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Ph.D. Thesis

Integro–Differential Problems Arising in Pricing Derivatives in Jump–Diffusion Markets

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"La forza trainante della Matematica
non è il ragionamento, ma l'immaginazione".
Augustus de Morgan

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Introduction

The aim of this Thesis is to investigate some analytical and numerical results for viscosity solutions to integro–differential problems arising in Mathematical Finance when derivatives are valuated in a market driven by general jump–diffusion processes.
The pricing problem has been an appealing task since the ′70s and a huge literature have been produced in recent years. Here we shall study models with jumps which allows for a more realistic description of Financial markets with the use of more sophisticated mathematical instruments.
The Thesis will be divided in three parts. The first one contains a basic description of financial markets with a particular regard to jump–diffusion models and to the ideas that shall be used in the following. In the second part we shall present our analytical results, while in the last one we shall present the numerical results concerning the problem studied in Part II.
Here, I shall briefly describe the three parts.

Financial Motivations. In Mathematical Finance, the Brownian motion has played until now a predominant role in describing the evolution of market prices, because of its simplicity of calculation.

A successful result for modeling is the Nobel prize awarded Black and Scholes' formula [28]: in a market described by a Brownian evolution model, it relates the stochastic problem of finding a fair price for derivatives with the solution to a deterministic partial differential equation with constant coefficient.
The approach proposed by Black and Scholes has gained great success among practitioners because it gives a closed form solution; moreover the PDE analysis allows simpler numerical approach to the problem. Unfortunately, various situations, as for instance the market crash of 1987, showed flaws in the Black and Scholes model. Real data appears to be not distributed in a lognormal way, but present fatter tails and asymmetry (skewness) with respect to the predicted gaussian distribution. In addition, a Brownian evolution implies continuity of the market evolution, which does not reflect real markets. It comes out, analyzing financial data, that the prices

process can jump: in particular jumps become more visible as one samples the path more frequently, making the assumption of high or infinite jump frequencies plausible.

A first outstanding attempt to overcome the lack of fit was made by Merton in [90]: he proposed to add a jump component to the continuous model, where the jumps are counted by a Poisson process and the jump amplitude is a lognormal variable. Using the instruments of stochastic calculus and most of the features of the Black and Scholes formula, he obtained a closed form solution for the jump-diffusion pricing problem. With more generality, in the works by Mandelbrot [83, 84] we can find a description of the markets which takes into account discontinuous, unforeseeable abrupt movements of markets.

This approach is based upon a Lévy modeling of the prices of the asset and gives a better fit to real-life data. In the paper by Madan and Seneta [80] a Lévy model was used to fit Australian stock datas. Lévy processes have been discovered by the French mathematician Paul Lévy as a generalization of the Brownian motion which drops the continuity requirement over time. They are the most general processes with stationary independent increments: they are characterized by means of their distribution and contain Brownian motion and the compound Poisson process, that is the continuous part and the jump component as particular cases.
In view of these considerations, the prices of the stocks are modeled in terms of exponential Lévy models, the choice of a particular Lévy process standing in the choice of its distribution, as it will be explained in Chapter 2. The discontinuous feature of Lévy processes has important consequences on the description of financial markets for what concerns the assumption of completeness of the market itself, see [25]. A need for any trader in the market is to cover his investment against the risk of the market (*hedge from the risk*), which is given by the random sources of the model. It will be shown that a way to hedge from that risk is to invest in the market product in a suitable way. The linear span of the products traded in the market gives the space of all the possibilities of investment in the market. If this space is a proper subspace of the linear span of the risky sources of the market, the trader will never be able to hedge perfectly against any risk. When this assumption is not met, the market becomes incomplete; this means that perfect hedging is impossible and the first consequence is in the correct definition of "price of the derivative", which is no more unique in the classical sense.

Several approaches can be used to overcome this difficulty, one of them being the *completion of the market* by adding as many new assets as the sources of uncertainty. Once the market is completed, one can proceed in the standard way to price derivatives. This approach cannot be used when the incompleteness of the market is determined by an infinite dimension process, such as the Lévy one.

In this case the *minimization of the risk* seems to be more appropriate. The price of any product depends on a parameter, the *market price for risk*, which indicates the “excess return” above the risk-free rate for accepting a certain level of risk. This value is a quantity depending on the market model and it is not uniquely determined in the case of incomplete markets: it is given once it has been selected an equivalent martingale measure. In an incomplete market there exist infinitely many equivalent martingale measures and over the year several approaches have been proposed for selecting one (see, e.g., [32, 107]); once the equivalent martingale measure corresponding to the minimal market price of risk has been selected, we can define the price of any product in the market as the *arbitrage free one* with respect to that chosen equivalent martingale measure, that is an expectation value with respect to that equivalent martingale measure. This approach is typically chosen by those agents that are “risk averse”, but a specular approach can be performed by those agents who are “risk taker”, the *super replication approach.*
All these criterions show that the pricing procedure in incomplete markets depends on the preference of the investor or on the knowledge of the expected return on the underlying stock, while in the Black and Scholes complete setting all the information were endogenous of the market.

Once one of these strategies has been chosen, using the generalized Ito’s calculus it is possible to rephrase the stochastic pricing problem in terms of an integro–partial differential problem, possibly non linear.

Analytical Results. In Part I we have seen that Lévy processes give an effective description of market price evolution. Using Ito’s calculus and other instruments from Probability theory, we can derive a deterministic nonlinear integro–partial differential problem to get the price of a prescribed financial product.
Notice that several features of this problem have a more natural formulation in the viscosity solutions setting. It will be shown that the incompleteness of the market reflects in the possible degeneration of the second order term, while the choice of a Lévy model introduces an integral term with respect to a measure possibly unbounded. In addiction if the market model describes a *large investor* setting, nonlinear effects arise and classical techniques do not apply any more.

This framework is described by the following class of nonlinear integro–partial differential equation:

$$-\partial_t u + H(x, t, u, \mathbf{x}\mathcal{D}u, \mathbf{x}\mathcal{D}^2 u \mathbf{x}^T, \mathcal{J}u, \mathcal{I}u) = 0, \tag{0.0.1}$$

where $\mathcal{J}u$ and $\mathcal{I}u$ are the following integro–differential operators:

$$\mathcal{J}u(x,t) \quad = \int\limits_{\mathbb{R}^M - \{0\}} [u(x + \mathbf{x}\beta(x,t,z), t) - u(x,t) - \mathbf{x}\beta(x,t,z) \cdot \mathcal{D}u(x,t)]\, m_{x,t}(dz),$$

$$\mathcal{I}u(x,t) \quad = \int\limits_{\mathbb{R}^{\text{M}}-\{0\}} [u(x+\mathbf{x}\beta(x,t,z),t)-u(x,t)]\,\gamma(x,t,z)m_{x,t}(dz).$$

H is a continuous function of its arguments and $m_{x,t}(dz)$ is the jump measure and could depend on the point (x,t). For the pricing problem, the H operator reads as

$$H\Big(x,t,u,\mathbf{x}\mathcal{D}u,\mathbf{x}\mathcal{D}^2u\mathbf{x}^T,\mathcal{J}u\Big) = -\frac{1}{2}\text{tr}\left[\sigma\sigma^T\mathbf{x}\mathcal{D}^2u\mathbf{x}^T\right] - b\mathbf{x}\mathcal{D}u + ru - \sum_{j=1}^{\text{N}}\mathcal{J}_j u$$

and the diffusion matrix $\frac{1}{2}\text{tr}\left[\sigma\sigma^T\mathbf{x}\mathbf{x}^T\right]$ could possibly have rank D strictly less than the space dimension N.

In the case of general Lévy processes, $m_{x,t}(dz)=\nu(dz)$ is an unbounded measure with a second order singularity at the origin, satisfying the following integral condition:

$$\int_{\mathbb{R}^{\text{M}}-\{0\}}(1\wedge|z|^2)\nu(dz)<\infty,$$

while in the case of bounded Lévy processes, or Poisson processes, $m_{x,t}(dz) = \mu_{x,t}(dz)$ is a bounded positive Radon measure and the $\mathcal{J}u$ could be reduced to $\mathcal{I}u$ on the domain $\mathbb{R}^{\text{M}}$.

The problems of interest are the Cauchy problem on $(0,+\infty)^{\text{N}}\times[0,T)=\Pi\times(0,T)$

$$\begin{cases} -\partial_t u + H(x,t,u,\mathbf{x}\mathcal{D}u,\mathbf{x}\mathcal{D}^2u\mathbf{x}^T,\mathcal{J}u,\mathcal{I}u)=0, \\ u(x,T)=u_T(x)\in\mathcal{C}(\Pi)\cap\mathcal{P}_n(\Pi), \end{cases} \tag{0.0.2}$$

which is related with the European option with payoff $G(X)=u_T\left(X^1,\dots,X^{\text{N}}\right)$ and maturity T, and the corresponding obstacle problem

$$\begin{cases} \min\Big\{-\partial_t u + H(x,t,u,\mathbf{x}\mathcal{D}u,\mathbf{x}\mathcal{D}^2u\mathbf{x}^T,\mathcal{J}u,\mathcal{I}u)=0, u-u_T\Big\}=0, \\ u(x,T)=u_T(x)\in\mathcal{C}(\Pi)\cap\mathcal{P}_n(\Pi), \end{cases} \tag{0.0.3}$$

which is related with the American option with the same payoff function and maturity, the set $\mathcal{P}_n(\Pi)$ indicating a class of growth which will be introduced in the following.

In the case of jump diffusion processes given by a Brownian motion plus a Poisson process, the set $\mathcal{P}_n(\Pi)$ reduces to $L^\infty_{\text{pol}}(\Pi)$; moreover the problems we have to deal with can be studied in the whole $\mathbb{R}^{\text{N}}$ after an exponential change of variable.

Integro–differential parabolic operators of type

$$\partial_t u - \text{div}\big(\mathcal{A}(x,t,u,\mathcal{D}u)\mathcal{D}u\big) + b(x,t,u,\mathcal{D}u) = \mathcal{J}u,$$

where

$$\mathcal{J}u = \int_{\mathbb{R}^{\mathrm{M}}} \big[u(x+z,t) - u(x,t) - z \cdot \mathcal{D}u(x,t)\big] m_{x,t}(dz),$$

and $\mathcal{A}$ is a positive definite matrix, namely

$$\mathcal{A} \geq \varepsilon I_{\mathrm{N}}$$

have been extensively studied by Garroni and Menaldi in the books [53, 54]. In [55, 57], Garroni, Solonnikov, and Vivaldi give particular attention to the initial–boundary value problem with oblique derivatives assigned at the boundary. Next, Garroni, Solonnikov and Vivaldi considered also the obstacle problem in [56] and [58] and Vivaldi substituted the Neumann type boundary condition with a boundary degeneracy of the operator in [114]. This choice of boundary condition guarantees that the processes of the returns satisfy their natural condition $X > 0$ without violating the principle of absence of arbitrage therefore is well fitting with problems arising from finance [37].
Application in Mathematical Finance of the previous results with different proofs have been obtained by Mastroeni and Matzeu in their works [86, 87].
This results cannot be applied to equation (0.0.1) because they use fixed point methods, therefore they deal at most with weak degeneracy

$$\mathcal{A} \geq \omega(x) I_{\mathrm{N}}, \quad \omega > 0 \text{ almost everywhere.}$$

Using the theory of viscosity solutions it is possible to deal with nonlinear dependence on the integral term and possibly degenerate second order terms.
The theory of viscosity solutions has been developed in the ′80s by Crandall and Lions to allow merely continuous function to be solution of first order Hamilton–Jacobi equations [35]. From that paper several other results have followed to extend the theory to nonlinear second order problems. The central aim of this theory is to produce a comparison principle among solutions from which derive a uniqueness result. Existence is proved by means of a monotone approximation of the equation, for example by means of the Perron's method.
The first attempt to deal with integral terms in this framework was made by Alvarez and Tourin [1] with a linear dependence on the term $\mathcal{I}u$, with bounded Radon measure.
Viscosity solution approach in the case of nonlinear dependence from the integral term, still with bounded Radon measure, was considered by Amadori [2, 4] for problems of the form:

$$\partial_t u + F(x,t,u,\mathcal{I}u,\mathcal{D}u,\mathcal{D}^2u) = 0. \tag{0.0.4}$$

Beside the usual assumption in the theory of viscosity solutions for differential problems, the operator F is assumed to be monotone with respect to the integral operator $\mathcal{I}u$ to get an existence and uniqueness result in this case. This assumption can be extended also to the operator H with respect to the two integral terms $\mathcal{I}u$ and $\mathcal{J}u$ in (0.0.1): it is naturally satisfied by the H operator appearing in financial problems and it is crucial for the proof of existence and uniqueness of the solution and for the construction of convergent approximation schemes.

In this part we shall prove our main analytical result:
Theorem *Assume that H satisfies suitable continuity and monotonicity conditions and take u_T continuous in a subset of $(0,+\infty)^{\mathrm{N}}$, satisfying a suitable rate of growth at infinity and in a neighborhood of the origin. Let u be a (possibly discontinuous) viscosity solution to (0.0.2) in a appropriate integrability space, satisfying the same rate of growth as the initial datum. Then u is the unique viscosity solution; moreover it is continuous on $(0,+\infty)\times[0,T)$ and can be extended continuously to $(0,+\infty)\times[0,T]$ by setting*

$$\mathbf{u}(x,t)=\begin{cases} u(x,t) & \text{if } (x,t)\in(0,+\infty)\times[0,T),\\ \lim\limits_{(0,+\infty)\times[0,T)\ni(y,s)\to(x,t)} u(y,s) & \text{if } (x,t)\in(0,+\infty)\times\{T\}.\end{cases}$$

The function $\mathbf{u}$ still solves (0.0.2), and satisfies $\mathbf{u}(x,T)=u_T(x)$ for all $x\in(0,+\infty)$.

A detailed statement and proof of this result can be found in Chapter 4, Theorem 4.2.14 and Corollary 4.2.15 and in the paper [5].

Analyzing the existence and uniqueness, particular attention is posed on the connection between the deterministic and stochastic problem. It shall be proved that under some suitable assumptions, the viscosity solution of the deterministic problem gives exactly the solution one would get from the corresponding stochastic one in terms of the expected value with respect to a suitable probability measure. An issue of particular interest in Financial Mathematics is the study of the regularity of the solution, because of its connection with the hedging problem. We shall establish a Lipschitz property for the solution of the deterministic problem, and this guarantees the existence of an hedging strategy, as it shall be pointed out.

Numerical Results. The Black and Scholes model has gained great popularity among Economists because it gives a closed form solution to the pricing problem of

financial derivatives. The use of more sophisticated model, such as the one proposed in this Thesis, while presenting a great resemblance to real markets, is related to a nonlinear problem which does not have a closed form solution. To overcome this difficulty a useful tool is given by the numerical approach which makes possible to deal with more complicated nonlinear problems.
The work by Barles and Souganidis [16] is a fundamental paper for the numerical approximation of viscosity solutions of purely differential problems. Starting from their result, we shall show convergence for monotone, stable, consistent schemes approximating integro–differential parabolic problems with bounded and unbounded Lévy measure.
The nonlocal nature of the integral term gives rise to a numerical difficulty, which requires the truncation of two domain: the problem one and the integral one.
For what concerns the integral domain, in case of bounded jump measure, the criterion to select the truncated domain is the following: fix a parameter $\epsilon > 0$ and choose the computational domain D_ε such that

$$\Big| \int_{\mathbb{R}^{\mathrm{M}}} \mu_{x,t}(dz) - \int_{D_\epsilon} \mu_{x,t}(dz) \Big| < \epsilon;$$

once the truncated domain D_ϵ is determined, it is possible to study a new problem where the $\mathcal{I}u$ term is replaced by the corresponding $\mathcal{I}_\epsilon u$.
Unfortunately, the nonlocal nature of the integral term has its influence even outside the computational domain, making it necessary to use some approximation for the solution outside that boundary. A general approach consists in replacing the original problem with the corresponding one without the integral term, or to use some information about the asymptotic behavior of the solution.
Here a new approach is proposed, using the diffusive effect of the integral term under appropriate assumptions of the measure $\mu_{x,t}$. First we show that the original problem can be well approximated by a pure differential problem with an artificial diffusion; then we apply this remark to implement an effective numerical boundary condition, giving as a consequence a full convergence result for the global approximation scheme.
In Chapter 8 we will extend the results obtained in Chapter 7 to unbounded Lévy measure. The difficulty we have to face in this case stands in the unboundedness of the measure $\nu(dz)$ in a neighborhood of the origin; we overcome this difficulty applying a truncation of the integration domain to couple with the truncation already described in Chapter 7. We fix two parameters $\varepsilon > 0$ and $\theta > 0$ and chose a computational domain $D_{\varepsilon,\theta}$ such that

$$\Big| \int_{\mathbb{R}^{\mathrm{M}}-\{0\}} (1 \wedge |z|^2)\nu(dz) - \int_{D_{\varepsilon,\theta}} (1 \wedge |z|^2)\nu(dz) \Big| < O(\varepsilon + \theta);$$

we point out that the domain $D_{\varepsilon,\theta}$ takes into account both the truncation owed to the unboundedness of the integration domain and the unboundedness of the measure near the origin.
Once the truncated domain $D_{\varepsilon,\theta}$ has been chosen, it is possible to substitute our problem with a truncated one, where the integral term $\mathcal{J}u$ is replaced by the corresponding one $\mathcal{J}_{\varepsilon,\theta}u$.

Theorem *Let us assume that the proposed approximation schemes is monotone, stable and consistent, and that the integral approximation is monotone; let us suppose that the problem* (0.0.4) *satisfies a strong uniqueness property. Then, as* $(h, k) \to 0$, *the solution* $\tilde{u}$ *of the scheme converges locally uniformly to the unique continuous viscosity solution of the problem* (0.0.4).

Precise statements and proof can be found in Theorem 7.2.2, for the bounded Lévy measures, and in Theorem 8.3.2 for the unbounded Lévy measures. For detailed examples and numerical test for bounded Lévy measure we refer to the paper [30].

In the framework of linear problems with constant coefficient and bounded measure, the integral term $\mathcal{I}u$ was already considered by Andersen and Andreasen [6] where an operator splitting method was proposed, compared with a pure Crank-Nicholson one. In that work the integral part is treated using a FFT method, which is lighter for what concerns numerical calculation, but could diminish the precision of the scheme in some areas.
For what concerns bounded jump measures, several work has been done by Forsyth and alt. [50, 39, 40], where implicit discretization is developed for American derivatives. Particular attention is devoted to the integral term, computed using an iterative method or a FFT method.
A different approach for the unbounded Lévy measures using variational inequalities and the semigroup theory in Sobolev spaces with exponential weights can be found in [89, 115], while an efficient numerical solution using a wavelet Galerkin discretization can be found in [116, 75, 88].
The results we prove in this part come out to be the first rigorous results of convergence in the viscosity solutions setting and, to the best of our knowledge, they are the first results concerning nonlinear integro–differential degenerate problems. Moreover we propose a way to deal with the boundary difficulties due to the truncation and the nonlocal nature of the integral term.

The original results of this thesis can be listed as follows:

1. we proved a result of existence and uniqueness of viscosity solutions to a class of nonlinear integro–partial differential problems related to general Lévy measure for unbounded set with non smooth boundary. This result generalizes all previous results and is based on a comparison principle among sub/supersolution without assigning a boundary data, but only a blow up rate at the boundary and at infinity;

2. we proved convergence results for numerical approximations to viscosity solutions to nonlinear integro–differential problems both for the case of bounded and unbounded Lévy measure. We proposed a new approach to deal with the nonlocal integral term: we prove that it can be well approximated by a diffusion term. This result will be useful to implement an effective numerical boundary condition.

Part I

A MATHEMATICAL DESCRIPTION OF FINANCIAL MARKETS

The Mathematical approach to financial markets started from the first years of the last century; the first attempt of modelization is by Bachelier, [9], who proposed to use probability models. Since the $'70$s, starting from the works by Black and Scholes [28] and Merton [90] deterministic partial differential approaches to describe the value of financial derivative have obtained greater success, especially among Economists. The reason of the wide use of these techniques has to be found in the possibility of obtaining closed form solution, which is not possible with probability models.
Moreover in low dimension problems a deterministic approach allows a more feasible numerical treatment which is fundamental from the point of view of practitioners.
On the other hand, the expansion of the market and the possibility to access to a larger set of datas, has required the supply of more sophisticated models describing with more accuracy the evolution of the prices of the derivatives, and, on the other side, has generated new kinds of financial instruments with which to deal with risk management. These are some of the reasons of the big impulse one can see in the evolution of instruments of stochastic calculus, mathematical and numerical analysis applied in Mathematical Finance.

This part will be organized as follows.
In Chapter 1 we shall focus our attention to the question of option pricing. It contains some basics of Mathematical Finance; the connection between the stochastic formulation and the deterministic one is explained. The standard Black and Scholes example is exposed in details, being the most important example of complete market. The Merton jump–diffusion example is exposed as a first example of incomplete market. It tries to capture the disagreements between the pure diffusion market and real data. Theoretical results concerning incomplete markets are proposed.
General Lévy models are presented in Chapter 2. These models give rise to incomplete markets, therefore the problem of finding an equivalent martingale measure becomes crucial. For a detailed discussion we refer to the paper by Chan [33]. The corresponding integro–partial differential problem can be degenerate and nonlinear; from the analytical viewpoint it has been widely studied in [3, 2, 4] in the viscosity

solutions setting and original results shall be presented in Part II. The numerical approach to that problem shall be presented in Part III.

We shall start by listing some notation that will be used in the following.

Notations.

There follows a list of notation used in this part

∂_t partial derivatives with respect to time,

∂_{x_i} partial derivatives with respect to a direction x_i,

in the Euclidean space $\mathbb{R}^{\mathrm{N}}$. Moreover

$\mathcal{D} = (\partial_{x_1}, \cdots, \partial_{x_{\mathrm{N}}})$ gradient of u w.r.t. x,

$\mathcal{D}^2 = \left(\partial^2_{x_i,x_j}\right)_{i,j=1\cdots\mathrm{N}}$ Hessian matrix of u w.r.t. x.

If a function $f(x,y,t)$ is defined on $\mathcal{Q} \subset \mathbb{R}^{\mathrm{N}} \times \mathbb{R}^{\mathrm{D}} \times (0,T)$, we denote

$$D_{x(y)} = \left(\partial_{x_1(y_1)}, \cdots, \partial_{x_{\mathrm{N}}(y_{\mathrm{D}})}\right) \quad D^2_{x(y)} = \left(\partial^2_{x_i,x_j(y_i,y_j)}\right)_{i,j=1\cdots\mathrm{N}(\mathrm{D})},$$

the gradient and the Hessian matrix with respect to x (y respectively).

Let $A \subset \mathbb{R}^{\mathrm{N}}$; we indicate with $\mathbf{1}_A : \mathbb{R}^{\mathrm{N}} \to \{0,1\}$ the characteristic function of the set A.

$\mathbf{1}_{\mathrm{N}} = (1, \ldots, 1)^T \in \mathbb{R}^{\mathrm{N}}$,

$I_{\mathrm{N}} = \mathrm{diag}\,(1, \ldots, 1) \in \mathbb{M}_{\mathrm{N}\times\mathrm{N}}$.

Let $\mathcal{O}$ be an subset of $\mathbb{R}^{\mathrm{N}}$. We denote

$$\mathcal{C}\,(\mathcal{O}) = \{f : \mathcal{O} \to \mathbb{R} \text{ s.t. } f \text{ is continuous}\},$$

and

$$L^p\,(\mathcal{O}) = \{f : \mathcal{O} \to \mathbb{R} \text{ s.t. } \|f\|_p < \infty\},$$

where

$$\|f\|_p = \begin{cases} \left(\int_{\mathcal{O}} \|f(x)\|^p dx\right)^{1/p}, & \text{if } p \in [1,\infty), \\ \text{ess sup}\Big\{|f(x)| \text{ s.t. } x \in \mathcal{O}\Big\} = \|f\|_\infty, & \text{if } p = \infty. \end{cases}$$

with the convention that

$$\|f\|_p^p = \|f\|_\infty \text{ when } p = \infty;$$

moreover we set p^* the conjugate exponent of p, namely

$$\begin{array}{ll} p^* = p/(p-1) & \text{if } p \in (1,\infty), \\ p^* = \infty & \text{if } p = 1, \\ p^* = 1 & \text{if } p = \infty. \end{array}$$

$W^{m,p}$ $(\mathcal{O})$ is the usual Sobolev space of functions whose derivatives till order m belong to $L^p(\mathcal{O})$ endowed with the norm

$$\|f\|_{m,p} = \left(\sum_{\|\alpha\|=0}^{k} \|D^\alpha f\|_p^p \right)^{1/p}.$$

L^p_{loc} $(\mathcal{O})$ (respectively, $W^{m,p}_{\text{loc}}(\mathcal{O})$) stands for the set of measurable functions f such that $f \in L^p(K)$ (respectively, $f \in W^{m,p}(K)$) for all compact set K contained in $\mathcal{O}$.

We indicate

$$\Pi = (0,\infty)^{\mathrm{N}}, \qquad \overline{\Pi} = [0,\infty)^{\mathrm{N}},$$

$$\mathcal{Q} = \Pi \times (0,T) \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}}.$$

For all integers $n \geq 0$ we define

$$L^p_n\ (\Pi) = \Big\{ f \in L^p_{\text{loc}}(\Pi) \text{ s.t. there exists c such that for all } R > 1$$
$$\|f\|_{L^p((1/R,R)^{\mathrm{N}})} \leq \mathrm{c}(1 + 1/R^n + R^n) \Big\},$$

$$L^p_{\text{pol}}\ (\Pi) = \bigcup_{n\geq 0} L^p_n(\overline{\Pi}),$$

$$W^{m,p}_n\ (\Pi) = \Big\{ f \in W^{m,p}_{\text{loc}}(\Pi) \text{ s.t. } D^\alpha f \in L^p_n(\Pi) \text{ for all } |\alpha| \leq m \Big\},$$

$$W^{m,p}_{\text{pol}}\ (\Pi) \bigcup_{n\geq 0} W^{m,p}_n(\Pi)$$

Let $\big(\Omega, \mathcal{F}, \mathcal{P}\big)$ be a probability space. We denote by X a random variable and by

$\mathbb{E}\ (X) = \displaystyle\int_\Omega X(\omega) d\mathcal{P}(\omega)$ the espectation of X,

$X\ = \big\{ X_t \ :\ t \in [0,T] \big\}$ a stochastic process.

Namely X represents a function which associates to any time t a random variable X_t. Once an event $\omega \in \Omega$ is selected, we define a *path* of the stochastic process X the function

$$[0, T] \ni t \to X_t(\omega) \in \mathbb{R}^{\mathrm{N}}.$$

If the probability space is endowed by a filtration $\big\{\mathcal{F}_t; t \in [0, T]\big\}$, we denote by

$$\mathbb{E}\,\Big(X \,|\, \mathcal{F}_t\Big) = \int_\Omega X(\omega) d\mathcal{P}_t(\omega)$$

the conditional expectation of X, conditioned to $\mathcal{F}_t$.

We define

$$\mathcal{L}^p\,(0,T) = \{\mathcal{F}_t\text{–adapted real processes } \{X_t\}_{t\in[0,T]} \text{ s.t. } X_t(\omega) \in L^p\,(0,T) \text{ a.s. w.r.t. } \mathcal{P}\}$$

namely

$$\mathbb{E}\left[\int_0^T |X_t(\omega)|^p dt\right] < \infty,$$

and

$$\mathcal{L}^p_+\ (0,T) = \Big\{X_t \in \mathcal{L}^p\,(0,T) \text{ such that } \mathcal{P}\Big(X_t \geq 0\Big) = 1, \text{ a.e. } t \in (0,T).\Big\}$$

Chapter 1

Markets and option pricing: stochastic models.

This chapter focuses on the mathematical modeling of financial markets. It starts with the basic financial concepts of interest rate, asset, portfolio and utility. Some examples are given to illustrate the theory. We turn then our attention to derivatives as new financial instruments for hedging against risk.
A whole section is devoted to the Black and Scholes model, proving that it describes a complete market; here we propose a pricing procedure based on the hedging from the risk technique. This model is taken as a starting point for several extensions: in fact, comparing this model with datas, it has been seen that the primary assets do not bring all the information about the market evolution, that is because the only Brownian source of uncertainty, proposed in [28], does not take into account for other abrupt movements of the market.
An extension was first proposed in a paper by Merton [90]: in the attempt to capture the jumps in the prices of the assets, a jump process was added obtaining an incomplete market; in this case it is no more possible to define an unique price for any financial product. Actually, the *market price of risk*, which was a quantity determined by the market itself in [28], is now something of exogenous from the market and has to be determined using some criterion.

One commonly used criterion consists in *completing the market*, adding some derivatives, behaving as a "virtual asset", in order to have so many asset as many are the sources of uncertainty. This approach has been widely studied in the works by Mancini, [81, 82], in a Poisson–Gaussian framework. However, this procedure, using derivatives as primary asset, leads to correlation in the new "virtual market". Other procedures have been proposed, which deal with the attitude of the agents in the market, one of them being the way of protecting from the worst, that is the *super replication* procedure, another being the *minimization of the risk*; both these

procedures consist in selecting an equivalent martingale measure, a maximal one and a minimal one respectively. Results in this frameset can be found in the works by Föllmer and Schweizer [48], and by Chan [33].
For all these approaches, once an equivalent martingale measure has been selected, the price of any derivative can be obtained in terms of a solution of a deterministic differential problem, possibly degenerate, by means of the use of Ito's calculus, see [28, 90, 2, 3, 4].
Here we propose a pricing procedure based on the completion of the market technique, which hides the choice of a particular price for risk. Once the market is completed one can price any derivative by arbitrage.
The deterministic approach is preferred to the stochastic one because it allows to deal with a large number of examples, such as the *large investor* model, that cannot be handled directly in the stochastic setting. When modeling this problem, all the parameters of the market depends on the strategy of the agents in the market itself, leading to some nonlinearity [37]. From the stochastic point of view, the resulting problem is a nonlinear forward–backward SDE; it can be solved using the four-step-scheme, as it has been proposed in [78], which consists in performing the study of a related partial differential equation.

All these procedures show that whichever procedure is used to price derivatives, one has to deal with a deterministic nonlinear possibly degenerate parabolic problem deriving from the probabilistic formulation of the problem itself.

1.1 Primary market, traders and economy.

A stochastic description of the market involves a probability space endowed with a filtration giving all the information about the market in evolving time. The *market theory* describes any market as a set of traded assets evolving following a stochastic process: the target of Mathematical Finance theory is to find the model which better performs the evolution of the market.
Another approach is present in the theory, completely opposite to this one; it is called *economic theory* and describes the economy as a set of agents characterized by their wealth and their attitude towards the market and consumption. The target is to chose a function, the *utility*, which better describes the traders behavior.
In the following we shall give all the background of definition and instruments necessary to go through the theory. We refer to the book by Øksendal [94] for the basic notions in stochastic differential theory and to the book by Jacod [67, 68, 29] for general results in stochastic calculus.
For what concerns Financial theory in pure diffusion models we refer to the book by

Lamberton and Lapeyre [76]; in that book the reader can also find a first approach to jump diffusion market model.

1.1.1 Market theory.

Definition 1.1.1 *A* **market place** *of duration T is a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ endowed with a filtration $\{\mathcal{F}_t : t \in [0,T]\}$ such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$.*

Ω is the sets of all the scenarios, the set of the parts of Ω is the set of all the events which may occur and $\mathcal{P}$ measures their occurrence. It is clear that the filtration represents the information available at time t: as time evolves, the information one gets grows. The information is available at the same rate from all the agents in the market, that means that $\mathcal{F}$ is intrinsic in the market itself.
Any financial product which is traded in the market is referred to as an *asset*: this word refers to stocks, commodities, indices, currencies and equity. It is assumed that the evolution of the asset is completely determined by its actual price: this assumption is referred to as *efficiency of the market.*

Definition 1.1.2 *A* **price of an asset** *or* **return** *is any X positive valued stochastic process such that $X(\omega) \in \mathcal{L}^p(0,T)$ for some $p \in [1,\infty]$ adapted to the filtration.*

Definition 1.1.3 *A (finite-dimensional)* **market** $\mathcal{M}(X)$ *is a couple composed by a market place and a vector $X = (X^0, \dots, X^{\mathrm{N}})$ of returns of assets which are called* **primary**. *The linear span of the returns of the primary assets is called* **marketed space**.

Definition 1.1.4 *A* **money market account** *or* **bond** *is a riskless asset whose price $X^0 = B \in \mathcal{L}^\infty$ evolves as*

$$B_t = \exp\Big(\int_0^t r_\tau d\tau\Big),$$

and $r_\tau \in \mathcal{L}^1_+(0,T)$ represents the **interest rate**.

In general $r = r_t$ is taken as a stochastic process, but in the following we suppose that is deterministic: in this case B is a solution of

$$dB = rBdt.$$

Generally in the Definition 1.1.3 $X^0 = B \in \mathcal{L}^\infty(0,T)$ represents the *money market account* while

$$\tilde{X}^j = \frac{X^j}{X^0}, \text{ for } j = 1, \dots, \mathrm{N},$$

are the **actualized prices**.
We denote by $\mathcal{M}_0(X)$ the market generated by the returns $X^0, \dots, X^{\mathrm{N}}$ such that

$$\mathbb{E}\Big[\|X^i_t(\omega)\|^{p_i}_{p_i}\Big] < \infty, \qquad p_i \in [1,\infty], \text{ for all } i = 1,\dots,\mathrm{N},$$

whose filtration is exactly the one given by the prices of the primary assets plus the set of $\mathcal{P}$–measure zero.

Definition 1.1.5 *A* **stock** *is any risky asset* $X^j = S \in \mathcal{L}^2(0,T)$.

Once a particular market is selected, the price of the stock is completely known as a stochastic process, as its process X_j evolves following a forward stochastic differential equation.

Example 1.1.6 [*Black and Scholes Market*] Let us consider the famous Black and Scholes model. The market is described by the stock evolution

$$dS_t = S_t\Big[\mu dt + \sigma dW_t\Big],$$

where W is a standard Brownian motion and μ and σ are the *drift* and *volatility* respectively. In the original model, they are supposed to be constant, but a such defined model does not have a good fit with the real data, in spite of its success among the Economist. ■

The Black and Scholes model can be used to derive the so called *implied volatility* from the prices of the options traded in the market. The resulting volatility depends from the maturity of the corresponding option, and it is not constant, as it was assumed. This result is known as the **smile of the implied volatility**.
Several approach have been proposed to overcome this disagreement with the real market data, one being the so called model with **stochastic volatility**, which introduces another random source in the market on which the volatility depends; in this model the market is described by the following stochastic differential equation

$$\begin{cases} dS_t &= S_t\Big[\mu dt + \sigma dW^1_t\Big], \\ d\sigma_t &= \alpha dt + \beta dW^2_t, \end{cases}$$

and W^1_t and W^2_t are two Brownian motion; in general they are uncorrelated.
Another possibility to get models more resemblant to real markets is to add some new processes to the standard Black and Scholes one in order to capture abrupt movements of the market. The resulting model can be described by the following price evolution equation

$$dS_t = S_t\Big[\mu dt + \sigma dW^1_t + \gamma dN_t\Big],$$

where N is a Poisson process and γ represents the *amplitude if the jump* and is assumed to be constant.

Example 1.1.7 [*Merton model*] The prototype of jump-diffusion market has been first proposed by Merton [90]. The dynamics of the underlying process are given by the following equation:

$$\frac{dS_t}{S_t} = (\mu - \lambda k)dt + \sigma dW_t + (\eta - 1)dN_t.$$

where dW_t is a standard Brownian motion, dN_t is a Poisson counting process of intensity λ, that is:

$$dN_t = \begin{cases} 0 & \text{with probability } 1 - \lambda dt \\ 1 & \text{with probability } \lambda dt. \end{cases}$$

Moreover we are assuming that η is a log-normally distributed jump amplitude with probability density:

$$\tilde{\Gamma}_\delta(\eta) = \frac{\exp(-\frac{1}{2}(\frac{\log \eta}{\delta})^2)}{\sqrt{2\pi}\delta\eta}, \tag{1.1.1}$$

k is the expectation $\mathbb{E}(\eta - 1)$, and the Brownian and the Poisson processes are uncorrelated. ∎

1.1.2 Economy theory.

Beside the description of the market place and of the product traded in the market itself, we can give a description of those who act in the market, the **traders**. They are identified with their position in the market.

Definition 1.1.8 *A* **financial strategy** *or* **dynamic portfolio** *in the market* $\mathcal{M}(X)$ *is a process* Δ *with value in* $\mathbb{R}^{N+1}$ *such that it is* $\mathcal{F}_t$ *predictable. Here* Δ_t^i *represents the number of shares of the* i^{th} *asset held by the trader at time* t*.*
A strategy is **self financing** *if its total wealth* $\Delta_t \cdot X_t$ *satisfies*

$$d(\Delta_t \cdot X_t) = \Delta_t \cdot dX_t \Longleftrightarrow \Delta_t \cdot X_t = \Delta_0 \cdot X_0 + \int_0^t \Delta_\tau \cdot dX_\tau.$$

Here the integrand is intended in the Ito's sense.
A strategy is called **admissible** *if it is self financing and has non negative total wealth,*

$$\Delta_t \cdot X_t \geq 0, \ \textit{for all } t \geq 0.$$

We say that if $\Delta_t^i > 0$ the trader is *long selling* the corresponding asset, otherwise he is *short selling*, at time t.
Any agent in the market is characterized by his attitude toward the risk; even if it is quite impossible to give a mathematical description of preferences of people, the economic theory tries to get the most information on agents and assets in order to attain equilibrium in the market. Under this logic any trader is characterized by his wealth and consumption.

Definition 1.1.9 *An* **economical agent** *is a couple* (e, U)*, where* e *is a process in* $\mathcal{L}_+^2(0,T)$ *is the* **endowment** *and* $U : \mathcal{L}_+^2(0,T) \to \mathbb{R}$ *is a concave increasing function called* **utility***.*
A **consumption policy** c *for the agent* (e, U) *is any process in* $\mathcal{L}_+^2(0,T)$*. His* **consumption–endowment** *process is given by* $e - c$*, a signed process even if both* e *and* c *are nonnegative.*

Remark 1.1.10 The utility function introduces a relation of order in the space of consumption $\mathcal{L}_+^2(0,T)$:

$$c \preceq c' \Longleftrightarrow U_t(c) \le U_t(c').$$

■

Definition 1.1.11 *An* **economy** $\mathcal{E}(e, U)$ *is a couple composed by a market place* $(\Omega, \mathcal{P}, \mathcal{F})$ *and a set of* D *agents defined by their couple endowment–utility* (e^k, U^k) *for* $k = 1, \dots, \mathrm{D}$*.*

Several utility function have been proposed during the year to describe attitude of the agents in the market. For a detailed discussion we refer to the paper by Duffie and Epstein [42] ant the reference therein.

1.2 Financial derivatives: pricing and hedging.

Until now the basic elements of a financial market have been described; nevertheless they are not the only elements of interest in the market transactions: other sophisticated instruments have been created to manage with the risk. They are called *derivatives* as their evolution depends on the evolution of some primary assets of the market.

Definition 1.2.1 *A* **contingent claim** *or* **derivative** *on the* **underlying assets** $X^1, \dots, X^{\mathrm{N}}$ *is a couple* (G, T)*, where* $G \in \mathcal{C}(\overline{\Pi}) \cup W^{1,\infty}_{\mathrm{pol}}(\Pi)$*,* $G \ge 0$ *is a random variable* $\mathcal{F}_T$*–measurable and it is the* **payoff***,* T *is the* **maturity** *or* **expiration time***. We assume it coincides with the duration of the market.*

A derivative is said **European** *if it can exercised only at expiry,* **American** *if it can be exercised at any time before expiry.*

Example 1.2.2 There are several examples of derivative actually traded in the market.

1. A **call option** on a stock S is a contract which gives the right but not the obligation to buy one unit of stock at a fixed time T and at a fixed price K, the *strike price*. In this case it easy to derive the payoff

$$G^{call}(S) = \max(S - K, 0) = (S - K)^+.$$

In the case of the call option, two parties are present, the *holder*, which buys the option at time $t = 0$ and has the right to exercise it, and the *writer*, which has the obligation to sell the asset if the holder wants to buy it. In the case of **put option**, the roles of the two parties are inverted, and the payoff function is

$$G^{put}(S) = \max(K - S, 0) = (K - S)^+.$$

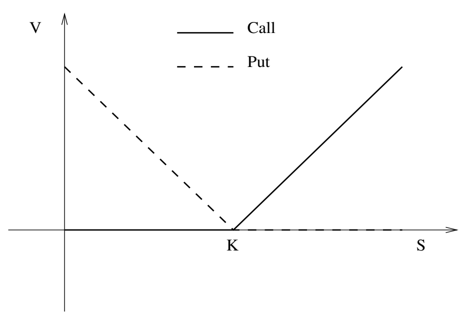

Figure 1.1: Example 1.2.2(1), payoff of European derivatives

2. A **forward contract** on the stock S is an agreement between two parties in which one of the two parties agrees to buy S from the other at a specified prices F, the *forward price*, on a specified date in the future, the *delivery price*. In this case the payoff is a constant function

$$G(S) = F,$$

therefore the main difference with the previous example of derivative is the lack of choice in this kind of contract.

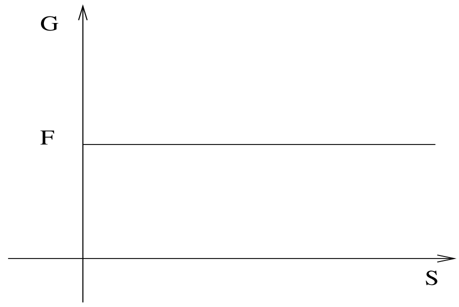

Figure 1.2: Example 1.2.2(2), payoff of Future contracts

3. An **option on option** is a derivative constructed on another derivative. The more used are the *exotic options*, whose value depends on the path of some stocks, [63].

4. A **fixed income derivative** is a forward contract on an interest rate. When the forward interest rate is given in terms of a stochastic process $f(s,t)$, see [26, 47, 70, 27], a *zero–coupon bond* paying 1 at time T can be seen as a derivative on the asset
$$R_t = \int_0^t f(\tau, t) d\tau,$$
with maturity T and payoff 1. In this kind of contract, the choice element is not more present.

■

It is clear that every derivative is completely determined once its payoff and its expiry has been determined, but this description does not show the risk element connected with the choice element. Moreover its stochastic representation does not appear to be fundamental, and therefore it cannot be described as an asset in the sense of Definition 1.1.2.
There are two main problems concerning derivatives:

pricing: this concept analyzes the option from the point of view of the holder; it concerns the problem of finding, if it does exist, a *fair price* for a derivative at any time $t < T$. Moreover an important question is about the existence of a *deterministic* function giving the value of the derivative.

hedging: this problem is quite opposite to the previous one and at the same time strictly connected with it. It regards the possibility for the writer to minimize the risk associated to the derivative he is going to sell. In this case one looks for a *deterministic* hedging strategy.

To give a good answer to both these problems, it is fundamental to find the model for the market which better perform the data of the underlying assets.
In the modeling procedure the following are standard, although quite unrealistic, assumptions:

1. no transaction costs in the market;

2. borrowing and short selling are indefinitely permitted;

3. assets are infinitely divisible.

1.2.1 Arbitrage theory.

A fundamental concept in Financial theory is the **absence of arbitrage** which states that there are no opportunity to make an instantaneous risk profit without any risk. More precisely, if such opportunities come to existence, they do not last for a long time, but there happens a movement of price that eliminates it.

Definition 1.2.3 *We say an* **arbitrage** *any portfolio* Δ *such that*

$$\Delta_0 \cdot X_0 = 0, \quad \mathcal{P}\{\Delta_T \cdot X_T \geq 0\} = 1, \quad \mathcal{P}\{\Delta_T \cdot X_T > 0\} > 0.$$

Definition 1.2.4 *A market* $\mathcal{M}(X)$ *is* **without arbitrage opportunities** *if for any self–financing strategy such that* $\Delta_t \cdot X_t \leq 0$ *for some time* t *then* $\Delta_s \cdot X_s \leq 0$ *for all* $s > t$.

This economic concept is strictly connected with mathematical results in the probability space determining the market place and with the number of assets and risk sources.

Theorem 1.2.5 *A market is without arbitrage opportunities if and only if there exists a probability measure* $\mathcal{P}^*$ *equivalent to* $\mathcal{P}$ *under which the actualized prices of the assets are martingales. Such a probability is called* **equivalent martingale** *or* **risk neutral** *measure.*

This classical result can be found in Harrison and Kreps [61] and in [41, 76] for the continuous and discrete case respectively. An intuitive proof of this result can be found in [25].
This concept is fundamental in pricing theory because it allows to price derivatives in terms of the so called *hedging strategy.*

Definition 1.2.6 *An admissible strategy* Δ *is called an* **hedging strategy** *for the*

(i) European derivative (G, T) *if*

$$\Delta_T \cdot X_T = G(X_T),$$

with probability 1*;*

(ii) American derivative (G, T) *if*

$$\Delta_T \cdot X_T = G(X_T), \text{ and } \Delta_t \cdot X_t \geq G(X_t) \text{ for all } t \in [0, T]$$

with probability 1*.*

The **value by arbitrage** *at time* $t \in [0, T]$ *is the process*

$$Y_t^* = \inf \Big\{ Y_t = \Delta_t \cdot X_t \,:\, \Delta \text{ is an hedging strategy for } (G, T) \Big\}.$$

This definition is motivated by the fact that the market obtained by $\mathcal{M}(X)$ adding the derivative is free of arbitrage opportunities, and any hedging strategy has Y^* as its total wealth.

Proposition 1.2.7 *Let* $\hat{Y}$ *be the price of the derivative* (G, T) *and assume that* $\hat{\mathcal{M}}(X) = \mathcal{M}(X) \cup \mathcal{M}_0(\hat{Y})$ *is free of arbitrage. Then, for any hedging strategy* Δ *we have* $\hat{Y}_t = \Delta_t \cdot X_t$*.*

In the risk neutral probability $\mathcal{P}^*$ the arbitrage value may be seen as a value function of a stochastic control problem.

Proposition 1.2.8 *Let* $\mathcal{M}(X)$ *a market free of arbitrage and* (G, T) *an European (American, resp.) derivative. Let* $\mathcal{P}^*$ *be an equivalent martingale measure,* $\mathbb{E}^*$ *the (conditional) expectation of* $\mathcal{P}^*$ *and* Y^{*E} *(*Y^{*A}*, resp.) the arbitrage value of* (G, T)*. Then*

$$Y_t^{*E} \geq X_t^0 \mathbb{E}^* \Big[\frac{G(X_T)}{X_T^0} \Big| \mathcal{F}_t \Big];$$

Furthermore if $\mathcal{T}_{t,T}$ *is the set of stopping time between* t *and* T*,*

$$Y_t^{*A} \geq X_t^0 \sup_{s \in \mathcal{T}_{t,T}} \mathbb{E}^* \Big[\frac{G(X_s)}{X_s^0} \Big| \mathcal{F}_t \Big];$$

Let us now suppose to describe the evolution of the market $\mathcal{M}(X)$ in terms of a stochastic forward system

$$X_t^i = X_0^i + \int_0^t \alpha^i d\tau + \sum_{j=1}^{\mathrm{P}} \int_0^t \beta_j^i dR_t^j, \tag{1.2.1}$$

where α^i and β^i_j, deterministic, are respectively the drift and the "volatility" of the process X^i, while R^j, $j = 1, \dots, \text{P}$ is the set of risky sources in the market.
In this setting, the pricing problem can be formulated as the problem of finding the minimal solution of a forward–backward stochastic differential equation:

Corollary 1.2.9 *Let (G, T) be an European (American, resp.) derivative in $\mathcal{M}(X)$. Let us suppose that there exists an hedging strategy Δ such that its total wealth $Y = \Delta \cdot X$ is the minimal solution of the following BSDE*

$$Y_t = G(X_T) - \int_t^T \Delta\alpha d\tau - \sum_{j=1}^{\text{P}} \int_t^T \Delta\beta_j dR^j_\tau, \tag{1.2.2}$$

for the European derivative, or the corresponding optimal stopping problem

$$Y_t = \sup_{s \in \mathcal{T}_{t,T}} \Big\{ G(X_s) - \int_t^s \Delta\alpha d\tau - \sum_{j=1}^{\text{P}} \int_t^s \Delta\beta_j dR^j_\tau \Big\}. \tag{1.2.3}$$

Then Y is the arbitrage value of (G, T).

More general models can be considered instead of this constant coefficient one, for example it is possible to choose the coefficients α^i and β^i_j depending on the time t or on the stock itself X.
In the case of *large investor economy*, these coefficients depend also on the strategy and we deal with a nonlinear backward–forward stochastic differential equation.
To prove that Y is the minimal solution of (1.2.2) ((1.2.3) resp.) we can go through a deterministic analysis, as is it possible to prove that $Y_t = U(X_t, t)$, with U smooth deterministic function.

Corollary 1.2.10 *Let $\mathcal{M}(X)$ be a free–arbitrage market and (G, T) a derivative. Let us suppose that there exists a hedging strategy Δ for (G, T) and a deterministic continuous function $U(X, t)$ such that $U(X_t, t) = Y_t = \Delta_t \cdot X_t$ for all $t \in [0, T]$. If U has the following regularity properties*

$$U \in L^\infty(0, T; W^{2,\infty}_{\text{pol}}(\Pi)),$$
$$\partial_t U \in L^\infty(0, T; L^\infty_{\text{pol}}(\Pi)),$$

then Y is the arbitrage value of the derivative (G, T).

Proof. Under the risk neutral $\mathcal{P}^*$ measure, Y_t can still be written as $Y_t = U(X_t, t)$. According to the definition of $\mathcal{P}^*$, X are martingales; U, by hypotheses is smooth, then, using Ito's formula we are guaranteed that Y_t previously defined is a martingale under $\mathcal{P}^*$. Then, for Theorem 1.2.5, the market $\hat{\mathcal{M}}(X) = \mathcal{M}(X) \cup \hat{\mathcal{M}}_0(Y)$ is free of arbitrage and the risk neutral measure is given by $\mathcal{P}^*$. Then the thesis follows from Proposition 1.2.7 ■

The stochastic equations (1.2.2) and (1.2.3) can be related to a deterministic problem for $U(X, t)$ applying formally Ito's calculus, thinking of X as an independent variable.
In case of a *small investor economy* in (1.2.1), U is given as a solution of a linear deterministic final value problem (obstacle problem, resp.), while, in the case of a *large investor economy* the problem we have to deal with is nonlinear and requires particular techniques to be handled.
Until now we have used the hedging strategy to determine the price of any derivative, saying nothing about its existence. This concept concerns the market and not only a particular derivative: it is indeed clear that if a new asset is added to a market, the resulting set of admissible strategies enlarges and an hedging strategy can be found where at the beginning was not possible.

Definition 1.2.11 *A market is* **complete** *if all derivatives have an hedging strategy.*

As it was done before, this property of markets can be characterized in terms of risk neutral measure.

Theorem 1.2.12 *Let us suppose that the market $\mathcal{M}(X)$ is free of arbitrage. Then it is complete if and only if the equivalent martingale measure is unique.*

This result is now a milestone in Mathematical Finance and can be found in [76] for a discrete time proof and in [41] for the continuous case. An intuitive proof is given in [25].
The completeness of a market is a concept with no real financial interpretation, but it is only an assumption to simplify the analysis. It is well known that real markets are far to be complete, and one simple example is given by the standard Black and Scholes market, where several asset are traded, but they all depend on a single risky source. The completeness assumption reads as a condition on the rank of the volatility matrix to which is not possible to give a financial interpretation.
This argument explains how in the upgrading of the Black and Scholes model, like the jump–diffusion one by Merton, or the stochastic volatility by Hull and White [62], the derivative price obtained by means of the hedging strategy and Ito's calculus is not unique. On the contrary it depends on the so called *price of risk* and it corresponds to a particular choice of the equivalent martingale measure.
This quantity can be obtained enlarging the market by adding some new assets in order to gain completeness, or picking up an equivalent martingale measure. Looking at the economic point of view, the price of risk can be determined performing the maximization of utility.

1.3 The classical Black and Scholes model.

The work by Black and Scholes [28] is a fundamental paper in Financial Mathematics. In this section we provide an introduction to this model, proving its completeness and absence of arbitrage. Moreover we deduce the deterministic partial differential operator related to the Δ–hedging technique for pricing derivative.
In their paper, Black and Scholes made some assumptions on the kind of market they were working in, as they are listed in the following:

1. The stock's price is described by a lognormal random walk:

$$dS_t = S_t[\mu dt + \sigma dW_t], \tag{1.3.1}$$

   where W_t is a standard Brownian motion, and μ and σ are respectively the *drift* and the *volatility* and are supposed to be constant. We assume that the filtration is generated by the Brownian motion W_t.

2. The interest rate r is constant,

3. There are no transaction cost in hedging a portfolio.

4. Short selling is always possible.

5. Assets are infinitely divisible.

The choice of a lognormal random walk in (1.3.1) implies important features of the market evolution, for example, the continuity of the path $t \to S_t(\omega)$ for almost all $\omega \in \Omega$ w.r.t. $\mathcal{P}$. Moreover the increments of S are independent and stationary, which implies that the process is $\mathcal{F}_t$–adapted and $\mathcal{F}_t$–predictable, see [94].

Proposition 1.3.1 *The Black and Scholes market is free of arbitrage.*

Proof. It suffices to find an equivalent martingale measure $\mathcal{P}^*$ under which the actualized price of the stock is a martingale. Let $\tilde{S}_t = e^{-rt}S_t$. As r is supposed to be constant

$$d\tilde{S}_t = \tilde{S}_t[(\mu - r)dt + \sigma dW_t] = \tilde{S}_t \sigma dW_t^*$$

where $W_t^* = \dfrac{\mu - r}{\sigma} t + W_t$. The process $\dfrac{\mu - r}{\sigma} t$ is clearly $\mathcal{F}_t$–adapted with paths belonging to $\mathcal{L}^2(\Omega)$ while the process

$$\zeta_t = \exp\Big(-\frac{1}{2}\Big(\frac{\mu - r}{\sigma}\Big)^2 t - \frac{\mu - r}{\sigma}\int_0^t dW_\tau \Big)$$

is a martingale. Then, by the Girsanov's Theorem [94] $\{W_t^*\}_{t\in[0,T]}$ is a standard Brownian motion in the probability $\mathcal{P}^* = \zeta_T \mathcal{P}$.

This measure $\mathcal{P}^*$ is absolutely equivalent to $\mathcal{P}$, because of the Radon–Nikodym's Theorem, and in that measure

$$\tilde{S}_t = \tilde{S}_0 \exp\Big(\int_0^t dW^*_\tau\Big)$$

is a martingale. ∎

It results from this proof that the measure $\mathcal{P}^*$ is uniquely determined. An exhaustive proof can be found in [76, 94, 25], therefore the resulting market is complete.
Practitioners in finance are indeed interested in finding an hedging strategy in order to price derivatives, but no information is available in that sense in the proof of completeness. We proceed in constructing directly an hedging strategy for a Black and Scholes market.

1.3.1 Pricing by Δ–hedging.

A fundamental tool in pricing derivatives in an incomplete market is the Δ–hedging technique. It is widely exposed in the book [117] by Wilmott, Howison and Dewinne. The existence of an unique equivalent martingale measure and the use of Ito's calculus turns the stochastic problem in the study of a deterministic partial differential final value problem. The solution is a deterministic function $U(S,t)$ which is exactly the value by arbitrage of the derivative constructed on that particular asset S at time t.
This approach is strictly dependent on the completeness of the market. In case of lack of completeness the Δ–hedging technique produces a class of deterministic problems depending on a parameter characterizing the market, the *price for risk*. Once this parameter is chosen it is possible to face the pricing problem with the differential approach.

Proposition 1.3.2 *The Black and Scholes market is complete. The arbitrage price of an European derivative (G,T) is given as the solution of the final value problem*

$$\begin{cases} -\partial_t U - \dfrac{1}{2}\sigma^2 S^2 \partial^2_{SS} U - rS\partial_S U + rU = 0, \\ U(S,T) = G(S), \end{cases} \tag{1.3.2}$$

and the hedging strategy is a deterministic function of (S_t,t) given by

$$\begin{cases} \Delta^0_t = e^{-rt}[U(S_t,t) - S_t\partial_S U(S_t,t)], \\ \Delta_t = \partial_S U(S_t,t). \end{cases} \tag{1.3.3}$$

The arbitrage price of the American derivative (G,T) *is given as the solution of the obstacle problem* $(0,\infty)\times(0,T)$*:*

$$\begin{cases} \min\{-\partial_t U - \frac{1}{2}\sigma^2 S^2 \partial^2_{SS} U - rS\partial_S U + rU, U_G\} &= 0, \\ U(S,T) &= G(S). \end{cases} \tag{1.3.4}$$

The corresponding hedging strategy is given by (1.3.3), for any $t\in(0,\tau^*)$*, where* τ^* *is the* **optimal exercise time***:*

$$\tau^* = \inf\{t\in\mathcal{T}_{0,T} \text{ such that } U(S_t,t)=G(S_t)\}.$$

Proof. Let us start analyzing the European case.

Let us suppose that the derivative (G,T) admits an hedging strategy (Δ^0_t,Δ_t), and let Y the total wealth of the corresponding portfolio

$$Y_t = \Delta^0_t e^{rt} + \Delta_t S_t; \tag{1.3.5}$$

the portfolio is assumed to be self-financing, therefore, Y_t solves the following BSDE:

$$Y_t = G(S_T) - \int_t^T [rY_\tau + \Delta_\tau S_\tau(\mu - r)]d\tau - \int_t^T \Delta_\tau \sigma S_\tau dW_\tau. \tag{1.3.6}$$

Let us now suppose that there exists a deterministic function U such that $U(S_t,t) = Y_t$ for any time $t\in[0,T]$. By Ito's formula, U solves the following BSDE:

$$\begin{aligned} U(S_t) &= G(S_T) - \int_t^T [\partial_t U + \frac{1}{2}\sigma^2 S^2\partial^2_{SS}U + \mu S\partial_S U]d\tau \\ &\quad - \int_t^T \sigma S\partial_S U dW_\tau. \end{aligned} \tag{1.3.7}$$

By the assumption on U, the equation (1.3.6) and (1.3.7) must be equal; identifying the coefficient of dW_τ we obtain $\Delta_t = \partial_S U(S_t,t)$, which, inserted in (1.3.5) gives (1.3.3). These equalities give really an hedging strategy in the Definition 1.1.8. Using these equalities and identifying the coefficient of $d\tau$, we obtain the famous Black and Scholes operator $(1.3.2)_1$. Therefore the value of the European derivative is the solution of (1.3.2). The nonnegativity of the total wealth of the hedging strategy may be attained applying the maximum principle to (1.3.2).

Let us study the case of the American derivative.

From Proposition 1.2.8, the arbitrage value of an American derivative Y^{A^*} is such that

$$Y^{A^*}_t \geq V_t$$

where V_t is the solution of the following optimal stopping problem

$$V_t = e^{rt} \sup_{t \in \mathcal{T}_{t,T}} \mathbb{E}^* \Big[e^{-r\tau} G(S_\tau) \Big| \mathcal{F}_t \Big].$$

From stochastic control theory ([113]) we obtain that V_t is invariant by change of equivalent probability and $V_t = U(S_t, t)$, where U is the classical solution of the obstacle problem (1.3.4). To prove the inverse inequality, the total wealth given by (1.3.3) is $Y_t = U(S_t, t)$; by Ito's formula, it is self financing and $Y_t \geq G(S_t)$ for all $t \in [0, T]$ and $Y_T = G(S_T)$ as U solves the obstacle problem. Hence

$$Y_t^{A^*} \leq U(S_t, t),$$

because Y^{A^*} is a arbitrage price. ■

1.4 Black and Scholes model and large investor economy.

In this section we expose a modification of the Black and Scholes proposed by Cvitaniĉ and Ma in [37], where they take into account the influence of the investor in the market.

We expose the pricing problem as a nonlinear backward–forward stochastic differential equation and we explain how it is rephrased as a nonlinear partial differential equation using the **four step scheme**.

The Cvitaniĉ and Ma model consider a market with N risky assets and N sources of uncertainty, given by standard N-dimensional Brownian motion with the following assumptions:

1. The prices of the assets evolve following

$$dS_t^i = S_t^i \Big[\mu^i(S_t, t, Y_t, \xi_t) dt + \sum_{j=1}^{\mathrm{N}} \sigma_j^i(S_t, t, Y_t, \xi_t) dW_t^j \Big], \tag{1.4.1}$$

   where $W_t = (W_t^1, \dots, W_t^{\mathrm{N}})^T$ is a standard N-dimensional Brownian motion, $\mu^i(\cdot)$ and $\sigma_h^i(\cdot)$, the drift and volatility respectively, are deterministic functions of $(S, t, Y, \xi) \in \Pi \times (0, T) \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}}$; Y is the total wealth of the portfolio of the agent and ξ^i is the wealth invested in the i^{th} asset, $\xi_t^i = \Delta_t^i S_t^i$.
   Let us assume that

$$\mu^i, \sigma_j^i \in \mathcal{C}(\overline{\Pi} \times [0, T] \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}}) \cap L^\infty(0, T; W^{1,\infty}(\Pi \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}})),$$

   and, writing $\sigma = (\sigma_j^i)_{i,j}$

$$\sigma \sigma^T \geq \varepsilon I_{\mathrm{D}} \text{ on } \Pi \times (0, T) \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}}. \tag{1.4.2}$$

2. The interest rate r is a deterministic function of (t, Y, ξ) with the following regularity
$$r \in \mathcal{C}([0,T] \times \mathbb{R} \times \mathbb{R}^{\text{N}}) \cap L^{\infty}([0,T] \times \mathbb{R} \times \mathbb{R}^{\text{N}}).$$

3. No transaction cost is associated with hedging a portfolio.

4. Short selling is permitted.

5. Assets are infinitely divisible.

A model of this kind verifies the consistency of the market, as the S given as solution of (1.4.1) is really a N-ple of returns; it suffices to study the solution of the equation obtained after a change of variable $x^i = \log S^i$.

This model include examples that were not considered in Black and Scholes' work.

Example 1.4.1 [*Large institutional investor economy*].
When too many wealth is invested in bond, the interest rate decreases as

$$r(t, Y, \xi) = R(t) f(Y - \sum_{i=1}^{\text{N}} \xi^i),$$

where $f \in \mathcal{C}(\mathbb{R})$ is positive, $f = 1$ in $(-\infty, \xi_0]$ and it is decreasing on $(\xi_0, +\infty)$. ■

Example 1.4.2 Assume that $\text{N} = 1$ and that there exists a time t_0 such that

$$\sigma(t, Y, \xi) = \begin{cases} \sigma & t < t_0, \\ \sigma X & t \geq t_0, \end{cases}$$

while μ and r remain constant. Let us suppose that at time $t = t_0$ a large investor sells a derivative for a price $\gamma(X_{t_0}, t_0) > U(X_{t_0}, t_0)$, where U is the price of the same derivative corresponding to σ always constant: this means that the investor is selling the derivative for more than its real worth.
This would create instabilities and arbitrage opportunities in the market if the volatilities were to remain the same. Let us suppose that the effect is in a change of the σ:

$$\sigma(t, Y, \xi) = \begin{cases} X[\sigma + f(\gamma(X,t)) - U(X,t)] & \gamma(X_{t_0}, t_0) \neq U(X_{t_0}, t_0) \\ \sigma X & \gamma(X_{t_0}, t_0) = U(X_{t_0}, t_0), \end{cases}$$

with f increasing function.
This phenomenon was unknown in classical models and says that selling a derivative for more than its fair price does not guarantees hedging. In this case the volatility increases and the minimal hedging price U changes. If $U < \gamma$ hedging might not be possible. See [37] for more details. ■

1.4.1 The four step scheme.

Arguing as in the previous section, we can prove that the density ζ_T provides an equivalent martingale measure, but in the large investor market it depends on the wealth and strategy of the agent. For what concerns the pricing of derivatives, this gives rise to a forward–backward stochastic differential equation. It is not clear, in this case, how to get completeness of the market in terms of unique martingale measure. We can get this property as a byproduct of the hedging strategy.

Let us consider the market

$$S_t = S_0 + \int_0^t \mathbf{S}_\tau \mu d\tau + \int_0^t \mathbf{S}_\tau \sigma dW_\tau, \tag{1.4.3}$$

where $\mathbf{S} = \mathrm{diag}\,(S^1, \dots, S^{\mathrm{N}})$. Let us suppose that (G, T) admits an hedging strategy (Δ_t^0, Δ_t). By definition

$$Y_t = \Delta_t^0 \exp\Big(\int_0^t r(\tau, U_\tau, \xi_\tau) d\tau\Big) + \Delta_t S_t, \qquad \xi_t^i = \Delta_t^i S_t^i;$$

to have the self–financing condition, Y_t solves

$$Y_t = G(S_T) - \int_t^T [rY_\tau + \xi_\tau(\mu - r\mathbf{1}_{\mathrm{N}})] d\tau - \int_t^T \xi_\tau \sigma dW_\tau. \tag{1.4.4}$$

In the following we refer to (1.4.3)–(1.4.4) as the FBSDE.

Definition 1.4.3 *An* **adapted solution** *of the FBSDE is a triplet* $(S, Y, \xi) \in (\mathcal{L}_+^2)^{\mathrm{N}} \times \mathcal{L}_+^2 \times (\mathcal{L}^2)^{\mathrm{N}}$ *which satisfies* (1.4.3)–(1.4.4).

Now the problem of completeness of the market turns in the well posedness of (1.4.3)–(1.4.4).

The notion of forward–backward stochastic differential equation is quite new: the proof of existence and uniqueness of solution requires techniques that are not only stochastic. Interesting results can be found in [98, 97], while Cvitanić and Ma proposed to use the "four step scheme" by Ma, Protter and Yong [78] to lead back the problem to a deterministic nonlinear PDE. An important point that we shall sketch is that the PDE approach allows to deal with nonlinear market with weaker assumption than the stochastic one.

Step 1. In order to match diffusion terms, find a change of variable $\vartheta : \Pi \times (0, T) \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}} \to \mathbb{R}^{\mathrm{N}}$ such that

$$\xi \mathbf{S}\sigma(S, t, Y, \vartheta(S, t, T, \xi)) = \vartheta(S, t, Y, \xi)\mathbf{S}\sigma(S, t, Y, \vartheta(S, t, T, \xi)),$$

namely $\vartheta = \xi$.

Step 2. Suppose that there exists a deterministic function $U(S,t)$ giving the arbitrage price of the derivative, $U(S,t) = Y_t$; then, by Ito's formula

$$\begin{aligned} U(S_t,t) &= G(S_T) - \int_t^T [\partial_t U + \frac{1}{2}\mathrm{tr}\left[(\mathbf{S}\sigma)(\mathbf{S}\sigma)^T \mathcal{D}^2 U \mathbf{S}\mu\right] d\tau \\ &\quad - \int_t^T \mathcal{D}U\mathbf{S}\sigma dW_\tau. \end{aligned} \tag{1.4.5}$$

At the same time U solves (1.4.4), therefore, imposing equality between coefficients of dW_τ^j

$$\xi = \vartheta(S,t,U,\mathcal{D}U\mathbf{S}) = \mathcal{D}U\mathbf{S}.$$

imposing equality on the coefficients of $d\tau$ we get that U is a solution of the following deterministic quasilinear partial differential problem on $\Pi \times (0,T)$:

$$\begin{cases} \partial_t U - \frac{1}{2}tr\left[(\mathbf{S}\sigma)(\mathbf{S}\sigma)^T(S,t,U,\mathcal{D}U\mathbf{S})\mathcal{D}^2 U\right] + r(t,U,\mathcal{D}U\mathbf{S})\times \\ \qquad \times(U - S\mathcal{D}U) = 0, \\ U(S,T) = G(S). \end{cases} \tag{1.4.6}$$

Step 3. Solve the purely forward stochastic differential equation obtained replacing Y and ξ by U and $\mathcal{D}U\mathbf{S}$ in (1.4.3)

$$S_t = S_0 + \int_0^t \mathbf{S}\mu(S,\tau,U,\mathcal{D}U\mathbf{S})d\tau + \int_0^t \mathbf{S}\sigma(S,\tau,U,\mathcal{D}U\mathbf{S})dW_\tau.$$

Step 4. Show that $(S_t, U(S_t,t), \mathcal{D}U(S_t,t)\mathbf{S}_t)$ is the solution of (1.4.3)–(1.4.4), with U solution of (1.4.6) and S_t solution of (1.4.3).

1.4.2 A PDE approach to nonlinear markets.

The result of existence and uniqueness for solution of BFSDE by Cvitanić and Ma has been obtained by rather restrictive assumption, either

$$G \in \mathcal{C}^2(\Pi) \cap L^\infty(\Pi) \text{ and } G \geq 0$$

or

$$\begin{aligned} &G \in \mathcal{C}^3(\Pi),\ G \geq 0, \text{ and } \lim_{\|S\|\to\infty} G(S) = \infty, \\ &\partial_{S^i} G \in W^{2,\infty}(\Pi) \text{ for all } i = 1,\dots,\mathrm{N}, \\ &\frac{S\mathcal{D}_S G}{1+G},\ \|S^2\mathcal{D}^2_{SS}G\| \in L^\infty(\Pi). \end{aligned}$$

In addiction, if there exists a positive constant R such that

$$\begin{aligned}
&\mu^i,\ \sigma^i_j \in \mathcal{C}(\mathcal{Q}) \cap L^\infty(0,T;W^{1,\infty}(\Pi\times\mathbb{R}\times\mathbb{R}^{\mathrm{N}})),\\
&S\mu,\ S\sigma_j \in \mathcal{C}^2(\mathcal{Q}),\ S\mathcal{D}_S\mu^i,\ S\mathcal{D}_S\sigma^i_j \in L^\infty(\mathcal{Q}),\\
&r \in \mathcal{C}^2([0,T]\times\mathbb{R}\times\mathbb{R}^{\mathrm{N}}),\ 0<r\le \mathrm{R},\\
&\limsup_{|Y|,\|\xi\|\to\infty} (|Y|+\|\xi\|)^2(|\partial_Y r| + \|\mathcal{D}_\xi r\|) < \infty,\ \text{uniformly w.r.t } t
\end{aligned}$$

In the PDE approach to the pricing problem in nonlinear markets, we shall not assume (1.4.2), but we shall consider market described by (1.4.3) where σ is a general $\mathrm{N}\times\mathrm{D}$ matrix. This allow us to deal with more general problems: the case $\mathrm{N}>\mathrm{D}$ corresponds to exotic options, while $\mathrm{N}<\mathrm{D}$, that is $\mathrm{rank}\,(\sigma)<\mathrm{D}$ we have an incomplete market.

To obtain consistency of the market, we need some regularity conditions, and the same for the absence of arbitrage condition. If we accept that the arbitrage price of a derivative is given in terms of a deterministic function $U(S_t,t)$, a well–defined notion of price of any derivative may be obtained whenever (1.4.6) is well posed for any payoff G.

Definition 1.4.4 *The* **weak arbitrage price** *of the European derivative* (G,T) *is the (unique) nonnegative solution of* (1.4.6) *provided that*

$$U \in L^\infty(0,T;W^{1,\infty}(\Pi)).$$

In this case the **weak hedging strategy** *is*

$$\begin{cases} \Delta^0_t &= \exp\Big(-\displaystyle\int_0^T r(\tau,U,\mathcal{D}U)d\tau\Big)[U(S_t,t) - S_t\mathcal{D}U(S_t,t)],\\ \Delta_t &= \mathcal{D}U(S_t,t). \end{cases} \tag{1.4.7}$$

The **weak arbitrage price** *of the American derivative* (G,T) *it the (unique) nonnegative solution of the obstacle problem*

$$\begin{cases} \min\Big\{\partial_t U - \dfrac{1}{2}tr\Big[(\mathbf{S}\sigma)(\mathbf{S}\sigma)^T(S,t,U,\mathcal{D}U\mathbf{S})\mathcal{D}^2U\Big] + r(t,U,\mathcal{D}U\mathbf{S}) \quad [U - S\mathcal{D}U], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad U-G\Big\} = 0,\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad U(S,T) = G(S), \end{cases}$$

provided that

$$U \in L^\infty(0,T;W^{1,\infty}(\Pi)).$$

In this case the hedging strategy is still given by (1.4.7) *on the interval* $(0,\tau^*)$*,* τ^* *being the* **optimal exercise price**

$$\tau^* = \inf\{t \in \mathcal{T}_{0,T}\ s.t.\ U(S_t,t) = G(S_t)\}.$$

The weak hedging strategy is well defined because it total wealth at time T is $U(S_T, T) = G(S_T)$ and at any time t it is given by $U(S_t, t)$ which is nonnegative.

Definition 1.4.5 *A market is* **weakly complete** *when any derivative has a well defined weak arbitrage price.*

We can state the following result to prove that the weak arbitrage price concept is consistent with the standard one under suitable regularity assumptions on U.

Proposition 1.4.6 *Let us suppose that*

(i) The quasilinear final value problem (1.4.6) *is well posed in the class* $\mathcal{C}(\overline{\Pi}) \cap W^{1,\infty}_{\text{pol}}(\Pi)$ *and its solution* U *is nonnegative if* $G \geq 0$*;*

(ii) For all $G \in \mathcal{C}(\overline{\Pi}) \cap W^{1,\infty}_{\text{pol}}(\Pi)$*, the solution* U *has the following regularity*

$$U \in \mathcal{C}^{2,1}(\mathbb{R}^{\text{N}} \times (0, T)).$$

Then the market is complete; the weak hedging strategy (1.4.7) *really hedge the derivative and* U *gives its arbitrage price.*

Proof. Under assumption (i), we can apply Ito's formula in Step 2. obtaining that the weak hedging strategy is self financing. Then the weak hedging strategy hedges the derivative and U gives the arbitrage price of (G, T) according to (1.2.6) ■

1.5 Incomplete markets.

In this section we shall go through the problem of describing an incomplete market: in these markets the traded assets do not bring all the information to cover all the possibility of investment, as the number of risky sources is not covered by the number of assets. The pricing procedure explained in the previous section cannot be used as the *market price for risk* is no more unique, but there are as many as the equivalent martingale measures.
Even though this difficulty in using the Δ–hedging technique, we can perform the pricing procedure in terms of a deterministic partial differential approach, once the market price of risk has been chosen. More details about the choice of this parameter, and the corresponding equivalent martingale measure, can be found in [49, 33].
The incompleteness assumption reads in the possible degeneracy of the second order term in the partial differential equation: if we refer to the model described in Section 1.4, the incomplete markets lack of the assumption (1.4.2).

In the following we shall describe the incomplete diffusion case in the large investor model.
Let us assume that the money market account evolves accordingly to

$$\begin{cases} dB_t &= B_t r dt, \quad t \in (0,T) \\ B_0 &= 1, \end{cases} \tag{1.5.1}$$

where r is the interest rate, and the assets evolve as

$$\begin{cases} dS_t^i &= S_t^i[\mu^i dt + \sum_{j=1}^{\mathrm{D}} \sigma_j^i dW_t^j], \quad t \in (0,T), \quad i = 1, \ldots, \mathrm{N} \\ S_0^i &= x_0^i > 0. \end{cases} \tag{1.5.2}$$

Here $W = (W^1, \ldots, W^{\mathrm{D}})$ is a D–dimensional standard Brownian motion, $\mu \in \mathbb{R}^{\mathrm{N}}$ is the vector of expectation, while $\sigma \in \mathbb{M}_{\mathrm{N}\times\mathrm{D}}$ is the matrix of volatility; the market is endowed with the filtration generated by W and the set of zero measure.
For the moment we do make no assumption on the volatility matrix, but we can observe that:

1. if $\operatorname{rank} \sigma < \mathrm{D}$, the linear span of the primary assets is strictly contained in the choice space generated by the Brownian motion; it means that the prices of the assets do not cover all the market possibilities and we have *lack of information.* The market in this case is said incomplete.

2. if $\operatorname{rank} \sigma < \mathrm{N}$ we can observe a correlation among the prices; it means that we are dealing with exotic option.

In the following we shall assume the *large investor economy*, therefore we assume that all the coefficients r, μ, and σ depend by $(S,t,Y,\xi) \in \mathcal{Q} = \Pi \times (0,T) \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}}$. We remember that $Y \in \mathcal{L}^1$ is the total wealth of the agent; if he is characterized by his portfolio (Δ^0, Δ):

$$\begin{aligned} \Delta_t^0 B_t &= \eta_t = Y_t - \xi_t \mathbf{1}_{\mathrm{N}}, \\ \Delta_t \mathbf{S}_t &= \xi_t. \end{aligned} \tag{1.5.3}$$

We assume that the following assumptions hold

$$\begin{aligned} &\mu^i,\, \sigma_j^i \in \mathcal{C}(\mathcal{Q}) \cap L^\infty(0,T; W^{1,\infty}(\Pi \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}})), \qquad (1.5.4)\\ &S\mathcal{D}_S \mu^i,\, S\mathcal{D}_S \sigma_j^i \in L^\infty(\mathcal{Q}; \mathbb{R}^{\mathrm{N}}), \\ &r \in \mathcal{C}(\overline{\mathcal{Q}} - \{Y = \xi \cdot \mathbf{1}_{\mathrm{N}}\}) \cap L^\infty(0,T; W^{1,\infty}(\Pi \times (\mathbb{R} \times \mathbb{R}^{\mathrm{N}} - \{Y = \xi \cdot \mathbf{1}_{\mathrm{N}}\})), \\ &S^i \mathcal{D}_S r \in L^\infty(\mathcal{Q}; \mathbb{R}^{\mathrm{N}}),\, (Y - \xi \cdot \mathbf{1}_{\mathrm{N}})(\partial_Y r - \mathcal{D}_\xi r \cdot \mathbf{1}_{\mathrm{N}}) \in L^\infty(\mathcal{Q}), \end{aligned}$$

and we refer to this market as the *diffusion market with large investor*. We notice that all the example of previous section verify all these assumptions. In this market we do not assume that the interest rate varies continuously with respect the bank deposit, as it is shown in the next example.

Example 1.5.1 [*Different rates for borrowing or lending*].

$$r(t,Y,\xi) = \begin{cases} R(t), & \text{if } Y \le \sum_i \xi_i, \quad R \text{ borrowing rate}, \\ \rho(t) & \text{if } Y > \sum_i \xi_i, \quad \rho \text{ lending rate}; \end{cases}$$

here R and ρ are continuous functions and $R(t) > \rho(t)$. ■

It can be shown that this market is consistent.

Lemma 1.5.2 *Under the previous assumptions on the coefficients μ and σ and for all fixed couples $(Y,\xi) \in (\mathcal{L}^1(0,T))^{\mathrm{N}+1}$, every solution of* (1.5.2) *with initial datum $x \in \Pi$ stays in Π.*
Furthermore all the paths $t \to X_t(\omega)$ are continuous for almost $\omega \in \Omega$ w.r.t. $\mathcal{P}$ and the relative increments of the processes S are independent and stationary.

The proof of this result can be found in [2]: it follows the line of the proof of well posedness of stochastic equations in $\mathbb{R}^{\mathrm{N}}$, see [94].

1.5.1 PDE approach: the arbitrage pricing

In Section 1.4 we have seen that the assumption (1.4.2) guarantees that the market is complete and there are no arbitrage opportunity. In this section we shall prove that assumption (1.4.2) can be weakened, leaving out the assumption of absence of arbitrage.
In this section we shall assume

$$\mu - r\mathbf{1}_{\mathrm{N}} \in \sigma(\mathbb{R}^{\mathrm{N}}), \tag{1.5.5}$$

and that there exists a matrix $\Sigma \in \mathbb{M}_{\mathrm{D}\times\mathrm{N}}$ with the same regularity as σ such that

$$\sigma\Sigma(\mu - r\mathbf{1}_{\mathrm{N}}) = \mu - r\mathbf{1}_{\mathrm{N}} \text{ identically on } \mathcal{Q}. \tag{1.5.6}$$

We can note that assumption (1.5.5) is implied by (1.4.2) in the case $\mathrm{N} = \mathrm{D}$.

Proposition 1.5.3 *The market described by the solutions of* (1.5.1)–(1.5.2) *satisfying assumption* (1.5.4)–(1.5.5) *is without arbitrage opportunities.*

The proof of this result can be achieved exhibiting an equivalent probability measure $\mathcal{P}^*$ under which the discounted prices are martingales. It is determined by means of its density with respect the objective measure; in this case, because of the incompleteness of the market, this density depends on a parameter, θ, given by

$$\theta = \Sigma(\mu - r\mathbf{1}_{\rm N}),$$

$\zeta_t(\theta_t)$. Details can be found in [2].

Remark 1.5.4 From Theorem 1.2.12 we know that a market is complete iff the density of the martingale $\zeta_t(\theta_t)$ is uniquely determined, so that the condition on the rank of the volatility σ is necessary and reads as $\operatorname{rank}\sigma = \rm D$.
If this uniqueness is not met, for any choice of the matrix Σ satisfying (1.5.6) we obtain a **price for risk**

$$\theta = \Sigma(\mu - r\mathbf{1}_{\rm N}),$$

and the corresponding equivalent martingale measure $\mathcal{P}^* = \zeta_T(\theta)\mathcal{P}$. ■

The market (1.5.1)–(1.5.2) can be completed adding some derivatives on the underlying $S^1, \dots, S^{\rm N}$, with suitable maturity and strike as primary assets; the resulting market is complete and all derivatives admit an hedging strategy and can be priced using the technique of Section 1.4.
In the following we shall describe how to price any derivative in the market (1.5.1)–(1.5.2) without completing; once a particular price for risk has been fixed it is possible relating the price of any derivative to a deterministic differential problem. The choice of the price for risk depends on the attitude of the trader toward the market: if he wants to minimize the risk, he would chose the minimal martingale measure, otherwise, if he wants to be solvable at expiry, he would chose the maximal one, see [49, 33].
In the following we shall assume that the attitude of the trader would not influence the equivalent martingale measure, that is

$$\sigma\Sigma \text{ depends only on } S,\ t. \tag{1.5.7}$$

The case of complete markets is contemplated in this assumption, as in that case $\operatorname{rank}\sigma\Sigma = \rm N = D$ and σ has an unique inverse Σ.
Let us assume that the derivative (G, T) admits an hedging strategy (Δ^0, Δ); ξ is given by (1.5.3) and

$$Y_t = \Delta^0_t \exp\Big(\int_0^t r(X, \tau, Y, \xi)d\tau\Big) + \xi_t \cdot \mathbf{1}_{\rm N}.$$

we want the portfolio to be self–financing, therefore Y solves the following BSDE

$$Y_t = G(S_T) - \int_t^T [rY + \xi(\mu - r\mathbf{1}_{\mathrm{N}})]d\tau - \int_t^T \xi\sigma dW_\tau,$$

Proceeding as in Subsection 1.4.1 we can derive a partial differential approach with which obtain the price of any derivative.
If we choose a change of variables $\vartheta : \mathcal{Q} \to \mathbb{R}^{\mathrm{D}}$

$$\vartheta(S,t,Y,\xi) = \xi\sigma(S,t,Y,\xi), \tag{1.5.8}$$

comparing the diffusion terms, equation (1.4.4) becomes

$$Y_t = G(S_T) - \int_t^T [rY + \vartheta\Sigma(\mu - r\mathbf{1})]d\tau - \int_t^T \vartheta dW_\tau. \tag{1.5.9}$$

Let us assume that there exist a deterministic function $U(S,t)$ which determines the arbitrage price of the derivative, $U(S,t) = Y_t$, by Ito's formula

$$\begin{aligned} U(S_t,t) &= G(S_T) - \int_t^T \Big[\partial_t U + \frac{1}{2}\mathrm{tr}\Big[(S\sigma)(S\sigma)^T\mathcal{D}^2 U\Big] + \mathcal{D}US\mu\Big]d\tau \\ &\quad - \int_t^T \mathcal{D}US\sigma dW_\tau. \end{aligned}$$

but, at the same time, it is given by (1.5.9), therefore, comparing the diffusion term and the drift term we get

$$\vartheta(S,t,U(S,t),\xi) = \mathcal{D}US\sigma(S,t,U(s,t),\xi),$$

$$-\partial_t U = \frac{1}{2}\mathrm{tr}\Big[(S\sigma)(S\sigma)^T\mathcal{D}^2 U\Big] - r(U - \vartheta\Sigma) + (\mathcal{D}US - \vartheta\Sigma)\mu.$$

This equation is not a PDE for U, as the coefficients do depend on ξ which is not uniquely determined by (1.5.8) unless $\mathrm{rank}\,\sigma = \mathrm{D}$; the choice of a particular price for risk determines an inverse for the change of variable (1.5.8). By assumption (1.5.7), we can set

$$\xi(S,t,\mathcal{D}U) = \vartheta(S,t,U,\xi)\Sigma(X,t,U,\xi) = \mathcal{D}US\sigma\Sigma(S,t),$$

and we can derive the hedging strategy

$$\begin{aligned} \Delta^0 &= \exp\Big(-\int_0^t r(S,\tau,U,\mathcal{D}US\sigma\Sigma)d\tau\Big)U - \mathcal{D}US\sigma\Sigma\mathbf{1}_{\mathrm{N}}, \\ \Delta &= \mathcal{D}US\sigma\Sigma \end{aligned} \tag{1.5.10}$$

and the corresponding pricing problem

$$-\partial_t U = \frac{1}{2}\text{tr}\left[(S\sigma)(S\sigma)^T(S,t,U,\mathcal{D}US\sigma\Sigma)\mathcal{D}^2 U\right] - \mathcal{D}US(\sigma\Sigma - I_{\text{N}})$$

$$\mu(S,t,U,\mathcal{D}US\sigma\Sigma) - r(S,t,U,\mathcal{D}US\sigma\Sigma)(U - \mathcal{D}US\sigma\Sigma\mathbf{1}_{\text{N}}), \quad (1.5.11)$$

$$U(S,T) \;=\; G(S_T). \quad (1.5.12)$$

If we are interested in American type derivatives, the pricing equation becomes

$$\min\Big\{ -\partial_t U - \frac{1}{2}\text{tr}\left[(S\sigma)(S\sigma)^T(S,t,U,\mathcal{D}US\sigma\Sigma)\mathcal{D}^2 U\right] + \mathcal{D}US(\sigma\Sigma - I_{\text{N}})\,\cdot$$

$$\cdot\mu(S,t,U,\mathcal{D}US\sigma\Sigma) - r(S,t,U,\mathcal{D}US\sigma\Sigma)(U - \mathcal{D}US\sigma\Sigma\mathbf{1}_{\text{N}}),$$

$$U - G\Big\} = 0 \quad (1.5.13)$$

Remark 1.5.5 If rank $\sigma = \text{N} = \text{D}$, there exists a unique right inverse Σ for σ, therefore there exists an unique equivalent martingale measure (the market is complete) and problem (1.5.11) coincides with (1.4.6). ■

Definition 1.5.6 *The* **weak arbitrage price** *of any European derivative* (G,T) *corresponding to the market price for risk* $\theta = \Sigma(\mu - r\mathbf{1}_{\text{N}})$ *is the (unique) nonnegative solution of the final value problem* (1.5.11)*, provided that* $U \in L^\infty(0,T;W^{1,\infty}_{\text{pol}}(\Pi))$. *In this case the system* (1.5.10) *determines the* **weak hedging strategy**.
The **weak arbitrage price** *of any American derivative* (G,T) *corresponding to the market price for risk* $\theta = \Sigma(\mu - r\mathbf{1}_{\text{N}})$ *is the (unique) nonnegative solution of the obstacle problem* (1.5.13)*, provided that* $U \in L^\infty(0,T;W^{1,\infty}_{\text{pol}}(\Pi))$. *In this case the* **weak hedging strategy** *is still given by* (1.5.10)*, but on the time interval* $(0,\tau^*)$*, where* τ^* *is the optimal exercise time:*

$$\tau^* = \inf\{t \in \mathcal{T}_{0,T} \,:\, U(S_t,t) = G(S_t)\}.$$

The hedging strategy obtained in this way is well defined and admissible, because its total wealth $U(S_t,t)$ is nonnegative and at time T it is equal to the strike price. When U satisfies the following regularity property

$$U \in L^\infty(0,T;W^{2,\infty}_{\text{pol}}(\Pi)),$$
$$\partial_t U \in L^\infty(0,T;L^\infty_{\text{pol}}(\Pi)),$$

the weak hedging strategy really hedges the derivative as it has been proved in Proposition 1.4.6.

Remark 1.5.7 The pricing equation (1.5.11) may present two kinds of degeneracies. As in the classical Black and Scholes' equation, at the boundary we have $\det(S\sigma) = 0$. On the other hand, because of the incompleteness of the market, the problem results strongly degenerate

$$\text{rank}\left[(S\sigma)(S\sigma)^T\right] < \text{N}, \text{ everywhere.}$$

■

The boundary degeneracy can be removed by the logarithmic smooth change of variable

$$x_i = \log S^i, \qquad u(x,t) = U(e^{x_1}, \dots, e^{x_\text{N}}, T-t),$$

therefore U solves (1.5.11) iff the function u defined above solves the following

$$\begin{aligned} \partial_t u &= \frac{1}{2}\text{tr}\left[\sigma\sigma^T(S(x), T-t, u, \mathcal{D}u\sigma\Sigma)\mathcal{D}^2 u\right] - \mathcal{D}u(\sigma\Sigma - I_\text{N})\mu(S(x), T-t, u, \mathcal{D}u\sigma\Sigma) \\ &\quad - r(S(x), T-t, u, \mathcal{D}u\sigma\Sigma)(u - \mathcal{D}u\sigma\Sigma\mathbf{1}_\text{N}), \end{aligned} \tag{1.5.14}$$

when pricing an European derivative, or

$$\begin{aligned} \min\Big\{ & \partial_t u - \frac{1}{2}\text{tr}\left[\sigma\sigma^T(S(x), T-t, u, \mathcal{D}u\sigma\Sigma)\mathcal{D}^2 u\right] \\ & + \mathcal{D}u(\sigma\Sigma - I_\text{N})\mu(S(x), T-t, u, \mathcal{D}u\sigma\Sigma) \\ & + r(S(x), T-t, u, \mathcal{D}u\sigma\Sigma)(u - \mathcal{D}u\sigma\Sigma\mathbf{1}_\text{N}), u - G(S(x))\Big\} = 0, \end{aligned} \tag{1.5.15}$$

in case of American derivatives. The coefficient $\sigma\Sigma$ is computed at $(S(x), T-t)$, and both the equation have to be coupled with the initial condition

$$u(x,0) = G(e^{x_1}, \dots, e^{x_\text{N}}).$$

Define

$$\begin{aligned} F(x,t,u,p,\mathcal{X}) &= -\frac{1}{2}\text{tr}\left[\sigma\sigma^T(S(x), T-t, u, p\sigma\Sigma(S(x), T-t))\mathcal{X}\right] \\ &\quad + p(\sigma\Sigma(S(x), T-t) - I_\text{N})\mu(S(x), T-t, u, p\sigma\Sigma(S(x), T-t)) \\ &\quad + r(S(x), T-t, u, p\sigma\Sigma(S(x), T-t))(u - p\sigma\Sigma(S(x), T-t)\mathbf{1}_\text{N}). \end{aligned}$$

Equations (1.5.14) and (1.5.15) can be studied with the viscosity solutions approach under the following monotonicity condition, as it will be shown in Part II:

There exists a continuous nonnegative function γ *with*

$$\int_0^{s_0} \frac{ds}{\gamma(s)} = \infty, \text{ for all } s_0 > 0,$$

such that for all $h > 0$

$$F(x,t,u+h,p,\mathcal{X}) - F(x,t,u,p,\mathcal{X}) \geq -\gamma(h),$$

for all $\mathcal{X} \in \mathcal{S}_{\rm N}$ *and for all* $(x,t,u,p) \in \mathbb{R}^{\rm N} \times [0,T] \times \mathbb{R} \times \mathbb{R}^{\rm N}$.

This condition is a requirement on the parameters μ, σ, and r and their dependence on the variable S: they may not decrease too fastly as the investment of the agent increases.
Standard theory of viscosity solutions does not apply to these problems because if $G \in L^\infty_n(\Pi)$ the initial datum $G(e^{x_1}, \ldots, e^{x_{\rm N}})$ grows as $e^{n\|x\|}$ for large x.
In [2] this problem is avoided applying a different change of variable

$$\tilde{x}_i = \begin{cases} \log S^i, & S^i \leq 1, \\ S^i, & X^i \geq 2, \end{cases} , \quad S^i(\tilde{x}_i) = \begin{cases} e^{\tilde{x}_i}, & \tilde{x}_i \leq 0, \\ \tilde{x}_i, & \tilde{x}_i \geq 2. \end{cases}$$

In this case U solves (1.5.11) iff the function

$$\tilde{u}(x,t) = U(S(\tilde{x}), T-t),$$

solves the following equation on $\mathbb{R}^{\rm N} \times (0,T)$:

$$\begin{aligned} \partial_t \tilde{u} &= \frac{1}{2}\left[\sigma\sigma^T(\tilde{x}, t, \tilde{u}, \mathcal{D}\tilde{u}\tilde{S}\sigma\Sigma(S(\tilde{x}), T-t))\mathcal{D}^2\tilde{u}\tilde{S}\tilde{S}^T\right] \\ & \quad -r\left(S(\tilde{x})T-t, \tilde{u}, \mathcal{D}\tilde{u}\tilde{S}\sigma\Sigma(S(\tilde{x}), T-t)\right)\left(\tilde{u} - \tilde{S}\cdot\mathcal{D}\tilde{u}\right), \end{aligned}$$

where

$$\tilde{S}^i = S^i \frac{\partial \tilde{x}_i}{\partial X^i} = \begin{cases} 1, & \tilde{x}_i \leq 0, \\ \\ \tilde{x}_1, & \tilde{x}_i \geq 2; \end{cases}$$

the final condition is transformed in

$$\tilde{u}(\tilde{x}, 0) = G(S(\tilde{x})),$$

where $G \circ S \in L^\infty_n(\mathbb{R}^{\rm N})$ whenever $G \in L^\infty_n(\Pi) \cap \mathcal{C}(\Pi)$.
The proof of the standard comparison principle can be adapted to this change of variable in order to get well posedness of the Cauchy problem, but the logarithmic change of variable defined before is more suitable to study the Lipschitz regularity of the solution and the related hedging procedure.
Let us assume the following:

(i) The volatility $\sigma \in \mathbb{M}_{\mathrm{N}\times\mathrm{D}}$ depends only on S and t and satisfies the following regularity assumptions

$$\begin{aligned}&\sigma^i_j \in \mathcal{C}(\Pi \times [0,T]) \cap L^\infty(0,T;W^{1,\infty}(\Pi)),\\ &S^i\mathcal{D}_S\sigma^i_j \in L^\infty(\Pi \times (0,T);\mathbb{R}^{\mathrm{N}}).\end{aligned}$$

(ii) The drift μ may depend, besides S and t, on the portion of wealth invested in the risky assets by the agent, ξ:

$$\begin{aligned}&\mu^i \in \mathcal{C}(\overline{\Pi} \times [0,T] \times \mathbb{R}^{\mathrm{N}}) \cap L^\infty(0,T;W^{1,\infty}(\Pi \times \mathbb{R}^{\mathrm{N}})),\\ &S^i\mathcal{D}_S\mu^i \in L^\infty(\Pi \times (0,T) \times \mathbb{R}^{\mathrm{N}};\mathbb{R}^{\mathrm{N}}),\\ &\xi^j\mathcal{D}_{\xi^h}\mu^i \in L^\infty(\Pi \times (0,T) \times \mathbb{R}^{\mathrm{N}}).\end{aligned}$$

(iii) The interest rate r may depend, besides S and t, on the wealth deposited in the bank by the agent, $\eta = \Delta^0 B = Y - \xi \cdot \mathbf{1}_{\mathrm{N}}$:

$$\begin{aligned}&r \in \mathcal{C}(\overline{\Pi} \times [0,T] \times (\mathbb{R} - \{0\})) \cap L^\infty(0,T;W^{1,\infty}(\Pi \times (\mathbb{R} - \{0\}))),\\ &S^i\mathcal{D}_S r \in L^\infty(\Pi \times (0,T) \times \mathbb{R};\mathbb{R}^{\mathrm{N}}),\\ &\eta\mathcal{D}_\eta r \in L^\infty(\Pi \times (0,T) \times \mathbb{R}).\end{aligned}$$

Under this assumption the following result holds.

Proposition 1.5.8 *Let us assume that under assumptions* $(i) - (iii)$ *and* (1.5.6) *are fulfilled. Then for any choice of the price for risk such that* (1.5.6) *is fulfilled, and for all final value*

$$G \in \mathcal{C}(\overline{\Pi}) \cap W^{1,\infty}_{\mathrm{pol}}(\Pi),$$

the problem (1.5.11)–(1.5.12) *(respectively* (1.5.13)–(1.5.12)*) admits an unique viscosity solution* $U \in L^\infty(0,T;W^{1,\infty}_{\mathrm{pol}}(\Pi))$. *Besides* $U \geq 0$ *whenever* $G \geq 0$. *In particular, any European (respectively American) derivative admits a well defined weak arbitrage price.*

1.6 The Merton model.

In this section we shall describe a general jump–diffusion model, with particular attention to the model proposed by Merton [90] to allow the prices of the assets to be discontinuous in order to catch the sudden and rare breaks of the datas. The aim is to avoid the discrepancies between the results given by the standard Black–Scholes model [28] and empirical evidence. For instance it is well-known that the implied volatility, fitted by using historical data in the Black–Scholes formula, is

not a constant, but depends on the strike price and on the expiration time. This behavior can be ascribed to exogenous information, such as governmental intervention in financial markets or natural disaster affecting the availability of materials or commodities and can be described by means of a point process N that counts the occurrences of rare and random events.
A point process is intended to describe events that occur randomly over time and can be represented by means of a sequence of random variables

$$0 = T_0 < T_1 < T_2 < \dots,$$

where T_n is the n–th instant of occurrence of an event and we assume that it is nonexplosive, that is

$$T_\infty = \lim_{n\to\infty} T_n = \infty.$$

The point process can be equivalently represented by means of its associated counting process N_t

$$N_t = n, \text{ if } t \in [T_n, T_{n+1}),\ n \geq 0, \quad \text{or, equivalently, } N_t = \sum_{n\geq 1} \mathbf{1}_{T_n \leq t}.$$

Here the process N_t counts the number of events occurred till time t; the non explosion assumptions here reads as

$$N_t < \infty, \text{ for } t \geq 0.$$

Both T_n and N_t are defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, with a filtration $\mathcal{F}_t$ to which N_t is adapted.

Definition 1.6.1 *A point process N_t is called* **Poisson point process** *if the following hold*

1. $N_0 = 0$;
2. N_t *has independent increments;*
3. $N_t - N_s$ *is a Poisson random variable with a given parameter* $\Lambda_{s,t}$.

It is usually asumed that the parameter $\Lambda_{s,t}$ is defined by

$$\Lambda_{s,t} = \int_s^t \lambda_u du,$$

λ_t a deterministic function called **intensity of the Poisson process**.
If the filtration $\mathcal{F}_t$ is the one defined by the point process and $\lambda_t \equiv 1$, then N_t is called **standard Poisson process**. More details about jump processes can be

found in [105].
We can note that the items in the definition of the Poisson process are parallel to those of a Wiener process and it will be shown in next chapter that they are parallel even to those of the Lévy process. It can be shown that while any Wiener process is a martingale, the Poisson process becomes a martingale only if one subtracts from N_t the process given by its mean: therefore it comes out that

$$M_t := N_t - \int_0^t \lambda_s ds,$$

is an $\mathcal{F}_t$ martingale, see [105, 29].

In the following we shall still consider a market whose money market account evolves as (1.5.1), while the risky assets $(S^1, \dots, S^{\mathrm{N}})$ evolve following a jump–diffusion dynamic

$$\begin{cases} dS_t^i &= S_{t-}^i \left[\mu^i dt + \sum_{j=1}^{\mathrm{D}} \sigma_j^i dW_t^j + \sum_{k=1}^{\mathrm{M}} \gamma_k^i dN_t^k \right], \quad t \in (0,T), \\ S_0^i &= x_0^i > 0. \end{cases} \tag{1.6.1}$$

Here $W = (W^1, \dots, W^{\mathrm{D}})^T$ is a D–Brownian motion and $N = (N^1, \dots, N^{\mathrm{M}})$ is a M–dimensional Poisson process, with intensity $\lambda = (\lambda^1, \dots, \lambda^{\mathrm{M}})$. The corresponding compensated martingale is

$$M_t^k = N_t^k - \int_0^t \lambda_\tau^k d\tau.$$

In order to assure the consistency of the market,we assume $\gamma_k^i > -1$.
We say that the market supports the *small investor pattern* if the coefficients μ, σ, γ and r are deterministic functions of (S,t) with the following regularity

$$f \in \mathcal{C}(\overline{\Pi} \times [0,T]) \cap L^\infty(0,T; W^{2,\infty}(\Pi)),$$
$$S^i \partial_{S^i} f \in L^\infty(0,T; W^{1,\infty}(\Pi)), \text{ as } i = 1, \dots, \mathrm{N},$$

where f plays the role of the coefficients.

Remark 1.6.2 If all the γ_{ik} are zero, and the functions μ and σ are constant, we have the standard Black–Scholes model, where the stock price follows a log-normal random walk:

$$dS_t = S_t(\mu dt + \sigma dW_t).$$

■

As it was done for the pure diffusion model, we can assume that the coefficients depends, beside (S,t), on the fraction of wealth invested in the stocks. If η denotes the bank deposit of the trader, we assume that the dependence of r from η is not continuous in order to deal with realistic examples. In general it is assumed that

$$\partial_\eta r \le 0, \qquad \partial_\eta (r\eta) \ge 0,$$

otherwise any trader is encouraged to withdraw money from the bank. The interest rate r is required to satisfy

$$\begin{aligned} &r \in \mathcal{C}(\Pi \times [0,T] \times (\mathbb{R} - \{0\})) \cap L^\infty((0,T) \times \mathbb{R}; W^{1,\infty}(\Pi)), \\ &S^i \partial_{S^i} r \in L^\infty(\Pi \times (0,T) \times \mathbb{R}), \end{aligned}$$

$\eta \to r(S,t,\eta)\eta$ is Lipschitz continuous and nondecreasing, uniformly w.r.t (S,t) that is

$$\eta \partial_\eta r \in L^\infty(\Pi \times (0,T) \times \mathbb{R}), \qquad \eta \partial_\eta r \ge -r \text{ a.e.}$$

Clearly the coefficients γ^i_j are not influenced by the strategy of the agent as jumps are caused by external events.

The prototype of jump-diffusion model is given by the Merton model [90], as it has been described in Example 1.1.7. This market is not complete as for any couple (θ,ϕ) such that

$$\begin{aligned} &\theta\sigma + \phi\gamma = r - \mu, \\ &1 + \phi > 0, \qquad \mathbb{E}[\zeta(\theta,\phi)] = 1, \end{aligned}$$

the density

$$\xi(\theta,\phi) = \exp\Big(- \frac{1}{2}\theta^2 T + \theta \int_0^T dW_t \Big) \exp\Big(- \phi T + \log(1+\phi) \int_0^T dN_t \Big),$$

provides an equivalent probability measure

$$\mathcal{P}^*(\theta,\phi) = \zeta(\theta,\phi)\mathcal{P},$$

under which the actualized returns of the asset are given by the following dynamic

$$d\tilde{S}_t = \tilde{S}^{t^-}[\sigma dW_t^+ + \gamma dM_t^*],$$

where $W_t^* = \theta t + W_t$ is a Brownian motion and $M_t^* = \phi t + M_t$ is the compensated martingale of the new Poisson process with intensity $\lambda + \phi$.
The parameter

$$\theta, \text{ and } 1 + \phi,$$

are the *market price for risk* concerning the diffusion and the jump respectively. For more details we refer to [70, 99].

It can be shown that under suitable assumption on the randomness matrix (σ, γ) the market given by (1.5.1)–(1.6.1) is without arbitrage opportunities, that is it is complete whenever the rank of the randomness matrix is large enough and that it is possible to price any derivative by means of integro–differential equations.
The study has been performed in [2] in the case of small investor and of large investor economy.
In the case the market is not complete, it can be completed by adding some derivative on the underlying assets $S^1, \dots, S^{\mathrm{N}}$ following a jump–diffusion dynamic as (1.6.1).
For a detailed discussion on this topic we refer to [82].

In a jump diffusion market with large investor economy we have the following results.

1.6.1 Pricing derivative in a jump–diffusion market with large investor economy.

In the case of large investor economy, the price for risk (θ, ϕ) depends on the strategy of the traders in terms of $\eta = \Delta^0 B$. They are determined by the relation

$$(\sigma, \gamma)(\theta^1, \dots, \theta^{\mathrm{D}}, \phi^1, \dots, \phi^{\mathrm{M}})^T = r\mathbf{1}_{\mathrm{N}} - \mu.$$

We assume that there are not strong correlated assets, that is $\mathrm{N} \leq \mathrm{D} + \mathrm{M}$.

In the following we shall use a technical lemma.

Lemma 1.6.3 *Suppose that the randomness matrix (σ, γ) has maximum rank N such that the submatrix $\mathrm{N} \times \mathrm{N}$ $(\sigma,\hat{\ } \gamma)$ composed by the row $j_1, \dots, j_{\mathrm{N}}$ is invertible. Let $w \in \mathbb{R}^{\mathrm{D} \times, \mathrm{M}}$, then*

$$(\sigma,\hat{\ }\gamma)w = v, \text{ iff } w_j^j = \det(\sigma,\hat{\ }\gamma)^{-1} \det(\sigma,\hat{\ }\gamma)_l[v],\ l = 1, \dots, \mathrm{N},$$

where $A_l[v]$ ($A^j[v]$) indicates the matrix obtained by the matrix A switching the l^{th} row (j^{th} column) with the vector v.

Let $\begin{pmatrix} \Sigma \\ \Gamma \end{pmatrix}$ be the inverse of (σ, γ), given by

$$\Sigma_j^i = \det(\sigma, \gamma)^{-1} \det(\sigma_i[e^j], \gamma), \quad \Gamma_j^i = \det(\sigma, \gamma)^{-1} \det(\sigma, \gamma_i[e^j]),$$

where $(e^1, \dots, e^{\mathrm{N}})$ is the canonical basis is $\mathbb{R}^{\mathrm{N}}$.
The market prices for risk are then determined by

$$\theta^i(S, t, \eta) = \det(\sigma, \gamma)^{-1} \det(\sigma_i[r\mathbf{1}_{\mathrm{N}} - \mu], \gamma) = r(S, t, \eta)\theta_0^i(S, t) - \Sigma^i(S, t)\mu(S, t),$$

$$\phi^j(S,t,\eta) = \det(\sigma,\gamma)^{-1}\det(\sigma,\gamma_j[r\mathbf{1}_{\rm N}-\mu]) = r(S,t,\eta)\phi_0^j(S,t) - \Gamma^j(S,t)\mu(S,t)$$

with proper θ_0^i and ϕ_0^j.
We assume that the model satisfies the following assumptions

(JDM.1) (σ,γ) has maximum rank equal to N and admits a maximal submatrix $(\widehat{\sigma,\gamma})$ such that for all $(S,t) \in \Pi \times [0,T]$

$$\|\det(\widehat{\sigma,\gamma})(S,t)\| \geq \varepsilon > 0.$$

Whenever the matrix (σ,γ) contains some rows of γ, that is $(\sigma,\gamma) = (\hat{\sigma},\hat{\gamma})$, we set

$$\begin{aligned}\hat{\theta}^{h_l} &= \det(\sigma,\gamma)^{-1}\det(\hat{\sigma}_l[r\mathbf{1}_{\rm N}-\mu],\hat{\gamma}), \quad ,l=1,\dots,{\rm N}-{\rm L},\\ \hat{\phi}^{k_l} &= \det(\sigma,\gamma)^{-1}\det(\hat{\sigma},\hat{\gamma}_l[r\mathbf{1}_{\rm N}-\mu]), \quad ,l=1\dots,{\rm L},\end{aligned}$$

where h_l is the generic row of $\hat{\sigma}$ and k_l the generic row of $\hat{\gamma}$, and we ask that there exist two predictable processes $\theta \in \mathbb{R}^{\rm D}$ and $\phi \in \mathbb{R}^{\rm M}$ such that

$$\begin{aligned}\theta_t^{h_l} &= \hat{\theta}^{h_l}(S,t), \quad ,l=1,\dots,{\rm N}-{\rm L},\\ \phi_t^{k_l} &= \hat{\phi}^{k_l}(S,t), \quad ,l=1\dots,{\rm L},\end{aligned}$$

fulfilling the following

(JDM.2) $\phi_0^k, \Gamma^k\mu \geq 0$ and $\max\{\phi_0^k,\Gamma^k\mu\} > 0$ for $k=1,\dots,{\rm M}$.

(JDM.3) $\mathbb{E}[\zeta_t(\theta,\phi)] = 1$, where

$$\begin{aligned}\zeta_t(\theta,\phi) &= \prod_{h=1}^{\rm D}\exp\Big(-\frac{1}{2}\int_0^t(\theta_\tau^h)^d\tau + \int_0^t\theta_\tau^h dW_\tau^h\Big)\\ &\quad \prod_{k=1}^{\rm M}\exp\Big(\int_0^t(\lambda_\tau^k-\phi_\tau^k\Big)d\tau - \int_0^t\log\frac{\lambda_\tau^k}{\phi_\tau^k}dN_\tau^k\Big).\end{aligned}$$

To go through the pricing problem in this market, we assume that an hedging strategy (Δ^0,Δ) does exists, Y is the total wealth and η is the investment in the bank. The hedging strategy should be self financing, therefore Y solves the following BSDE

$$\begin{aligned}Y_t &= G(S_t) - \int_t^T[\eta r(S,t,\eta) + \Delta S(\mu+\lambda\gamma)]d\tau\\ &\quad - \int_t^T\Delta S\sigma dW_\tau - \int_t^T\Delta S\gamma dM_\tau.\end{aligned}$$

We follow the line described in the case of a pure diffusion model: we assume that there exists a deterministic function $U(S,t)$ giving the price of the derivative, that is $U(S_t,t) = Y_t$ for all $t \in [0,T]$. Using the generalized Ito's formula, see [105], comparing the coefficients of dW_t and dM_t we get

$$\eta_t = \eta(S,t,U,\mathcal{D}U,\mathcal{J}U) = U - \mathcal{D}US\sigma\theta_0 - \mathcal{J}U\phi_0,$$

and U solves the following quasilinear final value problem

$$\begin{aligned} -\partial_t U &= -\frac{1}{2}\mathrm{tr}\left[(S\sigma)(S\sigma)^T\mathcal{D}^2U\right] + (\mathcal{D}US\gamma - \mathcal{J}U)\Gamma\mu \qquad (1.6.2)\\ &\quad +[U - \mathcal{D}US\sigma\theta_0 - \mathcal{J}U\phi_0]r(S,t,U - \mathcal{D}US\sigma\theta_0 - \mathcal{J}U\phi_0)\\ U(S_T,T) &= G(S_T), \qquad (1.6.3)\end{aligned}$$

where

$$\mathcal{J}_k U(S,t) = U(S+\gamma_k S,t) - U(S,t), \quad k = 1,\ldots,\mathrm{M},$$

and $\mathcal{J}U$ is the vector $(\mathcal{J}_1U,\ldots,\mathcal{J}_{\mathrm{M}}U)$.

Definition 1.6.4 *The* **weak arbitrage price** *of an European derivative* (G,T) *is any nonnegative solution of* (1.6.2)–(1.6.3), *provided that* $U \in L^\infty(0,T;W^{1,\infty}_{\mathrm{pol}}(\Pi))$. *In this case the* **weak hedging strategy** *is given by*

$$\begin{aligned}\Delta^0_t &= \exp\Big(-\int_0^t r(S_\tau,\tau)d\tau\Big)[U - \mathcal{D}US\sigma\theta_0 - \mathcal{J}U\phi_0](S_t,t),\\ \Delta^i_t &= \frac{1}{S^i_t}\Big[\mathcal{D}US(\sigma\Sigma)_i + \mathcal{J}U\Gamma_i\Big](S_t,t), \qquad i = 1,\ldots,\mathrm{N}.\end{aligned}$$

The **weak arbitrage price** *of an American derivative* (G,T) *is the (unique) nonnegative solution of the obstacle problem*

$$\begin{aligned}&\min\{-\partial_t U + \frac{1}{2}\mathrm{tr}\left[(S\sigma)(S\sigma)^T\mathcal{D}^2U\right] + (\mathcal{D}US\gamma + \mathcal{J}U)\Gamma\mu \qquad (1.6.4)\\ &-[U - \mathcal{D}US\sigma\theta_0 - \mathcal{J}U\phi_0]r(S,t,U - \mathcal{D}US\sigma\theta_0 - \mathcal{J}U\phi_0), U - G\} = 0\end{aligned}$$

with final datum (1.6.3), *provided that* $L^\infty(0,T;W^{1,\infty}_{\mathrm{pol}}(\Pi))$. *In this case the hedging strategy is the same as the European case, on the time interval* $(0,\tau^*)$, τ^* *being the optimal stopping time.*

Proposition 1.6.5 *[2, Proposition 2.20, page 59]Let us assume that*

(C.1) *the semilinear final value problem* (1.6.2)–(1.6.3) *and the corresponding obstacle problem* (1.6.4)–(1.6.3) *are well posed in the class* $\mathcal{C}(\Pi) \cap W^{1,\infty}_{\text{pol}}(\Pi)$ *and their solutions are nonnegative whenever the final datum G does;*

(C.2) *for all* $G \in \mathcal{C}(\Pi) \cap W^{1,\infty}_{\text{pol}}(\Pi)$ *the solution U of* (1.6.2)–(1.6.3) *(respectively* (1.6.4)–(1.6.3)*) has the following regularity property*

$$U \in \mathcal{C}^{1,2}(\Pi \times (0,T)).$$

Then the market is complete, the weak hedging strategy really hedges the derivative and U is its arbitrage price.

Remark 1.6.6 The operator

$$\begin{aligned}\mathcal{L}_{\mathcal{J}} &= -\frac{1}{2}\text{tr}\left[(S\sigma)(S\sigma)^T\mathcal{D}^2U\right] + (\mathcal{D}US\gamma - \mathcal{J}U)\Gamma\mu \\ &\quad +[U - \mathcal{D}US\sigma\theta_0 - \mathcal{J}U\phi_0]r(U - \mathcal{D}US\sigma\theta_0 - \mathcal{J}U\phi_0),\end{aligned}$$

is nonlinear in the case of large investor economy, as a nonlinear dependence on the integral term $\mathcal{J}U$ arises. ■

As in the previous cases, we can remove the boundary degeneracy by means of a logarithmic change of variable (1.5.8): in this case both the interest rate r and the Radon measure $\mu_{x,t}$ depend on η

$$\eta(x,t,u,\mathcal{I}^0u,\mathcal{D}u) = \left[u - (\sigma\theta_0)^T \cdot \mathcal{D}u - \mathcal{I}^0u\right](S(x), T-t),$$

and the measure $\mu_{x,t}$ is given by

$$\mu_{x,t,\eta} = r(x,t,\eta)\mu^0_{x,t} + \mu^1_{x,t}$$

where $\mu^i_{x,t}$ are the Radon measures defined by

$$\begin{aligned}\int_{\mathbb{R}^{\mathrm{N}}} f(z)\mu^0_{x,t}(dz) &= \sum_{k=1}^{\mathrm{M}} \phi^k_0 f(\log(1+\gamma^1_k), \ldots, \log(1+\gamma^{\mathrm{N}}_k)),\\ \int_{\mathbb{R}^{\mathrm{N}}} f(z)\mu^1_{x,t}(dz) &= \sum_{k=1}^{\mathrm{M}} -\Gamma^k\mu f(\log(1+\gamma^1_k), \ldots, \log(1+\gamma^{\mathrm{N}}_k)),\end{aligned}$$

for all continuous functions f; the parameters ϕ^k_0, γ^i_k and $\Gamma^k\mu$ are computed at $(e^{x_1}, \ldots, e^{x_{\mathrm{N}}}, T-t)$; the integral term becomes

$$\mathcal{I}^j u(x,t) = \int_{\mathbb{R}^{\mathrm{N}}} [u(x+z,t) - u(x,t)]\mu^j_{x,t}(dz), \quad j = 0,1.$$

We can conclude that U solves the nonlinear version of (1.6.2)–(1.6.3) (respectively (1.6.4)–(1.6.3)) iff

$$u(x,t) = U(e^{x_1}, \dots, e^{x_N}, T-t),$$

solves the following integro–differential Cauchy problem on $\mathbb{R}^N \times (0,T)$

$$\partial_t u + \mathcal{L}_{\mathcal{I}} u + H(x,t,u,\mathcal{D}u,\mathcal{I}^0 u) = 0,$$

$$u(x,0) = g(x) = G(e^{x_1}, \dots, e^{x_N}),$$

(respectively

$$\min\Big\{\partial_t u + \mathcal{L}_{\mathcal{I}} u + H(x,t,u,\mathcal{D}u,\mathcal{I}^0 u), u - g\Big\} = 0$$

with the same initial condition.) Here $\mathcal{L}$ is a linear degenerate elliptic operator

$$\mathcal{L}_{\mathcal{I}^1} u(x,t) = -\frac{1}{2}\mathrm{tr}\left[\sigma\sigma^T(S(x),T-t)\mathcal{D}^2 u\right] + (\gamma\Gamma\mu)^T(S(x),T-t)\cdot\mathcal{D}u - \mathcal{I}^1 u,$$

and H is a nonlinear first order integro–differential operator

$$\begin{aligned} H(x,t,u,\mathcal{D}u,\mathcal{I}^0) &= r(S(x),T-t,u-(\sigma\theta_0)^T(S(x),T-t)\cdot\mathcal{D}u - \mathcal{I}^0) \\ &\quad \Big[u-(\sigma\theta_0)^T(S(x),T-t)\cdot\mathcal{D}u - \mathcal{I}^0\Big]. \end{aligned}$$

The following result has been proved by Amadori in [2]:

Theorem 1.6.7 *[2, Theoerm 5.4, page 61 and page 141] Suppose that*

*(i) the coefficients μ^i, σ^i_j, γ^i_k satisfy assumptions (**JDM.1**)–(**JDM.2**) and have the following regularity properties*

$$\begin{aligned} &\mu^i,\ \sigma^i_j,\ \gamma^i_k \in \mathcal{C}(\overline{\Pi}\times[0,T]) \cap L^\infty(0,T;W^{1,\infty}(\Pi)), \\ &S\cdot\mathcal{D}\mu^i,\ S\cdot\mathcal{D}\sigma^i_j,\ S\gamma^i_k \in L^\infty(\Pi\times(0,T)); \end{aligned}$$

(ii) $r \in \mathcal{C}(\overline{\Pi}\times[0,T]\times(\mathbb{R}-\{0\}))$ is bounded and there exist

$$\lim_{\xi\to 0^+} r(S,t,\xi), \qquad \lim_{\xi\to 0^-} r(S,t,\xi)$$

for all S, t; moreover

$$\begin{aligned} &\|r(S,t,\xi)S - r(S',t,\xi)S'\| \le \mathrm{L}(1+|\xi|)\|S-S'\|, \\ &r(S,t,\xi)\xi - r(S,t,\xi')\xi' \le \mathrm{L}(\xi-\xi')^+. \end{aligned}$$

Then for all payoffs $G \in W^{1,\infty}_{\text{pol}}(\Pi)$*, the final value problem* (1.6.2)–(1.6.3) *(respectively, the obstacle problem* (1.6.4)–(1.6.3)*) has an unique viscosity solution* $U \in L^\infty(0,T;W^{1,\infty}_{\text{pol}}(\Pi))$*. In addiction* $U \geq 0$ *whenever* $G \geq 0$ *and*

$$U(s,t) + \|\mathcal{D}U(S,t)\| \leq \mathrm{B}(1+\|S\|^n),$$

for all $t \in (0,T)$ *if* $G(S) + \|\mathcal{D}G(S)\| \leq \mathrm{B}_0(1+\|S\|^n)$*.*
In particular all European derivatives (respectively, all American derivatives) have an unique weak arbitrage price and a weak hedging strategy.

This result guarantees that all the jump–diffusion markets with large investor is weakly complete: every derivatives admits a weak hedging strategy. A detailed discussion of this topic can be found in [2].

The first example of jump–diffusion market has been proposed by Merton [90], see Example 1.1.7. The price of any derivative in this market can be derived as the solution of a linear integro–differential problem, as it is exposed in the next example.

Example 1.6.8 Let us suppose to model the market as in Example 1.1.7, and consider an European call (G,T). If we set $\mathcal{J}U(S) = \lambda\Big(\int_D U(S\eta)\tilde{\Gamma}_\delta(\eta)d\eta - U\Big)$, we promptly obtain by the Ito formula the following pricing equation

$$\begin{aligned}&\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + (r-\lambda k)S\frac{\partial U}{\partial S} - rU + \mathcal{J}U \\ &= \frac{\partial U}{\partial t} + \mathcal{L}_{\mathcal{J}}U = 0,\end{aligned} \tag{1.6.5}$$

with $D = [0,+\infty)$ and the final condition

$$U(T,S_T) = G(S_T).$$

■

Jump–diffusion market are shown to be incomplete, therefore it is no more possible to price derivative with the hedging procedure. A possibility to price derivatives is to complete the market by adding some derivative on the same underlying. This procedure is explained in the next example.

Example 1.6.9 [*Completion of the market in the large investor model*] It is easily proved that a jump-diffusion market is incomplete because of the arbitrage opportunities arising at the jump time, see [25]. To overcome the difficulty of pricing a derivative it is possible to complete the jump-diffusion market by adding another derivative on the same underlying asset. A standard approach is to add a call whose parameters are taken directly from the market, and not by Ito rule. Therefore we can suppose that our market is described by

$$\begin{cases} dB_t &= B_t r(\mathbf{X}, t, \xi) dr, \\ \\ d\mathbf{X} &= \mathbf{X}\alpha dt + \mathbf{X}\beta dW_t + \mathbf{X}(\gamma - \mathbf{1}) dN_t, \end{cases}$$

where $\mathbf{X}_t = \operatorname{diag}(S_t, C_t)$ and the vectors of expectation α, volatility β, and jump γ are:

$$\alpha = \begin{pmatrix} \mu - \lambda k \\ \mu_C \end{pmatrix}, \quad \beta = \begin{pmatrix} \sigma \\ \sigma_C \end{pmatrix}, \quad \gamma = \begin{pmatrix} \eta \\ \eta_C \end{pmatrix}.$$

The jump amplitude γ is now lognormally distributed with density

$$\tilde{\Gamma}(\gamma) = \tilde{\Gamma}(\eta) \cdot \tilde{\Gamma}_C(\eta_C).$$

In this frameset the pricing equation is the extension of (1.6.5) to the multidimensional case:

$$\begin{cases} \partial_t U + L_{\mathcal{J}} U &= H(S, t, U, \mathcal{J}U, \mathcal{D}U), \\ \\ U(\mathbf{X}, T) &= G(\mathbf{X}). \end{cases}$$

Here, the operator

$$L_{\mathcal{J}} U = \frac{1}{2} \operatorname{tr}\left[(\mathbf{X}\beta)(\mathbf{X}\beta)^T \mathcal{D}^2 U\right] + \mathbf{X}[\alpha + \beta\theta] \cdot \mathcal{D}U - \phi \cdot \mathcal{J}U,$$

is linearly degenerate elliptic. Moreover

$$H(\mathbf{X}, t, U, \mathcal{J}U, \mathcal{D}U) = r(\mathbf{X}, t, \xi) \cdot \xi,$$

and $\mathcal{J}U$ as in Example 1.6.8 with $D = [0, +\infty) \times [0, +\infty)$. We can note that in this case the diffusion matrix is degenerate, since

$$rk((\mathbf{X}\beta)(\mathbf{X}\beta)^T) < 2.$$

If we apply a change of variable in order to have diffusion only in one direction:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \cdot \begin{pmatrix} \log S \\ \log C \end{pmatrix},$$

with $\vartheta = arctg \frac{\sigma_C}{\sigma}$, and proper coefficients A, B, C, D we obtain:

$$\begin{aligned} &\partial_t u + \frac{1}{2}(\sigma^2 + \sigma_C^2)\partial_{xx}^2 u + A\partial_x u + B\partial_y u - \phi \mathcal{I} u = \\ &= r\Big(x, y, t, u + C\partial_x u + D\partial_y u - \phi_0 \mathcal{I} u\Big) \times \end{aligned} \qquad (1.6.6)$$

$$\times\Big(u + C\partial_x u + D\partial_y u - \phi_0 \mathcal{I}u\Big),$$

where:

$$\mathcal{I}u = \lambda\Big(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u(x+\xi, y+\zeta, t)\frac{\exp\left(-\frac{\xi^2+\zeta^2}{2\delta^2}\right)}{2\pi\delta\rho}\cdot$$
$$\cdot\exp\Big(-\frac{(\delta^2-\rho^2)(\xi\sin\theta+\zeta\cos\theta)^2}{2\delta^2\rho^2}\Big)d\xi d\zeta - u(x,y,t)\Big).$$

This example shows the need for a theory of strongly degenerate nonlinear parabolic operators in financial applications. ■

The problem exposed in this example shall be studied in details from the numerical point of view in Chapter 7.

Chapter 2

Exponential Lévy markets.

In the previous chapter a general introduction to financial market has been given; for several years the classical Black and Scholes market has played a predominant role in financial modeling, because of its simplicity of calculation. Unfortunately this model does not have a good fit with real market datas: as it was observed by Mandelbrot [85] the logarithmic of relative price changes on financial markets exhibit a long-tailed distribution, which is not reflected by an exponential Brownian motion. This yields researcher to look for more sophisticated model to get more resemblance to the real evolution of markets.
Since from the non-normal exponential model by Mandelbrot, several other model have been proposed [95, 106, 103], and all of them are in the class of the so called Lévy models. They have been recently studied in [59, 43, 45, 46, 93, 104, 12, 20] and they seem to capture a peculiarity of financial markets as heavier tails of the prices distribution and they appear to fit better real datas. Models of that kind are clearly not complete because there are infinite sources of uncertainty, described by the Lévy measure, but only a finite number of assets with which hedge against risk. In the work by Madan and Seneta [80] a particular case of Lévy processes, the variance gamma model, was proposed to fit the Australian stock market data.
For what concerns stock price model, Eberlein and Keller [44] introduced an exponential hyperbolic Lévy motion, Barndorff-Nielsen [18] an exponential normal inverse Gaussian Lévy process. The whole family of generalized hyperbolic Lévy process has been studied by Eberlein and Prause [45] and Prause [102].

In this chapter we shall present a market model driven by an exponential Lévy model for the prices of the assets S_t.

2.1 Stock price models.

We remember from the previous chapter that a market place can be described by a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, endowed with a right–continuous filtration $(\mathcal{F}_r)_{t\geq 0}$, and

$$\mathcal{F} = \sigma\Big(\bigcup_{t\geq 0} \mathcal{F}_t\Big).$$

This assumption allow to specify a change of the underlying probability measure $\mathcal{P}$ to a measure $\mathcal{Q}$ by giving a density process ζ_t:

$$\zeta_t = \frac{d\mathcal{Q}_t}{d\mathcal{P}_t},$$

where $\mathcal{Q}_t$ and $\mathcal{P}_t$ are the restriction of $\mathcal{Q}$ and $\mathcal{P}$ to $\mathcal{F}_t$ respectively.
If $\zeta_t > 0$ for all $t \geq 0$, we say that $\mathcal{P}$ and $\mathcal{Q}$ are *locally equivalent* and we write $\mathcal{P} \overset{loc}{\sim} \mathcal{Q}$.

Definition 2.1.1 *An adapted process* $L = (L_t)_{t\geq 0}$ *with* $X_0 = 0$ *a.s. is a* **Lévy process** *if*

(i) L *has increments independent of the past, that is* $L_t - L_s$ *is independent of* $\mathcal{F}_s$ *for all* $0 \leq s < t$*;*

(ii) L *has stationary increments, that is* $L_t - L_s$ *has the same distribution as* L_{t-s} *for all* $0 \leq s < t$*;*

(iii) L_t *is continuous in probability, that is* $\lim_{t\to s} L_t = L_s$*, where the limit is taken in probability.*

The Lévy process is characterized by its distribution, which is determined by any of its one-dimensional marginal distribution; for simplicity we calculate the one corresponding to L_1. The characteristic function has a special structure given by the **Lévy–Khintchine formula**:

$$\mathbf{E}\left[\exp(i\theta L_1)\right] = \exp\left(i\theta b - \frac{c}{2}\theta^2 + \int \left(e^{i\theta z} - 1 - i\theta z\right)\nu(dz)\right)$$

From the independence and stationarity of the increments of L_t, it follows that the distribution of the Lévy process is infinitely divisible.

Definition 2.1.2 *The* **Lévy–Khintchine triplet** (b, c, ν) *of an infinite divisible distribution consists of the constants* $b \in \mathbb{R}$ *and* $c \geq 0$ *and the measure* $\nu(dz)$*, which appear in the Lévy–Khintchine representation of the characteristic function.*

In the following we consider stock price models of the form

$$S_t = S_0 \exp(L_t), \tag{2.1.1}$$

where L_t is a Lévy process satisfying some integrability conditions.

Remark 2.1.3 We can note that in the stock price model considered, we could include the interest rate r, since the process $\tilde{L}_t := rt + L_t$ is as well a Lévy process. This choice is motivated by the fact that in the pricing approach one would always consider discounted prices, that is the stock prices divided by the factor e^{rt}. ■

Example 2.1.4 [*Brownian model*] The classical Black and Scholes model describes the market by the following stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma dW_t),$$

with constant coefficients $\mu \in \mathbb{R}$ and $\sigma > 0$, and W_t a standard Brownian motion. The solution to this equation is

$$S_t = S_0 \exp\Big(\mu t - \frac{\sigma^2}{2} t + \sigma W_t\Big).$$

We can see that this one is a special case of (2.1.1): here the Lévy model is given by

$$L_t = \Big(\mu - \frac{\sigma^2}{2}\Big) t + \sigma W_t.$$

■

In mathematical finance the locally absolutely continuous transformations has the meaning to change the underlying probability measure $\mathcal{P}$, the objective probability measure, to the so called risk neutral measure $\mathcal{Q}$ under which all discounted prices processes are $\mathcal{Q}$–integrable and martingales. Therefore $\mathcal{Q}$ is called martingale measure.

In the derivative pricing problem, the price of an European derivative of maturity T and strike price K, we can calculate the price at any time $t < T$ just taking the conditional expectation of its discounted final value. In formulas, let $C(t)$ be the value of an European call at time t; by assumption the discounted value, $e^{-rt}C(t)$ is a $\mathcal{Q}$–martingale for any t, therefore

$$C(t) = e^{rt}\mathbb{E}^{\mathcal{Q}}\Big[e^{-rT}(S_T - K)^+ \Big| \mathcal{F}_t\Big].$$

In this way it is clear that to know the price of any derivative it suffices to know its final value and the risk–neutral measure $\mathcal{Q}$.

Lemma 2.1.5 *Let ζ_t be a density process, i.e. a non–negative $\mathcal{P}$–martingale with $\mathbb{E}[\zeta_t] = 1$ for all $t \geq 0$. Let $\mathcal{Q}$ be the measure defined by*

$$\frac{d\mathcal{Q}}{d\mathcal{P}}\bigg|_{\mathcal{F}_t} = \zeta_t, \qquad t \geq 0.$$

Any adapted process $(X_t)_{t\geq 0}$ is a $\mathcal{Q}$ martingale if and only if $(X_t\zeta_t)_{t\geq 0}$ is a $\mathcal{P}$–martingale. Furthermore, if we assume that $\zeta_t > 0$ for all $t \geq 0$, then, for any $t < T$ and any $\mathcal{Q}$–integrable, $\mathcal{F}_t$–measurable random variable X, we have

$$\mathbb{E}^{\mathcal{Q}}\left[X\middle|\mathcal{F}_t\right] = \mathbb{E}^{\mathcal{P}}\left[X\frac{\zeta_T}{\zeta_t}\middle|\mathcal{F}_t\right].$$

For what concerns the Lévy process, we can characterize the change of measure by the so call *Esscher transform.*

Assumption 2.1.6 *Let as assume the following:*

1. *The random variable L_1 is non–degenerate and posses a moment generating function*

   $$\text{mgf} : \theta \to \mathbb{E}[\exp(\theta L_1)],$$

   on some open interval (a_1, a_2) with $a_2 - a_1 > 1$.

2. *There exists a real number $\tilde{\theta} \in (a_1, a_2 - 1)$ such that*

   $$\text{mgf}(\tilde{\theta}) = \text{mgf}(\tilde{\theta} + 1).$$

Definition 2.1.7 *Let L be a Lévy process on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathcal{P})$. We define* **Esscher transform** *any change of measure from $\mathcal{P}$ to a locally equivalent measure $\mathcal{Q}$ with a density process*

$$\zeta_t = \frac{\mathcal{Q}}{\mathcal{P}}\bigg|_{\mathcal{F}_t} = \frac{\exp(\theta L_t)}{\text{mgf}(\theta)^t} \tag{2.1.2}$$

with $\theta \in \mathbb{R}$ and $\text{mgf}(\theta)$ is the moment generating function of L_t.

Proposition 2.1.8 *[104, Proposition 1.8, page 7] Equation (2.1.2) defines a density process for all $\theta \in \mathbb{R}$ such that $\mathbb{E}[\exp(\theta L_1)] < \infty$. L is a Lévy process under the new measure $\mathcal{Q}$.*

This result has an important consequence in option pricing problems, as it allows to find an equivalent probability measure under which discounted prices process are martingale.

Lemma 2.1.9 *Let the stock price process be given by* (2.1.1) *and let Assumption 2.1.6 holds. The basic probability measure $\mathcal{P}$ is locally equivalent to a measure $\mathcal{Q}$ such that the discounted stock price $e^{-rt}S_t$ is a $\mathcal{Q}$–martingale. A density process leading to such a martingale measure $\mathcal{Q}$ is given by the Esscher transform density:*

$$\zeta_t^{(\theta)} = \frac{\exp(\theta L_t)}{\mathrm{mgf}(\theta)^t},$$

with a suitable real constant θ. The value θ is uniquely determined as the solution of the following equation

$$\mathrm{mgf}(\theta) = \mathrm{mgf}(\theta+1), \qquad \theta \in (a_1, a_2).$$

A detailed proof of this result can be found in [104].

The focus problem we are analyzing is the one of finding a fair price for any derivative in a market driven by an exponential Lévy process. As we have explained before, the price of any derivative (G, T) can be derived as the expected value of the discounted strike price with respect to an equivalent martingale measure: let Y_t the value of the derivative, it is given by

$$Y_t = e^{-r(T-t)}\mathbb{E}^{\mathcal{P}}\Big[G(S_t e^{r(T-t)+L_{T-t}})\frac{\exp(\theta L_{T-t})}{\mathrm{mgf}(\theta)^{T-t}}\Big|\mathcal{F}_t\Big], \tag{2.1.3}$$

where we have used some manipulation and that the Lévy process has stationary and independent increments.

2.2 Differential approach to option pricing.

From equation (2.1.3) and from Lemma 2.1.5 we know that the price of any derivative (G, T) is given as an expected value of the discounted final value. If we assume that there exist a deterministic function U such that

$$Y_t = U(S_t, t),$$

we can ask if it is possible to derive, as in the Black and Scholes setting, a deterministic partial differential equation to which u is solution and which gives the fair price of the derivative.

Let us consider a market with one risky asset described by a Lévy process and one risk–free asset

$$\begin{cases} B_t &= B_0 e^{rt} \\ S_t &= S_0 e^{L_t} \end{cases}$$

where L_t is a one dimensional Lévy market.

As in the Black and Scholes settings, it turns out to be easy to express the option price in term of the logarithmic value of the stock prices; we therefore use the logarithmic change of variable $x = \log(S)$ and the unknown function $u(x,t) = U(e^x, t)$, given as

$$u(x,t) := e^{-r(T-t)}\mathbb{E}^{\mathcal{Q}}\left[G(e^{x+L_{T-t}})\Big|\mathcal{F}_t\right]. \tag{2.2.1}$$

Proposition 2.2.1 *Assume that the function defined in* (2.2.1) *is of class* $\mathcal{C}^{2,1}(\mathbb{R}\times[0,\infty))$. *Let us suppose that the law of* L_t *has support* $\mathbb{R}$*; then* u *satisfies the following integro–differential equation*

$$\begin{aligned} \partial_t u(x,t) + \frac{1}{2}\mathcal{D}^2 u(x,t)c + \mathcal{D}u(x,t)b - ru(x,t) & \\ + \int (u(x+z,t) - u(x,t) - \mathcal{D}u(x,t)z)\,\nu(dz) &= 0, \qquad (2.2.2) \\ u(x,T) &= G(e^x), \end{aligned}$$

for $(x,t) \in \mathbb{R}\times(0,T)$. *The only parameters entering in this equation are the short-term interest rate* r *and the Levy–Khintchine triplet* (b,c,ν) *of the Lévy process* L *under the pricing measure* $\mathcal{Q}$.

We refer to Chapter 5 to the general proof of this result in the multi-dimensional setting.

Remark 2.2.2 We can note that equation (2.2.2) is in some sense equivalent to the Feynman–Kač formula [72], even if they procede in two different direction, the last starting from the solution of some parabolic partial differential equation which can be represented as a conditional expectation value, provided it satisfies some regularity condition.
The general Feynmann–Kač formula for Lévy process was given in [33]. ∎

For what concerns the models considered in this thesis, we concentrate our attention on the so called *generalized hyperbolic distributions*: they have been widely studied in [45, 44, 17]. These processes are essential in the study of the singularity and the absolute continuity of the distribution of the Lévy processes near $z = 0$. The behavior of the Lévy measure in a neighborhood of $z = 0$ is important to derive information about the absolute change of measure.
To focus on that point, let us consider $\chi(\theta)$ denoting the characteristic function on an infinite divisible distribution; its Lévy–Khintchine representation is

$$\chi(\theta) = \exp\left(i\theta b - \frac{\theta^2}{2}c + \int_{\mathbb{R}-\{0\}}\left(e^{i\theta z} - 1 - i\theta h(z)\right)\nu(dz)\right),$$

where $b \in \mathbb{R}$, $c \geq 0$ and $\nu(dz)$ is the *Lévy measure.* This is a σ–finite measure on $\mathbb{R} - \{0\}$ that satisfies

$$\int_{\mathbb{R}-\{0\}} \left(z^2 \wedge 1\right) \nu(z) < \infty.$$

The function $h(z)$ is a *truncation function* that satisfies $h(x) = x$ in a neighbourhood of $x = 0$, [68]. In the following we shall use

$$h(z) = z\mathbf{1}_{\{|z|\leq 1\}}.$$

Definition 2.2.3 *Let $\nu(dz)$ be the Lévy measure of an infinitely divisible distribution. We define the* **modified Lévy measure** *the measure $\tilde{\nu}(dz)$ defined by*

$$\tilde{\nu}(dz) = z^2\nu(dz).$$

Lemma 2.2.4 *Let $\tilde{\nu}(dz)$ be the modified Lévy measure corresponding to the Lévy measure $\nu(dz)$ of an infinitely divisible distribution that possesses a second moment. Then $\tilde{\nu}(dz)$ is a finite measure.*

An important question is now concerned with change of equivalent martingale measure in a Lévy market. In the previous section we have seen that the Esscher transform is an important tool to obtain equivalent martingale measure.
Our aim is to obtain one specific equivalent martingale measure under which the stationarity and independence of increments is preserved, that is we are looking for change of measure which preserve Lévy processes.
The answer can be given in terms of the Lévy–Khintchine triplet of the process.

Proposition 2.2.5 *[104, Proposition 2.19, page 41] Let L be a Lévy process with a Lévy–Khintchine triplet (b, c, ν) under some probability measure $\mathcal{P}$. Then the following two condition are equivalent:*

1. *There is a probability measure $\mathcal{Q} \overset{loc}{\sim} \mathcal{P}$ such that L is a $\mathcal{Q}$–Lévy process with triplet (b', c', ν').*

2. *All of the following condition hold:*

   (i) $\nu'(dz) = k(z)\nu(dz)$ *for some Borel function* $k : \mathbb{R} \to (0, \infty)$*;*

   (ii) $b' = b + \int h(x)\left(k(x) - 1\right)\nu(dz) + \sqrt{c}\beta$ *for some* $\beta \in \mathbb{R}$*;*

   (iii) $\int \left(1 - \sqrt{k(z)}\right)^2 \nu(dz) < \infty$

A detailed proof of 1. ⇒ 2. can be found in [68, Theorem IV.4.39c.], the other implication in [104].

From this result we can see that there are infinitely many equivalent martingale measure which preserve the structure of a Lévy process. In financial application this would mean that is not more possible to define an unique arbitrage free price and that a market is not complete, as it has been shown in the previous chapter. In this thesis we are not interested in the study of the equivalent martingale measure in this kind of market, but only in the pricing problem, therefore in what follows, we suppose we are given an equivalent martingale measure and we derive our results in that setting.
The problem of selecting a particular equivalent martingale measure among the infinitely many is faced in the work by Chan [33].

2.3 Examples of Lévy processes.

In the following we will present some examples of the most used Lévy process, with particular attention to the one we will use in the numerical application of Chapter 8.

Example 2.3.1 [*Variance Gamma Process*] The variance gamma process is a Brownian motion with drift in which the calendaristic time has been turned in the "business" time modelled by a gamma process $\gamma(t;\upsilon)$ with mean rate unity and variance υ

$$L_{VG}(t;\sigma,\upsilon,\theta) = \theta\gamma(t;\upsilon) + \sigma W_{\gamma(t;\upsilon)}.$$

From the density of the gamma process, given by

$$f_{\gamma(t;\upsilon)}(x) = \frac{x^{t/\upsilon-1}e^{-x/\upsilon}}{\upsilon^{t/\upsilon}\Gamma(t/\upsilon)},$$

we can derive the characteristic function of the gamma process

$$\chi_{\gamma(t;\upsilon)}(u) = \mathbb{E}[e^{iu\gamma(t,\upsilon)}] = \Big(\frac{1}{1-i\upsilon u}\Big)^{t/\upsilon},$$

and the characteristic function of the variance gamma process

$$\chi_{L_{VG}(t;\sigma,\upsilon,\theta)}(u) = \mathbb{E}[e^{iuL_{VG}(t;\sigma,\upsilon,\theta)}] = \Big(\frac{1}{1-i\theta\upsilon u+\sigma^2\upsilon u^2/2}\Big)^{t/\upsilon}.$$

One can interpret the variance gamma process as the difference of two independent gamma process: if we indicate

$$\eta_p = \sqrt{\frac{\theta^2\upsilon^2}{4}+\frac{\sigma^2\upsilon}{2}}+\frac{\theta\upsilon}{2}, \qquad \eta_n = \sqrt{\frac{\theta^2\upsilon^2}{4}+\frac{\sigma^2\upsilon}{2}}-\frac{\theta\upsilon}{2},$$

then we have

$$\frac{1}{1-i\theta\upsilon u+\sigma^2\upsilon u^2/2}=\frac{1}{1-i\eta_p u}\cdot\frac{1}{1+i\eta_n u},$$

therefore

$$L_{VG}(t;\sigma,\upsilon,\theta)\overset{law}{=}\gamma_p(t;\eta_p/\upsilon,\eta_p^2\upsilon)-\gamma_n(t;\eta_n/\upsilon,\eta_n^2\upsilon).$$

This representation, together with the representation of the Lévy density for the gamma process lead to the following explicit form for the Lévy density for the VG process:

$$k_{VG}(x)=\begin{cases}\dfrac{1}{\upsilon}\cdot\dfrac{e^{-\frac{|x|}{\eta_n}}}{|x|} & \text{if } x<0,\\[2ex] \dfrac{1}{\upsilon}\cdot\dfrac{e^{-\frac{|x|}{\eta_p}}}{|x|} & \text{if } x<0.\end{cases}$$

■

Example 2.3.2 [*The CGMY Process*] The CGMY process [31] is a generalization of the previous VG process by adding a new parameter in the Lévy density that allows the resulting Lévy process to have both finite or infinite activity and finite or infinite variation. The Lévy density of this process is given by

$$K_{CGMY}(z)=\begin{cases}C\dfrac{e^{-G|z|}}{|z|^{1+Y}}, & z_i<0,\ i=1,2,\\[3ex] C\dfrac{e^{-M|z|}}{|z|^{1+Y}}, & z_i>0,\ i=1,2,\end{cases}\tag{2.3.1}$$

with $C>0$, $G,M\geq 0$ and $Y<2$. The case $Y=0$ is the special case of the VG process. The characteristic function corresponding to the density (2.3.1) is

$$\chi_{L_{CGMY}(t;C,G,M,Y)}(u)=\exp\Big(tC\Gamma(-Y)\{(M-iu)^Y-M^Y+(G+iu)^Y-G^Y\}\Big).$$

■

2.4 A IPDE approach to Lévy markets.

In this section we expose how to derive the pricing equation for the more general multidimensional Lévy markets. The pricing problem will be given as a nonlinear backward stochastic differential equation and we explain how to rephrase it as a deterministic problem.

More precise results about existence and uniqueness of solution to this kind of problem are given in Chapter 4.

Let us consider a market whose money market account evolves accordingly to (1.5.1), while the prices of the assets evolve as

$$S_t^i = e^{L_t^i}, \quad , i = 1, \dots, \text{N},$$

where $L_t = \{L_t^i\}_t$ is a N-dimensional Lévy process whose Lévy–Khintchine decomposition is

$$\begin{aligned} dL_t^i &= \mu^i(t)dt + \sum_{j=1}^{\text{M}} \sigma_j^i(t)dW_t^j + \sum_{j=1}^{\text{N}} \int_{|z|<1} \eta_j^i(t) z^j \tilde{N}^j(dt, dz) \\ &+ \sum_{j=1}^{\text{N}} \int_{|z|\geq 1} \eta_j^i(t) z^j N^j(dt, dz), \qquad i = 1, \dots, \text{N}, \end{aligned}$$

where $z \in E = \mathbb{R}^{\text{M}} - \{0\}$, $\sigma(t) \in \mathbb{R}^{\text{N}\times\text{M}}$ matrix, $1 \leq \text{M} \leq \text{N}$, $\sigma(t)\sigma^T(t) \geq 0$, $\eta(t) \in \mathbb{R}^{\text{N}\times\text{N}}$. Here W_t is a M-dimensional standard Brownian motion, $1 \leq \text{M} \leq \text{N}$, $\tilde{N}$ is the compensated martingale measure of a N-dimensional Poisson random measure N defined on $\mathbb{R}^+ \times E$ with compensator $\lambda(dt, dz) = dt \times \nu(dz)$, $\nu(dz)$ is its Lévy intensity, and $\nu : \mathcal{B}(E) \to \mathbb{R}^{\text{N}}$,

$$\nu(dz) = (\nu^1(dz), \nu^2(dz), \dots, \nu^{\text{N}}(dz)),$$

is the N-dimensional Lévy measure.
We assume the coefficients b and σ being globally bounded and globally Lipschitz continuous and that the Lévy measure satisfies the following

$$\int_E (1 \wedge |z|^2)\nu^j(dz) < \infty, \quad j = 1, \dots, \text{N},$$

$$\int_{|z|\geq 1} |e^z - 1|\nu^j(dz) < \infty, \quad j = 1, \dots, \text{N}.$$

Applying the generalized Ito's formula [68, 29] we can derive the dynamic of the prices S_t

$$\begin{aligned} dS_t^i &= S_{t^-}^i \Big\{ \Big[\mu^i + \frac{1}{2} \sum_{j=1}^{\text{N}} (\sigma_j^i)^2 + \sum_{j=1}^{\text{N}} \int_E \Big(e^{\eta_j^i z^j} - 1 - \eta_j^i z^j \mathbf{1}_{|z|<1} \Big) \Big] dt \\ &+ \sum_{j=1}^{\text{N}} \sigma_j^i dW_t^j + \sum_{j=1}^{\text{N}} \int_E \Big(e^{\eta_j^i z^j} - 1 \Big) \tilde{N}^j(dt, dz) \Big\}, \quad i = 1 \dots, \text{N}. \end{aligned}$$

In the calculation we have used that the component of the Poisson process are independent, therefore the discontinuous part of the process of the prices is given by the sum of the component owed to the different jump component. The previous equation can be written in a more compact way:

$$dS_t = \mathbf{S}_t\Big[b(S_t,t)dt + \sigma(S_t,t)dW_t + \int_E \beta(S_{t^-},t,z)\tilde{N}(dt,dz)\Big],$$

where $\mathbf{S}_t = \operatorname{diag}(S_t^1,\dots,S_t^{\mathrm{N}})$ and the coefficients being given, omitting the time dependence, by

$$\begin{aligned}
b_i(S_t) &= \mu^i + \frac{1}{2}\sum_{j=1}^{\mathrm{M}}(\sigma_j^i)^2 + \sum_{j=1}^{\mathrm{N}}\int_E \Big(e^{\eta_j^i z^j} - 1 - \eta_j^i z^j \mathbf{1}_{|z|<1}\Big)\nu^j(dz),\\
\sigma(S_t) &= \sigma,\\
\beta_j^i(S_t,z) &= e^{\eta_j^i z^j} - 1.
\end{aligned}$$

We assume to be in an equivalent martingale setting, otherwise, if a change of measure is needed, applying Proposition 2.2.5, the dynamics of the price will be changed only in the drift coefficient:

$$\begin{aligned}
(b^{\mathcal{Q}})^i(S_t,t) &= \mu^i + \frac{1}{2}\sum_{j=1}^{\mathrm{M}}(\sigma_j^i)^2(t) + \sum_{j=1}^{\mathrm{M}}\sigma_j^i\alpha^j\\
&\quad + \sum_{j=1}^{\mathrm{N}}\int_E \Big[e^{\eta_j^i(t)z^j} - 1 - \eta_j^i(t)z_j^i\mathbf{1}_{|z|<1}\Big]k^j(z)\nu^j(dz).
\end{aligned}$$

2.4.1 European derivatives in a Lévy market.

Let us now indicate with $U(S,t)$ the price of a derivative (G,T); from the general theory we know that this price is given by the following relation

$$u(S,t) = e^{-r(T-t)}\mathbb{E}\Big[G(S_T)\Big|\mathcal{F}_t\Big];$$

at the same time we can derive the dynamic of the price of the derivative by means of the generalized Ito's formula

$$\begin{aligned}
dU(S,t) &= \Big\{\partial_t U + \mathcal{D}U\mathbf{S}_t\Big(\mu + \frac{1}{2}\sigma\sigma^T\Big) + \frac{1}{2}(\mathbf{S}_t\sigma)\mathcal{D}^2U(\mathbf{S}_t\sigma)^T\\
&\quad + \int_E [U(S_te^{\eta z},t) - U(S_t,t) - \mathcal{D}U\mathbf{S}_t\eta z\mathbf{1}_{|z|<1}]\nu(dz)\Big\}dt\\
&\quad + \mathcal{D}U\mathbf{S}_t\sigma dW_t + \int_E [U(S_te^{\eta z},t) - U(S_t,t)]\tilde{N}(dt,dz).
\end{aligned}$$

By economical consideration, the discounted price of the derivative under the equivalent martingale measure is a martingale; if $\hat{U}(S,t) = e^{-r(T-t)}U(S,t)$ denotes the discounted price, its dynamic is given by

$$\begin{aligned} d\hat{U}(S,t) &= \Big\{\partial_t\hat{U} + \mathcal{D}\hat{U}\mathbf{S}_t\Big(\mu + \frac{1}{2}\sigma\sigma^T\Big) - r\hat{U} + \frac{1}{2}(\mathbf{S}_t\sigma)\mathcal{D}^2\hat{U}(\mathbf{S}_t\sigma)^T \\ &\quad + \int_E [\hat{U}(S_t e^{\eta z},t) - \hat{U}(S_t,t) - \mathcal{D}\hat{U}\mathbf{S}_t\eta z\mathbf{1}_{|z|<1}]\nu(dz)\Big\}dt \\ &\quad + \mathcal{D}\hat{U}\mathbf{S}_t\sigma dW_t + \int_E [\hat{U}(S_t e^{\eta z},t) - \hat{U}(S_t,t)]\tilde{N}(dt,dz). \end{aligned}$$

In order to be a martingale, $\hat{U}$ has to solve the following problem

$$\begin{cases} -\partial_t\hat{U} - \mathcal{D}\hat{U}\mathbf{S}_t\Big(\mu + \frac{1}{2}\sigma\sigma^T\Big) + r\hat{U} - \frac{1}{2}(\mathbf{S}_t\sigma)\mathcal{D}^2\hat{U}(\mathbf{S}_t\sigma)^T & \\ \quad - \int_E [\hat{U}(S_t e^{\eta z},t) - \hat{U}(S_t,t) - \mathcal{D}\hat{U}\mathbf{S}_t\eta z\mathbf{1}_{|z|<1}]\nu(dz) = 0, & \\ \hat{U}(S_T,T) = G(S_T). & \end{cases}$$

The solution of this linear integro–partial differential problem represents the price of a derivative in a Lévy market with small investor, as we have assumed the interest rate to be constant. More general market, such as the large investor model, can be described in a similar way. A more detailed discussion shall be given in Chapter 5.

2.4.2 American derivatives in a Lévy market.

Beside the European pricing problem, in Chapter 1 we have presented also the problem of finding the price for American derivatives. We have seen that both in the classical Black and Scholes setting and in the general jump–diffusion one, this problem is connected to an obstacle problem.
In the Lévy model under study we obtain the same result for an integro–partial differential equation.

Let us assume that $U(S,t)$ represents the price of an American derivative (G,T); the optimal stopping time for this problem is given by

$$\tau^* = \inf\{t \in [0,T] : U(S_t,t) = G(S_T)\}.$$

The price of the derivative is given by the following relation

$$U(S,t) = \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\Big[e^{-r(\tau-t)}G(S_T)\Big|\mathcal{F}_t\Big],$$

where $\mathcal{T}_{t,T}$ is the set of all stopping time between t and T. It has been proved (see [19, 100, 101]) that under suitable assumption on the coefficient of the dynamic of S, this optimal stopping problem is related to the following obstacle problem:

$$\begin{cases} \min\{-\partial_t \hat{U} - \mathcal{D}\hat{U}\mathbf{S}_t\Big(\mu + \frac{1}{2}\sigma\sigma^T\Big) + r\hat{U} - \frac{1}{2}(\mathbf{S}_t\sigma)\mathcal{D}^2\hat{U}(\mathbf{S}_t\sigma)^T & \\ \quad - \int_E [\hat{U}(S_t e^{\eta z}, t) - \hat{U}(S_t, t) - \mathcal{D}\hat{U}\mathbf{S}_t\eta z \mathbf{1}_{|z|<1}]\nu(dz), \hat{U} - G\} = 0, & \\ \hat{U}(S_T, T) = G(S_T). & \end{cases}$$

For a detailed discussion of that topic and for general features of the Lévy market we refer to Chapter 5.

Part II

Analytical results: Viscosity solutions for nonlinear integro–differential equations

Through Part 1, the relation between option pricing in incomplete markets and nonlinear possibly degenerate parabolic equations has been raised. This part is devoted to the analytical study of the equations arising from the arbitrage pricing in incomplete markets, especially of the integro–differential degenerate parabolic equations arising in the jump–diffusion case:

$$-\partial_t u + H(x,t,u,\mathbf{x}\mathcal{D}u,\mathbf{x}\mathcal{D}^2u\mathbf{x}^T,\mathcal{J}u,\mathcal{I}u) = 0, \qquad (2.4.1)$$

where $\mathcal{J}u$ and $\mathcal{I}u$ are the following integro–differential operators:

$$\mathcal{J}u(x,t) = \int_{\mathbb{R}^M-\{0\}} [u(x+\mathbf{x}\beta(x,t,z),t) - u(x,t) - \mathbf{x}\beta(x,t,z)\cdot\mathcal{D}u(x,t)]\, m_{x,t}(dz),$$

$$\mathcal{I}u(x,t) = \int_{\mathbb{R}^M-\{0\}} [u(x+\mathbf{x}\beta(x,t,z),t) - u(x,t)]\, \gamma(x,t,z)m_{x,t}(dz).$$

H is a continuous function of its arguments and $m_{x,t}(dz)$ is the jump measure and could depend on the point (x,t).

The viscosity solution approach seems to be the only instrument which allows to obtain general results for the pricing problem in incomplete markets with general Lévy processes.
The theory of viscosity solutions was first developed by Crandall and Lions [35] to study partial differential equations of the form

$$\partial_t u + F(x,t,u,\mathcal{D}u) = 0,$$

with $F \in \mathcal{C}\big(\mathbb{R}^N \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \times; \mathbb{R}\big)$. The stationary operator $F(\ ,t,\ )$ is possibly degenerate elliptic and fully nonlinear for every fixed t.
The most appealing property of this theory is that merely continuous functions can be proved to be solutions of nonlinear degenerate parabolic equations, in a quite general uniqueness framework. This is possible because the theory of viscosity solutions requires only L^∞ a priory estimates to obtain stability results and therefore there is great flexibility in passing to limits in various settings. In the American derivative pricing problem this property is crucial because it allows to deal with the obstacle problem by means of approximation given by suitable Cauchy problems.

Amadori [3, 2, 4] extended previous results to integro–partial differential problems allowing nonlinear dependence on the integral operator $\mathcal{I}u$. In particular, well posedness allowing exponential growth of the data for a general class of purely differential and integro–differential operator depending on $\mathcal{I}u$, including the ones of

financial interest, was proved, both for Cauchy and obstacle problem.
In Chapter 4 this theory shall be extended to a class of integro–differential equation (2.4.1), when the measure ν is a given Radon measure on $E = \mathbb{R}^{\mathrm{M}} - \{0\}$ (the so-called Lévy measure), which may possess a second order singularity at the origin, while β is a given function of $(x,t,z) \in \Pi \times [0,T) \times E$ with values in $(-1,+\infty)^{\mathrm{N}}$, and γ is a given function of $(x,t,z) \in \Pi \times [0,T) \times E$ with values in $(0,+\infty)$.

This part is organized as follows. In Chapter 3 we give an overview of the classical results by in the pure differential setting

$$\partial_t u + F(x,t,u,\mathcal{D}u,\mathcal{D}^2 u) = 0,$$

with $F \in \mathcal{C}(\mathbb{R}^{\mathrm{N}} \times [0,T] \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}} \times \mathcal{S}_{\mathrm{N}}; \mathbb{R})$. An existence and uniqueness result in the class of continuous function with exponential growth is given, with some financial examples.

In Chapter 4 we extend the theory of viscosity solution equations to nonlinear integro–partial differential problem with general Lévy measure.
Existence and uniqueness in the class of continuous function have been established by Amadori [2, 3] for problem of the form

$$\partial_t u + F(x,t,u,\mathcal{I}u,\mathcal{D}u,\mathcal{D}^2 u) = 0,$$

with $F \in \mathcal{C}(\mathbb{R}^{\mathrm{N}} \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}} \times \mathcal{S}_{\mathrm{N}}; \mathbb{R})$ and the integral term in a more general form

$$\mathcal{I}u(x,t) = \int_{\mathbb{R}^{\mathrm{N}}} M(u(x+z,t),u(x,t))\mu_{x,t}(dz),$$

$\mu_{x,t}$ is a positive Radon measure on $\mathbb{R}^{\mathrm{N}}$ for all (x,t) and $M \in \mathcal{C}(\mathbb{R}^2;\mathbb{R})$ is nondecreasing in its first argument. Particular attention is posed on the regularity of the solution as in financial application the typical initial datum is locally Lipschitz continuous, but not differentiable; moreover it has some kind of second order regularity related with convexity property.
The regularity of the price of the derivative with respect the prices of the assets X is of great interest being connected with the arbitrage strategy and the completeness of the market. It is possible to show that well posedness in the class $W^{1,\infty}_{\mathrm{pol}}(\mathbb{R}^{\mathrm{N}})$ is equivalent to the weak completeness of the market; the trader expects that, in order to hedge from the risk of his derivative, he needs to place a quantity of money which grows up exactly in the same way of the expected payoff, as the prices X increase. From relations (1.3.3) or (1.4.7) it is clear that the gradient of the solution is connected with the hedging portfolio, therefore the solutions must belong to $L^\infty\big(0,T;W^{1,\infty}_n(\Pi)\big)$ when the payoff is in $W^{1,\infty}_n(\Pi)$.

We shall briefly recall this theory as a starting point for the extension to more general unbounded Lévy measure, satisfying the following integrability condition

$$\int_E (1 \wedge |z|^2)\nu(dz) < \infty,$$

where $E = \mathbb{R}^{\mathrm{M}} - \{0\}$. We shall prove a comparison principle for unbounded semi-continuous viscosity sub- and supersolutions without assigning boundary data on $\partial\Pi$. This is possible because of the special structure of our problem:

$$\mathbf{x}Du = (x_1 u_{x_1}, \ldots, x_{\mathrm{N}} u_{x_{\mathrm{N}}}), \qquad \mathbf{x}D^2u\mathbf{x}^T = \begin{pmatrix} x_1^2 u_{x_1x_1} & \cdots & x_1 x_{\mathrm{N}} u_{x_1 x_{\mathrm{N}}} \\ \vdots & & \vdots \\ x_1 x_{\mathrm{N}} u_{x_1 x_{\mathrm{N}}} & \cdots & x_{\mathrm{N}}^2 u_{x_{\mathrm{N}} x_{\mathrm{N}}} \end{pmatrix}$$

occur respectively in the gradient and matrix slots of (2.4.1), while

$$\mathbf{x}\beta = (x_1\beta_1, \ldots, x_{\mathrm{N}}\beta_{\mathrm{N}})$$

occurs in the integral operators $\mathcal{J}u$ and $\mathcal{I}u$. This structure is typical of financial applications when a general geometric Lévy processes is the underlying stochastic processes for the assets dynamics. We shall prove a comparison result for a large class of integro–partial differential equations, dealing with an unbounded set with non smooth boundary and allowing solutions to blow up at $\partial\Pi$. It has to be mentioned that, due to the structure of β and the assumptions on ν, the boundary has to be studied with particular attention, because it has a part behaving as any interior point of Π, and another where the solutions possibly blows up.
We conclude this part with Chapter 5 where we present some applications of the previous results to problem of particular interest in Finance, such as the pricing of derivatives and the portfolio optimization problem.
All the results presented in Chapter 4 and 5 can be found in the paper [5].

Notations

We introduce some notations that shall be in use through all this Part.

$$\Pi = (0,\infty)^{\mathrm{N}}, \qquad \Pi_T = \Pi \times [0,T),$$
$$\overline{\Pi} = [0,\infty)^{\mathrm{N}}, \qquad \overline{\Pi}_T = \overline{\Pi} \times [0,T],$$
$$\Gamma = \{x \in \partial\Pi \,:\, x_i > 0 \text{ as } i = 1+\mathrm{N}', \cdots, \mathrm{N}\}, \qquad \tilde{\Pi} = \Pi \cup \Gamma,$$
$$\tilde{\Pi}_T = \tilde{\Pi} \times [0,T), \qquad \tilde{\Pi}_T^\star = \tilde{\Pi} \times [0,T],$$
$$\mathcal{Q}(r) = [0,r]^{\mathrm{N}'} \times [1/r, r]^{\mathrm{N}-\mathrm{N}'}.$$

Here N' stands for an integer between 0 and N that shall be selected later on; $\mathbf{x} =$ diag $(x_1, \cdots, x_{\text{N}})$.
For any $R > 0$ we indicate

$$B(0,R) = \{z \in \mathbb{R}^{\text{N}} : \|z\| < R\},$$

and

$$B^+(R) = \Pi \cap B(0,R), \text{ for any } R > 0.$$

We set $\mathcal{S}_{\text{N}}$ for the set of symmetrical $\text{N}\times\text{N}$ matrices with real coefficients, equipped with the usual partial order

$$\mathcal{X} \leq \mathcal{Y} \quad \text{if} \quad \mathcal{X}\xi \cdot \xi \leq \mathcal{Y}\xi \cdot \xi \text{ for all } \xi \in \mathbb{R}^{\text{N}}$$

and with the norm

$$\|\mathcal{X}\| = \sup\Big\{ |\mathcal{X}\xi \cdot \xi| : \xi \in \mathbb{R}^{\text{N}}, \|\xi\| = 1 \Big\}.$$

Moreover, $I_{\text{N}} = \text{diag}\,(1,\ldots,1)$ stands for the $\text{N}\times\text{N}$ identity matrix and $p \otimes q = \big(p_i q_j\big)_{i,j=1,\ldots,\text{N}}$ for the tensor product of two vectors of $\mathbb{R}^{\text{N}}$.

A **rate of growth** is a nonnegative function $g \in \mathcal{C}([0,\infty))$ which is sub-additive

$$g(s+s') \leq g(s) + g(s').$$

If moreover $g(0) = 0$, it is called a **modulus of continuity**.

We denote by $\mathcal{C}^{2,1}(\mathbb{R}^{\text{N}} \times [0,T))$ the set of functions which are twice continuously differentiable with respect to $x \in \mathbb{R}^{\text{N}}$ and once with respect to $t \in [0,T)$.

Let us now consider an arbitrary subset $\mathcal{O}$ of an Euclidean space $\mathbb{R}^{\text{N}}$.

Definition 2.4.1 *A real function f is* **upper semicontinuous** *(resp.,* **lower semicontinuous***) at $z \in \mathcal{O}$ if one of the following equivalent items is fulfilled:*

1. $$u(z') \leq u(z) + o(\|z - z'\|) \qquad \textit{(resp., } \geq\textit{)}$$
   as $\mathcal{O} \ni z' \to z$. Here, o stands for a modulus of continuity.

2. $$\limsup_{\mathcal{O}\ni z'\to z} u(z') \leq u(z) \qquad \textit{(resp., } \liminf_{\mathcal{O}\ni z'\to z} u(z') \geq u(z)\textit{).}$$

A trivial property of upper/lower semicontinuous functions is that they admit maximum (respectively, minimum) on every compact subset of $\mathcal{O}$ in which they are upper (respectively, lower) bounded.

Definition 2.4.2 *Given an arbitrary function $u : \mathcal{O} \to \mathbb{R}$, we define the* **upper semicontinuous envelope** u^* *and the* **lower semicontinuous envelope** u_* *of u as pointwise infimum (respectively, supremum) of continuous functions staying above (respectively, below) u, namely*

$$\begin{aligned} u^*(z) &= \inf\Big\{v(z) : v \in \mathcal{C}(\mathcal{O}),\, v \geq u \text{ near } z\Big\}, \\ u_*(z) &= \sup\Big\{v(z) : v \in \mathcal{C}(\mathcal{O}),\, v \leq u \text{ near } z\Big\}. \end{aligned}$$

Upper/lower semicontinuous envelopes can be easily characterized as follows

$$\begin{aligned} u^*(z) &= \limsup_{\mathcal{O}\ni z' \to z} u(z'), \\ u_*(z) &= \liminf_{\mathcal{O}\ni z' \to z} u(z'). \end{aligned}$$

To get well posedness for Cauchy problems and for the related obstacle problems, it is needed to fix a range of growth at infinity for solutions. So, for each rate of growth g, we set

$$\begin{aligned} L^\infty_g(\mathbb{R}^N) &= \Big\{f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N) : \text{ there is } B \geq 0 \text{ s.t. } |f(x)| \leq Bg(\|x\|) \text{ a.e. } \Big\} \\ &= \Big\{f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N) : \text{ there is } B \geq 0 \text{ s.t. for all } R > 0 \; \|f;\|_{L^\infty[-R,R]^N} \leq Bg(\|R\|) \Big\}, \end{aligned}$$

$$\begin{aligned} W^{k,\infty}_g(\mathbb{R}^N) &= \Big\{f \in W^{k,\infty}_{\mathrm{loc}}(\mathbb{R}^N): \text{ there is } B \geq 0 \text{ s.t. for all } R > 0 \\ &\qquad |D^\alpha f(x)| \leq Bg(\|x\|) \text{ for all } |\alpha| \leq k, \text{ a.e.} \Big\} \\ &= \Big\{f \in W^{k,\infty}_{\mathrm{loc}}(\mathbb{R}^N) : \text{ there is } B \geq 0 \text{ s.t. for all } R > 0 \; \|f\|_{W^{k,\infty}[-R,R]^N} \leq Bg(\|R\|) \Big\}. \end{aligned}$$

Theory of viscosity solution requires to work with sub/supersolution, therefore one sided estimates are requested. We introduce subsets of L^∞_g that fits this request:

$$\begin{aligned} L^\infty_{g,+}(\mathbb{R}^N) &= \Big\{ f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N) : \text{ there is } B \geq 0 \text{ s.t. } f(x) \leq Bg(\|x\|) \text{ a.e. } \Big\}, \\ L^\infty_{g,-}(\mathbb{R}^N) &= \Big\{ f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N) : \text{ there is } B \geq 0 \text{ s.t. } f(x) \geq -Bg(\|x\|) \text{ a.e. } \Big\}. \end{aligned}$$

If $g(r) = 1 + r$, the subscript "g" can be replaced with "lin", standing for linear growth at infinity.
If $g_n(r) = 1 + r^n$, the subscript "g_n" stands for polynomial growth, with degree n, at infinity. In the following the subscript "pol" shall denote the union over $n \geq 0$ of the correspondent sets subscripted by $1 + r^n$.
If $\tilde{g}_n(r) = e^{nr}$, the subscript "$\tilde{g}_n$" corresponds to an exponential growth, with weight

n. In the following the subscript "exp" shall stand for the union over $n \geq 0$ of the correspondent sets subscripted by e^{nr}.

In many financial applications, problem stated in Π can be turned in problems in the whole $\mathbb{R}^{\mathrm{N}}$ by a change of variables $x_i = \log X^i$, and any final datum problem can be turned in an initial datum problem by a change of variable $\tau = T - t$; for what concerns the rate of growth, the exponential function maps $L^\infty_{\text{pol}}(\Pi)$ into $L^\infty_{\text{exp}}(\mathbb{R}^{\mathrm{N}})$; hence, in order to deal with the option pricing problems described in Part I, we need of a theory in the class of exponential growth.

For all $n > 0$, we define the following function on $\tilde{\Pi}$

$$h_n(x) = \sum_{i=1}^{\mathrm{N}} x_i^n + \sum_{i=1+\mathrm{N}'}^{\mathrm{N}} x_i^{-n}.$$

We also introduce the set

$$\mathcal{P}_n\big(\Pi\big) = \left\{ u : \Pi \to \mathbb{R} \ : \ \frac{|u(x)|}{1 + h_n(x)} \text{ is bounded} \right\},$$

together with the norm $\|u\|_n = \sup_{x \in \Pi} \dfrac{|u(x)|}{1 + h_n(x)}$.

We can note that in the case of a market described by a Brownian motion or by a Brownian motion and a Poisson process the set $\mathcal{P}_n(\Pi)$ reduces to $L^\infty_{\text{pol}}(\Pi)$ as in that case $\mathrm{N}' = \mathrm{N}$.

With a little abuse of notation, we set

$$L^\infty\big(0, T; \mathcal{P}_n(\Pi)\big) = \left\{ u : \tilde{\Pi}^\star_T \to \mathbb{R} \ : \ \frac{|u(x,t)|}{1 + h_n(x)} \text{ is bounded} \right\}.$$

Chapter 3

Viscosity solutions for second order partial differential equations.

In this chapter we shall recall some backgrounds on the general theory of viscosity solutions to differential parabolic equations, possibly degenerate. It is an introduction to the more general theory for nonlinear integro–differential parabolic problem with unbounded Lévy measures presented in the next chapter; beside it contains some extensions to problems coming from option pricing.

The problem we are going to study are the following

$$\partial_t u + F(x,t,u,\mathcal{D}u,\mathcal{D}^2 u) = 0, \qquad (x,t) \in \mathbb{R}^{\mathrm{N}} \times (0,T) \tag{3.0.1}$$

$$u(x,0) = u_0(x) \qquad u_0 \in \mathcal{C}(\mathbb{R}^{\mathrm{N}}), \tag{3.0.2}$$

which is related to the pricing of an European derivative with payoff $G(X_T) = u_0(X_T)$ and expiration date T, and

$$\min\{\partial_t u + F(x,t,u,\mathcal{D}u,\mathcal{D}^2 u), u - u^0\} = 0, \qquad (x,t) \in \mathbb{R}^{\mathrm{N}} \times (0,T) \tag{3.0.3}$$

$$u(x,0) = u^0(x,0) \qquad u^0 \in \mathcal{C}(\mathbb{R}^{\mathrm{N}} \times [0,T]), \tag{3.0.4}$$

which is related ro the pricing of an American derivative with payoff $G(X_T) = u^0(X_T,T)$ and maturity T.

Both these problems are related to a pricing problem in an incomplete market driven by a diffusion process, as we have explained in Chapter 1.
The first definition of viscosity solution was given by Crandall and Lions [35] for first order Hamilton–Jacobi equations; from that paper several other results for more general equation have been obtained. This theory applies to partial differential equations (3.0.1) with $F \in \mathcal{C}\big(\mathbb{R}^{\mathrm{N}} \times [0,T] \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}} \times \mathcal{S}_{\mathrm{N}}; \mathbb{R}\big)$.
Viscosity solution theory applies to problems of that kind provided that F satisfies the following properties:

F.1 *F is degenerate elliptic, namely*

$$F(x,t,u,p,\mathcal{X}) \geq F(x,t,u,p,\mathcal{Y}) \text{ whenever } \mathcal{X} \leq \mathcal{Y},$$

F.2 *F is quasi–monotone as a function of u, uniformly with respect to the other variables, namely*

$$F(x,t,u,p,\mathcal{X}) \geq F(x,t,v,p,\mathcal{X}) - \gamma(u-v) \text{ whenever } u \geq v.$$

Here $\gamma \in \mathcal{C}([0,\infty))$, $\gamma(0)=0$, $\gamma(u)>0$ as $u>0$ and

$$\int_0^{\varepsilon} \frac{du}{\gamma(u)} = \infty \qquad \text{for all } \varepsilon > 0.$$

An operator F of such kind satisfying **F.1** and **F.2** is referred to as a *Hamilton–Jacobi operator*. Assumption F.2, is also known as *Osgood's condition* and it is the parabolic version of the *properness condition* in the elliptic case, namely that

$$F(x,u,p,\mathcal{X}) \geq F(x,v,p,\mathcal{X}) \text{ whenever } u \geq v.$$

It plays the same role of the standard assumptions $\inf c > -\infty$ for linear parabolic equations of the form

$$\partial_t u + \mathcal{L}u = \partial_t u - \sum_{ij} a_{ij}\partial^2_{x_i x_j} u + \sum_i b_i \partial_{x_i} u + cu = 0.$$

In particular, it is strictly connected with order preserving of solutions.
Previous assumptions allow to select the kind of problem to which theory of viscosity solutions applies: **F.1** allows to consider completely degenerate equations, i.e. first order evolution equations. On the other hand, **F.2** excludes from the study a great part of hyperbolic first order equations (for example, Burgers' equation) and all convection–diffusion equations.

Theory of viscosity solutions in the pure differential setting provides well posedness results both of the Cauchy and of the obstacle problem, for continuous initial data which grow up at most linearly for large $\|x\|$. In Financial applications datas typically grows up exponentially for large $\|x\|$, therefore this theory has to be modified in a suitable way.

An overview of viscosity solutions results for second order partial differential equation can be found in the User's guide [34].

3.1 Purely differential problems.

In this section we shall give an overview of the theory of viscosity solutions for the Cauchy and obstacle problem; existence is proved for the first one by the Perron method and the second one by the penalization method. Uniqueness is obtained via comparison principle. In both cases an application to financial models in the large investor model is given. Detailed proof can be found in [34].

The theory of viscosity solutions has its main point in the definition of the so called *parabolic semijets* which allows a definition of $\partial_t u$, $\mathcal{D}u$ and $\mathcal{D}^2 u$ for merely continuous function. Two equivalent definitions can be given, one local and one global.

Definition 3.1.1 *We say that* $(\tau, p, \mathcal{X}) \in \mathbb{R} \times \mathbb{R}^{\mathrm{N}} \times \mathcal{S}_{\mathrm{N}}$ *belongs to* $\mathcal{P}^+u(x,t)$*, the* **parabolic superjet** *of* u *at the point* (x,t) *if*

$$\begin{aligned} u(y,s) \quad \leq \quad & u(x,t) + \tau(t-s) + p\cdot(x-y) + \frac{1}{2}\mathcal{X}(x-y)\cdot(x-y) \\ & + o(|t-s| + \|x-y\|^2), \end{aligned}$$

as $\mathbb{R}^{\mathrm{N}} \times [0,T) \ni (y,s) \to (x,t)$*. Here* o *stands for a modulus of continuity.*
We say that $(\tau, p, \mathcal{X}) \in \mathbb{R} \times \mathbb{R}^{\mathrm{N}} \times \mathcal{S}_{\mathrm{N}}$ *belongs to* $\mathcal{P}^-u(x,t)$*, the* **parabolic subjet** *of* u *at the point* $(x.t)$ *if* $-(\tau,p,\mathcal{X}) \in \mathcal{P}^*(-u)(x,t)$.
We define the **closed parabolic semijets** *the set* $\overline{\mathcal{P}^+}$ *and* $\overline{\mathcal{P}^-}$ *given as*

$$\begin{aligned} \overline{\mathcal{P}^{\pm}}u(x,t) = \Big\{(\tau,p,\mathcal{X}) : \quad & \text{there are } (x_n,t_n) \to (x,t) \text{ and} \\ & (\tau_n,p_n,\mathcal{X}_n) \in \mathcal{P}^{\pm}u(x_n,t_n), (\tau_n,p_n,\mathcal{X}_n) \to (\tau,p,\mathcal{X})\Big\}. \end{aligned}$$

Remark 3.1.2 This definition is equivalent to requiring that there exists a function $\phi \in \mathcal{C}^{2,1}(\mathbb{R}^{\mathrm{N}} \times [0,T))$ such that $u - \phi$ has a local maximum (resp. minimum) at (x,t) and $(\tau,p,\mathcal{X}) = (\partial_t\phi, \mathcal{D}\phi, \mathcal{D}^2\phi)(x,t)$.
It can be noticed that replacing $\phi(y,s)$ by

$$\begin{aligned} &\phi(y,s) + (t-s)^2 + \|x-y\|^4 + [u(x,t) - \phi(x,t)], \\ &\phi(y,s) - (t-s)^2 - \|x-y\|^4 - [u(x,t) - \phi(x,t)], \end{aligned}$$

the local maximum (resp. minimum) is a *strict* maximum (resp. minimum). ■

By means of this remark we can give the global definition of the semijets.

Definition 3.1.3 *We call a* **good test function** *for the superjet (resp. subjet) of* u *at* (x,t) *any function* $\phi \in \mathcal{C}^{1,2}(\mathbb{R}^{\mathrm{N}} \times [0,T))$ *such that* $u - \phi$ *has a global strict maximum* *(resp. minimum) at (x,t) with $u(x,t) = \phi(x,t)$. We define $(\tau, p, \mathcal{X}) \in \mathcal{P}^{\pm}u(x,t)$ if there exists a good test function ϕ for the superjet (resp. subjet) of u at (x,t) such that*

$$(\tau, p, \mathcal{X}) = (\partial_t \phi, \mathcal{D}\phi, \mathcal{D}^2\phi)(x,t).$$

An important continuity property can be proved for the semijets.

Lemma 3.1.4 *Let u, $u_n \in \mathcal{USC}$ (resp. $\mathcal{LSC}$) and $(x,t) \in \mathbb{R}^N \times (0,T)$ such that $u_n \le u$ (resp. $u_n \ge u$) pointwise near (x,t) and there exists a sequence (y_n, s_n) converging to (x,t) such that*

$$\lim_{n\to\infty} u_n(y_n, s_n) = u(x,t).$$

Then for all $(\tau, p, \mathcal{X}) \in \mathcal{P}^+u(x,t)$ there exists a sequence (x_n, t_n) converging to (x,t) and $(\tau_n, p_n, \mathcal{X}_n) \in \mathcal{P}^+u_n(x_n, t_n)$ (resp. P^-) such that

$$\lim_{n\to\infty} u_n(x_n, t_n) = u(x,t), \tag{3.1.1}$$

$$\lim_{n\to\infty} (\tau_n, p_n, \mathcal{X}_n) = (\tau, p, \mathcal{X}). \tag{3.1.2}$$

Proof. We prove only the case of upper semicontinuous functions, the other one being the same.
Let $(\tau, p, \mathcal{X}) \in \mathcal{P}^+u(x,t)$. By Definition 3.1.3, there exists a good test function ϕ for the superjiet of u at (x,t) such that $(\tau, p, \mathcal{X}) = (\partial_t \phi, \mathcal{D}\phi, \mathcal{D}^2\phi)(x,t)$. By assumption, u_n are upper semicontinuous and locally upper bounded, therefore there exists a sequence (x_n, t_n) of maximum point for $u_n - \phi$ on the compact set

$$K = \{(y,s) : \|x-y\| \le 1 \text{ and } s \in [t/2, (T+t)/2]\}.$$

Extracting possibly a subsequence, (x_n, t_n) converges to a point $(x', t') \in K$. If $(x', t') \ne (x,t)$ then it would be

$$0 = \lim_{n\to\infty} (u_n - \phi)(y_n, s_n) \le \lim_{n\to\infty} (u_n - \phi)(x_n, t_n) \le (u - \phi)(x', t') < 0,$$

which contradicts the assumption. Therefore (x_n, t_n) converges to (x,t); as $u_n \le u$ pointiwise, relation (3.1.1) holds.
In particular $(x_n, t_n) \in \mathrm{Int}K$ are point of local maximum for $u_n - \phi$, therefore it follows from Remark 3.1.2 that $(\tau_n, p_n, \mathcal{X}_n) = (\partial_t \phi, \mathcal{D}\phi, \mathcal{D}^2\phi)(x_n, t_n) \in \mathcal{P}^+u_n(x_n, t_n)$. Being $\phi \in \mathcal{C}^{1,2}(\mathbb{R}^N \times [0,R))$ we get (3.1.2). ∎

Definition 3.1.5 *Given a function u and a point $(x,t) \in \mathbb{R}^N \times [0,T)$, we say that*

$$\partial_t u(x,t) + F(x,t,u,\mathcal{D}u,\mathcal{D}^2u) \le 0, \ (resp. \ge 0)$$

in **viscosity sense** *at (x,t) if one of the following equivalent condition is met:*

(i) *for all* $(\tau, p, \mathcal{X}) \in \mathcal{P}^+u(x,t)$ *(resp.* $\mathcal{P}^-u(x,t)$*)*

$$\tau + F(x,t,u(x,t),p,\mathcal{X}) \le 0, \ (resp. \ \ge 0)$$

(ii) *for each function* $\phi \in \mathcal{C}^{1,2}(\mathbb{R}^{\mathrm{N}} \times [0,T))$ *such that* $u - \phi$ *has a local maximum (resp. minimum) point at* (x,t)

$$\partial_t \phi(x,t) + F(x,t,u(x,t),\mathcal{D}\phi(x,t),\mathcal{D}^2\phi(x,t)) \le 0, \ (resp. \ \ge 0)$$

(iii) *for each good test function* $\phi \in \mathcal{C}^{1,2}(\mathbb{R}^{\mathrm{N}} \times [0,T))$ *touching from above (resp. below) the graph of* u *at* (x,t)

$$\partial_t \phi + F(x,t,\phi,\mathcal{D}\phi,\mathcal{D}^2\phi) \le 0, \ (resp. \ \ge 0)$$

holds in classical sense.

The continuity property of the semijets of semicontinuous function has important consequences when applied to differential inequality in viscosity sense.

Lemma 3.1.6 *Let* $F \in \mathcal{C}(\mathbb{R}^{\mathrm{N}} \times [0,T] \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}} \times \mathcal{S}_{\mathrm{N}}; \mathbb{R})$*, let* u*,* $u_n \in \mathcal{USC}$ *(resp.* $\mathcal{LSC}$*) and* $(x,t) \in \mathbb{R}^{\mathrm{N}} \times (0,T)$ *such that* $u_n \le u$ *(resp.* $u_n \ge u$*) pointwise near* (x,t)*. Let* (y_n, s_n) *be a sequence converging to* (x,t) *such that*

$$\lim_{n \to 0} u_n(y_n, s_n) = u(x,t).$$

Then for all $(\tau, p, \mathcal{X}) \in \mathcal{P}^+u(x,t)$ *there exists a sequence* (x_n, t_n) *and* $(\tau_n, p_n, \mathcal{X}_n) \in \mathcal{P}^+u_n(x_n,t_n)$ *(resp.* $\mathcal{P}^-$*) satisfying* (3.1.1) *and* (3.1.2) *such that*

$$\tau + F(x,t,u(x,t),p,\mathcal{X}) = \lim_{n \to \infty} \Big[\tau_n + F(x_n,t_n,u_n(x_n,t_n),p_n,\mathcal{X}_n)\Big].$$

Now that the semicontinuity property of the differential inequality is given, we can introduce *viscosity solutions.*

Definition 3.1.7 *A function* $u \in \mathcal{USC}$ *is a* **viscosity subsolution** *(resp.* $u \in \mathcal{LSC}$ *is a* **viscosity supersolution***) of equation* (3.0.1) *if*

$$\partial_t u + F(x,t,u,\mathcal{D}u,\mathcal{D}^2u) \le 0, \ (resp. \ \ge 0)$$

in viscosity sense for all $(x,t) \in \mathbb{R}^{\mathrm{N}} \times (0,T)$*. Moreover if*

$$u(x,0) \le u_0(x), \ (resp. \ \ge)$$

for all $x \in \mathbb{R}^{\mathrm{N}}$*, then* u *is a viscosity subsolution (resp. supersolution) of the Cauchy problem* (3.0.1)–(3.0.2).

An arbitrary function $u : \mathbb{R}^N \times [0,T) \to \mathbb{R}$ *is a* **weak viscosity solution** *for the Cauchy problem* (3.0.1)–(3.0.2) *if its upper/lower semicontinuous envelopes are sub/supersolutions, respectively.*
It is a **strong viscosity solution** *for the Cauchy problem* (3.0.1)–(3.0.2) *if it is both a sub/supersolution.*

By this definition it easily follows that a viscosity solution of the Cauchy problem (3.0.1)–(3.0.2) is continuous on $\mathbb{R}^N \times [0,T)$ and $u(x,0) = u_0(x)$ for all $x \in \mathbb{R}^N$.

Remark 3.1.8 If F is continuous in all its arguments, as it will always be in the following, the viscosity inequality in the definition can be replaced by

$$\tau + F(x,t,u,p,\mathcal{X}) \le 0,\ (resp.\ \ge 0) \qquad \text{for all } (\tau,p,\mathcal{X}) \in \overline{\mathcal{P}^+}u(x,t)\,(resp.\,\overline{\mathcal{P}^-}),$$

at all $(x,t) \in \mathbb{R}^N \times (0,T)$. ■

For what concerns an obstacle problem, the viscosity solutions are characterized by the following:

Definition 3.1.9 *A function* $u \in \mathcal{USC}$ *is a* **viscosity subsolution** *of the equation* (3.0.3) *if*

$$\partial_t u + F(x,t,u,\mathcal{D}u,\mathcal{D}^2u) \le 0,$$

in viscosity sense for all $(x,t) \in \mathbb{R}^N \times (0,T)$ *such that* $u(x,t) > u^0(x,0)$.
A function $u \in \mathcal{LSC}$ *is a* **viscosity supersolution** *of the equation* (3.0.3) *if*

$$\partial_t u + F(x,t,u,\mathcal{D}u,\mathcal{D}^2u) \ge 0,$$

in viscosity sense for all $(x,t) \in \mathbb{R}^N \times (0,T)$ *such that* $u(x,t) \ge u^0(x)$ *and* $\mathcal{P}^- u(x,t) = \emptyset$ *at all* $(x,t) \in \mathbb{R}^N \times (0,T)$ *such that* $u(x,t) < u^0(x,0)$.
If in addiction

$$u(x,0) \le u^0(x,0),\ (resp.\ \ge)$$

for all $x \in \mathbb{R}^N$*, then* u *is a* **viscosity subsolution** *(resp.* **supersolution***) of the obstacle problem* (3.0.3)–(3.0.4).
An arbitrary function $u : \mathbb{R}^N \times [0,T) \to \mathbb{R}$ *is a* **weak viscosity solution** *for the obstacle problem* (3.0.3)–(3.0.4) *if its upper/lower semicontinuous envelopes are sub/supersolution, respectively. It is a* **(strong) viscosity solution** *for the obstacle problem* (3.0.3)–(3.0.4) *if it is both a sub/supersolution.*

This definition is not symmetric, because a condition is imposed to any supersolution: a supersolution must satisfy $u \ge u^0$ in a set $\{(x,t) : \mathcal{P}^- u(x,t) = \emptyset\}$, which is dense in $\mathbb{R}^N \times (0,T)$ for any lower semicontinuous function. For more details, see [10]. From the definition it follows a continuity property for any viscosity solution. Moreover any viscosity solution satisfies $u \ge u^0$ on $\mathbb{R}^N \times (0,T)$ and the following:

(i) $\partial_t u + F(x, t, u, \mathcal{D}u, \mathcal{D}^2 u) = 0$ in viscosity sense for all $(x, t) \in \{u > u^0\}$;

(ii) $\partial_t u + F(x, t, u, \mathcal{D}u, \mathcal{D}^2 u) \geq 0$ in viscosity sense for all $(x, t) \in \{u = u^0\}$;

(iii) $\mathcal{P}^- u(x, t) = \emptyset$ if $(x, t) \in \{u < u^0\}$;

(iv) $u(x, 0) = u^0(x, 0)$ for all $x \in \mathbb{R}^{\mathrm{N}}$.

From this properties and the previous discussion it follows that $u \geq u^0$ pointwise.

To proceed in the proof of the main result in the theory of viscosity solutions to pure differential problems, we need some technical Lemmas. The first exploits a fundamental property for viscosity solutions to prove the comparison principle, the second extends the Osgood's condition in the viscosity setting, while the last one is a useful technical tool. For details we refer to [71, 34, 66].

Lemma 3.1.10 *[34, Theorem 8.3, page 48] Let* $u_i \in \mathcal{USC}$ *for* $i = 1, \ldots, \mathrm{P}$ *and* $\phi \in \mathcal{C}^{1,2}(\mathbb{R}^{\mathrm{NP}} \times [0, T])$*; suppose that*

(i) there exists a point $\hat{z} = (\hat{x}_1, \ldots, \hat{x}_{\mathrm{P}}, t) \in \mathbb{R}^{\mathrm{NP}} \times (0, T)$ *in which the function*

$$(x_1, \ldots, x_{\mathrm{P}}, t) \mapsto u_1(x_1, t) + \ldots + u_{\mathrm{P}}(x_{\mathrm{P}}, t) - \phi(x_1, \ldots, \mathbf{x}_{\mathrm{P}}, t),$$

attains its maximum;

(ii) there exists a constant $r > 0$ *such that for all* $\mathrm{M} > 0$

$$\max_{i=1\ldots,\mathrm{P}} \sup \Big\{ \tau_i \quad : \quad (\tau_i, p_i, \mathcal{X}_i) \in \mathcal{P}^+ u_i(x_i, t),\ \|x_i - \hat{x}_i\| + |t - \hat{t}| \leq r, \\ \text{and } |u_i(x_i, t)| + \|p_i\| + \|\mathcal{X}_i\| \leq \mathrm{M}. \Big\} < \infty.$$

Then, for each $\epsilon > 0$ *there are* P *real numbers* τ_i *and* P *matrices* $\mathcal{X}_i \in \mathcal{S}_{\mathrm{N}}$ *such that*

1. $\tau_1 + \ldots + \tau_{\mathrm{P}} = \partial_t \phi(\hat{z})$*;*

2. $-\Big(\dfrac{1}{\epsilon} + \|\mathcal{D}^2 \phi(\hat{z})\|\Big) I_{\mathrm{NP}} \leq \begin{pmatrix} \mathcal{X}_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \mathcal{X}_{\mathrm{P}} \end{pmatrix} \leq \mathcal{D}^2 \phi(\hat{z}) + \epsilon (\mathcal{D}^2 \phi(\hat{z}))^2.$

3. $(\tau_i, \mathcal{D}_{x_i} \phi(\hat{z}), \mathcal{X}_i) \in \overline{\mathcal{P}^+} u_i(\hat{x}_i, \hat{t}) \qquad$ *for* $i = 1, \ldots, \mathrm{P}$.

Lemma 3.1.11 *[66, Theorem 3, page 918] Let* $\gamma \in \mathcal{C}([0, T])$*,* $\gamma(0) = 0$*,* $\gamma(t) > 0$ *as* $t > 0$*,*

$$\int_0^{\epsilon} \frac{dt}{\gamma(t)} = \infty \text{ for all } \epsilon > 0.$$

Let $\theta \in \mathcal{USC}[0,T)$, nonnegative function, bounded from above and such that, in viscosity sense

$$\begin{cases} \theta \leq \gamma(\theta) & t \in (0,T), \\ \min\{\theta' - \gamma(\theta), \theta\} \leq 0 & t = 0. \end{cases}$$

Then $\theta = 0$.

Lemma 3.1.12 *Let $w \in \mathcal{USC}(\mathbb{R}^{2\mathrm{N}} \times [0,T))$ with $w(x,y,t) \leq \mathrm{B}(1 + \|x\| + \|y\|)$ and define*

$$\theta(t) = \lim_{r \to 0} \sup\{w(x,y,t) \,:\, \|x-y\| < r\},$$

and θ^* is its upper semicontinuous envelope. Let $t_0 \in [0,T)$ and $\phi \in \mathcal{C}^1([0,T))$ such that $\theta^* - \phi$ has a strict maximum at t_0 and $\phi(t_0) = \theta^*(t_0)$. Let now consider two positive parameters α and δ and construct an auxiliary function

$$\Phi_{\alpha\delta}(x,y,t) = w(x,y,t) - \frac{\alpha}{2}\|x-y\|^2 - \frac{\delta}{2}\|x|^2 - \phi(t).$$

Then, for all fixed α, δ there exist $(x_{\alpha\delta}, y_{\alpha\delta}, t_{\alpha\delta})$ at which $\Phi_{\alpha\delta}$ has a maximum such that

$$\lim_{\alpha \to +\infty} \lim_{\delta \to 0} \left[\frac{\alpha}{2}\|x-y\|^2 + \frac{\delta}{2}\|x|^2\right] = 0.$$

Moreover, extracting possibly a subsequence

$$\lim_{\alpha \to +\infty} \lim_{\delta \to 0} t_{\alpha\delta} = t_0,$$
$$\lim_{\alpha \to +\infty} \lim_{\delta \to 0} w(x_{\alpha\delta}, y_{\alpha\delta}, t_{\alpha\delta}) = \phi(t_0).$$

With these instruments it is possible to show the well posedness of the two problems. Classical theory of viscosity solutions proved existence and uniqueness of viscosity solution for the Cauchy (3.0.1)–(3.0.2) and obstacle problem (3.0.3)–(3.0.4) in the class of continuous function with linear growth at infinity. Unfortunately this class does not reflects the needs of financial problems, where typically functions present exponential growth at infinity. This extension was proved by Amadori [2, 3, 4].
Here we present these results both for the Cauchy and obstacle problem, using different techniques, the *Perron's method* for the former and the *penalization method* for the latter.

3.1.1 The Cauchy problem.

In this subsection the topic of well posedness of the Cauchy problem shall be analyzed, starting from the results by Barles and Perthame [14], where a relaxed notion of the initial condition is introduced. Particular attention is given to the extension to exponential growing solution.

Definition 3.1.13 *A function $u \in \mathcal{USC}$ is a subsolution to problem* (3.0.1)–(3.0.2) *with* **generalized initial condition** *(***GIC***) if the initial condition $u(\cdot,0) \le u_0(\cdot)$ is replaced by*

$$\min\{\partial_t + F(x,0,u,\mathcal{D}u,\mathcal{D}^2u), u - u^0\} \le 0,$$

in viscosity sense at $t = 0$.
A function $u \in \mathcal{LSC}$ is a supersolution to the Cauchy problem (3.0.1)–(3.0.2) *with* **generalized initial condition** *(***GIC***) if the initial condition $u(\cdot,0) \ge u_0(\cdot)$ is replaced by*

$$\min\{\partial_t + F(x,0,u,\mathcal{D}u,\mathcal{D}^2u), u - u^0\} \ge 0,$$

in viscosity sense at $t = 0$.

This definition can be interpreted as follows: the possibility that the condition $u(x,0) > u^0(x)$ holds at some x is allowed, but in that case u must be a subsolution till the boundary, that means that

$$\partial_t u + F(x,0,u,\mathcal{D}u,\mathcal{D}^2u) \le 0, \text{ in viscosity sense at } (x,0).$$

The same holds for the supersolution case.
A function u that is also a sub/supersolution with GIC is a sub/supersolution in the sense of Definition 3.1.7.

We introduce now the Perron's method to achieve the existence of viscosity solutions: this is based on the construction of a monotone approximation scheme and it allows to gain existence of weak viscosity solutions once continuity of sub/supersolution is proved. Notice that this will result a central point in the proof of numerical results in Part III.

Proposition 3.1.14 *Let $\{u_n\}_{n\in\mathbb{N}} \in \mathcal{USC}$ (resp. $\mathcal{LSC}$) be an increasing (resp. decreasing) sequence of subsolution (resp. supersolution) of equation* (3.0.1) *such that u_n converges pointwise to u. If the limit function u is locally bounded, then it is a subsolution (resp. supersolution) of equation* (3.0.1)*.*

This result follows easily from the continuity property of the semijets.

We can present the main result for the existence of solutions; for precise statements and proofs we refer to [34, 65].

Theorem 3.1.15 [Perron's Method] *Let h, $k \in \mathcal{C}(\mathbb{R}^N \times [0,T))$ be a viscosity subsolution and supersolution respectively of* (3.0.1)–(3.0.2) *such that $h \le k$ pointwise on $\mathbb{R}^N \times [0,T)$. Let now define the set*

$$\mathcal{S} = \{v : v \text{ is a subsolution of } (3.0.1)\text{–}(3.0.2),\ h \le v \le k \text{ pointwise on } \mathbb{R}^N \times [0,T)\}$$

and the function

$$u(x,t) = \sup\{v(x,t) \,:\, v \in \mathcal{S}\}. \tag{3.1.3}$$

Then u is a viscosity solution for (3.0.1)–(3.0.2).

From Theorem 3.1.15, no continuity property can be deduced for the solution u given by (3.1.3), but once the continuity is proved, u is a strong viscosity solution of the Cauchy problem.
We can now state the well posedness of the Cauchy problem in terms of the comparison principle; see [34] for a proof based on upper and lower envelopes.

Proposition 3.1.16 *Let $g \in \mathcal{C}[0,+\infty))$ be a rate of growth and assume that*

1. *for all initial data $u_0 \in \mathcal{C}(\mathbb{R}^{\mathrm{N}}) \cap L^\infty_g(\mathbb{R}^{\mathrm{N}})$, the Cauchy problem* (3.0.1)–(3.0.2) *admits a subsolution h and a supersolution k in $\mathcal{C}(\mathbb{R}^{\mathrm{N}} \times [0,T)) \cap L^\infty(0,T; L^\infty_g(\mathbb{R}^{\mathrm{N}}))$ such that $h \le k$ pointwise;*

2. *for any $u \in \mathcal{USC} \cap L^\infty(0,T; L^\infty_{g,+}(\mathbb{R}^{\mathrm{N}}))$ subsolution, and for any $v \in \mathcal{LSC} \cap L^\infty(0,T; L^\infty_{g,-}(\mathbb{R}^{\mathrm{N}}))$ supersolution of* (3.0.1)–(3.0.2) *there holds*

$$u \le v, \ \textit{pointwise},$$

   that is a comparison principle among sub/supersolutions holds.

Then the Cauchy problem (3.0.1)–(3.0.2) *is well posed in the class $L^\infty_g(\mathbb{R}^{\mathrm{N}})$: for all initial data $u_0 \in \mathcal{C}(\mathbb{R}^{\mathrm{N}}) \cap L^\infty_g(\mathbb{R}^{\mathrm{N}})$ there exists an unique viscosity solution of the Cauchy problem in $\mathcal{C}(\mathbb{R}^{\mathrm{N}} \times [0,T)) \cap L^\infty(0,T; L^\infty_g(\mathbb{R}^{\mathrm{N}}))$, and it is given by* (3.1.3).

From this results we can see that choosing a suitable class of growth and using a comparison principle, the well posedness of the Cauchy problem can be obtained quite easily. In the case of Cauchy problems related to the price of an European derivative, the typical rate of growth is exponential, because of the logarithmic change of variable, as it has been explained in Part I. Moreover the presence of nonlinearities due to the presence of large investor in the market requires the class of growth to be enlarged, under proper assumption on F.

Let us suppose that for the operator F there exists a modulus of continuity ω such that for all $\alpha > 0$

$$(\mathbf{F}^*) \quad \begin{aligned} &F(x,t,u,\alpha(x-y),\mathcal{X}) - F(y,t,u,\alpha(x-y),\mathcal{Y}) \\ &\ge -\omega(\|x-y\| + \alpha\|x-y\|^2), \end{aligned}$$

for all x, $y \in \mathbb{R}^{\mathrm{N}}$, $t \in [0,T]$, $u \in \mathbb{R}$ and $\mathcal{X}$, $\mathcal{Y} \in \mathcal{S}_{\mathrm{N}}$ such that

$$-3\alpha \begin{pmatrix} I_{\mathrm{N}} & 0 \\ 0 & I_{\mathrm{N}} \end{pmatrix} \le \begin{pmatrix} \mathcal{X} & 0 \\ 0 & -\mathcal{Y} \end{pmatrix} \le 3\alpha \begin{pmatrix} I_{\mathrm{N}} & -I_{\mathrm{N}} \\ -I_{\mathrm{N}} & I_{\mathrm{N}} \end{pmatrix}.$$

The standard comparison principle reads

Theorem 3.1.17 *[34, Th. 8.2]Let us suppose that* $\underline{u} \in \mathcal{USC} \cap L^\infty(0,T;L^\infty_{\mathrm{lin},+}(\mathbb{R}^{\mathrm{N}}))$ *and* $\overline{u} \in \mathcal{LSC} \cap L^\infty(0,T;L^\infty_{\mathrm{lin},-}(\mathbb{R}^{\mathrm{N}}))$ *are respectively a subsolution and supersolution of the Cauchy problem* (3.0.1)–(3.0.2)*. Then* $\underline{u} \leq \overline{u}$ *pointwise in* $\mathbb{R}^{\mathrm{N}} \times [0,T)$*.*

The previous result can be found in [34]. To extend this result to the class of growth that is of interest for financial application, assume that the operator F splits in two parts, one containing all the linear terms, the other one containing the nonlinear first order term:

$$F(x,t,u,\mathcal{D}u,\mathcal{D}^2u) = \mathcal{L}(x,t,\mathcal{D},\mathcal{D}^2)u + H(x,t,u,\mathcal{D}u),$$

where $\mathcal{L}$ is a linear degenerate second order operator

$$\mathcal{L}(x,t,\mathcal{D},\mathcal{D}^2)u = -\frac{1}{2}\mathrm{tr}\left[aa^T\mathcal{D}^2u\right] + b\mathcal{D}u + cu,$$

$a = (a^i_j)$ is a $\mathrm{N} \times \mathrm{D}$ matrix, with $\mathrm{D} \leq \mathrm{N}$ and

$$(\mathcal{L}.\mathbf{1}) \qquad \begin{array}{rcl} a^i_j,\, b_i, c & \in & \mathcal{C}(\mathbb{R}^{\mathrm{N}} \times [0,T]) \cap L^\infty(\mathbb{R}^{\mathrm{N}} \times (0,T)), \\ a^i_j,\, b_i & \in & L^\infty(0,T;W^{1,\infty}_{loc}(\mathbb{R}^{\mathrm{N}})). \end{array}$$

$H \in \mathcal{C}(\mathbb{R}^{\mathrm{N}} \times [0,T] \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}};\mathbb{R})$ is a nonlinear first order operator such that

$$(\mathbf{H}.1) \qquad H(\cdot,\cdot,0,0) \in L^\infty(0,T;L^\infty_{exp}(\mathbb{R}^{\mathrm{N}})),$$

that is that $H(\cdot,\cdot,0,0)$ plays the role of a source term and has the same rate of growth of the initial data.
Let us suppose that for each $R > 0$ there exists a modulus of continuity ω_R and a constant $\mathrm{L}_R > 0$ such that

$$(\mathbf{H}.\mathbf{2}.i) \qquad \begin{array}{l} |H(x,t,u,p)_H(y,t,u,p)| \\ \leq (1+|u|)\omega_R(\|x-y\|) + \mathrm{L}_R(1+\|p\|)\|x-y\|, \end{array}$$

for all x, $y \in B(0,R)$. Moreover there exists a constant $\mathrm{L}' \geq 0$ such that

$$(\mathbf{H}.\mathbf{2}.ii) \qquad H(x,t,u,p) - H(x,t,v,q) \leq \mathrm{L}'\left[|u-v| + \|p-q\|\right].$$

It can be proved that assumptions ($\mathcal{L}$.1) and (H.2.ii) imply that assumption F.2 is verified with

$$\gamma(u) = \max\{\mathrm{L}' - \inf c, 0\}u$$

These assumptions holding, a comparison result for exponential growing solution can be derived.

Theorem 3.1.18 *[2, Theorem 3.20, page 87]Let us assume that* $u_0 \in \mathcal{C}(\mathbb{R}^{\text{N}}) \cap L^\infty_{\text{exp}}(\mathbb{R}^{\text{N}})$*; let* $\underline{u} \in \mathcal{USC} \cap L^\infty(0,T; L^\infty_{\text{exp},+}(\mathbb{R}^{\text{N}}))$ *and* $\overline{v} \in \mathcal{LSC} \cap L^\infty(0,T; L^\infty_{\text{exp},-}(\mathbb{R}^{\text{N}}))$ *be respectively subsolution and supersolution of the Cauchy problem* (3.0.1)–(3.0.2). *Then*

$$\underline{u} \le \overline{u}, \text{ pointwise on } \mathbb{R}^{\text{N}} \times [0,T).$$

A detailed proof can be found in [2]: it is based on the choice of a good weight function by which multiply $\underline{u}$ and $\overline{u}$. A particular care is needed to deal with nonlinear terms.

Financial application. In the previous section we have seen that the class of existence of viscosity solutions for the Cauchy problem depends on the class of growth of the initial datum. For what concerns the Cauchy problem related to the pricing of an European derivative, the initial datum belongs to $\mathcal{C}(\mathbb{R}^{\text{N}}) \cap L^\infty_{\text{exp}}(\mathbb{R}^{\text{N}})$. It holds the following result.

Corollary 3.1.19 *Under the assumptions* ($\boldsymbol{\mathcal{L}}$.**1**)–(**H**.**2**) *the Cauchy problem* (3.0.1)–(3.0.2) *is well posed in the class* $\mathcal{C}(\mathbb{R}^{\text{N}}) \cap L^\infty_{\text{exp}}(\mathbb{R}^{\text{N}})$ *in the framework of viscosity solutions, that is for all* $u_0 \in \mathcal{C}(\mathbb{R}^{\text{N}}) \cap L^\infty_{\text{exp}}(\mathbb{R}^{\text{N}})$ *there exists a viscosity solution* u *such that*

$$|u(x,t)| \le \text{C}e^{n\|x\|},$$

with

$$n = \min\{m \in \mathbb{N} : e^{-m\|x\|}u_0(x), e^{-m\|x\|}H(x,t,0,0) \text{ are bounded}\}.$$

The obtained solution is unique in the class $L^\infty(0,T; L^\infty_{\text{exp}}(R^{\text{N}}))$.

Example 3.1.20 [*Large investor model*] Remember now the case of an incomplete market with large investor: the market model reads as

$$\begin{cases} X^0_t &= \exp\Big(\displaystyle\int_0^t r_\tau d\tau\Big), \\ X_t &= \mathbf{X}_0 \exp\Big(\displaystyle\int_0^t \mu d\tau + \int_0^t \sigma dW_\tau\Big). \end{cases}$$

Here all the parameters depends on X, t. Moreover the vector α may depend on the agent's portfolio wealth invested in the risky asset, ξ, while the interest rate r may be influenced by the wealth deposited in the bank, $\eta = Y - \xi \mathbf{1}_{\text{N}} \in \mathbb{R}^{\text{N}}$.
Let us relax the standard assumption of completeness by means of (1.5.5) and there exists a D × N matrix Σ such that $\sigma\Sigma$ only depends by X, t, satisfying (1.5.6) that guarantees that the market is without arbitrage opportunities, even if possibly not complete. Being $\operatorname{rank} \sigma \le \text{N}$, the matrix Σ is not uniquely determined and is related

to the choice of the *market price for risk*.
If we consider the price of an European derivative with payoff $G \in \mathcal{C}(\overline{\Pi}^{\mathrm{N}}) \cap L^\infty_{pol}(\Pi^{\mathrm{N}})$ and maturity $T > 0$, it is related to the solution of the following problem

$$\begin{cases} -\partial_t U &= \frac{1}{2}\mathrm{tr}\left[(\mathbf{X}\sigma)(\mathbf{X}\sigma)^T\mathcal{D}^2 U\right] - \mathcal{D}U\mathbf{X}(\sigma\Sigma - I_{\mathrm{N}})\alpha(X,t,\mathcal{D}U\mathbf{X}\sigma\Sigma) \\ & \quad -r(X,t,U-\mathcal{D}U\mathbf{X}\sigma\Sigma I_{\mathrm{N}})(U-\mathcal{D}U\mathbf{X}\sigma\Sigma I_{\mathrm{N}}), \\ U(x,t) &= G(X). \end{cases} \tag{3.1.4}$$

Applying a logarithmic change of variable we can see that the function $u(x,t) = U(e^{x_1}, \ldots, e^{x_{\mathrm{N}}}, T-t)$ is solution of the following

$$\begin{cases} \partial_t u &= \frac{1}{2}\mathrm{tr}\left[\sigma\sigma\mathcal{D}^2 u\right] - (\sigma\Sigma - I_{\mathrm{N}})\alpha(X(x), T-t, \mathcal{D}u\sigma\Sigma)^T\mathcal{D}u \\ & \quad -r(X(x), T-t, u-(\sigma\Sigma)^T\mathcal{D}u)(u-(\sigma\Sigma)^T\mathcal{D}u), \\ u(x,0) &= G\circ X(x) \in \mathcal{C}(\mathbb{R}^{\mathrm{N}}) \cap L^\infty_{\mathrm{exp}}(\mathbb{R}^{\mathrm{N}}). \end{cases}$$

The previous result can be applied even to this case, provided some assumptions hold. Therefore we have the following result:

Proposition 3.1.21 *[2, Proposition 3.22, page 92]Let us suppose that the financial coefficients have the following regularity*

$$\sigma^i_j \in \mathcal{C}(\overline{\Pi}^{\mathrm{N}} \times [0,T]) \cap L^\infty(0,T;W^{1,\infty}(\Pi^{\mathrm{N}})), X^i\mathcal{D}_X\sigma^i_j \in L^\infty(\Pi^{\mathrm{N}} \times (0,T);\mathbb{R}^{\mathrm{N}}),$$

$$\alpha^i \in \mathcal{C}(\overline{\Pi}^{\mathrm{N}} \times [0,T] \times \mathbb{R}^{\mathrm{N}}) \cap L^\infty(0,T;W^{1,\infty}(\Pi^{\mathrm{N}} \times \mathbb{R}^{\mathrm{N}})),$$

$$X^i\mathcal{D}_X\alpha^i \in L^\infty(\Pi^{\mathrm{N}} \times (0,T) \times \mathbb{R}^{\mathrm{N}};\mathbb{R}^{\mathrm{N}}), \xi^h\partial_{\xi^j}\alpha^i \in L^\infty(\Pi^{\mathrm{N}} \times (0,T) \times \mathbb{R}^{\mathrm{N}}),$$

$$r \in \mathcal{C}(\overline{\Pi}^{\mathrm{N}} \times [0,T] \times (\mathbb{R}-\{0\})) \cap L^\infty(0,T;W^{1,\infty}(\Pi^{\mathrm{N}} \times (\mathbb{R}-\{0\}))),$$

$$X^i\mathcal{D}_X r \in L^\infty(\Pi^{\mathrm{N}} \times (0,T) \times \mathbb{R};\mathbb{R}^{\mathrm{N}}), \eta\partial_\eta r \in L^\infty(\Pi^{\mathrm{N}} \times (0,T) \times \mathbb{R}).$$

Select a matrix Σ satisfying (1.5.6)*, that is equivalent to select a market price for risk. Then for any payoff $G \in \mathcal{C}(\overline{\Pi}^{\mathrm{N}}) \cap L^\infty_{\mathrm{pol}}(\Pi^{\mathrm{N}})$ problem* (3.1.4) *admits an unique viscosity solution U in the class $\mathcal{C}(\Pi^{\mathrm{N}} \times (0,T)) \cap L^\infty(0,T;L^\infty_{\mathrm{pol}}(\Pi^{\mathrm{N}})$ which satisfies*

$$0 \le U(X,t) \le \mathrm{B}(1+\|X\|^n), \text{ for all } (X,t) \in \Pi^{\mathrm{N}} \times (0,T),$$

provided that

$$0 \le G(X) \le \mathrm{B}_0(1+\|X\|^n).$$

■

3.1.2 The Obstacle problem.

In this subsection we shall present analogous results of well posedness, existence and uniqueness of viscosity solutions for the obstacle problem in the class of exponential growth, the one of financial interest.
The main tool will be the so call *penalization method* which consist in the construction of a monotone approximation which penalizes the obstacle.
At it as been done for the Cauchy problem, the initial condition can be relaxed even in this case:

Definition 3.1.22 *We say that $u \in \mathcal{USC}$ is a subsolution of* (3.0.3)–(3.0.4) *with* **generalized initial condition**, *(***GIC***), if the initial condition $u(\cdot,0) \leq u^o(\cdot,0)$ is replaced by*

$$\min\left\{\partial_t u + F(x,0,u,\mathcal{D}u,\mathcal{D}^2u), u-u^0\right\} \geq 0, \ \ \text{at } t=0.$$

A function $u \in \mathcal{LSC}$ is a supersolution with **generalized initial condition** *if the initial condition $u(\cdot,0) \geq u^0(\cdot,0)$ is replaced by*

$$\max\left\{u-u^0, \min\left\{\partial_t u + F(x,0,u,\mathcal{D}u,\mathcal{D}^2u), u-u^0\right\}\right\} \geq 0, \ \ \text{at } t=0.$$

It means that whenever $u(x,0) < u^0(x,0)$ then $\mathcal{P}^- u(x,0) = \emptyset$.

This definition is to interpret in the following sense: $u \in \mathcal{USC}$ is a subsolution of (3.0.3) till $t=0$, that is

$$\partial_t u + F(x,t,u,\mathcal{D}u,\mathcal{D}^2u) \leq 0,$$

holds in viscosity sense for all $(x,t) \in \mathbb{R}^N \times [0,T)$ with $u(x,t) > u^0(x,t)$.
On the other hand $u \in \mathcal{LSC}$ is a viscosity supersolution with GIC if

$$\partial_t u + F(x,t,u,\mathcal{D}u,\mathcal{D}^2u) \geq 0$$

in viscosity sense for all $(x,t) \in \mathbb{R}^N \times (0,T)$ with $u(x,t) > u^0(x,t)$ and $\mathcal{P}^- u(x,t) = \emptyset$ for all $(x,t) \in \mathbb{R}^N \times [0,T)$ with $u(x,t) < u^0(x,t)$.
In an analogous way as for the Cauchy problem, it can be proved that a sub/supersolution with generalized initial condition is a sub/supersolution in the sense of Definition 3.1.9.

Now we can state an important result to get a well posedness result for the obstacle problem. It is based on the penalization method, which can be seen as a monotone scheme in the sense of the one proposed by [16].
Let consider $\epsilon > 0$ and the penalized equation

$$(3.0.3)_\epsilon \qquad \partial_t u + F(x,t,y,\mathcal{D}u,\mathcal{D}^2u) = \frac{1}{\epsilon}(u^0-u)^+, \quad (x,t) \in \mathbb{R}^N \times (0,T),$$

with the initial condition (3.0.4). Define u_ϵ the solution of $(3.0.3)_\epsilon$–(3.0.4); equation $(3.0.3)_\epsilon$ is an approximation of the original obstacle problem as when ϵ goes to zero the penalization term blows up unless $u^0 \le u_\epsilon$. Intuitively one can assume that

$$u(x,t) = \lim_{(\epsilon,y,s)\to(0,x,t)} u_\epsilon(y,s),$$

satisfies $u \ge u^0$ and $\partial_t u + F(x,t,y,\mathcal{D}u,\mathcal{D}^2u) = 0$ on $u > u^0$ and therefore it solves the obstacle problem.
Unfortunately this assumption is not always verified, as, if the limiting function is proved to exists, it is not clear if it is continuous. As it has been made for the Cauchy problem, we deal with the upper and lower limit, proving that they are sub/supersolution and then we use a comparison principle to prove that a continuous solution does exist.

Theorem 3.1.23 [Stability of Penalization] *Suppose that for any $\epsilon > 0$ there exists a viscosity solution u_ϵ for the Cauchy problem $(3.0.3)_\epsilon$–(3.0.4) and that u_ϵ is locally bounded w.r.t. ϵ. Define*

$$\begin{aligned} u^*(x,t) &= \limsup_{(\epsilon,y,z)\to(0,x,t)} u_\epsilon(y,s), \\ u_*(x,t) &= \liminf_{(\epsilon,y,z)\to(0,x,t)} u_\epsilon(y,s). \end{aligned} \tag{3.1.5}$$

Then u^* and u_* are respectively sub/supersolution for the obstacle problem (3.0.3)–(3.0.4).

For a detailed proof of that point we refer to [2].
As in the Cauchy problem case, we show that a comparison principle among sub/supersolutions is sufficient to deduce well posedness for the obstacle problem.

Proposition 3.1.24 [Well posedness via penalization's method] *Let $g \in \mathcal{C}([0,+\infty))$ be a rate of growth; suppose that*

(i) for all obstacles $u^0 \in \mathcal{C}(\mathbb{R}^{\mathrm{N}} \times (0,T)) \cap L^\infty(0,T;L^\infty_g(\mathbb{R}^{\mathrm{N}}))$ and for all $\epsilon > 0$ the Cauchy problem $(3.0.3)_\epsilon$–(3.0.4) admit viscosity solutions $u_\epsilon \in \mathcal{C}(\mathbb{R}^{\mathrm{N}} \times (0,T)) \cap L^\infty(0,T;L^\infty_g(\mathbb{R}^{\mathrm{N}}))$ such that

$$|u_\epsilon(x,t)| \le \mathrm{B}g(\|x\|),$$

with B *independent from ϵ;*

(ii) comparison principle among subsolutions in $\mathcal{USC} \cap L^\infty(0,T;L^\infty_{g,+}(\mathbb{R}^{\mathrm{N}}))$ and supersolutions in $\mathcal{LSC} \cap L^\infty(0,T;L^\infty_{g,-}(\mathbb{R}^{\mathrm{N}}))$ of the obstacle problem (3.0.3)–(3.0.4) *holds.*

Then the obstacle problem (3.0.3)–(3.0.4) *is well posed in the class* $L^\infty_g(\mathbb{R}^N)$*, that is that for any obstacle* $u^0 \in \mathcal{C}(\mathbb{R}^N) \cap L^\infty_g(\mathbb{R}^N)$ *there exists a unique solution belonging to* $\mathcal{C}(\mathbb{R}^N \times [0,T)) \cap L^\infty(0,T; L^\infty_g(\mathbb{R}^N))$.
The viscosity solution is the pointwise limit of the approximating u_ϵ *and it is given by* (3.1.5).

As it has been done for the Cauchy problem, we shall recall at first the standard comparison principle for the obstacle problem in the class of linear growth at infinity; then we show how to enlarge the class of growth to take into account financial applications.

Theorem 3.1.25 *Consider the obstacle problem* (3.0.3)–(3.0.4) *such that the* F *operator satisfies* **F.1**–**F.2**–**F****; let*

$$\underline{u} \in \mathcal{USC} \cap L^\infty(0,T; L^\infty_{\text{lin},+}(\mathbb{R}^N) \text{ and } \overline{u} \in \mathcal{LSC} \cap L^\infty(0,T; L^\infty_{\text{lin},-}(\mathbb{R}^N),$$

be respectively a sub and a supersolution of the considered problem. Then

$$\underline{u} \leq \overline{u}, \text{ pointwise on } \mathbb{R}^N \times [0,T).$$

As in the Cauchy problem case, let us consider the obstacle problem related to the American derivative pricing problem:

$$\min\left\{\partial_t u + \mathcal{L}u + H(x,t,y,\mathcal{D}u), u - u^0\right\} = 0, \tag{3.1.6}$$

under the previous assumption ($\mathcal{L}$.1)–(H.2) and the following

$$u^0 \in \mathcal{C}(\mathbb{R}^N \times [0,T)) \cap L^\infty(0,T; L^\infty_{\exp}(\mathbb{R}^N)).$$

Theorem 3.1.26 *Let us suppose that* $\underline{u} \in \mathcal{USC} \cap L^\infty(0,T; L^\infty_{\exp,+}(\mathbb{R}^N))$ *and* $\overline{u} \in \mathcal{LSC} \cap L^\infty(0,T; L^\infty_{\exp,-}(\mathbb{R}^N))$ *are respectively a sub and a supersolution of the obstacle problem; then*

$$\underline{u} \leq \overline{u}, \text{ pointwise on } \mathbb{R}^N \times [0,T).$$

For a detailed proof of this result we refer to [2].

Financial application. As in the Cauchy case, the obstacle problem is related to the American derivative pricing problem. The initial datum of this problem belongs to $\mathcal{C}(\mathbb{R}^N \times [0,T)) \cap L^\infty(0,T; L^\infty_{\exp}(\mathbb{R}^N))$, therefore to get existence of the solution we need a comparison principle in this class of growth.

Corollary 3.1.27 *Under the assumption* ($\mathcal{L}$.1)*–(H.2) the obstacle problem* (3.1.6)*–*(3.0.4) *is well posed in the class* $\mathcal{C}(\mathbb{R}^{\mathrm{N}} \times [0,T)) \cap L^\infty(0,T; L^\infty_{\exp}(\mathbb{R}^{\mathrm{N}}))$ *in the framework of viscosity solutions. This means that for all obstacles* $u^0 \in \mathcal{C}(\mathbb{R}^{\mathrm{N}} \times [0,T)) \cap L^\infty(0,T; L^\infty_{\exp}(\mathbb{R}^{\mathrm{N}}))$ *there exists a viscosity solution* u *such that*

$$|u(x,t)| \leq \mathrm{C}e^{n\|x\|},$$

where

$$n = \min\left\{m \in \mathbb{N} : e^{-m\|x\|}u^0(x,t),\ e^{-m\|x\|}H(x,t,0,0) \text{ are bounded}\right\}.$$

The solution u *is the only one belonging to* $L^\infty(0,T; L^\infty_{\exp}(\mathbb{R}^{\mathrm{N}}))$*.*

Example 3.1.28 [*Incomplete diffusion market with large investor*] Let us consider the problem of pricing an American derivative with payoff $G \in \mathcal{C}(\overline{\Pi}^{\mathrm{N}}) \cap L^\infty_{\mathrm{pol}}(\Pi^{\mathrm{N}})$ and maturity $T > 0$: the fair price is related to the solution of the following obstacle problem

$$\begin{cases} \min\Big\{-\partial_t U - \frac{1}{2}\mathrm{tr}\left[(\mathbf{X}\sigma)(\mathbf{X}\sigma)^T\mathcal{D}^2U\right] + \mathcal{D}U\mathbf{X}(\sigma\Sigma - I_{\mathrm{N}})\alpha(X,t,\mathcal{D}U\mathbf{X}\sigma\Sigma) \\ \qquad +r(X,t,U - \mathcal{D}U\mathbf{X}\sigma\Sigma\mathbf{1}_{\mathrm{N}})(U - \mathcal{D}U\mathbf{X}\sigma\Sigma\mathbf{1}_{\mathrm{N}}), U - G\Big\} = 0, \\ U(X,T) = G(X). \end{cases} \tag{3.1.7}$$

If we apply, as before, a logarithmic change of variable and consider the function $u(x,t) = U(e^{x_1}, \ldots, e^{x_{\mathrm{N}}}, T-t)$; the obstacle problem, after the change of variable reads as

$$\begin{cases} \min\Big\{\partial_t u - \frac{1}{2}\mathrm{tr}\,[\sigma\sigma^T\mathcal{D}^2u] + ((\sigma\Sigma - I_{\mathrm{N}})\alpha(X(x),T-t,\mathcal{D}u\sigma\Sigma))^T\mathcal{D}u \\ \qquad +r(X(x),T-t,u-(\sigma\Sigma)^T\mathcal{D}u)(u-(\sigma\Sigma)^T\mathcal{D}u), u - G(X(x))\Big\} = 0, \\ u(x,0) = G(X(x)); \end{cases}$$

here $G(X(x)) \in \mathcal{C}(\mathbb{R}^{\mathrm{N}}) \cap L^\infty_{\exp}(\mathbb{R}^{\mathrm{N}})$ and $\sigma\Sigma$ is computed at $(X(x), T-t)$. Summing up the results obtained in this subsection for the obstacle problem we can derive the following well posedness result:

Proposition 3.1.29 *[2, Proposition 3.20, page 101]Let us suppose that the financial* *coefficients have the following regularity property:*

$$\sigma^i_j \in \mathcal{C}(\overline{\Pi}^{\mathrm{N}} \times [0,T]) \cap L^\infty(0,T;W^{1,\infty}(\Pi^{\mathrm{N}})), X^i \mathcal{D}_X \sigma^i_j \in L^\infty(\Pi^{\mathrm{N}} \times (0,T);\mathbb{R}^{\mathrm{N}}),$$

$$\alpha^i \in \mathcal{C}(\overline{\Pi}^{\mathrm{N}} \times [0,T] \times \mathbb{R}^{\mathrm{N}}) \cap L^\infty(0,T;W^{1,\infty}(\Pi^{\mathrm{N}} \times \mathbb{R}^{\mathrm{N}})),$$

$$X^i \mathcal{D}_X \alpha^i \in L^\infty(\Pi^{\mathrm{N}} \times (0,T) \times \mathbb{R}^{\mathrm{N}};\mathbb{R}^{\mathrm{N}}), \xi^h \partial_{\xi^j} \alpha^i \in L^\infty(\Pi^{\mathrm{N}} \times (0,T) \times \mathbb{R}^{\mathrm{N}}),$$

$$r \in \mathcal{C}(\overline{\Pi}^{\mathrm{N}} \times [0,T] \times (\mathbb{R} - \{0\})) \cap L^\infty(0,T;W^{1,\infty}(\Pi^{\mathrm{N}} \times (\mathbb{R} - \{0\}))),$$

$$X^i \mathcal{D}_X r \in L^\infty(\Pi^{\mathrm{N}} \times (0,T) \times \mathbb{R};\mathbb{R}^{\mathrm{N}}), \eta \partial_\eta r \in L^\infty(\Pi^{\mathrm{N}} \times (0,T) \times \mathbb{R}).$$

Select a matrix Σ *satisfying* (1.5.6)*, that is equivalent to select a market price for risk. Then for any payoff* $G \in \mathcal{C}(\overline{\Pi}^{\mathrm{N}}) \cap L^\infty_{\mathrm{pol}}(\Pi^{\mathrm{N}})$ *problem* (3.1.7) *admits an unique viscosity solution* U *in the class* $\mathcal{C}(\Pi^{\mathrm{N}} \times (0,T)) \cap L^\infty(0,T;L^\infty_{\mathrm{pol}}(\Pi^{\mathrm{N}})$ *which satisfies*

$$0 \le U(X,t) \le \mathrm{B}(1 + \|X\|^n), \text{ for all } (X,t) \in \Pi^{\mathrm{N}} \times (0,T)$$

whenever

$$0 \le G(X) \le \mathrm{B}_0(1 + \|X\|^n).$$

■

Chapter 4

Viscosity solutions to nonlinear integro–differential equations.

In the previous chapters we have seen the importance of the viscosity solution theory for integro–partial differential equations, in particular for the many applications in Mathematical Finance. Integro–differential problems come out as pricing problems in markets in which Lévy processes act as the underlying stochastic processes. Several works have been done in recent years [34, 48]; empirical work shows that the normal distribution poorly fits the logreturn data for, e.g., stock prices. Among other things the data show heavier tails than predicted by the normal distribution, and it has in recent years been suggested to model logreturns by generalized hyperbolic distributions (see the references in [21, 22, 23, 33, 104] for relevant works).

In this chapter we shall present theorems of existence and uniqueness for viscosity solutions to nonlinear degenerate parabolic integro–partial differential equations with a given terminal condition u_T:

$$-\partial_t u + H(x, t, u, \mathbf{x}\mathcal{D}u, \mathbf{x}\mathcal{D}^2 u\mathbf{x}^T, \mathcal{J}u) = 0, \qquad (x,t) \in \Pi_T, \quad (4.0.1)$$

$$u(x, T) = u_T(x), \qquad x \in \Pi, \quad (4.0.2)$$

where $\mathcal{J}u$ is an integro–differential operator given by

$$\mathcal{J}u(x,t) = \int_E [u(x + \mathbf{x}\beta(x,t,z), t) - u(x,t) - \mathbf{x}\beta(x,t,z) \cdot \mathcal{D}u(x,t)]\, \nu(dz). \quad (4.0.3)$$

In (4.0.3), ν is a given Radon measure on $E = \mathbb{R}^2 - \{0\}$ (the so-called Lévy measure), which may possess a second order singularity at the origin, while β is a given function of $(x,t,z) \in \Pi_T \times E$ with values in $(-1, \infty)^{\mathrm{N}}$.

The results we are going to present can be extended quite directly to the case of an operator H depending on another integral term of the type

$$\mathcal{I}u(x,t) = \int_E \left[u(x + \mathbf{x}\beta(x,t,z),t) - u(x,t)\right]\gamma(x,t,z)\nu(dz), \tag{4.0.4}$$

where γ is a given function of $(x,t,z) \in \Pi_T \times E$ with values in $(0,\infty)$.

Because (4.0.1) may be degenerate and ν is allowed to have a second order singularity at the origin ($\mathcal{I}$ and $\mathcal{J}$ are typically well defined on suitable subsets of the spaces of $\mathcal{C}^1$ and $\mathcal{C}^2$ functions, respectively), it seems natural to study (4.0.1) in the framework of viscosity solutions.
Integro–differential problem of the form (4.0.1) have been widely study by Garroni and Menaldi [53, 54] in the framework of Green functions and more regular solutions in Sobolev spaces. In their works, the nonlinear second order operator is assumed to be *uniformly elliptic*, while in the present case it is allowed to be degenerate elliptic; moreover in these works and reference therein the dependence upon the integral term is linear, while in the present case a more general dependence on the integral term is allowed.

Various existence and comparison/uniqueness results for viscosity solutions of integro–partial differential equations of first order (i.e., no local second order term) can be found in [108, 109, 7, 8, 21]. When the Lévy measure is bounded, general existence and comparison/uniqueness results for semicontinuous unbounded viscosity solutions of second order degenerate parabolic integro–partial differential equations are given in [1, 3, 4]; on the other hand, when the Lévy measure is unbounded near the origin, the existence and uniqueness of unbounded viscosity solutions of (systems of) semilinear degenerate parabolic integro–partial differential equations in $\mathbb{R}^N$ is proved in [12]. An existence result and a comparison principle among uniformly continuous and at most linearly growing viscosity sub- and supersolutions of fully nonlinear parabolic integro–partial differential equations of the Bellman type are proved in [101], see also [91] for some other existence results. The Bellman equations (variational inequalities) associated with some singular stochastic control problems arising in finance are studied in [22, 23]. In [69], the authors prove a "non-local" maximum principle for semicontinuous viscosity sub- and supersolutions of integro–partial differential equations, which should be compared with the "local" maximum principle for semicontinuous functions [34]. Such a result can be used to obtain various comparison results for integro–partial differential equations, and is also used herein.

In this chapter we shall prove a comparison principle for unbounded semicontinuous viscosity sub– and super solutions of(4.0.1)–(4.0.2) without assigning boundary data on $\partial\Pi$; this is possible because of the special structure of our problem, typical of finance application with geometric Brownian motion or more general Lévy

processes as the underlying stochastic process: the gradient and the matrix slots depend on $\mathbf{x}\mathcal{D}u$ and on $\mathbf{x}\mathcal{D}^2 u\mathbf{x}^T$, while $\mathbf{x}\beta$ occurs in the integral operators (4.0.3) and (4.0.4).
In the case of no integral operator, the problem (4.0.1)–(4.0.2) is equivalent to a Cauchy problem, up to a logarithmic change of variables: in this case it is sufficient to impose some blow-up rate both at $\partial\Pi$ and at infinity to get uniqueness.
The same change of variable in the present case modifies the structure of the integral operator (4.0.3), therefore we need other techniques to get existence and uniqueness of solutions.
Problems of the form (4.0.1) present two different kinds of degeneracies: one at the boundary $\partial\Pi$ (where $\mathbf{x}\mathcal{D}^2 u\mathbf{x}$ vanishes), and one at the interior points (where H is only assumed to be degenerate elliptic). Boundary value problems for non–uniformly parabolic equations have been studied by many authors along the lines of [92]. In this framework, we mention [114] for some different integro–partial differential equation. In the viscosity solution setting, comparison principles allowing degeneracy can be found in [15] in the context of Bellman equations (without an integro operator).
We shall prove a comparison result for a large class of integro–partial differential equations, dealing with an unbounded set with nonsmooth boundary and allowing some blow up condition at $\partial\Pi$ for the solutions.
It will be shown that the special structure of β and the assumptions on ν allows to deal with the boundary as two different regions: one which behaves as the interior of Π and one where the solutions possibly blow up.

4.1 The bounded Lévy case.

In this section we shall present some results of existence and uniqueness of viscosity solution to integro–partial differential equation with bounded Lévy measure. For details and proofs we refer to the works by Amadori [3, 2, 4].
The pricing equation for our model could be written as:

$$\begin{cases} \partial_t u + F(x,t,u,\mathcal{I}u,\mathcal{D}u,\mathcal{D}^2 u) &= 0, \\ u(x,0) &= u_0(x), \end{cases} \tag{4.1.1}$$

As for the general theory of viscosity solutions for pure differential equations, we shall make some assumptions on the function $F \in \mathcal{C}(\mathbb{R}^{\mathrm{N}} \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}} \times \mathcal{S}_{\mathrm{N}}, \mathbb{R})$.

F.1 F is degenerate elliptic:

$$F(x,t,u,\mathcal{I},p,\mathcal{X}) \geq F(x,t,u,\mathcal{I},p,\mathcal{Y}), \quad \text{for } \mathcal{X} \leq \mathcal{Y};$$

F.2 F is quasi-monotone with respect to u, uniformly with respect to the other variables:

$$F(x,t,u,\mathcal{I},p,\mathcal{X}) \geq F(x,t,v,\mathcal{I},p,\mathcal{X}) - \gamma(u-v), \quad u \geq v,$$

where $\gamma \in \mathcal{C}([0,\infty))$, $\gamma(0)=0$, $\gamma(u)>0$ as $u>0$ and:

$$\int_0^{\varepsilon} \frac{du}{\gamma(u)} = \infty \quad \text{for all } \varepsilon > 0.$$

To rely the dependency of F on the integral term, we shall also assume, following [4],

F.3 F is non-increasing with respect to $\mathcal{I}$:

$$F(x,t,u,\mathcal{I},p,\mathcal{X}) \geq F(x,t,u,\mathcal{J},p,\mathcal{X}) \quad , \text{ for } \mathcal{I} \leq \mathcal{J}.$$

Following [1, 2, 4], we can now give a modified notion of viscosity solutions, which makes use of the notion of upper and lower semijets $\mathcal{P}^{\pm}$, see [34].

Definition 4.1.1 *Given a function u and a point $(x,t) \in \mathbb{R}^N \times [0,T)$, we say that:*

$$\partial_t u + F(x,t,u,\mathcal{I}u,\mathcal{D}u,\mathcal{D}^2 u) \leq 0 \ (\textit{resp.} \ \geq 0)$$

in viscosity sense *at (x,t) if one of the following equivalent condition is met:*

1. *for all $(\tau,p,\mathcal{X}) \in \mathcal{P}^+ u(x,t)$ (respectively $\mathcal{P}^-(x,t)$):*

$$\tau + F(x,t,u(x,t),\mathcal{I}u(x,t),p,\mathcal{X}) \leq 0 \ (\textit{resp.} \ \geq 0);$$

2. *for each function $\phi \in \mathcal{C}^{2,1}(\mathbb{R}^N \times [0,T))$ such that $u-\phi$ has a local maximum at (x,t) (respectively, a minimum):*

$$\partial_t\phi(x,t) + F(x,t,u(x,t),\mathcal{I}\phi(x,t),\mathcal{D}\phi(x,t),\mathcal{D}^2\phi(x,t)) \leq 0 \ (\textit{resp.} \geq 0);$$

3. *for all test function $\phi \in \mathcal{C}^{2,1}(\mathbb{R}^N \times [0,T))$ such that $u-\phi$ has a global strict maximum at (x,t) (respectively, a minimum) and $\phi(x,t)=u(x,t)$:*

$$\partial_t\phi + F(x,t,\phi,\mathcal{I}\phi,\mathcal{D}\phi,\mathcal{D}^2\phi) \leq 0 \ (\textit{resp.} \ \geq 0)$$

holds in classical sense.

Here we denote by $\mathcal{C}^{2,1}(\mathbb{R}^N \times [0,T))$ the set of functions that are twice continuously differentiable with respect to $x \in \mathbb{R}^N$ and once with respect to $t \in [0,T)$. Let us notice that, the important difference between the purely differential case, treated in [34], and the integro-differential one is that the local continuity property of the semijets of semicontinuous functions does not imply the semicontinuity of the equation, because the new nonlocal term $\mathcal{I}u$ does not preserve semicontinuity in general. To overcome this difficulty, we have to define a new class of admissible functions.

Definition 4.1.2 *A function $f(y,s;z)$ has an* ***upper*** *(resp.* ***lower****)* μ***-bound*** *at (x,t) if there exist a neighborhood $V_{x,t}$ of (x,t) and a function $\Phi \in \mathcal{C}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N;\mu_{x,t})$ such that:*

- $\int \Phi(z)\mu_{y,s}(dz) \to \int \Phi(z)\mu_{x,t}(dz)$ *if* $(y,s) \to (x,t)$;
- $f(y,s;z) \leq \Phi(z)$ *(resp.,* $\geq$*)* $\mu_{y,s} - a.e.\ z,$ *for all* $(y,s) \in V_{x,t}$.

Remark 4.1.3 Let f be a locally bounded function on $\mathbb{R}^N \times [0,T) \times \mathbb{R}^N$ which has an upper (resp., lower) μ-bound at (x,t); then:

$$\limsup_{(y,s)\to(x,t)} \int_{\mathbb{R}^N} f(y,s;z)\mu_{y,s}(dz) \leq \int_{\mathbb{R}^N} \limsup_{(y,s)\to(x,t)} f(y,s;z)\mu_{x,t}(dz)$$

$$\Big(\text{resp. } \liminf_{(y,s)\to(x,t)} \int_{\mathbb{R}^N} f(y,s;z)\mu_{y,s}(dz) \geq \int_{\mathbb{R}^N} \liminf_{(y,s)\to(x,t)} f(y,s;z)\mu_{x,t}(dz)\Big).$$

■

In this way we can introduce some new classes of admissible functions.

$\mathcal{USC}^{\mathcal{I}}$ is the set of upper semicontinuous, locally bounded functions on $\mathbb{R}^N \times [0,T)$ such that $M(u(x+z,t),u(x,t))$ has an upper μ-bound at any (x,t);

$\mathcal{LSC}^{\mathcal{I}}$ is the set of lower semicontinuous, locally bounded functions on $\mathbb{R}^N \times [0,T)$ such that $M(u(x+z,t),u(x,t))$ has a lower μ-bound at any (x,t);

$\mathcal{C}^{\mathcal{I}} = \mathcal{USC}^{\mathcal{I}} \cap \mathcal{LSC}^{\mathcal{I}}$.

It can be observed that if the integral operator $\mathcal{I}$ has some more regularities, the admissible classes described before coincide with the classes of exponential growth at infinity, that are the classes of growth required in the framework of pure diffusion models; for details we refer to [2, 4].

Finally we can define viscosity sub/super solutions in the integro-differential framework.

Definition 4.1.4 *$u \in \mathcal{USC}^{\mathcal{I}}$ ($u \in \mathcal{LSC}^{\mathcal{I}}$) is a **viscosity subsolution** (resp. **viscosity supersolution**) of the equation (4.1.1) if:*

$$\partial_t u + F(x, t, u, \mathcal{I}u, \mathcal{D}u, \mathcal{D}^2 u) \leq 0 \ \ (resp. \ \ \geq)$$

holds in viscosity sense for all $(x, t) \in \mathbb{R}^{\mathrm{N}} \times (0, T)$. If in addition:

$$u(x, 0) \leq u_0(x) \ \ (resp. \ \ \geq)$$

*for all $x \in \mathbb{R}^{\mathrm{N}}$, then u is a viscosity subsolution (resp. viscosity supersolution) of the integro-differential Cauchy problem (4.1.1). An arbitrary function $u : \mathbb{R}^{\mathrm{N}} \times [0, T) \longrightarrow \mathbb{R}$ is a **weak viscosity solution** for the problem (4.1.1) if its upper/lower semicontinuous envelopes belong to $\mathcal{USC}^{\mathcal{I}}/\mathcal{LSC}^{\mathcal{I}}$, respectively, and they are sub/supersolutions. Besides, it is a (strong) **viscosity solution** for the problem (4.1.1) if it is both a sub/super solution.*

In this framework, using Perron method and comparison principles and with some further assumptions, it is possible to give a result of existence, uniqueness and regularity for the solutions of the integro-differential Cauchy problem associated to problem (7.1.1). For more precise statements and proofs we refer to [2].

Theorem 4.1.5 *Assume that the parameters α, σ, r satisfy some proper regularity conditions of continuity and Lipschitz continuity and the market is without arbitrage opportunities; then, for every final value $G \in \mathcal{C}((0, +\infty)^{\mathrm{N}}) \cap W^{1,\infty}_{\mathrm{pol}}((0, +\infty)^{\mathrm{N}})$, the integro-differential Cauchy problem* (7.1.1) *has an unique viscosity solution V, in the sense of Definition 4.1.4, which belongs to $L^{\infty}(0, T; W^{1,\infty}_{\mathrm{pol}}((0, +\infty)^{\mathrm{N}}))$. Moreover, comparison principle applies and we have that $U \geq 0$ whenever $G \geq 0$.*

For detailed assumptions and proof see Amadori [2].

4.2 The unbounded Lévy case.

In this section we prove a comparison principle for viscosity solutions of (4.0.1)–(4.0.2). Let us start by introducing some notation: in what follows, it will be useful to isolate the singularity of ν at the origin, therefore we split $\mathcal{J}u$ into

$$\mathcal{J}_{\kappa}\left(u, \varphi\right)(x, t) = \hat{\mathcal{J}}_{\kappa}\varphi(x, t) + \hat{\mathcal{J}}^{\kappa}\left(u, \mathcal{D}\varphi(x, t)\right)(x, t),$$

where

$$E_\kappa = \{z \in \mathbb{R}^2 \,:\, 0 < |z| < \kappa\}\,, \qquad E_\kappa^c = \mathbb{R}^2 - E_\kappa,$$

$$\hat{\mathcal{J}}_\kappa \varphi(x,t) \;=\; \int_{E_\kappa} [\varphi(x + \mathbf{x}\beta(x,t,z),t) - \varphi(x,t) - \beta(x,t,z)\cdot \mathbf{x}\mathcal{D}\varphi(x,t)]\,\nu(dz),$$

$$\hat{\mathcal{J}}^\kappa\,(u,p)\,(x,t) \;=\; \int_{E_\kappa^c} [u(x + \mathbf{x}\beta(x,t,z),t) - u(x,t) - \beta(x,t,z)\cdot \mathbf{x}p]\,\nu(dz),$$

We make the following assumptions on the integral operator $\mathcal{J}$:

A.1 ν is a Radon measure on E satisfying the integrability assumptions:

$$(A.1.i) \qquad \int_E (1 \wedge |z|^2)\nu(dz) < \infty,$$

$$(A.1.ii) \qquad \int_{E_\kappa^c} e^{\ell(z_1^+ + |z_2|)}\nu(dz) < \infty,$$

for some given $\ell > 0$ and for all $\kappa > 0$.

Here and henceforth z_1^+ and z_1^- stand for $(0 \vee z_1) = \max(0, z_1)$ and $(0 \vee -z_1)$, respectively.

A.2 $\beta \in \mathcal{C}\big(\overline{\Pi}_T \times E; \mathbb{R}^{\mathrm{N}}\big)$ and there exist $n_o \geq 1$ such that

$$(A.2.i) \qquad e^{-\frac{\ell}{n_o}(z_1^- + |z_2|)} \leq \beta_i(x,t,z) + 1 \leq e^{\frac{\ell}{n_o}(z_1^+ + |z_2|)}$$

for all $(x,t) \in \overline{\Pi}_T$, $z \in E$, and $i = 1, \ldots, \mathrm{N}$. Moreover for all $r > 1$, there exist $\mathrm{C}_r > 0$ such that

$$(A.2.ii) \qquad |\beta(x,t,z) - \beta(x',t,z)| \leq \mathrm{C}_r e^{\ell(z_1^+ + |z_2|)}|x - x'|,$$

for all $x, x' \in \mathcal{Q}(r)$, $t \in [0,T]$, $z \in E$.

Remark 4.2.1 All results can be trivially extended to the case $E = \mathbb{R}^{\mathrm{M}} - \{0\}$ for an arbitrary integer M. In that case $z_1^\pm$ and $|z_2|$ have to be replaced by $\sum_{i=1}^{\mathrm{M}'} z_i^\pm$ and $\sum_{i=1+\mathrm{M}'}^{\mathrm{M}} |z_i|$, respectively. ■

In force of assumption (A.2.i), β behaves like z near the origin. Thus the following lemma can be shown:

Lemma 4.2.2 *Under assumptions* **A.1** *and* **A.2**, $\hat{\mathcal{J}}_\kappa\varphi(x,t)$ *is finite for all* $\varphi \in \mathcal{C}^{2,1}\big(\Pi_T\big)$, $(x,t) \in \Pi_T$, *and* $\kappa > 0$. *Moreover* $\lim_{\kappa\to 0} \hat{\mathcal{J}}_\kappa(\varphi)(x,t) = 0$.

However in many applications the solution is not $\mathcal{C}^2$, and consequently solutions have to be interpreted in the viscosity sense.

Let us mention that $\mathbf{x}\beta$ stands for the jump of the underlying Lévy process, so that (i) $\beta_i \sim +\infty$ means that the process is jumping towards infinity, and (ii) $\beta_i \sim -1$ means that it is approaching the boundary $\partial\Pi$ at some point of the plane $\{x_i = 0\}$. The structure condition (A.2.i) yields that (i) can happen if $z_1 \sim +\infty$ or $z_2 \sim \pm\infty$, while (ii) can happen if $z_1 \sim -\infty$ or $z_2 \sim \pm\infty$. Up to rearranging the order of the variables, we may assume without loss of generality that there exists an integer $\mathrm{N}' \in \{0, 1, \ldots, \mathrm{N}\}$ and $\mathrm{B} > 0$ such that

$$\beta_i(x,t,z) + 1 \geq \mathrm{B}e^{-\frac{\ell}{n_o}|z_2|} \qquad \text{as } i = 1 + \mathrm{N}', \ldots, \mathrm{N},$$

for all $(x,t) \in \overline{\Pi}_T$, and $z \in E$. In other words, the underlying process approaches the region $\partial\Pi - \Gamma = \{x \in \partial\Pi : x_i = 0 \text{ for some } i = 1 + \mathrm{N}', \cdots, \mathrm{N}\}$ of the boundary only when $z_2 \sim \pm\infty$, and not when $z_1 \sim -\infty$. In force of assumption (A.1.ii), this matter is "unlikely", because the measure ν charges neighborhoods of $z_2 = \pm\infty$ less than neighborhoods of $z_1 = -\infty$. In some sense, the region $\partial\Pi - \Gamma$ is further away than Γ, at least as far as infinity is concerned. This suggests that the points of $\partial\Pi - \Gamma$ can be regarded as points at infinity, where the solutions are allowed to blow up. To be precise, the following lemma holds.

Lemma 4.2.3 *Under assumptions* **A.1** *and* **A.2**, $\hat{\mathcal{J}}^\kappa\,(u,p)\,(x,t)$ *is finite for any function* $u \in L^\infty\big(0,T;\,\mathcal{P}_{n_o}(\Pi_T)\big)$, $p \in \mathbb{R}^{\mathrm{N}}$, $(x,t) \in \Pi_T$, *and* $\kappa > 0$.

Remark 4.2.4 It is easily seen that the conclusions of Lemmas 4.2.2 and 4.2.3 also hold for $x \in \tilde{\Pi}$, if $\varphi \in \mathcal{C}^{2,1}(\tilde{\Pi}_T)$ and u is extended to $\tilde{\Pi}_T$ by giving any value between $\liminf_{\Pi\times[0,T)\ni(y,s)\to(x,t)} u(y,s)$ and $\limsup_{\Pi\times[0,T)\ni(y,s)\to(x,t)} u(y,s)$, respectively. ■

Both the limit situations $\mathrm{N}' = \mathrm{N}$ and $\mathrm{N}' = 0$ may happen in applications.

Remark 4.2.5 Take a market composed by two assets which always jump in the same direction. They can be modelled by pure jump Lévy processes with $z \in \mathbb{R}$, $\beta_i(x,t,z) = e^{\eta_i z} - 1$ and η_i are given positive constants, $i = 1, 2$, see Section 5.1. If, on the contrary, the two assets jump in opposite directions, then we must have $\eta_1 > 0$ and $\eta_2 < 0$. Both these matters are interesting in mathematical finance. The first one modeling perfect correlation between the assets, the second one modeling anti–correlation. They could be addressed in our framework by taking $\operatorname{supp}\nu$ as $\mathbb{R} \times \{0\}$ in the first case and as $\{0\} \times \mathbb{R}$ in the second case. Note that in any case assumption (A.1.ii) yields that the first momentum is finite. ■

Concerning the integro–differential equation (4.0.1), we suppose that H is degenerate elliptic, satisfies some continuity properties and is monotone with respect to the non–local term $\mathcal{J}u$. Denote by $\mathcal{S}^{\mathrm{N}}$ the set of symmetric $\mathrm{N}\times\mathrm{N}$ real matrices. More precisely, we make then the following assumptions:

A.3 $H \in \mathcal{C}\big(\tilde{\Pi}^{\star}_{T}\times\mathbb{R}\times\mathbb{R}^{\mathrm{N}}\times\mathcal{S}^{\mathrm{N}}\times\mathbb{R}\big)$ and there exist a real number L and a modulus of continuity ω such that

$$(A.3.i)\quad H(x,t,u,p,M,J)-H(x,t,u',p,M,J)\geq \mathrm{L}(u-u') \qquad \text{if } u\geq u',$$

$$(A.3.ii)\ |H(x,t,u,p,M,J)-H(x,t,u,p',M',J)|\leq \omega\left(|p-p'|+\|M-M'\|\right),$$

$$(A.3.iii)\quad 0\geq H(x,t,u,p,M,J)-H(x,t,u,p,M,J')\geq -\omega\left(J-J'\right) \quad \text{if } J\geq J',$$

for all $(x,t)\in\tilde{\Pi}^{\star}_{T}$, $u,u',J,J'\in\mathbb{R}$, $p,p'\in\mathbb{R}^{\mathrm{N}}$, and $M,M'\in\mathcal{S}^{\mathrm{N}}$.
Moreover there is a family of moduli of continuity $(o_{r,R})_{r>1,R>0}$ such that

$$(A.3.iv)\quad \begin{aligned} &H(x,t,u,\frac{1}{\varepsilon}\mathbf{x}(x-y),\mathbf{x}M\mathbf{x}^{T},J)-H(y,t,u,\frac{1}{\varepsilon}\mathbf{y}(x-y),\mathbf{y}N\mathbf{y}^{T},J)\\ &\qquad\geq o_{r,R}\Big(\frac{1}{\varepsilon}|x-y|^{2}+|x-y|\Big),\end{aligned}$$

for all $x,y\in\mathcal{Q}(r)$, $t\in[0,T]$, $u\in[-R,R]$, $J\in\mathbb{R}$ and $M,N\in\mathcal{S}^{\mathrm{N}}$ satisfying

$$\begin{pmatrix} M & 0\\ 0 & -N\end{pmatrix}\leq\frac{3}{\varepsilon}\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}.$$

Remark 4.2.6 It is possible to extend to the case when H also depends on an integral term of type (4.0.4): it suffices that (A.3.ii) holds also with respect to $\mathcal{I}u$ and that

A.4 $\gamma\in\mathcal{C}\big(\overline{\Pi}_{T}\times E\big)$ satisfies

$$(A.4.i)\qquad 0\leq\gamma(x,t,z)\leq \mathrm{B}'(1\wedge|z|),$$

for all $(x,t)\in\overline{\Pi}_{T}$, $z\in E$. Moreover for all $r>1$ there exist $\mathrm{C}'_{r}>0$ such that

$$(A.4.ii)\qquad |\gamma(x,t,z)-\gamma(y,t,z)|\leq \mathrm{C}'_{r}(1\wedge|z|^{2})|x-y|,$$

for all $x,y\in\mathcal{Q}(r)$, $t\in[0,T]$, $z\in E$.

■

Now we recall the notion of viscosity solutions for equation (4.0.1).

Definition 4.2.7 *Let $\mathcal{O} \subset \overline{\Pi}$ contain Π, and set $\mathcal{O}_T = \mathcal{O} \times [0, T)$. If u is any locally bounded function and $(x, t) \in \mathcal{O}_T$, we say that the inequality*

$$-\partial_t u + H(x, t, u, \mathbf{x}\mathcal{D}u, \mathbf{x}\mathcal{D}^2 u\mathbf{x}^T, \mathcal{J}u) \leq 0 \qquad \text{(respectively, } \geq 0\text{)}$$

holds true in the **viscosity sense** *at (x, t) (with respect to $\mathcal{O}_T$) if for any test function $\varphi \in \mathcal{C}^{2,1}(\mathcal{O}_T) \cap \bigcup_{n<n_o} L^\infty(0, T; \mathcal{P}_n(\Pi))$ such that $u(x, t) = \varphi(x, t)$ and (x, t) is a global maximum (respectively, minimum) point for $u - \varphi$ on $\mathcal{O}_T$, the inequality*

$$-\partial_t \varphi + H\big(x, t, u, \mathbf{x}\mathcal{D}\varphi, \mathbf{x}\mathcal{D}^2\varphi\mathbf{x}^T, \mathcal{J}_\kappa(u, \varphi)\big) \leq 0 \qquad \text{(respectively, } \geq 0\text{)} \tag{4.2.1}$$

holds true in classical sense at (x, t) for all $\kappa > 0$.

A locally bounded function u that is upper semicontinuous (respectively, lower semicontinuous) on Π_T is a **viscosity subsolution** *(respectively,* **viscosity supersolution***) of (4.0.1) if*

$$-\partial_t u + H(x, t, u, \mathbf{x}Du, \mathbf{x}D^2 u\mathbf{x}^T, \mathcal{J}u) \leq 0,$$

(respectively, ≥ 0) holds true in the viscosity sense for all $(x, t) \in \Pi_T$ (with respect to Π_T). Any locally bounded function u on Π_T is a **viscosity solution** *to (4.0.1) if its upper semicontinuous envelope is a viscosity subsolution and its lower semicontinuous envelope is a viscosity supersolution.*

We refer to [34] for definitions of the upper and lower semicontinuous envelopes.

Remark 4.2.8 By making use of assumption (A.3.v) and following [12, Lemma 3.4], it is clear that one may replace $\mathcal{J}_\kappa(u, \varphi)$ by $\mathcal{J}\varphi$ in the Definition 4.2.7. In the same way, one may replace global maximum (respectively, minimum) point by strict global maximum (respectively, minimum) point. ■

We emphasize that the study of equation (4.0.1) can not in general be reduced to the study of an equation in $\mathbb{R}^{\mathrm{N}} \times [0, T)$ of the type previously considered in [12] or in [101]. Indeed, the change of variables $\hat{x}_i = \log x_i$ as $i = 1, \cdots, \mathrm{N}$ maps (4.0.1) into

$$-\partial_t u + H(\hat{x}, t, u, \mathcal{D}u, \mathcal{D}^2 u, \hat{\mathcal{J}}u) = 0, \qquad (\hat{x}, t) \in \mathbb{R}^{\mathrm{N}} \times [0, T),$$

where the new non–local operator

$$\hat{\mathcal{J}}u(\hat{x}, t) = \int_E \Big[u(\hat{x} + \hat{\beta}, t) - u(\hat{x}, t) - \sum_{i=1}^{\mathrm{N}} \Big(e^{\hat{\beta}_i} - 1\Big)\, \partial_{\hat{x}_i} u(\hat{x}, t)\Big]\nu(dz),$$

$$\hat{\beta}_i(\hat{x}, t, z) = \log\big(1 + \beta_i(e^{\hat{x}_1}, \cdots, e^{\hat{x}_{\mathrm{N}}}, t, z)\big),$$

has a different structure than (4.0.3). Therefore we need to face two matters: (i) giving a suitable notion of terminal condition on $\partial\Pi \times \{T\}$ and (ii) dealing with the boundary $\partial\Pi_T$, where no data may feasibly be imposed. With respect to the notion of terminal condition, we choose here the naive one.

Definition 4.2.9 *A locally bounded function u that is upper semicontinuous (respectively, lower semicontinuous) on Π_T is a* **viscosity subsolution** *(respectively, a* **viscosity supersolution***) to (4.0.1)–(4.0.2) if it is a viscosity subsolution (respectively, a viscosity supersolution) to (4.0.1) in the sense of Definition 4.2.7 and*

$$\limsup_{\Pi_T \ni (y,s)\to(x,T)} u(y,s) \le \liminf_{\Pi\ni y\to x} u_T(y) \qquad \text{in } \tilde{\Pi},$$

$$\text{(respectively,} \quad \liminf_{\Pi_T \ni (y,s)\to(x,T)} u(y,s) \ge \limsup_{\Pi\ni y\to x} u_T(y) \qquad \text{in } \tilde{\Pi}\text{)}.$$

Any locally bounded function u on Π_T is a **viscosity solution** *to (4.0.1)–(4.0.2) if its upper semicontinuous envelope is a viscosity subsolution and its lower semicontinuous envelope is a viscosity supersolution.*

We devote the next subsection to an investigation of the behavior of sub- and supersolutions near the boundary of Π_T.

4.2.1 Behavior at the boundary.

The difficulty of dealing with a boundary where no data are assigned has been overcome in [21] by investigating constrained solutions. With respect to equations of type (4.0.1) such expedient is not necessary. In view of clarifying this issue, that shall play a central role in Theorem 4.2.14, we give a notion of constrained solution which fits with equations of type (4.0.1).

Definition 4.2.10 *A viscosity subsolution (respectively, viscosity supersolution) u to (4.0.1) is a* **constrained subsolution** *(respectively, a* **constrained supersolution***) if it is upper semicontinuous (respectively, lower semicontinuous) on $\tilde{\Pi}_T$ and the viscosity inequality (4.2.1) holds true also at the points $(x,t) \in \Gamma \times [0,T)$. A* **constrained solution** *to (4.0.1) is any locally bounded function u such that the upper and lower semicontinuous envelopes of u are respectively constrained sub- and supersolutions to (4.0.1).*

Notice that, if $\mathrm{N}' = \mathrm{N}$, then $\tilde{\Pi} = \overline{\Pi}$ and our definition of constrained subsolution reduces to [21, Definition 4.1], up to the fact that we deal with possibly discontinuous solutions.

As we have mentioned before, the points in $\partial\Pi - \Gamma$ have the character of being points at infinity. We will show that imposing a blow up rate is sufficient to pick up a unique solution. On the other hand, Γ is made up by points that cannot be reached by the trajectories of the underlying process by drift motion (because of the structure $\mathbf{x}Du$), nor by diffusion (because of the structure $\mathbf{x}D^2u\mathbf{x}^T$), nor by jumping (because of the structure $\mathbf{x}\beta$ with $\beta_i > -1$). Inspired by a similar result in [15], we shall establish that (4.0.1) holds also at Γ, up to boundary discontinuities, that is to say, any solution is constrained.

To this aim, for any upper semicontinuous function u (respectively, lower semicontinuous function v) we set

$$\tilde{u}(x,t) \;=\; \begin{cases} u(x,t) & \text{if } (x,t) \in \Pi\times [0,T), \\ \limsup\limits_{\Pi_T \ni (y,s)\to(x,t)} u(y,s) & \text{if } (x,t) \in \Gamma\times [0,T), \end{cases} \tag{4.2.2}$$

$$\tilde{v}(x,t) \;=\; \begin{cases} v(x,t) & \text{if } (x,t) \in \Pi\times [0,T), \\ \liminf\limits_{\Pi_T \ni (y,s)\to(x,t)} v(y,s) & \text{if } (x,t) \in \Gamma\times [0,T), \end{cases} \tag{4.2.3}$$

and then show some continuity of the non–local operators $\mathcal{I}$ and $\mathcal{J}$.

Lemma 4.2.11 *Assume* **A.1**–**A.3**. *With $(x,t) \in \tilde{\Pi}_T$, let $\varphi \in \mathcal{C}^{2,1}(\Pi_T) \cap L^\infty\big(0,T;\mathcal{P}_{n_o}(\Pi)\big)$ if $(x,t) \in \Pi_T$ and $\varphi \in \mathcal{C}^{2,1}(\tilde{\Pi}^\star_T) \cap L^\infty\big(0,T;\mathcal{P}_{n_o}(\Pi)\big)$ if $(x,t) \in \Gamma\times[0,T)$. Let u be an upper semicontinuous (respectively, a lower semicontinuous) function on Π_T belonging to $L^\infty\big(0,T;\mathcal{P}_{n_o}(\Pi)\big)$. Finally, let (x_n,t_n) be a sequence in Π_T such that $(x_n,t_n,u(x_n,t_n)) \to (x,t,\tilde{u}(x,t))$. Then*

$$\lim_{n\to\infty} \hat{\mathcal{J}}_\kappa\varphi(x_n,t_n) = \hat{\mathcal{J}}_\kappa\varphi(x,t), \;\; \textit{and} \;\; \limsup_{n\to\infty} \hat{\mathcal{J}}^\kappa(u,\varphi)(x_n,t_n) \le \hat{\mathcal{J}}^\kappa(\tilde{u},\varphi)(x,t),$$

(respectively, $\liminf\limits_{n\to\infty} \hat{\mathcal{J}}^\kappa(u,\varphi)(x_n,t_n) \ge \hat{\mathcal{J}}^\kappa(\tilde{u},\varphi)(x,t)$), for any fixed κ.

Lemma 4.2.11 is an immediate consequence of Lebesgue's dominated convergence theorem, that may be applied thanks to assumptions **A.1** and **A.2** .

Proposition 4.2.12 *Let u be an $\mathcal{USC}$ viscosity subsolution and v a $\mathcal{LSC}$ viscosity supersolution of (4.0.1), both belonging to $\bigcup\limits_{n<n_o} L^\infty\big(0,T;\mathcal{P}_n(\Pi)\big)$. Then the functions $\tilde{u}$ and $\tilde{v}$ defined in* (4.2.2) *and* (4.2.3) *are constrained sub and -supersolutions, respectively.*

Proof. We carry out the proof only for the subsolution case, the supersolution case being completely analogous.
Take $(x_o, t_o) \in \Gamma \times [0, T)$ and $\varphi \in \mathcal{C}^{2,1}\big(\tilde{\Pi}^\star_T\big) \bigcup\limits_{n<n_o} L^\infty\big(0, T; \mathcal{P}_n(\Pi)\big)$ such that $\varphi(x_o, t_o) = \tilde{u}(x_o, t_o)$ and (x_o, t_o) is a strict global maximum point for $\tilde{u} - \varphi$. We want to perturb the test function φ in order to achieve a sequence of maximum points in Π_T converging to (x_o, t_o). To this aim, we introduce the auxiliary functions $a, b \in \mathcal{C}^2(0, \infty)$

$$a(\xi) = \begin{cases} -\log(\xi) & \text{if } \xi \in (0, 1/e], \\ \text{is decreasing and polynomial} & \text{if } \xi \in (1/e, 1), \\ 0 & \text{if } \xi \in [1, \infty), \end{cases}$$

$$b(\xi) = \begin{cases} 0 & \text{if } \xi \in (0, 1], \\ \text{is increasing and polynomial} & \text{if } \xi \in (1, 2), \\ \xi^{n_o} & \text{if } \xi \in [2, \infty), \end{cases}$$

with the additional properties that $|\xi\, a'(\xi)|, |\xi a''(\xi)| \le \mathrm{A}_o$ and $|\xi b'(\xi)|, |\xi b''(\xi)| \le \mathrm{B}_o b(\xi)$. For any given $\alpha, \delta > 0$ we then set

$$\mathbf{a}(x) = \sum_{i=1}^{\mathrm{N}'} a(x_i), \qquad \mathbf{b}_\alpha(x) = \sum_{i=1}^{\mathrm{N}} b\left(\tfrac{x_i}{\alpha}\right),$$

$$\varphi_{\alpha\delta}(x, t) = \varphi(x, t) + \mathbf{b}_\alpha(x) + \delta\left[\mathbf{a}(x) + h_o(x)\right],$$

$$\Psi_{\alpha\delta}(x, t) = \tilde{u}(x, t) - \varphi_{\alpha\delta}(x, t).$$

Here and in the following we write h_o instead of h_{n_o}. We list in a lemma, to be proved later on, some relevant properties of $\Psi_{\alpha\delta}$.

Lemma 4.2.13 *For any given $\alpha > 1 + 2|x_o|$, there exists an infinitesimal sequence of parameters δ such that $\Psi_{\alpha\delta}$ achieves its global maximum at some point $(x_{\alpha\delta}, t_{\alpha\delta}) \in \Pi \times [0, T]$. In addition, up to an extracted subsequence,*

$$(x_{\alpha\delta}, t_{\alpha\delta}) \to (x_o, t_o), \qquad u(x_{\alpha\delta}, t_{\alpha\delta}) \to \tilde{u}(x_o, t_o), \qquad \textit{as } \delta \to 0, \tag{4.2.4}$$

$$\delta \mathbf{a}(x_{\alpha\delta}) \to 0, \qquad \delta h_o(x_{\alpha\delta}) \to 0, \qquad \textit{as } \delta \to 0. \tag{4.2.5}$$

As a consequence, we may suppose that (i) for α large enough, $x_{\alpha\delta} \in [0, \alpha]^{\mathrm{N}}$ and $t_{\alpha\delta} < T$ for small δ, and that (ii) $(x_{\alpha\delta}, t_{\alpha\delta}) \in \Pi_T$ is a global maximum point for $u - \varphi_{\alpha\delta}$ on Π_T. Therefore the viscosity inequality (4.2.1) is satisfied at $(x_{\alpha\delta}, t_{\alpha\delta})$ for any δ and α.

In view of passing to the limit, we recall that for all $x, q \in \Pi$ and we have

$$|\mathbf{x} D h_o(x)|\ ,\ \left|\mathbf{x} D^2 h_o(x) \mathbf{x}^T\right| \le \mathrm{C} h_o(x), \qquad h_o(\mathbf{x}q) \le h_o(x) h_o(q), \tag{4.2.6}$$

where the parameter C does not depend on (x,t).

By taking advantage of (4.2.4)–(4.2.6) and of assumption **A.3**, taking the limits with respect to $\delta \to 0$ and $\alpha \to \infty$ gives

$$-\partial_t\varphi(x_o,t_o) + H\Big(x_o,t_o,\tilde{u},\mathbf{x}_o\mathcal{D}\varphi,\mathbf{x}_o\mathcal{D}^2\varphi\mathbf{x}_o^T,\limsup_{\substack{\delta\to 0\\ \alpha\to\infty}}\mathcal{J}_\kappa(u,\varphi_{\alpha\delta})(x_{\alpha\delta},t_{\alpha\delta})\Big) \le 0.$$

Therefore, in force of (A.3.iii), we may conclude by checking that

$$\limsup_{\substack{\delta\to 0\\ \alpha\to\infty}}\mathcal{J}_\kappa(u,\varphi_{\alpha\delta})(x_{\alpha\delta},t_{\alpha\delta}) \le \mathcal{J}_\kappa(\tilde{u},\varphi)(x_o,t_o).$$

After some computations we obtain

$$\mathcal{J}_\kappa(u,\varphi_\delta)(x_\delta,t_\delta) \le \mathcal{J}_\kappa(u,\varphi)(x_\delta,t_\delta) + \mathrm{B}\int_{E_\kappa}\mathbf{b}_\alpha(\mathbf{x}_{\alpha\delta}\hat{q}_{\alpha\delta}(z))\sum_{i=1}^{\mathrm{N}}\frac{\beta_i(x_{\alpha\delta},t_{\alpha\delta},z)^2}{\hat{q}_{\alpha\delta\,i}(z)^2}\nu(dz)$$
$$+\delta\left(\mathrm{A}+\mathrm{C}h_o(x_\delta)\right)\Big[\int_{E_\kappa^c}|\beta(x_{\alpha\delta},t_{\alpha\delta},z)|\nu(dz) + \int_{E_\kappa}h_o(\hat{q}_{\alpha\delta}(z))\sum_{i=1}^{\mathrm{N}}\frac{\beta_i(x_{\alpha\delta},t_{\alpha\delta},z)^2}{\hat{q}_{\alpha\delta\,i}(z)^2}\nu(dz)\Big],$$

where $\hat{q}_{\alpha\delta}(z)$ stands for a suitable point in the segment line joining the points $(1,\cdots,1)\in\mathbb{R}^{\mathrm{N}}$ and $(1+\ \beta_1(x_{\alpha\delta},t_{\alpha\delta},z),\ \cdots,\ 1+\beta_{\mathrm{N}}(x_{\alpha\delta},t_{\alpha\delta},z))\in\mathbb{R}^{\mathrm{N}}$. Keeping in mind **A.2**, taking the limit with respect to δ and applying Lemma 4.2.11 gives

$$\limsup_{\delta\to 0}\mathcal{J}_\kappa(u,\varphi_{\alpha\delta})(x_\delta,t_\delta) \le \mathcal{J}_\kappa(u,\varphi)(x_o,t_o) + \mathrm{C}\max_{Q_\kappa}\mathbf{b}_\alpha\int_{E_\kappa}|z|^2\nu(dz),$$

where the parameter C does not depend by α, and $Q_\kappa = \big[0,2|x_o|\big(1+e^{\frac{\ell}{n_o}\kappa}\big)\big]^{\mathrm{N}}$. Eventually we may conclude by checking that $\max_{Q_\kappa}\mathbf{b}_\alpha \to 0$ as $\alpha\to\infty$, for any given $\kappa>0$. But this is trivial because $\mathbf{b}_\alpha\equiv 0$ on $[0,\alpha]^{\mathrm{N}}$ by construction. ∎

Proof of Lemma 4.2.13. Take $\alpha > 1+2|x_o|$ and set

$$M_{\alpha\delta} = \sup\left\{\tilde{u}(x,t)-\varphi_{\alpha\delta}(x,t)\right\}.$$

Notice that $M_{\alpha\delta}\le\max\{\tilde{u}-\varphi\}=0$. Next, take a maximizing sequence $(x_i,t_i)\in\Pi_T$ such that $(x_i,t_i,u(x_i,t_i)-\varphi(x_i,t_i))\to(x_o,t_o,0)$ and choose an infinitesimal sequence $\delta_i>0$ in such a way that $\delta_i\left[\mathbf{a}(x_i)+h_o(x_i)\right]\le 1/i$. Without loss of generality, we may suppose that $x_i\in[0,1+2|x_o|]$ for all i, so that $\mathbf{b}_\alpha(x_i)=0$ and therefore $M_{\alpha\delta_i}\to 0$ as $i\to\infty$. On the other hand, there exists $n<n_o$ such that $\tilde{u}-\varphi\in L^\infty\big(0,T;\mathcal{P}_n(\Pi)\big)$. Denoting by $\mathrm{C}=\sup_{t\in[0,T)}\|\tilde{u}(\cdot,t)-\varphi(\cdot,t)\|_n$ we have

$$\Psi_{\delta_j}(x,t) \le \sum_{i=1}^{\mathrm{N}}\left(\mathrm{C}-\delta_j x_{j_i}^{\,n_o-n}\right)x_{j_i}^{\,n} + \sum_{i=1+\mathrm{N}'}^{\mathrm{N}}(\mathrm{C}-\delta_j x_{j_i}^{\,-n_o+n})x_{j_i}^{\,-n} \to -\infty,$$

if $|x| \to \infty$ or $x \to \overline{x} \in \partial\Pi - \Gamma$. Hence standard semicontinuity arguments give that $\Psi_{\alpha\delta_i}$ has a maximum point $(x_{\alpha\delta_j}, t_{\alpha\delta_j}) \in \tilde{\Pi} \times [0,T]$. Moreover, $\Psi_{\alpha\delta_j}(x_{\alpha\delta_j}, t_{\alpha\delta_j}) \to 0$ as $j \to \infty$. As a consequence

$$\lim_{j\to\infty} \left\{ \delta_j \left[\mathbf{a}(x_{\alpha\delta_j}) + h_o(x_{\alpha\delta_j}) \right] + \mathbf{b}_\alpha(x_{\alpha\delta_j}) \right\} = \lim_{j\to\infty} \left[\tilde{u}(x_{\alpha\delta_j}, t_{\alpha\delta_j}) - \varphi(x_{\alpha\delta_j}, t_{\alpha\delta_j}) \right].\left.\right]\left.\right\}$$

Because the limit appearing in the right–hand side is less than or equal to zero by construction, (4.2.5) is established. In addition, $\lim_{j\to\infty} \mathbf{b}_\alpha(x_{\alpha\delta_j}) = 0$ allows to suppose that $x_{\alpha\delta_i}$ is bounded uniformly with respect to j (for fixed α). As a consequence, up to an extracted subsequence, $(x_{\alpha\delta_j}, t_{\alpha\delta_j})$ tends to some $(\hat{x}, \hat{t}) \in \tilde{\Pi} \times [0,T]$, so that standard semicontinuity arguments give (4.2.4). ■

4.2.2 Comparison Principle.

We now prove a comparison result between (semicontinuous) viscosity sub- and supersolutions which satisfy a suitable growth condition near $\partial\Pi$ and for large x.

Theorem 4.2.14 *Assume* **A.1**–**A.3** *and let* $u, v \in \bigcup_{n<n_o} L^\infty\big(0,T;\mathcal{P}_n(\Pi)\big)$ *be respectively an upper semicontinuous subsolution and a lower semicontinuous supersolution to (4.0.1) such that*

$$\limsup_{\Pi_T \ni (y,s)\to(x,T)} u(y,s) \le \liminf_{\Pi_T \ni (y,s)\to(x,T)} v(y,s) \quad \textit{for all } x \in \tilde{\Pi}. \tag{4.2.7}$$

Then $u \le v$ *on* $\Pi \times [0,T]$. *Moreover, the functions* $\tilde{u}$ *and* $\tilde{v}$ *defined in (4.2.2) and (4.2.3) satisfy* $\tilde{u} \le \tilde{v}$ *on* $\tilde{\Pi} \times [0,T]$.

Before proving Theorem 4.2.14, we explicitly mention an immediate consequence of it.

Corollary 4.2.15 *Assume* **A.1**–**A.3** *and take* $u_T \in \mathcal{C}(\tilde{\Pi}) \cap \mathcal{P}_n(\Pi)$ *for some* $n < n_o$. *Let* $u \in \bigcup_{n<n_o} L^\infty\big(0,T;\mathcal{P}_n(\Pi)\big)$ *be a (possibly discontinuous) viscosity solution to (4.0.1)–(4.0.2). Then* u *is the unique viscosity solution in the class* $\bigcup_{n<n_o} L^\infty\big(0,T;\mathcal{P}_n(\Pi)\big)$. *Moreover it is continuous on* Π_T *and can be extended continuously to* $\tilde{\Pi} \times [0,T]$ *by setting*

$$\mathbf{u}(x,t) = \begin{cases} u(x,t) & \textit{if } (x,t) \in \Pi_T, \\ \lim\limits_{\Pi_T \ni (y,s)\to(x,t)} u(y,s) & \textit{if } (x,t) \in \Gamma \times [0,T] \cup \Pi \times \{T\}. \end{cases}$$

The function $\mathbf{u}$ *still solves (4.0.1)–(4.0.2), and satisfies* $\mathbf{u}(x,T) = u_T(x)$ *for all* $x \in \tilde{\Pi}$.

Let us now give the proof of the comparison principle.

Proof of Theorem 4.2.14. Before entering into the details of the proof, we recall that we may assume that the parameter L appearing in hypothesis (A.3.i) is equal to -1 (if not, we simply multiply u and v by $\exp(-(1-\mathrm{L})(T-t))$).

The theorem follows if we can show that $\tilde{u} \le \tilde{v}$ on $\tilde{\Pi} \times [0,T]$. We argue by contradiction, and suppose that

$$\mathcal{M} = \sup\left\{\tilde{u}(x,t) - \tilde{v}(x,t) \,:\, (x,t) \in \tilde{\Pi} \times [0,T]\right\} > 0.$$

Because it is not known a priori whether $\mathcal{M}$ is finite or not, we approximate $\tilde{u}$ and $\tilde{v}$ by

$$\tilde{u}_\delta(x,t) = \tilde{u}(x,t) - \delta h_o(x), \qquad \tilde{v}_\delta(x,t) = \tilde{v}(x,t) + \delta h_o(x),$$

and we look at the upper semicontinuous function

$$\Psi_\delta(x,t) = \tilde{u}_\delta(x,t) - \tilde{v}_\delta(x,t), \qquad (x,t) \in \tilde{\Pi} \times [0,T].$$

Arguing as in the proof of Lemma 4.2.13, one may easily check that there exists an infinitesimal sequence of parameters δ (that we still denote by δ) such that Ψ_δ has a maximum point $(x_\delta, t_\delta) \in \tilde{\Pi} \times [0,T]$. Moreover

$$\tilde{u}(x_\delta, t_\delta) - \tilde{v}(x_\delta, t_\delta) \to \mathcal{M}, \qquad \delta h_o(x_\delta) \to 0, \tag{4.2.8}$$

as $\delta \to 0$.

Notice that our assumption $\mathcal{M} > 0$ allows us to suppose that $\tilde{u}(x_\delta, t_\delta) - \tilde{v}(x_\delta, t_\delta) > 0$ for all δ and therefore, recalling hypothesis (4.2.7), that $t_\delta < T$ for all δ. Besides $\tilde{u}_\delta$ and $\tilde{v}_\delta$ satisfy in the viscosity sense two modified integro–partial differential inequalities.

Lemma 4.2.16 *There exists* $\mathrm{D} \ge 0$ *not depending on* δ *such that* $\tilde{u}_\delta$ *and* $\tilde{v}_\delta$ *are respectively a constrained subsolution to*

$$-\partial_t \tilde{u}_\delta + H\left(x, t, \tilde{u}_\delta, \mathbf{x}\mathcal{D}\tilde{u}_\delta, \mathbf{x}\mathcal{D}^2\tilde{u}_\delta\mathbf{x}^T, \mathcal{J}\tilde{u}_\delta\right) = -\delta h_o(x) + \omega(\mathrm{D}\delta h_o(x)), \tag{4.2.9}$$

and a constrained supersolution to

$$-\partial_t \tilde{v}_\delta + H\left(x, t, \tilde{v}_\delta, \mathbf{x}\mathcal{D}\tilde{v}_\delta, \mathbf{x}\mathcal{D}^2\tilde{v}_\delta\mathbf{x}^T, \mathcal{J}\tilde{v}_\delta\right) = \delta h_o(x) - \omega(\mathrm{D}\delta h_o(x)), \tag{4.2.10}$$

for all $\delta > 0$.

We shall later on give a sketch of the proof of this statement. By now we prefer to show how to use these integro–partial differential inequalities to get a contradiction. To this aim, we double the x–variable by considering the function

$$\tilde{\Pi}^2 \times [0,T] \ni (x,y,t) \mapsto \tilde{u}_\delta(x,t) - \tilde{v}_\delta(y,t) - \psi(x,y),$$

where the penalization function $\psi(x,y)$ is defined by

$$\psi(x,y) = \frac{1}{2\varepsilon}|x-y|^2. \tag{4.2.11}$$

By a classical argument in the theory of viscosity solutions, for any fixed $\delta, \varepsilon > 0$ this function has a maximum point $(x^\varepsilon_\delta, y^\varepsilon_\delta, t^\varepsilon_\delta) \in \tilde{\Pi}^2 \times [0,T]$ which satisfies

$$x^\varepsilon_\delta,\, y^\varepsilon_\delta \to x_\delta, \qquad t^\varepsilon_\delta \to t_\delta, \qquad \frac{1}{2\varepsilon}|x^\varepsilon_\delta - y^\varepsilon_\delta|^2 \to 0, \tag{4.2.12}$$

$$\tilde{u}_\delta(x^\varepsilon_\delta, t^\varepsilon_\delta) - \tilde{v}_\delta(y^\varepsilon_\delta, t^\varepsilon_\delta) \to \tilde{u}_\delta(x_\delta, t_\delta) - \tilde{v}_\delta(y_\delta, t_\delta),$$

as $\varepsilon \to 0$. We remark that (4.2.12), together with (4.2.8), allows us to suppose without loss of generality that for all $\delta, \varepsilon > 0$

$$x^\varepsilon_\delta,\, y^\varepsilon_\delta \in \mathcal{Q}(r_\delta), \qquad\qquad t^\varepsilon_\delta < T, \tag{4.2.13}$$

$$|u_\delta(x^\varepsilon_\delta, t^\varepsilon_\delta)|,\, |v_\delta(y^\varepsilon_\delta, t^\varepsilon_\delta)| \le R_\delta, \qquad \tilde{u}_\delta(x^\varepsilon_\delta, t^\varepsilon_\delta) - \tilde{v}_\delta(y^\varepsilon_\delta, t^\varepsilon_\delta) > 0, \tag{4.2.14}$$

where $r_\delta > 1$ and $R_\delta > 0$ do not depend on ε.

To simplify notations in what follows, we replace the subscripts δ, ε by over-bars and we omit the time dependence.

It follows from [34, Theorem 8.3] that there exist $\overline{\tau} \in \mathbb{R}$ and two symmetric matrices $\overline{M}$ and $\overline{N}$ such that

$$\begin{pmatrix} \overline{M} & 0 \\ 0 & -\overline{N} \end{pmatrix} \le \frac{3}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

and, since $\tilde{u}_\delta$, $\tilde{v}_\delta$ solve (4.2.9) and (4.2.10) respectively,

$$-\overline{\tau} + H\Big(\overline{x}, \tilde{u}_\delta, \overline{\mathbf{x}p}, \overline{\mathbf{x}}\overline{M}\overline{\mathbf{x}}^T, \hat{\mathcal{J}}_\kappa\psi(\overline{x},\overline{y}) + \hat{\mathcal{J}}^\kappa\left(\tilde{u}_\delta, \overline{p}\right)(\overline{x})\Big) \le -\delta h_o(\overline{x}) + 2\omega(\mathrm{D}\delta h_o(\overline{x})),$$

$$-\overline{\tau} + H\Big(\overline{y}, \tilde{v}(\overline{y}), \overline{\mathbf{y}p}, \overline{\mathbf{y}}\overline{N}\overline{\mathbf{y}}^T, -\hat{\mathcal{J}}_\kappa\psi(\overline{x},\overline{y}) + \hat{\mathcal{J}}^\kappa\left(\tilde{v}_\delta, \overline{p}\right)(\overline{y})\Big) \ge \delta h_o(\overline{y}) - 2\omega(\mathrm{D}\delta h_o(\overline{y})),$$

where $\overline{p} = \frac{1}{\varepsilon}(\overline{x} - \overline{y})$. Subtracting the two inequalities yields

$$\overline{H}_\kappa \le \overline{G} \tag{4.2.15}$$

where

$$\overline{H}_\kappa := H\Big(\overline{x},\overline{t},\tilde{u}_\delta(\overline{x}),\overline{\mathbf{x}p},\overline{\mathbf{x}}\overline{M}\overline{\mathbf{x}}^T,\hat{\mathcal{J}}_\kappa\psi(\overline{x},\overline{y})+\hat{\mathcal{J}}^\kappa\left(\tilde{u}_\delta,\overline{p}\right)(\overline{x})\Big)$$

$$-H\Big(\overline{y},\overline{t},\tilde{v}(\overline{y}),\overline{\mathbf{y}p},\overline{\mathbf{y}}\overline{N}\overline{\mathbf{y}}^T,-\hat{\mathcal{J}}_\kappa\psi(\overline{x},\overline{y})+\hat{\mathcal{J}}^\kappa\left(\tilde{v}_\delta,\overline{p}\right)(\overline{y})\Big),$$

$$\overline{G} := -\delta\left[h_o(\overline{x})+h_o(\overline{y})\right]+\omega\left(\mathrm{D}\delta h_o(\overline{x})\right)+\omega\left(\mathrm{D}\delta h_o(\overline{y})\right).$$

Concerning $\overline{G}$, (4.2.8) and (4.2.12) imply that $\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\overline{G}=0$. Concerning $\overline{H}_\kappa$, by making use of hypothesis **A.3** and remembering (4.2.13), (4.2.14) we obtain

$$\overline{H}_\kappa \geq \left[u_\delta(\overline{x})-v_\delta(\overline{y})\right]+\omega\left((\overline{J}_\kappa)^+\right)-o_{r_\delta,R_\delta}\left(\frac{1}{\varepsilon}|\overline{x}-\overline{y}|^2+|\overline{x}-\overline{y}|\right), \tag{4.2.16}$$

where we have used the short-hand notation

$$\overline{J}_\kappa := 2\hat{\mathcal{J}}_\kappa\psi(\overline{x},\overline{y})+\hat{\mathcal{J}}^\kappa\left(\tilde{u}_\delta,\overline{p}\right)(\overline{x})-\hat{\mathcal{J}}^\kappa\left(\tilde{v}_\delta,\overline{p}\right)(\overline{y}).$$

Now (4.2.12) implies that the third term in the right–hand side of (4.2.16) tends to zero as $\varepsilon\to 0$. Regarding the estimation of the second term, we recall that $\hat{\mathcal{J}}_\kappa\psi(\overline{x},\overline{y})$ tends to zero as $\kappa\to 0$ for any fixed value of ε, by Lemma 4.2.2. Hence we only need to estimate the positive part of

$$\hat{J}_\kappa := \hat{\mathcal{J}}^\kappa\left(\tilde{u}_\delta,\overline{p}\right)(\overline{x})-\hat{\mathcal{J}}^\kappa\left(\tilde{v}_\delta,\overline{p}\right)(\overline{y}),$$

uniformly with respect to $\kappa\in(0,1)$. To this end, we split the integral into the two sets E_1^c, $A_\kappa=\{\kappa<|z|\leq 1\}$ and we write

$$\hat{J}_\kappa = \hat{\mathcal{J}}^1\left(\tilde{u}_\delta,\overline{p}\right)(\overline{x})-\hat{\mathcal{J}}^1\left(\tilde{v}_\delta,\overline{p}\right)(\overline{y})$$

$$+\int_{A_\kappa}\left[\left[\tilde{u}_\delta(\hat{x})-\tilde{v}_\delta(\hat{y})\right]-\left[\tilde{u}_\delta(\overline{x})-\tilde{v}_\delta(\overline{y})\right]-\frac{1}{\varepsilon}\left(\overline{\mathbf{x}}\beta(\overline{x},z)-\overline{\mathbf{y}}\beta(\overline{y},z)\right)\cdot(\overline{x}-\overline{y})\right]\nu(dz)$$

where we have used the shortened notations $\hat{x}=\overline{x}+\overline{\mathbf{x}}\beta(\overline{x},z)$, $\hat{y}=\overline{y}+\overline{\mathbf{y}}\beta(\overline{y},z)$.

With respect to the first term, we notice that

$$\hat{\mathcal{J}}^1\left(\tilde{u}_\delta,\overline{p}\right)(\overline{x})-\hat{\mathcal{J}}^1\left(\tilde{v}_\delta,\overline{p}\right)(\overline{y})\leq$$

$$\int_{E_1^c}\Big[\left[\tilde{u}_\delta(\hat{x})-\tilde{v}_\delta(\hat{y})\right]-\left[\tilde{u}_\delta(\overline{x})-\tilde{v}_\delta(\overline{y})\right]\Big]\nu(dz)+\int_{E_1^c}\frac{1}{\varepsilon}|\overline{x}-\overline{y}|\,|\overline{\mathbf{x}}\beta(\overline{x},z)-\overline{\mathbf{y}}\beta(\overline{y},z)|\,\nu(dz).$$

Assumptions **A.1** and **A.2** allow us to apply Lebesgue's dominated convergence theorem to both the integrals. Therefore (4.2.12) gives

$$\limsup_{\varepsilon\to 0}\left[\hat{\mathcal{J}}^1\left(\tilde{u}_\delta,\overline{p}\right)(\overline{x})-\hat{\mathcal{J}}^1\left(\tilde{v}_\delta,\overline{p}\right)(\overline{y})\right]\le\int_{E_1^c}\left[\Psi_\delta(\hat{x}_\delta,t_\delta)-\Psi_\delta(x_\delta,t_\delta)\right]\nu(dz)\le 0,$$

because (x_δ,t_δ) is a maximum point for Ψ_δ. On the other hand, arguing as in the proof of [22, Theorem 4.2], one obtains that also the second term is bounded from above by some infinitesimal quantity (with respect to ε) that does not depend on κ. By plugging the obtained estimates in (4.2.16) and making use of (4.2.8) we obtain $\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\lim_{\kappa\to 0}\overline{H}_\kappa\ge\mathcal{M}$. Eventually extracting the limit with respect to κ, ε and δ in (4.2.15) gives the contradiction $\mathcal{M}\le 0$. ∎

This section is concluded by checking the validity of Lemma 4.2.16.

Proof of Lemma 4.2.16. We give the proof only for (4.2.9), the proof of (4.2.10) being similar. By construction if φ is a test function for $\tilde{u}_\delta$, then $\varphi_\delta=\varphi+\delta h_o$ is a test function for $\tilde{u}$. In particular for all $\kappa>0$ we have $\mathcal{J}_\kappa(\tilde{u}_\delta,\varphi)=\mathcal{J}_\kappa(\tilde{u},\varphi_\delta)-\delta\mathcal{J}(h_o)$. Therefore, keeping in mind that $\tilde{u}$ is a constrained subsolution to (4.0.1), hypothesis **A.3** and the estimates (4.2.6) give

$$-\partial_t\tilde{u}_\delta+H\left(x,t,\tilde{u}_\delta,\mathbf{x}D\varphi,\mathbf{x}D^2\varphi\mathbf{x}^T,\mathcal{J}_\kappa(\tilde{u}_\delta,\varphi)\right)\le$$

$$-\delta h_o(x)+\omega\left(2\delta\mathrm{C}h_o(x)\right)+\omega\left(\delta(\mathcal{J}h_o(x))^+\right),$$

for all $\kappa>0$ and $(x,t)\in\tilde{\Pi}_T$. The thesis follows after computations similar to the ones in Proposition 4.2.12. ∎

Chapter 5

Lévy markets: Financial applications.

In Chapter 1 we have shown the link between the stochastic and the deterministic formulation of financial problems. Backward stochastic differential equations (BSDEs henceforth), or more generally forward-backward stochastic differential equations, have received a lot of attention in recent years due to their many applications in stochastic control and mathematical finance. For a general introduction to BSDEs the reader could refer to [79].

In this chapter we shall present many application of the theory developed in Chapter 4. Integro–differential equations of the type (4.0.1) occur in the applications presented in Sections 5.1 and 5.2, which concern a class of backward stochastic differential equations in a unbounded jump Lévy setting. In [12], the authors consider a BSDE with terminal condition and coefficients being functions of a Lévy process. Under certain conditions they prove that the solution of the BSDE gives the unique viscosity solution to a system of semilinear integro–partial differential equations set in $\mathbb{R}^N$. In Section 5.1 we prove an existence and uniqueness result for a class of "Lévy driven" BSDEs having $(0, \infty)^N$ as an invariant set. We then prove (under certain conditions) that the solution to the BSDE provides a unique viscosity solution of a semilinear integro–partial differential equation set in $(0, \infty)^N$. Here uniqueness follows from the result in Section 4.2, which does not require a specification of a boundary condition.
Here we consider more general problems than the one in [12], as we allow an unbounded domain with nonsmooth boundary, of a multidimensional Poisson random measure driving the model and a different structure for the jump amplitude.
Particular attention is given in the Financial application. In Section 5.2 we provide new results on pricing European and American derivatives in rather general Lévy markets via BSDEs and linear integro–partial differential equations, relying on the

results developed in Sections 4.2 and 5.1. For what concerns American derivatives we have to deal with an obstacle problem that is not directly included in the class described by (4.0.1)–(4.0.2). Nevertheless, it is well known that a comparison principle for obstacle problems can be easily derived from a corresponding principle for (4.0.1)–(4.0.2), as it has been proved in Chapter 3 in the pure differential setting; for more references see [3, 4, 101]. This argument shall be detailed in Subsection 5.2.2. We remember that Lévy markets are indeed incomplete, therefore we don't have a unique arbitrage free price for any derivative. Here we are not interested in any particular choice of an equivalent martingale measures, instead we shall simply assume that we are given one, and then we derive, via the theory of BSDEs, a linear integro–partial differential equation satisfied by the corresponding arbitrage free price.
Another immediate application of the result in Chapter 4, Section 4.2 to a classical problem from mathematical finance, namely Merton's optimal investment problem in a Lévy (jump-diffusion) market. The setup is the problem of an agent who invests in a financial market so as to maximize the expected utility of his terminal wealth, and it leads to an one-dimensional fully nonlinear integro–partial differential equation having a structure as in (4.0.1). For an overview of the use of viscosity solution theory in the area of portfolio management and derivative pricing (with emphasis on transaction costs) in financial markets driven by Brownian motions, we refer to [118]. In Lévy markets, more complicated portfolio models with singular consumption and state-constraint boundary conditions are studied by viscosity solution methods in [21, 22, 23], see also [51]. Although we do not pursue this here, the use of state-constraint boundary conditions in [21, 22, 23] can to some extent be avoided by using the techniques developed in Section 4.2.

5.1 Backward Stochastic Differential Equations.

This section is devoted to extending the results in [12]. More precisely, we present here an existence and uniqueness result for a class of BSDEs in a rather general Lévy setting. Moreover, we show how to relate, via viscosity solutions, this stochastic problem to a semilinear integro–partial differential equation on a domain with a boundary, however without specifying a boundary condition.

Let us consider a stochastic N-dimensional process X_t defined by means of its dynamics

$$dX_t = \mathbf{X}_t\Big[b(X_t,t)dt + \sigma(X_t,t)dW_t + \int_E \beta(X_{t^-},t,z)\tilde{N}(dt,dz)\Big], \tag{5.1.1}$$

where $\mathbf{X}_t = \operatorname{diag}(X_t^1,\dots,X_t^{\mathrm{N}})$. Here W_t is a M-dimensional standard Brownian motion, $1 \leq \text{M} \leq \text{N}$, $\tilde{N}$ is the compensated martingale measure of a N-dimensional Poisson random measure N defined on $\mathbb{R}^+ \times E$ with compensator $\lambda(dt, dz) = dt \times \nu(dz)$, $\nu(dz)$ is its Lévy intensity, and $\nu : \mathcal{B}(E) \to \mathbb{R}^{\text{N}}$,

$$\nu(dz) = (\nu^1(dz), \nu^2(dz), \ldots, \nu^{\text{N}}(dz)),$$

is the N-dimensional Lévy measure. Here $b : \overline{\Pi}_T \to \mathbb{R}^{\text{N}}$, $\sigma : \overline{\Pi}_T \to \mathbb{R}^{\text{N}\times\text{M}}$, $\sigma\sigma^T \geq 0$, $\beta : \overline{\Pi}_T \times E \to \mathbb{R}^{\text{N}\times\text{N}}$.

The X_t process is considered in the setting of a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$ such that $\mathcal{F}_0$ contains all $\mathcal{P}$-null elements of $\mathcal{F}$, and $\mathcal{F}_{t^+} = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$, $t > 0$. We suppose that the filtration is generated by the two mutually independent processes W_t and $N(dt, dz)$.

We shall assume that the measures $\nu^j(dz)$ satisfies assumption **(A.1)** and that β_j satisfies assumption **(A.2)**, for all $j = 1, \ldots, \text{N}$, and that

(B.1) $b : \overline{\Pi}_T \to \mathbb{R}^{\text{N}}$ and $\sigma : \overline{\Pi}_T \to \mathbb{R}^{\text{N}\times\text{M}}$ are bounded and globally Lipschitz continuous with respect to x, namely that for all $x, y \in \Pi$ there exists a positive constant C such that

$$|b(x,t) - b(y,t)| + |\sigma(x,t) - \sigma(y,t)| \leq \text{C}|x - y|,$$
$$|b(x,t)| + |\sigma(x,t)| \leq \text{C}.$$

We do not make any assumptions on the rank of the matrices σ and β.

Under the assumptions **(A.1)**, **(A.2)** of Chapter 4 and **(B.1)** on the coefficients, in [52] it is proved that there exists a unique solution of the problem (5.1.1) with initial data $X_0 = x_o \in \Pi$, which is denoted by $\{X_t^0(x_o)\}_{t\geq 0}$.

Lemma 5.1.1 *Let us suppose that assumptions* **(A.1)**, **(A.2)** *and* **(B.1)** *hold. Then the solution* $X_s^{t_o}(x)$ *of* (5.1.1) *with initial data* $X_{t_o}^{t_o}(x) \in \Pi$ *verifies* $X_s^{t_o}(x) \in \Pi$ *a.s. for all* $s \geq t_o \geq 0$.

Proof. By the assumptions on the coefficients, the solution of the ith component of equation (5.1.1) exists and it is unique in probability [72]. By Ito's formula, omitting the dependence of the coefficients on (X_t, t),

$$\begin{aligned} d(\log X_t^i) &= \Big[b^i + \sum_{j=1}^{\text{N}} \Big(\ln(1 + \beta_j^i(z)) - \beta_j^i(z)\Big)\nu^j(dz) - \frac{1}{2}\sum_{j=1}^{\text{M}} {\sigma_j^i}^2\Big]dt \\ &\quad + \sum_{j=1}^{\text{M}} \sigma_j^i dW_t^j + \sum_{j=1}^{\text{N}} \int_E \ln(1 + \beta_j^i(z))\tilde{N}^j(dt, dz). \end{aligned}$$

Integrating over $[t, s]$ and taking the exponential we get

$$(X_s^{t_o})^i(x) = X_{t_o}^{t_o} \exp\Big\{\int_{t_o}^s \Big[b^i - \frac{1}{2}\sum_{j=1}^{\mathrm{M}} {\sigma_j^i}^2 + \sum_{j=1}^{\mathrm{N}}\int_E \Big(\log(1+\beta_j^i(\tau, X_\tau, z)) - \beta_j^i(z)\Big)\nu^j(dz)\Big] d\tau$$
$$+ \sum_{j=1}^{\mathrm{M}} \sigma_j^i dW_\tau^j + \sum_{j=1}^{\mathrm{N}} \int_t^s \int_E \Big(\log(1+\beta_j^i(z)) - \beta_j^i(z)\Big)\tilde{N}^j(d\tau, dz)\Big\}.$$

Then, as $X_t^t \in \Pi$ by assumption, the lemma is proved. ■

Proposition 5.1.2 *Let $X_s^t(x)$ be the solution of problem* (5.1.1). *For all $p \geq 0$ and any terminal time $T > 0$ there exists a constant $\mathrm{K} = \mathrm{K}(p, \mathrm{C}, T)$ such that for all $s, t \in [0, T]$ and $x, y \in (\mathbb{R}^+)^{\mathrm{N}}$ the following hold:*

$$\begin{aligned}
&\mathbb{E}[|X_s^t(x)|^p] \leq \mathrm{K}(1 + |x|^p),\\
&\mathbb{E}[|X_s^t(x) - x|^p] \leq \mathrm{K}(1 + |x|^p)(s-t)^{\frac{p}{2}},\\
&\mathbb{E}\Big[\sup_{0\leq\tau\leq s} |X_\tau^t(x) - x|\Big]^p \leq \mathrm{K}(1 + |x|^p)(s-t)^{\frac{p}{2}},\\
&\mathbb{E}[|X_s^t(x) - X_s^t(y)|^p] \leq \mathrm{K}|x-y|^p.
\end{aligned}$$

Proof. These estimates follow as in [101, Lemma 3.1] and [52, Th.2.2, Th.2.3]. ■

5.1.1 Existence and uniqueness of a solution for a BSDE.

We introduce the following spaces:

$$\begin{aligned}
\mathcal{S}^2 &= \{Y_t \text{ an } \mathcal{F}_t- \text{ adapted processes, càdlàg, } 0 \leq t \leq T \text{ such that}\\
&\qquad ||Y||_{\mathcal{S}^2} = \Big|\Big| \sup_{0\leq t\leq T} |Y_t| \Big|\Big|_{L^2(\Omega)} < +\infty\}.\\
L^2(W) &= \{Z_t \text{ an } \mathcal{F}_t- \text{ progressively measurable such that}\\
&\qquad ||Z||_{L^2(W)} = \Big(\mathbb{E}\Big[\int_0^T |Z_t|^2 dt\Big]\Big)^{\frac{1}{2}} \leq +\infty\}.\\
L^2(\tilde{N}) &= \{U : \Omega \times [0,T] \times E \to \mathbb{R}^{\mathrm{N}}, \text{ measurable, such that}\\
&\qquad ||U||_{L^2(\tilde{N})} = \Big(\mathbb{E}\Big[\int_0^T \int_E U_t(z)^2 \nu(dz) dt\Big]\Big)^{\frac{1}{2}} \leq +\infty\}.\\
\mathcal{B}^2 &= \mathcal{S}^2 \times L^2(W) \times L^2(\tilde{N}),
\end{aligned}$$

where we say that a process Y_t is càdlàg if it is continuous from the right and has limits from the left.
The following result is a combination of results in [111, Lemma 2.4], [98, Prop. 2.2] and [97].

Theorem 5.1.3 *Let $G \in L^2(\Omega, \mathcal{F}_T, \mathcal{P})$ and $f : \Omega \times [0,T] \times \mathbb{R}^+ \times \mathbb{R}^M \times L^2(E, \nu; \mathbb{R}^N) \to \mathbb{R}$ be progressively measurable with respect to all its variables and satisfy:*

$$\mathbb{E}\Big[\int_0^T |f_t(0,0,0)|^2 dt\Big] < +\infty,$$

and there exists $K > 0$ such that

$$|f_t(y,\zeta,q) - f(y',\zeta',q')| \le K\Big(|y-y'| + |\zeta - \zeta'| + ||u-u'||_{L^2(E)}\Big),$$

for all $0 \le t \le T$ and $y, y' \in \mathbb{R}$, $\zeta, \zeta' \in \mathbb{R}^M$, $q, q' \in L^2(E, \nu; \mathbb{R}^N)$. Then there exists a unique triple $(Y, Z, U) \in \mathcal{B}^2$ which solves the BSDE:

$$Y_t = G + \int_t^T f_s(Y_s, Z_s, U_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(z) \tilde{N}(ds, dz), \qquad (5.1.2)$$

for $0 \le t \le T$.

We point out that the comparison result in [12] can be extended to the present case.

Proposition 5.1.4 *Let us consider a progressively measurable function $h : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}$ satisfying*

$$\mathbb{E}\Big[\int_0^T |h(\omega, t, 0, 0, 0)|^2 dt\Big] < +\infty,$$

h is globally Lipschitz continuous uniformly with respect to $(\omega, t) \in \Omega \times [0,T]$, and

$$q \to h(t, y, \zeta, q) \text{ is non decreasing for all } (t, y, \zeta) \in [0,T] \times \mathbb{R} \times \mathbb{R}^M.$$

Moreover, let $\gamma : \Omega \times [0,T] \times E \longrightarrow \mathbb{R}^{N \times N}$ be a measurable function satisfying assumption (A.4.i). Let us define

$$f(\omega, t, y, \zeta, \phi) = h\Big(\omega, t, y, \zeta, \int_E \phi(z) \gamma(\omega, t, z) \nu(dz)\Big)$$

for all $(\omega, t, y, \zeta, \phi) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^M \times (L^2(E,\nu))^N$. Here we use the notation

$$\int_E \phi(z)\gamma(\omega,t,z)\nu(dz) = \Big(\int_E \phi_1(z)\gamma^1(\omega,t,z)\cdot\nu(dz), \dots, \int_E \phi_N(z)\gamma^N(\omega,t,z)\cdot\nu(dz)\Big).$$

With $G, G' \in L^2(\Omega, \mathcal{F}_T, \mathcal{P})$, let (Y, Z, U) and (Y', Z', U') be the unique solutions of (5.1.2) corresponding respectively to G and G'. If $G \le G'$ then

$$Y_t \le Y'_t, \text{ for all } 0 \le t \le T.$$

This monotonicity result can be proved using Ito's formula, following the argument of the proof of the comparison principle for BSDEs without jumps, see [97].

In the following we analyze the case where both G and, for each t, y, ζ and q, the process $\{f_s(y,\zeta,q),\ t \le s \le T\}$ are given function of the stochastic process $X_s^t(x)$.

Let us suppose that

$$g : \Pi \to \mathbb{R}, \qquad f : \overline{\Pi}_T \times \mathbb{R} \times \mathbb{R}^{\mathrm{M}} \times \mathbb{R}^{\mathrm{N}} \to \mathbb{R}$$

verify the following conditions:

(B.2) there exist a positive constant C and an integer p such that, for all $(x,t) \in \overline{\Pi}_T$,

$$|f(x,t,0,0,0)| \le \mathrm{C}(1+|x|^p) \qquad |g(x)| \le \mathrm{C}(1+|x|^p).$$

(B.3) $f = f(x,t,y,\zeta,q)$ and g are globally Lipschitz in (y,ζ,q), uniformly in (x,t);

(B.4) for each $(x,t,y,\zeta) \in \overline{\Pi}_T \times \mathbb{R} \times \mathbb{R}^{\mathrm{M}}$ the function

$$q \mapsto f(x,t,y,\zeta,q)$$

is nondecreasing.

(B.5) there exists a continuous function $\gamma = (\gamma^1,\dots,\gamma^{\mathrm{N}}) : \overline{\Pi}_T \times E \longrightarrow \mathbb{R}^{\mathrm{N}}$ such that

$$|\gamma^j(x,t,z) - \gamma^j(x',t,z)| \le \mathrm{C}|x-x'|(1 \wedge |z|^2)$$

for all $(x,z) \in \Pi \times E$ uniformly with respect to t, $j = 1,\dots,\mathrm{N}$.

Moreover, we introduce some additional assumptions that shall be useful when proving the existence of a solution:

(B.6) there exists a modulus of continuity $\omega_R(\cdot)$ such that, for all $t \in [0,T]$, $x, x' \in \Pi \cap B(0,R)$, $|y| \le R$, $\zeta \in \mathbb{R}^{\mathrm{M}}$, $q \in \mathbb{R}^{\mathrm{N}}$, $R > 0$

$$|f(x,t,y,\zeta,q) - f(x',t,y,\zeta,q)| \le \omega_R(|x-x'|(1+|\zeta|)).$$

Let us consider the following BSDE:

$$Y_s^t(x) = g(X_T^t(x)) + \int_s^T f\Big(X_\tau^t(x), \tau, Y_\tau^t(x), Z_\tau^t(x), \int_E U_\tau^t(x,z)\gamma(X_\tau^t(x),\tau,z)\nu(dz)\Big)d\tau$$
$$- \int_s^T Z_\tau^t(x)dW_\tau - \int_s^T \int_E U_\tau^t(x,z)\tilde{N}(d\tau,dz), \quad t \le s \le T. \tag{5.1.3}$$

Arguing as in [12, Corollary 2.3] and using Theorem 5.1.3, Lemma 5.1.1 and Proposition 5.1.2, the following theorem can be proved.

Theorem 5.1.5 *For each* $(x,t) \in \overline{\Pi}_T$ *the BSDE* (5.1.3) *has a unique solution*

$$(Y^t(x), Z^t(x), U^t(x,\cdot)) \in \mathcal{B}^2,$$

and $(x,t) \mapsto Y^t_t(x)$ *is a deterministic mapping from* $\overline{\Pi}_T$ *into* $\mathbb{R}$.

Let us define

$$u(x,t) := Y^t_t(x). \tag{5.1.4}$$

This deterministic function has the following properties:

Theorem 5.1.6 *Under assumptions* **(B.1)**-**(B.5)** *the function* $u : \overline{\Pi}_T \to \mathbb{R}$ *is continuous. Moreover, there exist constants* C *and* p *such that:*

$$|u(x,t)| \leq \mathrm{C}(1+|x|^p)\ (x,t) \in \overline{\Pi}_T.$$

In addition, if g *and* $f(\cdot,t,y,\zeta,q)$ *are uniformly continuous, uniformly w.r.t.* (t,y,ζ,q) *and are bounded, then* u *is uniformly continuous and bounded.*

Proof. We first establish the growth of u, and then that u is Lipschitz continuous. As the solution $X^t_s(x)$ of (5.1.1) remains inside Π_T, omitting the dependence on the starting point x, by repeating the arguments of [12, Prop. 2.2] we find that

$$||(Y,Z,U)||^2_{\mathcal{B}^2} \leq c\mathbb{E}\Big[|g(X^t_s)|^2 + \int_0^T \Big|f\Big(X^t_\tau, \tau, Y^t_\tau, Z^t_\tau, \int_E U^t_\tau(\cdot,z)\gamma(X^t_\tau,\tau,z)\nu(dz)\Big)\Big|^2 ds\Big].$$

Using the assumptions **(B.2)** we obtain

$$|u(x,t)| \leq ||(Y,Z,U)||_{\mathcal{B}^2} \leq \mathrm{C}(T)(1+|x|^p)$$

The proof of the Lipschitz continuity of u follows as in [12, Prop. 2.5]. Define

$$\hat{Y}^t_s(x) = \begin{cases} Y^t_t(x) & 0 \leq s \leq t, \\ Y^t_s(x) & s > t. \end{cases}$$

It follows that

$$\begin{aligned}
|\hat{Y}^t_t(x) - \hat{Y}^{t'}_{t'}(x')|^2 &= |\hat{Y}^t_0(x) - \hat{Y}^{t'}_0(x')|^2 \leq\ ||(\hat{Y},\hat{Z},\hat{U})||^2_{\mathcal{B}^2} \\
&\leq \mathrm{C}\mathbb{E}\Big[|g(X^t_T(x)) - g(X^{t'}_T(x'))|^2 \\
&\quad + \int_0^T |\mathbf{1}_{[t,T]}(s) f\Big(X^t_s(x), s, Y^t_s(x), Z^t_s(x), U^t_s(x,z)\Big) \\
&\qquad - \mathbf{1}_{[t',T]} f\Big(X^{t'}_s(x'), s, Y^{t'}_s(x'), U^{t'}_s(x',z)\Big)|ds\Big] \\
&\leq \mathrm{K}\Big((1+|x|^2)^2(t-t') + |x-x'|^2\Big),
\end{aligned}$$

where we have used the assumptions **(B)**. Moreover, if we assume that both g and f are uniformly continuous, then we can prove that u is uniformly continuous. ■

5.1.2 Integro–differential equation. Existence and uniqueness.

We now prove that the function $u(x,t)$ introduced in (5.1.4) provides a viscosity solution for an integro–partial differential equation with terminal condition $g(x)$ and Hamiltonian

$$H(x,t,u,\mathbf{x}\mathcal{D}u,\mathbf{x}\mathcal{D}^2u\mathbf{x}^T,\mathcal{I}u,\mathcal{J}u) = -\frac{1}{2}tr\left[\sigma\sigma^T\mathbf{x}\mathcal{D}^2u\mathbf{x}^T\right] - b\mathbf{x}\mathcal{D}u \qquad (5.1.5)$$
$$-\sum_{j=1}^{\mathrm{N}}\mathcal{J}_ju - f(x,t,\mathbf{x}\mathcal{D}u\sigma,\mathcal{I}_1u,\ldots,\mathcal{I}_{\mathrm{N}}u),$$
$$\mathcal{J}_ju(x,t) = \int_E\left[u(x+\mathbf{x}\beta_j(x,z),t) - u(x,t) - \mathbf{x}\beta_j(x,z)\mathcal{D}u(x,t)\right]\nu^j(dz),$$
$$\mathcal{I}_ju(x,t) = \int\limits_E\left[u(x+\mathbf{x}\beta_j(x,t,z),t) - u(x,t)\right]\gamma^j(x,t,z)\nu^j(dz),$$

and the coefficients b, σ, β and γ satisfy all the assumptions in **(A)**-**(B)**.

It will be shown that this problem can be considered as a particular case of the problem (4.0.1)-(4.0.2) where $u_T(x) = g(x)$. The techniques of Section 4.2 can easily be applied to this case even if the Hamiltonian depends explicitly on two integro operators, namely $\mathcal{J}u$ and $\mathcal{I}u$. This is not a complication, as it is explained in Remark 4.2.6 and in the introduction.

In the next theorem we show the link between (5.1.3)-(5.1.4) and (4.0.1) by proving that the unique solution of the stochastic problem is a viscosity solution of the integro–partial differential problem.

Theorem 5.1.7 *The function $u(x,t) = Y_t^t(x)$, $(x,t) \in \Pi_T$, is a viscosity solution of (4.0.1) with terminal condition (4.0.2).*

Proof. We have previously shown that $u(x,t)$ is deterministic, is continuous, and has polynomial growth. Moreover, it is easily seen that u satisfies the terminal condition

$$u(x,T) = Y_T^t(x) = u_T(X_T^t(x)) = u_T(x).$$

We will show that u is a subsolution, the supersolution case being similar.

Let $\phi \in \mathcal{C}^2(\overline{\Pi}_T)$, (x,t) be a global maximum point for $u-\phi$ such that $u(x,t) - \phi(x,t) = 0$. For simplicity, we suppose that ϕ has the same polynomial growth as u. It is easy to see that

$$Y_s^t(x) = Y_s^s(X_s^t(x)) = u(X_s^t(x),s).$$

Choose now $h > 0$ such that $t + h < T$. Omitting for simplicity the dependence on the starting point x, for $t \leq s \leq t + h$ we obtain

$$Y_s^t = u(X_{t+h}^t, t+h) + \int_s^{t+h} f\Big(X_\tau^t, \tau, Y_\tau^t, Z_\tau^t, \int_E U_\tau^t(x,z)\gamma(X_\tau^t, z)\nu(dz)\Big)d\tau$$
$$- \int_s^{t+h} Z_\tau^t dW_\tau - \int_s^{t+h}\int_E U_\tau^t(x,z)\tilde{N}(d\tau, dz).$$

Still omitting the dependence on x, consider now the following process

$$\overline{Y}_s^t = \phi(X_{t+h}^t(x), t+h) + \int_s^{t+h} f\Big(X_\tau^t, \tau, \overline{Y}_\tau^t, \overline{Z}_\tau^t, \int_E \overline{U}_\tau^t(x,z)\gamma(X_\tau^t, z)\nu(dz)\Big)d\tau$$
$$- \int_s^{t+h} \overline{Z}_\tau^t dW_\tau - \int_s^{t+h}\int_E \overline{U}_\tau^t(x,z)\tilde{N}(d\tau, dz).$$

As (x, t) is a global maximum point, from Proposition 5.1.4, comparison holds

$$\overline{Y}_s^t(x) \geq Y_s^t(x), \;\; t \leq s \leq t+h.$$

In particular, for $s = t$ there holds

$$\overline{Y}_t^t(x) \geq u(x,t) = \phi(x,t).$$

Let us now define

$$\begin{aligned}
\psi(y,s) &= \partial_s\phi(y,s) + \frac{1}{2}\mathrm{tr}\left[\sigma(y,t)\sigma(y,t)^T\mathbf{y}\mathcal{D}^2\phi(y,s)\mathbf{y}^T\right] + b(y,s)\mathbf{y}\mathcal{D}\phi(y,s)\\
&\quad + \sum_{j=1}^{\mathrm{N}} \mathcal{J}_j\phi(y,s),\\
\Phi_j(y,s,z) &= \phi(y + \beta_j(y,s,z), s) - \phi(y,s),
\end{aligned}$$

with $\Phi(y,s,z) = (\Phi_1(y,s,z), \ldots, \Phi_{\mathrm{N}}(y,s,z)) \in \mathbb{R}^{\mathrm{N}}$. By the generalized Ito's formula,

$$\phi(X_s^t(x), s) = \phi(X_{t+h}^t(x), t+h) - \int_s^{t+h} \psi(X_\tau^t(x), \tau)d\tau$$
$$- \int_s^{t+h} \mathbf{X}_\tau^t(x)\cdot\mathcal{D}\phi(X_\tau^t(x),\tau)\sigma(X_\tau^t(x),\tau)dW_\tau - \int_s^{t+h}\int_E \Phi(\tau, X_\tau^t(x))\tilde{N}(d\tau, dz).$$

Let us now define the new processes

$$\begin{aligned}
\hat{Y}_s^t(x) &:= \overline{Y}_s^t - \phi(X_s^t(x), s),\\
\hat{Z}_s^t(x) &:= \overline{Z}_s^t(x) - \mathbf{X}_s^t\mathcal{D}\phi(X_s^t(x), s)\sigma(X_s^t(x), s),\\
\hat{U}_s^t(x,z) &:= \overline{U}_s^t(x,z) - \Phi(X_s^t(x), s, z).
\end{aligned}$$

Then the triplet $(\hat{Y},\hat{Z},\hat{U})$ is the solution of the following BSDE:

$$\hat{Y}^t_s(x) = \int_s^{t+h} \Big[f\Big(X^t_\tau(x),\tau,\overline{Y}^t_\tau(x),\overline{Z}^t_\tau(x),\int_E \overline{U}^t_\tau(x,z)\gamma(X^t_\tau(x),z)\nu(dz)\Big)$$
$$+\psi(\tau,X^t_\tau(x))\Big]d\tau - \int_s^{t+h}\hat{Z}^t_\tau(x)dW_\tau - \int_s^{t+h}\int_E \hat{U}^t_\tau(x)\tilde{N}(d\tau,dz).$$

Using standard techniques it is easy to prove that

$$\mathbb{E}[|\hat{Y}^t_s|^2] + \mathbb{E}\Big[\int_s^{t+h}|\hat{Z}^t_\tau|d\tau\Big] + \mathbb{E}\Big[\int_s^{t+h}\int_E|\hat{U}^t_\tau(z)|^2\nu(dz)d\tau\Big]$$
$$\leq 2\mathbb{E}\Big[\int_s^{t+h}|\hat{Y}^t_\tau|\Big|f\Big(X^t_\tau(x),\tau,\overline{Y}^t_\tau(x),\overline{Z}^t_\tau(x),\int_E\overline{U}^t_\tau(x,z)\gamma(X^t_\tau(x),z)\nu(dz)\Big)$$
$$+\psi(X^t_\tau(x),\tau)\Big|d\tau\Big]$$
$$\leq 2\kappa\mathbb{E}\Big[\int_s^{t+h}\Big(|\hat{Y}^t_\tau| + |\hat{Y}^t_\tau|^2\Big)d\tau\Big],$$

where we have used **(B.3)** for f and the polynomial growth of ϕ. Using now assumptions **(B)** and Proposition 5.1.2 we obtain

$$\mathbb{E}[|\hat{Y}^t_s|^2] \leq 3\kappa\Big(h + \mathbb{E}\Big[\int_s^{t+h}|\hat{Y}^t_\tau|^2 d\tau\Big]\Big).$$

By Gronwall's lemma,

$$\mathbb{E}[|\hat{Y}^t_\tau|^2] \leq \kappa h,$$

for some $\kappa > 0$. In the same way, if $0 < h < 1$, we obtain

$$\frac{1}{h}\Big(\mathbb{E}\Big[\int_s^{t+h}|\hat{Z}^t_\tau|^2 d\tau\Big] + \mathbb{E}\Big[\int_s^{t+h}\int_E|\hat{U}^t_\tau(z)|^2\nu(z)d\tau\Big]\Big) \leq \kappa\sqrt{h}.$$

Let us suppose now that it holds

$$\partial_t\phi(x,t) - H(x,t,u(x,t),\mathbf{x}\mathcal{D}\phi(x,t),\mathbf{x}\mathcal{D}^2\phi(x,t)\mathbf{x}^T,\mathcal{J}\phi(x,t),\mathcal{I}\phi(x,t)) < 0.$$

Omitting the dependence of ψ and ϕ on $(X^t_\tau(x),\tau)$, let us define

$$\xi_h := \frac{1}{h}\mathbb{E}\Big[\int_t^{t+h}\Big[\psi + f\Big(X^t_\tau(x),\tau,\phi,\mathcal{D}\phi\sigma,\mathcal{I}\phi\Big)\Big]d\tau\Big].$$

From the previous hypotheses there exist $\delta > 0$ and $\overline{h} > 0$ such that for all $0 < h \leq \overline{h}$:

$$\xi_h \leq -\delta < 0.$$

Remembering the definition of $\hat{Y}^t_s(x)$, that (x,t) is a global maximum point for $u-\phi$ and the monotonicity of the solution of (5.1.2) with respect the initial data, we have

$$\hat{Y}^t_t(x) = \overline{Y}^t_t(x) - \phi(x,t) \geq u(x,t) - \phi(x,t) = 0.$$

Therefore, omitting the dependence on the starting point x,

$$0 \leq \frac{1}{h}\hat{Y}^t_t = \frac{1}{h}\mathbb{E}\Big[\int_t^{t+h} \Big[\psi(X^t_\tau,\tau) + f\Big(X^t_\tau,\tau,\overline{Y}^t_\tau,\overline{Z}^t_\tau,\int_E \overline{U}^t_\tau\gamma(X^t_\tau,z)\nu(dz)\Big)\Big]d\tau\Big].$$

Then, still omitting the dependence on x, for $0 < h \leq \overline{h}$ we have

$$\begin{aligned}\delta \leq \frac{1}{h}\hat{Y}^t_t - \xi_h = \frac{1}{h}\mathbb{E}\Big[\int_t^{t+h} \Big[f\Big(X^t_\tau,\tau,\overline{Y}^t_\tau,\overline{Z}^t_\tau,\int_E \overline{U}^t_\tau\gamma(X^t_\tau,z)\nu(dz)\Big)\\ -f\Big(X^t_\tau,\tau,\phi(X^t_\tau,\tau),(\mathcal{D}\phi\sigma)(X^t_\tau,\tau),\mathcal{I}\phi(X^t_\tau,\tau)\Big)\Big]d\tau\Big]\\ \leq \frac{\mathrm{K}}{h}\mathbb{E}\Big[\int_t^{t+h} \Big(|\hat{Y}^t_\tau| + |\hat{Z}^t_\tau| + \Big(\int_E |\hat{U}^t_\tau(z)|^2\nu(dz)\Big)^2\Big)d\tau\Big] \leq \mathrm{K}[c_1 h + c_2\sqrt[4]{h}] \leq \mathrm{C}\sqrt[4]{h},\end{aligned}$$

which is a contradiction. Hence we must have

$$-\partial_t\phi(x,t) + H(x,t,u(x,t),\mathbf{x}\mathcal{D}\phi(x,t),\mathbf{x}\mathcal{D}^2\phi(x,t)\mathbf{x}^T,\mathcal{J}\phi(x,t),\mathcal{I}\phi(x,t)) \leq 0,$$

which means that u is a subsolution. ■

Using the result of Section 4.2 we also have the uniqueness of the solution of the integro–partial differential problem.

Theorem 5.1.8 *Let us suppose that assumptions* **(B)** *hold. Then the function* $u(x,t)$ *defined in* (5.1.4) *is the unique viscosity solution of the integro–partial differential problem (4.0.1)-(4.0.2), with Hamiltonian* (5.1.5) *and terminal data* $g(x)$*. The viscosity solution* u *has the same polynomial growth as the function* f *and the terminal data* $g(x)$*.*

Proof. By Corollary 4.2.15 it suffices to verify the assumptions **(A.1)**-**(A.3)** of Section 4.2 for the Hamiltonian (5.1.5). As a consequence of the presence of the integral term $\mathcal{I}u$, we also have to take into account assumption **(A.4)**, as explained in Remark 4.2.6. This does not introduce new difficulties and follows straightforward from **(B)**. This proves the result. ■

5.2 The pricing problem.

In this section we apply the results from Sections 4.2 and 5.1 to the pricing of European and American derivatives in Lévy markets. To this end, let us consider N Lévy processes with the following dynamics

$$\begin{aligned} dL_t^i &= \mu^i(t)dt + \sum_{j=1}^{M} \sigma_j^i(t)dW_t^j + \sum_{j=1}^{N} \int_{|z|<1} \eta_j^i(t) z^j \tilde{N}^j(dt, dz) \\ &+ \sum_{j=1}^{N} \int_{|z|\geq 1} \eta_j^i(t) z^j N^j(dt, dz), \qquad i = 1, \ldots, N, \end{aligned}$$

where $\sigma(t) \in \mathbb{R}^{N\times M}$ matrix, $1 \leq M \leq N$, $\eta(t) \in \mathbb{R}^{N\times N}$. Moreover we assume that μ, σ satisfy the assumption (**B.1**), while η is such that $\eta \cdot z$ satisfy assumption (**A.2**) Consider a financial market where the risk-free asset, B_t, and the risky assets, $X_t = (X_t^1, \ldots, X_t^N)$, evolve according to

$$\begin{cases} B_t = e^{rt}, \\ X_t^i = e^{L_t^i}, \quad i = 1, \ldots, N, \end{cases} \tag{5.2.1}$$

where $r > 0$ and $t > 0$. By the generalized Ito's formula, the X_t dynamics are given by (5.1.1) with parameters $b(X_t, t) \in \mathbb{R}^N$, $\sigma(X_t, t) \in \mathbb{R}^{N\times M}$, $\beta(X_t, t, z) \in \mathbb{R}^{N\times N}$ taking the following form

$$\begin{aligned} b^i(X_t) &= \mu^i + \frac{1}{2}\sum_{j=1}^{M} (\sigma_j^i)^2 + \sum_{j=1}^{N} \int_E \left(e^{\eta_j^i z^j} - 1 - \eta_j^i z^j \mathbf{1}_{|z|<1} \right) \nu^j(dz), \\ \sigma(X_t) &= \sigma, \\ \beta_j^i(X_t, z) &= e^{\eta_j^i z^j} - 1, \end{aligned}$$

where, for simplicity, we have omitted the t dependence.

In the following we shall assume that the coefficients b, σ verify assumption **(B.1)**, β and the measures $\nu = (\nu^1, \ldots, \nu^N)$ verify assumptions **(A.2)** and **(A.1)** of Section 4.2 respectively, and that the interest rate r is bounded from below.

The presence of the jump components in the price dynamics makes the market incomplete. Here we assume to be in an equivalent martingale setting. This assumption can be made without loss of generality, as it has been proved in Chapter 2, Proposition 2.2.5

We conclude that if a change of measure is needed, the X_t dynamics with respect to the new measure are still given by (5.1.1), modulo a change in the drift coefficient:

$$(b^Q)^i(X_t, t) = \mu^i + \frac{1}{2}\sum_{j=1}^{M} (\sigma_j^i)^2(t) + \sum_{j=1}^{M} \sigma_j^i \alpha^j$$

$$+\sum_{j=1}^{\mathrm{N}}\int_E \left[e^{\eta_j^i(t)z^j} - 1 - \eta_j^i(t)z_j^i\mathbf{1}_{|z|<1}\right]k^j(z)\nu^j(dz),$$

and with a new compensated martingale measure $\tilde{N}^{\mathbb{Q}}$. We point out that the coefficient $b^{\mathbb{Q}}$ still satisfies assumption **(B.1)**.

In an equivalent martingale setting, the discounted prices of the assets are martingales, and therefore the parameters have to satisfy

$$b^i(X_t,t) - r = 0, \text{ for all } i = 1,\ldots,\mathrm{N}.$$

5.2.1 The European derivatives.

Consider the price $u(X_t,t)$ of a European derivative constructed on the assets X_t with maturity T and payoff $u(X_T,T) = u_T(X_T)$. By the generalized Ito formula, this price solves the following BSDE

$$u(X_t,t) = e^{-r(T-t)}u_T(X_T) + \int_t^T \Big[-re^{-r(s-t)}u(X_s,s)\Big]ds \tag{5.2.2}$$
$$-\int_t^T e^{-r(s-t)}\mathbf{X}_s\mathcal{D}u\,\sigma(s)dW_s - \int_t^T\int_E e^{-r(s-t)}\Big[u(\mathbf{X}_s e^{\eta(s)z},s) - u(X_s,s)\Big]\tilde{N}(ds,dz).$$

Proposition 5.2.1 *The price u of a European derivative with maturity T and payoff u_T is a viscosity solution of the following equation*

$$-\partial_t u + H\Big(x,t,u,\mathbf{x}\mathcal{D}u,\mathbf{x}\mathcal{D}^2u\mathbf{x}^T,\mathcal{J}u\Big) = 0, \tag{5.2.3}$$

with terminal condition $u_T(x)$, $x\in\Pi$. Here

$$H\Big(x,t,u,\mathbf{x}\mathcal{D}u,\mathbf{x}\mathcal{D}^2u\mathbf{x}^T,\mathcal{J}u\Big) = -\frac{1}{2}\mathrm{tr}\left[\sigma\sigma^T\mathbf{x}\mathcal{D}^2u\mathbf{x}^T\right] - b\mathbf{x}\mathcal{D}u + ru - \sum_{j=1}^{\mathrm{N}}\mathcal{J}_j u$$

and

$$\mathcal{J}_j u(x,t) = \int_E \Big[u(\mathbf{x}e^{(\eta z)_j},t) - u(x,t) - \mathbf{x}\Big(e^{\eta z} - \mathbf{1}_{\mathrm{N}}\Big)\mathcal{D}u\Big]\nu^j(dz).$$

where $\mathbf{1}_{\mathrm{N}}$ is the N*-dimensional vector* $(1,\ldots,1)$.

Proof. This result is a consequence of the theory presented in Section 5.1, since we have the following identifications:

$$\begin{aligned}
Y_s^t(x) &= e^{-r(s-t)}u(X_s^t(x),t),\\
Z_s^t(x) &= e^{-r(s-t)}\mathbf{X}_s^t(x)\mathcal{D}u\sigma(t),\\
U_s^t(x,z) &= e^{-r(s-t)}\Big(u(\mathbf{X}_s^t(x)e^{\eta(t)z},t) - u(X_s^t(x),t)\Big),\\
f(X_t,t,Y_t,Z_t,U_t) &= -rY_t.
\end{aligned}$$

■

We can now state the following result:

Theorem 5.2.2 *There exists a unique viscosity solution $u(x,t)$ of* (5.2.3) *with terminal data $u(x,T) = u_T(x)$. Moreover, $u(x,t)$ is given as the solution of the BSDE* (5.2.2).

Proof. We have just shown that the pricing problem can be formulated as a special case of the more general problem described in Section 5.1. Hence to prove that a solution in the viscosity sense exists and is unique, we need only to check that it satisfies the assumptions of Sections 4.2 and 5.1. Moreover, the present integro–partial differential equation does not depend on the integro operator $\mathcal{I}u$, and therefore its study reduces exactly to Corollary 4.2.15 and Theorem 5.1.7. Assumptions **(B)** are easily checked as r is supposed to be bounded from below. To prove uniqueness it suffices to verify that the Hamiltonian H satisfies assumption **(A.3)**. Assumptions $(A.1.i) - (A.1.iii)$ are consequences of (**B**), of the boundedness of r and of the linearity of the Hamiltonian H with respect to its arguments.

To check assumption $(A.3.iv)$, suppose $x, y \in \mathcal{Q}(r)$, $t \in [0,T]$, $u \in [-R,R]$, $\mathcal{J} \in \mathbb{R}$, and $M, N \in \mathcal{S}^{\mathrm{N}}$ satisfy

$$\begin{pmatrix} M & 0 \\ 0 & -N \end{pmatrix} \leq \frac{3}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Then

$$\begin{aligned} &H(x,t,u,\frac{1}{\varepsilon}\mathbf{x}(x-y),\mathbf{x}M\mathbf{x}^T,\mathcal{J}) - H(y,t,u,\frac{1}{\varepsilon}\mathbf{y}(x-y),\mathbf{y}N\mathbf{y}^T,\mathcal{J}) \\ &\geq -\frac{3}{2\varepsilon}tr(\sigma\sigma^T)|\mathbf{x}-\mathbf{y}|^2 - \frac{1}{2}(x-y)(\mathbf{x}-\mathbf{y})\Big(\mu + \sigma\sigma^T\Big), \end{aligned}$$

and the calculations are carried out as in [34, Example 3.6]. ■

5.2.2 The American derivatives.

Let us suppose we are in a market described by (5.2.1) and that we are equipped with an equivalent martingale measure. Let us consider an American derivative with maturity T and exercise price u_T. For the general theory about the market of American derivatives and the related control problem we refer to [48, 72, 76].

We recall that the optimal stopping time for this problem is the random variable

$$\tau^* = \inf\{t \in [0,T) \text{ such that } u(X_t,t) = u_T(X_t)\}.$$

Let $\mathcal{T}_{t,T}$ denote the set of all stopping times between t and T. The price of the American derivative is given by the BSDE

$$u(x,t) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\Big[e^{-r(\tau-t)}u_T(X_\tau)\Big|\mathcal{F}_t\Big]. \tag{5.2.4}$$

The related integro–partial differential equation is

$$\min\Big(-\partial_t u + H\Big(x,t,u,\mathbf{x}\mathcal{D}u,\mathbf{x}\mathcal{D}^2u\mathbf{x}^T,\mathcal{J}u\Big), u-u_T\Big)=0, \tag{5.2.5}$$

for $(x,t)\in\Pi_T$, with terminal data

$$u(x,T)=u_T(x), \tag{5.2.6}$$

for $x\in\Pi$. We intend to show that the function u defined in (5.2.4) is a viscosity solution of the integro–partial differential equation (5.2.5).

We recall here a definition of viscosity solutions that is suitable for the obstacle problem (5.2.5)-(5.2.6). It is the analogue of Definition 3.1.9 for pure differential problems.

Definition 5.2.3 *An upper semicontinuous function $u:\Pi_T\to\mathbb{R}$ is a viscosity subsolution of equation* (5.2.5) *if*

$$-\partial_t u + H(x,t,u,\mathbf{x}\mathcal{D}u,\mathbf{x}\mathcal{D}^2u\mathbf{x}^T,\mathcal{J}u)\le 0,$$

in the viscosity sense for all $(x,t)\in\Pi_T$ such that $u(x,t)>u_T(x)$.

A lower semicontinuous function $u:\Pi_T\to\mathbb{R}$ is a viscosity supersolution of equation (5.2.5) *if*

$$-\partial_t u + H(x,t,u,\mathbf{x}\mathcal{D}u,\mathbf{x}\mathcal{D}^2u\mathbf{x}^T,\mathcal{J}u)\ge 0,$$

in the viscosity sense for all $(x,t)\in\Pi_T$ such that $u(x,t)\ge u_T(x)$ and the parabolic subjet $\mathcal{P}^-=\emptyset$ for all $(x,t)\in(\mathbb{R}^+)^{\mathrm{N}}\times[0,T]$ such that $u(x,t)<u_T(x)$.

If, in addition,

$$u(x,T)\le u_T(x) \ \textit{(respectively, } \ge \textit{)},$$

for all $x\in\Pi$, then u is a viscosity subsolution (respectively, viscosity supersolution) of the obstacle problem (5.2.5)-(5.2.6).

An arbitrary function $u:\Pi_T\to\mathbb{R}$ is a viscosity solution for the obstacle problem (5.2.5)-(5.2.6) *if the upper and lower semicontinuous envelopes of u are sub- and supersolutions, respectively.*

Because of Lemma 5.1.1, the Dynamic Programming Principle (DPP) established in [101, Prop. 3.1 and Prop. 3.2] holds for our problem (5.2.5)-(5.2.6). Hence the following results can be proved along the lines of [101, Prop. 3.3] and [101, Th. 3.1].

Proposition 5.2.4 *Under assumptions* **(B)** *of Section 5.1, $u\in\mathcal{C}^0(\Pi_T)$. In particular there exists a constant $\mathrm{C}>0$ such that for all $t,s\in[0,T]$ and for all $x,y\in(\mathbb{R}^+)^{\mathrm{N}}$*

$$|u(x,t)-u(y,s)|\le \mathrm{C}\Big[(1+|x|)|t-s|^{\frac{1}{2}}+|x-y|\Big].$$

Proof. This result is a consequence of the DPP. It can be proved along the lines of [101] and proceeding as in Proposition 5.1.6. ■

Theorem 5.2.5 *Under assumptions* **(B)** *of Section 5.1 the function* u *in* (5.2.4) *is a viscosity solution of* (5.2.5)-(5.2.6).

Proof. We divide the proof in two steps, proving at first that u is a supersolution and afterwards that it is a subsolution.

Let $\phi \in \mathcal{C}^2(\overline{\Pi}_T)$ and suppose that $(x,t) \in \Pi_T$ is a global minimum point of $u-\phi$ such that $(u-\phi)(x,t) = 0$. By definition, for $t < h < T$

$$\begin{aligned} u(x,t) \;\; &\geq \;\; \mathbb{E}\Big[e^{-r(h-t)}u(X_h^t(x),h) \\ &\qquad + \int_t^h e^{-r(s-t)} f(X_s^t(x), s, u(X_s^t(x),s), \mathbf{X}_s^t(x)\mathcal{D}u(X_s^t(x),s))ds\Big]. \end{aligned}$$

Subtracting $\phi(x,t)$ and using the hypotheses

$$\begin{aligned} &0 \geq \mathbb{E}\Big[e^{-r(h-t)}\phi(X_h^t(x),h) - \phi(x,t) \\ &+ \int_t^h e^{-r(s-t)} f(X_s^t(x), s, u(X_s^t(x),s), \mathbf{X}_s^t(x)\mathcal{D}u(X_s^t(x),s))ds\Big]. \end{aligned}$$

On the other hand, applying Ito's formula to $e^{-rt}\phi(X_t,t)$ we obtain

$$\frac{1}{h}\mathbb{E}\Big[\int_t^h \Big(\partial_t\phi - H(X_s^t(x), s, \phi, \mathbf{X}_s^t\mathcal{D}\phi, \mathbf{X}_s^t\mathcal{D}^2\phi\, \mathbf{X}_s^{t\,T}, \mathcal{J}\phi)\Big)ds\Big] \leq \omega(h-t),$$

where ω is a modulus of continuity and the function ϕ is evaluated at $(X_s^t(x), s)$. Sending now h to t we obtain

$$-\partial_t\phi(x,t) + H(x,t,u(x,t), \mathbf{x}\mathcal{D}u(x,t), \mathbf{x}\mathcal{D}^2u(x,t)\mathbf{x}^T, \mathcal{J}u(x,t)) \geq 0.$$

On the other hand, by definition of American option we have

$$u(x,t) \geq u_T(x),$$

and therefore we have proved the viscosity supersolution inequality.

To prove that u is a subsolution we shall use the equivalent definition of u by the DPP. Let $\phi \in \mathcal{C}^2(\overline{\Pi}_T)$ and suppose that $(x,t) \in \Pi_T$ is a global maximum point of $u-\phi$ such that $(u-\phi)(x,t) = 0$. By definition we already know that $u(x,t) \geq u_T(x)$.

Let us suppose that the equality is met, then the subsolution inequality is obviously satisfied and we are done. On the contrary, suppose that the strict inequality holds and define

$$\varepsilon = \frac{u(x,t) - u_T(x)}{2} > 0$$

and the related stopping time

$$\tau^{\varepsilon}_{x,t} = \inf\{t \le s \le T, u(X^t_s(x), s) \le u_T(X^t_s(x)) + \varepsilon\}.$$

Using [101, Prop. 3.2], for all $h > t$ we have

$$\begin{aligned} u(x,t) &= \mathbb{E}\Big[e^{-r(h\wedge\tau^{\varepsilon}_{x,t}-t)}u(h\wedge\tau^{\varepsilon}_{x,t}, X^t_{h\wedge\tau^{\varepsilon}_{x,t}}(x)) \\ &\quad + \int_t^{h\wedge\tau^{\varepsilon}_{x,t}} e^{-r(s-t)} f(X^t_s(x), s, u, \mathbf{X}^t_s \mathcal{D}u)ds\Big], \end{aligned}$$

where the function u in the integral part is evaluated in $(X^t_s(x), s)$. Subtracting $\phi(x,t)$, as (x,t) is a maximum point, we get

$$\begin{aligned} 0 &\le \frac{1}{h-t}\mathbb{E}\Big[e^{-r(h\wedge\tau^{\varepsilon}_{x,t}-t)}\phi(h\wedge\tau^{\varepsilon}_{x,t}, X^t_{h\wedge\tau^{\varepsilon}_{x,t}}(x)) - \phi(x,t) \\ &\quad + \int_t^{h\wedge\tau^{\varepsilon}_{x,t}} e^{-r(s-t)} f(X^t_s(x), s, u(X^t_s(x), s), \mathbf{X}^t_s(x)\mathcal{D}u(X^t_s(x), s))ds\Big]. \end{aligned}$$

As before, using Ito's formula on $e^{-rt}\phi(X_t, t)$ and properties of the process X_t we obtain

$$\begin{aligned} \omega(h-t) &\le \frac{1}{h-t}\mathbb{E}\Big[\int_t^{h\wedge\tau^{\varepsilon}_{x,t}} \Big(\partial_t\phi - H(X^t_s(x), s, \phi, \mathbf{X}^t_s\mathcal{D}\phi, \mathbf{X}^t_s\mathcal{D}^2\phi\, \mathbf{X}^{t\,T}_s, \mathcal{J}\phi)\Big)ds\Big] \\ &\le \sup_{t\le s\le h\wedge\tau^{\varepsilon}_{x,t}} \Big(\partial_t\phi(x,s) - H(x, s, \phi(s,x), \mathbf{x}\mathcal{D}\phi, \mathbf{x}\mathcal{D}^2\phi\mathbf{x}^T, \mathcal{J}\phi(s,x))\Big) \\ &\quad \times\mathbb{E}\Big[\frac{h\wedge\tau^{\varepsilon}_{x,t} - t}{h-t}\Big], \end{aligned}$$

where the function ϕ on the first line is evaluated at $(X^t_s(x), s)$. To estimate the last quantity we shall use the definition of $\tau^{\varepsilon}_{x,t}$. Consider the function $\tilde{u}(x,s) = u(x,s) - u_T(x)$, and note that this function has the same properties as u. Moreover,

$$\begin{aligned} \mathcal{P}[\tau^{\varepsilon}_{x,t} \le h] &\le \mathcal{P}\Big[\sup_{t\le s\le h} |\tilde{u}(X^t_s(x), t) - \tilde{u}(x,t)| \ge \varepsilon\Big] \\ &\le \frac{1}{\varepsilon^2}\mathbb{E}\Big[\sup_{t\le s\le h} |\tilde{u}(X^t_s(x), t) - \tilde{u}(x,t)|\Big]^2 \\ &\le \frac{\mathrm{C}}{\varepsilon^2}\mathbb{E}\Big[(1+|x|)(h-t)^{\frac{1}{2}} + \sup_{t\le s\le h} |X^t_s(x) - x|\Big]^2 \le \mathrm{C}'(h-t). \end{aligned}$$

Sending h to t we obtain

$$0 \le \partial_t\phi(x,t) - H(x, t, \phi(x,t), \mathbf{x}\mathcal{D}\phi(x,t), \mathbf{x}\mathcal{D}^2\phi(x,t)\mathbf{x}^T, \mathcal{J}\phi(x,t)),$$

which is the desired subsolution inequality. ■

The uniqueness of the viscosity solution of the obstacle problem (5.2.5)-(5.2.6) does not follow immediately from the results in Section 4.2, as the obstacle problem is not directly included in the class described by (4.0.1)-(4.0.2). However, as is well known, a comparison principle for an obstacle problem can be easily established starting off from the corresponding comparison principle for the problem without an obstacle, see, for example, [3, 4, 2, 101].

Theorem 5.2.6 *Suppose that assumptions* **(B)** *hold. Then the function u given by* (5.2.4) *is the unique viscosity solution of the problem* (5.2.5)-(5.2.6).

Proof. The proof is an extension of Theorem 4.2.14 to the obstacle problem case.

Let us define $\tilde{u}$ and $\tilde{v}$ as in (4.2.2)-(4.2.3) respectively. We would like to prove the comparison result

$$\tilde{u} \le \tilde{v} \text{ on } \tilde{\Pi} \times [0,T],$$

arguing by contradiction. Let us suppose that

$$\mathcal{M} = \sup_{(x,t)\in\tilde{\Pi}\times[0,T]} \Big\{\tilde{u}(x,t) - \tilde{v}(x,t)\Big\} > 0.$$

An a priori bound on $\mathcal{M}$ is not given, therefore we proceed with the approximated function

$$\tilde{u}_\delta(x,t) = \tilde{u}(x,t) - \delta h_o(x), \qquad \tilde{v}_\delta(x,t) = \tilde{v}(x,t) + \delta h_o(x).$$

Defining the function

$$\Psi_\delta(x,t) = \tilde{u}_\delta(x,t) - \tilde{v}_\delta(x,t), \qquad (x,t) \in \tilde{\Pi} \times [0,T],$$

it can be proved that there exists an infinitesimal sequence of parameters $\delta = \delta(n)$ such that (x_δ, t_δ) is a maximum point for Ψ_δ, and, as $\delta \to 0$,

$$\tilde{u}_\delta(x_\delta,t_\delta) - \tilde{v}_\delta(x_\delta,t_\delta) \to \mathcal{M}, \qquad \delta h_o(x_\delta) \to 0.$$

Using Definition 5.2.3, Lemma 4.2.16 holds even in the case of the obstacle problem. To get the desired contradiction, we double the x variable and consider the function

$$(x,y,t) \mapsto \tilde{u}_\delta(x,t) - \tilde{v}_\delta(y,t) - \psi(x,y),$$

where $\psi(x,y)$ is defined in (4.2.11). Using classical arguments in the theory of viscosity solutions, for any fixed δ, $\varepsilon > 0$, this function attains its maximum at the point $(x^\varepsilon_\delta, y^\varepsilon_\delta, t^\varepsilon_\delta) \in \tilde{\Pi}^2 \times [0,T]$ such that

$$x^\varepsilon_\delta,\, y^\varepsilon_\delta \to x_\delta, \qquad t^\varepsilon_\delta \to t_\delta, \qquad \frac{1}{2\varepsilon}|x^\varepsilon_\delta - y^\varepsilon_\delta|^2 \to 0,$$

$$\tilde{u}_\delta(x^\varepsilon_\delta, t^\varepsilon_\delta) - \tilde{v}_\delta(y^\varepsilon_\delta, t^\varepsilon_\delta) \to \tilde{u}_\delta(x_\delta, t_\delta) - \tilde{v}_\delta(y_\delta, t_\delta),$$

as $\varepsilon \to 0$. Hence we can suppose, without loss of generality, that for all δ, $\varepsilon > 0$

$$\begin{array}{cc} x^\varepsilon_\delta,\, y^\varepsilon_\delta \in \mathcal{Q}(r_\delta), & t^\varepsilon_\delta < T, \\ |u_\delta(x^\varepsilon_\delta, t^\varepsilon_\delta)|,\, |v_\delta(y^\varepsilon_\delta, t^\varepsilon_\delta)| \le R_\delta, & \tilde{u}_\delta(x^\varepsilon_\delta, t^\varepsilon_\delta) - \tilde{v}_\delta(y^\varepsilon_\delta, t^\varepsilon_\delta) > 0, \end{array}$$

where $r_\delta > 1$ and $R_\delta > 0$ do not depend on ε. To simplify notations in what follows, the indices δ, ε are replaced with an over-bar and time dependence is omitted.

By an extension of Lemma 4.2.16, $\tilde{u}_\delta$, $\tilde{v}_\delta$ solve

$$\min\Big(-\partial_t \tilde{u}_\delta + H(x,t,\tilde{u}_\delta, \mathbf{x}\mathcal{D}\tilde{u}_\delta, \mathbf{x}\mathcal{D}^2\tilde{u}_\delta\mathbf{x}^T, \mathcal{J}\tilde{u}_\delta) + \delta h_o(x) - 2\omega(\mathrm{D}\delta h_o(x)),$$
$$\tilde{u}_\delta(x,t) - u_T(x)\Big) = 0,$$
$$\min\Big(-\partial_t \tilde{v}_\delta + H(x,t,\tilde{v}_\delta, \mathbf{x}\mathcal{D}\tilde{v}_\delta, \mathbf{x}\mathcal{D}^2\tilde{v}_\delta\mathbf{x}^T, \mathcal{J}\tilde{v}_\delta) - \delta h_o(x) + 2\omega(\mathrm{D}\delta h_o(x)),$$
$$\tilde{v}_\delta(x,t) - u_T(x)\Big) = 0,$$

Proceeding as in Section 4.2 there exist $\overline{\tau} \in \mathbb{R}$ and symmetric matrices $\overline{M}$ and $\overline{N}$ such that

$$\begin{pmatrix} \overline{M} & 0 \\ 0 & -\overline{N} \end{pmatrix} \le \frac{3}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

and

$$\min\Big(-\overline{\tau} + H\Big(\overline{x}, \tilde{u}_\delta, \overline{\mathbf{x}p}, \overline{\mathbf{x}}\overline{M}\overline{\mathbf{x}}^T, \hat{\mathcal{J}}_\kappa\psi(\overline{x},\overline{y}) + \hat{\mathcal{J}}^\kappa(\tilde{u}_\delta,\overline{p})(\overline{x})\Big)$$
$$+\delta h_o(\overline{x}) - 2\omega(\mathrm{D}\delta h_o(\overline{x})), \tilde{u}_\delta(\overline{x}) - u_T(\overline{x})\Big) \qquad \le 0,$$

$$\min\Big(-\overline{\tau} + H\Big(\overline{y}, \tilde{v}(\overline{y}), \overline{yp}, \overline{\mathbf{y}}\overline{N}\overline{\mathbf{y}}^T, -\hat{\mathcal{J}}_\kappa\psi(\overline{x},\overline{y}) + \hat{\mathcal{J}}^\kappa(\tilde{v}_\delta,\overline{p})(\mathbf{y})\Big)$$
$$-\delta h_o(\mathbf{y}) + 2\omega(\mathrm{D}\delta h_o(\underline{y})), \tilde{v}_\delta(\overline{y}) - u_T(\overline{y})\Big) \qquad \ge 0,$$

where $\overline{p} = \frac{1}{\varepsilon}(\overline{x} - \overline{y})$. Subtracting the two inequalities yields

$$\min\Big(\overline{H}_\kappa - \overline{G}, \tilde{u}(\overline{x}) - \tilde{v}(\overline{y})\Big) \le 0,$$

where

$$\begin{array}{rcl} \overline{H}_\kappa & = & H\Big(\overline{x}, \tilde{u}_\delta(\overline{x}), \overline{\mathbf{x}p}, \overline{\mathbf{x}}\overline{M}\overline{\mathbf{x}}^T, \hat{\mathcal{J}}_\kappa\psi(\overline{x},\overline{y}) + \hat{\mathcal{J}}^\kappa(\tilde{u}_\delta,\overline{p})(\overline{x})\Big) \\ & & -H\Big(\overline{y}, \tilde{v}(\overline{y}), \overline{\mathbf{y}p}, \overline{\mathbf{y}}\overline{N}\overline{\mathbf{y}}^T, -\hat{\mathcal{J}}_\kappa\psi(\overline{x},\overline{y}) + \hat{\mathcal{J}}^\kappa(\tilde{v}_\delta,\overline{p})(\overline{y})\Big), \\ \overline{G} & = & -\delta[h_o(\overline{x}) + h_o(\overline{y})] + 2\omega(\mathrm{D}\delta h_o(\overline{x})) + 2\omega(\mathrm{D}\delta h_o(\overline{y})). \end{array}$$

From now on the proof proceeds as in Section 4.2, and the desired contradiction $\mathcal{M} \le 0$ is obtained. ■

5.3 Merton's problem.

In this section we shall consider the celebrated Merton's optimal portfolio management problem. We consider a market with two securities, a bond whose price evolves according to

$$dB_s = rB_s ds,$$

where $r > 0$ is the interest rate, and a stock whose price process satisfies

$$S_t = S_0 e^{L_t},$$

where L_t is a Lévy process, i.e., L_t has independent and stationary increments, is continuous in probability, càdlàg (continuous from the right, with limit from the left), and $L(0) = 0$, see [24]. For such processes we have the Lévy-Khintchine decomposition

$$L_t = \mu t + \sigma W_t + \int_0^t \int_{|z|<1} z\, \tilde{N}(ds, dz) + \int_0^t \int_{|z|\geq 1} z\, N(ds, dz),$$

where μ and σ are constants, W_t is a Wiener process, $N(dt, dz)$ is Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity measure $dt \times \nu(dz)$, $\nu(dz)$ is a σ-finite Borel measure on $\mathbb{R}\backslash\{0\}$ with the property

$$\int_{\mathbb{R}-\{0\}} (1 \wedge z^2)\, \nu(dz) < \infty,$$

and $\tilde{N}(dt, dz) = N(dt, dz) - dt \times \nu(dz)$ is the compensated Poisson random measure. We assume W_t and $N(dt, dz)$ are independent stochastic processes. If $\sigma = 0$, then L is said to be a Lévy *jump process* and if also $\mu = 0$ it is a Lévy *pure jump process*. If L has only non-negative increments, then it is a *subordinator*, see [24]. Under the additional integrability condition on the Lévy measure

$$\int_{|z|\geq 1} \left| e^z - 1 \right| \nu(dz) < \infty,$$

we can write the differential of the stock price dynamics as (using Itô's Formula [64])

$$dS_t = \hat{\mu} S_t\, dt + \sigma S_t\, dW_t + S_{t-} \int_{\mathbb{R}\backslash\{0\}} \left(e^z - 1\right) \tilde{N}(dt, dz).$$

Here we have introduced the short-hand notation

$$\hat{\mu} = \mu + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}\backslash\{0\}} \left(e^z - 1 - z\mathbf{1}_{|z|<1}\right) \nu(dz).$$

Let $\pi(s)$ be the fraction of wealth invested in the stock at time t. Assume that there are no transaction costs in the market. The wealth $R(t)$ of the investor is defined as

$$R(t) = \frac{\pi(t)R(t)}{S(t)}S(t) + \frac{(1-\pi(t))R(t)}{B(t)}B(t),$$

where $\frac{\pi(t)R(t)}{S(t)}$ and $\frac{\pi(t)R(t)}{B(t)}$ are respectively the number of stocks and bonds owned by the investor. The self-financing hypothesis gives

$$dR(t) = \frac{\pi(t)R(t)}{S(t)}\,dS(t) + \frac{(1-\pi(t))R(t)}{B(t)}\,dB(t). \tag{5.3.1}$$

Inserting the expression for $dS(t)$ and $dR(t)$ into (5.3.1) gives the following dynamics for the wealth process:

$$dR_s = (\pi_s(\mu - r) + r)R_s\,ds + \pi_s \sigma R_s\,dW_s + \pi_s R_s \int_{\mathbb{R}-\{0\}} (e^z - 1)\tilde{N}(ds, dz),$$

with initial wealth $R(t) = \rho$. The control $\pi_s, t \leq s \leq T$ is admissible if it satisfies

c1 $\mathcal{F}_s$ -progressively measurable, with $\mathcal{F}_s = \sigma\{W_u,\ N(u,A);\ t \leq u \leq s,\ \nu(A) < \infty\}$

c2 $\mathbb{E}\Big[\int_t^T \pi_s^2\,ds\Big] < \infty$.

We denote the set of admissible policies by $\mathcal{A}_t$.

The value function is defined by

$$V(t,\rho) = \sup_{\mathcal{A}_t} \mathbb{E}[U(R_T)], \tag{5.3.2}$$

where $U : [0,\infty) \to \mathbb{R}$ is a utility function, i.e., a continuous, nondecreasing, concave, and sublinearly growing function. For examples and details about the choice of an utility function see Chapter 1 and the work by Duffie and Epstein [42].

The dynamic programming method reduces the study of the value function to the study of a nonlinear integro–partial differential Bellman equation:

$$\begin{aligned} &v_t + \max_{\pi \in [0,1]} \Bigg[(r + (\hat{\mu} - r)\pi)xv_x + \frac{1}{2}\sigma^2\pi^2x^2v_{xx} \\ &\quad + \int_{\mathbb{R}\setminus\{0\}} \Big(v(x + \pi x(e^z - 1), t) - v(x,t) - \pi x v_x(x,t)(e^z - 1) \Big)\,\nu(dz) \Bigg] = 0, \\ &v(x,T) = U(x), \end{aligned} \tag{5.3.3}$$

for $(x,t) \in \Pi_T = (0,\infty) \times [0,T)$. Note that we do not specify a boundary condition at $x = 0$. Since clearly (5.3.3) can be written in the form (4.0.1)-(4.0.2), we have the following theorem:

Theorem 5.3.1 *The value function defined in* (5.3.2) *is non-negative, non-decreasing, concave, has the same sublinear growth as the utility function, and is uniformly continuous on* $[0,\infty) \times [0,T]$*, Moreover,* V *is the unique viscosity solution (in the sense of Section 4.2) of* (5.3.3)*.*

The first part of this theorem follows from arguments that are standard in the pure PDE setting and, since they can easily be transferred to the integral setting, we omit them, see instead [118, 48] for similar proofs. Although the value function is uniformly continuous on $[0,\infty) \times [0,T]$, the comparison principle in Section 4.2 for semicontinuous viscosity sub- and supersolutions is very useful for giving simple convergence proofs for numerical methods for the Merton model, see [34, 48].

We could have also treated Merton's problem in a multi-asset setting giving raise to multi-dimensional integro–partial differential equations of the type (4.0.1).

Part III

Numerical approximation of the integro–differential models for financial markets

This part is devoted to the numerical analysis of the problem presented in Part II. We shall present the numerical approximation of the nonlinear integro–differential problem

$$-\partial_t u + H(x, t, u, \mathbf{x}\mathcal{D}u, \mathbf{x}\mathcal{D}^2 u \mathbf{x}^T, \mathcal{J}u, \mathcal{I}u) = 0,$$

where H is a continuous function of all its arguments and $\mathcal{J}u$ and $\mathcal{I}u$ are the following integro–differential operators:

$$\mathcal{J}u(x,t) = \int\limits_{\mathbb{R}^M - \{0\}} \left[u(x + \mathbf{x}\beta(x,t,z), t) - u(x,t) - \mathbf{x}\beta(x,t,z) \cdot \mathcal{D}u(x,t)\right] m_{x,t}(dz),$$

$$\mathcal{I}u(x,t) = \int\limits_{\mathbb{R}^M - \{0\}} \left[u(x + \mathbf{x}\beta(x,t,z), t) - u(x,t)\right] \gamma(x,t,z) m_{x,t}(dz),$$

$m_{x,t}(dz)$ is the jump measure and could depend on the point (x, t).

In the previous chapters we have seen that problems of this form arise in financial market modeling when the market evolution is expressed in terms of a general Lévy process.
A great deal has been done for the numerical approximation of viscosity solutions, starting from [36].
For what concerns nonlinear second order problems, several work has been done by Barles and alt., [16, 13, 11]: starting from the monotonicity property of the differential operator and the ordering property of viscosity solutions, they derived a convergence result for a large class of numerical schemes.

The numerical approximation of nonlinear strongly degenerated integro–differential parabolic problems is the main objective of our investigation.
The integral term was already considered in [6] in the framework of linear problems with constant coefficients. The authors proposed to use an operator splitting method compared with the drawbacks of a pure Crank-Nicholson one. In that context, the method was shown to be quite effective as it had a lighter computational burden and allowed to couple the differential part, with an implicit finite difference method, and the integral part, with an FFT method. The FFT method requires a constant grid step, however it could diminish the numerical precision of the scheme in some areas; to overcome this difficulty we can use an asymptotic profile of the solution or a particular feature of the integral operator. A closer discussion of this method is done in Chapter 7.
For what concerns bounded jump measures, several work has been done by Forsyth and alt. [50, 39, 40], where implicit discretization is developed for American derivatives. Particular attention is devoted to the integral term, computed using an iterative method or a FFT method.

In the unbounded Lévy measure framework, a different approach using variational inequalities and the semigroup theory in Sobolev spaces with exponential weights can be found in [89, 115], while an efficient numerical solution using a wavelet Galerkin discretization can be found in [116, 75, 88]. Another difficulty stems from the non local nature of the integral term and the possibly unboundedness of the general Lévy measure. Therefore to deal with the numerical approximation of the integro–differential problem it is necessary to truncate the problem domain on one hand and the integral domain on the other.

This part is organized as follows.

In Chapter 6 we present the fundamental result by Barles and Souganidis: in the paper [16] they showed convergence results for a large class of numerical schemes to the solution of fully nonlinear second order elliptic or parabolic differential equation of the form

$$\begin{cases} \partial_t u + F(x,t,u,\mathcal{D}u,\mathcal{D}^2 u) &= 0, \\ u(x,0) &= u_0(x). \end{cases}$$

Their theory is a starting point for the numerical approximation of viscosity solution to nonlinear integro–partial differential problems.

In Chapter 7 we present an original convergence result for monotone consistent schemes to viscosity solutions of nonlinear integro–differential Cauchy problems, when the Lévy measure is bounded:

$$\begin{cases} \partial_t u - \mathcal{L}_{\mathcal{I}}(x,t,\mathcal{I},\mathcal{D},\mathcal{D}^2)u + H(x,t,\mathcal{D}u,\mathcal{I}u) &= 0, \\ u(x,0) &= u_0(x), \end{cases} \tag{5.3.4}$$

where u_0 is a continuous initial data, $\mathcal{L}_{\mathcal{I}}$ is a linear degenerate elliptic operator and H is a nonlinear first order operator. Here $\mathcal{I}u$ is an integral term given by

$$\mathcal{I}u = \int_{\mathbb{R}^N} M(u(x+z,t),u(x,t))\mu_{x,t}(dz),$$

where $\mu_{x,t}$ is a positive bounded measure and M is a function which is non decreasing in the first argument, $M(u,u)=0$ and such that

$$M(u,v) - M(w,z) \leq c((u-w)_+ + |v-z|).$$

We will propose a criterion to select a bounded domain in which calculate the integral term and a way to control the error we make applying this truncation.

In Chapter 8 we present the extension of the previous results to the general unbounded Lévy measure:

$$\begin{cases} -\partial_t u + H(x,t,u,\mathbf{x}\mathcal{D}u,\mathbf{x}\mathcal{D}^2 u\mathbf{x}^T,\mathcal{J}u,\mathcal{I}u) &= 0, \\ u(x,T) &= u_T(x), \end{cases}$$

$\mathcal{J}u$ and $\mathcal{I}u$ integral operator

$$\begin{aligned}\mathcal{J}u(x,t) &= \int\limits_{\mathbb{R}^{\mathrm{M}}-\{0\}} [u(x+\mathbf{x}\beta(x,t,z),t)-u(x,t)-\mathbf{x}\beta(x,t,z)\cdot\mathcal{D}u(x,t)]\,\nu(dz),\\ \mathcal{I}u(x,t) &= \int\limits_{\mathbb{R}^{\mathrm{M}}-\{0\}} [u(x+\mathbf{x}\beta(x,t,z),t)-u(x,t)]\,\gamma(x,t,z)\nu(dz),\end{aligned}$$

and the Lévy measure $\nu(dz)$ satisfying the following integrability condition

$$\int_E (1\wedge|z|^2)\nu(dz)<\infty.$$

The difficulty in this case stands in the unboundedness of the Lévy measure in a neighborhood of the origin, which requires another truncation of the integration domain.
Here, as in the bounded Lévy case, we propose a criterion to select the bounded integration domain and we estimate the error we incur applying that truncation.

A particular care is needed when calculating the solution near to the boundary points. A common approach consists in replacing the original problem with an homogeneous one or to use an asymptotic representation formula of the solution.
In Chapter 7 we propose a different approach analyzing the diffusive effect of the integral term. With particular attention, this approach can be extended to the unbounded Lévy measure.

Notations.

We define a **numerical grid** in $\mathbb{R}^{\mathrm{N}}\times(0,T)$ using the following notation:

h $=(h_1,...,h_{\mathrm{N}})$ is the spatial grid size;

k is the time grid size;

$(x_j,\ t_n)=(jh,nk)$, $j\in\mathbb{Z}$ and $n\in\mathbb{N}$, are the grid points;

v_j^n is the value of the function v, defined on the grid or defined for continuously varying (x,t), at the grid point (x_j,t_n);

$\tilde{v}$ is the vector of v values, $(v_j^n)_j$ for j varying on a subset of $\mathbb{Z}$ and $n\in\mathbb{N}$.

Chapter 6

Numerical approximation of purely differential problems.

In this chapter we would like to give an overview of the convergence results for degenerate fully nonlinear second order equations, with particular care for problems arising in Finance Theory. Several work has been done in this field [16, 11, 13] in the viscosity solution setting.
We shall present a convergence result for the following problem

$$\partial_t u + F(x, t, u, \mathcal{D}u, \mathcal{D}^2 u) = 0, \tag{6.0.1}$$

with an initial data

$$u(x, 0) = u_0(x); \tag{6.0.2}$$

where $u_0 \in \mathcal{C}(\mathbb{R}^{\text{N}})$ and $F \in \mathcal{C}(\mathbb{R}^{\text{N}} \times [0, T] \times \mathbb{R} \times \mathbb{R}^{\text{N}} \times \mathcal{S}_{\text{N}})$. These kind of problem arise in Finance when pricing a derivative in a market driven by a continuous process, like in the Black and Scholes market [28]. Nonlinearity effects arise when the interest rate is no more a constant value, but changes as a function of the wealth invested in the market. Typical examples of nonlinear market are the incomplete market and the market with large investors, as it has been explained in Chapter 1.

In [16] a convergence result for monotone stable and consistent scheme is given, provided that suitable assumptions on the pure differential problem hold. The convergence result relies on a passage to the limit in fully nonlinear second order partial differential equations based only on L^∞ estimates and on the notion of viscosity solution.

Furthermore we shall present some simple numerical schemes for the specific pricing problem examined.
For all the definition and results in viscosity solution theory, we refer to Chapter 3 and [34].

6.1 Financial examples.

Let us recall the classical Black and Scholes example introduced in Section 1.3 evolving in the risk–neutral probability.
The price of an European call can be given as an expected value of its discounted payoff:

$$U(S,t) = \mathbb{E}\left[e^{-r(T-t)}(S_t - K)^+ \middle| \mathcal{F}\right];$$

Black and Scholes proved that the function U defined by means of that equality is a solution of the following partial differential equation on $\mathbb{R}^+ \times (0,T)$

$$-\partial_t U - \frac{1}{2}\sigma^2 S^2 \partial^2_{SS} U - rS\partial_S U + rU = 0,$$

with final datum

$$U(S,T) = (S-K)^+.$$

In the simple constant coefficient case we have an explicit solution to the pricing problem, but it is not always the case, as to get resemblance to real data we need to consider more sophisticated model, such as non–constant interest rate r and volatility σ.
The value $S = 0$ is a degeneration point for the equation: to overcome this difficulty we apply a change of variable $x = \log S$; the function $u(x, T-t) = U(S,t)$ is solution of equation (6.0.1) with

$$F(x,t,u,\mathcal{D}u,\mathcal{D}^2 u) = \frac{1}{2}\sigma^2 \partial^2_{xx} u + \left(r - \frac{1}{2}\sigma^2\right)\partial_x u - ru,$$

on $\mathbb{R} \times (0,T)$.

On the other hand, the price of an American put with strike price K and maturity T is given as the solution of an optimal stopping problem on $\mathbb{R}^+ \times (0,T)$

$$U(S,t) = \inf_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}\left[e^{-r(\tau-t)}(K-S)^+ \middle| \mathcal{F}_t\right].$$

This problem can be rephrased in terms of a solution of a deterministic obstacle problem

$$\min\left(-\partial_t U - \frac{1}{2}\sigma^2 S^2 \partial^2_{SS} U - rS\partial_S U + rU, U - (K-S)^+\right) = 0,$$

with final datum

$$U(S,T) = (K-S)^+.$$

Compared to the European pricing problem, in this case there is no explicit solution, even for the constant coefficient problem; the same singularity at $S = 0$ is present in this problem and can be faced as in the European case, applying a logarithmic change of variable to the original problem..

Generalizing these equation to the cases exposed in Chapter 1 we obtain fully nonlinear and degenerate equation, which imply particular care in the theoretical and numerical approach.
These equations do not have classical solutions because of their features, therefore the viscosity solution approach is needed.

6.2 Discontinuous viscosity solutions and numerical schemes.

Let us consider the following problem

$$\partial_t u + \overline{F}(x, t, u, \mathcal{D}u, \mathcal{D}^2 u) = 0, \text{ in } \mathbb{R}^{\mathrm{N}} \times [0, T] \tag{6.2.1}$$

where $\overline{F} : \mathbb{R}^{\mathrm{N}} \times [0, T] \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}} \times \mathcal{S}_{\mathrm{N}} \to \mathbb{R}$ and u are locally bounded, possibly discontinuous. The problem (6.2.1) is stated on the closed set $[0, T]$, as it has been done in [16] because we would like to write both the equation and the "boundary condition" in one expression: let define $\overline{F}$ as follows

$$\overline{F} = \begin{cases} F(x, t, u, \mathcal{D}u, \mathcal{D}^2 u), & \text{if } (x, t) \in \mathbb{R}^{\mathrm{N}} \times (0, T], \\ \\ u - u_0 & (x, t) \in \mathbb{R}^{\mathrm{N}} \times \{0\}, \end{cases}$$

from which derives the possibly discontinuity of the operator $\overline{F}$.
We assume that F is an *elliptic operator*, that is

$$F(x, t, u, p, \mathcal{X}) \le F(x, t, u, p, \mathcal{Y}), \text{ for all } \mathcal{X}, \mathcal{Y} \in \mathcal{S}_{\mathrm{N}} \text{ such that } \mathcal{X} \ge \mathcal{Y}. \tag{6.2.2}$$

Because of the possible discontinuity of u and $\overline{F}$, we have to restate the definition of viscosity solution as follows.

Definition 6.2.1 *A locally bounded function $u : \mathbb{R}^{\mathrm{N}} \to \mathbb{R}$ is a* **viscosity supersolution** *(respectively* **subsolution***) of* (6.2.1) *if for all $\phi \in \mathcal{C}^\infty(\mathbb{R}^{\mathrm{N}} \times [0, T])$ and all $(x, t) \in \mathbb{R}^{\mathrm{N}} \times [0, T]$ such that $u^* - \phi$ (respectively, $u_* - \phi$) has a local maximum point (respectively, minimun) at (x, t) we have*

$$\overline{F}_*(x, t, u^*(x, t), \mathcal{D}\phi(x, t), \mathcal{D}^2\phi(x, t)) \le 0,$$

(respectively

$$\overline{F}^*(x, t, u_*(x, t), \mathcal{D}\phi(x, t), \mathcal{D}^2\phi(x, t)) \ge 0.)$$

The function u is a **viscosity solution** *if it is both a sub/supersolution of* (6.2.1).

In this way the problem to study becomes

$$\partial_t u + \overline{F}(x,t,u,\mathcal{D}u,\mathcal{D}^2u) = 0, \text{ on } \mathbb{R}^{\mathrm{N}} \times [0,T].$$

A locally bounded function u is a viscosity solution of (6.2.1)-(6.0.2) in the sense of Definition 6.2.1 if it satisfies in viscosity sense the following

$$\partial_t u + F(x,t,u,\mathcal{D}u,\mathcal{D}^2u) = 0, \text{ in } \mathbb{R}^{\mathrm{N}} \times (0,T], \tag{6.2.3}$$

$$\max\left\{\partial_t u + F(x,t,u,\mathcal{D}u,\mathcal{D}^2u), u - u_0\right\} \geq 0, \text{ on } \mathbb{R}^{\mathrm{N}} \times \{0\}, \tag{6.2.4}$$

$$\min\left\{\partial_t u + F(x,t,u,\mathcal{D}u,\mathcal{D}^2u), u - u_0\right\} \leq 0, \text{ on } \mathbb{R}^{\mathrm{N}} \times \{0\}. \tag{6.2.5}$$

It has been noticed that the previous equation (6.2.3)–(6.2.4)–(6.2.5) seems to be the natural replacement of problem (6.2.1)–(6.0.2), which does not have in general a solution which assumes continuously the boundary condition.

6.3 A general convergence result.

Let us consider an approximation scheme of the form

$$\tilde{Q}(h,k,j,n,u_j^n,\tilde{u}) = 0, \text{ in } \mathbb{R}^{\mathrm{N}} \times [0,T]; \tag{6.3.1}$$

here $S : \mathbb{R} \times \mathbb{R} \times \mathbb{Z} \times \mathbb{N} \times \mathbb{R} \times B(\mathbb{R}^{\mathrm{N}} \times [0,T]) \to \mathbb{R}$, where $B(\mathbb{R}^{\mathrm{N}} \times [0,T])$ indicates the set of all bounded functions on $\mathbb{R}^{\mathrm{N}} \times [0,T]$.

It will be proved that as far as the proposed scheme is monotone, stable and consistent, its solution converges to the unique continuous viscosity solution of (6.2.1), provided a comparison principle for this equation has been proved.

Let us suppose the following assumptions on the scheme S hold:

PROPERTIES OF THE SCHEME.

S.1 *Monotonicity of the approximation* If $\tilde{u} \leq \tilde{v}$ and $u_j^n = v_j^n$ for all h, k and $0 \leq n \leq N$ then:

$$\tilde{Q}(h,k,j,n,u_j^n,\tilde{u}) \leq \tilde{Q}(h,k,j,n,v_j^n,\tilde{v}). \tag{6.3.2}$$

S.2 *Stability*

For all h, k a solution $\hat{u}$ does exist that is bounded independently from (h,k); (6.3.3)

S.3 *Consistency*
For all $\phi \in \mathcal{C}_b^\infty(\mathbb{R}^N \times [0,T])$ and for all $(x,t) \in \mathbb{R}^N \times (0,T)$ we have:

$$\liminf_{\substack{(h,k)\to 0\\(jh,nk)\to(x,t)\\ \xi\to 0}} \frac{\tilde{Q}(h,k,j,n,\phi_j^n+\xi,\tilde{\phi}+\xi)}{\rho(h,k)} \geq \partial_t u + F_*(x,t,u,\mathcal{D}u,\mathcal{D}^2 u);$$

$$\limsup_{\substack{(h,k)\to 0\\(jh,nk)\to(t,x)\\ \xi\to 0}} \frac{\tilde{Q}(h,k,j,n,\phi_j^n+\xi,\tilde{\phi}+\xi)}{\rho(h,k)} \leq \partial_t u + F^*(x,t,u,\mathcal{D}u,\mathcal{D}^2 u);$$

PROPERTY OF THE EQUATION

S.4 *Maximum principle or Strong uniqueness property.* If $u \in B(\mathbb{R}^N \times [0,T])$ is an upper semicontinuous solution of (6.2.1) and $v \in B(\mathbb{R}^N \times [0,T])$ is a lower semicontinuous solution of (6.2.1), then

$$u \leq v, \text{ on } \mathbb{R}^N \times [0,T].$$

Under this theoretical assumptions the following convergence result holds.

Theorem 6.3.1 *[16, Theorem 2.1, page 275] Let assumptions (**S.1**)–(**S.4**) hold true. Then as* $(h,k) \to 0$*, the solution of the scheme* (6.3.1) *converges locally uniformly to the unique continuous viscosity solution of the problem* (6.2.1)–(6.0.2).

Remark 6.3.2 The monotonicity assumption (6.3.2) is the discrete analogue of the ellipticity condition (6.2.2). It can be noticed that it can be relaxed in several way as inequality (6.3.2) needs only to hold up to $o(\rho(h,k))$ terms. ■

Proof. Let $\overline{u}$, $\underline{u} \in B(\mathbb{R}^N \times [0,T])$ be defined respectively by

$$\underline{u}(x,t) := \liminf_{\substack{(\Delta x,\Delta t)\to 0\\(j\Delta x,n\Delta t)\to(x,t)}} u_j^n \tag{6.3.4}$$

$$\overline{u}(x,t) := \limsup_{\substack{(\Delta x,\Delta t)\to 0\\(j\Delta x,n\Delta t)\to(x,t)}} u_j^n. \tag{6.3.5}$$

We would like to prove that $\overline{u}$ and $\underline{u}$ are respectively sub– and supersolutions of (6.2.1); if this claim is proved to be true, by definition we have $\underline{u} \leq \overline{u}$, the other inequality holding because of lower semicontinuity of $\underline{u}$ and upper semicontinuity $\overline{u}$, therefore

$$u = \underline{u} = \overline{u},$$

is the unique continuous solution to (6.2.1). Local uniform convergence of the solution of the scheme to the solution of the problem is gained by (6.3.4)–(6.3.5) and the continuity of u.

To prove the claim we shall consider only the case of $\overline{u}$, the other being the same. Let $\phi \in \mathcal{C}_b^\infty(\mathbb{R}^N \times [0,T])$ and assume (x_0, t_0) be a local maximum of $\overline{u} - \phi$ on $\mathbb{R}^N \times [0,T]$; without loss of generality we can assume that the maximum point is a strict local maximum and $u(x_0, t_0) = \phi(x_0, t_0)$; moreover we can assume that

$$\phi \geq 2 \sup_{j \in \mathbb{Z}, n \in \mathbb{N}} \|u_j^n\|, \text{ outside the ball } B((x_0, t_0), r),$$

where $r > 0$ is such that

$$\overline{u}(x,t) - \phi(x,t) \leq 0 = \overline{u}(x_0, t_0) - \phi(x_0, t_0), \text{ in } B((x_0, t_0), r).$$

From these assumptions it follows that there exist a sequence $(\Delta x_k, \Delta t_k) \in \mathbb{R}^{+2}$ and $(y_k, s_k) \in \mathbb{R}^N \times [0,T]$ such that, as $k \to \infty$

$$\begin{aligned} &(\Delta x_k, \Delta t_k) \to 0, \quad (y_k, s_k) \to (x_0, t_0), \quad u^{(\Delta x_k, \Delta t_k)}(y_k, s_k) \to \overline{u}(x_0, t_0), \\ &(y_k, s_k) \text{ is a global maximum point of } u^{(\Delta x_k, \Delta t_k)}(\cdot, \cdot) - \phi(\cdot, \cdot). \end{aligned} \tag{6.3.6}$$

Denoting by $\xi_k = u^{(\Delta x_k, \Delta t_k)}(y_k, s_k) - \phi(y_k, s_k)$, we have

$$\begin{aligned} &\xi_k \to 0 \text{ and} \\ &u^{(\Delta x_k, \Delta t_k)}(x,t) \leq \phi(x,t) + \xi_k, \end{aligned}$$

for all $(x,t) \in \mathbb{R}^N \times [0,T]$. By the definition of $u^{(\Delta x_k, \Delta t_k)}$, the hypotheses (6.3.2), (6.3.6)

$$\tilde{Q}(\Delta x, \Delta t, j_k, n_k, \phi(y_k, s_k) + \xi_k, \tilde{\phi} + \xi_k) \leq 0.$$

Now, taking limits in the previous inequality, using the consistency of the scheme, we obtain:

$$\begin{aligned} 0 &\geq \liminf_{k \to \infty} \frac{\tilde{Q}(\Delta x_k, \Delta t_k, j_k, n_k, \phi(y_k, s_k) + \xi_k, \tilde{\phi} + \xi_k)}{\rho_k(\Delta x, \Delta t)} \\ &\geq \liminf_{\substack{(\Delta x, \Delta t) \to 0 \\ (j\Delta x, n\Delta t) \to (x,t) \\ \xi \to 0}} \frac{\tilde{Q}(\Delta x, \Delta t, j, n, \phi(y,s) + \xi, \tilde{\phi} + \xi)}{\rho(\Delta x, \Delta t)} \\ &\geq \partial_t \phi + F_*(x, t, \phi, \mathcal{D}\phi, \mathcal{D}^2\phi), \end{aligned}$$

which is the desired result, because of the assumption $\overline{u}(x_0, t_0) = \phi(x_0, t_0)$. ■

6.4 Financial applications.

In this section we consider some application of the previous result to the pricing problem, which is the main interest of this paper.
As we have previously shown in Chapter 3, the class of growth in which establish an existence and uniqueness result is of fundamental importance: solutions do exist in the class of growth determined by the initial (final) datum of the problem. Classical results have been established in the class of solution with linear growth at infinity, while the problem of financial interest have initial datas which typically grow up exponentially at infinity.
The convergence result by Barles and Souganidis [16] has been established in the class of locally bounded solution, but it can be easily extended to the class of exponential growth at infinity, as it suffices to consider a suitable monotonicity result for the differential problem.

Property of the equation

S.4 *Maximum principle or Strong uniqueness property.* Let $u_0 \in \mathcal{C}(\mathbb{R}^N) \cap L^\infty_{\exp}(\mathbb{R}^N)$ be the initial data of (6.2.1), (6.0.2), such that there exists $m > 0$:

$$|u_0(x)| \le Be^{n_0\|x\|} \text{ for } n_0 \le m;$$

if $u \in \bigcup_{n<m} L^\infty(0,T; L^\infty_{e^{n\|\cdot\|}}(\mathbb{R}^N))$ is an upper semicontinuous subsolution of (6.2.1) and $v \in \bigcup_{n<m} L^\infty(0,T; L^\infty_{e^{n\|\cdot\|}}(\mathbb{R}^N))$ is a lower semicontinuous supersolution of (6.2.1), then

$$u \le v \text{ on } \mathbb{R}^N \times [0,T].$$

With this property the general convergence result 6.3.1 still holds even in the financial setting.

Theorem 6.4.1 *[16, Theorem 2.1, page 275]Let assumptions (***S.1***)–(***S.4***) hold true. Then as* $(h,k) \to 0$, *the solution of the scheme* (6.3.1) *converges locally uniformly to the unique continuous viscosity solution of the problem* (6.2.1)–(6.0.2).

Proof. Let $\underline{u}$, $\overline{u} \in \bigcup_{n<m} L^\infty(0,T; L^\infty_{e^{n\|\cdot\|}}(\mathbb{R}^D))$ be defined by:

$$\underline{u}(x,t) := \liminf_{\substack{(\Delta x,\Delta t)\to 0\\ (j\Delta x, n\Delta t)\to(x,t)}} u_j^n,$$

$$\overline{u}(x,t) := \limsup_{\substack{(\Delta x,\Delta t)\to 0\\ (j\Delta x, n\Delta t)\to(x,t)}} u_j^n.$$

We want to prove that $\underline{u}$ and $\overline{u}$ are respectively supersolution and subsolution of the problem (6.2.1). If this claim is proved to be true, then by definition we have $\underline{u} \leq \overline{u}$, while the other inequality holds because of lower semicontinuity of $\underline{u}$ and upper semicontinuity of $\overline{u}$, hence:

$$u = \underline{u} = \overline{u},$$

is the unique continuous solution of the problem (6.2.1). This result, together with the definition of $\underline{u}$ and of $\overline{u}$ leads to the local uniform convergence of the solution of the scheme to the solution of the problem. To prove the claim, we shall consider only the case of $\overline{u}$, the other being the same.
We want to prove that $\overline{u} \in \mathcal{USC}$ and that it is a subsolution for the problem, i.e. for all $\phi \in \mathcal{C}^{2,1}(\mathbb{R}^N \times [0,T])$ such that $\overline{u} - \phi$ has a local maximum in (x_0, t_0) we have:

$$\partial_t \phi(x_0, t_0) + F(x_0, t_0, \overline{u}(x_0, t_0), \mathcal{D}\phi(x_0, t_0), \mathcal{D}^2\phi(x_0, t_0)) \leq 0.$$

We start by proving that $\overline{u} \in \mathcal{USC}$.

1. $\overline{u}$ is upper semicontinuous: it follows from the definition itself: we want to prove that the following inequality holds:

$$\limsup_{(y,s)\to(x,t)} \overline{u}(y,s) \leq \overline{u}(x,t),$$

   By definition:

$$\overline{u}(y,s) = \limsup_{\substack{(\Delta x, \Delta t)\to 0 \\ (j\Delta x, n\Delta t)\to(y,s)}} u_j^n,$$

   therefore, by definition of limsup, there exist an $\varepsilon > 0$ and (n, j) such that:

$$\overline{u}(y,s) - \varepsilon \leq u_j^n;$$

   now, taking limsup for $(\Delta x, \Delta t) \to 0$ and $(j\Delta x, n\Delta t) \to (x,t)$, we obtain:

$$\overline{u}(y,s) - \varepsilon \leq \overline{u}(x,t);$$

   Now, as ε is arbitrarily chosen, we obtain the desired result.

2. $\overline{u}$ is locally bounded: by definition it is obtained by the solutions of the scheme. By hypothesis (6.3.3) they are bounded, independently from $\Delta t, \Delta x$. So let $K \in \mathbb{R}^D \times [0,T]$ be a compact set, then there exists a constant A_K such that:

$$|u_j^n| \leq A_K \ \ \forall n, j \text{ s.t. } (j\Delta x, n\Delta t) \in K \ \Rightarrow \ |\overline{u}(x,t)| \leq A_K \ \ \forall (x,t) \in K;$$

We have proved that $\overline{u} \in \mathcal{USC}$, and now we need only to prove that $\overline{u}$ is a viscosity subsolution. To this aim let (x_0, t_0) be a global strict maximum for $\overline{u} - \phi$ on $\mathbb{R}^{\rm N} \times [0, T]$ for some $\phi \in \mathcal{C}_b^\infty(\mathbb{R}^{\rm N} \times [0, T])$. We could assume that $\overline{u}(x_0, t_0) = \phi(x_0, t_0)$ and that :

$$\overline{u}(x, t) - \phi(x, t) \leq 0 = \overline{u}(x_0, t_0) - \phi(x_0, t_0) \text{ in } \mathbb{R}^{\rm N} \times [0, T].$$

From these hypothesis it follows that there exists a sequence $(\Delta x_k, \Delta t_k) \in \mathbb{R}^{+2}$ and $(y_k, s_k) \in \mathbb{R}^{\rm N} \times [0, T]$ such that, as $k \to \infty$:

$$\begin{aligned} &(\Delta x_k, \Delta t_k) \to 0, \ \ (y_k, s_k) \to (x_0, t_0), \ \ u^{(\Delta x_k, \Delta t_k)}(y_k, s_k) \to \overline{u}(x_0, t_0), \\ &(y_k, s_k) \text{ is a global maximum point of } u^{(\Delta x_k, \Delta t_k)}(\cdot, \cdot) - \phi(\cdot, \cdot). \end{aligned} \tag{6.4.1}$$

Denoting by $\xi_k = u^{(\Delta x_k, \Delta t_k)}(y_k, s_k) - \phi(y_k, s_k)$, we have

$$\begin{aligned} &\xi_k \to 0 \text{ and} \\ &u^{(\Delta x_k, \Delta t_k)}(x, t) \leq \phi(x, t) + \xi_k, \end{aligned}$$

for all $(x, t) \in \mathbb{R}^{\rm N} \times [0, T]$. By the definition of $u^{(\Delta x_k, \Delta t_k)}$, the hypotheses (6.3.2), (6.4.1)

$$\tilde{Q}(\Delta x, \Delta t, j_k, n_k, \phi(y_k, s_k) + \xi_k, \tilde{\phi} + \xi_k) \leq 0.$$

Now, taking limits in the previous inequality, using the consistency of the scheme, we obtain:

$$\begin{aligned} 0 &\geq \liminf_{k \to \infty} \frac{\tilde{Q}(\Delta x_k, \Delta t_k, j_k, n_k, \phi(y_k, s_k) + \xi_k, \tilde{\phi} + \xi_k)}{\rho_k(\Delta x, \Delta t)} \\ &\geq \liminf_{\substack{(\Delta x, \Delta t) \to 0 \\ (j\Delta x, n\Delta t) \to (x,t) \\ \xi \to 0}} \frac{\tilde{Q}(\Delta x, \Delta t, j, n, \phi(y, s) + \xi, \tilde{\phi} + \xi)}{\rho(\Delta x, \Delta t)} \\ &\geq \partial_t \phi + F_*(x, t, \phi, \mathcal{D}\phi, \mathcal{D}^2\phi), \end{aligned}$$

which is the desired result, because of the assumption $\overline{u}(x_0, t_0) = \phi(x_0, t_0)$. ∎

Let us now shortly introduce a standard explicit 3-points finite-difference scheme for the Black-Scholes equation

$$u_t + \mathcal{L}u = u_t - bu_{xx} + au_x + cu = 0, \tag{6.4.2}$$

where $a = -(r - \sigma^2/2)$, $b = \sigma^2/2$ and $c = r > 0$, is

$$\tilde{Q}(h, k, j, n, u_j^n, \tilde{u}) = \frac{u_j^{n+1} - u_j^n}{k} + a\frac{u_{j+1}^n - u_{j-1}^n}{2h}$$

$$-(\frac{q}{2k}+\frac{b}{h^2})(u^n_{j+1}-2u^n_j+u^n_{j-1})+cu^n_j=0. \qquad (6.4.3)$$

The q parameter is connected with the numerical viscosity of the scheme. In order to verify the monotonicity and stability hypotheses, the scheme has to satisfy the following Courant-Friedrichs-Levy (CFL) condition

$$\frac{|a|k}{h}\le\frac{2bk}{h^2}+q\le 1-ck.$$

Usual values of q are given by: $q=0$, the standard central scheme, which is second order, but stable only under the CFL condition $k\le\min(\frac{2b}{a^2+2bc},\frac{h^2}{2b+ch^2})$; $q=\frac{|a|k}{h}$, the upwind scheme, which is first order, but stable for $k\le\frac{h^2}{2b+|a|h+ch^2}$.

The most elementary way to avoid the CFL conditions is to use an implicit scheme in time, such as a Crank-Nicholson scheme, given by

$$\tilde{Q}(h,k,j,n,u^n_j,\tilde{u})=\frac{u^{n+1}_j-u^n_j}{k}+\mathcal{L}\Big[\theta u^n_j+(1-\theta)u^{n+1}_j\Big]=0. \qquad (6.4.4)$$

with $\theta=1/2$.

Chapter 7

Convergence of numerical schemes in the bounded Lévy case.

In this chapter we study the numerical approximation of a class of semilinear strongly degenerate parabolic integro-differential Cauchy problems of the following form:

$$\begin{cases} \partial_t u - \mathcal{L}_{\mathcal{I}}(x,t,\mathcal{I},\mathcal{D},\mathcal{D}^2)u + H(x,t,\mathcal{D}u,\mathcal{I}u) &= 0, \\ u(x,0) &= u_0(x), \end{cases} \tag{7.0.1}$$

where u_0 is a continuous initial data, $\mathcal{L}_{\mathcal{I}}$ is a linear degenerate elliptic operator and H is a nonlinear first order operator. Here $\mathcal{I}u$ is an integral term given by

$$\mathcal{I}u = \int_{\mathbb{R}^M} M(u(x+z,t),u(x,t))\mu_{x,t}(dz), \tag{7.0.2}$$

where $\mu_{x,t}$ is a positive bounded measure and M is a function which is non decreasing in the first argument, $M(u,u)=0$ and such that

$$M(u,v) - M(w,z) \leq \mathrm{C}((u-w)_+ + |v-z|).$$

Problems in this form have been described in Part I and II: they arise in market evolving following a jump–diffusion dynamic, as in Merton [90], instead of the classical diffusion dynamics, in order to have a more realistic description of the market than the one obtained with the Black–Scholes model [28].

One of the main difficulty stands in the nonlinearity of the differential operator. Here we extend the results by Barles and Souganidis, summed up in the previous chapter, to the integro–differential problem.

Another difficulty stems from the nonlocal nature of the integral term. It is necessary to truncate the problem domain on one hand, and the integral domain on the other. As $\mu_{x,t}$ is a bounded measure, for a fixed $\nu > 0$ we can choose a bounded computational domain D_ν for the integral term, such that

$$\Big| \int_{\mathbb{R}^M} \mu_{x,t}(dz) - \int_{D_\nu} \mu_{x,t}(dz) \Big| < \nu,$$

and we can consider a new problem with $\mathcal{I}_\nu u = \int_{D_\nu} M(u(x+z,t), u(x,t))\mu_{x,t}(dz)$ instead of $\mathcal{I}u$; after that, we have to truncate the domain of the problem.

Unfortunately, due the non-local nature of the integral term, once we have found a given domain, we still need to use some approximation of the solution in a larger computational domain. The common approach consists in replacing the original problem with an homogeneous one, i.e. without the integral term, or to use some asymptotic representation formula for the solution.

Here we try a different approach. First we show that our original problem can be well approximated by a pure differential problem with an artificial diffusion. We apply this remark to implement an effective numerical boundary condition, giving as a consequence a full convergence result for the global approximation scheme.

7.1 The financial model - option pricing with jump-diffusion processes.

We briefly recall here the general pricing equation for the jump–diffusion case in the large investor case. For details see Chapter 1.

Let $\xi(U, \mathcal{D}U, \mathcal{J}U) = U - S\sigma\theta_0 \cdot \mathcal{D}U - \phi_0 \cdot \mathcal{J}U$ denote the amount of money invested in stocks by the agent, obtained choosing a proper replicating portfolio; then the interest rate r is influenced by the agents by means of ξ. Usually r is a non-increasing function of ξ, but we shall suppose that $r(\cdot, \xi)\xi$ is non decreasing with respect to ξ. In addition, we do not impose that r is continuous with respect to ξ at zero, in order to include interesting examples. For details about the parameters θ_0 and ϕ_0 we refer to Chapter 1 and [2].

In this setting the function U must solve the following quasi-linear final value problem:

$$\begin{cases} \partial_t U + L_{\mathcal{J}} U &= H(S,t,U,\mathcal{J}U,\mathcal{D}U), \\ \\ U(S,T) &= G(S). \end{cases} \tag{7.1.1}$$

where we have simply rewritten equation 1.6.5 with the addition of the nonlinear first order operator:

$$H(S,t,U,\mathcal{J}U,\mathcal{D}U) = r(S,t,\xi) \cdot \xi.$$

Obviously, if all the parameters of the model r, μ_i, σ_{ij} and γ_{ij} are deterministic function of (S,t), the problem is linear and we obtain the so called *small investor economy.*

7.2 A general convergence result.

We want to approximate the following problem:

$$\partial_t u + F(x,t,u,\mathcal{I}u,\mathcal{D}u,\mathcal{D}^2u) = 0. \tag{7.2.1}$$

A numerical scheme approximating (7.2.1) can be written as

$$Q(h,k,j,n,u_j^n,\mathcal{I}_h\tilde{u},\tilde{u}) = 0, \tag{7.2.2}$$

where $\mathcal{I}_h\tilde{u}$ denotes the integral approximation. We want to prove that, under suitable conditions, this scheme converges to the solution of the problem (7.2.1), provided that this problem satisfies proper conditions.

Properties of the scheme

H1 *Monotonicity of the approximating integral.*
If $\tilde{u} \geq \tilde{v}$ and $u_j^n = v_j^n$ we have the following inequality:

$$\mathcal{I}_h\tilde{u} \geq \mathcal{I}_h\tilde{v}; \tag{7.2.3}$$

H2 *Stability.*

For all h,k a solution $\hat{u}$ does exist that is bounded independently from (h,k); (7.2.4)

H3 *Consistency.*
For all $\phi \in \mathcal{C}_b^\infty(\mathbb{R}^{\mathrm{N}} \times [0,T])$ and for all $(x,t) \in \mathbb{R}^{\mathrm{N}} \times (0,T)$ we have:

$$\liminf_{\substack{(h,k)\to 0\\ (jh,nk)\to(x,t)\\ \xi\to 0}} \frac{Q(h,k,j,n,\phi+\xi,\mathcal{I}_h(\phi+\xi),\tilde{\phi}+\xi)}{\rho(h,k)}$$

$$\geq \partial_t u + F(x,t,u,\mathcal{I}u,\mathcal{D}u,\mathcal{D}^2u); \tag{7.2.5}$$

$$\limsup_{\substack{(h,k)\to 0\\ (jh,nk)\to(t,x)\\ \xi\to 0}} \frac{Q(h,k,j,n,\phi+\xi,\mathcal{I}_h(\phi+\xi),\tilde{\phi}+\xi)}{\rho(h,k)}$$

$$\leq \partial_t u + F(x, t, u, \mathcal{I}u, \mathcal{D}u, \mathcal{D}^2 u); \qquad (7.2.6)$$

H4 *Monotonicity.*
If $\tilde{u} \geq \tilde{v}$ and $u_j^n = v_j^n$ for all $h,\ k \geq 0$ and $1 \leq n \leq N$, we have:

$$Q(h, k, n, j, u_j^n, \mathcal{I}_h \tilde{u}, \tilde{u}) \leq Q(h, k, n, j, v_j^n, \mathcal{I}_h \tilde{v}, \tilde{v}). \qquad (7.2.7)$$

Remark 7.2.1 The theory of numerical approximation of fully nonlinear degenerate parabolic problems presented in Chapter 6, see [16], could be considered as a special case of the present one. We define the numerical scheme approximating the parabolic problem:

$$\partial_t u + F(x, t, u, 0, \mathcal{D}u, \mathcal{D}^2 u) = 0 \text{ in } \mathbb{R}^{\mathrm{N}} \times (0, T), \qquad (7.2.8)$$

as:

$$\tilde{Q}(h, k, j, n, u_j^n, \tilde{u}) = Q(h, k, n, j, u_j^n, 0, \tilde{u}); \qquad (7.2.9)$$

in this way the scheme $\tilde{Q}$ clearly satisfies the properties (**S.1**)–(**S.3**) of Chapter 6, required by Barles and Souganidis in [16], and therefore the approximation scheme (7.2.9) converges to the viscosity solutions of (7.2.8). ■

PROPERTIES OF THE EQUATION

H5 *Maximum Principle* or *Strong Uniqueness Property* Let $u_0 \in \mathcal{C}(\mathbb{R}^{\mathrm{N}}) \cap L^\infty_{\exp}(\mathbb{R}^{\mathrm{N}})$ be the initial data of (7.2.1), $(7.0.1)_2$, such that there exists $m > 0$:

$$|u_0(x)| \leq B e^{n_0 ||x||} \text{ for } n_0 \leq m;$$

if $u \in \bigcup_{n<m} L^\infty(0, T; L^\infty_{e^{||\cdot||}}(\mathbb{R}^{\mathrm{N}}))$ is an $\mathcal{USC}^{\mathcal{I}}$ subsolution of (7.2.1) and $v \in \bigcup_{n<m} L^\infty(0, T; L^\infty_{e^{||\cdot||}}(\mathbb{R}^{\mathrm{N}}))$ is a $\mathcal{LSC}^{\mathcal{I}}$ supersolution of (7.2.1), then

$$u \leq v \text{ on } \mathbb{R}^{\mathrm{N}} \times [0, T].$$

Under these assumptions we shall prove our main theoretical result.

Theorem 7.2.2 *Let assumption (***H1***)–(***H5***) hold true. Then, as* $(h, k) \to 0$*, the solution* $\tilde{u}$ *of the scheme* (7.2.2) *converges locally uniformly to the unique continuous viscosity solution of the problem* (7.2.1)

Proof. Let $\underline{u}$, $\overline{u} \in \bigcup_{n<m} L^\infty(0,T;L^\infty_{e^{n\|\cdot\|}}(\mathbb{R}^N))$ be defined by:

$$\underline{u}(x,t) = \liminf_{\substack{(\Delta x,\Delta t)\to 0\\(j\Delta x,n\Delta t)\to(x,t)}} u_j^n,$$
$$\overline{u}(x,t) = \limsup_{\substack{(\Delta x,\Delta t)\to 0\\(j\Delta x,n\Delta t)\to(x,t)}} u_j^n.$$

We want to prove that $\underline{u}$ and $\overline{u}$ are respectively supersolution and subsolution of the problem (7.2.1). If this claim is proved to be true, then by definition we have $\underline{u} \le \overline{u}$, while the other inequality holds because of lower semicontinuity of $\underline{u}$ and upper semicontinuity of $\overline{u}$, hence:

$$u = \underline{u} = \overline{u},$$

is the unique continuous solution of the problem (7.2.1). This result, together with the definition of $\underline{u}$ and of $\overline{u}$ leads to the local uniform convergence of the solution of the scheme to the solution of the problem. To prove the claim, we will consider only the case of $\overline{u}$, the other being the same.
We want to prove that $\overline{u} \in \mathcal{USC}^{\mathcal{I}}$ and that it is a subsolution for the problem, i.e. for all $\phi \in \mathcal{C}^{2,1}(\mathbb{R}^N \times [0,T])$ such that $\overline{u} - \phi$ has a local maximum in (x_0,t_0) we have:

$$\partial_t\phi(x_0,t_0) + F(x_0,t_0,\overline{u}(x_0,t_0),\mathcal{I}\phi(x_0,t_0),\mathcal{D}\phi(x_0,t_0),\mathcal{D}^2\phi(x_0,t_0)) \le 0.$$

We start by proving that $\overline{u} \in \mathcal{USC}^{\mathcal{I}}$.

1. $\overline{u}$ is upper semicontinuous: it follows from the definition itself: we want to prove that the following inequality holds:

   $$\limsup_{(y,s)\to(x,t)} \overline{u}(y,s) \le \overline{u}(x,t),$$

   By definition:

   $$\overline{u}(y,s) = \limsup_{\substack{(\Delta x,\Delta t)\to 0\\(j\Delta x,n\Delta t)\to(y,s)}} u_j^n,$$

   therefore, by definition of limsup, there exist an $\varepsilon > 0$ and (j,n) such that:

   $$\overline{u}(y,s) - \varepsilon \le u_j^n;$$

   now, taking limsup for $(\Delta x,\Delta t) \to 0$ and $(j\Delta x,n\Delta t) \to (x,t)$, we obtain:

   $$\overline{u}(y,s) - \varepsilon \le \overline{u}(x,t);$$

   Now, as ε is arbitrarily chosen, we obtain the desired result.

2. $\overline{u}$ is locally bounded: by definition it is obtained by the solutions of the scheme. By hypothesis (7.2.4) they are bounded, independently from $\Delta x, \Delta t$. So let $K \in \mathbb{R}^{\mathrm{N}} \times [0,T]$ be a compact set, then there exists a constant A_K such that:
$$|u_j^n| \le A_K \ \ \forall n,j \text{ s.t. } (j\Delta x, n\Delta t) \in K \ \Rightarrow \ |\overline{u}(x,t)| \le A_K \ \ \forall (x,t) \in K;$$
3. $M(\overline{u}(x+z,t), \overline{u}(x,t))$ has an upper μ-bound in (x,t). From the hypotheses on M, it clearly follows that M is a Lipschitz function with constant C, so we have:
$$M(\overline{u}(x+z,t), \overline{u}(x,t)) \le \mathrm{C}|\overline{u}(x+z,t) - \overline{u}(x,t)|;$$
it is easily shown that in a compact neighborhood $V_{x,t}$ of (x,t) we have:
$$M(\overline{u}(x+z,t), \overline{u}(x,t)) \le 2\mathrm{C}A_{V_{x,t}}.$$
It is then sufficient to choose Φ as a constant to be the wanted μ-bound.

We have proved that $\overline{u} \in \mathcal{USC}^{\mathcal{I}}$, and now we need only to prove that $\overline{u}$ is a viscosity subsolution. To this aim let (x_0,t_0) be a global strict maximum for $\overline{u}-\phi$ on $\mathbb{R}^{\mathrm{N}} \times [0,T]$ for some $\phi \in \mathcal{C}_b^\infty(\mathbb{R}^{\mathrm{N}} \times [0,T])$. We could assume that $\overline{u}(x_0,t_0) = \phi(x_0,t_0)$ and that :
$$\overline{u}(x,t) - \phi(x,t) \le 0 = \overline{u}(x_0,t_0) - \phi(x_0,t_0) \text{ in } \mathbb{R}^{\mathrm{N}} \times [0,T].$$
From these hypothesis it follows that there exists a sequence $(\Delta x_k, \Delta t_k) \in \mathbb{R}^{+2}$ and $(y_k, s_k) \in \mathbb{R}^{\mathrm{N}} \times [0,T]$ such that, as $k \to \infty$:
$$\begin{aligned} &(\Delta x_k, \Delta t_k) \to 0, \ \ (y_k,s_k) \to (x_0,t_0), \ \ u^{(\Delta x_k, \Delta t_k)}(y_k,s_k) \to \overline{u}(x_0,t_0), \\ &(y_k,s_k) \text{ is a global maximum point of } u^{(\Delta x_k,\Delta t_k)}(\cdot,\cdot) - \phi(\cdot,\cdot). \end{aligned} \tag{7.2.10}$$
Denoting by $\xi_k = u^{(\Delta x_k,\Delta t_k)}(y_k,s_k) - \phi(y_k,s_k)$, we have
$$\begin{aligned} &\xi_k \to 0 \text{ and} \\ &u^{(\Delta x_k,\Delta t_k)}(x,t) \le \phi(x,t) + \xi_k, \end{aligned}$$
for all $(x,t) \in \mathbb{R}^{\mathrm{N}} \times [0,T]$. By the definition of $u^{(\Delta x_k,\Delta t_k)}$, the hypotheses (7.2.7), (7.2.10) and the hypotheses on M we obtain:
$$Q(\Delta x, \Delta t, j_k, n_k, \phi(y_k,s_k) + \xi_k, \mathcal{I}_{n_k,j_k}(\tilde{\phi} + \xi_k), \tilde{\phi} + \xi_k) \le 0.$$
Now, taking limits in the previous inequality, using the consistency of the scheme, we obtain:
$$0 \ge \liminf_{k\to\infty} \frac{Q(\Delta x_k, \Delta t_k, j_k, n_k, \phi(y_k,s_k) + \xi_k, \mathcal{I}_{n_k,j_k}(\tilde{\phi} + \xi_k), \tilde{\phi} + \xi_k)}{\rho_k(\Delta x, \Delta t)}$$

$$\geq \liminf_{\substack{(\Delta x,\Delta t)\to 0\\(j\Delta x,n\Delta t)\to(x,t)\\ \xi\to 0}} \frac{Q(\Delta x,\Delta t,j,n,\phi(y,s)+\xi,\mathcal{I}(\tilde{\phi}+\xi),\tilde{\phi}+\xi)}{\rho(\Delta x,\Delta t)}$$

$$\geq \partial_t\phi + F(x,t,\phi,\mathcal{I}\phi,\mathcal{D}\phi,\mathcal{D}^2\phi),$$

which is the desired result, because of the assumption $\overline{u}(x_0,t_0)=\phi(x_0,t_0)$. ∎

7.3 The numerical approximation of the integral term.

According to the classical theory of approximated integration, see for instance [38], we use the compound Newton-Cotes formulas to approximate the integral term on the interval $[a,b]$:

$$(R_S)(f) = \frac{b-a}{2S}\sum_{s=0}^{S-1}\sum_{i=1}^{\rho}\alpha_i f(x_{is}) \approx \int_a^b f(x)dx, \tag{7.3.1}$$

where S is the number of subinterval in which we have divided $[a,b]$, $a=y_0<y_1<\ldots<y_S=b$, ρ is the number of point in each subinterval $[y_s,y_{s+1}]$ and

$$x_{is} = y_s + \frac{b-a}{2S}(1+t_i), \quad s=0,\ldots,S-1.$$

The most used simple Newton-Cotes formulas reads as follows: divide the interval $[a,b]$ in ρ subintervals of equal length $h=(b-a)/\rho$ and interpolate the function by the Lagrange interpolation:

$$\int_a^b f(x)dx \approx \frac{b-a}{\rho}\sum_{i=0}^{\rho}\omega_i f(x_i), \tag{7.3.2}$$

where

$$\omega_i = (-1)^{\rho-i}\frac{\rho h}{b-a}\int_0^{\rho}\frac{t(t-1)\ldots(t-\rho)}{(t-i)i!(\rho-i)!}dt.$$

The simple trapezoidal rule, which integrates linear functions exactly, and Simpson's rule, which integrates cubics exactly, are the first two rules in the Newton-Cotes family.
The notation is as follows:

$$a=x_0<x_1<\ldots<x_\rho=b \quad f(x_i)=f_i.$$

Trapezoidal rule, $\rho = 1$

$$\int_a^b f(x)dx \approx h\frac{f_0 + f_1}{2} \tag{7.3.3}$$

Simpson's rule, $\rho = 2$

$$\int_a^b f(x)dx \approx \frac{h}{6}(f_0 + 4f_1 + f_2) \tag{7.3.4}$$

Simpson's $\frac{3}{8}$ rule, $\rho = 3$

$$\int_a^b f(x)dx \approx \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3) \tag{7.3.5}$$

Newton-Cotes 6-point rule, $\rho = 5$

$$\int_a^b f(x)dx \approx \frac{5h}{288}(19f_0 + 75f_1 + 50f_2 + 50f_3 + 75f_4 + 19f_5) \tag{7.3.6}$$

We decide to use Newton–Coates formulas in a compound rule as it is a feature of these formulas that when ρ is large, the Newton-Cotes ρ-points coefficients are large and are of mixed sign. Since this may lead to large losses of significance by cancellation, a high-order Newton-Cotes rule must be used with caution, usually it is not suitable to choose the order grater than 7.
In practice, all these formulas are applied in a compound rule.

Definition 7.3.1 *If R designates a fixed rule of approximate integration utilizing ρ points, then $S \times R = R_S$ will designate the rule of ρS points which results from dividing the interval of integration into S subintervals and applying R to each of them.*

The errors which occur in approximate integration formulas are conventionally expressed in terms of the higher derivatives of the integrand function f and they are valid only if the integrand is sufficiently smooth. It is a feature of the Newton-Cotes formulas that, if the number of point is $2k-1$ or $2k$, the error is of the form $\mathcal{E}_R(f) = ch^{2k+1}f^{2k}(\xi)$, for $a < \xi < b$. To obtain the error estimates when the integrand function f has a low-order continuity, we can approximate the function f by a suitable polynomial, according to the following standard result, see for instance [38].

Proposition 7.3.2 *Let $f(x)$ be of class $\mathcal{C}[a,b]$, and let $\omega(\delta)$ be its modulus of continuity. Then for each $n \in \mathbb{N}$, there exists a polynomial of degree $\leq n$, $p_n(x)$, such that*

$$| f(x) - p_n(x) | \leq 2\omega\Big(\frac{b-a}{2n}\Big), \quad a \leq x \leq b.$$

If $f(x)$ is of class $\mathcal{C}[a,b]$ and has a bounded derivative,

$$| f'(x) | \leq M, \quad a \leq x \leq b,$$

then for each $n \in \mathbb{N}$ there exists a polynomial $p_n(x)$ of degree $\leq n$ such that

$$| f(x) - p_n(x) | \leq \frac{3(b-a)M}{n}, \quad a \leq x \leq b.$$

7.3.1 The numerical approximation in the one dimensional case.

Here we want to consider the integro-differential equation (4.1.1) in one dimension. We suppose F to be linear in the integral part,

$$\partial_t u + F(x,t,u,\mathcal{D}u,\mathcal{D}^2 u) - \mathcal{I}u = 0. \tag{7.3.7}$$

The first step to approximate the integral operator (7.0.2) using the numerical integrations formula described in the previous section is to truncate the integral domain; let us choose the interval $[z_m, z_M]$ such that

$$\int_{z_m}^{z_M} \mu_{x,t}(dz) \approx \int_{-\infty}^{+\infty} \mu_{x,t}(dz) - \nu = 1 - \nu, \quad \nu \ll 1. \tag{7.3.8}$$

Assume $u(\cdot,t) \in L^\infty(\mathbb{R})$ and let $\textsc{u}$ be its L^∞-norm. The error due to the truncation of the domain is estimated as follows.

$$\begin{aligned}
&\Big| \int_{-\infty}^{+\infty} M(u(x+z,t),u(x,t))\mu_{x,t}(dz) - \int_{z_m}^{z_M} M(u(x+z,t),u(x,t))\mu_{x,t}(dz) \Big| \\
&= \Big| \int_{-\infty}^{z_m} M(u(x+z,t),u(x,t))\mu_{x,t}(dz) \Big| + \Big| \int_{z_M}^{+\infty} M(u(x+z,t),u(x,t))\mu_{x,t}(dz) \Big| \\
&\leq \textsc{c} \int_{-\infty}^{z_m} | u(x+z,t) - u(x,t) | \, \mu_{x,t}(dz) + \textsc{c} \int_{z_M}^{+\infty} | u(x+z,t) - u(x,t) | \, \mu_{x,t}(dz)
\end{aligned}$$

$$\begin{aligned}
&\leq 2\textsc{uc}\Big[\int_{-\infty}^{z_m} \mu_{x,t}(dz) + \int_{z_M}^{+\infty} \mu_{x,t}(dz) \Big] \\
&= 2\textsc{uc} \int_{\mathbb{R}-[z_m,z_M]} \mu_{x,t}(dz) = 2\textsc{uc}\nu.
\end{aligned} \tag{7.3.9}$$

7.3.2 The case of the Gaussian distribution.

In the estimation (7.3.9), we have supposed that the function $u(\cdot,t)$ is $L^\infty(\mathbb{R})$, but it is possible to suppose even more regularity, under particular hypothesis on the Radon measure. Let us consider, for example, on the tracks of Merton, the Gaussian distribution. It leads to a probability measure with the property of being symmetric with respect to the origin of the real line. In this case it is possible to assume that $u(\cdot,t) \in Lip(\mathbb{R})$ with constant L; we remember that in this particular case, the integral term is:

$$\mathcal{I}u = \lambda \int_{-\infty}^{+\infty} [u(x+z,t) - u(x,t)]\mu_{x,t}(z)dz,$$

where $\mu_{x,t} = \Gamma_\delta$ is the Gaussian probability density:

$$\mu_{x,t}(dz) := \Gamma_\delta(z)dz = \frac{1}{\sqrt{2\pi}\delta} \exp\Big(-\frac{z^2}{2\delta^2}\Big)dz.$$

As was previously shown, the calculation of the integral term could be simplified by considering a finite interval instead of the whole real line. Thanks to the particular shape of the density measure Γ_δ, we can select the finite interval considering only those points for which the density has a significant value and this choice would not introduce big errors. Choose a parameter $\varepsilon > 0$ and select the interval $[z_m, z_M]$ as the set of all the points z that verify:

$$\Gamma_\delta(z) \geq \varepsilon \iff \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{z^2}{2\delta^2}} \geq \varepsilon;$$

by simple calculation we can derive z_m and z_M:

$$-\sqrt{-2\delta^2 \log(\varepsilon\delta\sqrt{2\pi})} \leq z \leq \sqrt{-2\delta^2 \log(\varepsilon\delta\sqrt{2\pi})}.$$

As Γ_δ is a symmetric function with respect to its axis (that in this case is the line $z = 0$), we define:

$$z_M = \sqrt{-2\delta^2 \log(\varepsilon\delta\sqrt{2\pi})}, \quad z_m = -z_M.$$

Under these hypotheses we have the following estimate:

$$\Big| \int_{-\infty}^{+\infty} M(u(x+z,t), u(x,t))\Gamma_\delta(dz) - \int_{z_m}^{z_M} M(u(x+z,t), u(z,t))\Gamma_\delta(dz) \Big|$$
$$\leq \text{L} \left(\int_{-\infty}^{z_m} |z|\Gamma_\delta(dz) + \int_{z_M}^{+\infty} |z|\Gamma_\delta(dz) \right),$$

therefore

$$\int_{-\infty}^{z_m} |z|\Gamma_\delta(dz) + \int_{z_M}^{+\infty} |z|\Gamma_\delta(dz)$$
$$= 2\int_{z_M}^{+\infty} z\frac{1}{\sqrt{2\pi}\delta}\exp\Big(-\frac{z^2}{2\delta^2}\Big)dz = \frac{2\delta^2}{\sqrt{2\pi}}\exp\Big(-\frac{z_M^2}{2\delta}\Big) = 2\delta^2\varepsilon.$$

Let us now apply the compound rule (7.3.1) to the truncated integral.

$$\mathcal{I}_h u = \lambda(R_S)(M\Gamma_\delta) = \lambda\frac{z_M - z_m}{2S}\sum_{s=0}^{S-1}\sum_{i=0}^{\rho}\alpha_i M(u(x+z_{is},t), u(x,t))\Gamma_\delta(z_{is}). \quad (7.3.10)$$

Since the function $g(z) = M(u(x+z,t), u(x,t))\Gamma_\delta(z)$ has a low-order continuity, to get an error estimate for the approximation (7.3.10), we apply Proposition 7.3.2. Recall that for a generic function f we have

$$\mathcal{E}_R(f) = \int_a^b f(x)dx - R(f) = \int_a^b (f(x) - p_n(x))dx + \int_a^b p_n(x)dx - R(f)$$
$$= \int_a^b (f(x) - p_n(x))dx + R(p_n - f).$$

Then

$$\mid \mathcal{E}_R(f) \mid \leq \Big((b-a) + \sum_{i=0}^{\rho} \mid \alpha_i \mid\Big) \mid f(x) - p_n(x) \mid .$$

If $\alpha_i > 0$, using formula (7.3.1), we obtain

$$\mid \mathcal{E}_R(f) \mid \leq 2(b-a) \mid f(x) - p_n(x) \mid .$$

An (R_S) compound rule, applied to our function g, yields

$$\mathcal{E}_{R_S}(g) = \sum_{s=0}^{S-1}\mathcal{E}_R(g) = \sum_{s=0}^{S-1} 2\Big(\frac{b-a}{S}\Big) \mid g(x_s) - p_n(x_s) \mid .$$

Then there exists a polynomial of degree $\leq S\rho$, $p_{S\rho}(z)$, such that

$$\mid g(z) - p_{S\rho}(z) \mid \leq 2\omega\Big(\frac{z_M - z_m}{2S\rho}\Big).$$

There follows

$$\Big|\int_{z_m}^{z_M} g(z)dz - (R_S)(g(z))\Big| \quad (7.3.11)$$
$$\leq \int_{z_m}^{z_M}\Big|g(z) - p_{Sp}(z)\Big|dz + \Big|(R_S)(p_{Sp}(z) - g(z))\Big|$$

$$\leq \Big((z_M - z_m) + \frac{z_M - z_m}{2S}\sum_{s=0}^{S-1}\sum_{i=0}^{p} \mid \alpha_i \mid \Big) 2\omega\Big(\frac{z_M - z_m}{2Sp}\Big)$$
$$\leq 2(z_M - z_m)\Big(1 + \frac{1}{2}\sum_{i=0}^{p} \mid \alpha_i \mid \Big)\omega\Big(\frac{h}{2}\Big),$$

where ω is the modulus of continuity for g.

7.3.3 Check of the hypotheses of Theorem 7.2.2 for the integral part.

First, we have to approximate the differential operator $\partial_t + F$: we take a numerical scheme $\tilde{Q}$ that verifies the convergence (differential) conditions (**S.1**)-(**S.3**) of Chapter 6 (see [16]). In particular, to keep the order of the convergence of the integration formula (7.3.10), we assume that the space discretization grid of the numerical operator $\tilde{Q}$ coincides with the integral one, i.e. we set the common space step h such that

$$h \leq \frac{z_M - z_m}{\rho \cdot S}.$$

Then the approximation of the integro-differential equation (7.3.7) is given by:

$$Q(h, k, j, n, u_j^n, \mathcal{I}_h\tilde{u}, \tilde{u}) = \tilde{Q}(h, k, j, n, u_j^n, \tilde{u}) - \mathcal{I}_h\tilde{u} = 0,$$

We want to show that under the above assumption, this scheme satisfies conditions (7.2.3)-(7.2.7).

1. *Monotonicity of the approximating integral*
   Since the function M is such that

   $$M(u, w) \leq M(v, w), \text{ if } u \leq v,$$

   to get the monotonicity of the integral approximation it is sufficient that the weights α_i are greater than zero for all i. Clearly, if $\tilde{u} \leq \tilde{v}$ and $u_j^n = v_j^n$, we have

   $$\lambda\frac{z_M - z_m}{2S}\sum_{s=0}^{S-1}\sum_{i=0}^{\rho} \alpha_i M(u^n(x_j + z_{is}), u_j^n)\Gamma_\delta(z_{is})$$
   $$\leq \lambda\frac{z_M - z_m}{2S}\sum_{s=0}^{S-1}\sum_{i=0}^{\rho} \alpha_i M(v^n(x_j + z_{is}), v_j^n)\Gamma_\delta(z_{is}),$$

   for all $j \in \mathbb{Z}$ and $n \in \mathbb{N}$.

2. *Stability*
   It is a trivial consequence of the $\tilde{Q}$ stability (7.2.4) and the monotonicity of the integral approximation.

3. *Consistency*
   Let $\phi \in \mathcal{C}^{\infty}(\mathbb{R} \times (0,T))$, from the consistency condition (7.2.5) on $\tilde{Q}$, we get the following inequality:

$$\liminf_{\substack{(h,k)\to 0\\ (jh,nk)\to(x,t)\\ \xi\to 0}} \frac{\tilde{Q}(h,k,j,n,\phi+\xi,\tilde{\phi}+\xi) - \mathcal{I}_h(\tilde{\phi}+\xi)}{\rho(h,k)}$$

$$\geq \partial_t u + F(x,t,u,\mathcal{D}u,\mathcal{D}^2 u) - \liminf_{\substack{(h,k)\to 0\\ (jh,nk)\to(x,t)\\ \xi\to 0}} \frac{\mathcal{I}_h(\tilde{\phi}+\xi)}{\rho(h,k)}.$$

   From the error estimate of the integral approximation (7.3.11), we have

$$\liminf_{\substack{(h,k)\to 0\\ (jh,nk)\to(x,t)\\ \xi\to 0}} \frac{\mathcal{I}_h(\tilde{\phi}+\xi)}{\rho(h,k)} = \mathcal{I}u - \lim_{\substack{(h,k)\to 0\\ \xi\to 0}} \frac{\mathcal{E}_{R_S}(\phi+\xi)}{\rho(h,k)} = \mathcal{I}\phi.$$

   then, we get condition (7.2.5). Condition (7.2.6) follows by analogous considerations.

4. *Monotonicity*
   It is a trivial consequence of the $\tilde{Q}$ monotonicity (6.3.2) and the monotonicity of the integral approximation (point 1).

Remark 7.3.3 In our numerical test, we have always considered a Radon measure absolutely continuous with respect to the Lebesgue measure, i.e:

$$\mu_{x,t}(dz) = \lambda \Gamma_\delta(z) dz.$$

It is even possible to consider a discrete measure, for example the Dirac measure:

$$\mu_{x,t}(dz) = \delta_{z_0}(z) dz.$$

In that case the numerical approximation is even simpler, thanks to the absence of the integral term. ■

7.4 The diffusive effect of the integral operator.

An important point in the numerical simulation for the problem we have presented, is the behaviour of the solution at the limiting point of the truncated numerical domain. In this particular framework, the presence of the integral term which convolutes "internal" and "external" points requires a particular tool to deal with such a difficulty. One possibility is to look at the particular form of the integral term $\mathcal{I}u$ with respect to the Gaussian parameter δ: we show that a convenient way to deal with the integral operator is to replace it (locally) by an effective diffusion term. This result will be useful in the numerical simulations, as is shown next in Subsection 7.5.2. The following discussion, which is presented only in the linear case, has the main purpose of rigorously investigating the error generated by this approximation. Let us consider the two following one dimensional equations, for $(x,t) \in \mathbb{R} \times (0,T)$:

$$u_t + au_x - bu_{xx} + cu = \mathcal{I}u, \tag{7.4.1}$$

$$v_t + av_x - bv_{xx} + cv = \frac{\lambda\delta^2}{2} v_{xx}, \tag{7.4.2}$$

with the same initial condition

$$u(x,0) = v(x,0) = u_0(x), \qquad x \in \mathbb{R}.$$

It is possible to prove that, under proper hypotheses on the density distribution Γ_δ and on the solutions u and v, the integral problem (7.4.1) is well approximated by the advection-diffusion one (7.4.2).

Proposition 7.4.1 *Let u be the solution of problem* (7.4.1) *and v the solution of problem* (7.4.2) *with the same initial condition $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, if $\delta \ll 1$, there holds*

$$||u - v||_{L^\infty(0,T;L^1(\mathbb{R}))} \leq O(T\delta^3).$$

Proof. The function $w = u - v$ is a solution of the following problem (written in the weak formulation):

$$-\int_0^T \int_{-\infty}^{+\infty} \Big[\phi_t(x,t) + a\phi_x(x,t) + b\phi_{xx}(x,t) - c\phi(x,t)\Big] w(x,t)dxdt \tag{7.4.3}$$
$$= \lambda \int_0^T \int_{-\infty}^{+\infty} \frac{\delta^2}{2}\phi_{xx}(x,t)w(x,t)dxdt$$
$$+\lambda \int_0^T \int_{-\infty}^{+\infty} u(x,t) \int_{-\infty}^{+\infty} \Big[\phi(x+z,t) - \phi(x,t) - \frac{\delta^2}{2}\phi_{xx}(x,t)\Big]\Gamma_\delta(z)dzdxdt,$$

for every test function $\phi \in \mathcal{C}_0^\infty(\mathbb{R} \times [0,T])$.

To estimate the inner integral in the second member of the RHS we can take the Taylor expansion of ϕ, which leads to

$$\begin{aligned}&\phi(x+z,t)-\phi(x,t)-\frac{\delta^2}{2}\phi_{xx}(x,t)\\&=z\phi(x,t)+\frac{z^2-\delta^2}{2}\phi_{xx}(x,t)+\frac{z^3}{6}\int_0^1(1-k)^3\phi_{xxx}(x+(1-k)z,t)dk.\end{aligned}$$

We can estimate in term of the norm of u, the error made by using this expansion:

$$\begin{aligned}&\Big|\lambda\int_0^T\int_{-\infty}^{+\infty}u(x,t)\int_{-\infty}^{+\infty}\Big[\phi(x+z,t)-\phi(x,t)-\frac{z^2}{2}\phi_{xx}(x,t)\Big]\Gamma_\delta(z)dzdxdt\Big|\\&\le\lambda\Big|\int_0^T\int_{-\infty}^{+\infty}u(x,t)\int_{-\infty}^{+\infty}\frac{z^3}{6}\Big[\int_0^1(1-k)^3\phi_{xxx}(x+(1-k)z,t)dk\Big]\Gamma_\delta(z)dzdxdt\Big|\\&\le\int_0^T||\phi(\cdot,t)||_{C_0^3(\mathbb{R})}||u(\cdot,t)||_{L^1(\mathbb{R})}dt\int_{-\infty}^{+\infty}\frac{|z|^3}{6}\Gamma_\delta(z)dz\\&\le\lambda\frac{\delta^3}{\sqrt{2\pi}}||\phi||_{C_0^3(\mathbb{R}\times[0,T])}T||u||_{L^\infty(0,T;L^1(\mathbb{R}))}=O(T\delta^3).\end{aligned}$$

The RHS of (7.4.3) can be rewritten taking into account the last estimate:

$$\begin{aligned}&-\int_0^T\int_{-\infty}^{+\infty}\Big[\phi_t(x,t)+a\phi_x(x,t)+b\phi_{xx}(x,t)-c\phi(x,t)\Big]w(x,t)dxdt\\&=\lambda\int_0^T\int_{-\infty}^{+\infty}\frac{\delta^2}{2}\phi_{xx}w(x,t)dxdt\\&+\lambda\int_0^T\int_{-\infty}^{+\infty}w(x,t)\int_{-\infty}^{+\infty}\Big[\phi_x(x,t)z+\frac{z^2-\delta^2}{2}\phi_{xx}(x,t)\Big]\Gamma_\delta(z)dzdxdt+O(T\delta^3).\end{aligned}$$

Since for the inner integral, there holds

$$\int_{-\infty}^{+\infty}\Big[\phi_x(x,t)z+\frac{z^2-\delta^2}{2}\phi_{xx}(x,t)\Big]\Gamma_\delta(z)dz=0,$$

w is just the weak solution to problem

$$w_t+aw_x-\Big(b+\lambda\frac{\delta^2}{2}\Big)w_{xx}+cw=O(T\delta^3),$$

with initial datum

$$w(x,0)=0,\qquad x\in\mathbb{R},$$

which yields

$$||w||_{L^\infty(0,T;L^1_{loc}(\mathbb{R}))}\le O(T\delta^3).$$

Now, taking a suitable sequence of test functions such that $\overline{\text{supp }\phi(x)}=[-R,R]$, and letting $R\to+\infty$, gives the result. ∎

7.5 Finite difference methods for the one dimensional jump-diffusion model.

In this section we introduce an explicit approximation for the linear PIDE arising from the jump-diffusion models and we give a convenient way to deal with the problem of the numerical boundary conditions.

We remember that a huge literature exists for the pure diffusion Black-Scholes problem (1.3.2) within the subject of numerical approximation for the linear convection-diffusion equations. We turn the reader to Chapter 6, Section 6.4 to a short review of numerical schemes in the pure diffusion setting.

We can extend the result of Chapter 6 to the PIDE. After appropriate logarithmic transformations the Merton problem (1.6.5) becomes

$$\begin{cases} u_t + au_x = bu_{xx} - cu + \lambda\Big(\int_{-\infty}^{\infty} u(x+z,t)\Gamma_\delta(z)dz - u\Big), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad u(x,0) = \psi(x), \end{cases} \tag{7.5.1}$$

where

$$a = -(r - \lambda\bar{k} - \tfrac{1}{2}\sigma^2), \quad b = \tfrac{1}{2}\sigma^2, \quad c = r, \quad \bar{k} = \mathbb{E}(\eta - 1),$$

and the initial data $\psi(x)$ is the payoff function of the European contingent claim. Let the exercise price E be given, we have

$$\psi(x) = (e^x - E)_+ \quad \text{and} \quad \psi(x) = (E - e^x)_+,$$

for the call and the put option respectively.
As done in (6.4.4), we can write the time approximation of the PIDE (7.5.1) in the following "θ-form":

$$\frac{u_j^{n+1} - u_j^n}{k} + \mathcal{L}\Big[\theta_1 u_j^n + (1-\theta_1)u_j^{n+1}\Big] + \theta_2 \mathcal{I}u_j^n + (1-\theta_2)\mathcal{I}u_j^{n+1} = 0, \tag{7.5.2}$$

where $\theta_1,\ \theta_2 \in [0,1]$. The choice $\theta_1 = \theta_2 = 0$ gives the explicit scheme, while $\theta_1 = \theta_2 = 1$ gives an implicit time differencing scheme, unconditionally stable, but not practically feasible. Actually the convolution integral introduces a significant complication for the numerical solution, since it couples grid points over an extended range, leading to a dense system of equations which is hard to be solved. In fact, after discretizing the x-space into N points the inversion of a full $N \times N$ matrices is required. For $\theta_1 = 1/2$, $\theta_2 = 0$ it gives an asymmetric treatment (implicit-explicit) of the differential and integral part. This is a way to avoid dense systems, but it is only first order in time.

In the book [112], Tavella and Randall propose an iterative approach to avoid dense systems and to increase the convergence order in time. They write the time-discretized equation as

$$\frac{u^{m+1}-u^n}{k}+\mathcal{L}\frac{u^{m+1}+u^n}{2}+$$
$$-\lambda\Big(\int_{-\infty}^{\infty}\frac{u^m(x+z)+u^n(x+z)}{2}\Gamma_\delta(z)dz-\frac{u^m+u^n}{2}\Big)=0.$$

At each time step, the iteration begins with $u^m = u^n$, then proceeds by solving for u^{m+1} and substituting the new u^{m+1} for u^m. The iteration proceeds until a convergence criterion is met. Here they set $u^{m+1} \approx u^{n+1}$ and a new time step begins.
Due to the iteration procedure, this method turns out to be computationally heavy and it is still not clear how to select a good stop criterion.

In the article [6], Andersen and Andreasen proposed an FFT-ADI (Fast Fourier Transform - Alternating Directions Implicit) to avoid the conditional stability of explicit methods. The FFT technique is applied to the convolution integral and coupled with an ADI method where each time step is split into two half steps: the idea is to choose in the time approximation (7.5.2), $\theta_1 = 1$ and $\theta_2 = 0$ for the half time step $t_n \to t_{n+1/2}$ and $\theta_1 = 0$ and $\theta_2 = 1$ for $t_{n+1/2} \to t_{n+1}$. Then, the discrete version of (7.5.1) is

$$\begin{cases} (\frac{2}{k}+\mathcal{L})u^{n+\frac{1}{2}} = (\frac{2}{k}-\lambda+\lambda\bar{\Gamma}*)u^n \\ \\ (\frac{2}{k}-\lambda+\lambda\bar{\Gamma}*)u^{n+1} = (\frac{2}{k}-\mathcal{L})u^{n+\frac{1}{2}}, \end{cases} \tag{7.5.3}$$

where $\bar{\Gamma}*u^n$ is the FFT approximation of the convolution term. As shown in [6], this scheme has the following good properties: (i) it is unconditionally stable in the von Neumann sense; (ii) for the case of deterministic parameters, the numerical solution of the scheme is locally accurate of order $O(k^2+h^2)$; (iii) if M is the number of time steps and N is the number of steps in spatial direction, the computational burden is $O(MNlog_2N)$.

Notice that this method is only proposed for the linear constant coefficient one dimensional case, namely for the original Merton equation. We point out that the main difference from the scheme (7.5.4) that we will present in the next section, is not the FFT approximation of the convolution term. Actually, our integral approximation formula in (7.5.4), can be easily substituted by the FFT technique without changing the general behaviour of the scheme.

Instead, the main feature of that scheme is an original decomposition to solve the implicit part. Actually, in the second half time step of (7.5.3), the values $\{u_j^{n+1}\}$

are first computed in the Fourier space as

$$< u^{n+1} >_j = \frac{< \Big(\frac{2}{k} - \mathcal{L}\Big) u^{n+\frac{1}{2}} >_j}{\Big(\frac{2}{k} - \lambda + \lambda < \Gamma >_j\Big)},$$

and then transformed back by the inverse FFT. However, this procedure turns out to be of difficult implementation and even the monotonicity property of the problem is far from being clear. Moreover, due to the nonlinearities and degeneracies of the equations considered, the effectiveness of these methods in the general case has still to be established.

7.5.1 An explicit finite difference method.

In this section, we give an exhaustive description of the explicit scheme. To solve the integro-differential equation (7.5.1), first, we truncate the integral domain. As we have previously described in Subsection 7.3.2, we choose the interval $[z_m, z_M]$ such that (7.3.8) holds and we point out that a positive constant C exists such that

$$\int_{-\infty}^{\infty} [u(x+z,t) - u(x,t)]\Gamma_\delta(z)dz = \int_{z_m}^{z_M} u(x+z,t)\Gamma_\delta(z)dz - u(x,t) + C\delta^2\epsilon.$$

We apply a compound rule to the integral term and a standard explicit finite-difference scheme for the differential part as done in (6.4.3). Then, our approximation of the equation (7.5.1) is given by,

$$Q(h,k,j,n,u_j^n,\mathcal{I}_h\tilde{u},\tilde{u}) = \frac{u_j^{n+1} - u_j^n}{k} + a\frac{u_{j+1}^n - u_{j-1}^n}{2h} \qquad (7.5.4)$$

$$-\Big(\frac{q}{2k} + \frac{b}{h^2}\Big)\Big(u_{j+1}^n - 2u_j^n + u_{j-1}^n\Big) + cu_j^n + \lambda u_j^n - \lambda \sum_{p\in P} \alpha_p u_{j+p}^n (\Gamma_\delta)_p,$$

where P is the index set of the integral approximation.

Proposition 7.5.1 *The scheme* (7.5.4) *is accurate to order* $O\Big(h^2 + \frac{qh^2}{2k}\Big)$ *under the CFL stability condition*

$$\frac{|a|k}{h} \leq \frac{2bk}{h^2} + q \leq 1 - (c+\lambda)k. \qquad (7.5.5)$$

Proof. The condition (7.5.5) is easily checked by looking at the monotonicity of the function Q. To study the accuracy of the scheme, we use the symbol analysis [110]. Let $p(s,\xi)$ be the symbol of the integro-differential operator (7.5.1)

$$p(s,\xi) = s + ia\xi + b\xi^2 + c - \lambda\Big(\int_{z_m}^{z_M} e^{i\xi z}\Gamma_\delta(z)dz - 1 + C\delta^2\epsilon\Big).$$

Substituting $u_j^n = e^{skn}e^{ijh\xi}$ in (7.5.4), we get the symbol $p_{k,h}(s,\xi)$ of the difference scheme,

$$p_{k,h}(s,\xi) = \frac{e^{sk}-1}{k} + ia\frac{\sin h\xi}{h} + 2(\frac{qh^2}{2k} - b)\frac{\cos h\xi - 1}{h^2} + c - \lambda\Big(\sum_{p\in P}\alpha_p e^{iph\xi}(\Gamma_\delta)_p - 1\Big).$$

Taking into account that our integral approximation verifies

$$\sum_{p\in P}\alpha_p e^{iph\xi}(\Gamma_\delta)_p = \int_{z_m}^{z_M} e^{i\xi z}\Gamma_\delta(z)dz + O(h^l) \quad \text{with} \quad l \geq 2,$$

we have, by the Taylor expansion

$$p_{k,h}(s,\xi) = s + ia\xi + b\xi^2 + r - \lambda\Big(\int_{z_m}^{z_M} e^{i\xi z}\Gamma_\delta(z)dz - 1\Big) + \frac{qh^2}{2k}\xi^2 + O(k+h^2).$$

Then, we look for a symbol $r_{k,h}(s,\xi)$ such that the difference $p_{k,h}(s,\xi) - r_{k,h}(s,\xi)p(s,\xi)$ gives the order of accuracy. We have that $r_{k,h}(s,\xi) = 1 + o(1)$ and

$$p_{k,h}(s,\xi) - p(s,\xi) = +\frac{qh^2}{2k}\xi^2 + \lambda C\delta^2\epsilon + O(k+h^2).$$

Since $\epsilon \ll 1$ and, from the CFL condition (7.5.5), $k = O(h^2)$, the scheme is accurate of order $O\Big(h^2 + \frac{qh^2}{2k}\Big)$. ■

7.5.2 Numerical boundary conditions.

To apply the scheme (7.5.4) we have to specify a numerical bounded domain. Let $\Omega \subset \mathbb{R}$ be the interval where we want to calculate the numerical solution. We set

$$\Omega_h = \Big\{x_j = jh, j \in \mathbb{Z} \mid x_j \in \Omega\Big\},$$

and we define the numerical domain $\bar{\Omega}$ for the problem (7.5.1) in the following way. For every fixed x, set

$$\Omega_x = \Big\{z \in [z_m, z_M] \mid x + z \in \Omega\Big\}$$

and

$$\bar{\Omega} = \Omega \cup \Big\{\cup_{x\in\Omega}\Omega_x^C\Big\}.$$

We point out that, since the integral is a nonlocal term, its approximation will be split in two parts. For every $x \in \Omega$ fixed, we integrate on the union of the “inside” set Ω_x and the “outside” set Ω_x^C. Since accurate representation of the integral term will generally require a very wide grid, the “outside” set Ω_x^C must contains many grid points. Then, as we have defined our numerical problem on Ω, we need a limiting form for the solution u on the external set Ω^C. If $v(x,t)$ is any given analytic approximation of $u(x,t)$, the integral term will be approximated by

$$\begin{aligned}\mathcal{I}u(x,t) &= \lambda\Big(\int_{-\infty}^{\infty} u(x+z,t)\Gamma_\delta(z)dz - u(x,t)\Big)\\ &\approx \lambda\Big(\int_{\Omega_x} u(x+z,t)\Gamma_\delta(z)dz + \int_{\Omega_x^C} v(x+z,t)\Gamma_\delta(z)dz - u(x,t)\Big).\end{aligned}$$

If the option price is linear in e^x, the simplest choice is to use as approximation function v the payoff function ψ.

To give a more general scheme, not depending on the initial data of the problem or on a special form of the solution, we approximate on the external set Ω^C the problem (7.5.1) by the diffusive one (7.4.2). We define

$$j_- = \inf_j \Big\{jh \in \Omega_h\Big\}, \quad j_+ = \sup_j \Big\{jh \in \Omega_h\Big\},$$

$$P_{in} = P \cap \{j_-, ..., j_+\}, \qquad P_{out} = P - P_{in}.$$

We modify the scheme (7.5.4) with $q = 0$ fixed, as follows

$$\begin{aligned} v_j^{n+1} &= kw_{-1}v_{j-1}^n + \Big(1 - kw_0\Big)v_j^n + kw_1 v_{j+1}^n \\ &\quad +\lambda k\Big[h\sum_{p\in P_{in}} \alpha_p v_{j+p}^n(\Gamma_\delta)_p + h\sum_{p\in P_{out}} \alpha_p \tilde{v}_{j+p}^n(\Gamma_\delta)_p - v_j^n\Big], \end{aligned} \tag{7.5.6}$$

where

$$w_{-1} = \frac{b}{h^2} + \frac{a}{2h}, \quad w_0 = \frac{2b}{h^2} + c, \quad w_1 = \frac{b}{h^2} - \frac{a}{2h},$$

and where the values $\{\tilde{v}_i^n\}$ are given by the approximation of the diffusive equation (7.4.2) with a general diffusion coefficient D, to be fixed later,

$$\begin{aligned} \tilde{v}_i^n &= kw_{-1}v_{i-1}^{n-1} + \Big(1 - kw_0\Big)v_i^{n-1} + kw_1 v_{i+1}^{n-1} \\ &\quad +\lambda k\frac{D}{h^2}\Big(v_{i-1}^{n-1} - 2v_i^{n-1} + v_{i+1}^{n-1}\Big). \end{aligned} \tag{7.5.7}$$

Let us rewrite the scheme (7.5.4) in the following form

$$\begin{aligned} u_j^{n+1} &= kw_{-1}u_{j-1}^n + \Big(1 - kw_0\Big)u_j^n + kw_1 u_{j+1}^n \\ &\quad +\lambda k\Big[h\sum_{p\in P} \alpha_p u_{j+p}^n(\Gamma_\delta)_p - u_j^n\Big] - k\tau_{h,k}^\epsilon, \end{aligned} \tag{7.5.8}$$

where $\tau^{\epsilon}_{h,k}$ is the truncation error estimate in Proposition 7.5.1, which is $\tau^{\epsilon}_{h,k} = O(k+h^2) + \lambda C\delta^2\epsilon$. We want to estimate the global difference between the two numerical solution (7.5.6) and (7.5.8). Define

$$e_j^{n+1} = v_j^{n+1} - u_j^{n+1}, \quad \text{and} \quad E^n = \sup_j \mid e_j^n \mid .$$

We can prove the following result.

Proposition 7.5.2 *If $\epsilon = O(h^4)$ and for the time step there holds a standard CFL condition, $k = O(h^2)$, then $E^n = O(h^2)$, for $h \to 0$.*

Proof. Subtracting the equation (7.5.8) from the (7.5.6), we have

$$\begin{aligned} e_j^{n+1} &= kw_{-1}e_{j-1}^n + \Big(1 - kw_0\Big)e_j^n + kw_1 e_{j+1}^n \qquad (7.5.9)\\ &+\lambda k\Big[h\sum_{p\in P_{in}} \alpha_p e_{j+p}^n(\Gamma_\delta)_p + h\sum_{p\in P_{out}} \alpha_p\Big[\tilde{v}_{j+p}^n - u_{j+p}^n\Big](\Gamma_\delta)_p - e_j^n\Big] + k\tau^{\epsilon}_{h,k}. \end{aligned}$$

By (7.5.7) and (7.5.8), we obtain

$$\begin{aligned} &\tilde{v}_{j+p}^n - u_{j+p}^n = k\Big(w_{-1} + \frac{\lambda D}{h^2}\Big)e_{j+p-1}^{n-1} + \Big[1 - k\Big(w_0 + \frac{2\lambda D}{h^2}\Big)\Big]e_{j+p}^{n-1}\\ &+k\Big(w_1 + \frac{\lambda D}{h^2}\Big)e_{j+p+1}^{n-1} + \lambda k\Big[\frac{D}{h^2}\Big(u_{j+p-1}^{n-1} - 2u_{j+p}^{n-1} + u_{j+p+1}^{n-1}\Big) \qquad (7.5.10)\\ &-\Big(h\sum_{\tilde{p}\in P}\alpha_{\tilde{p}} u_{j+p+\tilde{p}}^{n-1}(\Gamma_\delta)_{\tilde{p}} - u_{j+p}^{n-1}\Big)\Big] + k\tau^{\epsilon}_{h,k}. \end{aligned}$$

We have to estimate the difference between the central second order finite difference approximation and the integral approximation.

For every $p \in P$ fixed

$$u_{j+p} - u_j = (ph)u_x + \frac{(ph)^2}{2}u_{xx} + \frac{(ph)^3}{3!}u_{xxx} + \frac{(ph)^4}{4!}u_{xxxx} + \dots .$$

We call $z_\epsilon = z_M = \delta\sqrt{-2\log(\epsilon\delta\sqrt{2\pi})}$ as described in Subsection 7.3.2, and, for the compound rule (7.3.1), with $\rho = 1$, a point $\xi \in [-z_\epsilon, z_\epsilon]$ exists such that

$$h\sum_{p\in P}\alpha_p (ph)^\beta(\Gamma_\delta)_p = \int_{-z_\epsilon}^{z_\epsilon} z^\beta \Gamma_\delta(z)dz + \frac{h^2}{6}z_\epsilon \frac{d^2(z^\beta\Gamma_\delta(z))}{dz^2}\Big|_{z=\xi}.$$

Then,

$$h\sum_{p\in P}\alpha_p\Big(u_{j+p} - u_j\Big)(\Gamma_\delta)_p \;=\; \frac{1}{2}\frac{d^2u(x)}{dx^2}\Big|_{x=x_j}\int_{-z_\epsilon}^{z_\epsilon} z^2\Gamma_\delta(z)dz + \frac{h^2}{6}z_\epsilon R_j.$$

This yields

$$D\frac{u_{j-1}-2u_j+u_{j+1}}{h^2} - h\sum_{p\in P}\alpha_p\Big(u_{j+p}-u_j\Big)(\Gamma_\delta)_p$$
$$= \frac{1}{2}\frac{d^2u(x)}{dx^2}\Big|_{x=x_j}\Big(2D - \int_{-z_\epsilon}^{z_\epsilon} z^2\Gamma_\delta(z)dz\Big) - \frac{h^2}{6}z_\epsilon R_j.$$

Now, we choose the diffusion coefficient D such that

$$D = \frac{1}{2}\int_{-z_\epsilon}^{z_\epsilon} z^2\Gamma_\delta(z)dz. \tag{7.5.11}$$

Under the CFL condition (7.5.5) and for

$$1 - kw_0 - k\frac{2\lambda D}{h^2} \geq 0, \tag{7.5.12}$$

from (7.5.9) and (7.5.10), we have

$$E^{n+1} \leq \Big[1 - kc + \lambda kh\sum_{p\in P_{in}}(\Gamma_\delta)_p - \lambda k\Big]E^n + \Big[\lambda k\Big(1-kc\Big)h\sum_{p\in P_{out}}\alpha_p(\Gamma_\delta)_p\Big]E^{n-1}$$
$$+\lambda^2k^2\frac{h^2}{6}z_\epsilon h\sum_{p\in P_{out}}\alpha_p \mid R_{j+p}\mid (\Gamma_\delta)_p + \lambda k^2 \mid \tau^\epsilon_{h,k}\mid h\sum_{p\in P_{out}}\alpha_p(\Gamma_\delta)_p + k\mid \tau^\epsilon_{h,k}\mid .$$

This is, for some coefficients A, B and C,

$$E^{n+1} \leq AE^n + BE^{n-1} + C.$$

As a consequence, for any $\frac{n+1}{2} \leq m < n$ we have

$$E^{n+1} \leq E^{n-2m+1}\sum_{k=0}^{m}\binom{m}{k}(AE)^kB^{m-k} + C\sum_{k=0}^{m-1}(A+B)^k.$$

When $n-2m+1=0$, $E^0=0$ it yields

$$E^{n+1} \leq C\frac{1-(A+B)^{\frac{n+1}{2}}}{1-(A+B)}. \tag{7.5.13}$$

Now, we have

$$h\sum_{p\in P_{in},P_{out}}(\Gamma_\delta)_p \leq 2z_\epsilon \max_z \Gamma_\delta = 2z_\epsilon\Gamma \quad \text{and} \quad \mid R_j \mid\leq R.$$

The CFL condition (7.5.12) gives $k = O(h^2)$ and $\tau^\epsilon_{h,k} = O(h^2+\epsilon)$. Then, for $N = T/k$, the global error (7.5.13) is estimated by

$$E^N \leq \Big(1 - \frac{T}{N}c - 2\frac{T^2}{N^2}\lambda z_\epsilon\Gamma\Big)^{N/2} g(h,\epsilon),$$

where

$$g(h,\epsilon) = O\Big(\frac{h^4 z_\epsilon^2 + h^4 z_\epsilon + h^2 \epsilon z_\epsilon + h^2 + \epsilon}{1 + h^2 z_\epsilon}\Big).$$

As h and k go to zero, we obtain

$$\lim_{N\to\infty} (A+B)^{N/2} = \Big(1 - \frac{T}{N}c - 2\frac{T^2}{N^2}\lambda z_\epsilon \Gamma\Big)^{N/2} = e^{\frac{TC}{2}}.$$

Then, to get the rate of convergence as $h \to 0$, we observe that the minimal value of the function $g(\cdot,\epsilon)$ is achieved for $\epsilon = O(h^4)$.

Therefore, the conclusion follows, since

$$\lim_{N\to\infty} E^N \le e^{\frac{TC}{2}} g(h,h^4)$$

$$\le e^{\frac{TC}{2}} \Big(- h^4 \log h^4 + h^4\sqrt{-\log h^4} + h^6\sqrt{-\log h^4} + h^2 + h^4\Big) = O(h^2).$$

■

Remark 7.5.3 We point out that, for the Gaussian probability density (1.1.1) we have

$$\bar{k} = \mathbb{E}(\eta - 1) = \exp\Big(\frac{\delta^2}{2}\Big) - 1 \approx \frac{\delta^2}{2} + O(\delta^4), \qquad \delta \ll 1,$$

then, solving the approximated problem (7.4.2) in Ω^C is just solving the Black-Scholes equation (6.4.2) with coefficients

$$a = \frac{\sigma^2}{2} - r + \lambda\bar{k} \approx \frac{\sigma^2}{2} - r + \lambda\frac{\delta^2}{2}, \qquad b = \frac{\sigma^2}{2} + \lambda\frac{\delta^2}{2}.$$

■

Even if the scheme (7.5.6) needs for a CFL condition and its convergence in time is only first order accurate, we shall see in Subsection 7.6.1 that it is of simple practice application and computationally fast. It is easy to obtain a scheme which is second order in time, by applying the SSP (Strong Stability Preserving) Runge-Kutta technique, as in [60] and references therein, but we observe no real advantages for the total accuracy at least for the second order case. Then in what follows, we just use scheme (7.5.6).

7.6 Examples and Numerical tests.

In this section we compute the order γ of the error in the following form

$$\gamma = \log_2\Big(\frac{e_1}{e_2}\Big), \tag{7.6.1}$$

with

$$e_p = \frac{\| u(\frac{h}{p}, T) - u(\frac{h}{2p}, T) \|_{1,\infty}}{\| u(\frac{h}{2p}, T) \|_{1,\infty}}, \qquad p = 1, 2,$$

where $u(h)$ denotes the numerical solution obtained with the space step discretization equal to h, under the discrete norm l^1 and l^∞, respectively

$$\| u(\cdot, T) \|_1 = h \sum_i | u(x_i, T) |, \qquad \| u(\cdot, T) \|_\infty = \max_i | u(x_i, T) | .$$

If not specified, in tables that follow we give the average convergence order.

7.6.1 European option.

Let us consider the problem of pricing an European option according to the problem (7.5.1). As we showed in Subsection 7.5.1, we solve the integro-differential equation on the numerical domain $\bar{\Omega}$. We apply the second order scheme (7.5.6) under the CFL condition

$$h \le \frac{2b}{a}, \qquad k \le \min\Big(\frac{h^2}{2b + 2\lambda D + ch^2}, \frac{h^2}{2b + ch^2 + \lambda h^2}\Big), \tag{7.6.2}$$

with D given by (7.5.11), $D = \lambda\delta^2/2$.

Let us fix the parameters as follows: $E = 100$, $r = 0.05$, $\sigma = 0.2$, $\delta = 0.2$ and $T = 1$. In Figure 7.1 we present the value of the option given by the jump-diffusion model (dotted-solid curve), the pure diffusion model ($\lambda = 0$) (solid curve), and the payoff value (dotted curve) respectively. The difference between the two models is clear: the value in the jump diffusion model is larger than the one in the pure diffusion setting in a neighborhood of the exercise price, according to the theoretical results in [90].

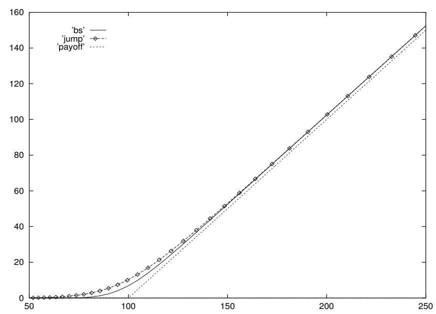

Figure 7.1: Subsection 7.6.1, jump-diffusion model ($\diamond$), pure diffusion model ($-$) and payoff value ($\cdots$), with Simpson compound rule and $h = 0.05$.

In Figure 7.2, we show the variation of the solution according to the jump intensity λ. We compare the solution with $\lambda = 0.5$ ($\diamond$), $\lambda = 2$ (+) and $\lambda = 8$ (−) and we observe that the solutions increase with λ. This is what we expect from the model, because as the intensity of the jump increases the risk of the investment increases and consequently the price of the derivative needs to be higher.

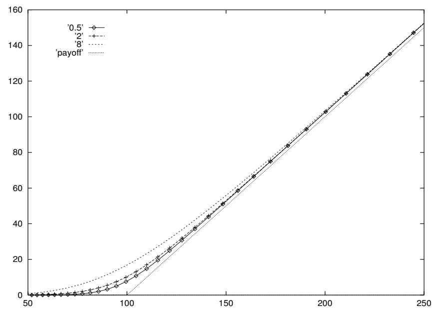

Figure 7.2: Subsection 7.6.1, jump-diffusion model with different values of jump intensity, $\lambda = 0.5$ ($\diamond$), $\lambda = 2$ (+), $\lambda = 8$ (−).

In Tables $T2$ we show the l^∞ errors and the convergence order (7.6.1) for the European call option initial data. This confirms experimentally that the scheme is second order accurate.

N	$T = 0.1$	$T = 1$	$T = 10$
129	0.000262	0.000778	0.002118
257	0.000123	0.000122	0.000645
513	0.000044	0.000036	0.000038
1025	0.000011	0.000016	0.000130
2049	0.000000	0.000008	0.000003
Convergence-order	2.40096	1.663561	2.379236

Table T2: Example 7.6.1, l^∞ errors and convergence orders of the European call option computed using the scheme (7.5.6). The process parameters are $E = 100$, $r = 0.05$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.8$. The number of time steps is given by the CFL condition (7.6.2).

Since we might be interested in obtaining the value u of the option for a given stock price $S^* = \exp(x^*)$, we fix $x^* = \ln(100)$ and $\sigma = 0.2$, $\delta = 0.8$, $\lambda = 0.1$. We compute the "exact" option price P by the analytical solution given in [90]. Then, in Table $T3$ we show the convergence order,

$$\tilde{\gamma} = \log_2 \Big(\frac{\mid u(h; x^*, T) - P \mid}{\mid u(\frac{h}{2}; x^*, T) - P \mid} \Big), \tag{7.6.3}$$

where $u(h; x^*, T)$ is the numerical solution of (7.5.6) with space step h, valued in x^* at time T. We stress out that we construct the interval Ω centered on x^*, then the values on the Table $T3$ are few influenced by the boundary error.

	$T = 0.1$		$T = 1$		$T = 10$	
N	PUT	CALL	PUT	CALL	PUT	CALL
64	1.562877	1.876357	7.809220	13.640541	13.902732	67.105924
128	2.356042	2.826212	8.167357	13.579657	14.916689	56.847952
256	2.572710	3.075204	8.268306	13.378464	14.911828	56.876483
512	2.625636	3.128717	8.319940	13.286915	15.179737	54.510145
1024	2.628921	3.122007	8.337027	13.223001	15.179249	54.513929
Analytical solution	2.633642	3.132394	8.341444	13.218501	15.179245	54.525989
Convergence-order $\tilde{\gamma}$	1.956285	2.916243	1.728216	1.637843	4.602457	2.506671

TABLE T3: Example 7.6.1, convergence orders (7.6.3) with respect to the analytical solution, see [90], of the European put and call option prices of Merton model computed using the explicit scheme (7.5.6). The process parameters are $E = 100$, $r = 0.05$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.8$, $x = \ln(100)$. The number of time steps is given by the CFL condition (7.6.2).

Table $T4$ shows CPU times on a 1, 6 GHz Pentium IV PC for various number of space steps. Although the scheme (7.5.6) is of explicit type, it is computationally fast.

N	64	128	256	512	1024
CPU time (seconds)	N.A.	N.A.	0.02s	0.16s	4.4s

TABLE T4: Example 7.6.1, CPU times on 1.6 GHz Pentium IV PC for the scheme (7.5.6) when $T = 1$ and the number of time steps is given by the CFL condition (7.6.2). The CPU times for 64 and 128 nodes are not available.

7.6.2 A two-dimensional example.

In this section we present an operator splitting method for the two-dimensional degenerate equation (1.6.6). For an extensive description of operator splitting methods, we refers to the paper [73].

The main difficulty is given by the presence of a hyperbolic direction y. For simplicity, we set $r = 0$ and we write the equation in the short form

$$\partial_t u(x, y, t) + Du(x, y, t) = \lambda J u(x, y, t),$$

where

$$\begin{aligned} Du &= -b\partial_{xx}^2 u + a_1\partial_x u + a_2\partial_y u, \\ Ju &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u(x+\xi, y+\eta, t)\Gamma(\xi,\eta)d\xi d\eta - u(x,y,t), \\ \Gamma(\xi,\eta) &= \frac{1}{2\pi\delta^2}\exp\Big(-\frac{1}{2\delta^2}(\xi^2+\eta^2)\Big), \end{aligned}$$

and b, a_1 and a_2 are constants.
The operator splitting method can be summarized as follows: let $v^{n+1} = \mathcal{D}v^n$ be the numerical solution of

$$\partial_t v(x,y,t) + Dv(x,y,t) = 0, \tag{7.6.4}$$

and let $w^{n+1} = \mathcal{J}w^n$ be the numerical solution of

$$\partial_t w(x,y,t) = \lambda Jw(x,y,t). \tag{7.6.5}$$

Then the operator splitting is based on the following approximation

$$u^{n+1} = [\mathcal{J}\mathcal{D}]u^n.$$

To approximate the differential part (7.6.4), we shall apply an ADI method that combine Crank-Nicholson scheme in the two directions. To approximate the integral part (7.6.5), we shall apply the Euler rule for the time discretization and the compound Simpson's product rule to the two-dimensional integral.
Let us define the two following discrete operators,

$$\mathcal{D}_x = \Big[\frac{k}{2}\alpha_1\delta_x - \frac{k}{2}\beta\delta_{xx}\Big], \quad \mathcal{D}_y = \Big[\frac{k}{2}\alpha_2\delta_y\Big],$$

where

$$\alpha_1 = \frac{a_1}{2h_1}, \quad \alpha_2 = \frac{a_2}{h_2}, \quad \beta = \frac{b}{h_1^2} \qquad \delta_{xx}u_{i,j} = (u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

$$\delta_y u_{i,j} = \begin{cases} (u_{i,j} - u_{i,j-1}) & a_2 \geq 0 \\ (u_{i,j+1} - u_{i,j}) & a_2 < 0 \end{cases} \qquad \delta_x u_{i,j} = (u_{i+1,j} - u_{i-1,j})$$

We observe that we are using a central finite difference scheme for the x-direction and an upwind approximation for the degenerate one. We can now write the complete scheme. To calculate the numerical solution u^{n+1} from u^n we have to solve the following two steps:

1. We compute the values $\tilde{u}_{i,j}^{n+1}$, for $i,j = 0,...,N$, by solving two tridiagonal system,
$$\begin{cases} (I+\mathcal{D}_x)\tilde{u}^{n+\frac{1}{2}} &= (I-\mathcal{D}_y)u^n \\ \\ (I+\mathcal{D}_y)\tilde{u}^{n+1} &= (I-\mathcal{D}_x)\tilde{u}^{n+\frac{1}{2}} \end{cases} \tag{7.6.6}$$
where I denotes the identity matrix.

2. We obtain the solution $u_{i,j}^{n+1}$, for $i,j = 0,...,N$, by the expression
$$u_{i,j}^{n+1} = (1-\lambda k)\tilde{u}_{i,j}^{n+1} + \lambda k h_1 h_2 \sum_{l,m} \alpha_l \alpha_m \tilde{u}_{i+l,j+m}^{n+1} \Gamma_{l,m}. \tag{7.6.7}$$

We consider the following example,
$$\begin{cases} \partial_t u(x,y,t) + Du(x,y,t) = \lambda J u(x,y,t) & (x,y,t) \in Q \times [0,T] \\ \\ u(x,y,0) = u_0(x,y) & (x,y) \in Q, \end{cases} \tag{7.6.8}$$
where we fix the parameters $\lambda = 1$, $b = 1$, $a_1 = -a_2 = 0.5$ and the x space discretization h_1 equal to the y space discretization h_2.

In Table $T5$, we show the γ order (7.6.1) under the norm l^1 of the scheme (7.6.6)-(7.6.7) applied to the problem (7.6.8) with a regular initial data $u_0(x,y) = \sin(\pi(x+y))$. We point out that we have chosen an upwind approximation to deal with the pure hyperbolic direction y, then the scheme is at most first order accurate, as well verified in Table $T5$.

	$\delta = 10^{-4}$		$\delta = 10^{-2}$	
$h_1 = h_2$	γ	e_p	γ	e_p
0.025		0.099739		0.089381
0.0125	1.308349	0.040273	1.485643	0.031917
0.00625	0.557111	0.027372	0.9416737	0.016169
0.003125	0.9288881	0.014378	0.447856	0.011854

TABLE $T5$: Convergence order γ, defined in (7.6.1), and errors, for the solution of the problem (7.6.8) with $u_0(x,y) = \sin(\pi(x+y))$, $b = 1$, $a_1 = -a_2 = 0.5$.

7.6.3 The nonlinear case.

As we have already seen in Section 7.1, the option pricing in large investor economy leads to a quasilinear differential problem. From equation (7.1.1), by the standard change of variable $x = \log S$, we get the following general equation
$$u_t + \mathcal{L}_{\mathcal{I}u} u = H(x,t,u,\mathcal{I}u,\mathcal{D}u),$$

where $\mathcal{L}_{\mathcal{I}}$ is a linear degenerate elliptic integro-differential operator and H is a nonlinear integro-differential Hamilton-Jacobi operator.

The numerical approximation of Hamilton-Jacobi equations has been intensively studied, both for first and second order equations. We refer again to [36, 16] for classical results and to [96, 77, 74] for recent developments of high order accurate schemes, such as ENO, WENO, and central schemes.

Let us introduce some standard notations:

$$u^{\pm} = \Delta_{\pm} u_j = \frac{\pm(u_{j\pm 1} - u_j)}{h}, \qquad \Delta^2 u_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}, \qquad \hat{H}(u^+, u^-),$$

where $\hat{H}$ is a Lipschitz continuous numerical flux, which is monotone and consistent with H [36], i.e.:

$$\hat{H}(p,p) = H(p).$$

Monotonicity here means that $\hat{H}$ in non-increasing in its first argument and nondecreasing in the other one. Two of the most useful admissible numerical fluxes are the local Lax-Friedrichs (LLF) flux and the Godunov flux, [96].

Example 7.6.1 [*Large institutional investor*]. Let us consider the Merton model for the large investor economy. As we have seen in the Example 1.6.8, the interest rate r depends on the wealth ξ invested in stocks and the price function solves the quasi-linear final value problem (7.1.1). In the specific case of the large institutional investor, the interest rate decreases when too much wealth is invested in bonds, according to the law $r(S,t,\xi) = R(S,t)f(\xi)$ where f is a positive continuous function such that, for a given wealth $\xi_0 \geq 0$ fixed, $f(\xi) = 1$ as $\xi \leq \xi_0$ and f is decreasing as $\xi > \xi_0$, but $f(\xi)\xi$ non decreasing. A good prototype of such type function f is given by

$$f(\xi) = \begin{cases} 1, & \xi \leq \xi_0 \\ \alpha + \beta\xi_0\xi^{-\gamma} & \xi > \xi_0, \end{cases}$$

for all α, β and γ such that $\alpha,\ \beta > 0$, $0 < \gamma \leq 1$ and $\alpha + \beta\xi_0^{-\gamma+1} = 1$. We select,

$$\gamma = \frac{1}{2}, \quad \beta = \frac{1}{2\sqrt{\xi_0}} \quad \Rightarrow \quad \alpha = \frac{1}{2},$$

and we fix constant the interest rate $R(S,t) = R$.

We want to solve the following one dimensional quasi-linear problem,

$$\begin{cases} u_t - bu_{xx} + au_x + H\Big(u, u_x, \displaystyle\int u(x+z,t)\Gamma_\delta(z)dz\Big) & = \mathcal{I}u, \\ u(x,0) & = \psi(x), \end{cases} \tag{7.6.9}$$

where $a = (\lambda\bar{k} + \frac{\sigma^2}{2})$, $b = \sigma^2/2$ and the non linear term H is given by

$$H(u,p,q) = Rf\Big(u - \tilde{a}u_x - \tilde{b}(q-u)\Big)\Big(u - \tilde{a}u_x - \tilde{b}(q-u)\Big),$$

We point out that the H operator verify the general assumptions **F1**, **F2**, **F3**, given in Section 4.1, then the Cauchy problem (7.6.9) has a unique viscosity solution in the sense of Definition 4.1.4. Moreover, $H(\cdot,p,\cdot)$ is a decreasing monotone function, convex for $\xi > \xi_0$.

To discretize the equation (7.6.9) we approximate the nonlinear term by

$$\hat{H}_J\Big(u_j^n, u^+, u^-, \sum_{p\in P}\alpha_p u_{j+p}^n(\Gamma_\delta)_p\Big) = H\Big(u_j^n, \frac{u^+ + u^-}{2}, \sum_{p\in P}\alpha_p u_{j+p}^n(\Gamma_\delta)_p\Big),$$

This is of course a Lipschitz continuous numerical flux, monotone and consistent with $H(\cdot,p,\cdot)$. Applying the explicit scheme (7.5.4) for the linear part, we get the following approximation: for $j = j_-, ..., j_+$,

$$\begin{aligned} u_j^{n+1} &= u_j^n - \frac{ak}{2h}\Delta_- u_j^n + \frac{bk}{h^2}\Delta^2 u_j^n - \lambda k u_j^n \\ &\quad + \lambda k \sum_{p\in P}\alpha_p u_{j+p}^n(\Gamma_\delta)_p - k\hat{H}_J\Big(u_j^n, u^+, u^-, \sum_{p\in P}\alpha_p u_{j+p}^n(\Gamma_\delta)_p\Big). \end{aligned} \tag{7.6.10}$$

The scheme verifies the general convergence result (7.2.2) under the following CFL condition,

$$h \le \frac{2b}{a}, \qquad \frac{2bk}{h^2} + \lambda k + k\max_u\Big[\frac{dH}{du}(u,\cdot,\cdot)\Big] - k\min_q\Big[\frac{dH}{dq}(\cdot,\cdot,q)\Big] \le 1,$$

As it has been done for the linear problem (7.5.1), on the numerical boundary domain Ω^C we approximate the integral term $\mathcal{I}u$ in (7.6.9) by the diffusive one Du_{xx} and we solve the following equation,

$$u_t - bu_{xx} + au_x + H\Big(u, u_x, Du_{xx}\Big) = \lambda D u_{xx}, \qquad (x,t) \in \Omega^C \times (0,T],$$

under the condition

$$\frac{2bk}{h^2} + \frac{2\lambda Dk}{h^2} + k\max_u\Big[\frac{dH}{du}(u,\cdot,\cdot)\Big] - Dk\min_q\Big[\frac{dH}{dq}(\cdot,\cdot,q)\Big] \le 1. \tag{7.6.11}$$

We fix $\tilde{a} = a$, $\tilde{b} = b$, the parameters $E = 100$, $R = 0.05$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.4$ and the initial data $\psi(x) = (e^x - E)_+$ as the call option payoff function.

In Tables $T6$ we show the l^∞ errors and the convergence order (7.6.1) for $\xi_0 = 10^2$. This experimentally shows that the scheme is second order accurate.

N	$T = 0.5$	$T = 1$	$T = 10$
65	0.007924	0.013589	0.032857
129	0.000850	0.002836	0.007711
257	0.000610	0.000072	0.000522
513	0.000071	0.000056	0.000052
1025	0.000033	0.000068	0.000013
Convergence-order	1.977426	1.911010	2.839315

TABLE $T6$: Example 7.6.1, l^∞ errors and convergence orders of the European call option computed using the scheme (7.6.10). The process parameters are $\xi_0 = 10^2$, $E = 100$, $r = 0.05$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.4$. The number of time steps is given by the CFL condition (7.6.11).

Figure 7.3 shows the call option payoff function compared with the solution of (7.6.9) at time $T = 1$, with $\xi_0 = 10^2$ fixed.

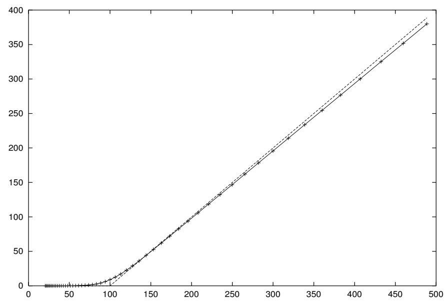

Figure 7.3: Example 7.6.1, we show the call option payoff function compared with the solution of (7.6.9) ($\times$) at time $T = 1$, with $\xi_0 = 10^2$ fixed.

■

Chapter 8

Convergence of numerical schemes in the unbounded Lévy case

8.1 Introduction

In this chapter we study the numerical approximation of the class of nonlinear parabolic integro-differential Cauchy problems introduced in Chapter 4:

$$\begin{cases} -\partial_t u + H(x,t,u,\mathbf{x}\mathcal{D}u,\mathbf{x}\mathcal{D}^2u\mathbf{x}^T,\mathcal{J}u) &= 0, \\ u(x,T) &= u_T(x), \end{cases} \tag{8.1.1}$$

where u_T is a continuous initial data and H is a nonlinear second order operator. We recall that $\mathcal{J}u$ is an integro–differential operator given by

$$\mathcal{J}u = \int_E \Big[u(x+\mathbf{x}\beta(x,t,z),t) - u(x,t) - \mathbf{x}\beta(x,t,z)\cdot\mathcal{D}u(x,t)\Big]\nu(dz), \tag{8.1.2}$$

where $\nu(dz)$ is a given Radon measure on $E = \mathbb{R}^2 - \{0\}$, the so-called Lévy measure, satisfying

$$\int_E (1\wedge |z|^2)\nu(dz) < \infty. \tag{8.1.3}$$

The theory can be extended to more general H operator depending on another integral term:

$$\mathcal{I}u(x,t) = \int_E \left[u(x+\mathbf{x}\beta(x,t,z),t) - u(x,t)\right]\gamma(x,t,z)\nu(dz), \tag{8.1.4}$$

using the techniques of Chapter 7.

Problems in this form arise when considering a financial derivative constructed on an underlying asset evolving as an exponential Lévy process instead of the classical diffusion dynamics, as it has been explained in Chapter 2 and Chapter 5 in order to have a more realistic description of the market than the one obtained with the Black–Scholes model [28].

8.2 The Financial model - option pricing with Lévy processes

In Chapter 5 we have proposed several examples in which the exponential Lévy model arises and which are described by problem (8.1.1). We recall here the problem of our main interest: let us consider a market whose money market account evolves according to

$$dB_t = B_t r dt,$$

where r is the deterministic interest rate in the market, while $S = (S^1, \ldots, S^{\mathrm{N}})$, the risky assets, are described by a stochastic differential system of equations

$$dS_t = \mathbf{S}_t\Big[b(S_t,t)dt + \sigma(S_t,t)dW_t + \int_E \beta(S_{t^-},t,z)\tilde{N}(dt,dz)\Big],$$

where $\mathbf{S}_t = \operatorname{diag}(S_t^1, \ldots, S_t^{\mathrm{N}})$, W_t is a M–dimensional standard Brownian motion, $1 \le \mathrm{M} \le \mathrm{N}$, $\tilde{N}$ is the compensated martingale measure of a N–dimensional Poisson random measure N defined on $\mathbb{R}^+ \times E$ with compensator $\lambda(dt,dz) = dt \times \nu(dz)$ is its Lévy intensity, and $\nu : \mathcal{B}(E) \to \mathbb{R}^{\mathrm{N}}$,

$$\nu(dz) = (\nu^1(dz), \ldots, \nu^{\mathrm{N}}(dz)),$$

is the N–dimensional Lévy measure. We assume that $X_t^i = e^{L_t^i}$, where L_t is a N–Lévy process described by

$$\begin{aligned} dL_t^i &= \mu^i(t)dt + \sum_{j=1}^{\mathrm{M}} \sigma_j^i dW^j + \sum_{j=1}^{\mathrm{N}} \int_{|z|<1} \eta_j^i(t) z^j \tilde{N}^j(dt,dz) \\ &\quad + \sum_{j=1}^{\mathrm{N}} \int_{|z|\ge 1} \eta_j^i(t) z^j N^j(dt,dz), \qquad i = 1, \ldots, \mathrm{N}, \end{aligned}$$

where $\sigma(t) \in \mathbb{R}^{\mathrm{N}\times\mathrm{M}}$, $1 \le \mathrm{M} \le \mathrm{N}$, $\sigma(t)\sigma^T(t) \ge 0$, $\eta(t) \in \mathbb{R}^{\mathrm{N}\times\mathrm{N}}$. By the generalized Ito's formula we derive

$$\begin{aligned} b_i(S_t) &= \mu^i + \frac{1}{2}\sum_{j=1}^{\mathrm{M}} (\sigma_j^i)^2 + \sum_{j=1}^{\mathrm{N}} \int_E \Big(e^{\eta_j^i z^j} - 1 - \eta_j^i z^j \mathbf{1}_{|z|<1}\Big)\nu^j(dz), \\ \sigma(S_t) &= \sigma, \end{aligned}$$

$$\beta^i_j(S_t, z) \quad = \quad e^{\eta^i_j z^j} - 1,$$

where we have omitted the t dependence for simplicity.

Following the line of Chapters 4 and 5 we can derive the price of any derivative $U(S_t, t)$ as the solution of the following nonlinear integro–differential parabolic problem:

$$\begin{cases} -\partial_t U + H\Big(x, t, U, \mathbf{x}\mathcal{D}U, \mathbf{x}\mathcal{D}^2U\mathbf{x}^T, \mathcal{J}U\Big) & = \quad 0, \\ \\ U(S_T, T) & = \quad U_T(S_T), \end{cases}$$

where

$$\begin{aligned} H\Big(x, t, U, \mathbf{x}\mathcal{D}U, \mathbf{x}\mathcal{D}^2U\mathbf{x}^T, \mathcal{J}U\Big) \quad &= \quad -\frac{1}{2}\mathrm{tr}\left[\sigma(t)\sigma(t)^T\mathbf{x}\mathcal{D}^2U\mathbf{x}^T\right] - b\mathbf{x}\mathcal{D}U \\ &\quad +rU - \sum_{j=1}^{\mathrm{N}} \mathcal{J}_j U, \end{aligned}$$

and

$$\mathcal{J}_j U(x,t) = \int_E \left[U(\mathbf{x}e^{(\eta(t)z)_j}, t) - U(x,t) - \mathbf{x}\Big(e^{\eta(t)z} - \mathbf{1}_{\mathrm{N}}\Big)\mathcal{D}U\right]\nu^j(dz).$$

This is the more general model one can consider in order to modeling a Lévy economy: if the parameters of the model are deterministic functions of (S_t, t), the problem reduces to a linear problem describing a *small investor economy*, otherwise we have a pricing problem in the *large investor economy*.

It is well known that in the case of no integral operator, or with an integral term of the form (8.1.4) with a bounded measure, the problem (8.2.1)–(8.2.2) is equivalent to a Cauchy problem on an unbounded domain, up to a logarithmic change of variables.

It can be proven that under suitable hypotheses on the β function, a change of variable of that kind can be applied even in this case, because of the special structure of the integro–differential operator, as it has been already described in Chapter 4, Section 4.2.

8.2.1 Change of variable

The problem we have to deal with is established as follows:

$$-\partial_t U + H(x, t, U, \mathbf{x}DU, \mathbf{x}D^2U\mathbf{x}^T, \mathcal{J}U) = 0, \qquad (x,t) \in \Pi_T, \quad (8.2.1)$$

$$U(x,T) = U_T(x), \qquad x \in \Pi, \qquad (8.2.2)$$

where $\Pi = (0,\infty)^{\mathrm{N}}$, $\Pi_T = \Pi \times [0,T)$, $\mathbf{x} = \mathrm{diag}\ (x_1, \cdots, x_{\mathrm{N}})$, and $\mathcal{J}u$ is the integro–differential operator previously defined, (8.1.2).

From what concerns the main object of this Thesis, the jump amplitude β satisfies all the requirements in order to apply a change of variable as in the pricing problem we have

$$\beta(t,x,z) = \exp(\eta(t)z) - \mathbf{1}_{\mathrm{N}} \approx \eta(t)x,$$

near the origin.The problem reads as

$$-\partial_t U - \frac{1}{2} tr\Big[\sigma\sigma^T \mathbf{x}\mathcal{D}^2 U \mathbf{x}^T\Big] - b\mathbf{x}\mathcal{D}U + rU - \sum_{j=1}^{\mathrm{N}} \mathcal{J}_j U = 0,$$

with

$$\mathcal{J}_j U(x,t) = \int_E \Big[U(\mathbf{x}e^{(\eta(t)z)_j}, t) - U(x,t) - \mathbf{x}(e^{\eta(t)z} - \mathbf{1}_{\mathrm{N}})\mathcal{D}U\Big]\nu^j(dz).$$

Applying a change of variable $u(x,t) = U(e^x, T-t)$, the pricing equation for any derivative could be written as

$$\begin{cases} u_t + H(x,t,u,\mathcal{D}u,\mathcal{D}^2 u, Ju) &= 0, \\ u(x,0) &= u_T(x); \end{cases}$$

omitting the x dependence in the coefficients the operator H reads

$$H(x,t,u,Ju,\mathcal{D}u,\mathcal{D}^2u) = -\frac{1}{2}\sigma(T-t)\sigma^T(T-t)\mathcal{D}^2 u - \Big(b(T-t) - c_{exp}(T-t)\Big)\mathcal{D}u$$
$$+ru - \sum_{j=1}^{\mathrm{N}} J_j u = 0,$$

where

$$c^i_{exp}(t) = \sum_{j=1}^{\mathrm{M}} \int_E \Big[e^{\eta^i_j(t)z^j} - 1 - \eta^i_j(t) z^j \mathbf{1}_{|z|<1}\Big]\nu^i(dz),$$

and the new integral operator becomes

$$J_j u(x,t) = \int_E \Big[u(\mathbf{x} + (\eta(T-t)z)^j, t) - u(x,t) - (\eta(T-t)z)\cdot\mathcal{D}u(x,t)\mathbf{1}_{|z|<1}\Big]\nu^j(dz).$$

We can note that

$$b - c_{exp} = \mu + \frac{1}{2}\sigma\sigma^T. \qquad (8.2.3)$$

8.3 A general convergence result

We want to approximate the following problem:

$$\partial_t u + H(x,t,u,\mathcal{D}u,\mathcal{D}^2u,Ju) = 0. \tag{8.3.1}$$

A numerical scheme approximating (5.3.4) can be written as

$$Q(h,k,j,n,u_j^n,J_h\tilde{u},\tilde{u}) = 0, \tag{8.3.2}$$

where $J_h\tilde{u}$ denotes the integral approximation. We want to prove that, under suitable conditions, this scheme converges to the unique solution of the problem (5.3.4), provided that the following conditions are satisfied:

PROPERTIES OF THE SCHEME

Q1 *Monotonicity of the approximating integral.*
If $\tilde{u} \geq \tilde{v}$ and $u_j^n = v_j^n$ we have the following inequality:

$$J_h\tilde{u} \geq J_h\tilde{v};$$

Q2 *Stability.*

$$\text{For all } h,k \text{ a solution } \hat{u} \text{ does exist that is bounded independently from } (h,k); \tag{8.3.3}$$

Q3 *Consistency.*
For all $\phi \in \mathcal{C}_b^\infty(\mathbb{R}^N \times [0,T])$ and for all $(x,t) \in \mathbb{R}^N \times (0,T)$ we have:

$$\liminf_{\substack{(h,k)\to 0\\(jh,nk)\to(x,t)\\\xi\to 0}} \frac{Q(h,k,j,n,\phi_j^n+\xi,J_h(\tilde{\phi}+\xi),\tilde{\phi}+\xi)}{\rho(h,k)}$$

$$\geq \partial_t u + H(x,t,u,\mathcal{D}u,\mathcal{D}^2u,Ju);$$

$$\limsup_{\substack{(h,k)\to 0\\(jh,nk)\to(t,x)\\\xi\to 0}} \frac{Q(h,k,j,n,\phi_j^n+\xi,J_h(\tilde{\phi}+\xi),\tilde{\phi}+\xi)}{\rho(h,k)}$$

$$\leq \partial_t u + H(x,t,u,\mathcal{D}u,\mathcal{D}^2u,Ju);$$

Q4 *Monotonicity.*
If $\tilde{u} \geq \tilde{v}$ and $u_j^n = v_j^n$ for all $h,\ k \geq 0$ and $1 \leq n \leq N$, we have:

$$Q(h,k,n,j,u_j^n,J_h\tilde{u},\tilde{u}) \leq Q(h,k,n,j,v_j^n,J_h\tilde{v},\tilde{v}). \tag{8.3.4}$$

Remark 8.3.1 The theory of numerical approximation of fully nonlinear degenerate parabolic problems presented in Chapter 6 could be considered as a special case of the present one. We define the numerical scheme approximating the parabolic problem:

$$\partial_t u + H(x,t,u,\mathcal{D}u,\mathcal{D}^2u,0) = 0 \text{ in } \mathbb{R}^{\mathrm{N}} \times (0,T), \tag{8.3.5}$$

as:

$$\tilde{Q}(h,k,j,n,u_j^n,\tilde{u}) = Q(h,k,n,j,u_j^n,0,\tilde{u}); \tag{8.3.6}$$

in this way the scheme $\tilde{Q}$ satisfies clearly all the properties (**S.1**)–(**S.3**) of Chapter 6, and therefore the approximation scheme (8.3.6) converges to the viscosity solutions of (8.3.5). ■

PROPERTIES OF THE EQUATION

Q5 *Maximum Principle* or *Strong Uniqueness Property* Assume **A.1**–**A.3** of Chapter 4 and take $u_T \in \mathcal{C}(\tilde{\Pi}) \cap \mathcal{P}_n(\Pi)$ for some $n < n_o$. Let $u \in \bigcup_{n<n_o} L^\infty\big(0,T;\mathcal{P}_n(\Pi)\big)$ be a (possibly discontinuous) viscosity solution to (5.3.4). Then u is the unique viscosity solution in the class $\bigcup_{n<n_o} L^\infty\big(0,T;\mathcal{P}_n(\Pi)\big)$. Moreover it is continuous on Π_T and can be extended continuously to $\tilde{\Pi} \times [0,T]$ by setting

$$\mathbf{u}(x,t) = \begin{cases} u(x,t) & \text{if } (x,t) \in \Pi_T, \\ \lim\limits_{\Pi_T \ni (y,s) \to (x,t)} u(y,s) & \text{if } (x,t) \in \Gamma \times [0,T] \cup \Pi \times \{T\}. \end{cases}$$

The function $\mathbf{u}$ still solves (8.1.1), and satisfies $\mathbf{u}(x,T) = u_T(x)$ for all $x \in \tilde{\Pi}$.

Under these assumptions we shall prove our main theoretical result.

Theorem 8.3.2 *Let assumption (***Q1***)–(***Q5***) hold true. Then, as* $(h,k) \to 0$*, the solution* $\tilde{u}$ *of the scheme* (8.3.2) *converges locally uniformly to the unique continuous viscosity solution of the problem* (8.3.1)*.*

Proof. We can note that the property (**Q5**) is given in the original variable, while the approximation scheme is given after having applied the change of variable discussed in the previous section. This is not a difficulty as the previous change of

variable is monotone, therefore every required property concerning an ordering relation is preserved.
Let us now rephrase the Maximum Principle property after the exponential change of variable. Let

$$k_n(x) = h_n(e^x) = \sum_{i=1}^{N} e^{nx_i} + \sum_{i=N'+1}^{N} e^{-n|x_i|};$$

the set $\mathcal{P}_n(\Pi)$ becomes the set

$$\mathcal{P}_{e^n}(\mathbb{R}^N) = \left\{u : \mathbb{R}^N \to \mathbb{R} : \frac{u(x)}{1+k_n(x)} \text{ is bounded}\right\},$$

and

$$L^\infty(0,T;\mathcal{P}_{e^n}(\mathbb{R}^N)) = \left\{u : \mathbb{R}^N \times [0,T] \to \mathbb{R} : \frac{u(x,t)}{1+k_n(x)} \text{ is bounded}\right\}.$$

Now we can prove the main convergence result.

Let $\underline{u}$, $\overline{u} \in \bigcup_{n<n_o} L^\infty(0,T;\mathcal{P}_{e^n}(\mathbb{R}^N))$ be defined by:

$$\underline{u}(x,t) = \liminf_{\substack{(\Delta x,\Delta t)\to 0\\ (j\Delta x, n\Delta t)\to(x,t)}} u_j^n,$$

$$\overline{u}(x,t) = \limsup_{\substack{(\Delta x,\Delta t)\to 0\\ (j\Delta x, n\Delta t)\to(x,t)}} u_j^n.$$

We want to prove that $\underline{u}$ and $\overline{u}$ are respectively supersolution and subsolution of the problem (8.3.1). If this claim is proved to be true, then by definition we have $\underline{u} \leq \overline{u}$, while the other inequality holds because of lower semicontinuity of $\underline{u}$ and upper semicontinuity of $\overline{u}$, hence:

$$u = \underline{u} = \overline{u},$$

is the unique continuous solution of the problem (8.3.1). This result, together with the definition of $\underline{u}$ and of $\overline{u}$ leads to the local uniform convergence of the solution of the scheme to the solution of the problem. To prove the claim, we shall consider only the case of $\overline{u}$, the other being the same.

We want to prove that $\overline{u} \in \mathcal{USC}$ and that it is a subsolution for the problem, i.e. for all $\phi \in \mathcal{C}^{2,1}(\mathbb{R}^N \times [0,T]) \cap \bigcup_{n<n_0} L^\infty(0,T;\mathcal{P}_{e^n}(\mathbb{R}^N))$ such that $\overline{u}(x_0,t_0) = \phi(x_0,t_0)$, (x_0,t_0) is a global maximum point and the following inequality

$$\partial_t\phi + H(x,t,\overline{u},\mathcal{D}\phi,\mathcal{D}^2\phi,J\phi) \leq 0,$$

holds in classical sense for all $(x,t) \in \mathbb{R}^N \times [0,T]$.
The main difference with the case studied in Chapter 7 stands in the class of growth, which has been modified in order to take into account the unboundedness of the Lévy measure near the origin. This feature reflects on the integral term that has to be treated with particular care, as it has been shown in Chapter 4, Section 4.2, isolating the singularity point. Fortunately, in force of Remark 4.2.8, for regular function the definition of the integral term does not need this particular care. (For a detailed discussion we refer to Chapter 4.)

We start by proving that $\overline{u} \in \mathcal{USC}$.

1. $\overline{u}$ is upper semicontinuous: it follows from the definition itself: we want to prove that the following inequality holds:

$$\limsup_{(y,s)\to(x,t)} \overline{u}(y,s) \le \overline{u}(x,t),$$

   By definition:

$$\overline{u}(y,s) = \limsup_{\substack{(\Delta x,\Delta t)\to 0\\ (j\Delta x, n\Delta t)\to(s,y)}} u_j^n,$$

   therefore, by definition of limsup, there exist an $\varepsilon > 0$ and (n,j) such that:

$$\overline{u}(y,s) - \varepsilon \le u_j^n;$$

   now, taking limsup for $(\Delta x, \Delta t) \to 0$ and $(j\Delta x, n\Delta t) \to (x,t)$, we obtain:

$$\overline{u}(y,s) - \varepsilon \le \overline{u}(x,t);$$

   Now, as ε is arbitrarily chosen, we obtain the desired result.

2. $\overline{u}$ is locally bounded: by definition it is obtained by the solutions of the scheme. By hypothesis (8.3.3) they are bounded, independently from $\Delta t, \Delta x$. So let $K \in \mathbb{R}^N \times [0,T]$ be a compact set, then there exists a constant A_K such that:

$$|u_j^n| \le A_K \ \ \forall n,j \text{ s.t. } (j\Delta x, n\Delta t) \in K \ \Rightarrow\ |\overline{u}(x,t)| \le A_K \ \ \forall (x,t) \ \in K;$$

   from this property it easily follows that $\overline{u} \in \mathcal{P}_{e^n}(\mathbb{R}^N)$:

We have proved that $\overline{u} \in \mathcal{USC}$, and now we need only to prove that $\overline{u}$ is a viscosity subsolution. To this aim let (x_0,t_0) be a global strict maximum for $\overline{u}-\phi$ on $\mathbb{R}^N \times [0,T]$ for some $\phi \in \mathcal{C}^{2,1}(\mathbb{R}^N \times [0,T]) \cap \bigcup_{n<n_0} L^\infty(0,T;\mathcal{P}_{e^n}(\mathbb{R}^N))$ such that $\overline{u}(x_0,t_0) = \phi(x_0,t_0)$, then:

$$\overline{u}(x,t) - \phi(x,t) \le 0 = \overline{u}(x_0,t_0) - \phi(x_0,t_0) \text{ in } \mathbb{R}^N \times [0,T].$$

From these hypothesis it follows that there exists a sequence $(\Delta x_k, \Delta t_k) \in \mathbb{R}^{+2}$ and $(y_k, s_k) \in \mathbb{R}^N \times [0,T]$ such that, as $k \to \infty$:

$$(\Delta x_k, \Delta t_k) \to 0, \quad (y_k, s_k) \to (x_0, t_0), \quad u^{(\Delta x_k, \Delta t_k)}(y_k, s_k) \to \overline{u}(x_0, t_0),$$
$$(y_k, s_k) \text{ is a global maximum point of } u^{(\Delta x_k, \Delta t_k)}(\cdot,\cdot) - \phi(\cdot,\cdot). \tag{8.3.7}$$

Denoting by $\xi_k = u^{(\Delta x_k, \Delta t_k)}(y_k, s_k) - \phi(y_k, s_k)$, we have

$$\xi_k \to 0 \text{ and}$$
$$u^{(\Delta x_k, \Delta t_k)}(x,t) \le \phi(x,t) + \xi_k,$$

for all $(x,t) \in \mathbb{R}^N \times [0,T]$. By the definition of $u^{(\Delta x_k, \Delta t_k)}$, the hypotheses (8.3.4) and (8.3.7) we obtain:

$$Q(\Delta x, \Delta t, j_k, n_k, \phi(y_k, s_k) + \xi_k, J_{n_k, j_k}(\tilde{\phi} + \xi_k), \tilde{\phi} + \xi_k) \le 0.$$

Now, taking limits in the previous inequality, using the consistency of the scheme, we obtain:

$$\begin{aligned} 0 &\ge \liminf_{k\to\infty} \frac{Q(\Delta x_k, \Delta t_k, j_k, n_k, \phi(y_k, s_k) + \xi_k, J_{n_k, j_k}(\phi + \xi_k), \tilde{\phi} + \xi_k)}{\rho_k(\Delta x, \Delta t)} \\ &\ge \liminf_{\substack{(\Delta x, \Delta t)\to 0 \\ (j\Delta x, n\Delta t)\to(x,t) \\ \xi\to 0}} \frac{Q(\Delta x, \Delta t, j, n, \phi(y,s) + \xi, J(\tilde{\phi} + \xi), \tilde{\phi} + \xi)}{\rho(\Delta x, \Delta t)} \\ &\ge \partial_t \phi + H(x, t, \phi, \mathcal{D}\phi, \mathcal{D}^2\phi, J\phi), \end{aligned}$$

which is the desired result, because of the assumption $\overline{u}(x_0, t_0) = \phi(x_0, t_0)$. ∎

8.4 The numerical approximation of the integral term

According to the classical theory of approximated integration, see for instance [38], we use the compound Newton-Cotes formulas to approximate the integral term on the interval $[a, b]$:

$$(R_S)(f) = \frac{b-a}{2S} \sum_{s=0}^{S-1} \sum_{i=1}^{\rho} \alpha_i f(x_{is}) \approx \int_a^b f(x)dx, \tag{8.4.1}$$

where S is the number of subinterval in which we have divided $[a, b]$, $a = y_0 < y_1 < \dots < y_S = b$, ρ is the number of point in each subinterval $[y_s, y_{s+1}]$ and

$$x_{is} = y_s + \frac{b-a}{2S}(1 + t_i), \quad s = 0, \dots, S-1.$$

The errors which occur in approximate integration formulas are conventionally expressed in terms of the higher derivatives of the integrand function f and they are valid only if the integrand is sufficiently smooth. It is a feature of the Newton-Cotes formulas that, if the number of point is $2k-1$ or $2k$, the error is of the form $\mathcal{E}_R(f) = ch^{2k+1} f^{2k}(\xi)$, for $a < \xi < b$.
Moreover we point out that we are using compound Newton–Coates formula because when ρ is large, the coefficients α_i are large and of mixed sign. For our purpose we require the coefficients α_i to be positive, therefore it is not suitable to choose ρ bigger than 7. We refer to Chapter 7, Section 7.3

To obtain the error estimates when the integrand function f has a low-order continuity, we can approximate the function f by a suitable polynomial, as we have already done in Chapter 7, Proposition 7.3.2, according to the following standard result, see for instance [38].

8.4.1 The numerical approximation in the one dimensional case.

Here we want to consider the integro-differential equation $(8.1.1)_1$ in one dimension. We suppose H to be linear in the integral part, which is the case of derivative pricing in small investor economy

$$\partial_t u + H(x,t,u,\mathcal{D}u,\mathcal{D}^2 u) - Ju = 0.$$

We can split the integral operator in two terms:

$$Ju(x,t) = J^{in}u(x,t) + J^{out}u(x,t),$$

where

$$J^{in}u(x,t) = \int_{|z|<1} \Big[u(x+z,t) - u(x,t) - z u_x(x,t) \Big] \Gamma_\delta(z) dz,$$
$$J^{out}u(x,t) = \int_{|z|\geq 1} \Big[u(x+z,t) - u(x,t) \Big] \Gamma_\delta(z) dz.$$

To construct a good approximation for this problem we have to deal with two difficulties, one being completely new. The singularity of the measure requires a truncation of the domain in a neighborhood of the origin; moreover we need to apply a truncation of the unbounded integration domain, as it has already been done in Chapter 7.

Let $0 < \theta \ll 1$ and define a domain D_θ such that

$$\Big| \int_{|z|<1} |z|^2 \nu(dz) - \int_{D_\theta} |z|^2 \nu(dz) \Big| < O(\theta).$$

Assume that $u(\cdot,t) \in W^{2,\infty}((-1,1))$, then the error we incur in this truncation is

$$\Big|\int_{\{|z|<1\}-D_\theta}\Big[u(x+z,t)-u(x,t)-zu_x(x,t)\Big]\nu(dz)\Big]\nu(dz)\Big|$$
$$\leq ||u(\cdot,t)||_{W^{2,\infty}((-1,1))}\int_{\{|z|<1\}-D_\theta}|z|^2\nu(dz) = ||u(\cdot,t)||_{W^{2,\infty}((-1,1))}O(\theta).$$

Let us now focus on the second truncation difficulty, that is concerned with the unboundedness of the integration domain; let us fix a parameter $0 < \varepsilon \ll 1$ and a domain $D_\varepsilon = [z_m(\varepsilon), -1] \cup [1, z_M(\varepsilon)]$ such that

$$\Big|\int_{|z|\geq 1}\nu(dz) - \int_{D_\varepsilon}\nu(dz)\Big| \leq \varepsilon.$$

The error we incur in this truncation can be estimated in term of the norm of u. Let us suppose that $u \in L^\infty(\mathbb{R}\times[0,T])$, then

$$\Big|\int_{\{|z|\geq 1\}}\Big[u(x+z,t)-u(x,t)\Big]\nu(dz) - \int_{D_\varepsilon}\Big[u(x+z,t)-u(x,t)\Big]\nu(dz)\Big|$$
$$= \Big|\int_{\{|z|<z_m(\varepsilon)\}\cup\{|z|>z_M(\varepsilon)\}}\Big[u(x+z,t)-u(x,t)\Big]\nu(dz)\Big|$$
$$\leq 2U\int_{\{|z|<z_m(\varepsilon)\}\cup\{|z|>z_M(\varepsilon)\}}\nu(dz) = 2U\varepsilon.$$

8.4.2 The case of the CGMY distribution

In the previous estimates we have assumed $u(\cdot,t) \in L^\infty(\mathbb{R}) \cap W^{2,\infty}((-1,1))$, but it is possible to assume even more regularity because of the results in Theorem 4.2.14 and Corollary 4.2.15; under particular choices of the Lévy measure it is possible to give explicit expression of the domains D_θ and D_ε and more precise estimates on the truncation errors.

Let us assume the Lévy measure expressed in terms of a CGMY symmetric density, as we have explained in Chapter 2, Equation (2.3.1), see [31], $\nu^j(dz) = K_{CGMY}(z)dz$, where

$$K_{CGMY}(z) = \begin{cases} C\dfrac{e^{-G|z|}}{|z|^{1+Y}}, & z_i < 0,\ i = 1,2, \\ \\ C\dfrac{e^{-M|z|}}{|z|^{1+Y}}, & z_i > 0,\ i = 1,2, \end{cases}$$

with $C > 0$, $G, M \geq 0$ and $Y < 2$. We start considering the case where $E = \mathbb{R} - \{0\}$ and a symmetric density:

$$\nu(dz) = \Gamma_\delta(z)dz,$$

with

$$\Gamma_\delta(z) = \frac{\exp\left(-\dfrac{z}{\delta}\right)}{z^2}.$$

Let $0 < \theta \ll 1$ and $D_\theta = \{z \in \mathbb{R} \text{ s.t. } \theta < |z| < 1\}$. Performing the calculation we obtain

$$\Big|\int_{0\le|z|\le\theta} \exp\Big(-\frac{|z|}{\delta}\Big)dz\Big| = 2\int_0^\theta \exp\Big(-\frac{z}{\delta}\Big)dz = 2\delta\Big(1-\exp\Big(-\frac{\theta}{\delta}\Big)\Big) = O(\theta).$$

Define

$$J^{in,\theta}u(x,t) := \int_{\theta<|z|<1} \Big[u(x+z,t) - u(x,t) - zu_x(x,t)\Big]\Gamma_\delta(z)dz, \tag{8.4.2}$$

and

$$J^{in}_\theta u(x,t) := \int_{|z|\le\theta} \Big[u(x+z,t) - u(x,t) - zu_x(x,t)\Big]\Gamma_\delta(z)dz.$$

For all $\phi \in \mathcal{C}^2(\mathbb{R}\times[0,T])$, then the error we incur using this truncation is given by

$$\begin{aligned}&|J^{in}\phi(x,t) - J^{in,\theta}\phi(x,t)| = |J^{in}_\theta\phi(x,t)|\\ &\le ||\phi||_{\mathcal{C}^2}\int_{|z|\le\theta}\frac{z^2}{2}\Gamma_\delta(z)dz \le ||\phi||_{\mathcal{C}^2}\delta\Big(1-e^{-\frac{\theta}{\delta}}\Big).\end{aligned}$$

We construct an approximation of this integral term using the Newton–Coates formula for the integral and a finite difference for the integrand; the approximated integral becomes

$$\begin{aligned}J^{in,\theta}_h u(x,t) &= R_{S^{in}}\Big[\Big(u(x+\cdot,t) - u(x,t) - \cdot\, u_x(x,t)\Big)\Gamma_\delta(\cdot)\Big]\\ &= \frac{1-\theta}{2S^{in,-}}\sum_{s=0}^{S^{in,-}}\sum_{i=1}^{\rho^{in,-}}\alpha_i^{in,-}\Big[u(x+z_{is},t) - u(x,t) - z_{is}u_x(x,t)\Big]\Gamma_\delta(z_{is})\\ &\quad+\frac{1-\theta}{2S^{in,+}}\sum_{s=0}^{S^{in,+}}\sum_{i=1}^{\rho^{in,+}}\alpha_i^{in,+}\Big[u(x+z_{is},t) - u(x,t) - z_{is}u_x(x,t)\Big]\Gamma_\delta(z_{is}).\end{aligned}$$

We can note that if we choose a symmetric approximation formula we have

$$\begin{aligned}&\Big[\frac{1-\theta}{2S^{in,-}}\sum_{s=0}^{S^{in,-}}\sum_{i=1}^{\rho^{in,-}}\alpha_i^{in,-} + \frac{1-\theta}{2S^{in,+}}\sum_{s=0}^{S^{in,+}}\sum_{i=1}^{\rho^{in,+}}\alpha_i^{in,+}\Big]z_{is}u_x(x,t)\Gamma_\delta(z_{is})\\ &= R_{S^{in}}\Big(z\Gamma_\delta(z)\Big)u_x(x,t) = 0,\end{aligned}$$

therefore the previous formula can be read in a simplified way:

$$J^{in,\theta}_h u(x,t) = R_{S^{in}}\Big[\Big(u(x+\cdot,t) - u(x,t) - \cdot u_x(x,t)\Big)\Gamma_\delta(\cdot)\Big] \tag{8.4.3}$$

$$= \frac{1-\theta}{2S^{in,-}} \sum_{s=0}^{S^{in,-}} \sum_{i=1}^{\rho^{in,-}} \alpha_\rho^{in,-} \Big[u(x+z_{is},t) - u(x,t) \Big] \Gamma_\delta(z_{is})$$
$$+ \frac{1-\theta}{2S^{in,+}} \sum_{s=0}^{S^{in,+}} \sum_{i=1}^{\rho^{in,+}} \alpha_\rho^{in,+} \Big[u(x+z_{is},t) - u(x,t) \Big] \Gamma_\delta(z_{is}).$$

Then for any $\phi \in \mathcal{C}^2(\mathbb{R} \times [0,T])$ we can write

$$J^{in}\phi(x,t) = J_h^{in,\theta}\phi(x,t) + O(h^l) + O(\theta), \qquad l \geq 2.$$

We can now focus our attention in the truncation of the unbounded domain. Let us fix a parameter $0 < \varepsilon \ll 1$ and a domain D_ε such that

$$\Big| \int_{|z|\geq 1} \Gamma_\delta(z)dz - \int_{D_\varepsilon} \Gamma_\delta(z)dz \Big| \leq \varepsilon;$$

in this case the points $z_M(\varepsilon)$ and $z_m(\varepsilon)$ can be given explicitly. As $\Gamma_\delta(z)$ has been chosen as a symmetric density, we have $D_\varepsilon = \{z \in \mathbb{R} \text{ s.t } 1 \leq |z| \leq z_M\}$ defined by the following relation: for all $z \in D_\varepsilon$

$$\exp\Big(- \frac{|z|}{\delta} \Big) \geq \varepsilon;$$

in this case a simple calculation gives

$$z_M = -z_m = -\delta \log(\varepsilon) \stackrel{\varepsilon \to 0}{\longrightarrow} +\infty.$$

Define

$$J_\varepsilon^{out} u(x,t) = \int_{D_\varepsilon} \Big[u(x+z,t) - u(x,t) \Big] \Gamma_\delta(z)dz. \tag{8.4.4}$$

Under these assumptions we can give an estimate of the error in which we incur if we suppose that $u(\cdot,t) \in Lip(\mathbb{R})$ with constant L:

$$\Big| J^{out}u(x,t) - J_\varepsilon^{out}u(x,t) \Big| = \Big| \int_{|z|>z_M} \Big[u(x+z,t) - u(x,t) \Big] \Gamma_\delta(z)dz \Big|$$
$$\leq \text{L} \int_{|z|>z_M} |z|\Gamma_\delta(z)dz \leq 2\text{L} \int_{z_M}^{\infty} \exp\Big(- \frac{z}{\delta} \Big) = 2\text{L}\delta \exp\Big(- \frac{z_M}{\delta} \Big) = 2\text{L}\delta\varepsilon.$$

A realistic assumption for functions u in financial application is they have an exponential rate of growth; if $u \in \mathcal{P}_{e^n}(\mathbb{R})$,

$$u(x+z,t) - u(x,t) \sim e^x(e^z - 1),$$

and

$$\Big| \int_{|z|\geq z_M} [u(x+z,t) - u(x,t)]\Gamma_\delta(z)dz \Big| \leq e^x \int_{|z|\geq z_M} |e^z - 1|\Gamma_\delta(z)dz$$

$$\leq \frac{2e^x}{\delta}(1-\delta)\exp\Big(\frac{-z_M(1-\delta)}{\delta}\Big) = \frac{2e^x}{\delta}(1-\delta)\varepsilon^{1-\delta}.$$

At this stage we can approximate the integral operator (8.4.4) using the numerical integration formulas of Section 8.4:

$$\begin{aligned} J_h u(x,t) &= R_{S^{out}}\Big[(u(x+\cdot)-u(x))\Gamma_\delta(\cdot)\Big] \qquad (8.4.5)\\ &= \frac{z_m-1}{2S^{out,-}}\sum_{s=0}^{S^{out,-}-1}\sum_{i=0}^{\rho^{out,-}}\alpha_i^{out,-}\Big[u(x+z_{is},t)-u(x,t)\Big]\Gamma_\delta(z_{is})\\ &\quad+\frac{z_M-1}{2S^{out,+}}\sum_{s=0}^{S^{out,+}-1}\sum_{i=0}^{\rho^{out,+}}\alpha_i^{out,+}\Big[u(x+z_{is},t)-u(x,t)\Big]\Gamma_\delta(z_{is}); \end{aligned}$$

Since the functions $g_1(z) = \Big(u(x+z,t)-u(x,t)-zu_x(x,t)\Big)\Gamma_\delta(z)$ and $g_2(z) = \Big(u(x+z,t)-u(x,t)\Big)\Gamma_\delta(z)$ have a low-order continuity, to get an error estimate for the approximations (8.4.3) and (8.4.5) we apply Proposition 7.3.2. Recall that for a generic function f we have

$$\mathcal{E}_R(f) = \int_a^b f(x)dx - R(f) = \int_a^b (f(x)-p_n(x))dx + \int_a^b p_n(x)dx - R(f)$$
$$= \int_a^b (f(x)-p_n(x))dx + R(p_n - f).$$

Then

$$\mid \mathcal{E}_R(f) \mid \leq \Big((b-a)+\sum_{i=0}^{p}\mid \alpha_i \mid\Big) \mid f(x)-p_n(x) \mid .$$

If $\alpha_i > 0$, using formula (8.4.1), we obtain

$$\mid \mathcal{E}_R(f) \mid \leq 2(b-a) \mid f(x)-p_n(x) \mid .$$

An R_S compound rule, applied to our functions $f = g_1$, yields

$$\mathcal{E}_{R_{S^{out}}}(g_1) = \sum_{s=0}^{S^{out}-1}\mathcal{E}_R(g_1) = \sum_{s=0}^{S^{out}-1} 2\Big(\frac{b-a}{S^{out}}\Big) \mid g_1(x_s)-p_n(x_s) \mid .$$

Then there exists a polynomial of degree $\leq S^{out}\rho^{out}$, $p_{S^{out}\rho^{out}}(z)$, such that

$$\mid g_1(z) - p_{S^{out}\rho^{out}}(z) \mid \leq 2\omega_{g_1}\Big(\frac{z_M-1}{S^{out}\rho^{out}}\Big).$$

There follows

$$\Big| \int_{1\leq|z|\leq z_M} g_1(z)dz - (R_{S^{out}})(g_1(z))\Big|$$

$$
\begin{aligned}
&\leq \int_{1\leq|z|\leq z_M} \Big| g_1(z) - p_{S^{out}\rho^{out}}(z) \Big| dz + \Big| (R_{S^{out}})(p_{S^{out}\rho^{out}}(z) - g_1(z)) \Big| \\
&\leq \Big(2(z_M - 1) + \frac{z_M - 1}{S^{out}} \sum_{s=0}^{S^{out}-1} \sum_{i=0}^{\rho^{out}} \mid \alpha_i \mid \Big) 2\omega_{g_1}\Big(\frac{z_M - 1}{S^{out}\rho^{out}}\Big) \\
&\leq 2(z_M - 1)\Big(1 + \sum_{i=0}^{\rho^{out}} \mid \alpha_i \mid \Big)\omega_{g_1}(h),
\end{aligned}
$$

where ω_{g_1} is the modulus of continuity for g_1.

An R_S compound rule, applied to our function $f = g_2$, yields

$$
\mathcal{E}_{R_{S^{in}}}(g_2) = \sum_{s=0}^{S^{in}-1} \mathcal{E}_R(g_2) = \sum_{s=0}^{S^{in}-1} 2\Big(\frac{b-a}{S^{in}}\Big) \mid g_2(x_s) - p_n(x_s) \mid .
$$

Then there exists a polynomial of degree $\leq S^{in}\rho^{in}$, $p_{S^{in}\rho^{in}}(z)$, such that

$$
\mid g_2(z) - p_{S^{in}\rho^{in}}(z) \mid \leq 2\omega_{g_2}\Big(\frac{1-\theta}{S^{in}\rho^{in}}\Big).
$$

There follows

$$
\begin{aligned}
&\Big| \int_{\theta<|z|<1} g_2(z)dz - (R_{S^{in}})(g_2(z)) \Big| \\
&\leq \int_{\theta<|z|<1} \Big| g_2(z) - p_{S^{in}\rho^{in}}(z) \Big| dz + \Big| (R_{S^{in}})(p_{S^{in}\rho^{in}}(z) - g_2(z)) \Big| \\
&\leq \Big(2(1-\theta) + \frac{1-\theta}{S^{in}} \sum_{s=0}^{S^{in}-1} \sum_{i=0}^{\rho^{in}} \mid \alpha_i \mid \Big) 2\omega_{g_2}\Big(\frac{1-\theta}{S^{in}\rho^{in}}\Big) \\
&\leq 2(1-\theta)\Big(1 + \sum_{i=0}^{\rho^{in}} \mid \alpha_i \mid \Big)\omega_{g_2}(h),
\end{aligned}
$$

where ω_{g_2} is the modulus of continuity for g_2.

As previously, we choose a formula such that $\alpha_i^{in(out)} > 0$ for all i.

To give an exhaustive description of the approximation scheme, we point out that for all $\phi \in \mathcal{C}^2(\mathbb{R} \times [0,T])$

$$
J\phi(x,t) = J^{in,\theta}\phi(x,t) + J_{\varepsilon}^{out}\phi(x,t) + O(\theta) + O(\varepsilon),
$$

therefore we apply a compound rule to the integral term and a standard explicit difference scheme for the differential part, as it has been done in Chapter 7.

8.4.3 Check of the hypotheses

Following the line of the paper by Barles and Souganidis [16], summarized in Chapter 6, and the result exposed in Chapter 7 we approximate at first the differential part with a scheme $\tilde{Q}$ that verifies the differential condition of convergence (**S.1**)–(**S.3**) of Chapter 6. Moreover we assume that the integral approximation formula is symmetric and that the space discretization grid of the numerical operator $\tilde{Q}$ coincides with the ones of the two integral approximation, that is we set the common space step h such that

$$h \leq \min\left\{\frac{1-\theta}{\rho^{in}S^{in}}, \frac{z_M - 1}{\rho^{out}S^{out}}\right\}.$$

Then the approximation of the integro–differential equation (8.5.1) reads

$$Q(h,k,j,n,u_j^n,J_h\tilde{u},\tilde{u}) = \tilde{Q}(h,k,j,n,u_j^n,\tilde{u}) - J_h\tilde{u} = 0,$$

where $J_h\tilde{u}$ is the approximation of the integral term:

$$\begin{aligned} J_h\tilde{u} &= \sum_{p\in P^{in}} \alpha_p^{in}\Big[u_{j+p}^n - u_j^n\Big](\Gamma_\delta)_p + \sum_{p\in P^{out}} \alpha_p^{out}\Big[u_{j+p}^n - u_j^n\Big](\Gamma_\delta)_p \\ &= \sum_{p\in P} \alpha_p\Big[u_{j+p}^n - u_j^n\Big](\Gamma_\delta)_p \end{aligned}$$

where $P = P^{in} \cup P^{out}$.

1. *Monotonicity of the approximated integral.*
   To get the monotonicity of the integral approximation it is sufficient to choose a Newton–Coates formula such that the weights α_p are greater than zero for all p. Clearly, if $\tilde{u} \leq \tilde{v}$ and $u_j^n = v_j^n$ then we have
   $$\sum_{p\in P} \alpha_p\Big[u_{j+p}^n - u_j^n\Big](\Gamma_\delta)_p \leq \sum_{p\in P} \alpha_p\Big[v_{j+p}^n - v_j^n\Big](\Gamma_\delta)_p,$$
   for all $j \in \mathbb{Z}$ and $n \in \mathbb{N}$.

2. *Stability*
   It is a trivial consequence of the $\tilde{Q}$ stability and the monotonicity of the approximation of the integral term.

3. *Consistency*
   Let $\phi \in \mathcal{C}^\infty(\mathbb{R} \times [0,T])$; from the consistency condition on $\tilde{Q}$ we get the following:
   $$\liminf_{\substack{(h,k)\to 0\\ (jh,nk)\to(x,t)\\ \xi\to 0}} \frac{\tilde{Q}(h,k,j,n,\phi_j^n + \xi, \tilde{\phi} + \xi) - J_h(\tilde{\phi} + \xi)}{\rho(h,k)}$$

$$\geq \partial_t u + H(x,t,u,\mathcal{D}u,\mathcal{D}^2u) - \liminf_{\substack{(h,k)\to 0\\(jh,nk)\to(x,t)\\ \xi\to 0}} \frac{J_h(\tilde{\phi}+\xi)}{\rho(h,k)}.$$

Performing the calculation for $\phi \in \mathcal{C}^2(\mathbb{R}\times(0,T))$, writing $\phi^n_{j\pm 1}$ in terms of its first and second order derivative, using the Taylor expansion, we get

$$J_h(\tilde{\phi}+\xi) = \sum_{p\in P} \alpha_p[\phi^n_{j+p} - \phi^n_j](\Gamma_\delta)_p$$
$$= J^{in,\theta}(\phi+\xi) + J^{out}_\varepsilon(\phi+\xi) + \mathcal{E}_{R_S}(\phi+\xi) + O(h^l) + O(\varepsilon) + O(\theta), \quad l \geq 2,$$

therefore

$$\liminf_{\substack{(h,k)\to 0\\(jh,nk)\to(x,t)\\ \xi\to 0}} \frac{J_h(\tilde{\phi}+\xi)}{\rho(h,k)}$$
$$= J\phi - \lim_{\substack{(h,k)\to 0\\(jh,nk)\to(x,t)\\ \xi\to 0}} \frac{\mathcal{E}_{R_S}(\phi+\xi) + O(h^2) + O(\varepsilon) + O(\theta)}{\rho(h,k)} = J\phi,$$

as ε and θ are arbitrary chosen.

4. *Monotonicity*
It is a trivial consequence of the $\tilde{Q}$ monotonicity and the monotonicity of the integral approximation (point 1).

8.5 Finite difference methods for the one dimensional Lévy model

In this section we introduce an explicit approximation for the linear IPDE arising from the Lévy models in financial markets. We remember that lot of work has been done for the pure diffusion problem of the Black and Scholes type; for what concerns IPDE coming from bounded Lévy models, we refer to Chapter 7 for a detailed discussion and for some numerical tests, see also [30].

Let us consider the following constant coefficient Cauchy problem

$$u_t - \frac{1}{2}\sigma^2 u_{xx} - (\mu + \frac{1}{2}\sigma^2)u_x + ru - Ju = 0, \tag{8.5.1}$$

$$u(x,T) = \psi(x);$$

As we have previously described, we apply a compound rule to the truncated integral terms (8.4.4) and (8.4.2), while we use a standard explicit finite–difference scheme for the differential part. Then an approximation scheme can be read as

$$\begin{aligned} Q(h,k,j,n,u_j^n,J_h\tilde{u},\tilde{u}) = \; & \frac{u_j^{n+1}-u_j^n}{k} - (b-c_{exp})\frac{u_{j+1}^n - u_{j-1}^n}{2h} \\ & -\Big(\frac{q}{2k}+\frac{\sigma^2}{2h^2}\Big)\Big(u_{j+1}^n - 2u_j^n + u_{j-1}^n\Big) + ru_j^n \\ & -\sum_{p\in P}\alpha_p\Big[u_{j+p}^n - u_j^n\Big](\Gamma_\delta)_p. \end{aligned} \tag{8.5.2}$$

Proposition 8.5.1 *The scheme* (8.5.2) *is accurate to order* $O\Big(h^2+\frac{qh^2}{2k}\Big)$ *under the CFL stability condition*

$$\Big|\mu+\frac{1}{2}\sigma^2\Big|\frac{k}{h} \le \sigma^2\frac{k}{h^2}+q \le 1-k\Big(r+\sum_{p\in P}\alpha_p(\Gamma_\delta)_p\Big). \tag{8.5.3}$$

Proof. The CFL condition can be easily derived by the monotonicity condition. To find out the order of accuracy, we perform the study of the symbol of the discretized operator [110]. Let $p(s,\xi)$ the symbol of the integro–differential operator (8.5.1):

$$p(s,\xi) = s+\frac{1}{2}\sigma^2\xi^2 - \Big(\mu+\frac{1}{2}\sigma^2\Big)i\xi + r - \int_E\Big[e^{iz\xi}-1-iz\xi\mathbf{1}_{|z|<1}\Big]\Gamma_\delta(z)dz.$$

If $u_j^n = e^{skn}e^{ijh\xi}$, we can derive the symbol of the discretized scheme (8.5.2):

$$\begin{aligned} p_{k,h}(s,\xi) \;=\; & \frac{e^{sk}-1}{k} - i(b-c_{exp})\frac{\sin(h\xi)}{h} - \Big(\frac{qh^2}{k}+\sigma^2\Big)\frac{cos(h\xi)-1}{h^2} + r \\ & -\sum_{p\in P}\alpha_p[e^{iph\xi}-1](\Gamma_\delta)_p. \end{aligned}$$

To perform the calculation we remember the choice of the approximation of the integral terms (8.4.2) and (8.4.4) and that it holds the relation (8.2.3).

From the Newton–Coates formula

$$\sum_{p\in P}\alpha_p[e^{iph\xi}-1](\Gamma_\delta)_p = \int_{\theta<|z|<z_M}(e^{iz\xi}-1)\Gamma_\delta(z)dz + O(h^l), \qquad l\ge 2;$$

therefore, using the Taylor expansion in $p_{k,h}$ and the previous relation we get

$$p_{k,h}(s,\xi) \;=\; s - i(b-c_{exp})\xi - \Big(\frac{qh^2}{k}+\sigma^2\Big)\frac{\xi^2}{2} + r$$

$$-\sum_{p\in P}\alpha_p[e^{jph\xi}-1](\Gamma_\delta)_p+O(k+h^2).$$

We now look for a symbol $r_{k,h}(s,\xi)$ such that

$$r_{k,h}(s,\xi)p(s,\xi)-p_{k,h}(s,\xi)$$

gives the order of accuracy; we have that $r_{k,h}(s,\xi)=1+o(1)$ and

$$p(s,\xi)-p_{k,h}(s,\xi)=\frac{qh^2}{2k}\xi^2+O(\theta+\varepsilon)+O(k+h^2),$$

where we have used the relation (8.2.3); from the CFL condition (8.5.3) we get $k=O(h^2)$; moreover ε and θ are arbitrary, then we chose them as $O(h^2)$; therefore the scheme is accurate to order

$$O\Big(h^2+\frac{qh^2}{2k}\xi^2\Big).$$

■

8.6 Conclusions.

In this chapter we have proved that the general convergence result of Chapter 7 can be extended also to general unbounded Lévy measure satisfying assumption (8.1.3). A particular care is due to the singularity point: here we have proposed a way to deal with the unboundedness of the measure and how to couple this result with the one obtained in Chapter 7 for the unboundedness of the domain of integration.
An important point in the numerical simulation is concerned with the boundary: the nonlocal nature of the integral term is such that it couples grid points which are "inside" and "outside" the computational domain.
In Section 7.4 this problem was studied for bounded Lévy measures, where the integral term has been compared to a diffusion.
The technique proposed in that section could be extended also in this case, with particular care to the singularity points: the particular form of the integrand function suggests to replace the integral operator with an effective diffusion terms. Performing similar calculation to the one of Section 7.5.2 it is possible to obtain an estimate of the committed error.

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