set of reflections of B_n is

$$T = \{(i, -i) : i \in [n]\} \cup \{(i, j)(-i, -j) : 1 \leq i < |j| \leq n\}.$$

It is useful to extend a notation introduced for the symmetric group: for $\sigma \in B_n$ and for $(h, k) \in [\pm n]^2$ we set

$$\sigma[h, k] = |\{i \in [-n, h] : \sigma(i) \in [k, n]\}|.$$

Definition 1.6.6 *Let $\sigma, \tau \in B_n$. We say that the pair (σ, τ) satisfies the B-*condition *if*

$$\sigma[h, k] \leq \tau[h, k]$$

for every $h, k \in [\pm n]^2$.

The following result gives a combinatorial characterization of the Bruhat order relation in B_n (see, e.g., [BB, Theorem 8.1.8]).

Theorem 1.6.7 *Let $\sigma, \tau \in B_n$. Then $\sigma \leq \tau$ if and only if the pair (σ, τ) satisfies the B-condition.*

Comparing Theorems 1.6.4 and 1.6.7, we can conclude the following.

Proposition 1.6.8 *Let $\sigma, \tau \in B_n$. Then $\sigma \leq \tau$ in the Bruhat order of B_n if and only if $\sigma \leq \tau$ in the Bruhat order of the symmetric group $S_{\pm n}$.*

The maximum of B_n is

$$w_0 = \underline{1}\,\underline{2} \ldots \underline{n}$$

and the effects on the diagram of a signed permutation of taking the inverse, composing with w_0 and conjugating by it are the same as described for the symmetric group.

1.6.3 The even-signed permutation group

We denote by D_n the *even-signed permutation group*, defined by

$$D_n = \{\sigma \in B_n : neg(\sigma) \text{ is even}\}.$$

Notation and terminology are ineherited from the hyperoctahedral group. For example the signed permutation $\sigma = \underline{3}\,\underline{2}\,1$, whose diagram is represented in Figure 1.2, is also in D_3.

As a set of generators for D_n we take

$$S = \{s_0, s_1, \ldots, s_{n-1}\},$$

where $s_0 = (1, -2)(-1, 2)$ and $s_i = (i, i+1)(-i, -i-1)$ for every $i \in [n-1]$.

The even-signed permutation group is a representative for Coxeter groups of type $\mathbf{D}_n$ (see, e.g., [BB]).

Theorem 1.6.9 *(D_n, S) is a Coxeter system of type $\mathbf{D}_n$.*

In order to make no confusion between signed permutations and even-signed permutations, we will denote if necessary by l_B and l_D the length functions and by $\leq_B$ and $\leq_D$ the Bruhat order relations in the respective groups B_n and D_n.

About the length function in D_n, it is known (see, e.g., [BB]) that

$$l_D(\sigma) = l_B(\sigma) - neg(\sigma).$$

Thus, by (1.1), the length of $\sigma \in D_n$ is given by

$$l(\sigma) = \frac{inv(\sigma) - neg(\sigma)}{2}. \tag{1.2}$$

It is known (see, e.g., [BB]) that the set of reflections of D_n is

$$T = \{(i, j)(-i, -j) : 1 \leq i < |j| \leq n\}.$$

In order to give a combinatorial characterization of the Bruhat order relation in D_n, we introduce the following notation: for $\sigma \in D_n$ and $(h, k) \in [-n] \times [n]$, we set

$$\begin{aligned}
\sigma_{center}[h,k] &= \sigma_{[\pm|h|]\times[\pm k]},\\
\sigma_{NW}[h,k] &= \sigma_{[-n,h-1]\times[k+1,n]},\\
\sigma_{N}[h,k] &= \sigma_{[\pm|h|]\times[k+1,n]},\\
\sigma_{Nleft}[h,k] &= \sigma_{[h]\times[k+1,n]},\\
\sigma_{Wup}[h,k] &= \sigma_{[-n,h-1]\times[k]}.
\end{aligned}$$

We say that $(h,k) \in [-n] \times [n]$ is *free* for σ if

$$\sigma_{center}[h,k] = 0.$$

Definition 1.6.10 *Let $\sigma, \tau \in D_n$. We say that $(h,k) \in [-n] \times [n]$ is a D-*cell *of the pair (σ,τ) if it is free for both σ and τ and*

$$\sigma_{NW}[h,k] = \tau_{NW}[h,k].$$

If (h,k) is a D-cell of (σ,τ), then we say that it is valid *if*

$$\sigma_{Nleft}[h,k] \equiv \tau_{Nleft}[h,k],$$

or, equivalently, if

$$\sigma_{Wup}[h,k] \equiv \tau_{Wup}[h,k].$$

*Finally, we say that the pair (σ,τ) satisfies the D-*condition *if every D-cell of (σ,τ) is valid.*

The following result gives a combinatorial characterization of the Bruhat order relation in D_n (see [BB, Theorem 8.2.8]).

Theorem 1.6.11 *Let $\sigma, \tau \in D_n$. Then $\sigma \leq \tau$ if and only if the pair (σ,τ) satisfies both the B-condition and the D-condition.*

Note that $\sigma \leq_D \tau$ implies $\sigma \leq_B \tau$, while the converse is not true.

For example, consider the two even-signed permutations $\sigma = 6\,\underline{4}\,3\,\underline{2}\,1\,\underline{7}\,\underline{5}$ and $\tau = \underline{4}\,\underline{5}\,\underline{3}\,1\,\underline{2}\,\underline{6}\,\underline{7}$ in D_7, whose diagrams are shown in Figure 1.3.

It's easy to check that the pair (σ,τ) satisfies the B-condition, so $\sigma \leq_B \tau$.

The D-cells of the pair (σ,τ) are $(-3,1)$, $(-3,2)$, $(-2,3)$ and $(-1,3)$. Among these, $(-3,1)$ and $(-3,2)$ are valid, while $(-2,3)$ and $(-1,3)$ are not valid. Thus the pair (σ,τ) does not satisfy the D-condition, so $\sigma \not\leq_D \tau$.

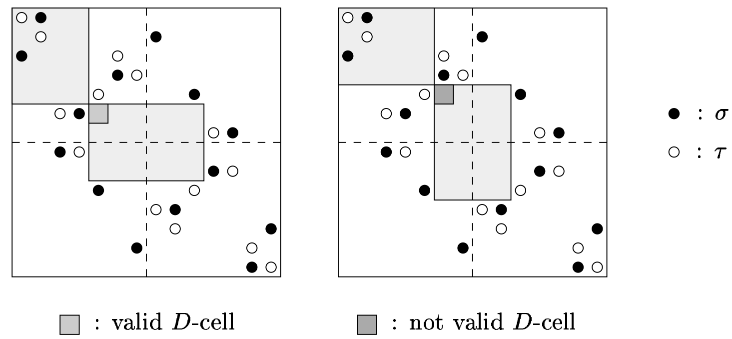


Figure 1.3: D-cells.

The maximum of D_n is

$$w_0 = \begin{cases} \underline{1}\,\underline{2}\,\ldots\,\underline{n}\,, & \text{if } n \text{ is even,} \\ 1\,\underline{2}\,\ldots\,\underline{n}\,, & \text{if } n \text{ is odd.} \end{cases}$$

Chapter 2

The main problem

Every Coxeter group W, partially ordered by the Bruhat order, is a graded poset, with rank function given by the length, and it is EL-shellable, hence Cohen-Macaulay, and Eulerian.

The aim of this work is to investigate whether a particular subposet of W, namely that induced by the set of involutions of W, that is

$$Invol(W) = \{w \in W : w^2 = 1\},$$

is endowed with similar properties.

2.1 Motivation

The problem arises from a geometric question.

In fact it is known that the symmetric group, partially ordered by the Bruhat order, encodes the cell decomposition of Schubert varieties (see, e.g., [Ful]). In [RS1], [RS2] Richardson and Springer consider a vast generalization of this partial order, in relation to the cell decomposition of certain symmetric varieties, obtaining, in a particular case, the poset $Invol(S_n)$.

Here we give an outline of their work. Let G be a connected and reductive linear algebraic group. Assume that G is defined over an algebraically closed field F of characteristic $\neq 2$. Let θ be an automorphism of G of order 2. Let T be a maximal torus of G and let B be a Borel subgroup of G containing T. We assume that B and T are θ-stable. Let K be the fixed point subgroup of θ.

It is known that K is a (not necessarily connected) reductive group. We denote by X the quotient variety G/K and call it the *symmetric variety* defined by (G,θ). Then B, acting by left translations, has finitely many orbits in X. The authors study the ordered set V of the orbits, whose order is the following: if $u, v \in V$, then $u \leq v$ if u is contained in the Zariski closure of v.

They give some concrete examples of this order on V, and one is the following. Let $G = GL_n(F)$ and define the involution θ_1 by $\theta_1(g) = {}^t g^{-1}$. The fixed point subgroup K of θ_1 is the orthogonal group $O_n(F)$, and the symmetric variety $X = GL_n(F)/O_n(F)$ can be identified with the variety of non-singular quadrics on F^n. Let B (resp. T) be the group of all upper triangular (resp. diagonal) matrices in G. Actually, θ_1 is replaced by another involutive authomorphism θ, conjugate to θ_1 by an inner automorphism, such that B and T are θ-stable.

In this case, the authors show that there is a natural bijection of V onto the set of involutions of the symmetric group S_n, and that this bijection is an anti-isomorphism of posets (where the involutions of S_n have the induced Bruhat order).

2.2 The main result

The following is the main result of this work.

Theorem 2.2.1 *Let W be a classical Weyl group. The poset $Invol(W)$ is*

1. *graded, with rank function given by*

$$\rho(w) = \frac{l(w) + al(w)}{2},$$

   for every $w \in Invol(W)$;

2. *EL-shellable;*

3. *Eulerian.*

Theorem 2.2.1 will be proved separately for the symmetric group, for the hyperoctahedral group and for the even-signed permutation group, respectively in Chapters 4, 5, 6. In next section we discuss some algebraic and topological consequence of Theorem 2.2.1.

2.3 Algebraic and topological consequences

Let W be a classical Weyl group. Our result have the following algebraic and topological consequences.

Theorem 2.3.1 *The poset $Invol(W)$ is Cohen-Macaulay.*

Proof. By Theorem 1.2.2, it is a consequence of the EL-shellability. □

Theorem 2.3.2 *Let $\sigma, \tau \in Invol(W)$, with $\sigma < \tau$, be such that $\rho(\tau) - \rho(\sigma) = d + 2 \geq 2$. Let Δ be the order complex of the open interval (σ, τ). Then Δ triangulates a d-dimensional sphere.*

Proof. The complex Δ is pure d-dimensional. Since $Invol(W)$ is EL-shellable and Eulerian, by the definition of EL-labelling and by Theorem 1.2.1, we have that every interval in $Invol(W)$ of length 2 has exactly two maximal chains. Thus every $(d-1)$-face of Δ is included in exactly two d-faces. Since Δ is shellable, by a result of Danaraj and Klee [DK, p. 444], it follows that Δ triangulates a d-dimensional sphere. □

Theorem 2.3.3 *Let $\sigma, \tau \in Invol(W)$, with $\sigma < \tau$ and $\rho(\tau) - \rho(\sigma) \geq 2$. Let $P = [\sigma, \tau]$ or $P = (\sigma, \tau)$. Then P is Gorenstein.*

Proof. It is a consequence of Theorems 2.3.2 and 1.2.3. □

2.4 A conjecture

It is natural to conjecture that our main result actually holds for every Coxeter group. In infinite cases, we mean that every interval $[\hat{0}, x]$ of the poset has the mentioned properties.

Conjecture 2.4.1 *Let W be a Coxeter group. The poset $Invol(W)$ is*

1. *graded, with rank function given by*

$$\rho(w) = \frac{l(w) + al(w)}{2},$$

   for every $w \in Invol(W)$;

2. *EL-shellable, hence Cohen-Macaulay;*

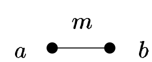
3. *Eulerian.*

After a preliminary investigation on the affine Weyl groups (which also have nice combinatorial descriptions), we fill that our techniques may be applicable also to this class of Coxeter groups.

There is another class of Coxeter groups, which are not Weyl groups, for which the conjecture is true, namely the class of dihedral groups. To this class is dedicated next subsection.

2.4.1 The dihedral group

Definition 2.4.2 *The* dihedral group $I_2(m)$, *with* $m \geq 3$, *is the Coxeter group generated by two elements* a *and* b, *whose Coxeter graph is*



By the subword property (Theorem 1.4.1), it's easy to reconstruct the Hasse diagram of the poset $I_2(m)$ with the Bruhat order. We denote by e the empty word, which is the minimum of the poset. Note that, since

$$(ab)^m = e,$$

the poset is finite and the maximum is

$$w_0 = \underbrace{abab\dots}_{m} .$$

As an example, in Figure 2.1 the Hasse diagram of $I_2(5)$ is represented.

Note that in the dihedral group $I_2(m)$ the involutions are the minimum, the maximum and all the elements of odd rank. So the subposet of $I_2(m)$ induced by the involutions has the same Hasse diagram as another dihedral group. Precisely (up to poset isomorphism):

$$Invol(I_2(m)) = I_2\left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right). \tag{2.1}$$

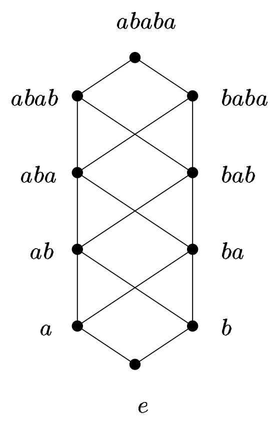


Figure 2.1: Hasse diagram of $I_2(5)$.

The reflections are exactly the elements of odd rank. So the reflections are all the involutions except the minimum and, only if m is even, except the maximum. Thus the absolute length is given by

$$al(w) = \begin{cases} 2, & \text{if } l(w) \text{ is even,} \\ 1, & \text{if } l(w) \text{ is odd,} \end{cases}$$

for every $w \in I_2(m) \setminus \{e\}$, and $al(e) = 0$.

Theorem 2.4.3 *The poset $Invol(I_2(m))$ is*

1. *graded, with rank function given by*

$$\rho(w) = \frac{l(w) + al(w)}{2},$$

   for every $w \in Invol(I_2(m))$;

2. *EL-shellable, hence Cohen-Macaulay;*

3. *Eulerian.*

Proof. Since the results are known to hold for every Coxeter group, in particular for dihedral groups, by (2.1) it follows that $Invol(I_2(m))$ is graded, EL-shellable and Eulerian.

On the other hand it's easy to check that the rank function ρ of $Invol(I_2(m))$ and the average between the length and the absolute length actually are the same function. In fact for every $w \in Invol(I_2(m))$ we have

$$\rho(w) \;=\; \frac{l(w)+al(w)}{2} \;=\; \begin{cases} 0, & \text{if } w = e, \\ \dfrac{l(w)+1}{2}, & \text{if } w \neq e \text{ and } w \neq w_0, \\ \dfrac{m+2}{2}, & \text{if } w = w_0 \text{ and } m \text{ is even}, \\ \dfrac{m+1}{2}, & \text{if } w = w_0 \text{ and } m \text{ is odd}. \end{cases}$$

□

Chapter 3

General techniques

In this section we expose some general techniques that we will follow to prove that a poset is graded and EL-shellable.

3.1 Gradedness

Let P be a finite bounded poset.

Definition 3.1.1 *A* successor system *of P is a subset*

$$H \subseteq \{(x, y) \in P^2 : x < y\}.$$

An insertion system *of P is a successor system H of P such that*

> **(insertion property)** *for every $x, y \in P$, with $x < y$, there exists $z \in P$ such that*
>
> $$(x, z) \in H \quad and \quad z \leq y.$$

A covering system *of P is a pair (H, ρ), where H is an insertion system of P and $\rho : P \to \mathbf{N} \cup \{0\}$ is a statistic on P such that*

> **(ρ-base property)** $\rho(\hat{0}) = 0$*;*
>
> **(ρ-increasing property)** *for every $(x, y) \in H$, we have*
>
> $$\rho(y) = \rho(x) + 1.$$

Next theorem gives a general method to prove that a poset is graded with a given rank function: it suffices to find a covering system of P.

Theorem 3.1.2 *If there exists a covering system (H, ρ) of P, then P is graded with rank function ρ.*

Proof. By the ρ-base property, $\rho(\hat{0}) = 0$. Now let $x, y \in P$, with $x \lhd y$. By the insertion property, there is $z \in P$, such that $(x, z) \in H$ and $z \leq y$. Since H is a successor system, we have $x < z$, and since $x \lhd y$, necessarily $z = y$. By the ρ-increasing property, we have $\rho(y) = \rho(z) = \rho(x) + 1$, $\square$

By Theorem 3.1.2, to prove that a poset P is graded, it suffices to find a covering system of P, and this can be done by the following steps:

1. exhibit a successor system H of P;

2. prove that H is an insertion system, by showing that the insertion property holds;

3. exhibit a candidate rank function ρ and prove that (H, ρ) actually is a covering system, by showing that the ρ-base property and the ρ-increasing property hold.

A covering system (H, ρ) also gives a complete description of the covering relation in P: the pairs of elements which are in covering relation are exactly the pairs in H.

Theorem 3.1.3 *Let (H, ρ) be a covering system of P. Let $x, y \in P$. Then*

$$x \lhd y \quad \Leftrightarrow \quad (x, y) \in H.$$

Proof. If $x \lhd y$, then we have already observed, in the proof of Theorem 3.1.2, that $(x, y) \in H$. On the other hand, for every $(x, y) \in H$ we have $x \lhd y$. In fact, from the insertion property and the ρ-increasing property, it follows that ρ is order-preserving, that is, for every $s, t \in P$, $s < t$ implies $\rho(s) < \rho(t)$. If we suppose, by contradiciton, that there is $z \in P$ such that $x < z < y$, then we have $\rho(y) \geq \rho(x) + 2$, which is in contradiction with $\rho(y) = \rho(x) + 1$. $\square$

3.2 Labels

Let Q be a totally ordered set, the set of *labels*.

Definition 3.2.1 *Let H be a successor system of P. A* good labelling *of H is a function $\lambda : H \to Q$ such that*

> **(injectivity property)** *for every $(x,y),(x,z) \in H$, we have*
>
> $$\lambda(x,y) = \lambda(x,z) \quad \Rightarrow \quad y = z.$$

Let H be a successor system of P and let λ be a good labelling of H. Let $x \in P$. An element $i \in Q$ is a *suitable label* of x if there is $y \in P$ such that $(x,y) \in H$ and $\lambda(x,y) = i$. By the injectivity property, such a y is unique, and we call it the *transformation* of x with respect to the label i, and denote it by

$$t_i^P(x).$$

The set of all suitable labels of x is denoted by $\Lambda(x)$.

To show that H is an insertion system we give a good labelling of H, and we use the following equivalent version of the insertion property:

> **(insertion property)** *for every $x, y \in P$, with $x < y$, there exists a label $i \in \Lambda(x)$ such that*
>
> $$t_i^P(x) \leq y.$$

If (H, ρ) is a covering system of P, then by Theorem 3.1.3 we have $x \lhd y$ if and only if $(x,y) \in H$. In this case a good labelling λ of H is an edge-labelling of P. It is useful to introduce the following terminology: if $x \in P$ and $i \in \Lambda(x)$ then we call $t_i^P(x)$ the *covering transformation* of x with respect to the label i, and denote it by

$$ct_i^P(x).$$

Thus, for every $x \in P$, $i \in \Lambda(x)$ we have $x \lhd ct_i^P(x)$. On the other hand, if $x \lhd y$, then $y = ct_i^P(x)$ for a unique $i \in \Lambda(x)$, and we write also

$$x \underset{i}{\lhd} y.$$

3.3 EL-shellability

In each of the posets that we consider we define a particular edge-labelling, which we call "standard". Then we prove that the posets are EL-shellable by showing that the standard labelling of P is an EL-labelling, and this is done with the following general considerations.

Note that, if (H, ρ) is a covering system of P, then by the insertion property, for every $x, y \in P$, with $x < y$, the set

$$\{i \in \Lambda(x) : ct_i^P(x) \leq y\}$$

is not empty. This allows to give the following definition.

Definition 3.3.1 *Let (H, ρ) be a covering system of P. Let $x, y \in P$, with $x < y$. The* minimal label *of x with respect to y, denoted by $mi_y(x)$ (or simply mi), is*

$$mi_y(x) = \min\{i \in \Lambda(x) : ct_i(x) \leq y\}.$$

The minimal covering transformation *of x with respect to y, denoted by $mct_y^P(x)$, is the covering transformation of x with respect to the minimal label:*

$$mct_y^P(x) = ct_{mi}^P(x).$$

It is useful to state the following, which is a consequence of the definitions.

Theorem 3.3.2 *Let $x, y \in P$, with $x < y$. Then*

$$x \lhd mct_y^P(x) \leq y.$$

By Theorem 3.3.2, the following definition is well-posed.

Definition 3.3.3 *Let $x, y \in P$, with $x < y$. The* minimal chain *from x to y is the saturated chain*

$$x = x_0 \lhd x_1 \lhd \ldots \lhd x_k = y,$$

defined by

$$x_i = mct_y^P(x_{i-1}),$$

for every $i \in [k]$.

By the definition of minimal covering transformation, this chain has, among all the saturated chains from x to y, the lexicographically minimal labelling. To prove the EL-shellability of the poset, it remains to show that this chain has increasing labels (**increasing property**) and that any other saturated chain from x to y, has at least one decrease in the labels (**decreasing property**).

Chapter 4

Preliminary results

In this chapter we discuss some preliminary results, which play a crucial role in the proof of the main result of this work. Precisely, we give a combinatorial description of the absolute length of the involutions in classical Weyl groups, and we describe the minimal covering transformation (in the sense of Definition 3.3.1) in these groups, in particular discovering a characterization of the covering relation in the groups B_n and D_n.

4.1 Absolute length of involutions in classical Weyl groups

In classical Weyl groups there is a nice combinatorial description for the absolute length of the involutions, as we show in this section.

We recall that the absolute length of an element $w \in W$, denoted by $al(w)$, is the minimal k such that w can be written as a product of k reflections.

4.1.1 The symmetric group

We recall that in the symmetric group the reflections are the transpositions:

$$T = \{(i, j) \in [n]^2 : i < j\}.$$

Let $\sigma \in S_n$. The number of *excedances* of σ is

$$exc = |\{i \in [n] : \sigma(i) > i\}|,$$

that is, the number of dots of the diagram which are above the diagonal.

In the symmetric group the absolute length of an involution is simply given by the number of excedances. Note that an involution of S_n has the diagram symmetric with respect to the diagonal.

Theorem 4.1.1 *Let $\sigma \in Invol(S_n)$. Then*

$$al(\sigma) = exc(\sigma).$$

Proof. Let $\{i_1, \dots, i_e\}$ be the excedances of σ. If we set $j_p = \sigma(i_p)$, for $p \in [e]$, then we have

$$\sigma = (i_1, j_1) \dots (i_e, j_e). \tag{4.1}$$

Since $al(\sigma)$ is the minimal number of transpositions in which σ can be decomposed, it follows that

$$al(\sigma) \leq exc(\sigma).$$

On the other hand, for every $\chi \in S_n$ and for every transposition (i, j) we have $exc(\chi(i, j)) \leq exc(\chi) + 1$, as it can be easily checked. So

$$exc(\sigma) \leq al(\sigma).$$

Thus $al(\sigma) = exc(\sigma)$ and (4.1) gives a minimal decomposition of σ as a product of reflections. □

For example, for $\sigma = 32154 \in Invol(S_5)$, we have $al(\sigma) = exc(\sigma) = 2$. In fact

$$\sigma = \underbrace{(1,3)}_{t_1} \cdot \underbrace{(4,5)}_{t_2}$$

is a minimal decomposition of σ as a product of reflections of S_5.

4.1.2 The hyperoctahedral group

We recall that in the hyperoctahedral group the set of reflections is

$$T = \{(i, -i) : i \in [n]\} \cup \{(i, j)(-i, -j) : 1 \leq i < |j| \leq n\}.$$

We define a new statistic on a signed permutation σ.

Definition 4.1.2 *Let $\sigma \in B_n$. The number of* deficiencies-not-antideficiencies *of σ is*

$$dna(\sigma) = |\{i \in [n] : -i \leq \sigma(i) < i\}|,$$

that is, the number of dots of the diagram of σ which are below the main diagonal (deficiencies) and not below the antidiagonal (not-antideficiencies).

For example, consider the signed permutation $\sigma = 4\,\underline{7}\,\underline{3}\,1\,5\,\underline{6}\,\underline{2} \in B_7$, whose diagram is shown in Figure 4.1. Looking at the picture, $dna(\sigma)$ is the number of dots which lie in the gray area. In this case $dna(\sigma) = 4$.

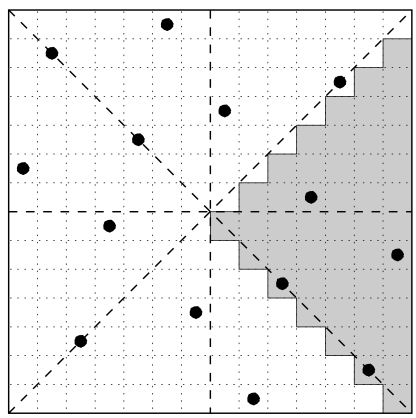


Figure 4.1: The *dna* statistic.

Note that, if σ is an involution of B_n, as in our example, an alternative description of $dna(\sigma)$ is the following:

$$dna(\sigma) = \frac{n - fix^+(\sigma) + afix^+(\sigma)}{2},$$

where

$$fix^+(\sigma) = |\{i \in [n] : \sigma(i) = i\}| \quad \text{and} \quad afix^+(\sigma) = |\{i \in [n] : \sigma(i) = -i\}|$$

are, respectively, the number of *positive fixed points* and *positive antifixed points* of σ, that is, the number of dots of the diagram of σ which are, respectively, in the cells (i, j), with $i > 0$, of the main diagonal and of the antidiagonal.

In the hyperoctahedral group the absolute length of an involution is given by the *dna* statistic. Note that an involution of B_n has the diagram symmetric with respect to both the diagonals.

Theorem 4.1.3 *Let $\sigma \in Invol(B_n)$. Then*

$$al(\sigma) = dna(\sigma).$$

Proof. Let $\{i_1, \ldots, i_r\}$ be the deficiencies-antiexcedances of σ, that is, the indices $i > 0$ such that $-i < \sigma(i) < i$. Let $\{h_1, \ldots, h_s\}$ be the positive antifixed points of σ, that is, the indices $i > 0$ such that $\sigma(i) = -i$. Obviously $dna(\sigma) = r + s$. If we set $j_p = \sigma(i_p)$, for every $p \in [r]$, then we have

$$\sigma = \prod_{p=1}^{r} (i_p, j_p)(-i_p, -j_p) \cdot \prod_{q=1}^{s} (h_q, -h_q). \tag{4.2}$$

Since $t_p = (i_p, j_p)(-i_p, -j_p)$, for $p \in [r]$, and $t_{r+q} = (h_q, -h_q)$, for $q \in [s]$, are all reflections of B_n, we have

$$al(\sigma) \leq r + s = dna(\sigma).$$

On the other hand for every $\chi \in B_n$ and for every reflection t of B_n we have $dna(\chi t) \leq dna(\chi) + 1$. So

$$dna(\sigma) \leq al(\sigma).$$

Thus $al(\sigma) = dna(\sigma)$ and (4.2) gives a minimal decomposition of σ as a product of reflections. □

For example, for the involution of Figure 4.1, we have $al(\sigma) = dna(\sigma) = 4$. In fact

$$\sigma = \underbrace{(1,4)(-1,-4)}_{t_1} \cdot \underbrace{(7,-2)(-7,2)}_{t_2} \cdot \underbrace{(3,-3)}_{t_3} \cdot \underbrace{(6,-6)}_{t_4} \tag{4.3}$$

is a minimal decomposition of σ as a product of reflections of B_7.

4.1.3 The even-signed permutation group

We recall that in the even-signed permutation group, the set of reflections is

$$T = \{(i,j)(-i,-j) : 1 \leq i < |j| \leq n\}.$$

A surprising fact is that in the hyperoctahedral group and in the even-signed permutation group, the combinatorial description for the absolute length of an involution is exactly the same: in both cases it is given by the *dna* statistic. But the reasons are different.

Theorem 4.1.4 *Let $\sigma \in Invol(D_n)$. Then*

$$al(\sigma) = dna(\sigma).$$

Proof. Let $\{i_1, \ldots, i_r\}$ be the deficiencies-antiexcedances of σ. Note that an even-signed permutation which is an involution must have an even number of positive antifixed points, so we can consider them in pairs: let $\{h_1, k_1, \ldots, h_s, k_s\}$ be the positive antifixed points of σ. We now have $dna(\sigma) = r + 2s$. If we set $j_p = \sigma(i_p)$, for every $p \in [r]$, then we have

$$\sigma = \prod_{p=1}^{r} (i_p, j_p)(-i_p, -j_p) \cdot \prod_{q=1}^{s} (h_q, k_q)(-h_q, -k_q) \cdot \prod_{q=1}^{s} (h_q, -k_q)(-h_q, k_q). \quad (4.4)$$

Since $t_p = (i_p, j_p)(-i_p, -j_p)$, for $p \in [r]$, $t_{r+2q-1} = (h_q, k_q)(-h_q, -k_q)$ and $t_{r+2q} = (h_q, -k_q)(-h_q, k_q)$, for $q \in [s]$, are all reflections of D_n, we have

$$al(\sigma) \leq r + 2s = dna(\sigma).$$

On the other hand for every $\chi \in B_n$ and for every reflection t of B_n we have $dna(\chi t) \leq dna(\chi) + 1$. So

$$dna(\sigma) \leq al(\sigma).$$

Thus $al(\sigma) = dna(\sigma)$ and (4.4) gives a minimal decomposition of σ as a product of reflections. □

For example, for the involution of Figure 4.1, which is also in $Invol(D_7)$, we have $al(\sigma) = dna(\sigma) = 4$. Note that the decomposition in (4.3) does not work in D_7, since $(3, -3)$ and $(6, -6)$ are not elements of D_7. But

$$\sigma = \underbrace{(1,4)(-1,-4)}_{t_1} \cdot \underbrace{(7,-2)(-7,2)}_{t_2} \cdot \underbrace{(3,6)(-3,-6)}_{t_3} \cdot \underbrace{(3,-6)(-3,6)}_{t_4}$$

is a minimal decomposition of σ as a product of reflections of D_7.

4.2 Minimal covering transformation in classical Weyl groups

In this section we apply the general techniques introduced in Chapter 3 to classical Weyl groups, obtaining results which give new proofs of known facts and which will be the starting points for the proofs of next chapters.

4.2.1 The symmetric group

By Proposition 1.6.3, we can define a natural edge-labelling of S_n (the same introduced by Edelman in [Ede] to prove the EL-shellability of S_n).

Definition 4.2.1 *The* standard labelling *of S_n is the edge-labelling*

$$\lambda : \{(x,y) \in S_n^2 : x \lhd y\} \to \{(i,j) \in [n]^2 : i < j\}$$

defined in the following way: for every $\sigma, \tau \in S_n$, with $\sigma \lhd \tau$, we set

$$\lambda(\sigma, \tau) = (i,j),$$

where (i,j) is the free rise of σ such that

$$\tau = \sigma(i,j).$$

With the terminology introduced in Section 3.2, we can say that the *suitable labels* of σ are its free rises and that, if (i,j) is a free rise of σ, then the *covering transformation* of σ with respect to (i,j) is

$$ct_{(i,j)}^{S_n}(\sigma) = \sigma(i,j).$$

In order to describe the minimal covering transformation, we give the following definitions.

Definition 4.2.2 *Let $\sigma, \tau \in S_n$, with $\sigma < \tau$. The* difference index *of σ with respect to τ, denoted by $di_\tau(\sigma)$ (or simply di), is the minimal index on which σ and τ differ:*

$$di_\tau(\sigma) = \min\{i \in [n] : \sigma(i) \neq \tau(i)\}.$$

We write di, instead of $di_\tau(\sigma)$, when there is no ambiguity about the permutations σ and τ which we are referring to.

Lemma 4.2.3 *Let $\sigma, \tau \in S_n$, with $\sigma < \tau$. Then*

$$\sigma(di) < \tau(di).$$

Proof. First note that, by definition, we have

$$\sigma(di) \neq \tau(di).$$

Now suppose, by contradiction, that $\sigma(di) > \tau(di)$. In this case we would have $\sigma[di, \sigma(di)] = \tau[di, \sigma(di)] + 1$. But $\sigma < \tau$ and, by Theorem 1.6.4, this implies $\sigma[di, \sigma(di)] \leq \tau[di, \sigma(di)]$, which is a contradiction. □

Lemma 4.2.4 *Let $\sigma, \tau \in S_n$, with $\sigma < \tau$. Then the set*

$$\{j \in [di+1, n] : \sigma(j) \in [\sigma(di)+1, \tau(di)]\} \tag{4.5}$$

is not empty.

Proof. Set $k = \sigma^{-1}(\tau(di))$. If $k \in [di-1]$, then $\sigma(k) = \tau(k)$, that is $k = di$, which is a contradiction. If $k = di$, then σ and τ agree at the index di, which is also a contradiction. Thus $k \in [di+1, n]$. Also, $\sigma(k) = \tau(di)$, so k belongs to the set (4.5). □

Previous lemmas ensure that next definition is well-posed.

Definition 4.2.5 *Let $\sigma, \tau \in S_n$, with $\sigma < \tau$. The* covering index *of σ with respect to τ, denoted by $ci_\tau(\sigma)$ (or simply ci), is*

$$ci_\tau(\sigma) = \min\{j \in [di+1, n] : \sigma(j) \in [\sigma(di)+1, \tau(di)]\}.$$

By definition (di, ci) is a free rise of σ, so it is one of its suitable labels. In next two propositions we prove that it is the *minimal label* of σ with respect to τ, in the sense of Definition 3.3.1.

Proposition 4.2.6 *Let $\sigma, \tau \in S_n$, with $\sigma < \tau$. Then*

$$ct^{S_n}_{(di,ci)}(\sigma) = \sigma(di, ci) \leq \tau.$$

Proof. Let $\chi = \sigma(di, ci)$. We may assume, without loss of generality, that $di = 1$. Set $R = [1, ci-1] \times [\sigma(1)+1, \sigma(ci)]$. For every $(h,k) \in [n]^2$, we have

$$\chi[h,k] = \begin{cases} \sigma[h,k]+1, & \text{if } (h,k) \in R, \\ \sigma[h,k], & \text{if } (h,k) \notin R. \end{cases}$$

Thus, by Theorem 1.6.4, to prove that $\chi \leq \tau$, we only have to show that $\tau[h,k] \geq \sigma[h,k]+1$ for every $(h,k) \in R$. But if $(h,k) \in R$, then we have

$$\sigma[h,k] = \sigma[h, \tau(1)+1] \leq \tau[h, \tau(1)+1] \leq \tau[h,k] - 1,$$

so $\chi \leq \tau$. □

Proposition 4.2.7 *Let* $\sigma, \tau \in S_n$*, with* $\sigma < \tau$*. Then*

$$mi_\tau(\sigma) = (di, ci).$$

Proof. Let (i,j) be a free rise of σ such that

$$ct^{S_n}_{(i,j)}(\sigma) = \sigma(i,j) \le \tau.$$

We want to prove that $(di, ci) \le (i,j)$. Suppose, by contradiction, that $(i,j) < (di, ci)$, so either $i < di$, or $i = di$ and $j < ci$. If $i < di$, since σ and τ must differ at the index i, the minimality of di is contradicted. If $i = di$ and $j < ci$, set $\xi = \sigma(i,j)$. We have $\xi(di) = \sigma(j)$ and, since $\xi \le \tau$, by Lemma 4.2.3, $\xi(di) \le \tau(di)$. So $\sigma(j) \le \tau(di)$ and this contradicts the minimality of ci. □

Thus in the symmetric group the *minimal covering transformation* of σ with respect to τ is

$$mct^{S_n}_\tau(\sigma) = \sigma(di, ci),$$

and we have the following.

Theorem 4.2.8 *Let* $\sigma, \tau \in S_n$*, with* $\sigma < \tau$*. Then*

$$\sigma \lhd mct^{S_n}_\tau(\sigma) \le \tau.$$

Starting from Theorem 4.2.8, with the techniques described in Section 3.3, it could be proved that the standard labelling of S_n is an EL-labelling, rediscovering the result proved by Edelman.

4.2.2 The hyperoctahedral group

Definition 4.2.9 *Let* $\sigma \in B_n$*. A rise* (i,j) *of* σ *is* central *if*

$$(0,0) \in [i,j] \times [\sigma(i), \sigma(j)].$$

A central rise (i,j) *of* σ *is* symmetric *if* $j = -i$*.*

In order to find a characterization of the covering relation in B_n, we start defining a successor system.

Definition 4.2.10 *Let $\sigma, \tau \in B_n$. We say that (σ, τ) is a* good pair *in B_n if either*

1. $\tau = \sigma(i,j)(-i,-j)$*, where (i,j) is a not central free rise of σ, or*

2. $\tau = \sigma(i,j)$*, where (i,j) is a central symmetric free rise of σ.*

Definition 4.2.10 is illustrated in Figure 4.2 (whose caption will be made clear later), where black and white circles denote respectively σ and τ, inside the gray areas there are no other dots of σ and τ, and the diagrams of the two permutations are supposed to be the same anywhere else.

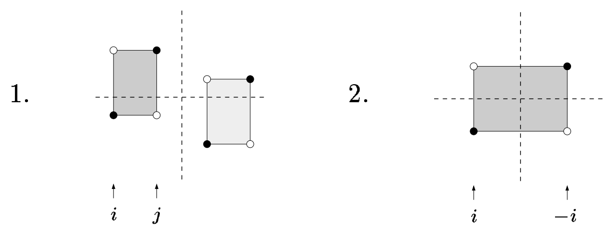


Figure 4.2: Covering relation in B_n.

We set

$$H_{B_n} = \{(\sigma,\tau) \in B_n^2 : (\sigma,\tau) \text{ is a good pair in } B_n\},$$

and define the *standard labelling* λ of B_n by associating with every good pair $(\sigma,\tau) \in H_{B_n}$, the pair $(i,j) \in [\pm n]^2$ mentioned in Definition 4.2.10, which is obviously unique.

By Propositions 1.6.8 and 1.6.3, it follows that H_{B_n} is a successor system of B_n and, since τ is uniquely determined by σ and by the label (i,j), λ is a good labelling.

Given $\sigma \in B_n$, the *suitable labels* of σ are then the not central free rises of σ and the central symmetric free rises of σ. If (i,j) is a suitable label of σ then the *transformation* of σ with respect to (i,j) is

$$t^{B_n}_{(i,j)}(\sigma) = \begin{cases} \sigma(i,j)(-i,-j), & \text{if } (i,j) \text{ is not central,} \\ \sigma(i,j), & \text{if } (i,j) \text{ is central symmetric.} \end{cases}$$

Now let $\sigma, \tau \in B_n$, with $\sigma < \tau$. In order to prove that the insertion property holds, we define the label

$$i_{B_n}(\sigma,\tau) = \begin{cases} (di, ci), & \text{if } (di, ci) \text{ is not central}, \\ (di, -di), & \text{if } (di, ci) \text{ is central}. \end{cases}$$

Note that $i_{B_n}(\sigma,\tau)$ is always a suitable label of σ. So we can define the signed permutation

$$\chi_{B_n}(\sigma,\tau) = t^{B_n}_{i_{B_n}(\sigma,\tau)}(\sigma) = \begin{cases} \sigma(di, ci)(-di, -ci), & \text{if } (di, ci) \text{ is not central}, \\ \sigma(di, -di), & \text{if } (di, ci) \text{ is central}. \end{cases}$$

Proposition 4.2.11 *Let $\sigma, \tau \in B_n$, with $\sigma < \tau$. Then*

$$\chi_{B_n}(\sigma,\tau) \leq \tau.$$

Proof. Let $\chi = \chi_{B_n}(\sigma,\tau)$. If (di, ci) is not central then

$$\chi = \sigma(di, ci)(-di, -ci) = mct^{S_{\pm n}}_{\tau}(w_0(mct^{S_{\pm n}}_{\tau}(\sigma))w_0).$$

Thus, by Proposition 1.6.8, Proposition 1.4.3 and Theorem 4.2.8 (applied twice), we have

$$\chi \leq \tau.$$

If (di, ci) is central, then $\chi = \sigma(di, -di)$. We may assume, without loss of generality, that $di = -n$. So necessarily $\sigma(di) = -1$. Set $R = [\pm n] \times \{1\}$. For every $(h, k) \in [\pm n]^2$ we have

$$\chi[h,k] = \begin{cases} \sigma[h,k] + 1, & \text{if } (h,k) \in R, \\ \sigma[h,k], & \text{if } (h,k) \notin R. \end{cases}$$

Thus to prove that $\chi \leq \tau$ it suffices to show that $\tau[h,k] \geq \sigma[h,k] + 1$ for every $(h,k) \in R$. By the simmetry of the diagram, it's enough to show that $\tau[h,1] \geq \sigma[h,1] + 1$ for every $h \in [-n]$. But, if $h \in [-n]$ we have

$$\sigma[h,1] = \sigma[h, \tau(di) + 1] \leq \tau[h, \tau(di) + 1] \leq \tau[h,1] - 1.$$

□

We recall that the length of $\sigma \in B_n$ is given by

$$l(\sigma) = \frac{inv(\sigma) + neg(\sigma)}{2}.$$

Proposition 4.2.12 *The pair* (H_{B_n}, l) *is a covering system of* B_n.

Proof. By Proposition 4.2.11, H_{B_n} is an insertion system of B_n. The ρ-base property is trivial. It remains to prove the ρ-increasing property. Consider $(\sigma, \tau) \in H_{B_n}$ and let $\lambda(\sigma, \tau) = (i, j)$. We have

$$inv(\tau) = \begin{cases} inv(\sigma) + 2, & \text{if } (i,j) \text{ is not central,} \\ inv(\sigma) + 1, & \text{if } (i,j) \text{ is central syemmtric,} \end{cases}$$

and

$$neg(\tau) = \begin{cases} neg(\sigma), & \text{if } (i,j) \text{ is not central} \\ neg(\sigma) + 1, & \text{if } (i,j) \text{ is central symmetric.} \end{cases}$$

Thus in each case $l(\tau) = l(\sigma) + 1$. □

We have found a covering system of B_n. So we have a characterization of the covering relation in B_n, which we state in the following.

Theorem 4.2.13 *Let* $\sigma, \tau \in B_n$. *Then* $\sigma \lhd \tau$ *if and only if either*

1. $\tau = \sigma(i,j)(-i,-j)$, *where* (i,j) *is a not central free rise of* σ, *or*
2. $\tau = \sigma(i,j)$, *where* (i,j) *is a central symmetric free rise of* σ.

Theorem 4.2.13 is illustrated in Figure 4.2.

If $\sigma \in B_n$ and (i,j) is a suitable label of σ, then the covering transformation of σ with respect to (i,j) actually is a *covering transformation*, denoted by

$$ct^{B_n}_{(i,j)}(\sigma).$$

In next proposition we prove that $i_{B_n}(\sigma, \tau)$ is the *minimal label* of σ with respect to τ, in the sense of Definition 3.3.1.

Proposition 4.2.14 *Let* $\sigma, \tau \in B_n$, *with* $\sigma < \tau$. *Then*

$$mi_\tau(\sigma) = i_{B_n}(\sigma, \tau) = \begin{cases} (di, ci), & \text{if } (di, ci) \text{ is not central,} \\ (di, -di), & \text{if } (di, ci) \text{ is central.} \end{cases}$$

Proof. If (di, ci) is not central (case 1) then $i_{B_n}(\sigma, \tau) = (di, ci)$, otherwise (case 2) $i_{B_n}(\sigma, \tau) = (di, -di)$. Let (i,j) be a suitable label of σ such that $ct^{B_n}_{(i,j)}(\sigma) \leq \tau$. We want to prove that

$$i_{B_n}(\sigma, \tau) \leq (i, j).$$

Necessarily $i \geq di$. If $i > di$ then $i_{B_n}(\sigma,\tau) < (i,j)$. So, suppose $i = di$.

In case 1, we have to prove that $j \geq ci$. Suppose, by contradiction, that $j < ci$ and set $\xi = \sigma(i,j)$. We have $\xi(di) = \sigma(j)$ and, since $\xi \leq \tau$, by Lemma 4.2.3, $\xi(di) \leq \tau(di)$. So $\sigma(j) \leq \tau(di)$ and this contradicts the minimality of ci.

In case 2, we have to prove that $j \geq -di$ (actually, the only possibility is $j = -di$). If we suppose $j < ci$, as in case 1 we get a contradiction. Thus $j \geq ci$. Since (di, j) is a suitable label of σ and it is central, it has to be symmetric, that is $j = -di$. □

Thus in the hyperoctahedral group the *minimal covering transformation* of σ with respect to τ is

$$mct_\tau^{B_n}(\sigma) = \begin{cases} \sigma(di, ci)(-di, -ci), & \text{if } (di, ci) \text{ is not central,} \\ \sigma(di, -di), & \text{if } (di, ci) \text{ is central,} \end{cases}$$

and we have the following.

Theorem 4.2.15 *Let $\sigma, \tau \in B_n$, with $\sigma < \tau$. Then*

$$\sigma \lhd mct_\tau^{B_n}(\sigma) \leq \tau.$$

Starting from Theorem 4.2.15, it could be proved that the standard labelling of B_n is an EL-labelling, rediscovering a known result.

4.2.3 The even-signed permutation group

Definition 4.2.16 *Let $\sigma \in D_n$. A central rise (i, j) is* semifree *if*

$$\{k \in [i,j] : \sigma(k) \in [\sigma(i), \sigma(j)]\} = \{i, -j, j\},$$

in other words if the only dots of the diagram of σ lying in the rectangle $[i,j] \times [\sigma(i), \sigma(j)]$ are those in the cells $(i, \sigma(i))$, $(-j, -\sigma(j))$ and $(j, \sigma(j))$.

An example of central semifree rise is illustrated in Figure 4.3 (3).

As we did for B_n, we start defining a successor system of D_n.

Definition 4.2.17 *Let $\sigma, \tau \in D_n$. We say that (σ, τ) is a* good pair *in D_n if*

$$\tau = \sigma(i,j)(-i,-j),$$

where (i, j) is either

1. *a not central free rise of* σ*, or*

2. *a central not symmetric free rise of* σ*, or*

3. *a central semifree rise of* σ*.*

Definition 4.2.17 is illustrated in Figure 4.3 (whose caption will be made clear later), where we use the same notation as in Figure 4.2.

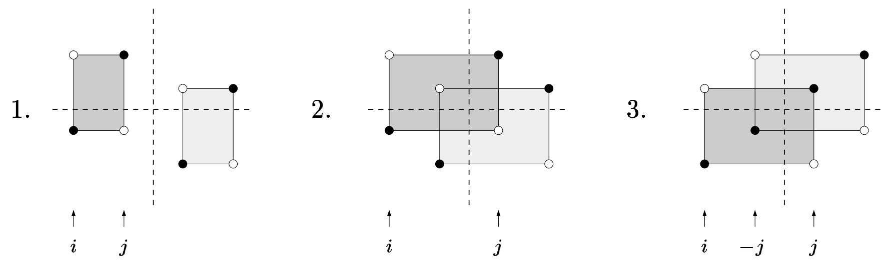


Figure 4.3: Covering relation in D_n.

We set

$$H_{D_n} = \{(\sigma,\tau) \in D_n^2 : (\sigma,\tau) \text{ is a good pair in } D_n\},$$

and define the *standard labelling* λ of D_n by associating with every good pair $(\sigma,\tau) \in H_{D_n}$, the pair $(i,j) \in [\pm n]^2$ mentioned in Definition 4.2.17, which is obviously unique.

It's easy to see that H_{D_n} is a successor system of D_n and, since τ is uniquely determined by σ and by the label (i,j), λ is a good labelling.

Given $\sigma \in D_n$, the *suitable labels* of σ are then the not central free rises of σ, the central not symmetric free rises of σ and the central semifree rises of σ. If (i,j) is a suitable label of σ then the *transformation* of σ with respect to (i,j) is

$$t^{D_n}_{(i,j)}(\sigma) = \sigma(i,j)(-i,-j).$$

In order to prove that the insertion property holds, we need the following definition, which can be given in general for the symmetric group, and which the hyperoctahedral group and the even-signed permutation group inherit.

Definition 4.2.18 *Let* $\sigma,\tau \in S_n$*, with* $\sigma < \tau$*. Suppose that the set*

$$\{j \in [ci+1,n] : \sigma(j) \in [\sigma(di)+1, \sigma(ci)-1]\}$$

is not empty. Then the second covering index *of σ with respect to τ, denoted by $sci_\tau(\sigma)$ (or simply sci) is*

$$sci_\tau(\sigma) = \min\{j \in [ci+1, n] : \sigma(j) \in [\sigma(di)+1, \sigma(ci)-1]\}.$$

Definition 4.2.19 *Let $\sigma, \tau \in D_n$, with $\sigma < \tau$. We say that (σ, τ) is a D-*special pair *if*

1. $(di <)\ ci < 0$;
2. $(\sigma(di) <)\ \sigma(ci) < 0$;
3. $\tau(di) = -\sigma(ci)$;
4. $[ci+1, -ci-1] \times [\sigma(ci), -\sigma(ci)]$ *is empty for σ.*

Moreover, a special pair (σ, τ) can be either of the first kind, *if*

5'. $[ci+1, -ci-1] \times [\sigma(di), -\sigma(di)]$ *is not empty for σ,*

or of the second kind, *if*

5". $[ci+1, -ci-1] \times [\sigma(di), -\sigma(di)]$ *is empty for σ.*

Let $\sigma, \tau \in D_n$, with $\sigma < \tau$. We define the label

$$i_{D_n}(\sigma,\tau) = \begin{cases} (di, ci), & \text{if } (\sigma,\tau) \text{ is not a } D\text{-special pair,} \\ (di, sci), & \text{if } (\sigma,\tau) \text{ is a } D\text{-special pair of the first kind,} \\ (di, -ci), & \text{if } (\sigma,\tau) \text{ is a } D\text{-special pair of the second kind,} \end{cases}$$

Note that, if (σ, τ) is a D-special pair of the first kind, then, by 4 and 5', sci necessarily exists. Also note that $i_{D_n}(\sigma, \tau)$ is always a suitable label of σ, so we can define the even-signed permutation

$$\chi_{D_n}(\sigma, \tau) = t^{D_n}_{i_{D_n}(\sigma,\tau)}(\sigma).$$

All cases are shown in Figure 4.4, where σ, τ and $\chi = \chi_{D_n}(\sigma, \tau)$ are represented. Black circles denote σ, white squares τ and white circles χ. Only the dots in columns di and $-di$ of τ are represented, possibly with a gray rectangle around, denoting the range of variation of $\tau(di)$. Inside the gray rectangles there are no other dots of σ and χ than those indicated and the diagrams of σ and χ are supposed to be the same anywhere else.

If (σ, τ) is not a D-special pair, we distinguish between the following cases:

1. $(di <)\ 0 < ci,\ \sigma(di) < 0 < \sigma(ci);$
2. $(di <)\ 0 < ci,\ 0 < \sigma(di)\ (< \sigma(ci));$
3. $(di <)\ 0 < ci,\ (\sigma(di) <)\ \sigma(ci) < 0;$
4. $(di <)\ ci < 0,\ 0 < \sigma(di)\ (< \sigma(ci));$
5. $(di <)\ ci < 0,\ \sigma(di) < 0 < \sigma(ci),\ \sigma(ci) > -\sigma(di);$
6. $(di <)\ ci < 0,\ \sigma(di) < 0 < \sigma(ci),\ \sigma(ci) < -\sigma(di);$
7. $(di <)\ ci < 0,\ (\sigma(di) <)\ \sigma(ci) < 0,\ \tau(di) \neq -\sigma(ci);$
8. $(di <)\ ci < 0, (\sigma(di) <)\ \sigma(ci) < 0,\ \tau(di) = -\sigma(ci),$
   $[ci + 1, -ci - 1] \times [\sigma(ci), -\sigma(ci)]$ is not empty for σ.

Otherwise (σ, τ) can be either a D-special pair of the first kind:

9. $(di <)\ ci < 0, (\sigma(di) <)\ \sigma(ci) < 0,\ \tau(di) = -\sigma(ci),$
   $[ci + 1, -ci - 1] \times [\sigma(ci), -\sigma(ci)]$ is empty for σ,
   but $[ci + 1, -ci - 1] \times [\sigma(di), -\sigma(di)]$ is not,
   and we distinguish between

   9a. $sci < 0$ and

   9b. $sci > 0;$

or a D-special pair of the second kind:

10. $(di <)\ ci < 0, (\sigma(di) <)\ \sigma(ci) < 0,\ \tau(di) = -\sigma(ci),$
    $[ci + 1, -ci - 1] \times [\sigma(di), -\sigma(di)]$ is empty for σ.

Theorem 4.2.20 *Let $\sigma, \tau \in D_n$, with $\sigma < \tau$. Then*

$$\chi_{D_n}(\sigma, \tau) \leq \tau.$$

The proof of Theorem 4.2.20 is rather technical and will be exposed at the end of this section.

We recall that the length of $\sigma \in D_n$ is given by

$$l(\sigma) = \frac{inv(\sigma) - neg(\sigma)}{2}.$$

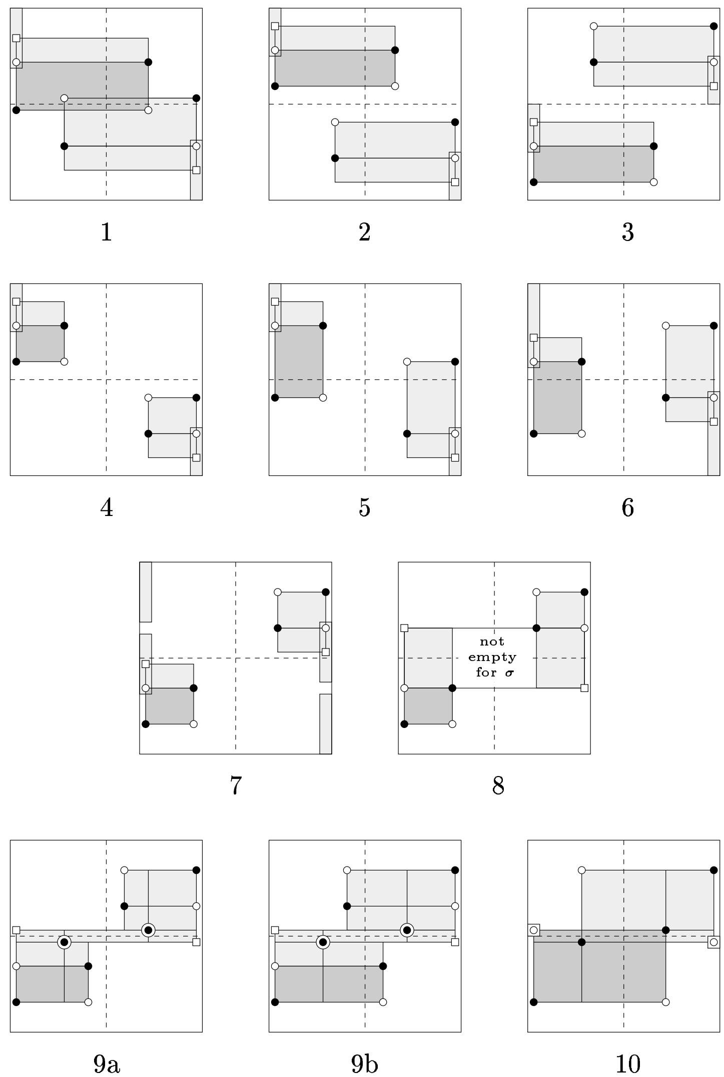


Figure 4.4: Minimal covering transformation in D_n.

Proposition 4.2.21 *The pair (H_{D_n}, l) is a covering system of D_n.*

Proof. By Theorem 4.2.20, H_{D_n} is an insertion system of D_n. The ρ-base property is trivial. It remains to prove the ρ-increasing property. Consider $(\sigma, \tau) \in H_{D_n}$. We refer to the cases as in Figure 4.4. We have

$$inv(\tau) = \begin{cases} inv(\sigma) + 4, & \text{in cases 1 and 10,} \\ inv(\sigma) + 2, & \text{in all other cases.} \end{cases}$$

and

$$neg(\tau) = \begin{cases} neg(\sigma) + 2, & \text{in cases 1 and 10,} \\ neg(\sigma), & \text{in all other cases.} \end{cases}$$

Thus in each case $l(\tau) = l(\sigma) + 1$. □

We have found a covering system of D_n. So we have a characterization of the covering relation in D_n, which we state in the following.

Theorem 4.2.22 *Let $\sigma, \tau \in D_n$. Then $\sigma \lhd \tau$ if and only if*

$$\tau = \sigma(i,j)(-i,-j),$$

where (i, j) is either

1. *a not central free rise of σ, or*
2. *a central not symmetric free rise of σ, or*
3. *a central semifree rise of σ.*

Theorem 4.2.22 is illustrated in Figure 4.3.

If $\sigma \in D_n$ and (i, j) is a suitable label of σ, then the transformation of σ with respect to (i, j) actually is a *covering transformation*, denoted by

$$ct^{D_n}_{(i,j)}(\sigma) = \sigma(i,j)(-i,-j).$$

We now prove that $i_{D_n}(\sigma, \tau)$ is the *minimal label* of σ with respect to τ.

Proposition 4.2.23 *Let $\sigma, \tau \in D_n$, with $\sigma < \tau$. Then*

$$mi_\tau(\sigma) = i_{D_n}(\sigma, \tau).$$

Proof. Let $\chi = \chi_{D_n}(\sigma,\tau)$. If (σ,τ) is not a D-special pair (case 1), then $i_{D_n}(\sigma,\tau) = (di, ci)$, if (σ,τ) is a D-special pair of the first kind (case 2), then $i_{D_n}(\sigma,\tau) = (di, sci)$, and if (σ,τ) is a D-special pair of the second kind (case 3), then $i_{D_n}(\sigma,\tau) = (di, -ci)$. Let (i,j) be a suitable label of σ such that $ct^{D_n}_{(i,j)}(\sigma) \le \tau$ and let $\xi = ct^{D_n}_{(i,j)}(\sigma)$. We want to prove that

$$i_{D_n}(\sigma,\tau) \le (i,j).$$

Necessarily $i \ge di$. If $i > di$ then $i_{D_n}(\sigma,\tau) < (i,j)$. So suppose $i = di$.

In case 1, we have to prove that $j \ge ci$. Suppose, by contradiction, that $j < ci$. By the definition of ci, we have $\sigma(j) > \tau(di)$. But $\xi(di) = \sigma(j)$, so $\xi(di) > \tau(di)$, which contradicts $\xi \le \tau$.

In cases 2 and 3, we have to prove, respectively, that $j \ge sci$ and $j \ge -ci$. Suppose that the contrary is true. Looking at Figure 4.4 (9a, 9b, 10), it's easy to see that, in both cases, the only possibilities are $j = ci$ or $j < ci$ and $\sigma(j) > \tau(di)$. But if $j = ci$, then $\xi = \sigma(di, ci)(-di, -ci)$ and the pair (ξ, τ) does not satisfy the D-condition, since $(di + 1, -\sigma(ci) + 1)$ is a not valid D-cell of (ξ, τ), contradicting $\xi \le \tau$. On the other hand, if $j < ci$ and $\sigma(j) > \tau(di)$, then the conclusion is the same as in case 1. □

Thus in the even-signed permutation group the *minimal covering transformation* of σ with respect to τ is

$$mct^{D_n}_{\tau}(\sigma) = \begin{cases} \sigma(di, ci)(-di, -ci), & \text{if } (\sigma,\tau) \text{ is not a } D\text{-special pair,} \\ \sigma(di, sci)(-di, -sci), & \text{if } (\sigma,\tau) \text{ is a } D\text{-special pair} \\ & \qquad \text{of the first kind,} \\ \sigma(di, -ci)(-di, ci), & \text{if } (\sigma,\tau) \text{ is a } D\text{-special pair} \\ & \qquad \text{of the second kind,} \end{cases}$$

and we have the following.

Theorem 4.2.24 *Let $\sigma, \tau \in D_n$, with $\sigma < \tau$. Then*

$$\sigma \lhd mct^{D_n}_{\tau}(\sigma) \le \tau.$$

As for S_n and B_n, starting from Theorem 4.2.24, it could be proved that the standard labelling of D_n is an EL-labelling, rediscovering a known result.

To prove Theorem 4.2.20 we need two preliminary lemmas.

Lemma 4.2.25 *Let $\sigma, \tau \in D_n$ be such that (σ, τ) satisfies the B-condition. Let $(h, k) \in [-n] \times [n]$ be such that $\sigma_{NW}[h, k] = \tau_{NW}[h, k]$. If (h, k) is free for σ, then (h, k) is also free for τ, hence it is a D-cell of (σ, τ).*

Proof. Consider the equality $\sigma_{[h]\times[\pm n]} = \tau_{[h]\times[\pm n]}\ (= h)$. We have $\sigma_{[h]\times[\pm n]} = \sigma_N[h, k]$ (since (h, k) is free for σ) and $\tau_{[h]\times[\pm n]} = \tau_N[h, k] + \tau_{[h]\times[\pm k]}$. So

$$\tau_{[h]\times[\pm k]} = \sigma_N[h, k] - \tau_N[h, k],$$

and

$$\sigma_N[h, k] \geq \tau_N[h, k].$$

On the other hand, by the B-condition, we have $\sigma[|h|, k+1] \leq \tau[|h|, k+1]$, that is

$$\sigma_{NW}[h, k] + \sigma_N[h, k] \leq \tau_{NW}[h, k] + \tau_N[h, k].$$

So, by $\sigma_{NW}[h, k] = \tau_{NW}[h, k]$, we have

$$\sigma_N[h, k] \leq \tau_N[h, k].$$

Thus $\sigma_N[h, k] = \tau_N[h, k]$ and $\tau_{[h]\times[\pm k]} = 0$, that is, (h, k) is free for τ. □

For the second lemma, we introduce the following notation: for $\sigma \in D_n$, $(h, k) \in [-n] \times [n]$ and $k_1 \in [k]$, we set

$$\begin{aligned} \sigma_{left}[h; k_1, k] &= \sigma_{[-n,h-1]\times[k_1,k]}, \\ \sigma_{right}[h; k_1, k] &= \sigma_{[h,n]\times[k_1,k]}. \end{aligned}$$

Lemma 4.2.26 *Let $\sigma, \tau \in D_n$ be such that (σ, τ) satisfies the B-condition. Let $(h, k) \in [-n] \times [n]$ be such that $\sigma_{NW}[h, k] = \tau_{NW}[h, k]$. Let $k_1 \in [k]$. Set*

$$\sigma_{left} = \sigma_{left}[h; k_1, k],$$

and similarly for σ_{right}, τ_{left} and τ_{right}. Then

$$\begin{cases} \tau_{right} \leq \sigma_{right}, \\ \sigma_{left} \leq \tau_{left} \leq \sigma_{left} + \sigma_{right}. \end{cases}$$

Moreover, if $\sigma_{right} \leq \tau_{right}$, in particular if $\sigma_{right} = 0$, then

$$\begin{cases} \tau_{right} = \sigma_{right}, \\ \tau_{left} = \sigma_{left}, \end{cases}$$

and if $k_1 \in [2, k]$ *we have*

$$\sigma_{NW}[h, k_1 - 1] = \tau_{NW}[h, k_1 - 1],$$

otherwise, if $k_1 = 1$*, we have*

$$\sigma_{Wup}[h, k] = \tau_{Wup}[h, k].$$

Proof. By the B-condition, we have $\sigma[h-1, k_1] \leq \tau[h-1, k_1]$, that is

$$\sigma_{NW}[h, k] + \sigma_{left} \leq \tau_{NW}[h, k] + \tau_{left}.$$

So, by $\sigma_{NW}[h, k] = \tau_{NW}[h, k]$, we have $\sigma_{left} \leq \tau_{left}$.

Consider the equality $\sigma_{[\pm n]\times[k_1,k]} = \tau_{[\pm n]\times[k_1,k]}$ $(= k - k_1 + 1)$, that is

$$\sigma_{left} + \sigma_{right} = \tau_{left} + \tau_{right}.$$

It follows

$$\sigma_{right} - \tau_{right} = \tau_{left} - \sigma_{left} \geq 0.$$

So $\tau_{right} \leq \sigma_{right}$ and $\sigma_{left} \leq \tau_{left} \leq \sigma_{left} + \sigma_{right}$.

If $\sigma_{right} \leq \tau_{right}$ then obviously $\tau_{right} = \sigma_{right}$ and $\tau_{left} = \sigma_{left}$.

In this case, if $k_1 \in [2, k]$ we have

$$\sigma_{NW}[h, k_1 - 1] = \sigma_{NW}[h, k] + \sigma_{left} = \tau_{NW}[h, k] + \tau_{left} = \tau_{NW}[h, k_1 - 1],$$

and if $k_1 = 1$ we have

$$\sigma_{Wup}[h, k] = \sigma_{left} = \tau_{left} = \tau_{Wup}[h, k].$$

□

We can now prove Theorem 4.2.20

Proof of Theorem 4.2.20. Let $\chi = \chi_{D_n}(\sigma, \tau)$. We recall that

$$\chi = \begin{cases} \sigma(di, ci)(-di, -ci), & \text{if } (\sigma, \tau) \text{ is not a } D\text{-special pair,} \\ \sigma(di, sci)(-di, -sci), & \text{if } (\sigma, \tau) \text{ is a } D\text{-special pair of the first kind,} \\ \sigma(di, -ci)(-di, ci), & \text{if } (\sigma, \tau) \text{ is a } D\text{-special pair of the second kind.} \end{cases}$$

We refer to the cases as in Figure 4.4. Let us show, case by case, that $\chi \leq \tau$. In every case we may assume, without loss of generality, that $di = -n$.

In all cases, except 1, 9 and 10, we have

$$\chi = mct_{\tau}^{B_n}(\sigma),$$

so, by Theorem 4.2.15, the pair (χ, τ) satisfies the B-condition.

In case 1, in order to prove that (χ, τ) satisfies the B-condition, we only have to show that $\sigma[h,k] \leq \tau[h,k] - 2$, when $h \in [-ci]$ and $k = -\sigma(di)$. We have

$$\begin{aligned} \sigma[h,k] &= \sigma[h, \tau(di)+1] \\ &\leq \tau[h, \tau(di)+1] \\ &= \tau[h,k] - 1 - \tau_{[di,h]\times[k,\tau(di)-1]} \\ &\leq \tau[h,k] - 1. \end{aligned}$$

Suppose, by contradiction, that $\sigma[h,k] = \tau[h,k]-1$. It follows that $\sigma[h,\tau(di)+1] = \tau[h,\tau(di)+1]$ and $\tau_{[di,h]\times[k,\tau(di)-1]} = 0$. The pair (σ,τ) satisfies the B-condition, $\sigma_{NW}[h+1,\tau(di)] = \tau_{NW}[h+1,\tau(di)]$ and $(h+1,\tau(di))$ is free for σ, thus, by Lemma 4.2.25, $(h+1,\tau(di))$ is a D-cell of (σ,τ). Since (σ,τ) satisfies the D-condition, it has to be valid, that is, $\sigma_{Wup}[h+1,\tau(di)] \equiv \tau_{Wup}[h+1,\tau(di)]$. But $\sigma_{Wup}[h+1,\tau(di)] = 0$ and, since $\tau_{[di,h]\times[k,\tau(di)-1]} = 0$, we have $\tau_{Wup}[h+1,\tau(di)] = 1$, a contradiction. Thus $\sigma[h,k] \leq \tau[h,k] - 2$.

In case 10, for the B-condition of (χ,τ), we have again to show that $\sigma[h,k] \leq \tau[h,k]-2$, when $h \in [ci]$ and $k = -\sigma(ci)$. As before we have $\sigma[h,k] \leq \tau[h,k]-1$ and, supposing by contradiction that $\sigma[h,k] = \tau[h,k]-1$, we get $\sigma_{NW}[h+1,\tau(di)] = \tau_{NW}[h+1,\tau(di)]$. Now $(h+1,\tau(di))$ is obviously free for both σ and τ. So $(h+1,\tau(di))$ is a D-cell of (σ,τ) and the conclusion is the same as before.

In case 9a, the B-condition of (χ,τ) is proved if we show that $\sigma[h,k] \leq \tau[h,k]-1$, when $(h,k) \in [ci, sci-1] \times [\sigma(di)+1, \sigma(sci)]$. If we suppose, by contradiction, that $\sigma[h,k] = \tau[h,k]$, we get $\sigma_{NW}[h+1,\tau(di)] = \tau_{NW}[h+1,\tau(di)]$, with the same conclusion as in previous cases.

Finally, in case 9b, we have to show that $\sigma[h,k] \leq \tau[h,k]-1$, when $(h,k) \in [ci] \times [\sigma(di)+1, \sigma(sci)]$ or $(h,k) \in [-sci] \times [-\sigma(sci)+1, -\sigma(di)]$. If $(h,k) \in [ci] \times [\sigma(di)+1, \sigma(sci)]$ this is proved as in case 9a. If $(h,k) \in [-sci] \times [-\sigma(sci)+1, -\sigma(di)]$, suppose by contradiction that $\sigma[h,k] = \tau[h,k]$, that is, $\sigma_{NW}[h+1,k-1] = \tau_{NW}[h+1,k-1]$. Then, by the D-condition of (σ,τ), we get $\sigma_{Wup}[h+1,k-1] \equiv \tau_{Wup}[h+1,k-1]$. On the other hand, since $\sigma_{right}[h+1;\tau(di)+1,k] = 0$, by Lemma 4.2.26 we get $\sigma_{left}[h+1;\tau(di)+1,k] = \tau_{left}[h+1;\tau(di)+1,k]$, which implies $\tau_{Wup}[h+1,k-1] = 1 + \sigma_{Wup}[h+1,k-1]$, a contradiction.

It remains to prove that (χ, τ) satisfies the D-condition.

If (h, k) is a D-cell of (χ, τ) which is also a D-cell of (σ, τ), then it has to be valid for (σ, τ), and this necessarily implies that it is also valid for (χ, τ), as it can be easily checked in every case. So, case by case, we have to look for the D-cells of (χ, τ) which are *not* D-cells of (σ, τ) (we call them *new* D-cells) and show that they are valid for (χ, τ).

In case 1, if (h, k) is a new D-cell, then $(h, k) \in [-ci + 1] \times [-\sigma(di), \sigma(ci) - 1]$ and $\sigma_{NW}[h, k] = \tau_{NW}[h, k] - 1$. From

$$\sigma_{NW}[h, k] = \sigma_{NW}[h, \tau(di)] \leq \tau_{NW}[h, \tau(di)] \leq \tau_{NW}[h, k] - 1,$$

it follows that $\sigma_{NW}[h, \tau(di)] = \tau_{NW}[h, \tau(di)]$. Moreover $(h, \tau(di))$ is free for σ. So, by Lemma 4.2.25, $(h, \tau(di))$ is a D-cell of (σ, τ). By the D-condition of (σ, τ), it has to be valid, that is, $\sigma_{Nleft}[h, \tau(di)] \equiv \tau_{Nleft}[h, \tau(di)]$. Since $(h, \tau(di))$ is free for both σ and τ, this implies $\chi_{Nleft}[h, k] = \sigma_{Nleft}[h, k] \equiv \tau_{Nleft}[h, k]$. Thus (h, k) is valid for (σ, χ).

In case 2, if (h, k) is a new D-cell, then $(h, k) \in [-ci + 1] \times [\sigma(di), \sigma(ci) - 1]$ and $\sigma_{NW}[h, k] = \tau_{NW}[h, k] - 1$. In this case the reasoning is exactly the same as in case 1.

In case 3, if (h, k) is a new D-cell, then there are two possibilities: either $(h, k) \in [di + 1, -ci] \times [-\sigma(ci), -\sigma(di) - 1]$ or $(h, k) \in [-ci + 1] \times [-\sigma(ci), -\sigma(di) - 1]$ and $\sigma_{NW}[h, k] = \tau_{NW}[h, k] - 1$.

In the first one we have $\sigma_{NW}[h, k] = \tau_{NW}[h, k]$ and $\sigma_{right}[h; -\tau(di), k] = 1 \leq \tau_{right}[h; -\tau(di), k]$. So, by Lemma 4.2.26, we have

$$\sigma_{left}[h; -\tau(di), k] = \tau_{left}[h; -\tau(di), k] \tag{4.6}$$

and $\sigma_{NW}[h, -\tau(di) - 1] = \tau_{NW}[h, -\tau(di) - 1]$. Thus $(h, -\tau(di) - 1)$ is a D-cell of (σ, τ), which has to be valid, that is, $\sigma_{Wup}[h, -\tau(di) - 1] \equiv \tau_{Wup}[h, -\tau(di) - 1]$. This, together with (4.6), implies $\chi_{Wup}[h, k] = \sigma_{Wup}[h, k] \equiv \tau_{Wup}[h, k]$.

In the second possibility we have $\chi_{NW}[h, k] = \tau_{NW}[h, k]$ and $\chi_{right}[h; -\tau(di), k] = 1 \leq \tau_{right}[h; -\tau(di), k]$. So, by Lemma 4.2.26, we have

$$\chi_{left}[h; -\tau(di), k] = \tau_{left}[h; -\tau(di), k] \tag{4.7}$$

and $\chi_{NW}[h, -\tau(di) - 1] = \tau_{NW}[h, -\tau(di) - 1]$. But $\sigma_{NW}[h, -\tau(di) - 1] = \chi_{NW}[h, -\tau(di) - 1]$, thus $(h, -\tau(di) - 1)$ is a D-cell of (σ, τ), which has to be valid, so $\chi_{Wup}[h, -\tau(di) - 1] = \sigma_{Wup}[h, -\tau(di) - 1] \equiv \tau_{Wup}[h, -\tau(di) - 1]$. This, together with (4.7), implies $\chi_{Wup}[h, k] \equiv \tau_{Wup}[h, k]$.

In cases 4 and 5 there are no new D-cells.

In case 6, if (h,k) is a new D-cell, then $(h,k) \in [di+1, ci] \times [\sigma(ci), -\sigma(di)-1]$. The pair (χ,τ) satisfies the B-condition, we have $\chi_{NW}[h,k] = \tau_{NW}[h,k]$ and $\chi_{right}[h;1,k] = 0$. So, by Lemma 4.2.26, $\chi_{Wup}[h,k] = \tau_{Wup}[h,k]$.

In case 7, if $\tau(di) > 0$ then there are no new D-cells. In fact, if $\tau(di) \in [-\sigma(ci)-1]$, then the presence of a new D-cells implies that $(di+1, \tau(di))$ is a not valid D-cell of (σ,τ), contradicting $\sigma < \tau$. If $\tau(di) \in [-\sigma(ci)+1, n]$ and (h,k) is a new D-cell, then $(h,k) \in [di+1, ci] \times [-\sigma(ci), -\sigma(di)-1]$. In particular, if $\tau(di) \in [-\sigma(ci)+1, -\sigma(di)-1]$, since $\sigma^{-1}(\tau(di)) \in [ci+1, -ci-1]$, then $k \in [-\sigma(ci), \tau(di)-1]$. So

$$\sigma_{NW}[h,k] = \sigma[h-1, \tau(di)+1] \leq \tau[h-1, \tau(di)+1] \leq \tau_{NW}[h,k]-1,$$

contradicting $\sigma_{NW}[h,k] = \tau_{NW}[h,k]$.

It remains to consider case 7, when $\tau(di) < 0$, that is, when $\tau(di) \in [\sigma(ci)]$. If (h,k) is a new D-cell, then $(h,k) \in [di+1, ci] \times [-\sigma(ci), -\sigma(di)-1]$. We have $\sigma_{right}[h; -\tau(di), k] = 1 \leq \tau_{right}[h; -\tau(di), k]$. By Lemma 4.2.26, this implies

$$\sigma_{left}[h; -\tau(di), k] = \tau_{left}[h; -\tau(di), k] \tag{4.8}$$

and $\sigma_{NW}[h, -\tau(di)-1] = \tau_{NW}[h, -\tau(di)-1]$. Thus $(h, -\tau(di)-1)$ is a D-cell of (σ,τ), which has to be valid, that is, $\sigma_{Wup}[h, -\tau(di)-1] \equiv \tau_{Wup}[h, -\tau(di)-1]$. This, together with (4.8), implies $\chi_{Wup}[h,k] = \sigma_{Wup}[h,k] \equiv \tau_{Wup}[h,k]$.

In case 8 there are no new D-cells.

In case 9a, if (h,k) is a new D-cell, then $(h,k) \in [ci+1, sci] \times [-\sigma(sci), -\sigma(di)-1]$. We have $\sigma_{right}[h; \tau(di)+1, k] = 1$, so, by Lemma 4.2.26, either $\tau_{left} = \sigma_{left}$ or $\tau_{left} = \sigma_{left}+1$. If we suppose, by contradiction, that $\tau_{left} = \sigma_{left}$, then $(h, \tau(di)+1)$ is a D-cell of (σ,τ), which has to be valid. But $\sigma_{Wup}[h, \tau(di)+1] = 0 \not\equiv 1 = \tau_{Wup}[h, \tau(di)+1]$, a contradiction. So $\tau_{left} = \sigma_{left}+1$ and $\sigma_{Wup}[h,k] = \tau_{Wup}[h,k]+2$. Thus $\chi_{Wup}[h,k] = \sigma_{Wup}[h,k] \equiv \tau_{Wup}[h,k]$.

In case 9b, if (h,k) is a new D-cell, then either $(h,k) \in [ci+1, -sci] \times [-\sigma(sci), -\sigma(di)-1]$ or $(h,k) \in [-sci+1] \times [-\sigma(sci), -\sigma(di)-1]$. In the first hypotesis the reasoning is the same as in case 9a. In the second one we have $\chi_{NW}[h,k] = \tau_{NW}[h,k]$ and $\sigma_{right}[h; \tau(di)+1, k] = 1$. So, by Lemma 4.2.26, either $\tau_{left} = \chi_{left}$ or $\tau_{left} = \chi_{left}+1$. If we suppose, by contradiction, that $\tau_{left} = \chi_{left}$, then $\sigma_{NW}[h, \tau(di)+1] = \chi_{NW}[h, \tau(di)+1] = \tau_{NW}[h, \tau(di)+1]$. So, by the D-condition of (σ,τ), we get $\sigma_{Wup}[h, \tau(di)+1] \equiv \tau_{Wup}[h, \tau(di)+1]$. But $\sigma_{Wup}[h, \tau(di)+1] = 0 \not\equiv 1 = \tau_{Wup}[h, \tau(di)+1]$, a contradiction.

So $\tau_{left} = \chi_{left} + 1$, which implies $\tau_{Wup}[h,k] = \chi_{Wup}[h,k] + 2$, that is, (h,k) is valid for (χ,τ).

In case 10, if (h,k) is a new D-cell, then $(h,k) \in [di+1] \times [-\sigma(di)-1]$. The pair (χ,τ) satisfies the B-condition, we have $\chi_{NW}[h,k] = \tau_{NW}[h,k]$ and $[h,n] \times [k]$ is empty for χ. Thus, as in case 6, by Lemma 4.2.26 we get $\chi_{Wup}[h,k] = \tau_{Wup}[h,k]$, that is, once again, (h,k) is valid for (χ,τ). □

Chapter 5

Bruhat order on the involutions of S_n

In this chapter we study the poset $Invol(S_n)$.

In Figure 5.1 the example of the poset S_4, with the induced subposet $Invol(S_4)$, is illustrated. Even in this simple case it is not obvious why the poset $Invol(S_4)$ is graded and who the rank function is. Note that, for example, the involutions 2143 and 4231 have distance 3 in the Hasse diagram of S_4, while they are in covering relation in $Invol(S_4)$.

5.1 Successor system

Definition 5.1.1 *Let $\sigma, \tau \in Invol(S_n)$. We say that (σ, τ) is a* good pair *in $Invol(S_n)$ if there exists a rectangle $R = [i, j] \times [\sigma(i), \tau(i)]$ such that σ and τ have the same diagram except for the dots in R, and in its symmetric with respect to the diagonal, for which the situation, depending on the position of R with respect to the diagonal, is described in Figure 5.2: black and white circles denote respectively σ and τ, and the rectangle R (darker gray rectangle) contains no other dots of σ and τ than those indicated.*

Let (σ, τ) be a good pair in $Invol(S_n)$. The *case* of the pair (σ, τ), denoted by $case(\sigma, \tau)$, is $h \in [6]$, referring to the pictures of Figure 5.2.

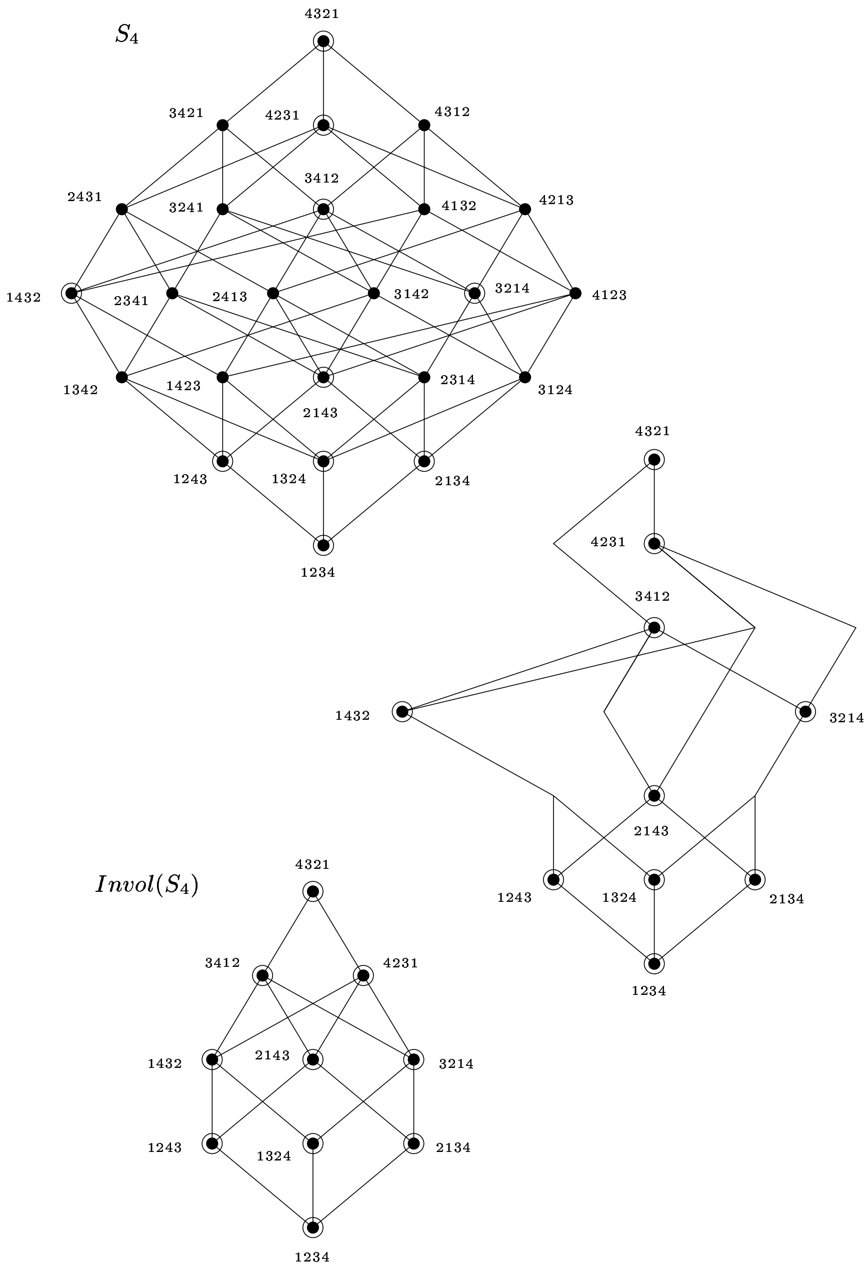


Figure 5.1: From S_4 to $Invol(S_4)$.

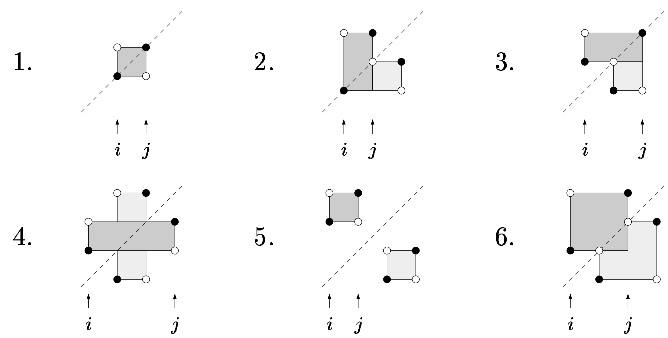


Figure 5.2: Covering relation in $Invol(S_n)$.

The *main rectangle* of (σ,τ), denoted by $R(\sigma,\tau)$ is the rectangle $R=[i,j]\times[\sigma(i),\tau(i)]$ mentioned in Definition 5.1.1 (darker gray rectangle in the pictures).

We set

$$H_{Invol(S_n)}=\{(\sigma,\tau)\in Invol(S_n)^2 : (\sigma,\tau) \text{ is a good pair in } Invol(S_n)\}$$

and define the *standard labelling* λ of $Invol(S_n)$ by associating, with every good pair $(\sigma,\tau)\in H_{Invol(S_n)}$, the pair $(i,j)\in[n]^2$, if $R=[i,j]\times[\sigma(i),\tau(i)]$ is the main rectangle of (σ,τ).

By Proposition 1.6.3, it follows that $H_{Invol(S_n)}$ is a successor system and, since τ is uniquely determined by σ and by the label (i,j), λ is a good labelling.

Given $\sigma\in Invol(S_n)$, the *suitable labels* of σ are the pairs $(i,j)\in[n]^2$ such that exists $\tau\in Invol(S_n)$, with $\lambda(\sigma,\tau)=(i,j)$. Such a τ, obviously unique, is called the *transformation* of σ with respect to (i,j) and it is denoted by

$$t_{(i,j)}^{Invol(S_n)}(\sigma).$$

In order to give a more explicit description of the suitable labels in $Invol(S_n)$, we introduce the following notation. Let $\sigma\in S_n$. We denote by

$$\begin{aligned} I_f(\sigma) &= \{i\in[n] : \sigma(i)=i\},\\ I_e(\sigma) &= \{i\in[n] : \sigma(i)>i\},\\ I_d(\sigma) &= \{i\in[n] : \sigma(i)<i\}, \end{aligned}$$

respectively the sets of *fixed points*, of *excedances* and of *deficiencies* of σ.

The *type* of a rise (i, j) is the pair (a, b), where $a, b \in \{f, e, d\}$ are such that $i \in I_a(\sigma)$ and $j \in I_b(\sigma)$. A rise of type (a, b) is also called an *ab-rise*. Furthermore, we distinguish between two kinds of ee-rises: an ee-rise (i, j) is *crossing* if $i < \sigma(i) < j < \sigma(j)$, *non-crossing* if $i < j < \sigma(i) < \sigma(j)$. In other words, an ee-rise (i, j) is crossing if the cells $(i, \sigma(j))$ and $(j, \sigma(i))$ are on opposite sides of the diagonal, and non-crossing otherwise.

For example, for $\sigma = 321654 \in Invol(S_6)$, the rises of σ are $(1, 4)$, $(1, 5)$, $(1, 6)$, $(2, 4)$, $(2, 5)$, $(2, 6)$, $(3, 4)$, $(3, 5)$, $(3, 6)$, and their types are, respectively, (e, e), (e, f), (e, d), (f, e), (f, f), (f, d), (d, e), (d, f), (d, d) (all nine possible types) and the ee-rise is crossing.

The following is an easy consequence of the definitions.

Proposition 5.1.2 *Let $\sigma \in Invol(S_n)$. The suitable labels of σ are the free rises whose type is one of the following: (f, f), (f, e), (e, f), (e, d), (e, e).*

5.2 Insertion property

Lemma 5.2.1 *Let $\sigma, \tau \in Invol(S_n)$, with $\sigma < \tau$. Then*

$$di \leq \sigma(di).$$

Proof. Fist note that $\sigma(di) \neq \tau(di)$. Since σ and τ are involutions, they must differ also at the index $\sigma(di)$. But di is the minimal index at which they differ, thus $di \leq \sigma(di)$. □

We have already observed that (di, ci) is a free rise of σ, but if σ and τ are involutions, more is true.

Proposition 5.2.2 *Let $\sigma, \tau \in Invol(S_n)$, with $\sigma < \tau$. Then (di, ci) is a suitable label of σ.*

Proof. By Proposition 5.1.2 and Lemma 5.2.1, we only have to show that (di, ci) is not an fd-rise. We may assume, without loss of generality, that $di = 1$. Suppose, to the contrary, that $(1, ci)$ is an fd-rise, so $\sigma(1) = 1$ and $\sigma(ci) < ci$. By the definition of ci, there is no $k \in [2, ci - 1]$ such that $\sigma(k) \in [1, \tau(1)]$. In particular, since $\sigma(ci) \in [2, ci - 1]$, we have $ci > \tau(1)$. Thus $|\{k \in [\tau(1)] : \sigma(k) \in [\tau(1)]\}| = 1$, which implies

$$\sigma[\tau(1), \tau(1) + 1] = \tau(1) - 1.$$

Also, since di is an excedance of τ and τ is an involution, we have $|\{k \in [\tau(1)] : \tau(k) \in [\tau(1)]\}| \geq 2$, which implies

$$\tau[\tau(1), \tau(1)+1] \leq \tau(1) - 2.$$

So $\sigma[\tau(1), \tau(1)+1] > \tau[\tau(1), \tau(1)+1]$, but $\sigma < \tau$ and, by Theorem 1.6.4, this is a contradiction. Thus $(1, ci)$ cannot be an fd-rise. □

By Proposition 5.2.2 it makes sense to consider the involution

$$\chi_{Invol(S_n)}(\sigma, \tau) = t_{(di,ci)}^{Invol(S_n)}(\sigma).$$

In order to prove that the insertion property holds, it is useful to introduce the following definition.

Definition 5.2.3 *Let* $(\sigma, \tau) \in H_{Invol(S_n)}$. *The* multiplicity *of* (σ, τ) *is*

$$mult(\sigma, \tau) = \begin{cases} 0, & \text{if } case(\sigma, \tau) = 1, \\ 1, & \text{if } case(\sigma, \tau) = 2, 3, 4, 5, \\ 2, & \text{if } case(\sigma, \tau) = 6. \end{cases}$$

We introduce the following notation. Given $\sigma \in S_n$, we set

$$\sigma i = \sigma^{-1}.$$

Given $\sigma, \tau \in S_n$, with $\sigma < \tau$, we set

$$\sigma m = mct_{\tau}^{S_n}(\sigma).$$

Proposition 5.2.4 *Let* $\sigma, \tau \in Invol(S_n)$, *with* $\sigma < \tau$, *and let*

$$\chi = \chi_{Invol(S_n)}(\sigma, \tau).$$

Then

$$\chi = \begin{cases} \sigma m, & \text{if } mult(\sigma, \chi) = 0, \\ \sigma mim, & \text{if } mult(\sigma, \chi) = 1, \\ \sigma mimm, & \text{if } mult(\sigma, \chi) = 2. \end{cases}$$

Proof. It can be easily checked case by case, using the description of the minimal covering transformation in S_n. For example, if (di, ci) is a non-crossing ee-rise, that is, $case(\sigma, \chi) = 5$, then $mult(\sigma, \chi) = 1$ and we have

$$\begin{aligned} \sigma m &= \sigma(di, ci); \\ \sigma mi &= \sigma(\sigma(di), \sigma(ci)); \\ \sigma mim &= \sigma(di, ci)(\sigma(di), \sigma(ci)) = \chi. \end{aligned}$$

□

We can now prove the insertion property of $Invol(S_n)$.

Proposition 5.2.5 *Let $\sigma, \tau \in Invol(S_n)$, with $\sigma < \tau$. Then*

$$\chi_{Invol(S_n)}(\sigma, \tau) \leq \tau.$$

Proof. For every $\sigma \in S_n$, such that $\sigma < \tau$, by Proposition 1.4.2, since τ is an involution, we have

$$\sigma i = \sigma^{-1} \leq \tau$$

and by Theorem 4.2.8 we have

$$\sigma m \leq \tau.$$

Thus $\sigma \leq \tau$ is a consequence of Proposition 5.2.4. □

5.3 $Invol(S_n)$ is graded

We recall that the length of $\sigma \in S_n$ is

$$l(\sigma) = inv(\sigma),$$

and that the absolute length of $\sigma \in Invol(S_n)$ is

$$al(\sigma) = exc(\sigma).$$

So the average between the length and the absolute length of $\sigma \in Invol(S_n)$ is

$$\rho(\sigma) = \frac{inv(\sigma) + exc(\sigma)}{2}.$$

Proposition 5.3.1 *The pair $(H_{Invol(S_n)}, \rho)$ is a covering system of $Invol(S_n)$.*

Proof. By Proposition 5.2.5, $H_{Invol(S_n)}$ is an insertion system of $Invol(S_n)$. The ρ-base property is trivial. It remains to prove the ρ-increasing property. Let $(\sigma, \tau) \in H_{Invol(S_n)}$. We have to prove that

$$\Delta\rho = \frac{\Delta inv(\sigma) + \Delta exc(\sigma)}{2} = 1,$$

where $\Delta x = x(\tau) - x(\sigma)$. It's easy to check that

$$(\Delta inv, \Delta exc) = \begin{cases} (1,1), & \text{if } mult(\sigma,\tau) = 0, \\ (2,0), & \text{if } mult(\sigma,\tau) = 1, \\ (3,-1), & \text{if } mult(\sigma,\tau) = 2. \end{cases}$$

Thus in every case $\Delta\rho = 1$. □

We are now able to state and prove the gradedness of $Invol(S_n)$.

Theorem 5.3.2 *The poset $Invol(S_n)$ is graded, with rank function given by*

$$\rho(\sigma) = \frac{inv(\sigma) + exc(\sigma)}{2},$$

for every $\sigma \in Invol(S_n)$. In particular $Invol(S_n)$ has rank

$$\rho(Invol(S_n)) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Proof. By Theorem 3.1.2, the first part is a consequence of Proposition 5.3.1.

For the second part, note that w_0, the maximum of S_n, which is also the maximum of $Invol(S_n)$, has $n(n-1)/2$ inversions and $\lfloor n/2 \rfloor$ excedances. □

We also have a characterization of the covering relation in $Invol(S_n)$: if $\sigma, \tau \in Invol(S_n)$, then $\sigma \lhd \tau$ if and only if (σ, τ) is a good pair in $Invol(S_n)$. And a transformation of σ with respect to a suitable label (i, j) actually is a *covering transformation*, denoted by

$$ct_{(i,j)}^{Invol(S_n)}(\sigma).$$

As an example consider the involution $\sigma = 321654 \in Invol(6)$. The suitable labels of σ are $(1,4)$, $(1,5)$, $(1,6)$, $(2,4)$ and $(2,5)$, and we have

(i,j)	$ct_{(i,j)}^{Invol(S_6)}(\sigma)$
$(1,4)$	623451
$(1,5)$	523614
$(1,6)$	426153
$(2,4)$	361452
$(2,5)$	351624

Thus

$$\{\tau \in Invol(S_6) : \sigma \lhd \tau\} = \{623451, 523614, 426153, 361452, 351624\}.$$

5.4 $Invol(S_n)$ is EL-shellable

Proposition 5.4.1 *Let $\sigma, \tau \in Invol(S_n)$, with $\sigma < \tau$. Then*

$$mi_\tau(\sigma) = (di, ci).$$

Proof. The proof is exactly the same as in Proposition 4.2.7. □

Thus, if $\sigma, \tau \in Invol(S_n)$, with $\sigma < \tau$, then the *minimal covering transformation* of σ with respect to τ is

$$mct_\tau^{Invol(S_n)}(\sigma) = ct_{(di,ci)}^{Invol(S_n)}(\sigma),$$

and we have the following.

Theorem 5.4.2 *Let $\sigma, \tau \in Invol(S_n)$, with $\sigma < \tau$. Then*

$$\sigma \lhd mct_\tau^{Invol(S_n)}(\sigma) \leq \tau.$$

Theorem 5.4.2 ensures that next definition is well-posed.

Definition 5.4.3 *Let $\sigma, \tau \in Invol(S_n)$, with $\sigma < \tau$. The* minimal chain *in $Invol(S_n)$ from σ to τ is the saturated chain*

$$\sigma = \sigma_0 \lhd \sigma_1 \lhd \ldots \lhd \sigma_k = \tau,$$

defined by

$$\sigma_i = mct_\tau^{Invol(S_n)}(\sigma_{i-1}), \tag{5.1}$$

for every $i \in [k]$.

In order to prove that $Invol(S_n)$ is EL-shellable, we prove the increasing and the decreasing properties, introduced in Section 3.3.

Proposition 5.4.4 (Increasing property) *Let $\sigma, \tau \in Invol(S_n)$, with $\sigma < \tau$. The minimal chain*

$$\sigma = \sigma_0 \lhd \sigma_1 \lhd \ldots \lhd \sigma_k = \tau,$$

defined in (5.1) has increasing labels.

Proof. Suppose, by contradiction, that at a certain step there is a decrease in the labels. We may assume, without loss of generality, that this happens at the first step. Thus

$$\sigma_1 = mct_\tau^{Invol(S_n)}(\sigma) = ct_{(di,ci)}^{Invol(S_n)}(\sigma)$$

and

$$\sigma_2 = ct_{(i,j)}^{Invol(S_n)}(\sigma_1),$$

with $(i,j) < (di,ci)$. So either $i < di$ or $i = di$ and $j < ci$. If $i < di$, since σ and τ must differ at the index i, the minimality of di is contradicted. If $i = di$ and $j < ci$, since $\sigma(j) \in [\sigma(di)+1, \tau(di)]$, the minimality of ci is contradicted. Thus the chain has increasing labels. □

Proposition 5.4.5 (Decreasing property) *Let $\sigma, \tau \in Invol(S_n)$, with $\sigma < \tau$, and let*

$$\sigma = \sigma_0 \lhd \sigma_1 \lhd ... \lhd \sigma_k = \tau,$$

be the minimal chain defined in (5.1)*. Every saturated chain from σ to τ, different from the minimal one, say*

$$\sigma = \tau_0 \lhd \tau_1 \lhd \ldots \lhd \tau_k = \tau,$$

has at least one decrease in the labels.

Proof. We may assume, without loss of generality, that $\sigma_1 \neq \tau_1$. Then

$$\sigma_1 = mct_\tau^{Invol(S_n)}(\sigma) = ct_{(di,ci)}(\sigma)$$

and

$$\tau_1 = ct_{(i,j)}(\sigma),$$

for some suitable label (i,j) of σ different from (di,ci) and lexicographically greater than it. So either $di < i$ or $di = i$ and $ci < j$.

If $di < i$, then in the covering relations

$$\tau_1 \lhd \tau_2 \lhd ... \lhd \tau_k = \tau$$

there must be at least one with label containing di, so lower than (i,j).

Suppose $di = i$ and $ci < j$. Since the dot in column di has to move from row $\sigma(di)$ to row $\tau(di)$ and because of the presence in the diagram of σ of the dot in the cell $(ci, \sigma(ci))$, in the covering relations

$$\tau_1 \lhd \tau_2 \lhd ... \lhd \tau_k = \tau$$

either there is one with label (di, ci), so lower than (i, j), or there is one with label starting with ci, followed by one with label starting with di, so again with a decrease. $\square$

The following is a consequence of the increasing and the decreasing properties.

Theorem 5.4.6 *The poset $Invol(S_n)$ is EL-shellable, having the standard labelling as an EL-labelling.*

5.5 $Invol(S_n)$ is Eulerian

In this section we prove that the poset $Invol(S_n)$ is Eulerian. In order to do this, we introduce some notions which, in some sense, invert those introduced in the preceding sections.

Definition 5.5.1 *Let $\tau \in Invol(S_n)$. A pair (i, j) is an* inv-suitable label *of τ if (i, j) is a suitable label of some $\sigma \in Invol(S_n)$ and*

$$ct_{(i,j)}^{Invol(S_n)}(\sigma) = \tau.$$

We call such a σ (obviously unique) the inverse covering transformation *of τ with respect to (i, j) and we denote it by*

$$ict_{(i,j)}^{Invol(S_n)}(\tau).$$

Definition 5.5.2 *Let $\sigma, \tau \in Invol(S_n)$, with $\sigma < \tau$. The* minimal inverse label *of τ with respect to σ, denoted by $mi_\sigma(\tau)$ is the minimal (in the lexicographic order) inv-suitable label (i, j) of τ such that*

$$\sigma \leq ict_{(i,j)}^{Invol(S_n)}(\tau).$$

The minimal inverse covering tranformation *of τ with respect to σ, denoted by $mict_\sigma^{Invol(S_n)}(\tau)$, is the inverse covering transformation of τ with respect to the minimal inverse label:*

$$mict_\sigma^{Invol(S_n)}(\tau) = ict_{mi_\sigma(\tau)}^{Invol(S_n)}(\tau).$$

We can now prove that the condition of Theorem 1.2.1 holds for the poset $Invol(S_n)$, and thus that it is Eulerian.

Theorem 5.5.3 *The poset $Invol(S_n)$ is Eulerian.*

Proof. Suppose we label the edges of the Hasse diagram of $Invol(S_n)$ with the standard labelling. Since the standard labelling is an EL-labelling, by theorem 1.2.1, we have to prove that, given $\sigma, \tau \in Invol(S_n)$, with $\sigma < \tau$, there is exactly one saturated chain from σ to τ with decreasing labels.

In this proof we use the following terminology: if (i, j) is an inv-suitable label of τ we call it simply a *move* for τ, precisely an h-move, with $h \in [6]$, if we are in case h of Figure 5.2. Furthermore, if $ict_{(i,j)}^{Invol(S_n)}(\tau) = \sigma$, then we write

$$\tau \underset{(i,j)}{\triangleright} \sigma.$$

We first prove that there is at least one saturated chain from σ to τ with decreasing labels. Consider the *descending* chain

$$\tau = \sigma_0 \triangleright \sigma_1 \triangleright \ldots \triangleright \sigma_k = \sigma,$$

defined by

$$\sigma_i = mict_{\sigma}^{Invol(S_n)}(\sigma_{i-1}),$$

for every $i \in [k]$. We claim that it has increasing labels (so the corresponding *ascending* chain will have decreasing labels). Suppose, by contradiction, that at a certain step there is a decrease in the labels. We may assume, without loss of generality, that this happens at the first step. So

$$\sigma_0 \underset{(i,j)}{\triangleright} \sigma_1 \underset{(i',j')}{\triangleright} \sigma_2,$$

with $(i', j') < (i, j)$. So either $i' < i$ or $i' = i$ and $j' < j$.

If $i' < i$, then (i, j) cannot be the minimal choice for σ_0, since σ_0 must have an inv-suitable inversion containing i'. This contradicts the definition of the chain.

If $i' = i$ and $j' < j$, again (i, j) cannot be the minimal choice for σ_0. The proof of this fact is a case-by-case verification, depending on the type of (i, j). We show some cases, leaving the others to the reader.

First of all note that (i, j) cannot be a 1-move or a 2-move, in fact in this case we could not apply to σ_1 a move (i, j'), with $j' < j$. If (i, j) is a 3-move and (i, j') is a 1-move, the situation is illustrated in Figure 5.3 (a): if we apply to σ_0 the two moves (i, j') and (j', j) (in this order), we again reach σ_2:

$$\sigma_0 \underset{(i,j')}{\triangleright} \sigma_1' \underset{(j',j)}{\triangleright} \sigma_2.$$

But $(i, j') < (i, j)$, so (i, j) is not the minimal choice for σ_0. In the picture we represent a pair of moves by colouring the areas “enclosed” in the moves, with a lighter grey for the first move and a darker grey for the second one; the arrow represents the possibility of substituting a pair of moves with another pair reaching the same involution. Figure 5.3 (b) is the synthetic version of Figure 5.3 (a). If (i, j) is a 3-move, all other cases are synthetically described in Figure 5.3 (c, d, e, f), with the notation described above. If (i, j) is a 4, 5 or 6-move the reasoning is similar. In each case we get a contradiction.

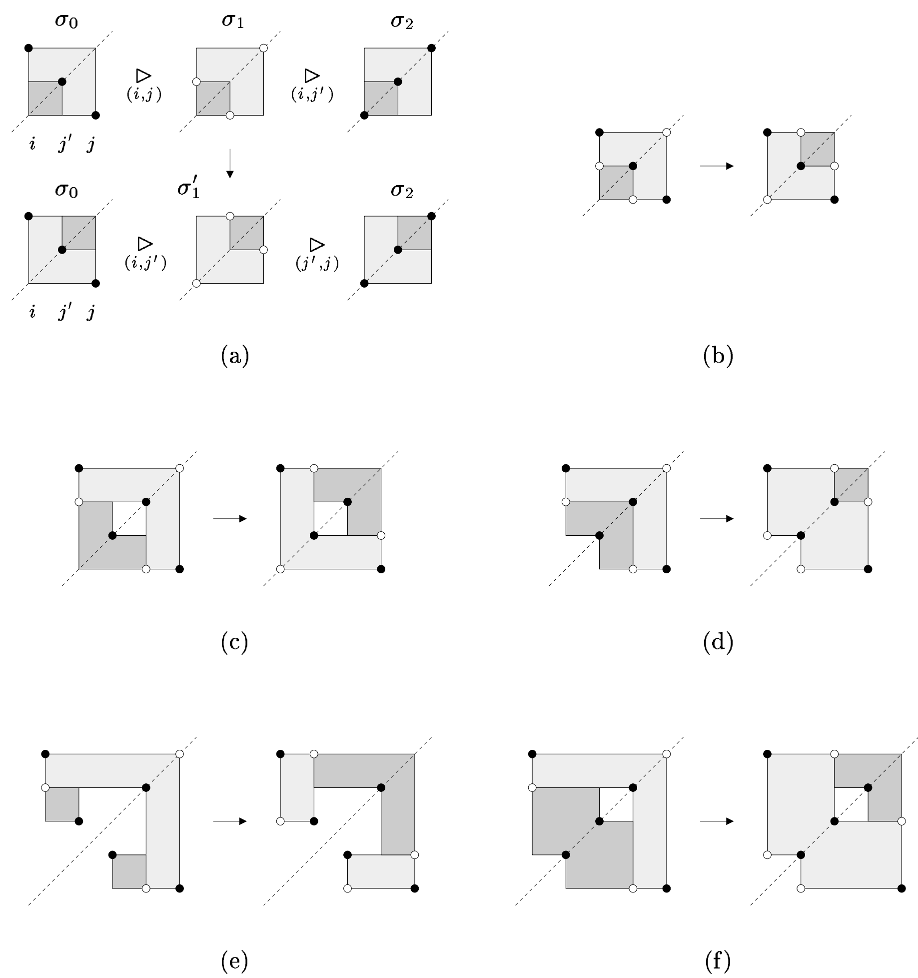


Figure 5.3: Proof of Theorem 5.5.3.

We now prove that any other saturated chain from σ to τ, different from the minimal one, has at least one increase. Let

$$\tau = \tau_0 \rhd \tau_1 \rhd ... \rhd \tau_k = \sigma$$

be a saturated descending chain from τ to σ, different from the minimal one. We will prove that in it there is at least one decrease (so the corresponding ascending chain will have at least one increase). We may assume, without loss of generality, that $\sigma_1 \neq \tau_1$. Then

$$\sigma_1 = ict_{(i,j)}(\tau)$$

and

$$\tau_1 = ict_{(i',j')}(\tau),$$

for some inv-suitable inversion (i', j') of τ different from (i, j) and lexicographically greater than it. So either $i < i'$ or $i = i'$ and $j < j'$.

If $i < i'$, then in the covering relations $\tau_1 \rhd \tau_2 \rhd ... \rhd \tau_k = \sigma$ there must be one with label containing i, so lower than (i', j').

Suppose $i = i'$ and $j < j'$ and suppose, by contradiction, that the chain $\tau = \tau_0 \rhd \tau_1 \rhd \tau_2 \rhd \dots \tau_k = \sigma$ has increasing labels. If $l = min\{s \in [k] : \tau_s(i) = \sigma(i)\}$, then

$$\tau \underset{(i,j')}{\rhd} \tau_1 \underset{(i,j_2)}{\rhd} \tau_2 \underset{(i,j_3)}{\rhd} ... \rhd \tau_{l-1} \underset{(i,j_l)}{\rhd} \tau_l,$$

with $(j <)j' < j_2 < j_3 < ... < j_l$. But this is in contradiction with the fact that (i, j) is an inv-suitable inversion of τ. The proof of this fact is again a case-by-case verification, depending on the type of (i, j).

If (i, j) is a 1-move, a 3-move or a 4-move, then the (i, j_r)'s can only be 4-moves, and none of these can send the dot in column i on or below row $\sigma_1(i)$. But $\sigma(i) \leq \sigma_1(i)$, and we get a contradiction.

If (i, j) is a 2-move, a 5-move or a 6-move, then the sequence of the (i, j_r)'s is made of a sequence (possibly empty) of 5-moves, possibly followed by a 3 or a 6-move (but not both) and then by a sequence (possibly empty) of 4-moves. If (i, j) is a 5-move, then none of these moves can send the dot in column i on or below row $\sigma_1(i)$. If (i, j) is a 2-move or a 6-move, then the 3 or 6-move is the only one that can move the dot in column i. In this case the dot is moved in row j and none of the following 4-moves can move it on or below row $\sigma_1(i)$. But, as before, $\sigma(i) \leq \sigma_1(i)$, so in each case we get a contradiction. □

Chapter 6

Bruhat order on the involutions of B_n

In this chapter we study the poset $Invol(B_n)$ of the involutions of B_n.

In Figure 6.1 the posets B_3 and $Invol(B_3)$ are illustrated.

6.1 Successor system

Looking at the diagram of a signed permutation, with *orbit* of an object (which can be a dot, a cell, or a rectangle of cells), we mean the set made of that object and its symmetric with respect to the main diagonal, the antidiagonal and the center.

Definition 6.1.1 *Let $\sigma, \tau \in Invol(B_n)$. We say that (σ, τ) is a* good pair *in $Invol(B_n)$ if there exists a rectangle $R = [i, j] \times [\sigma(i), \tau(i)]$ such that σ and τ have the same diagram except for the dots in R, and in the rectangles of its orbit, for which the situation, depending on the position of R with respect to the antidiagonal and to the main diagonal, is described in Figures 6.2 and 6.3: black and white circles denote, respectively, σ and τ, and the rectangle R (gray rectangle) contains no other dots of σ and τ than those indicated.*

The *case* of the pair (σ, τ) is (Ah, Mk), with $h, k \in [6]$, where Ah (*anticase*) and Mk (*main case*) refer to the pictures of Figures 6.2 and 6.3.

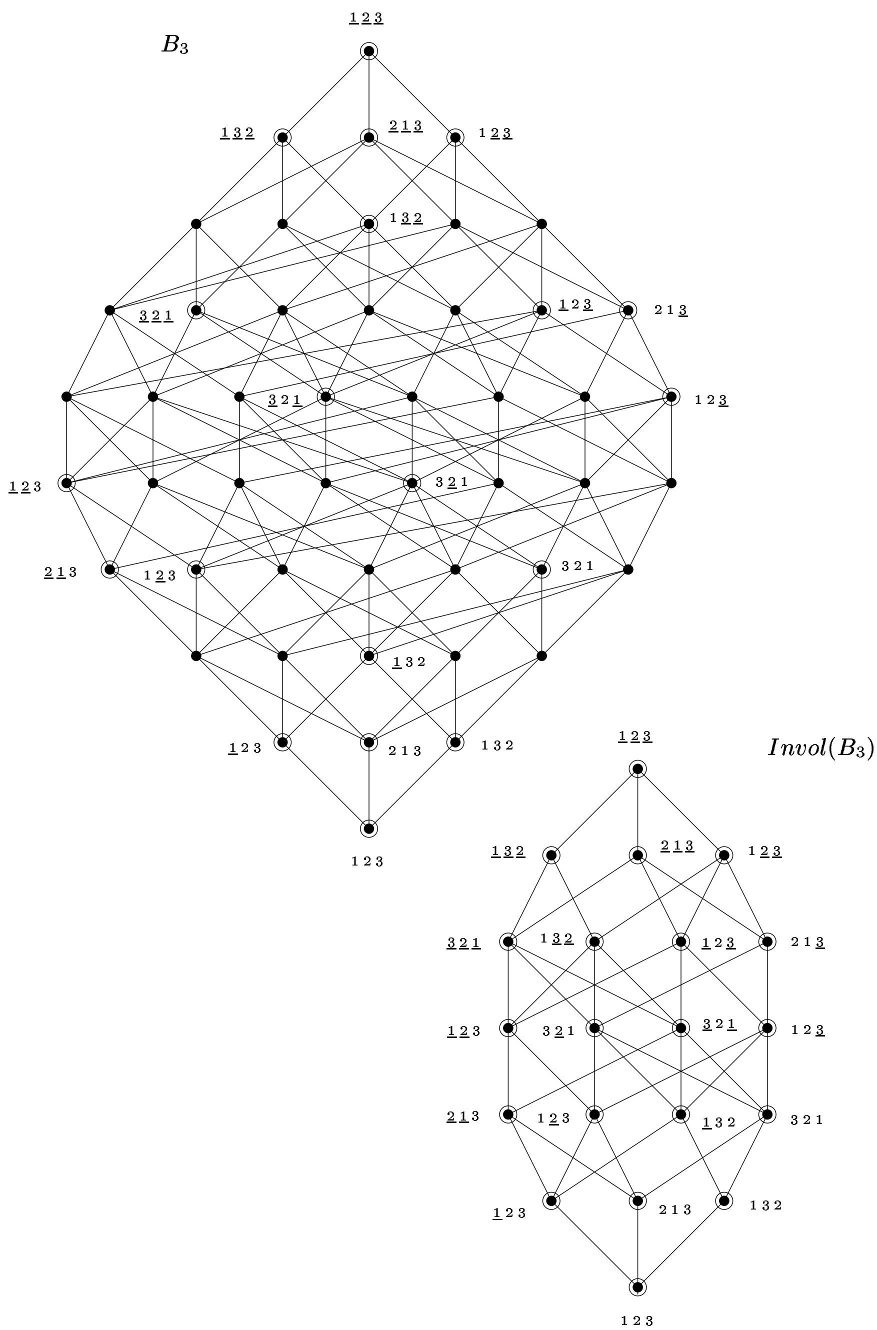


Figure 6.1: From B_3 to $Invol(B_3)$.

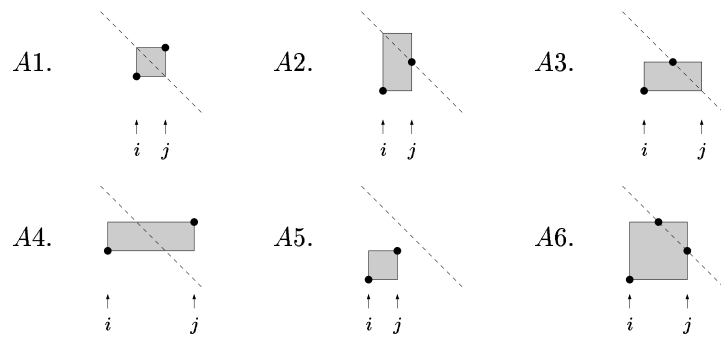


Figure 6.2: Covering relation in $Invol(B_n)$: anticases.

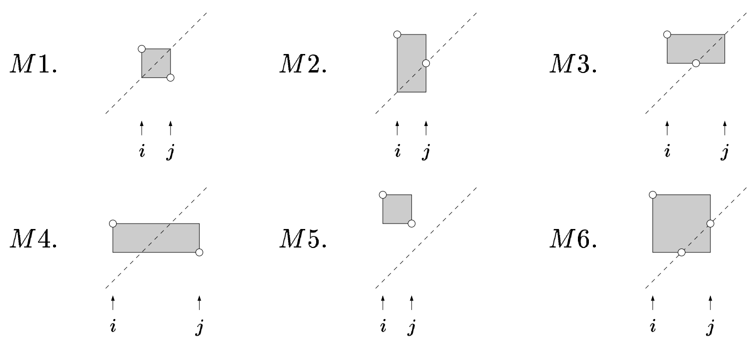


Figure 6.3: Covering relation in $Invol(B_n)$: main cases.

Note that for geometrical reasons not all the 36 pairs are possible cases.

Let (σ, τ) be a good pair in $Invol(B_n)$. The *main rectangle* of (σ, τ), denoted by $R(\sigma, \tau)$ is the rectangle $R = [i, j] \times [\sigma(i), \tau(i)]$ mentioned in Definition 6.1.1.

It is useful to consider separately the cases in which $R(\sigma, \tau)$ is *central*, that is $(0, 0) \in R(\sigma, \tau)$: not central cases are illustrated in Table 6.1, and central cases in Table 6.2. In every case, black circles denote σ and white circles denote τ. The darker gray rectangle is the main rectangle $R(\sigma, \tau)$ and the complete gray area is the union of the rectangles of its orbit.

In Tables 6.1 and 6.2 almost all cases are illustrated. In fact in some cases there is more than one possibility, as we show in the following tables.

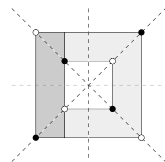
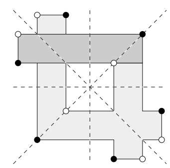
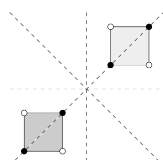
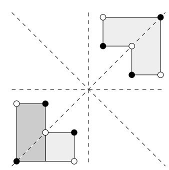
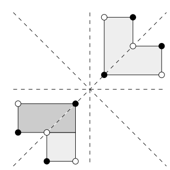
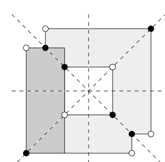
	M1	M2	M3
A1	-	-	-
A2	-		-
A3	-	-	-
A4	-	-	
A5			
A6	-		-

Table 6.1: Covering relation in $Invol(B_n)$: not central cases - part 1.

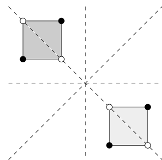
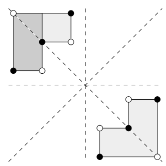
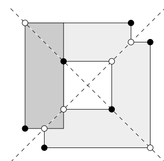
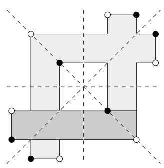
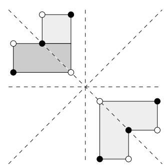
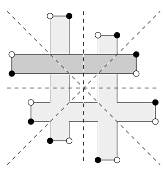
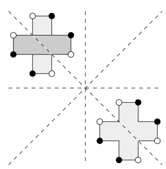
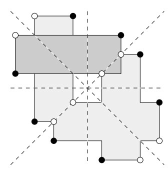
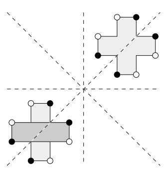
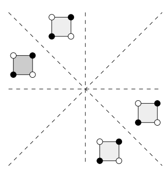
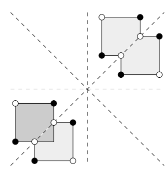
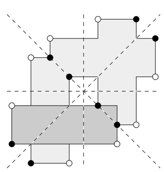
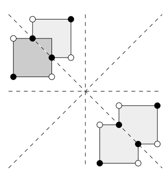
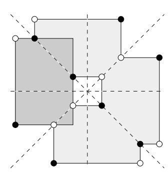
	M4	M5	M6	
	-		-	A1
	-			A2
			-	A3
				A4
				A5
				A6

Table 6.1: Covering relation in $Invol(B_n)$: not central cases - part 2.

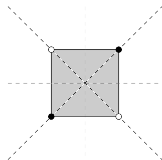
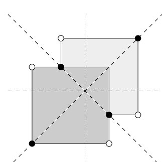
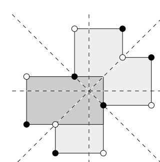
	M1	M2	M3
A1		-	-
A2	-	-	-
A3	-	-	-
A4	-	-	-
A5	-	-	-
A6		-	

Table 6.2: Covering relation in $Invol(B_n)$: central cases - part 1.

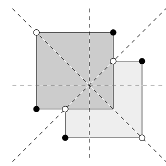
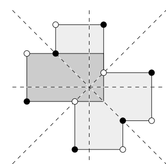
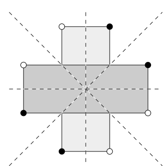
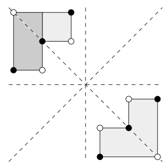
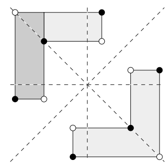
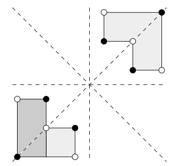
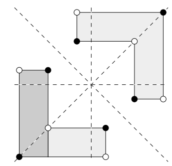
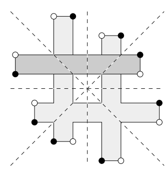
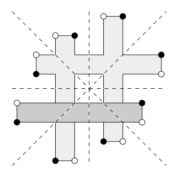
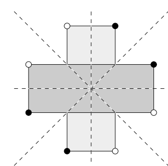
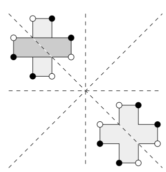
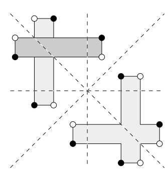
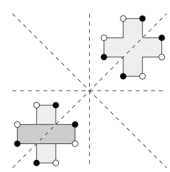
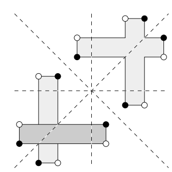
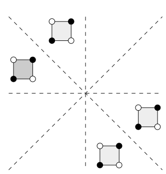
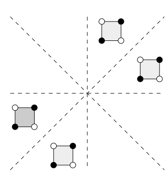
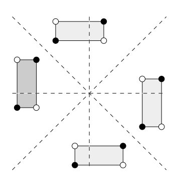
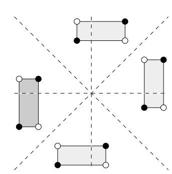
	M4	M5	M6	
	-	-		A1
	-	-	-	A2
	-	-		A3
		-	-	A4
	-	-	-	A5
	-	-	-	A6

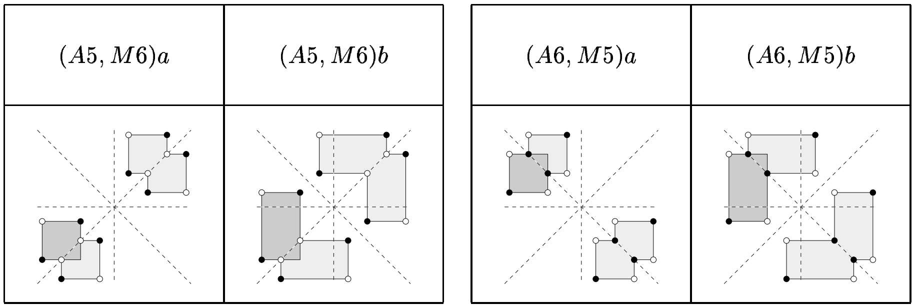
Table 6.2: Covering relation in $Invol(B_n)$: central cases - part 2.

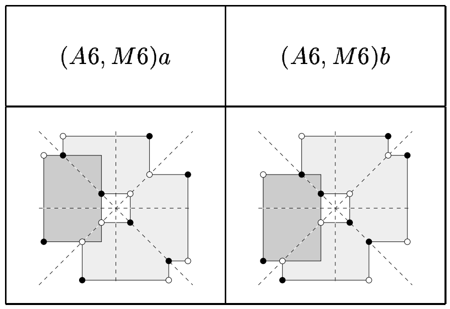
$(A2, M5)a$	$(A2, M5)b$	$(A5, M2)a$	$(A5, M2)b$
			

$(A4, M4)a$	$(A4, M4)b$	$(A4, M4)c$
		

$(A4, M5)a$	$(A4, M5)b$	$(A5, M4)a$	$(A5, M4)b$
			

$(A5, M5)a$	$(A5, M5)b$	$(A5, M5)c$	$(A5, M5)d$
			





We set

$$H_{Invol(B_n)} = \{(\sigma, \tau) \in Invol(B_n)^2 : (\sigma, \tau) \text{ is a good pair in } Invol(B_n)\},$$

and define the *standard labelling* λ of $Invol(B_n)$ by associating, with every good pair $(\sigma, \tau) \in H_{Invol(B_n)}$, the pair $(i, j) \in [\pm n]^2$, if $R = [i, j] \times [\sigma(i), \tau(i)]$ is the main rectangle of (σ, τ).

By Theorem 4.2.13, $H_{Invol(B_n)}$ is a successor system and, since τ is uniquely determined by σ and by the label (i, j), λ is a good labelling.

Given $\sigma \in Invol(B_n)$, a pair $(i, j) \in [\pm n]^2$ is a *suitable label* of σ if there exists $\tau \in Invol(B_n)$, with $\lambda(\sigma, \tau) = (i, j)$. Such a τ, obviously unique, is called the *transformation* of σ with respect to (i, j) and it is denoted by

$$t^{Invol(B_n)}_{(i,j)}(\sigma).$$

6.2 Insertion property

Definition 6.2.1 *Let $\sigma, \tau \in Invol(B_n)$, with $\sigma < \tau$. Consider the following seven properties of the pair (σ, τ):*

1. $ci \in \{-di, -\sigma(di)\}$*;*

2. (di, ci) *is a central rise of* σ *(not necessarily symmetric);*

3. $\sigma(ci) \neq -ci$;

4. $\tau(di) = -di$;

5. sci *does not exist, or* $\sigma(sci) > -sci$;

6. sci *exists, and* $sci = -ci$;

7. sci *exists, and* $\sigma(sci) = -sci$.

The B-type *of the pair* (σ, τ) *is*

$$B\text{-}type(\sigma, \tau) = Bh,$$

where $h = 8$ *if* (σ, τ) *does not satisfy any of the above properties, otherwise*

$$h = \min\{k \in [7] : (\sigma, \tau) \text{ satisfies property } k\}.$$

All cases are represented in Tables 6.3 to 6.10. The notation used in the pictures will be soon described. But we first need the following definition.

Definition 6.2.2 *Let* $\sigma, \tau \in Invol(B_n)$, *with* $\sigma < \tau$. *The* B-covering index *of* σ *with respect to* τ, *denoted by* $Bci_\tau(\sigma)$ *(or simply* Bci*), is the minimal index* $j \in [di + 1, n]$ *such that there exists* $\chi \in Invol(B_n)$, *with*

1. $(\sigma, \chi) \in H_{Invol(B_n)}$;

2. $\lambda(\sigma, \chi) = (di, j)$;

3. $\chi(di) \leq \tau(di)$.

As it comes out from the pictures, Bci is always well defined and precisely:

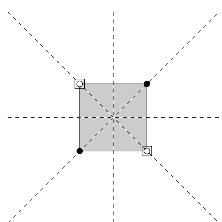
$$Bci = \begin{cases} ci, & \text{if } B\text{-}type(\sigma, \tau) = B1, B3, B4, \\ -di, & \text{if } B\text{-}type(\sigma, \tau) = B2, \\ -\sigma(di), & \text{if } B\text{-}type(\sigma, \tau) = B5, \\ sci, & \text{if } B\text{-}type(\sigma, \tau) = B6, B7, B8. \end{cases}$$

By definition, (di, Bci) is a suitable label of σ, so we can consider the involution

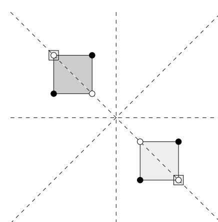
$$\chi_{Invol(B_n)}(\sigma, \tau) = t^{Invol(B_n)}_{(di, Bci)}(\sigma).$$

In the pictures the involutions σ, τ and $\chi = \chi_{Invol(B_n)}(\sigma, \tau)$ are represented: black circles denote σ, white squares denote τ and white circles denote χ. Only the dot in column di of τ, with its orbit, is indicated, possibly with a gray rectangle around, indicating the range of variation of $\tau(di)$. The involutions σ and χ are supposed to have the same diagram anywhere else. Finally the darker gray rectangle is the main rectangle of (σ, χ) and inside the complete gray area there are no other dots of σ (hence of χ) than those indicated.

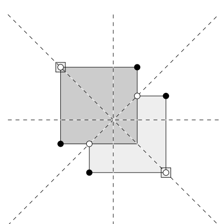
B1	► $ci \in \{-di, -\sigma(di)\}$	$Bci = ci$



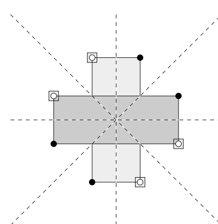
B1.1



B1.2



B1.3



B1.4

Table 6.3: B-type B1.

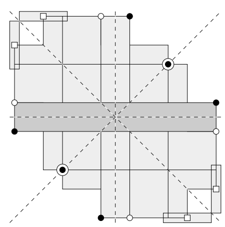
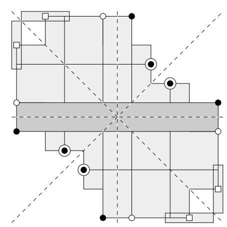
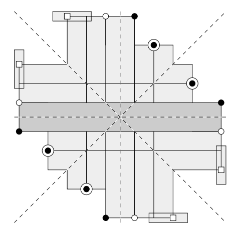
B2	▶ $ci \notin \{-di, -\sigma(di)\}$ ▶ $(0,0) \in [di, ci] \times [\sigma(di), \sigma(ci)]$	$Bci = -di$
 B2.1	 B2.2	 B2.3

Table 6.4: B-type B2.

B3	▸ $ci \notin \{-di, -\sigma(di)\}$ ▸ $(0,0) \notin [di, ci] \times [\sigma(di), \sigma(ci)]$ ▸ $\sigma(ci) \neq -ci$	$Bci = ci$

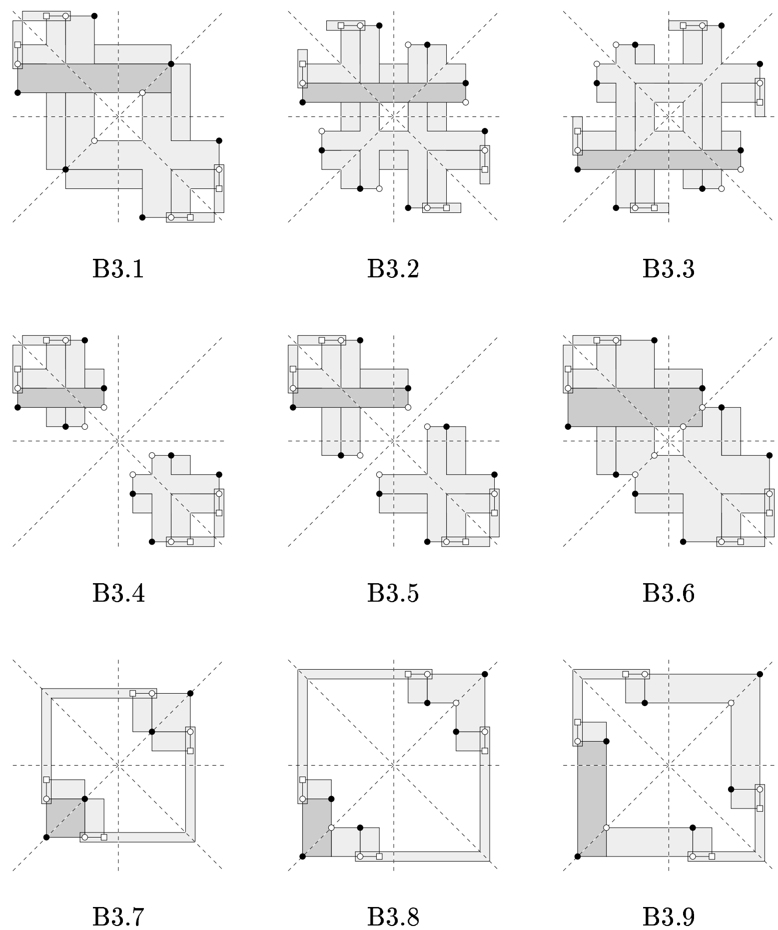
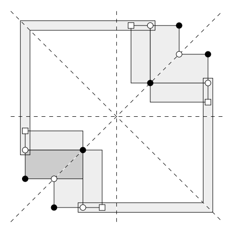
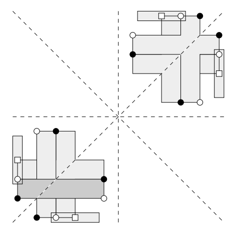
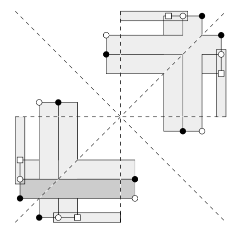


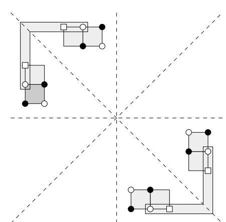
Table 6.5: B-type B3 - part 1.

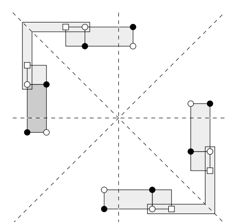
B3	▶ $ci \notin \{-di, -\sigma(di)\}$ ▶ $(0,0) \notin [di, ci] \times [\sigma(di), \sigma(ci)]$ ▶ $\sigma(ci) \neq -ci$	$Bci = ci$

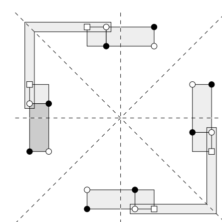

B3.10

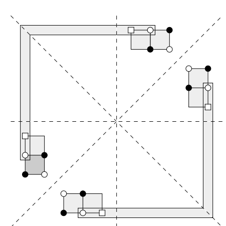

B3.11

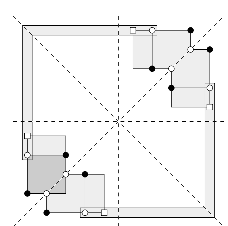

B3.12


B3.13


B3.14


B3.15


B3.16


B3.17

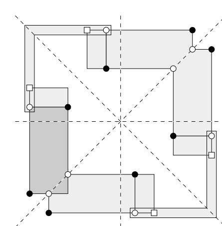
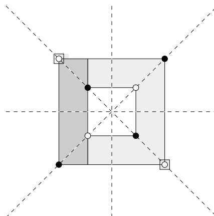
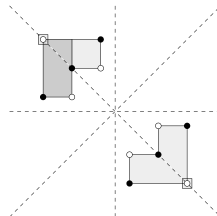

B3.18

Table 6.5: B-type B3 - part 2.

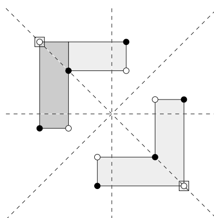
B4	▶ $\sigma(ci) = -ci$ ▶ $\tau(di) = -di$	$Bci = ci$



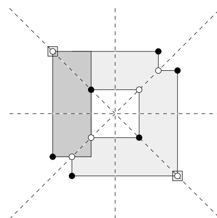
B4.1



B4.2



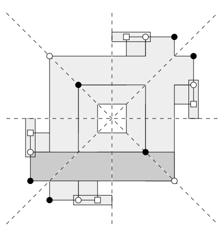
B4.3



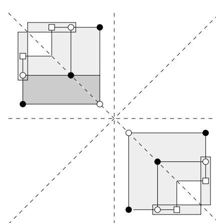
B4.4

Table 6.6: B-type B4.

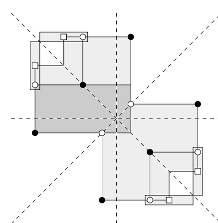
B5	▶ $\sigma(ci) = -ci$ ▶ $\tau(di) < -di$ ▶ sci does not exist, or $\sigma(sci) > -sci$	$Bci = -\sigma(di)$



B5.1



B5.2



B5.3

Table 6.7: B-type B5.

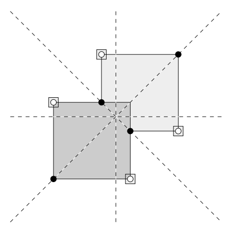
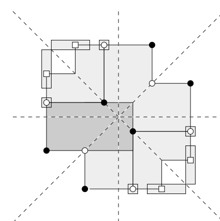
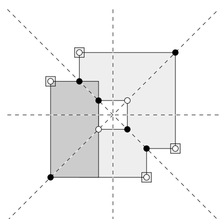
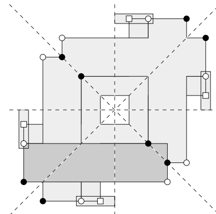
B6	▶ $\sigma(ci) = -ci$ ▶ $\tau(di) < -di$ ▶ sci exists ▶ $sci = -ci$	$Bci = sci$
 B6.1	 B6.2	

Table 6.8: B-type B6.

B7	▶ $\sigma(ci) = -ci$ ▶ $\tau(di) < -di$ ▶ sci exists ▶ $\sigma(sci) = -sci$ ▶ $sci \neq -ci$	$Bci = sci$



B7.1



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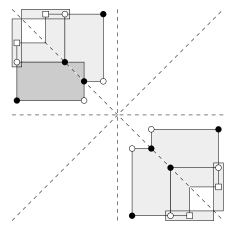

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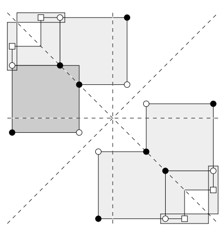

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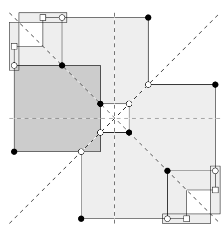

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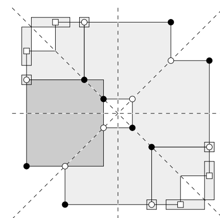
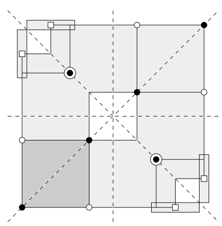


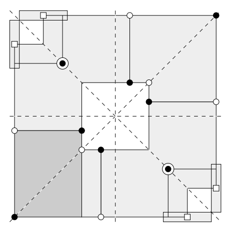
B7.6

Table 6.9: B-type B7.

B8	▸ $\sigma(ci) = -ci$ ▸ $\tau(di) < -di$ ▸ sci exists ▸ $\sigma(sci) < -sci$	$Bci = sci$



B8.1



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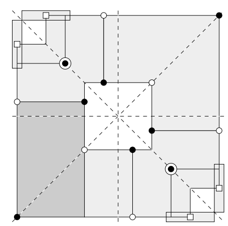
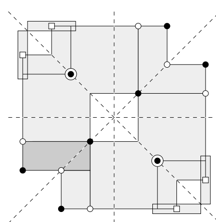


B8.3



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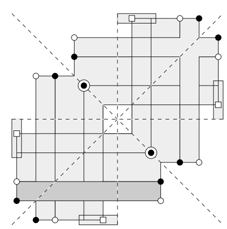

B8.5

Table 6.10: B-type B8 - part 1.

<table>
<tr><td>B8</td><td>

- $\sigma(ci) = -ci$
- $\tau(di) < -di$
- sci exists
- $\sigma(sci) < -sci$

</td><td>$Bci = sci$</td></tr>
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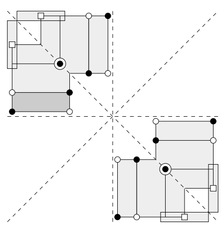
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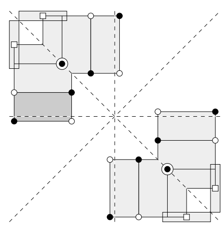
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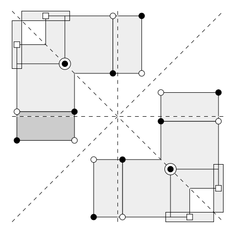
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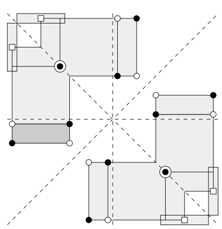
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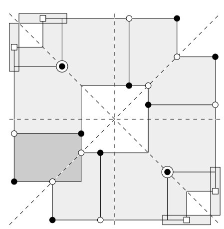
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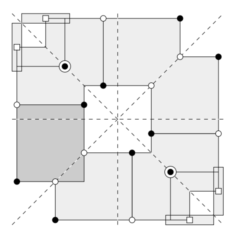
B8.11
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Table 6.10: B-type B8 - part 2.

In order to prove that the insertion property holds, we need the following definitions.

Definition 6.2.3 *Let (σ,τ) be a good pair in $Invol(B_n)$. The* anti-B-multiplicity *of (σ,τ), denoted by a-B-mult(σ,τ), is*

$$a\text{-}B\text{-}mult(\sigma,\tau)=\begin{cases}0, & \text{if } a\text{-}case(\sigma,\tau)=A1,\\ 1, & \text{if } a\text{-}case(\sigma,\tau)=A2,A3,A4,A5,\\ 2, & \text{if } a\text{-}case(\sigma,\tau)=A6.\end{cases}$$

Similarly, the main B-multiplicity *of (σ,τ), denoted by m-B-mult(σ,τ), is*

$$m\text{-}B\text{-}mult(\sigma,\tau)=\begin{cases}0, & \text{if } m\text{-}case(\sigma,\tau)=M1,\\ 1, & \text{if } m\text{-}case(\sigma,\tau)=M2,M3,M4,M5,\\ 2, & \text{if } m\text{-}case(\sigma,\tau)=M6.\end{cases}$$

The B-multiplicity *of (σ,τ), denoted by B-mult(σ,τ), is*

$$B\text{-}mult(\sigma,\tau)=a\text{-}B\text{-}mult(\sigma,\tau)+m\text{-}B\text{-}mult(\sigma,\tau).$$

We summarize the multiplicities in all cases:

		$M1$	$M2$	$M3$	$M4$	$M5$	$M6$
		0	1	1	1	1	2
$A1$	0	0	-	-	-	1	2
$A2$	1	-	2	-	-	2	3
$A3$	1	-	-	-	2	2	3
$A4$	1	-	-	2	2	2	3
$A5$	1	1	2	2	2	2	3
$A6$	2	2	3	3	3	3	4

The following definitions are valid in general for signed permutations (not necessarily involutions).

Definition 6.2.4 *Let $\sigma,\tau\in B_n$, with $\sigma<\tau$. We say that (σ,τ) is a* B-exceptional pair *if*

1. $\tau(di)<di$;
2. $\tau(\tau(di))=di$;

3. $\sigma(ci) = -ci$;
4. sci *exists;*
5. $\sigma(sci) < -sci$;
6. $[-\tau(di), sci] \times [\tau(di)+1, -di]$ *is empty for* σ;

Examples of B-exceptional pairs are the pairs $(\sigma, \tau) \in Invol(B_n)^2$, with $\sigma < \tau$, whose B-type is $B8$ (see Table 6.10).

Definition 6.2.5 *Let* (σ, τ) *be a* B*-exceptional pair. The* second covering transformation *in* B_n *of* σ *with respect to* τ*, denoted by* $sct_\tau^{B_n}(\sigma)$ *is*

$$sct_\tau^{B_n}(\sigma) = \sigma(di, sci)(-di, -sci).$$

Proposition 6.2.6 *Let* (σ, τ) *be a* B*-exceptional pair. Then*

$$sct_\tau^{B_n}(\sigma) \leq \tau.$$

Proof. By the symmetry of the diagram and by Proposition 1.4.3, it suffices to prove that

$$\sigma(di, sci) \leq \tau.$$

We may assume, without loss of generality, that $di = -n$. Let $\chi = \sigma(di, sci)$ and let $R = [di, sci-1] \times [\sigma(di)+1, \sigma(sci)]$. For every $(h,k) \in [\pm n]^2$, we have

$$\chi[h,k] = \begin{cases} \sigma[h,k]+1, & \text{if } (h,k) \in R, \\ \sigma[h,k], & \text{if } (h,k) \notin R. \end{cases}$$

Thus we have to show that $\tau[h,k] \geq \sigma[h,k]+1$, for every $(h,k) \in R$. Let $(h,k) \in R$. If $h < ci$, then

$$\sigma[h,k] = \sigma[h, \tau(di)+1] \leq \tau[h, \tau(di)+1] \leq \tau[h,k] - 1.$$

If $h \geq ci$, by property 6 of the B-exceptional pairs, we have

$$\sigma[h,k] = \sigma[-\tau(di)-1, \tau(di)+1] + 1 \leq \tau[-\tau(di)-1, \tau(di)+1] + 1,$$

and, by properties 1 and 2, we have

$$\tau[-\tau(di)-1, \tau(di)+1] \leq \tau[h,k] - 2,$$

so again $\tau[h,k] \geq \sigma[h,k]+1$. □

We introduce the following notation. Given $\sigma \in B_n$, we set

$$\sigma i = \sigma^{-1}.$$

Given $\sigma, \tau \in B_n$, with $\sigma < \tau$, we set

$$\sigma m = \begin{cases} sct_\tau^{B_n}(\sigma), & \text{if } (\sigma,\tau) \text{ is a } B\text{-exceptional pair,} \\ mct_\tau^{B_n}(\sigma), & \text{otherwise.} \end{cases}$$

Proposition 6.2.7 *Let $\sigma, \tau \in Invol(B_n)$, with $\sigma < \tau$, and let*

$$\chi = \chi_{Invol(B_n)}(\sigma, \tau).$$

Then

$$\chi = \begin{cases} \sigma m, & \textit{if } B\textit{-mult}(\sigma,\chi) = 0, 1, \\ \sigma mim, & \textit{if } B\textit{-mult}(\sigma,\chi) = 2, \\ \sigma mimm, & \textit{if } B\textit{-mult}(\sigma,\chi) = 3, \\ \sigma mimmm, & \textit{if } B\textit{-mult}(\sigma,\chi) = 4. \end{cases}$$

Proof. It can be checked case by case, looking at the pictures of Tables 6.3 to 6.10 and using the description of the minimal covering transformation in B_n.

For example, if $B\text{-}type(\sigma,\tau) = B6.2$, then we have $case(\sigma,\chi) = (A6, M4)$, so $B\text{-}mult(\sigma,\chi) = 3$, and $\chi = \sigma mimm$ is illustrated in Figure 6.4.

□

We can now prove the insertion property of $Invol(B_n)$.

Proposition 6.2.8 *Let $\sigma, \tau \in Invol(B_n)$, with $\sigma < \tau$. Then*

$$\chi_{Invol(B_n)}(\sigma, \tau) \leq \tau.$$

Proof. For every $\sigma \in B_n$, with $\sigma < \tau$, by Proposition 1.4.2, since τ is an involution, we have

$$\sigma i = \sigma^{-1} \leq \tau,$$

and, by Theorem 4.2.15 and Proposition 6.2.6, we have

$$\sigma m \leq \tau.$$

Thus $\chi_{Invol(B_n)}(\sigma, \tau) \leq \tau$ is a consequence of Proposition 6.2.7. □

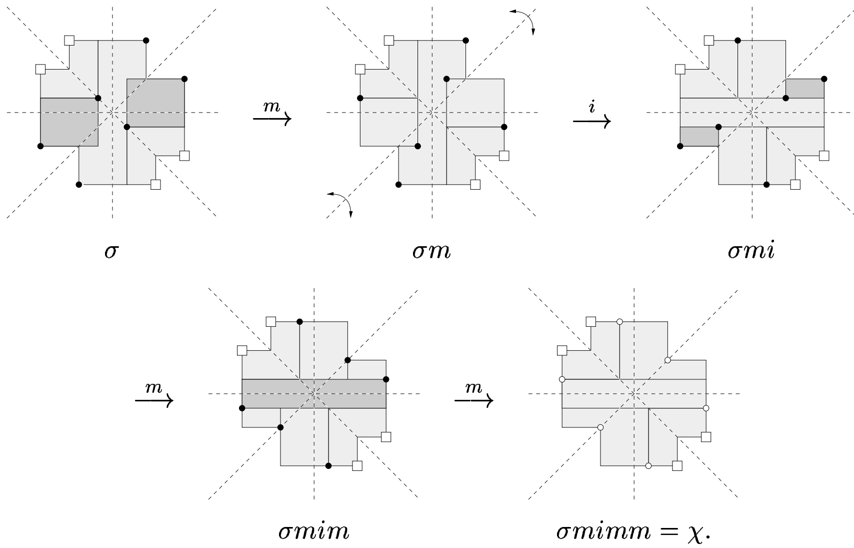


Figure 6.4: Proof of Proposition 6.2.7.

6.3 $Invol(B_n)$ is graded

We recall that the length of $\sigma \in B_n$ is

$$l(\sigma) = \frac{inv(\sigma) + neg(\sigma)}{2},$$

and that the absolute length of $\sigma \in Invol(B_n)$ is

$$al(\sigma) = dna(\sigma).$$

So the average between the length and the absolute length of $\sigma \in Invol(B_n)$ is

$$\rho(\sigma) = \frac{inv(\sigma) + neg(\sigma) + 2dna(\sigma)}{4}.$$

Proposition 6.3.1 *The pair $(H_{Invol(B_n)}, \rho)$ is a covering system of $Invol(B_n)$.*

Proof. By Proposition 6.2.8, $H_{Invol(B_n)}$ is an insertion system of $Invol(B_n)$. The ρ-base property is trivial. It remains to prove the ρ-increasing property. Consider $(\sigma, \tau) \in H_{Invol(B_n)}$. We have to prove that

$$\Delta\rho = \frac{\Delta inv + \Delta neg + 2\Delta dna}{4} = 1,$$

where $\Delta x = x(\tau) - x(\sigma)$.

In case $(A1, M1)$ we have

$$(\Delta inv, \Delta neg, \Delta dna) = (1, 1, 1),$$

and in case $(A4, M4)c$

$$(\Delta inv, \Delta neg, \Delta dna) = (2, 2, 0).$$

All other cases can be unified. We set $mult = B\text{-}mult(\sigma, \tau)$. If Δneg is odd, and this happens in cases $(A1, M6)$, $(A6, M1)$, $(A3, M6)$, $(A6, M3)$, then

$$(\Delta inv, \Delta neg, \Delta dna) = (2mult - 1, 1, 2 - mult).$$

If Δneg is even, that is in all the remaining cases, we have

$$(\Delta inv, \Delta neg, \Delta dna) = (2mult, 0, 2 - mult).$$

Thus in every case $\Delta\rho = 1$. □

We are now able to state and prove the gradedness of $Invol(B_n)$.

Theorem 6.3.2 *The poset $Invol(B_n)$ is graded, with rank function given by*

$$\rho(\sigma) = \frac{inv(\sigma) + neg(\sigma) + 2dna(\sigma)}{4},$$

for every $\sigma \in Invol(B_n)$. In particular $Invol(B_n)$ has rank

$$\rho(Invol(B_n)) = \frac{n^2 + n}{2}.$$

Proof. By Theorem 3.1.2, the first part is a consequence of Proposition 6.3.1.

For the second part, note that the maximum w_0 of B_n, which is also the maximum of $Invol(B_n)$, is such that

$$(inv(w_0), neg(w_0), dna(w_0)) = (n(2n - 1), n, n).$$

□

We also have a characterization of the covering relation in $Invol(B_n)$: if $\sigma, \tau \in Invol(B_n)$ then $\sigma \lhd \tau$ if and only if (σ, τ) is a good pair in $Invol(B_n)$. And the transformation of σ with respect to a suitable label (i, j) actually is a *covering transformation*, denoted by

$$ct_{(i,j)}^{Invol(B_n)}(\sigma).$$

6.4 $Invol(B_n)$ is EL-shellable and Eulerian

Proposition 6.4.1 *Let $\sigma, \tau \in Invol(B_n)$, with $\sigma < \tau$. Then*

$$mi_\tau(\sigma) = (di, Bci).$$

Proof. It is similar to the proof of Proposition 4.2.7, using the definitions of the indices di and Bci. □

Thus, if $\sigma, \tau \in Invol(B_n)$, with $\sigma < \tau$, then the *minimal covering transformation* of σ with respect to τ is

$$mct_\tau^{Invol(B_n)}(\sigma) = ct_{(di,Bci)}^{Invol(B_n)}(\sigma),$$

and we have the following.

Theorem 6.4.2 *Let $\sigma, \tau \in Invol(B_n)$, with $\sigma < \tau$. Then*

$$\sigma \lhd mct_\tau^{Invol(B_n)}(\sigma) \leq \tau.$$

Theorem 6.4.2 ensures that next definition is well-posed.

Definition 6.4.3 *Let $\sigma, \tau \in Invol(B_n)$, with $\sigma < \tau$. The* minimal chain *in $Invol(B_n)$ from σ to τ is the saturated chain*

$$\sigma = \sigma_0 \lhd \sigma_1 \lhd \ldots \lhd \sigma_k = \tau,$$

defined by

$$\sigma_i = mct_\tau^{Invol(B_n)}(\sigma_{i-1}), \tag{6.1}$$

for every $i \in [k]$.

In order to prove that $Invol(B_n)$ is EL-shellable, we prove the increasing and the decreasing properties.

Proposition 6.4.4 (Increasing property) *Let $\sigma, \tau \in Invol(B_n)$, with $\sigma < \tau$. The minimal chain*

$$\sigma = \sigma_0 \lhd \sigma_1 \lhd \ldots \lhd \sigma_k = \tau,$$

defined in (6.1) *has increasing labels.*

Proof. Suppose, by contradiction, that at a certain step there is a decrease in the labels. We may assume, without loss of generality, that this happens at the first step. So

$$\sigma \underset{(di,Bci)}{\lhd} \sigma_1 \underset{(i,j)}{\lhd} \sigma_2,$$

with $(i,j) < (di, Bci)$. So either $i < di$ or $i = di$ and $j < Bci$. If $i < di$, since σ and τ must differ at the index i, the minimality of di is contradicted. So suppose $i = di$ and $j < Bci$. Let $v = \sigma_2(di)$. Since $\sigma_2 \leq \tau$, we have $v \leq \tau(di)$. But $[di, j] \times [\sigma(di), v]$ is the main rectangle of (σ, χ), for some $\chi \in Invol(B_n)$. Thus the minimality of Bci is contradicted. □

Proposition 6.4.5 (Decreasing property) *Let $\sigma, \tau \in Invol(B_n)$, with $\sigma < \tau$, and let*

$$\sigma = \sigma_0 \lhd \sigma_1 \lhd ... \lhd \sigma_k = \tau,$$

be the minimal chain defined in (6.1). *Every saturated chain from σ to τ, different from the minimal one, say*

$$\sigma = \tau_0 \lhd \tau_1 \lhd \ldots \lhd \tau_k = \tau,$$

has at least one decrease in the labels.

Proof. Suppose, by contradiction, that this chain has increasing labels. We may assume, without loss of generality, that $\tau_1 \neq \sigma_1$. Let $l = \min\{h \in [k] : \tau_h(di) = \tau(di)\}$. So, if we set $j = Bci_\tau(\sigma)$, we have

$$\sigma = \sigma_0 \underset{(di,j)}{\lhd} \sigma_1 \lhd \ldots \lhd \sigma_k = \tau$$

and

$$\sigma = \tau_0 \underset{(di,j_1)}{\lhd} \tau_1 \underset{(di,j_2)}{\lhd} \tau_2 \lhd \ldots \lhd \tau_{l-1} \underset{(di,j_l)}{\lhd} \tau_l \underset{(i_{l+1},j_{l+1})}{\lhd} \tau_{l+1} \lhd \ldots \lhd \tau_k = \tau,$$

with $j < j_1 < j_2 < \ldots < j_l$ and $i_{l+1} > di$.

Let $v_h = \tau_h(di)$ for every $h \in [0, l]$. The situation is described in Figure 6.5.

For simplicity, we call *σ-move* and *σ-rect*, respectively, the move (di, j) and the corresponding main rectangle $[di, j] \times [\sigma(di), \sigma_1(di)]$ (gray rectangle in the picture), and we call *τ-moves* and *τ-rects*, respectively, the moves (di, j_r) and the corresponding main rectangles $[di, j_r] \times [v_{r-1}, v_r]$, for $r \in [l]$.

We also call *fix-moves* the moves of anti-cases $A1$ and $A2$, *semifix-moves* the moves of anti-cases $A3$ and $A6$, *exc-moves* the moves of anti-case $A4$, and *def-moves* the moves of anti-case $A5$.

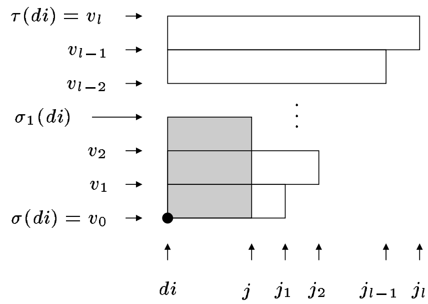


Figure 6.5: Proof of Proposition 6.4.5 (1).

Note that in every case the sequence of the τ-moves is made of a (possibly empty) sequence of def-moves, possibily followed by a semifix-move, and by a (possibly empty) sequence of exc-moves.

Suppose that the anti-case of the σ-move is $A1$. In this case the τ-moves are all exc-moves, and none of them could bring the dot in column di on the row $-di$. But the σ-move brings that dot on that row, so we get a contradiction.

Suppose that the anti-case of the σ-move is $A2$. In this case the only τ-move which could move the dot in the cell $(j, -j)$ is the semifix-move, say (di, j_m). In all cases, except $(A3, M4)$ and $(A6, M4)$, the τ-move (di, j_m) would bring the dot in column di on the row $-j$. In case $(A3, M4)$, after this τ-move there would be a dot in the cell $(v_{m-1}, -v_{m-1})$. In case $(A6, M4)$, after this τ-move there would be a dot in the cell $(v_{m-1}, -j)$. In every case the following τ-moves are exc-moves, and none of them could bring the dot in column di on the row $-di$. Again we get a contradiction.

Suppose that the anti-case of the σ-move is $A3$. In this case the τ-moves are all exc-moves, and none of them could move the dot in the row $\sigma_1(di)$. Thus none of them could bring the dot in column di on or above the row $\sigma_1(di)$. But the σ-move brings that dot on or above that row, so we get a contradiction.

If the anti-case of the σ-move is $A4$, then reasoning is the same as in case $A3$, if the main case is $M3$, $M4$ or $M5$, and the same as in next case $A5$, if the main case is $M6$.

Suppose that the anti-case of the σ-move is $A5$ (see Figure 6.6). In this case none of the τ-moves could move the dot in the cell $(j, \sigma_1(di))$, unless the symmetric of that dot with respect to the main diagonal is the up-right corner of one of the τ-rects, say $[di, j_m] \times [v_{m-1}, j]$. After this move there would be a dot in the cell $(v_{m-1}, \sigma_1(di))$. The following τ-moves could not move that dot. So no τ-move could bring the dot in column di on or above the row $\sigma_1(di)$, again with a contradiction.

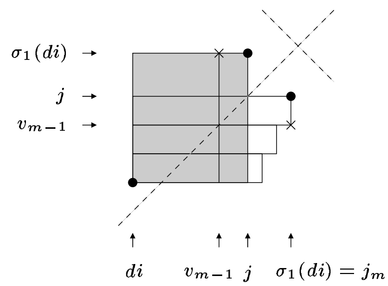


Figure 6.6: Proof of Proposition 6.4.5 (2).

Finally, suppose that the anti-case of the σ-move is $A6$ (see Figure 6.7). In this case the only τ-move which could move the dots in rows $-j$ and $\sigma_1(di)$ is the semifix-move. The only possibility is that this move is the one corresponding to the B-rectangle $[di, \sigma_1(di)] \times [v_{m-1}, j]$. After this move there would be a dot in the cell $(v_{m-1}, \sigma_1(di))$ and the conclusion is the same as in previous case.

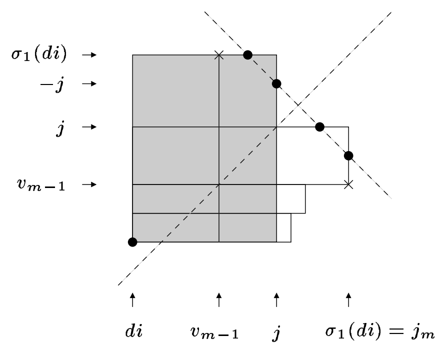


Figure 6.7: Proof of Proposition 6.4.5 (3).

□

Next theorem is an immediate consequence of the increasing and the decreasing properties.

Theorem 6.4.6 *The poset $Invol(B_n)$ is EL-shellable, having the standard labelling as an EL-labelling.*

We can now prove that the condition of Theorem 1.2.1 holds for the poset $Invol(B_n)$, and thus that it is Eulerian.

Theorem 6.4.7 *The poset $Invol(B_n)$ is Eulerian.*

Proof. Suppose we label the edges of the Hasse diagram of $Invol(B_n)$ with the standard labelling.

Let $\sigma, \tau \in Invol(B_n)$, with $\sigma < \tau$. By Theorem 1.2.1, we only have to show that there is exactly one saturated chain from σ to τ with decreasing labels. Consider the two involutions $w_0\sigma$ and $w_0\tau$ of B_n. By Proposition 1.4.3, we have

$$w_0\tau < w_0\sigma.$$

Since the standard labelling of $Invol(B_n)$ is an EL-labelling, there is exactly one saturated chain from $w_0\tau$ to $w_0\sigma$ with non decreasing labels (it actually has strictly increasing labels), say

$$w_0\tau = \chi_0 \lhd \chi_1 \lhd \ldots \lhd \chi_k = w_0\sigma.$$

Then

$$\sigma = w_0\chi_k \lhd \ldots \lhd w_0\chi_1 \lhd w_0\chi_0 = \tau$$

is the unique saturated chain from σ to τ with decreasing labels. □

Chapter 7

Bruhat order on the involutions of D_n

In this chapter we study the poset $Invol(D_n)$ of the involutions of D_n.

In Figure 7.1 the poset $Invol(D_4)$ is illustrated.

7.1 Successor system

Definition 7.1.1 *Let $\sigma, \tau \in Invol(D_n)$. We say that (σ, τ) is a* good pair *in $Invol(D_n)$ if there exists a rectangle $R = [i, j] \times [\sigma(i), \tau(i)]$, either not central or central not symmetric, such that the same conditions as in Definition 6.1.1 are satisfied, with the exceptions, if R is central not symmetric, that:*

1. *in cases $(A6, M1)$ and $(A6, M3)$, picture $A6$ is replaced by picture $A6'$, and in cases $(A1, M6)$ and $(A3, M6)$, picture $M6$ is replaced by picture $M6'$, as shown in Figure 7.2;*

2. *in the remaining cases, $(A3, M4)$, $(A4, M3)$, $(A4, M4)$, $(A4, M6)$, $(A6, M4)$, the presence in R of one more dot either of σ or of τ, which is in the orbit of one of those indicated in the pictures, is allowed.*

Let (σ, τ) be a good pair in $Invol(D_n)$. The *main rectangle* of (σ, τ), denoted by $R(\sigma, \tau)$ is the rectangle $R = [i, j] \times [\sigma(i), \tau(i)]$ mentioned in Definition 7.1.1.

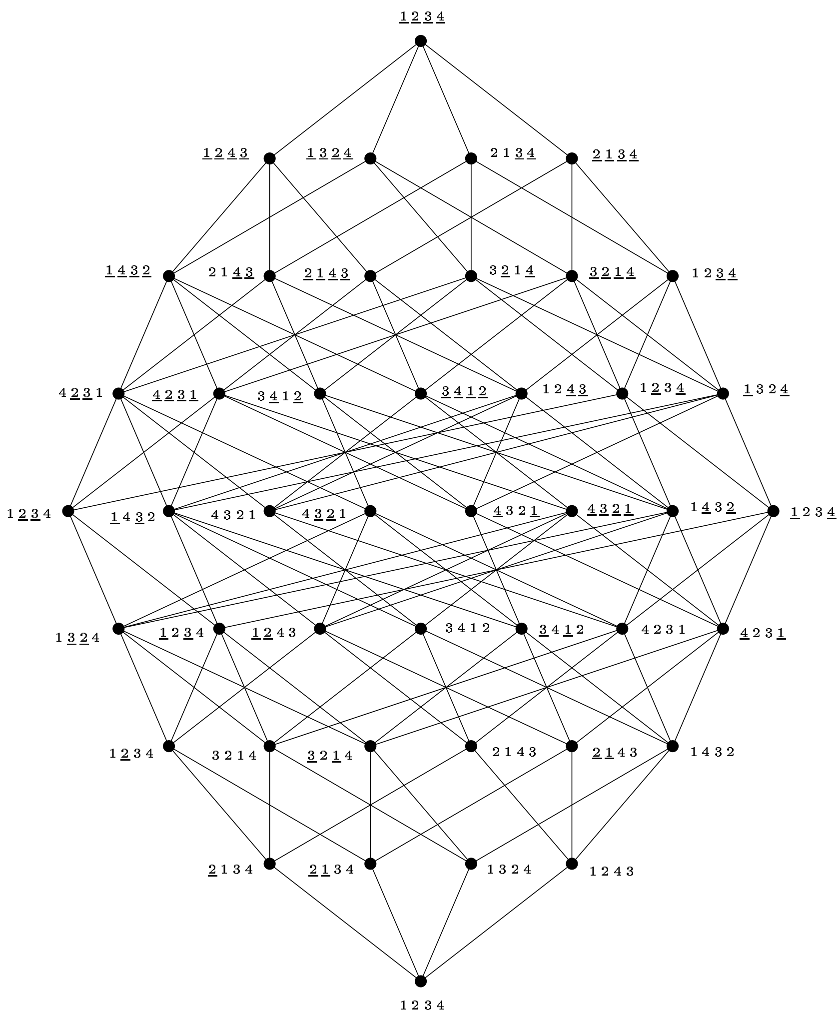


Figure 7.1: $Invol(D_4)$.

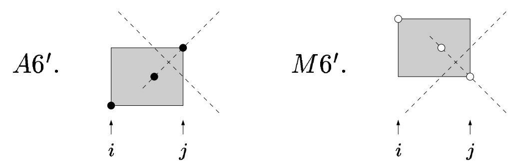
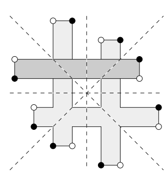
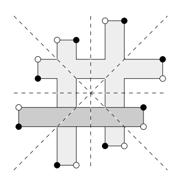
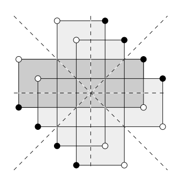
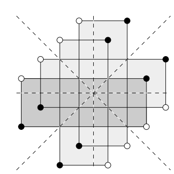


Figure 7.2: Covering relation in $Invol(D_n)$: new cases.

Note that not central cases are exactly the same as in $Invol(B_n)$, so they are described in Table 6.1 (p. 76). Central cases are new, and they are described in Table 7.1. The notation is the same used in Table 6.1.

The only case in which there is more than one possibility is $(A4, M4)$, as we show in the following table.

$(A4, M4)a$	$(A4, M4)b$	$(A4, M4)c$	$(A4, M4)d$
			

We set

$$H_{Invol(D_n)} = \{(\sigma, \tau) \in Invol(D_n)^2 : (\sigma, \tau) \text{ is a good pair}\},$$

and define the *standard labelling* λ of $Invol(D_n)$ by associating, with every good pair $(\sigma, \tau) \in H_{Invol(D_n)}$, the pair $(i, j) \in [\pm n]^2$, if $R = [i, j] \times [\sigma(i), \tau(i)]$ is the main rectangle of (σ, τ).

By Theorem 4.2.22, $H_{Invol(D_n)}$ is a successor system and, since τ is uniquely determined by σ and by the label (i, j), λ is a good labelling.

Given $\sigma \in Invol(D_n)$, a pair $(i, j) \in [\pm n]^2$ is a *suitable label* of σ if there exists $\tau \in Invol(D_n)$, with $\lambda(\sigma, \tau) = (i, j)$. Such a τ, obviously unique, is called the *transformation* of σ with respect to (i, j) and it is denoted by

$$t^{Invol(D_n)}_{(i,j)}(\sigma).$$

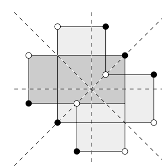
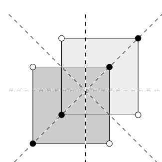
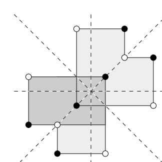
	M1	M2	M3
A1	-	-	-
A2	-	-	-
A3	-	-	-
A4	-	-	
A5	-	-	-
A6		-	

Table 7.1: Covering relation in $Involv(D_n)$: central cases - part 1.

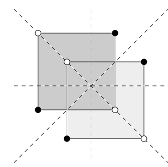
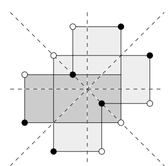
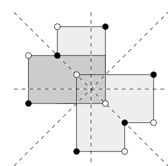
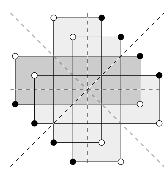
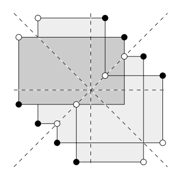
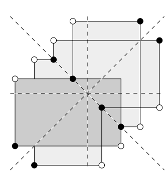
	M4	M5	M6	
	-	-		A1
	-	-	-	A2
		-		A3
		-		A4
	-	-	-	A5
		-	-	A6

Table 7.1: Covering relation in $Invol(D_n)$: central cases - part 2.

7.2 Insertion property

Since an element of $Invol(D_n)$ is also in $Invol(B_n)$, we can consider the same types as defined in 6.2.1.

Definition 7.2.1 *Let $\sigma, \tau \in Invol(D_n)$. The* D-type *of the pair (σ, τ) is*

$$D\text{-}type(\sigma, \tau) = Dh.k,$$

if the B-type of (σ, τ) is $Bh.k$, referring to the cases as in Tables 6.3 to 6.10.

Note that in $Invol(D_n)$ some B-types cannot occur.

Proposition 7.2.2 *Let $\sigma, \tau \in Invol(D_n)$, with $\sigma < \tau$. The B-type of (σ, τ) cannot be $B1.1$, $B1.4$ (see Table 6.3, p. 83) or $B6.1$ (see Table 6.8, p. 88).*

Proof. If $B\text{-}type(\sigma, \tau) = B1.1$ or $B1.4$, then $(di + 1, \sigma(ci))$ is a not valid D-cell of (σ, τ), contradicting $\sigma < \tau$.

Now let $B\text{-}type(\sigma, \tau) = B6.1$. We may assume, without loss of generality, that $di = -n$. We have $ci = -1$ and $\sigma[-2, 2] = n - 2$, which implies $\tau[-2, 2] = n - 2$. So $neg(\sigma) = n - 1$ and $neg(\tau) = n$, contradicting $\sigma, \tau \in D_n$. □

We want to define the D-covering index Dci and the involution $\chi_{Invol(D_n)}(\sigma, \tau)$, which are the analogs of Bci and $\chi_{Invol(B_n)}(\sigma, \tau)$ in $Invol(B_n)$. In almost all cases the behaviour in $Invol(D_n)$ is exactly the same as in $Invol(B_n)$, that is, $Dci = Bci$ and $\chi_{Invol(D_n)}(\sigma, \tau) = \chi_{Invol(B_n)}(\sigma, \tau)$. Only in some cases the approach in $Invol(D_n)$ is different with respect to $Invol(B_n)$. These are the cases represented in Tables 7.2 to 7.10.

In particular we consider the cases in which (σ, τ) is a D-special pair, in the sense of Definition 4.2.19 (p. 48), corresponding to cases 9a, 9b and 10 of Figure 4.4 (p. 50). This can occur in D-types $D3.7$, $D3.8$, $D3.10$, $D3.16$, $D3.17$.

Definition 7.2.3 *Let $\sigma, \tau \in Invol(D_n)$, with $\sigma < \tau$. The* D-covering index *of σ with respect to τ, denoted by $Dci_\tau(\sigma)$ (or simply Dci), is the minimal index $j \in [di + 1, n]$ such that there exists $\chi \in Invol(D_n)$, with*

1. $(\sigma, \chi) \in H_{Invol(D_n)}$;
2. $\lambda(\sigma, \chi) = (di, j)$;

3. $\chi(di) \leq \tau(di)$.

As it comes out from the pictures, Dci is always well defined.

By definition, (di, Dci) is a suitable label of σ, so we can consider the involution

$$\chi_{Invol(D_n)}(\sigma,\tau) = t^{Invol(D_n)}_{(di,Dci)}(\sigma).$$

In the pictures the involutions σ, τ and $\chi = \chi_{Invol(D_n)}(\sigma,\tau)$ are represented, with the same notation used in Tables 6.3 to 6.10.

In order to prove that the insertion property holds, we need the following definitions.

Definition 7.2.4 *Let (σ,τ) be a good pair in $Invol(D_n)$. The* anti-D-multiplicity *of (σ,τ), denoted by a-D-$mult(\sigma,\tau)$, is*

$$a\text{-}D\text{-}mult(\sigma,\tau) = \begin{cases} 0, & \text{if } a\text{-}case(\sigma,\tau) = A1, \\ 1, & \text{if } a\text{-}case(\sigma,\tau) = A2, A3, A4, A5, A6', \\ 2, & \text{if } a\text{-}case(\sigma,\tau) = A6. \end{cases}$$

Similarly, the main D-multiplicity *of (σ,τ), denoted by m-D-$mult(\sigma,\tau)$, is*

$$m\text{-}D\text{-}mult(\sigma,\tau) = \begin{cases} 0, & \text{if } m\text{-}case(\sigma,\tau) = M1, \\ 1, & \text{if } m\text{-}case(\sigma,\tau) = M2, M3, M4, M5, M6', \\ 2, & \text{if } m\text{-}case(\sigma,\tau) = M6. \end{cases}$$

The D-multiplicity *of (σ,τ), denoted by D-$mult(\sigma,\tau)$, is*

$$D\text{-}mult(\sigma,\tau) = a\text{-}D\text{-}mult(\sigma,\tau) + m\text{-}D\text{-}mult(\sigma,\tau).$$

The following definitions are valid in general for even-signed permutations (not necessarily involutions).

If $\sigma, \tau \in D_n$, with $\sigma < \tau$, are such that (σ,τ) is a B-exceptional pair (see Definition 6.2.4, p. 92), then the *second covering transformation* in D_n of σ with respect to τ, denoted by $sct^{D_n}_{\tau}(\sigma)$, is defined in the same way as in B_n:

$$sct^{D_n}_{\tau}(\sigma) = sct^{B_n}_{\tau}(\sigma) = \sigma(di, sci)(-di, -sci).$$

The following result is the analog in D_n of Proposition 6.2.6.

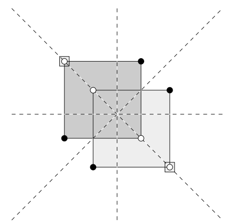
D1.3	
 D1.3	

Table 7.2: D-type D1.3.

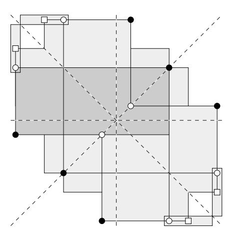
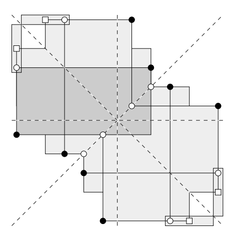
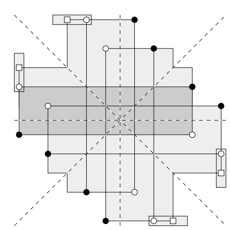
D2		
 D2.1	 D2.2	 D2.3

Table 7.3: D-type D2.

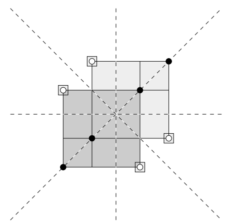
D3.7	
 D3.7b	

Table 7.4: *D*-type D3.7.

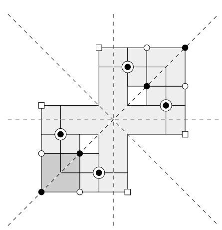
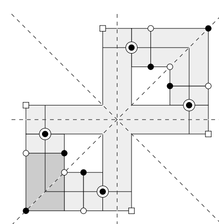
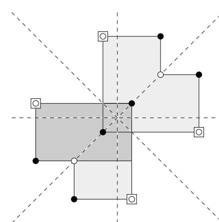
D3.8	
 D3.8b	 D3.8c

Table 7.5: *D*-type D3.8.

D3.10	



D3.10b

Table 7.6: D-type D3.10.

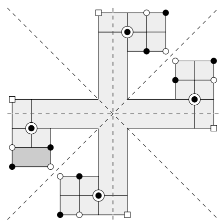
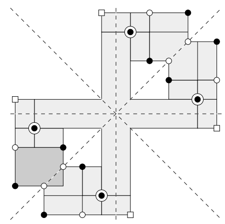
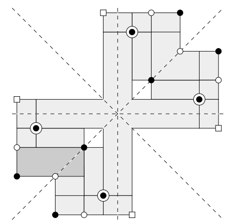
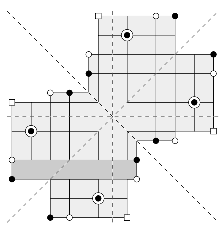
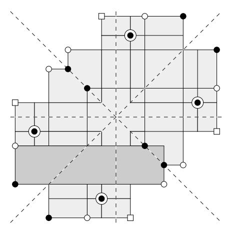
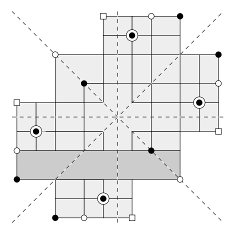
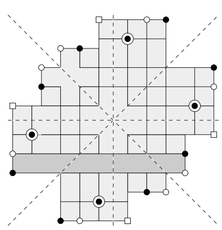
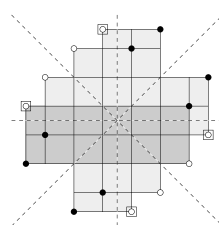
D3.16		
 D3.16b	 D3.16c	 D3.16d
 D3.16e	 D3.16f	 D3.16g
 D3.16h	 D3.16i	

Table 7.7: *D*-type D3.16.

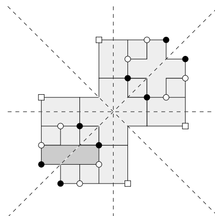
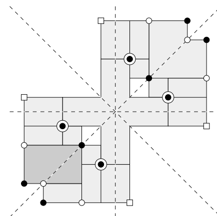
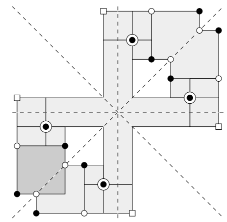
D3.17	
 D3.17b  D3.17c  D3.17d	

Table 7.8: D-type D3.17.

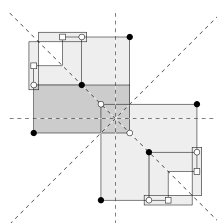
D5.3	
 D5.3	

Table 7.9: D-type D5.3.

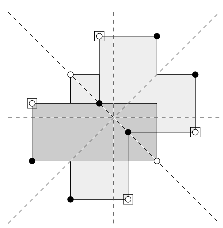
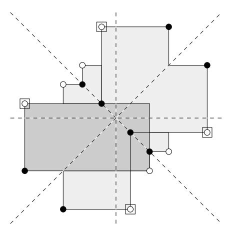
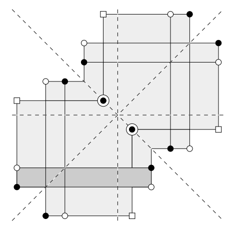
D6.2		
		
D6.2a	D6.2b	D6.2c

Table 7.10: *D*-type D6.2.

Proposition 7.2.5 *Let $\sigma, \tau \in D_n$, with $\sigma < \tau$, be such that (σ, τ) is a B-exceptional pair. Then*

$$sct_\tau^{D_n}(\sigma) \leq \tau.$$

Proof. Let $\chi = sct_\tau^{D_n}(\sigma)$. By Proposition 6.2.6, we have that (χ, τ) satisfies the B-condition. On the other hand, in every case the pair (χ, τ) has no new D-cells with respect to the pair (σ, τ). So (χ, τ), as well as (σ, τ), satisfies the D-condition. □

In D_n there is one more *exceptional* case to consider.

Definition 7.2.6 *Let $\sigma, \tau \in D_n$. Suppose that sci exists and that the set*

$$\{j \in [sci + 1, n] : \sigma(j) \in [\sigma(di) + 1, \sigma(sci) - 1]\}$$

is not empty. Then the third covering index *of σ with respect to τ, denoted by $tci_\tau(\sigma)$ (or simply tci), is*

$$tci_\tau(\sigma) = \min\{j \in [sci + 1, n] : \sigma(j) \in [\sigma(di) + 1, \sigma(sci) - 1]\}.$$

Definition 7.2.7 *Let $\sigma, \tau \in D_n$, with $\sigma < \tau$. We say that (σ, τ) is a D-*exceptional pair *if*

1. $\tau(di) = \sigma(ci)$;
2. *sci exists;*
3. $sci = -ci$;
4. *tci exists;*
5. $\sigma(tci) < -tci$;

An example of D-exceptional pair is $(\sigma, \tau) \in Invol(D_n)^2$, with $\sigma < \tau$, whose D-type is $D6.2c$ (see Table 7.10).

Definition 7.2.8 *Let $\sigma, \tau \in D_n$, with $\sigma < \tau$, be such that (σ, τ) is a D-exceptional pair. The* third covering transformation *in D_n of σ with respect to τ, denoted by $tct_\tau^{D_n}(\sigma)$, is*

$$tct_\tau^{D_n}(\sigma) = \sigma(di, tci)(-di, -tci).$$

Proposition 7.2.9 *Let $\sigma, \tau \in D_n$, with $\sigma < \tau$, be such that (σ, τ) is a D-exceptional pair. Then*

$$tct_{\tau}^{D_n}(\sigma) \leq \tau.$$

Proof. By the symmetry of the diagram and by Proposition 1.4.3, it suffices to prove that

$$\sigma(di, tci) \leq \tau.$$

Let $\chi = \sigma(di, tci)$. In order to prove that (χ, τ) satisfies the B-condition, we may assume, without loss of generality, that $di = -n$. Consider the rectangle $R = [di, tci - 1] \times [\sigma(di) + 1, \sigma(tci)]$. For every $(h, k) \in [\pm n]^2$, we have

$$\chi[h,k] = \begin{cases} \sigma[h,k] + 1, & \text{if } (h,k) \in R, \\ \sigma[h,k], & \text{if } (h,k) \notin R. \end{cases}$$

So we have to show that $\tau[h,k] \geq \sigma[h,k] + 1$, for every $(h,k) \in R$. Let $(h,k) \in R$. If $h < sci$, then the proof is the same as in Proposition 6.2.6. Now let $h \in [sci, tci - 1]$. Since $\sigma, \tau \in D_n$, $\sigma[-\tau(di), \tau(di)]$ and $\tau[-\tau(di), \tau(di)]$ have the same parity, so we have

$$\sigma[-\tau(di) - 1, \tau(di) + 1] \leq \tau[-\tau(di) - 1, \tau(di) + 1] + 1.$$

Then

$$\begin{aligned} \sigma[h,k] &= \sigma[-\tau(di) - 1, \tau(di) + 1] + 2 \\ &\leq \tau[-\tau(di) - 1, \tau(di) + 1] + 3 \\ &\leq (\tau[h,k] - 2) + 3. \end{aligned}$$

Thus $\tau[h,k] \geq \sigma[h,k] + 1$.

Finally, the pair (χ, τ) has no new D-cells with respect to the pair (σ, τ). So (χ, τ), as well as (σ, τ), satisfies the D-condition. □

We introduce the following notation. Given $\sigma \in D_n$, we set

$$\sigma i = \sigma^{-1}.$$

Given $\sigma, \tau \in D_n$, with $\sigma < \tau$, we set

$$\sigma m = \begin{cases} sct_{\tau}^{D_n}(\sigma), & \text{if } (\sigma, \tau) \text{ is a } B\text{-exceptional pair}, \\ tct_{\tau}^{D_n}(\sigma), & \text{if } (\sigma, \tau) \text{ is a } D\text{-exceptional pair}, \\ mct_{\tau}^{D_n}(\sigma), & \text{otherwise}. \end{cases}$$

Proposition 7.2.10 *Let $\sigma, \tau \in Invol(D_n)$, with $\sigma < \tau$, and let*

$$\chi = \chi_{Invol(D_n)}(\sigma, \tau).$$

Then

$$\chi = \begin{cases} \sigma m, & \text{if } D\text{-}mult(\sigma,\chi) = 0, 1, \\ \sigma mim, & \text{if } D\text{-}mult(\sigma,\chi) = 2, \\ \sigma mimm, & \text{if } D\text{-}mult(\sigma,\chi) = 3, \\ \sigma mimmm, & \text{if } D\text{-}mult(\sigma,\chi) = 4. \end{cases}$$

Proof. It can be checked case by case, looking at the pictures of Tables 6.3 to 6.10 and those of Tables 7.2 to 7.10, and using the description of the minimal covering transformation in D_n.

For example, if $D\text{-}type(\sigma, \tau) = D6.2b$, then we have $case(\sigma, \chi) = (A6, M4)$, so $D\text{-}mult(\sigma, \chi) = 3$, and $\chi = \sigma mimm$ is illustrated in Figure 7.3.

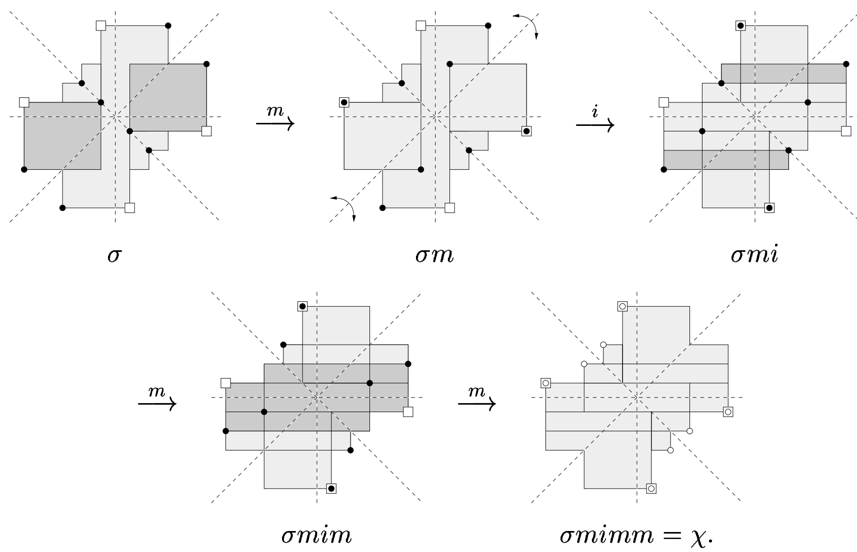


Figure 7.3: Proof of Proposition 7.2.10.

□

We can now prove the insertion property of $Invol(D_n)$.

Proposition 7.2.11 *Let $\sigma, \tau \in Invol(D_n)$, with $\sigma < \tau$. Then*

$$\chi_{Invol(D_n)}(\sigma, \tau) \leq \tau.$$

Proof. For every $\sigma \in D_n$, with $\sigma < \tau$, by Proposition 1.4.2, since τ is an involution, we have

$$\sigma i = \sigma^{-1} \leq \tau,$$

and by Theorem 4.2.24, Proposition 7.2.5 and Proposition 7.2.9, we have

$$\sigma m \leq \tau.$$

Thus $\chi_{Invol(D_n)}(\sigma, \tau) \leq \tau$ is a consequence of Proposition 7.2.10. □

7.3 $Invol(D_n)$ is graded

We recall that the length of $\sigma \in D_n$ is

$$l(\sigma) = \frac{inv(\sigma) - neg(\sigma)}{2},$$

and that the absolute length of $\sigma \in Invol(D_n)$ is

$$al(\sigma) = dna(\sigma).$$

So the average between the length and the absolute length of $\sigma \in Invol(D_n)$ is

$$\rho(\sigma) = \frac{inv(\sigma) - neg(\sigma) + 2dna(\sigma)}{4}.$$

Proposition 7.3.1 *The pair $(H_{Invol(D_n)}, \rho)$ is a covering system of $Invol(D_n)$.*

Proof. By Proposition 7.2.11, $H_{Invol(D_n)}$ is an insertion system of $Invol(D_n)$. The ρ-base property is trivial. It remains to prove the ρ-increasing property. Consider $(\sigma, \tau) \in H_{Invol(D_n)}$. We have to prove that

$$\Delta\rho = \frac{\Delta inv - \Delta neg + 2\Delta dna}{4} = 1,$$

where $\Delta x = x(\tau) - x(\sigma)$.

In not central cases (see Table 6.1, p. 76) we have $\Delta neg = 0$, so $\Delta\rho$ is the same as in $Invol(B_n)$, and it has already been proved that $\Delta\rho = 1$.

In central cases (see Table 7.1, p. 106) we have

$$(\Delta inv, \Delta neg, \Delta dna) = \begin{cases} (4,2,1), & \text{in cases } (A1, M6'), (A6', M1), \\ (6,2,0), & \text{in cases } (A3, M4), (A4, M3), (A3, M6'), (A6', M3), \\ (8,2,-1), & \text{in cases } (A4, M6), (A6, M4), \\ (8,4,0), & \text{in case } (A4, M4). \end{cases}$$

Thus in every case $\Delta\rho = 1$. □

We are now able to state and prove the gradedness of $Invol(D_n)$.

Theorem 7.3.2 *The poset $Invol(D_n)$ is graded, with rank function given by*

$$\rho(\sigma) = \frac{inv(\sigma) - neg(\sigma) + 2dna(\sigma)}{4},$$

for every $\sigma \in Invol(D_n)$. In particular $Invol(D_n)$ has rank

$$\rho(Invol(D_n)) = \left\lfloor \frac{n^2}{2} \right\rfloor.$$

Proof. By Theorem 3.1.2, the first part is a consequence of Proposition 7.3.1.

For the second part, note that the maximum w_0 of D_n, which is also the maximum of $Invol(D_n)$, is such that

$$(inv(w_0), neg(w_0), dna(w_0)) = \begin{cases} (n(2n-1), n, n), & \text{if } n \text{ is even}, \\ (n(2n-1)-1, n-1, n-1), & \text{if } n \text{ is odd}. \end{cases}$$

So

$$\rho(Invol(D_n)) = \rho(w_0) = \begin{cases} n^2/2, & \text{if } n \text{ is even}, \\ (n^2-1)/2, & \text{if } n \text{ is odd}. \end{cases}$$

□

We also have a characterization of the covering relation in $Invol(D_n)$: if $\sigma, \tau \in Invol(D_n)$, then $\sigma \lhd \tau$ if and only if (σ, τ) is a good pair in $Invol(D_n)$. And the transformation of σ with respect to a suitable label (i,j) actually is a *covering transformation*, denoted by

$$ct_{(i,j)}^{Invol(D_n)}(\sigma).$$

7.4 $Invol(D_n)$ is EL-shellable and Eulerian

Proposition 7.4.1 *Let $\sigma, \tau \in Invol(D_n)$, with $\sigma < \tau$. Then*

$$mi_\tau(\sigma) = (di, Dci).$$

Proof. It is similar to the proof of Proposition 4.2.7, using the definitions of the indices di and Dci. □

Thus, if $\sigma, \tau \in Invol(D_n)$, with $\sigma < \tau$, then the *minimal covering transformation* of σ with respect to τ is

$$mct_\tau^{Invol(D_n)}(\sigma) = ct_{(di,Dci)}^{Invol(D_n)}(\sigma),$$

and we have the following.

Theorem 7.4.2 *Let $\sigma, \tau \in Invol(D_n)$, with $\sigma < \tau$. Then*

$$\sigma \lhd mct_\tau^{Invol(D_n)}(\sigma) \leq \tau.$$

Theorem 7.4.2 ensures that next definition is well-posed.

Definition 7.4.3 *Let $\sigma, \tau \in Invol(D_n)$, with $\sigma < \tau$. The* minimal chain *in $Invol(D_n)$ from σ to τ is the saturated chain*

$$\sigma = \sigma_0 \lhd \sigma_1 \lhd \ldots \lhd \sigma_k = \tau,$$

defined by

$$\sigma_i = mct_\tau^{Invol(D_n)}(\sigma_{i-1}), \tag{7.1}$$

for every $i \in [k]$.

In order to prove that $Invol(D_n)$ is EL-shellable, we prove the increasing and the decreasing properties.

Proposition 7.4.4 (Increasing property) *Let $\sigma, \tau \in Invol(D_n)$, with $\sigma < \tau$. The minimal chain*

$$\sigma = \sigma_0 \lhd \sigma_1 \lhd \ldots \lhd \sigma_k = \tau,$$

defined in (7.1) *has increasing labels.*

Proof. Suppose, by contradiction, that at a certain step there is a decrease in the labels. We may assume, without loss of generality, that this happens at the first step. So

$$\sigma \underset{(di,Dci)}{\lhd} \sigma_1 \underset{(i,j)}{\lhd} \sigma_2,$$

with $(i,j) < (di, Dci)$. So either $i < di$ or $i = di$ and $j < Dci$. If $i < di$, since σ and τ must differ at the index i, the minimality of di is contradicted. So suppose $i = di$ and $j < Dci$. Let $v = \sigma_2(di)$. Since $\sigma_2 \leq \tau$, we have $v \leq \tau(di)$. But $[di, j] \times [\sigma(di), v]$ is the main rectangle of (σ, χ), for some $\chi \in Invol(D_n)$. Thus the minimality of Dci is contradicted. □

Proposition 7.4.5 (Decreasing property) *Let $\sigma, \tau \in Invol(D_n)$, with $\sigma < \tau$, and let*

$$\sigma = \sigma_0 \lhd \sigma_1 \lhd ... \lhd \sigma_k = \tau,$$

be the minimal chain defined in (7.1). *Every saturated chain from σ to τ, different from the minimal one, say*

$$\sigma = \tau_0 \lhd \tau_1 \lhd \ldots \lhd \tau_k = \tau,$$

has at least one decrease in the labels.

Proof. The proof is essentially the same as in Proposition 6.4.5. There are a few further cases to be considered, and they can be handled with the same techniques. □

Next theorem is an immediate consequence of the increasing and the decreasing properties.

Theorem 7.4.6 *The poset $Invol(D_n)$ is EL-shellable, having the standard labelling as an EL-labelling.*

We can now prove that the condition of Theorem 1.2.1 holds for the poset $Invol(D_n)$, and thus that it is Eulerian.

Theorem 7.4.7 *The poset $Invol(D_n)$ is Eulerian.*

Proof. Suppose we label the edges of the Hasse diagram of $Invol(D_n)$ with the standard labelling.

Let $\sigma, \tau \in Invol(D_n)$, with $\sigma < \tau$. By Theorem 1.2.1, we only have to show that there is exactly one saturated chain from σ to τ with decreasing labels.

Note that, if $\chi \in D_n$, then $w_0\chi$ not necessarily is in D_n. More precisely, since

$$neg(w_0\chi) = n - neg(\chi),$$

we have that $w_0\chi \in D_n$ if and only if n is even.

But, if $\chi \in D_n$, we can consider the signed permutation of B_{n+1}, which we denote by $ext(\chi)$, whose diagram is obtained from the diagram of χ, by adding rows and columns $\pm(n+1)$, and either the two dots in cells $(n+1, n+1)$ and $(-n-1, -n-1)$, if n is even, or the two dots in cells $(n+1, -n-1)$ and $(-n-1, n+1)$, if n is odd. In every case $w_0 ext(\chi) \in D_{n+1}$.

Consider the two even-signed permutations $w_0 ext(\sigma)$ and $w_0 ext(\tau)$. They are involutions of D_{n+1} and it's easy to see that

$$w_0 ext(\tau) < w_0 ext(\sigma).$$

Since the standard labelling of $Invol(D_{n+1})$ is an EL-labelling, there is exactly one saturated chain from $w_0 ext(\tau)$ to $w_0 ext(\sigma)$ with non decreasing labels (it actually has strictly increasing labels), and necessarily all the elements of this chain have the form $w_0 ext(\chi)$, for some $\chi \in Invol(D_n)$:

$$w_0 ext(\tau) = w_0 ext(\chi_0) \lhd w_0 ext(\chi_1) \lhd \ldots \lhd w_0 ext(\chi_k) = w_0 ext(\sigma).$$

Then

$$\sigma = \chi_k \lhd \ldots \lhd \chi_1 \lhd \chi_0 = \tau$$

is the unique saturated chain from σ to τ with decreasing labels. □

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