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An analytical and computational study of a model for the actin based movement of bacteria

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Introduction

In recent years the interest in mathematical and computational study of biological phenomena is rapidly increased. Models for this kind of processes are spreading in many branches of biology like oncology [5], epidemiology [15], natural pattern formation [22], ecology [1], and morphogenesis [23]. There is a wide literature about these subjects; in particular, for an extensive overview we sugggest the reading of the articles [2], [10], [14] and the references therein.

This rising interest is mainly due to two reasons. On one hand, mathematics provides a powerful tool in the description of biological processes. In fact, these phenomena generally involve the spatio-temporal evolution of one or more physical quantities, like for example, the density of a microbial population or the concentration of a substance. So, partial differential equations seem to be a good instrument to describe them. On the other hand, a good mathematical model could be used to predict the evolution of the biological phenomenon it refers to, and it could also predict how the system is going to react to a change in environmental conditions, like a change in the concentration of a chemical agent or of a microbial nutrient.

The following example in epidemiology field can better clarify this concept; let's suppose that a certain disease is transmitted by direct physical contact. A good mathematical model for the spread of that infection could predict how the disease will diffuse. But it could also predict how the diffusion would be affected by the closure of the school, or the isolation of infected people. So it could suggest how to behave in order to limit the spread of the infection.

In these years some efforts in biomathematical modelling are directed to the understanding of cell and microorganism motility. This problem can be studied from two different points of view. One is referred to the direction of the motion; the other to how this motion happens, from which forces it is generated and which parts of the cells are involved.

Cell and microorganism movement direction is essentially controlled by the chemotaxis process; that is, cells and microorganisms direct their movement according to the presence and the concentration of certain chemical agents in their environment. These processes are generally described by evolution and transport equations. In particular we suggest to the reader the models introduced in [2] [14], [22] and [23].

On the other hand, the models for the description of cell motility are still at the beginning. From the observation, it is known that cell motion is divided into three different processes: *protrusion*, *adhesion* and *contraction*. More in detail, first the cell pulls out the front, then it adheres at the surface tightly by the leading edge and weakly by the rear one and finally it develops a contraction that pulls up the rear, completing the motile cycle (for further information see [20]). In particular, cell motion is mostly due to the use of the so called *lamellipodia* as motile appendages. *Lamellipodia* are structures very similar to the actin tails that bacteria like *Lysteria Monicytogenes*, *Shigella Flexenari* and *Rickettsia Rickettsii* use for their motion inside host cells. The characterization of the movement of these microorganisms is easier than the one of the cells; for this reason, modelling the motion of these organisms can be a step in the modelling of the protrusion at the leading edge.

Actin tail is composed by a large number of crosslinked actin filaments. As every protein the actin can polymerize and the process of polymerization drives protrusive forces generation in the motion. The mechanism by which the movement occurs is still not well understood, see [10]. In literature there are two different ways of modelling the actin based movement. The first approach, see for example [10], considers all the actin filaments attached to the surface of the bacterium or of the cell and it assumes that the polymerization process produces a compression in the filaments. So, there are some filaments under compression and some other under tension. The opposition between these two classes of filaments makes the bacterium moving.

The other approach is to suppose that after a short period, the filaments detached from the bacterium and their place is taken from new ones. The detachment of these filaments causes a compression in the tail which pushes the bacterium. In this kind of model, (see for example the ones in [21] and [12]), the motion is due to the opposition between the force of pushing filaments and a frictional force resulting from the hydrodynamic drag and/or the resisting force that is necessary to break the link between the filaments and the surface. Models obtained by this approach are the so called *Brownian and tethered ratched* models.

In this work we have studied a 1-dimensional model of Brownian and tethered ratched type from an analytical and a numerical point of view. This model describes the spatio-temporal evoution of two physical quantities of the actin tail in bacteria movement. In particular we have studied the evolution of the density and of the number of filaments per unit volume of the tail, which we denote respectively by ρ and u. The model is obtained from the one introduced by B.Bazaliy, Y.Bazaliy and A.Friedman in [3]. It is a system of two partial differential equations with moving boundaries. The equation for the evolution of u is of porous media type, while the one for ρ is of linear transport type with the flux velocity and the source that depend both from spatial and time variables.

In the first section of the present thesis we describe the model introduced in [3] and a generalization we have introduced in order to better represent some physical properties of the actin tail. More in detail, in [3], the filament length decreases with a constant rate as the distance between the rear side of the bacterium and the filament increases. In this way the filament elongation due to the polymerization process is not taken into account. In order to include this phenomenon, we assume that the variation of the filaments length is controlled by a C^2 not increasing function, defined in equation (1.20), that is positive in a small interval near the bacterium and negative far from it.

The second section is devoted to the proof of the existence of a travelling wave solution for the problem. From observation, the length of the tail of a bacterium increases for a short time after the infection of a cell and then it becomes constant. From a mathematical point of view this means that the physical quantities of the tail reach a steady state, and a travelling wave solution actually describes a steady state for our problem. Our result, summarized in Theorem 1, is an improvement of that obtained in [3] with variational arguments. In fact, it provides some estimates on upper and lower bounds for the parameters of the problem in order to guarantee the existence of such a special solution.

In the third section we prove a local existence and uniqueness theorem for the model. In order to prove the theorem we have transformed the moving domain in a fixed one. In the new domain the equation for the evolution of u is transformed from a porous media to a nonlinear degenerate parabolic equation with coefficients depending on the spatial and the temporal variables. We show that, if the initial data verify some conditions, in particular hypoteses (3.2)-(3.10) of Theorem 3, then there exists a solution for the problem and it is unique. Moreover, we show that $u \in C^{2,1}$ and that ρ is a continuous function.

As for every parabolic degenerate equation, the global existence of the solution is harder to prove than the local one. In fact, for this proof an estimate of the coefficients of the equations for all time $t > 0$ is needed and this estimate cannot be given, due to the nature of the problem.

Finally, in the fourth section, we describe and discuss two numerical schemes that we have also implemented in C++ programs. The first one is for the numerical characterization of the travelling wave solution, which is not known in explicit form, the second one for the approximation of the solution of the general problem. In this case, in order to verify the efficiency of the method, we set the travelling wave solution as the initial datum of the problem and we use the numerical scheme to find the corresponding approximated solution. Then we compare it and the approximated travelling wave solution at every time $t > 0$. We obtain that in a finite time the approximated solution becomes a travelling wave.

This result can be interpreted as a stability property of the travelling wave. In fact, starting from a perturbed travelling wave solution, the system reaches in a finite time the steady state configuration of the non perturbed travelling wave.

In conclusion, we have proposed a generalization of the model for the actin based movement introduced in [3]. We have shown that, under suitable choices of the initial data and of the parameters, it has a unique solution and in particular we have determined an implicit expression for a travelling wave solution of the problem. Moreover, we have implemented a numerical scheme for the approximation of the solution of the system and we have proved its efficiency using the travelling wave solution as initial datum for the model.

1 The model

Listeria monocytogenes, *Shigella Flexenari* and *Rickettsia Rickettsii*, are particular bacteria that may cause serious diseases, such as meningitis, typhus and Rocky Mountain fever. Their virulence is strictly connected to the high speed of their movement inside a single host cell and their ability to spread out, infecting many other cells. Unlike many bacteria, they don't move using flagella, but they exploit a cytosol protein: the actin.

As a protein, actin can polymerize, and as a consequence of this process, several actin monomers aggregate in a chain. In particular, actin polymers look like gelatinous and elastic filaments. Their peculiarity is their polar structure; monomers, in fact, can only attach to one end of a filament while they can only detach from the other. Polymerization and depolymerization processes are the cause of the movement of *Listeria moncytogenes*, *Shigella Flexenari* and *Rickettsia Rickettsii*.

In fact, on the outer membranes of these bacteria, there is an enzyme, which attracts actin causing its polymerization. Polymerization ends of actin filaments tie at the bacterium enzyme site. Monomers addiction compresses the filament, until it leaves the bacterium. This makes the bacterium and the filament move in opposite directions. New filaments now tie to the bacterium while the detached filament completely depolymerizes after several chemical reactions until it vanishes. During these processes, detached actin monomers are free to polymerize again, creating new filaments.

A 1-dimensional model for the spatio-temporal evolution of two physical quantities of the actin tail. In particular, the model is a system of two partial differential equations with two moving boundaries, that describes the evolution of the actin density and of the numerical filaments density of the bacterium tail.

The model is a generalization of the one introduced by B.Bazaliy, Y. Bazaliy and A.Friedman in [3]. So, we first describe their model, then we will explain our changes.

1.1 The original model

In [3] it is assumed that at any point of the tail, the physical quantities like density, number density, velocity, filaments length and so on, depend only on the distance of that point from the bacterium. Let x denote the spatial variable and t the temporal one. We assume that the motion happens in the x-axis negative direction. So, since the tail and the bacterium movement directions are opposite, the actin tail moves in the direction of increasing x. Moreover we assume that at time $t = 0$, the bacterium position is $x = 0$.

In [3], the following variables are introduced:

- $w(x, t)$ velocity of the tail;
- $u(x, t)$ numerical filaments density of the tail;
- $l_f(x, t)$ length of filaments;
- $\rho(x, t)$ actin density of the tail, and

$$\rho(x, t) = C_\rho l_f(x, t) u(x, t) \tag{1.1}$$

with C_ρ positive constant;

- $p(x,t)$ pressure of the tail;
- $l(t)$ left end of the tail;
- $r(t)$ right end of the tail;
- $V(t)$ velocity of the bacterium;

we will define our problem in the region occupied by the actin gel, $\Omega(t) = (l(t), r(t))$.
So, we have the bacterium which moves with velocity $V(t)$ togheter with the attached filaments, and the tail, composed by the detached filaments that moves in the opposite direction with velocity $w(x,t)$. We remark that, in our notation, $l(t)$ is the end of the tail in touch with the bacterium, so $V(t)$ is the speed of the front $l(t)$ or easily $\frac{dl}{dt} = V(t)$. It must be noted that $V(t)$ is different from $w(l(t),t)$; in fact, $V(t)$ denotes the velocity at which the front is moving, while $w(l(t),t)$ denotes the velocity at which the part of the tail in $l(t)$ moves. In the above the tail is considered as the set of detached filaments, so $w(l(t),t)$ and $V(t)$ identify, respectively, the velocity of detached and of attached filaments.
From a physical point of view this problem can be regarded as a motion in a viscous fluid. In particular, in [3] the authors assume that:

- the motion happens for low Reynolds number, so that it is laminar;
- each filament doesn't affect the motion of the other filaments;
- a deformation of a piece of gel is only due to a change in the numerical filaments density. So that the pressure of the tail verifies the following constitutive law:

  $$p(x,t) = p_0(t) + \overline{E}(u - u_0) \tag{1.2}$$

  where u_0 is the numerical filaments density at the bacterium surface, $\overline{E}$ is a positive constant and $p_0(t)$ is the pressure at the bacterium surface which is defined as follows:

  $$p_0(t) = \beta - \alpha\,(w(l(t),t) + V(t)) \tag{1.3}$$

  with α and β positive constants;
- the viscosity coefficients for the bacterium and the actin tail, denoted respectively by μ and b, are both positive.

Note that under these assumptions, the balance between the force that pushes the gel and the drag force can be written in this way:

$$bw = -\frac{\partial p}{\partial x} = -\overline{E}\frac{\partial u}{\partial x}$$

and so:

$$w(x,t) = -\frac{\overline{E}}{b}\frac{\partial u}{\partial x}. \tag{1.4}$$

As we said previously:

$$\frac{dl}{dt} = V(t)$$

and since the motion is laminar:

$$V(t) = -\frac{p_0(t)}{\mu} \tag{1.5}$$

So, from (1.3):

$$V(t) = -\frac{1}{\mu}\left[\beta - \alpha\left(w(l(t),t) + V(t)\right)\right]$$

where we assume that

$$\mu > \alpha$$

and then:

$$V(t) = \frac{\alpha}{\mu-\alpha} w(l(t),t) - \frac{\beta}{\mu-\alpha}$$

Replacing $V(t)$ by dl/dt and using (1.4) we obtain the following evolution equation for $l(t)$:

$$\frac{dl}{dt} = -E\frac{\alpha}{\mu-\alpha} u_x(l(t),t) - \frac{\beta}{\mu-\alpha} \tag{1.6}$$

With regard to the evolution of $r(t)$ we assume, as in [3], that at $x = r(t)$ the average distance between the tail filaments is proportional both to $u(x,t)^{-\frac{1}{3}}$ and to $l_f(x,t)$. So there exists a constant $\overline{C}$ such that:

$$u(r(t),t)^{-\frac{1}{3}} = \overline{C} l_f(r(t),t)$$

and from (1.1):

$$u(r(t),t)^{\frac{2}{3}} = C\overline{C}\rho(r(t),t)$$

Set:

$$\rho_d(t) = du(r(t),t)^{\frac{2}{3}}$$

with $d > 0$.

Following [3]:

$$\frac{dr}{dt} = w(r(t),t) - \frac{\nu}{\rho(r(t),t) - \rho_d(t)}$$

and so:

$$\frac{dr}{dt} = -Eu_x(r(t),t) - \frac{\nu}{\rho(r(t),t) - \rho_d(t)}. \tag{1.7}$$

As far as the boundary data is concerned

$$u(l(t),t) = u_0 > 0 \tag{1.8}$$

Due to the disintegration of the tail at $x = r(t)$, in [3] it is assumed that $p(r(t),t) = 0$. So, from equation (1.2):

$$p(r(t),t) = p_0(t) + \overline{E}(u(r(t),t) - u_0) = -\mu\frac{dl}{dt} + \overline{E}(u(r(t),t) - u_0) = 0$$

where last equation is derived using (1.5).
Then:

$$u(r(t),t) = u_0 + \frac{\mu}{\overline{E}}\frac{dl}{dt} \tag{1.9}$$

Also for ρ in [3] a boundary condition is given:

$$\rho(l(t),t) = \rho_0 > 0 \tag{1.10}$$

Let's now turn to the derivation of the system of partial differential equations for u and ρ. In order to study the evolution of u we write down the conservation law for the numerical density of the filaments in the following way:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(wu) = 0 \quad \forall x \in \Omega(t), t > 0 \tag{1.11}$$

and replacing w by (1.4), we find:

$$\frac{\partial u}{\partial t} - \frac{\overline{E}}{b}\frac{\partial}{\partial x}(u_x u) = 0 \quad \forall x \in \Omega(t), t > 0$$

that is:

$$\frac{\partial u}{\partial t} - \frac{\overline{E}}{2b}\frac{\partial^2}{\partial x^2}(u^2) = 0 \quad \forall x \in \Omega(t), t > 0$$

B.Bazaliy, Y.Bazaliy and A.Friedman in [3] assume that the filaments length decreases with a constant rate $\overline{K}$ as the filaments distance from the bacterium increases. So they have the following equation for $l_f(x,t)$

$$\frac{\partial l_f}{\partial t} + w\frac{\partial l_f}{\partial x} = -\overline{K} \quad \forall x \in \Omega(t), t > 0 \tag{1.12}$$

Using (1.1), they obtain the following conservation law for ρ:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(w\rho) = -C_\rho \overline{K} u \tag{1.13}$$

In fact, from (1.1):

$$\frac{\partial \rho}{\partial t} = C\left(\frac{\partial l_f}{\partial t}u + \frac{\partial u}{\partial t}l_f\right)$$

and

$$\frac{\partial \rho}{\partial x} = C\left(\frac{\partial l_f}{\partial x}u + \frac{\partial u}{\partial x}l_f\right)$$

So,

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(w\rho) &= C_\rho\left[\left(\frac{\partial l_f}{\partial t} + w\frac{\partial l_f}{\partial x}\right)u + \left(\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(wu)\right)l_f\right] = \\ &= C_\rho\left(\frac{\partial l_f}{\partial t} + w\frac{\partial l_f}{\partial x}\right)u = -C\overline{K}u\end{aligned}$$

Finally they define an initial data for the problem. That is:

$$\begin{cases} u(x,0) = \tilde{u}_0(x) & x \in (l(t), r(t)) \\ \rho(x,0) = \tilde{\rho}_0(x) & x \in (l(t), r(t)) \end{cases} \tag{1.14}$$

with $\tilde{u}_0(l(t)) = u_0$ and $\tilde{\rho}_0(l(t)) = \rho_0$
From (1.6), (1.7), (1.8), (1.9), (1.10), (1.11), (1.13) and (1.14) the original model could be summarized as follows:

$$\begin{cases} u_t - \dfrac{E}{2}(u^2)_{xx} = 0 & x \in (l(t), r(t)), t > 0 \\ \\ \rho_t - E(u_x \rho)_x = -K_2 u & x \in (l(t), r(t)), t > 0 \end{cases} \tag{M_o}$$

subject to the following initial conditions:

$$\begin{cases} u(x, 0) = \tilde{u}_0(x) & x \in (0, r_0) \\ \rho(x, 0) = \tilde{\rho}_0(x) & x \in (0, r_0) \end{cases} \tag{I}$$

and to the boundaries conditions:

$$\begin{cases} u(l(t), t) = u_0 \\ u(r(t), t) = u_0 + \dfrac{\mu}{bE}\dfrac{dl}{dt} \\ \rho(l(t), t) = \rho_0 \end{cases} \tag{B}$$

with the following equations for the evolution of the two moving boundaries:

$$\begin{cases} \dfrac{dl}{dt} = -E\dfrac{\alpha}{\mu - \alpha} u_x(l(t), t) - \dfrac{\beta}{\mu - \alpha} & l(0) = 0 \\ \\ \dfrac{dr}{dt} = -Eu_x(r(t), t) - \dfrac{\nu}{\rho(r(t), t) - du(r(t), t)^{\frac{2}{3}}} & r(0) = r_0 > 0 \end{cases} \tag{LR}$$

Moreover, in order to ensure that the problem is well-posed and consistent from a physical point of view, u and ρ have to verify the following conditions:

$$u(x, t), \rho(x, t) > 0 \quad \forall x \in (l(t), r(t)), \quad t > 0 \tag{1.15}$$

$$\rho(r(t), t) - du(r(t), t)^{\frac{2}{3}} > 0 \quad \forall t > 0 \tag{1.16}$$

$$r(t) - l(t) > 0 \quad \forall t > 0 \tag{1.17}$$

where $E = \overline{E}/b$ and $K_2 = CK$

1.2 Change to the model

Let's describe how we have changed the model. We have included in our model the effect of filaments elongation on the bacterium surface due to the polymerization process. This means that the filaments length increases for $x = l(t)$ while it decreases for all $x > l(t)$. In fact, the limit of the model introduced in [3], is that the filaments length can only decrease. We introduced the following C^2 function, $\overline{K}(x,t)$:

$$\overline{K}(x,t) = \begin{cases} \overline{K}_1 & x \in (l(t), l(t)+\delta_l), t > 0 \\ \text{decreasing} & x \in (l(t)+\delta_l, l(t)+\delta_r), t > 0 \\ -\overline{K}_2 & x \in (l(t)+\delta_r, r(t)) \end{cases}$$

with $\overline{K}_1$ and $\overline{K}_2$ positive constants and $0 \leq \delta_l \leq \delta_r$. Moreover, since the filaments length increases only for $x = l(t)$, we assume that δ_l and δ_r are small and enough and in particular we assume that:

$$0 < \delta_l << \delta_r$$

Then we derive the following evolution equation for $l_f(x,t)$:

$$\frac{\partial l_f}{\partial t} + w\frac{\partial l_f}{\partial x} = -\overline{K}(x,t) \quad \forall x \in \Omega(t), t > 0 \tag{1.18}$$

As in the original method we derive the equation for the evolution of ρ using (1.1). So, in our case the equation for ρ becomes:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(w\rho) = -\overline{K}(x,t)u \tag{1.19}$$

As for the original model from (1.6), (1.7), (1.8), (1.9), (1.10), (1.11), (1.19) and (1.14) we can write the new model as:

$$\begin{cases} u_t - \dfrac{E}{2}(u^2)_{xx} = 0 & x \in (l(t), r(t)), t > 0 \\ \\ \rho_t - E(u_x\rho)_x = K(x,t)u & x \in (l(t), r(t)), t > 0 \end{cases} \tag{M}$$

$$K(x,t) = C\overline{K}(x,t) = \begin{cases} K_1 & x \in (l(t), l(t)+\delta_l), t > 0 \\ C^2 \text{ and decreasing} & x \in (l(t)+\delta_l, l(t)+\delta_r), t > 0 \\ -K_2 & x \in (l(t)+\delta_r, r(t)) \end{cases} \tag{1.20}$$

subject (I), (B), (LR) and also conditions (1.15)-(1.17).

As we said previously, the model is a system of two partial differential equation with two moving boundaries.

We remark that model (M_o) can be obtained from model (M) simply setting $K_1 = -K_2$, $\delta_l = \delta_r = 0$. In fact, with these choices of the parameters the two intervals $(l(t), l(t)+\delta_l)$ and $(l(t)+\delta_l, l(t)+\delta_r)$ are empties and on $x = l(t)$, $K(x,t) = -K_2$. So $K(x,t) \equiv -K_2$ for all $x \in \Omega(t), t > 0$. So, all of the theoretical results we show in next sections for (M) also hold for (M_o).

2 Travelling wave solution

In this section we will show that under suitable choices for the parameters (M) together with (I), (B), (LR) and (1.15)-(1.17) admits a travelling wave solution. Since (M_o) is the special case of (M) for $K_1 = -K_2$ and $\delta_l = \delta_r = 0$, our result holds also for (M_o). The existence of a travelling wave solution was shown previously in [3] using variational arguments. In the following, we will prove the existence of the travelling wave solution for (M), and so for (M_o), using a different approach. In particular our result provides a set of conditions the parameters of the problem have to satisfy, in order to guarantee the existence of such a special solution.

$(l(t), r(t), u, \rho)$ is a travelling wave solution for (M) with (I), (B), (LR) and (1.15)-(1.17) if:

$$\frac{dl}{dt} = \frac{dr}{dt} = -W \tag{2.1}$$

with $W > 0$ and:

$$\begin{cases} u(x,t) = u(x+Wt) = u(z) \\ \rho(x,t) = \rho(x+Wt) = \rho(z) \end{cases} \tag{2.2}$$

where $z = x + Wt$.

So, we define a travelling wave solution as follows:

Definition 1 *(r_0, W, u, ρ) is a* travelling wave solution *for (M) with (I), (B), (LR) and* (1.15)-(1.17) *if it solves:*

$$\begin{cases} ((\varepsilon W - u')u)' = 0 & z \in (0, r_0) \\ ((\varepsilon W - u')\rho)' = K(z)u & z \in (0, r_0) \\ \varepsilon W = \dfrac{\alpha}{\mu - \alpha} u'(0) + \dfrac{\beta\varepsilon}{\mu - \alpha} \\ \varepsilon W = u'(r_0) + \dfrac{\nu\varepsilon}{\rho(r_0) - du(r_0)^{\frac{2}{3}}} \end{cases} \tag{2.3}$$

subject to:

$$\begin{cases} u(0) = u_0 \\ u(r_0) = u_0 - \dfrac{\mu\varepsilon}{b} W \\ \rho(0) = \rho_0 \end{cases} \tag{2.4}$$

and

$$u(z), \rho(z) > 0 \quad \forall z \in (0, r_0) \tag{2.5}$$

$$\rho(r_0) - du(r_0)^{\frac{2}{3}} > 0 \tag{2.6}$$

$$r_0 > 0 \tag{2.7}$$

where $\varepsilon = 1/E$ and $K(z) = K(x, 0)$.

This definition is obtained by replacing (2.1) and (2.2) in (M), (I), (B) and (1.15)-(1.17) and noting that:

$$\begin{cases} u_t(x,t) = Wu'(z) \\ \rho(x,t) = W\rho'(z) \end{cases}$$

Our aim is to prove the following theorem:

Theorem 1 *Let $\alpha, \beta, \mu, \nu, \varepsilon, K_1, K_2, b, d, u_0, \rho_0, h, R_0$ be positive constants such that:*

$$\mu > 5\alpha \tag{2.8}$$

$$h \in \left(1, \frac{5\mu - 9\alpha}{4(\mu - \alpha)}\right) \tag{2.9}$$

$$u_0 \in \left(2\varepsilon \frac{\mu}{\mu - \alpha} \frac{\beta}{b}, \frac{\mu\varepsilon\beta}{2b(h(\mu - \alpha) - (\mu - 2\alpha))}\right) \tag{2.10}$$

$$R_0 = \frac{\mu}{b} \frac{h}{h - 1} \tag{2.11}$$

$$(\mu - \alpha)K_1\delta_l < 8K_2(R_0 - \delta_r) \tag{2.12}$$

$$\rho_{0_{min}} = \frac{\alpha}{\beta}\left[-\frac{1}{2}K_1\delta_l + K_2(R_0 - \delta_r)\frac{h(\mu - \alpha)}{h(\mu - \alpha) - (\mu - 2\alpha)}\right] \tag{2.13}$$

$$\rho_{min} = \frac{h(\mu - \alpha)}{\beta(h(\mu - \alpha) - (\mu - 2\alpha))}\rho_{0_min} + d\left(u_0 - \frac{\mu\varepsilon\beta}{bh(\mu - \alpha)}\right)^{-\frac{1}{3}} \tag{2.14}$$

$$\rho_0 > \frac{bu_0h(\mu - \alpha) - \mu\varepsilon\beta}{bu_0(\mu - 2\alpha) - \mu\varepsilon\beta}\rho_{min}u_0 \tag{2.15}$$

$$\nu \in I_\nu = (\nu_1, \nu_2) \tag{2.16}$$

with

$$\nu_1 = \frac{\beta}{\alpha}\rho_0 \frac{bu_0(h(\mu - \alpha) - (\mu - 2\alpha))}{bhu_0(\mu - \alpha) - \mu\varepsilon\beta} \tag{2.17}$$

and

$$\nu_2 = \frac{\beta}{\alpha}(\rho_0 - \rho_{min}) \tag{2.18}$$

Then the system (2.3) with (2.4) has a solution (u, ρ, r_0, W) that verifies also conditions (2.5)-(2.7).
Moreover u and ρ are decreasing functions and

$$W \in \left(0, \frac{\beta}{h(\mu - \alpha)}\right) \tag{2.19}$$

Such a theorem allows us to use the travelling wave solution as an initial data for system (M). In fact, with this choice, at every time t, the solution will be such that:

$$\begin{cases} l(t) = -Wt & t > 0 \\ r(t) = r_0 - Wt & t > 0 \\ u(x,t) = u(x + Wt) = u(z) & x \in \Omega(t), t > 0 \\ \rho(x,t) = \rho(x + Wt) = \rho(z) & x \in \Omega(t), t > 0 \end{cases}$$

So, at each time, we can test the efficiency of a numerical scheme for the approximation of (M) with (I), (B), (LR) and (1.15)-(1.17), comparing the approximated solution and the travelling wave.

2.1 Tools for the proof of Theorem 1

In order to prove Theorem 1 we rearranged the equations of system (2.3) and their boundary conditions as follows:

- a Cauchy problem for the unknown u:

$$\begin{cases} ((\varepsilon W - u')u)' = 0 & z \in (0, r_0) \\ u(0) = u_0 \\ u'(0) = \dfrac{\varepsilon}{\alpha}(W(\mu - \alpha) - \beta) \end{cases} \tag{C_u}$$

- a Cauchy problem for the unknown ρ:

$$\begin{cases} ((\varepsilon W - u')\rho)' = -K\varepsilon u & z \in (0, r_0) \\ \rho(0) = \rho_0 \end{cases} \tag{C_ρ}$$

- an implicit equation for r_0:

$$u(r_0) = u_0 - \frac{\mu\varepsilon}{b} W \tag{Eq_{r_0}}$$

- an implicit equation for W:

$$\varepsilon W = u'(r_0) + \frac{\nu\varepsilon}{\rho(r_0) - du(r_0)^{\frac{2}{3}}} \tag{Eq_W}$$

The outline of the proof is the following. First we study Cauchy problem (C_u) togheter with (Eq_{r_0}). We will show that under suitable conditions they admit a solution (r_0, u) for all W in $(0, \beta/(\mu - \alpha))$. Then we show that these conditions guarantee also the existence of a solution for the Cauchy problem (C_ρ). Finally we will show that under the hypoteses of Theorem (1), there exists $W \in \left(0, \dfrac{\beta}{\mu - \alpha}\right)$ that verifies (Eq_W). Moreover, we will prove that those hypoteses guarantee also that (2.5)-(2.7) are verified.
Let's focus on the study of (C_u) and (Eq_{r_0}). In order to prove the existence of a solution for them, we need the following lemma:

Lemma 1 *Let $\alpha, \beta, \mu, b, \varepsilon, u_0$ be positive constants such that:*

$$\mu > 2\alpha \quad \textit{and} \quad u_0 > 2\frac{\mu\varepsilon}{b}\frac{\beta}{\mu - \alpha}$$

If

$$r_0(W) = -\frac{\mu}{b} + u_0 \frac{\beta - W(\mu - 2\alpha)}{\varepsilon\alpha W^2} \log\left(1 + \frac{\mu\varepsilon\alpha}{bu_0}\frac{W^2}{\beta - W(\mu - \alpha)}\right) \tag{2.20}$$

then

$$\lim_{W \to 0} r_0(W) = 0, \quad \lim_{W \to \beta/(\mu - \alpha)} r_0(W) = +\infty$$

and

$$r_0 > 0 \quad \forall W \in \left(0, \frac{\beta}{\mu - \alpha}\right)$$

Proof.

$$\begin{aligned}\lim_{W\to 0} r_0(W) &= -\frac{\mu}{b} + u_0 \lim_{W\to 0} \frac{\beta - W(\mu-2\alpha)}{\varepsilon\alpha W^2} \log\left(1 + \frac{\mu\varepsilon\alpha}{bu_0}\frac{W^2}{\beta - W(\mu-\alpha)}\right) = \\ &= -\frac{\mu}{b} + u_0 \lim_{W\to 0} \frac{\beta - W(\mu-2\alpha)}{\varepsilon\alpha W^2}\frac{\mu\varepsilon\alpha}{bu_0}\frac{W^2}{\beta - W(\mu-\alpha)} = \\ &= -\frac{\mu}{b} + \frac{\mu}{b}\lim_{W\to 0}\frac{\beta - W(\mu-2\alpha)}{\beta - W(\mu-\alpha)} = 0\end{aligned}$$

$$\lim_{W\to\beta/(\mu-\alpha)} r_0(W) = -\frac{\mu}{b} + \frac{u_0(\mu-\alpha)}{\varepsilon\beta}\log\left(1 + \frac{\mu\varepsilon\alpha\beta^2}{u_0(\mu-\alpha)^2}\lim_{W\to\frac{\beta}{\mu-\alpha}}\frac{1}{\beta - W(\mu-\alpha)}\right) = +\infty$$

So $\lim_{W\to 0} r_0(W) = 0$ and $\lim_{W\to\frac{\beta}{\mu-\alpha}} r_0(W) = +\infty$.

From (2.20) we obtain:

$$\begin{aligned}\frac{dr_0}{dW} &= \frac{u_0}{\varepsilon\alpha}\left[\frac{-2\beta + W(\mu-2\alpha)}{W^3}\log\left(1 + \frac{\mu\varepsilon\alpha}{bu_0}\frac{W^2}{\beta - W(\mu-\alpha)}\right) +\right.\\ &\quad\left. + \frac{\beta - W(\mu-2\alpha)}{\beta - W(\mu-\alpha)}\frac{\mu\varepsilon\alpha}{W}\frac{2\beta - W(\mu-\alpha)}{bu_0(\beta - W(\mu-\alpha)) + \mu\varepsilon\alpha W^2}\right] = \\ &= \frac{u_0}{\varepsilon\alpha}\frac{2\beta - W(\mu-2\alpha)}{W^3}\left[-\log\left(1 + \frac{\mu\varepsilon\alpha}{bu_0}\frac{W^2}{\beta - W(\mu-\alpha)}\right) +\right.\\ &\quad\left. + \frac{\beta - W(\mu-2\alpha)}{\beta - W(\mu-\alpha)}\frac{2\beta - W(\mu-\alpha)}{2\beta - W(\mu-2\alpha)}\frac{\mu\varepsilon\alpha W^2}{bu_0(\beta - W(\mu-\alpha)) + \mu\varepsilon\alpha W^2}\right] \geq\end{aligned}$$

and since $\log\left(1 + \frac{\mu\varepsilon\alpha}{bu_0}\frac{W^2}{\beta - W(\mu-\alpha)}\right) \leq \frac{\mu\varepsilon\alpha}{bu_0}\frac{W^2}{\beta - W(\mu-\alpha)}$:

$$\begin{aligned}\frac{dr_0}{dW} &\geq \frac{u_0}{\varepsilon\alpha}\frac{2\beta - W(\mu-2\alpha)}{W^3}\left[-\frac{\mu\varepsilon\alpha}{bu_0}\frac{W^2}{\beta - W(\mu-\alpha)} +\right.\\ &\quad\left. + \frac{\beta - W(\mu-2\alpha)}{\beta - W(\mu-\alpha)}\frac{2\beta - W(\mu-\alpha)}{2\beta - W(\mu-2\alpha)}\frac{\mu\varepsilon\alpha W^2}{bu_0(\beta - W(\mu-\alpha)) + \mu\varepsilon\alpha W^2}\right] = \\ &= \frac{2\beta - W(\mu-\alpha)}{W(\beta - W(\mu-\alpha))}[-1+ \\ &\quad\left. + \frac{\beta - W(\mu-2\alpha)}{\beta - W(\mu-\alpha)}\frac{2\beta - W(\mu-\alpha)}{2\beta - W(\mu-2\alpha)}\frac{bu_0(\beta - W(\mu-\alpha))}{bu_0(\beta - W(\mu-\alpha)) + \mu\varepsilon\alpha W^2}\right] \\ &\geq \frac{\mu(2\beta - W(\mu-2\alpha))}{bW(\beta - W(\mu-\alpha))}(-1 + \\ &\quad\left. + \frac{2\beta - W(\mu-\alpha)}{2\beta - W(\mu-2\alpha)}\frac{\beta - W(\mu-2\alpha)}{\beta - W(\mu-\alpha)}\frac{bu_0(\beta - W(\mu-\alpha))}{\mu\varepsilon\alpha W^2 + bu_0(\beta - W(\mu-\alpha))}\right)\end{aligned}$$

$W \in \left(0, \frac{\beta}{\mu-\alpha}\right)$ implies $\frac{2\beta - W(\mu-2\alpha)}{W(\beta - W(\mu-\alpha))} > 0$.

So $r_0'(W) > 0$ if and only if:

$$\frac{2\beta - W(\mu-\alpha)}{2\beta - W(\mu-2\alpha)}\frac{\beta - W(\mu-2\alpha)}{\beta - W(\mu-\alpha)}\frac{bu_0(\beta - W(\mu-\alpha))}{\mu\varepsilon\alpha W^2 + bu_0(\beta - W(\mu-\alpha))} > 1 \qquad (2.21)$$

and the last inequality is verified thanks to the condition (2.10) on u_0 and to the condition on W, $W \in \left(0, \dfrac{\beta}{\mu-\alpha}\right)$.

Hence, in the hypoteses of the lemma $\dfrac{dr_0}{dW} > 0$ and

$$r_0(0) = 0 \text{ implies } r_0(W) > 0 \quad \forall W \in \left(0, \frac{\beta}{\mu-\alpha}\right)$$

We use this lemma and the implicit function's theorem to prove the following theorem:

Theorem 2 *Let $\alpha, \beta, \mu, b, \varepsilon$ be positive constants such that:*

$$\mu > 2\alpha \tag{2.22}$$

$$u_0 > 2\varepsilon \frac{\mu}{\mu-\alpha} \frac{\beta}{b} \tag{2.23}$$

then (C_u) and (Eq_{r_0}) have a solution (u, r_0) if and only if $W \in \left(0, \dfrac{\beta}{\mu-\alpha}\right)$.
Moreover, the solution is such that:

u is positive and decreasing

and

$$r_0 > 0$$

Direct implication's proof. From the first equation of the system (C_u) we obtain:

$$(\varepsilon W - u')u = C \tag{2.24}$$

and

$$C = (\varepsilon W - u'(0))u(0) = \frac{\varepsilon}{\alpha}(\beta - W(\mu - 2\alpha))u_0 \tag{2.25}$$

Let

$$\overline{u} = \frac{C}{\varepsilon W} \tag{2.26}$$

then:

$$u' = \begin{cases} 0 & u = \overline{u} \\ \dfrac{\varepsilon W u - C}{u} & \text{otherwise} \end{cases} \tag{2.27}$$

For $u \neq \overline{u}$ the equation (2.27) is an autonomous one and:

$$\frac{u\,du}{\varepsilon W u - C} = dz$$

From the integration of the last equation we obtain:

$$u(z) + \frac{C}{\varepsilon W} \log|\varepsilon W u(z) - C| = \varepsilon W z + C'$$

Let's compute C' using the initial condition for the system:

$$C' = u_0 + \frac{C}{\varepsilon W} \log |\varepsilon W u_0 - C|$$

Then, we obtain the following implicit expression for u:

$$u(z) - u_0 + \frac{C}{\varepsilon W} \log \left| \frac{\varepsilon W u(z) - C}{\varepsilon W u_0 - C} \right| = \varepsilon W z \tag{2.28}$$

First of all we note that (2.7) is verified, that is $r_0 > 0$. The hypoteses of the theorem on μ, α and u_0 are the same of Lemma 1 and from (2.28) we obtain:

$$r_0(W) = \frac{1}{\varepsilon W} \left[u(r_0) - u_0 + \frac{C}{\varepsilon W} \log \left| \frac{\varepsilon W u(r_0) - C}{\varepsilon W u_0 - C} \right| \right] \tag{2.29}$$

Thus, $r_0(W)$ is also defined like in Lemma 1. Hence:

$$r_0 > 0$$

Now, let:

$$F(u) = u + \frac{C}{\varepsilon W} \log \left| \frac{\varepsilon W u - C}{\varepsilon W u_0 - C} \right|.$$

and $G(u)$:

$$G(u) = \varepsilon W z = F(u) - F(u_0)$$

with $G(u_0) = 0$.

$$F'(u) = \frac{\varepsilon W u}{|\varepsilon W u - C|} \qquad \begin{cases} > 0 & u > \overline{u} \\ < 0 & u \in (0, \overline{u}) \\ > 0 & u < 0 \end{cases}$$

We need to invert $G(u)$ in a neighborhood of $u = u_0 < \overline{u}$. Moreover for (2.5), we want u to be positive.

So, we look at the invertibility of $G(u)$ in the interval $u \in (0, \overline{u})$. In this interval $G(u)$ is invertible because of its monotony. Then there exists $u(z) = G^{-1}(z)$.

$G(u)$ is a decreasing function in $[0, \overline{u}]$ and $G(u_0) = 0$. Moreover, since (2.3) is defined for $z \in (0, r_0)$ then:

$$\forall z \in (0, r_0) \quad u(z) < u_0$$

Then $u(r_0) < u(z) < u_0$ and since:

$$u(r_0) = u_0 - \frac{\mu \varepsilon}{b} W$$

from (2.23), (2.22) and $W < \beta/(\mu - \alpha)$:

$$u(r_0) > 2 \frac{\mu \varepsilon}{b} \frac{\beta}{\mu - \alpha} - \frac{\mu \varepsilon}{b} W > \frac{\mu \varepsilon}{b} \frac{\beta}{\mu - \alpha} > 0$$

So u is also positive $\forall z \in (0, r_0)$.

To prove the converse implication of the Theorem 2 we suppose $W \geq \dfrac{\beta}{\mu - \alpha}$ or $W = 0$ and we prove that the solution doesn't exist.

Converse implication's proof. Let $W = 0$, then the system (C_u) becomes:

$$\begin{cases} (-u'u)' = 0 & z \in (0, r_0) \\ u(0) = u_0 \\ u'(0) = -\dfrac{\beta}{\alpha}\varepsilon \end{cases}$$

and the equation (Eq_{r_0}):

$$u(r_0) = u_0 \tag{2.30}$$

From the first equation we obtain that

$$-u'u = C$$

and

$$C = -u'(0)u_0 = \frac{\varepsilon\beta}{\alpha}u_0$$

Hence

$$udu = -\frac{\varepsilon\beta}{\alpha}u_0 dz$$

and integrating the previous equation in $(0, z)$ we obtain:

$$u(z)^2 = u_0^2 - 2\frac{\varepsilon\beta}{\alpha}u_0 z$$

but then for $z = r_0$:

$$u(r_0^2) = u_0^2 - 2\frac{\varepsilon\beta}{\alpha}u_0 r_0 < u_0$$

and the previous inequality contradicts the boundary condition on r_0, $u(r_0) = u_0$.
If $W = \dfrac{\beta}{\mu - \alpha}$ then $u'(0) = 0$. Hence the costant function $u \equiv u_0$ is a solution of the problem (C_u) but it doesn't verify the condition (Eq_{r_0}).
Let $W > \dfrac{\beta}{\mu - \alpha}$ then $u'(0) > 0$ and since $u(r_0) < u_0$, there exists $\overline{z} \in (0, r_0)$ such that:

$$u(\overline{z}) > u_0 \text{ and } u'(\overline{z}) = 0 \tag{2.31}$$

From the first equation and from the definition of C,(2.25) , we obtain:

$$(\varepsilon W - u')u = \frac{\varepsilon}{\alpha}(W(2\alpha - \mu) + \beta)u_0$$

and in particular in $\overline{z}$:

$$u(\overline{z}) = \frac{W(2\alpha - \mu) + \beta}{\alpha W}u_0 \tag{2.32}$$

and $u(\overline{z}) < u_0$. In fact, $W > \dfrac{\beta}{\mu - \alpha}$ and so:

$$W(\alpha - \mu) + \beta < 0$$

Then:

$$W(2\alpha - \mu) + \beta < \alpha W$$

and so:

$$\frac{u(\overline{z})}{u_0} = \frac{W(2\alpha - \mu) + \beta}{\alpha W} < 1$$

and from (2.32) we obtain

$$u(\overline{z}) < u_0$$

that is a contradiction within the hypotesis (2.31) on $u(\overline{z})$.
Now we study the Cauchy problem for ρ with (u, r_0) solution of (C_u) with (Eq_{r_0}). We prove the following lemma:

Lemma 2 *Let $\alpha, \beta, \mu, b, \varepsilon$ be positive constants such that:*

$$\mu > 2\alpha$$

and

$$u_0 > 2\frac{\mu\varepsilon\beta}{\mu - \alpha}$$

then a solution (u, ρ, r_0) of (C_u), together with (C_ρ) and (Eq_{r_0}) exists $\forall W \in \left(0, \frac{\beta}{\mu - \alpha}\right)$.

Proof. From Theorem 2 we know that $\forall W \in \left(0, \frac{\beta}{\mu - \alpha}\right)$, it exists a solution of the Cauchy problem (C_u) with (Eq_{r_0}), (u, r_0). Then the Cauchy problem (C_ρ) has the following solution:

$$\rho(z) = u(z)\left(\frac{\rho_0}{u_0} + \frac{\varepsilon}{C}\int_0^z K(\xi)u(\xi)d\xi\right) \tag{2.33}$$

with C defined in (2.25) and $K(z) = K(x, 0)$. Thus, $\forall W \in (0, \beta/(\mu - \alpha))$, (u, ρ, r_0) is the required solution.

2.2 Proof of Theorem 1

From Lemma 2, we know that if $\mu > 2\alpha$ and $u_0 > 2\mu\varepsilon\beta/(\mu - \alpha)$ then a solution of (C_u) with (C_ρ) and (Eq_{r_0}) exists.
The hypoteses of the theorem verify the hypoteses of the lemma.
In fact:

$$\mu > 5\alpha > 2\alpha$$

and

$$u_0 > 2\frac{\mu\varepsilon\beta}{\mu - \alpha}$$

So, for all $W \in (0, \beta/(\mu - \alpha))$ it exists (u, ρ, r_0) solution of (C_u), with (C_ρ) and (Eq_{r_0}).
Our aim is to show that there exists $W \in (0, \beta/(\mu - \alpha))$ that verifies the equation (Eq_W).
First of all we have to verify if the hypoteses of the theorem are well posed. Hence we have to verify that the intervals defined in the hypoteses are not empty.
That is:

$$\frac{5\mu - 9\alpha}{4(\mu - \alpha)} > 1 \tag{2.34}$$

$$\frac{\mu\varepsilon\beta}{2b(h(\mu-\alpha)-(\mu-2\alpha))} > 2\varepsilon\frac{\mu}{\mu-\alpha}\frac{\beta}{b} \tag{2.35}$$

and

$$\nu_1 < \nu_2 \tag{2.36}$$

Condition (2.8), guarantees that inequality (2.34) is satisfied, while (2.9) implies the (2.35) to be satisfied.

Replacing ν_1 and ν_2 as defined in (2.17) and (2.18), (2.36) can be written as:

$$\frac{\beta}{\alpha}\rho_0\frac{bu_0(h(\mu-\alpha)-(\mu-2\alpha))}{bhu_0(\mu-\alpha)-\mu\varepsilon\beta} < \frac{\beta}{\alpha}(\rho_0-\rho_{min})$$

that is:

$$\rho_0\left(1-\frac{bu_0(h(\mu-\alpha)-(\mu-2\alpha))}{bhu_0(\mu-\alpha)-\mu\varepsilon\beta}\right) > \rho_{min} \tag{2.37}$$

Note that the coefficient of ρ in the previous inequality is positive; in fact:

$$1-\frac{bu_0(h(\mu-\alpha)-(\mu-2\alpha))}{bhu_0(\mu-\alpha)-\mu\varepsilon\beta} = \frac{bu_0(\mu-2\alpha)-\mu\varepsilon\beta}{bhu_0(\mu-\alpha)-\mu\varepsilon\beta} > 0$$

for (2.10).

Thus, the inequality (2.37) is equivalent to:

$$\rho_0 > \frac{bu_0h(\mu-\alpha)-\mu\varepsilon\beta}{bu_0(\mu-2\alpha)-\mu\varepsilon\beta}\rho_{min}$$

that is the hypotesis (2.15). So, the inequality (2.36) is verified and the three intervals defined in the hypoteses of the theorem are not empty.

We have to impose the boundedness of r_0.

From Lemma 1 we know that

$$r_0(W) = -\frac{\mu}{b} + \frac{(\beta-W(\mu-2\alpha))u_0}{\alpha\varepsilon W^2}\log\left(1+\frac{\mu\varepsilon\alpha W^2}{bu_0(\beta-W(\mu-\alpha))}\right)$$

is a positive, increasing function such that:

$$\lim_{W\to 0} r_0(W) = 0 \quad \text{and} \quad \lim_{W\to\beta/(\mu-\alpha)} = +\infty$$

In order to have $r_0 < +\infty$, W can't be close to $\beta/(\mu-\alpha)$.

Thus, we search a solution of the system for $W \in (0, \beta/h(\mu-\alpha))$ with h defined in (2.9).

From Lemma 2 we obtain that for all $W \in (0, \beta/(\mu-\alpha))$ a solution exists for (C_u), (C_ρ) and (Eq_{r_0}). The same Lemma holds for all $W \in (0, \beta/h(\mu-\alpha))$ since $h > 1$ and so $(0, \beta/(h(\mu-\alpha))) \subset (0, \beta/(\mu-\alpha))$.

We still have to prove that ρ is positive for all $z \in (0, r_0)$ and then condition (2.6) holds.

From Lemma (2.33):

$$\rho(z) = u(z)\left[\frac{\rho_0}{u_0} + \frac{\varepsilon}{C}\int_0^z K(\xi)u(\xi)d\xi\right]$$

Let $\delta_0 \in (\delta_l, \delta_r)$ such that $K(\delta_0) = 0$. Since $K(z) \geq 0$ for all $z \in (0, \delta_0)$, $\rho(z) > 0$, while for $z \in (\delta_0, r_0)$ $\rho(z) \geq \rho(r_0)$. In fact,

$$
\begin{aligned}
\rho(r_0) &= u(r_0)\left[\frac{\rho_0}{u_0} + \frac{\varepsilon}{C}\int_0^{r_0} K(\xi)u(\xi)d\xi\right] < \\
&< u(z)\left[\frac{\rho_0}{u_0} + \frac{\varepsilon}{C}\int_0^{r_0} K(\xi)u(\xi)d\xi\right] = \\
&= u(z)\left[\frac{\rho_0}{u_0} + \frac{\varepsilon}{C}\int_0^{z} K(\xi)u(\xi)d\xi - \int_z^{r_0}|K(\xi)|u(\xi)d\xi\right] = \\
&= \rho(z) - u(z)\frac{\varepsilon}{C}\int_z^{r_0}|K(\xi)|u(\xi)d\xi < \rho(z)
\end{aligned}
$$

So if $\rho(r_0) > 0$ then $\rho(z) > 0$ for all $z \in (0, r_0)$. Moreover, from the hypoteses of the theorem, (2.6) holds.
Setting δ_0 such that $K(\delta_0) = 0$ we can prove that $\rho(r_0) < \rho(z)$ for all $z \in (0, r_0)$.
To prove the previous inequality we first derive an upper bound for r_0. Since $r_0(W)$ is an increasing function:

$$
\begin{aligned}
r_0 &\leq r_0\left(\frac{\beta}{h(\mu-\alpha)}\right) = \\
&= -\frac{\mu}{b} + \frac{h^2(\mu-\alpha)^2}{\alpha\varepsilon\beta}u_0\left(1 - \frac{\mu - 2\alpha}{h(\mu-\alpha)}\right)\log\left(1 + \frac{\mu\varepsilon\beta\alpha}{bu_0h^2(\mu-\alpha)^2}\frac{h}{h-1}\right) \leq \\
&\leq -\frac{\mu}{b} + \frac{\mu}{b}\frac{h}{h-1}\left(1 - \frac{\mu-2\alpha}{h(\mu-\alpha)}\right) < \\
&< \frac{\mu}{b}\frac{h}{h-1} = R_0
\end{aligned}
$$

that is

$$
r_0 < R_0 \tag{2.38}
$$

In particular we will prove $\rho(r_0) > du(r_0)^{\frac{2}{3}}$; in this way we show both condition (1.15) and (2.6).

$$
\rho(r_0) - du(r_0)^{\frac{2}{3}} = u(r_0)\left[\frac{\rho_0}{u_0} + \frac{\varepsilon}{C}\int_0^{r_0} K(\xi)u(\xi)d\xi - du(r_0)^{-\frac{1}{3}}\right]
$$

and

$$
\begin{aligned}
\int_0^{r_0} K(\xi)u(\xi)d\xi &= K_1\int_0^{\delta_l} u(\xi)d\xi + \int_{\delta_l}^{\delta_0} K(\xi)u(\xi)d\xi + \\
&\quad - \int_{\delta_0}^{\delta_r}|K(\xi)|u(\xi)d\xi - K_2\int_{\delta_r}^{r_0} u(\xi)d\xi > \\
&> K_1\int_0^{\delta_l} u(\xi)d\xi - \int_{\delta_0}^{\delta_r}|K(\xi)|u(\xi)d\xi - K_2\int_{\delta_r}^{r_0} u(\xi)d\xi > \\
&> K_1\delta_l u(r_0) - K_2(r_0 - \delta_0)u_0 > \\
&> K_1\delta_l u(r_0) - K_2 R_0 u_0
\end{aligned}
$$

Then

$$\frac{\varepsilon}{C}\int_0^{r_0} K(\xi)u(\xi)d\xi > \frac{\varepsilon}{C}[K_1\delta_l u(r_0) - K_2(R_0-\delta_r)u_0] >$$

$$> \frac{\alpha}{(\beta - W(\mu-2\alpha))u_0}[K_1\delta_l u(r_0) - K_2(R_0-\delta_r)u_0] >$$

$$> \frac{\alpha}{\beta}\left[K_1\delta_l\frac{u(r_0)}{u_0} - K_2(R_0-\delta_r)\frac{h(\mu-\alpha)}{h(\mu-\alpha)-(\mu-2\alpha)}\right] >$$

$$> \frac{\alpha}{\beta}\left[\frac{1}{2}K_1\delta_l - K_2(R_0-\delta_r)\frac{h(\mu-\alpha)}{h(\mu-\alpha)-(\mu-2\alpha)}\right] =$$

$$= -\frac{\rho_{0_{min}}}{u_0}$$

where in previous inequalities we have used (2.19) and (2.10).
Then, from (2.15):

$$\rho(r_0) - du(r_0)^{\frac{2}{3}} > u(r_0)\left[\frac{\rho_0}{u_0} - \frac{\rho_{0_{min}}}{u_0} - du(r_0)^{-\frac{1}{3}}\right] >$$

$$> u(r_0)\left[\frac{bu_0h(\mu-\alpha)-\mu\varepsilon\beta}{bu_0(\mu-2\alpha)-\mu\varepsilon\beta}\rho_{min} - \frac{\rho_{0_{min}}}{u_0} - du(r_0)^{-\frac{1}{3}}\right] > 0$$

since $\dfrac{bu_0h(\mu-\alpha)-\mu\varepsilon\beta}{bu_0(\mu-2\alpha)-\mu\varepsilon\beta} > 1$.
Now, we want to prove that there exists W which verifies (Eq_W). From (Eq_W):

$$\varepsilon W = u'(r_0) + \frac{\nu\varepsilon}{\rho(r_0) - du(r_0)^{\frac{2}{3}}}$$

From (2.24) and (2.25), (Eq_W) is equivalent to:

$$W = \frac{1}{\mu-2\alpha}\left[\beta - \frac{\nu\alpha}{\rho(r_0)-du(r_0)^{\frac{2}{3}}}\frac{u(r_0)}{u_0}\right] \tag{2.39}$$

From (2.6) we obtain that:

$$\frac{\beta}{\alpha}(\rho(r_0) - du(r_0)^{\frac{2}{3}})\frac{u_0}{u(r_0)} > \frac{\beta}{\alpha}(\rho_0 - \rho_{min}) = \nu_2$$

So,

$$\frac{\beta}{\alpha}(\rho(r_0) - du(r_0)^{\frac{2}{3}})\frac{u_0}{u(r_0)} > \nu \quad \forall\nu \in (\nu_1, \nu_2)$$

The last inequality implies that $\forall\nu \in (\nu_1,\nu_2)$, $W > 0$ with W defined as in (2.39).
Finally $W < \dfrac{\beta}{h(\mu-\alpha)}$. In fact:

$$\beta\left(1 - \frac{\mu-2\alpha}{h(\mu-\alpha)}(\rho(r_0)-du(r_0)^{\frac{2}{3}})\right)\frac{u_0}{u(r_0)} < \frac{\beta}{\alpha}\rho_0\frac{u_0}{u_0 - \dfrac{\mu\varepsilon\beta}{h(\mu-\alpha)}}\left(1 - \frac{\mu-2\alpha}{h(\mu-\alpha)}\right) =$$

$$= \frac{\beta}{\alpha}\rho_0 bu_0\frac{h(\mu-\alpha)-(\mu-2\alpha)}{bhu_0(\mu-\alpha)-\mu\varepsilon\beta} = \nu_1$$

So

$$\beta\left(1-\frac{\mu-2\alpha}{h(\mu-\alpha)}(\rho(r_0)-du(r_0)^{\frac{2}{3}})\right)\frac{u_0}{u(r_0)}<\nu \quad \forall\nu\in(\nu_1,\nu_2)$$

and hence:

$$W=\frac{1}{\mu-2\alpha}\left(\beta-\frac{\nu\alpha}{\rho(r_0)-du(r_0)^{\frac{2}{3}}}\frac{u(r_0)}{u_0}\right)<\frac{\beta}{h(\mu-\alpha)}$$

This theorem doesn't prove the uniqueness of the travelling wave solution. It only proves the existence of at least one travelling wave solution; it's enough for our purpose. In fact, as we will show later, we can approximate one of these solutions, and then use it as the initial data for system (M) to test the numerical scheme.

Noting that (M_o) is a special case of (M) with $K_1 = -K_2$ and $\delta_l = \delta_r = 0$, we can generalize Theorem 1 for (M_o), simply setting $\rho_{0_{min}} = K_2R_0\dfrac{h(\mu-\alpha)}{h(\mu-\alpha)-(\mu-2\alpha)}$.

Note that Theorem 1 gives a lower bound for ρ_0 and this bound increases as δ_l decreases.

3 Theoretical results

This chapter is focused on the study of system (M) with (I), (B), (LR) and (1.15)-(1.17), from a theoretical point of view.
In particular we will show the following local existence theorem:

Theorem 3 *Let r_0 be a positive constant and let $\tilde{u}_0(x)$ and $\tilde{\rho}_0(x) : [0, r_0] \to \mathbb{R}$ be respectively $C^{4+\alpha}[0, r_0]$ and $C[0, r_0]$ functions such that:*

$$\tilde{u}_0(x) \text{ and } \tilde{\rho}_0(x) \quad \forall x \in [0, r_0] \tag{3.1}$$

$$\tilde{\rho}_0(r_0) - d\tilde{u}_0(r_0)^{\frac{2}{3}} > 0 \tag{3.2}$$

$$(\tilde{u}_0)_x < 0 \quad \forall x \in [0, r_0] \tag{3.3}$$

$$(\tilde{u}_0)_x(0) > -\frac{1}{E}\frac{\beta}{\alpha} \tag{3.4}$$

$$(\tilde{u}_0)_x(r_0) > -\frac{1}{E}\frac{\nu}{\tilde{\rho}_0(r_0) - d\tilde{u}_0(r_0)^{\frac{2}{3}}} \tag{3.5}$$

$$\tilde{u}_0(l(t)) = u_0 \tag{3.6}$$

$$\tilde{u}_0(r(t)) = u_0 + \frac{\mu}{bE} l'(0) \tag{3.7}$$

$$0 = \Big[(E\tilde{u}_0(\tilde{u}_0)_x)_x + l'(0)(\tilde{u}_0)_x\Big]_{x=0} \tag{3.8}$$

$$\frac{\mu}{b}\frac{\alpha}{\mu - \alpha}[E(\tilde{u}_0(\tilde{u}_0)_x)_{xx} + l'(0)(\tilde{u}_0)_{xx}] = -\Big[E(\tilde{u}_0(\tilde{u}_0)_y)_y + r'(0)(\tilde{u}_0)_y\Big]_{y=r_0} \tag{3.9}$$

$$0 = E^2(\tilde{u}_0(\tilde{u}_0(\tilde{u}_0)_x)_x)_{xx} + 2El'(0)(\tilde{u}_0(\tilde{u}_0)_x)_{xx} + l'(0)^2(\tilde{u}_0)_{xx} + l''(0)(\tilde{u}_0)_x \tag{3.10}$$

where $l'(t)$ and $r'(t)$ are defined in LR and

$$l''(t) = -E\frac{\alpha}{\mu - \alpha}\left[\frac{r_0}{r(t) - l(t)}u_{yt}(0, t) - r_0\frac{r'(t) - l'(t)}{(r(t) - l(t))^2}u_y(0, t)\right]$$

Then there exists $T > 0$ such that system (M) with (I), (B), (LR) and (1.15)-(1.17) *has a unique solution for all $(x, t) \in \Omega_T(t) = (l(t), r(t)) \times [0, T]$.*
Moreover u and ρ are respectively $C^{2,1}(\Omega_T(t))$ and $C(\Omega_T(t))$ functions.

The hypoteses of this theorem guarantee the solvability of system (M), with (B), (I), (LR) and (1.15)-(1.17) for all $(x, t) \in [0, T]$. In fact:

- hypoteses (3.1) and (3.2) together with the positivity of r_0 imply that the initial data verify conditions (1.15)-(1.17);
- hypoteses (3.3)-(3.5) guarentee, as we will show in the proof of the theorem, that the equation for ρ together with its boudary value at $x = l(t)$ is well posed;
- hypoteses (3.6)-(3.10) make $u \in C^{2,1}$ and $\rho \in C$. So, they also guarantee that l and r are C^1 functions.

This last condition ensures that there exists a neighborhood of $t = 0$, $I_T = [0, T]$, such that, for all $t \in I_T$, (1.15)-(1.17) hold and the equation for ρ is well posed.

Unfortunately we can't estimate T. In fact for a fixed $\bar{t} > 0$, the equation for ρ with its boundary condition on $x = l(t)$ is well posed if the following inequalities are satisfied:

$$\begin{cases} u_x(x,\bar{t}) < 0 & \forall (x,t) \in (0,r_0) \times [0,T] \\ u_x(0,\bar{t}) > -\dfrac{1}{E}\dfrac{\beta}{\alpha} & \forall t \in [0,T] \\ u_x(r_0,\bar{t}) > -\dfrac{1}{E}\dfrac{\nu}{\rho(r_0,\bar{t}) - du(r_0,\bar{t})^{\frac{2}{3}}} & \forall t \in [0,T] \end{cases} \tag{3.11}$$

Since T is defined as:

$$T = \sup_{t>0}\{(1.15)\text{-}(1.17) \text{ and } (3.11) \text{ are verified for all } (x,t) \in (l(t), r(t)) \times [0,T]\} \tag{3.12}$$

its value strictly depends on the evolution of the solution. In order to estimate it, we should have more information on the evolution of $r(t) - l(t)$, $u(x,t)$, $u_x(x,t)$ and $\rho(x,t)$. For the same reason, it is not possible to prove the global existence of the solution for our problem.

So, we focus our attention on the proof of the local solvability of the system, assuming that T is such that, for all $t \in [0,T]$, $r(t) - l(t)$ and $\rho(r(t),t) - du(r(t),t)^{\frac{2}{3}}$ are positive and (3.11) holds. With regard to the positivity of u in $\Omega_T(t) = (l(t), r(t)) \times [0,T]$, we will show that it is implied from $\tilde{u}_0(x) > 0$ for all $x \in (0, r_0)$. While, as we will prove in the next section, in order to have $\rho(y,t) > 0$ in $\Omega_T(t)$, it is sufficient to have $\rho_0 > 0$.

3.1 The model in a fixed domain

From the literature about moving boundaries, see for example [7], in order to study this kind of problem, it is useful to eliminate the moving boundaries transforming the moving domain $\Omega(t) = (l(t), r(t))$ in a fixed one. In [3], the existence of the solution of system (M_o) with its initial and boudary conditions is shown passing from $\Omega(t)$ to the fixed domain $Q = [0, r_0]$ through the following transformation of the spatial variable x:

$$x = y + R(y,t)$$

with $R(y,t) = l(t)\chi(y) + (r(t) - r_0)\chi(y - r_0)$, $R(y,0) = 0$ and $\chi(z) \in C^\infty$ such that $\chi(z) = 0$ for $|z| > \delta_0/4$ and $\chi(z) = 1$ for $|z| < \delta_0/8$, with a fixed $\delta_0 \in (0, r_0/2)$.

For the sake of semplicity we have preferred to use the following linear transformation:

$$y = r_0 \frac{x - l(t)}{r(t) - l(t)} \tag{3.13}$$

In this way $\Omega(t)$ once again is transformed in the fixed domain $Q = [0, r_0]$.

We remark that this transformation is well posed, since, as we said at the end of the previous section, we have assumed that T is such that $r(t) - l(t) > 0$ for all $t \in [0,T]$.

The passage from the moving domain to the fixed one corresponds to an increase of the difficulty in the equations. In fact, since:

$$\frac{\partial y}{\partial x} = \frac{r_0}{(r(t) - l(t))}$$

and

$$\frac{\partial y}{\partial t} = -\frac{l'(t)(r_0 - y) + r'(t)y}{r(t) - l(t)}$$

system (M) becomes:

$$\begin{cases} u_t - \dfrac{l'(t)(r_0 - y) + r'(t)y}{r(t) - l(t)} u_y - \dfrac{E}{2} \dfrac{r_0^2}{(r(t) - l(t))^2} (u^2)_{yy} = 0 & (y, t) \in Q_T \\ \rho_t - \dfrac{l'(t)(r_0 - y) + r'(t)y}{r(t) - l(t)} \rho_y - E \dfrac{r_0^2}{(r(t) - l(t))^2} (u_y \rho)_y = K(y, t) u(y, t) & (y, t) \in Q_T \end{cases} \tag{M_F}$$

where $Q_T = Q \times [0, T]$. Moreover, $K(y, t)$ is the C^2 function corresponding to $K(x, t)$. That is:

$$K(y, t) = \begin{cases} K_1 & y \in \left(0, \dfrac{r_0}{r(t) - l(t)} \delta_l\right) \\ \text{is decreasing} & y \in \left(\dfrac{r_0}{r(t) - l(t)} \delta_l, \dfrac{r_0}{r(t) - l(t)} \delta_r\right) \\ K_2 & y \in \left(\dfrac{r_0}{r(t) - l(t)} \delta_r, r_0\right) \end{cases} \tag{3.14}$$

(M_F) is subject to the same initial and boundary conditions of system (M):

$$\begin{cases} u(y, 0) = \tilde{u}_0(y) & y \in (0, r_0) \\ \rho(y, 0) = \tilde{\rho}_0(y) & y \in (0, r_0) \end{cases} \tag{I_F}$$

$$\begin{cases} u(0, t) = u_0 \\ u(r_0, t) = u_0 + \dfrac{\mu}{bE} \dfrac{dl}{dt} \\ \rho(0, t) = \rho_0 \end{cases} \tag{B_F}$$

and to the following evolution equations for $l(t)$ and $r(t)$:

$$\begin{cases} \dfrac{dl}{dt} = -E \dfrac{\alpha}{\mu - \alpha} \dfrac{r_0}{r(t) - l(t)} u_y(0, t) - \dfrac{\beta}{\mu - \alpha} & l(0) = 0 \\ \dfrac{dr}{dt} = -E \dfrac{r_0}{r(t) - l(t)} u_y(r_0, t) - \dfrac{\nu}{\rho(r_0, t) - du(r_0, t)^{\frac{2}{3}}} & r(0) = r_0 \end{cases} \tag{LR_F}$$

Note that $l(t)$ and $r(t)$ lose their meaning of moving boundary positions. However their computation at every time is needed to define the coefficients of system (M_F).

Moreover we want the solution to satisfy:

$$u, \rho > 0 \quad \forall (y, t) \in Q_T \tag{3.15}$$

$$\tag{3.16}$$

3.2 Local existence and uniqueness of the solution

Since $\Omega_T(0) = \Omega(0)$ the hypoteses (3.1)-(3.10) of Theorem 3 are not affected from the change of variable (3.13), provided they are related to the variable y instead of x. For the sake of simplicity, in the following we will refer to them without specify at which variable they are related, except when it will not clear from the context.

So, in the new variable Theorem 3 is equivalent to the following one:

Theorem 4 *Let r_0 be a positive constant and $\tilde{u}_0$, $\tilde{\rho}_0 : [0, r_0] \to \mathbb{R}$ respectively be $C^{4+\alpha}([0, r_0])$ and $C([0, r_0])$ functions such that the hypoteses* (3.1)-(3.10) *of Theorem 3 are verified. Then there exists $T > 0$ such that system (M_F) with (I_F), (B_F) and (LR_F) has a unique solution for all $(y, t) \in Q_T = (0, r_0) \times [0, T]$.*
Moreover u and ρ are respectively a $C^{2,1}(Q_T)$ and a $C(Q_T)$ positive function.

In order to study the existence and the uniqueness of the solution for (M_F) with (I_F), (B_F) and (LR_F) we split system (M_F) in two easier coupled systems and then we will show the existence and the uniqueness of the solution of both systems.

In particular, we split the system in the two following ones:

$$\begin{cases} u_t - (a(t,u)u_y)_y - b(y,t)u_y = 0 & (y,t) \in Q_T \\ u(y,0) = \tilde{u}_0(y) & y \in (0, r_0) \\ u(0,t) = u_0 & t \in (0,T] \\ u(r_0,t) = u_0 + \dfrac{\mu}{bE} l'(t) & t \in (0,T] \end{cases} \tag{S_u}$$

$$\begin{cases} \rho_t - (a(t,u)u_y\rho)_y - b(y,t)\rho_y = K(y,t)u & (y,t) \in Q_T \\ \rho(y,0) = \tilde{\rho}_0(y) & y \in [0, r_0] \\ \rho(0,t) = \rho_0 & t \in (0,T] \end{cases} \tag{S_ρ}$$

where:

$$a(t,u) = E\frac{r_0^2}{(r(t) - l(t))^2}u = a_1(t)a_2(u) \tag{3.17}$$

with

$$a_1(t) = E\frac{r_0^2}{(r(t) - l(t))^2} \quad \text{and} \quad a_2(u) = u \tag{3.18}$$

$$b(y,t) = \frac{l'(t)(r_0 - y) + r'(t)y}{r(t) - l(t)} \tag{3.19}$$

and $K(y,t)$ is defined in (3.14).

3.2.1 Existence and uniqueness for S_ρ

The partial differential equation of system (S_ρ) is of transport type with a flux velocity and a source term depending on y, t and u.

$$\rho_t - a(t)(u_y\rho)_y - b(y,t)\rho_y = K(y,t)u \quad \forall (y,t) \in Q_T$$

Following the classical literature, see for example [16], we can determine its solution using the method of characteristics. In particular, the characteristic curves are defined in the following way:

$$\begin{cases} s = t \\ \dfrac{dy}{dt} = -a(t)u_y - b(y,t) \quad y(0) = y_0 \end{cases} \tag{3.20}$$

Since the boundary data for ρ is defined on $y = 0$, S_ρ is well posed if:

$$\frac{dy}{dt} = a(t)u_y - b(y,t) > 0$$

and for the definition of $a(t)$, (3.17), and $b(y,t)$, (3.19), at $t = 0$:

$$\frac{dy}{dt}(0) = a(0)u_y - b(y,0) = E(\tilde{u_0})_y + \frac{l'(0)(r_0 - y) - r'(0)y}{r_0}$$

and this quantity is negative. In fact from (3.3) $(\tilde{u}_0)_y < 0$ and, from (LR_F), hypoteses (3.4) and (3.5) imply that $l'(0)$ and $r'(0)$ are also negative. So, for $t = 0$ (S_ρ) is well posed and for the definition of T, (3.12), it is well posed for all $t \in [0, T]$.
Along the characteristic curves ρ verifies:

$$\frac{d\rho}{dt} = a(t)u_{yy}(y(t),t)\rho(y(t),t) + K(y(t))u(y(t),t) \quad t \in (0,T] \tag{3.21}$$

Then, along that curves:

$$\rho(t) = e^{A(t)}\left(\overline{\rho}_0(y_0) + B(t)\right) \tag{3.22}$$

where:

$$A(s) = \int_0^s a(s')u_{yy}(y(s'),s')ds' \tag{3.23}$$

and

$$B(s) = \int_0^s e^{-A(s')}K(y(s'),s')u(y(s'),s')ds' \tag{3.24}$$

This prove the existence of the solution for (S_ρ) for all $(y,t) \in Q_T$.
Moreover, this solution is a positive and continuous function.
In fact, let $\delta_0(t) : [0,T] \to [0,r_0]$ such that:

$$K(\delta_0(t),t) = 0 \quad \forall t \in [0,T]$$

Then, since $\rho_t - (a(t,u)\rho)_y - b(y,t)\rho_y = K(y,t)u$, for every fixed $t \in [0,T]$, $\rho(y,t)$ is increasing where $K(y,t)$ is positive and decreasing otherwise. So:

$$\rho(y,t) \geq \begin{cases} \rho_0 & y \in [0,\delta_0(t)] \\ \rho(r_0,t) & y \in [\delta_0(t),r_0] \end{cases}$$

that is, for all $t \in [0,T]$, $\rho(y,t) \geq \min\{\rho_0, \rho(r_0,t)\}$.
As a consequence $\rho_0 > 0$ and $\rho(r_0) > 0$ are sufficient conditions for the positivity of ρ. From hypotesis (3.1), $\rho_0 > 0$. Moreover, from the definition of T, (3.12), for all $t \in [0,T]$, $\rho(r_0) - du(r_0)^{\frac{2}{3}} > 0$. Then, $\rho(r_0) > du(r_0)^{\frac{2}{3}} > 0$.
As far as the regularity of ρ is concerned, we note that from Theorem 3, u is a $C^{2,1}$ function and it is a sufficient condition for the continuity of ρ. We summarize the result obtained in this secton as follows:

Theorem 5 *Let r_0 be a positive constant and $\tilde{u}_0$, $\tilde{\rho}_0 : [0,r_0] \to \mathbb{R}$ respectively be $C^{4+\alpha}([0,r_0])$ and $C([0,r_0])$ functions such that the hypoteses (3.1)-(3.10) of Theorem 3 are verified. Then there exists $T > 0$ such that (S_ρ) has a unique solution which is positive and continuous.*

3.2.2 Existence and uniqueness for (S_u)

The study of the existence and the uniqueness of system (S_u) is very difficult. This is due on one hand to the degeneracy of the equation and on the other to the time dependent coefficients of the equation.

There is a wide literature about porous media equations, e.g. [4] or [30], and quasilinear parabolic degenerate equations, i.e. [8], [9] or [13]. But the general case for coefficients depending on y, t, u and u_y is not treated. A masterpiece in the literature about quasi-linear parabolic equation is [17] and it has been our reference for the understanding of the problem.

In order to prove the existence and the uniqueness of the solution for (S_u), we will study the following more general system:

$$\begin{cases} u_t - \left(g(t,u)u_y\right)_y - b(y,t)u_y = 0 & y \in (\eta_1, \eta_2), t > 0 \\ u(y,0) = \tilde{u}_0(y) \geq 0 & y \in (\eta_1, \eta_2) \\ u(\eta_1, t) = \psi_1(t) & t > 0 \\ u(\eta_2, t) = \psi_2(t) & t > 0 \\ (\psi_1)_t = a(t)(uu_y)_y + b(y,t)(u)_y & \text{for } (y,t) = (\eta_1, 0) \\ (\psi_2)_t = a(t)(uu_y)_y + b(y,t)(u)_y & \text{for } (y,t) = (\eta_2, 0) \end{cases} \tag{3.25}$$

where, $\psi_1(t)$ and $\psi_2(t)$ are C^1 functions and

$$g(t,u) = g_1(t)g_2(u)$$

Moreover, we will assume that the following conditions hold:

$$0 < \lambda_1 \leq g_1(t) \leq \Lambda_1 \quad \forall t \in (0,T] \tag{3.26}$$

$$\begin{cases} g_2(u) \in C^1 \\ g_2(0) = 0 \quad \text{and } g_2'(s) > 0 \quad \forall s > 0 \\ ug_2(u) \text{ is a locally Holder continue function} \end{cases} \tag{3.27}$$

and

$$b(y,t), b_y(y,t) \in L^\infty(Q_T) \tag{3.28}$$

In particular, we will prove the following local existence theorem:

Theorem 6 *Let $\tilde{u}_0 : [0, r_0] \to \mathbb{R}^+$ be a $C^2(0, r_0)$ functions, let $g_1(t) : [0,T] \to \mathbb{R}^+$ $g_2(u) : \mathbb{R}^+ \to \mathbb{R}$ be $C^1(\mathbb{R}^+)$ functions and let λ, Λ be positive constants such that conditions* (3.26)-(3.28) *are verified.*
Then (3.25) has at least one weak solution.

We remark that in (3.25)

$$\begin{cases} u(\eta_1, t) = \psi_1(t) & t > 0 \\ u(\eta_2, t) = \psi_2(t) & t > 0 \end{cases} \tag{3.29}$$

are the so called, *compatibility conditions* of zero order and

$$\begin{cases} (\psi_1)_t = g(t)(uu_y)_y + b(y,t)(u)_y & \text{for } (y,t) = (\eta_1, 0) \\ (\psi_2)_t = g(t)(uu_y)_y + b(y,t)(u)_y & \text{for } (y,t) = (\eta_2, 0) \end{cases} \tag{3.30}$$

are those of the first one. From classical theory about parabolic equation, see [17], these conditions are needed to have u in $C^{2,1}$.
First of all we note if $\tilde{u}_0(y)$ verifies the hypoteses (3.1)-(3.10) of Theorem 3, then (S_u) with (B_F) and (LR_F) is the special case of (3.25) obtained setting:

$$\begin{cases} \eta_1 & = 0 \\ \eta_2 & = r_0 \\ g(t,u) & = E\dfrac{r_0^2}{(r(t)-l(t))}u \\ \psi_1(t) & = u_0 \\ \psi_2(t) & = u_0 + \dfrac{\mu\varepsilon}{b}l'(t) \end{cases}$$

where $a(t,u)$ is defined in (3.17).
Setting $g(t,u) = a(t,u)$ conditions (3.27)-(3.28) are also satisfied.
In fact, since for all $t \in [0,T]$, $r(t) - l(t)$ is a positive and continuous function, setting:

$$m = \min_{t\in[0,T]} r(t) - l(t) \text{ and } M = \max_{t\in[0,T]} r(t) - l(t)$$

and

$$\lambda = E\frac{r_0^2}{M} \quad \text{and} \quad \Lambda = E\frac{r_0^2}{m}$$

we have that:

$$\lambda \le a_1(t) \le \Lambda$$

where $a_1(t)$ is defined in (3.18). Moreover $a_2(u) = u$ verifies (3.27). Finally:

$$b(y,t) = \frac{l'(t)(r_0 - y) + r'(t)y}{r(t) - l(t)}$$

and

$$b_y(y,t) = \frac{-l'(t) + r'(t)}{r(t) - l(t)}$$

are continuous functions and so they are bounded on Q_T. Then, hyotesis (3.28) is also satisfied.
Hypoteses (3.6)-(3.9) correspond to the compatibility conditions of zero and first order for (S_u) and they are verified.
In fact:

$$\tilde{u}_0(0) = u_0 \quad \text{and} \quad \tilde{u}_0(r_0) = u_0 + \frac{\mu\varepsilon}{b}l'(0)$$

So:

$$\begin{cases} \psi_1(0) = \overline{u}_0(0) \\ \psi_2(0) = \overline{u}_0(r_0) \end{cases}$$

With regard to the first order compatibility condition in $y = 0$, from (3.30):

$$\psi_{1t}(0) = \left|(a(t,u)(uu_y)_y + b(y,t)u_y)\right|_{y=0,t=0}$$

Since $\psi_1(t) = u_0$ then $\psi_{1t} = 0$ and

$$\left|(a(t,u)(uu_y)_y + b(y,t)u_y)\right|_{y=0,t=0} = a(0,\overline{u}_0)(\overline{u}_0(\overline{u}_0)_y)_y + b(0,0)(\overline{u}_0)_y = \\ = E(\overline{u}_0(\overline{u}_0)_y)_y + l'(0)(\overline{u}_0)_y$$

So hypotesis (3.8) implies that this compatibility condition holds.
In $y = r_0$, the first compatibility condition is:

$$\psi_{2t}(0) = \left|(a(t,u)(uu_y)_y + b(y,t)u_y)\right|_{y=r_0,t=0}$$

From the definition of $\psi_2(t)$ we obtain that:

$$\psi_{2t}(t) = \frac{\mu}{bE} l''(t)$$

and deriving $l'(t)$, defined in (LR_F), we can write $l''(t)$ as follows:

$$l''(t) = E\frac{\alpha}{\mu-\alpha}\left[\frac{r_0(r'(t)-l'(t))}{(r(t)-l(t))^2}u_y - \frac{r_0}{r(t)-l(t)}u_{yt}\right]_{y=0} = \\ = E\frac{\alpha}{\mu-\alpha}\frac{r_0}{r(t)-l(t)}\left[\frac{r'(t)-l'(t)}{r(t)-l(t)}u_y - \frac{r_0}{r(t)-l(t)}(u_t)_y\right]_{y=0}$$

and

$$(u_t)_y = \left[E\frac{r_0^2}{(r(t)-l(t))^2}(uu_y)_y + \frac{l'(t)(r_0-y)+r'(t)y}{r(t)-l(t)}u_y\right]_y = \\ = \left[E\frac{r_0^2}{(r(t)-l(t))^2}(uu_y)_{yy} + \frac{r'(t)-l'(t)}{r(t)-l(t)}u_y + \frac{l'(t)(r_0-y)+r'(t)y}{r(t)-l(t)}u_{yy}\right]$$

Then

$$l''(t) = E\frac{\alpha}{\mu-\alpha}\frac{r_0}{r(t)-l(t)}\left[\frac{r'(t)-l'(t)}{r(t)-l(t)}\left(1-\frac{r_0}{r(t)-l(t)}\right)u_y + \right. \\ \left. -E\frac{r_0^3}{(r(t)-l(t))^3}(uu_y)_{yy} - r_0\frac{l'(t)(r_0-y)+r'(t)y}{r(t)-l(t)}u_{yy}\right]_{y=0} = \\ = E\frac{\alpha}{\mu-\alpha}\frac{r_0}{r(t)-l(t)}\left[\frac{r'(t)-l'(t)}{r(t)-l(t)}\left(1-\frac{r_0}{r(t)-l(t)}\right)u_y(0,t) + \right. \\ \left. -E\frac{r_0^3}{(r(t)-l(t))^3}(uu_y)_{yy}(0,t) - \frac{r_0^2}{(r(t)-l(t))^2}l'(t)u_{yy}(0,t)\right]$$

So

$$l''(0) = -E\frac{\alpha}{\mu-\alpha}\left[E(\tilde{u}_0(\tilde{u}_0)_y)_{yy} + l'(0)(\tilde{u}_0)_{yy}\right] \tag{3.31}$$

and replacing l'' in the first order compatibility condition we obtain:

$$-\frac{\mu}{b}\frac{\alpha}{\mu-\alpha}\left[E(\overline{u}_0(\overline{u}_0)_y)_{yy} + l'(0)(\overline{u}_0)_{yy}\right] = \\ = \left[E\frac{r_0^2}{(r(t)-l(t))^2}(\overline{u}_0(\overline{u}_0)_y)_y + \frac{l'(t)(r_0-y)+r'(t)y}{r(t)-l(t)}(\overline{u}_0)_y\right]_{y=r_0,t=0} = \\ = \left[E(\tilde{u}_0(\tilde{u}_0)_y)_y + r'(0)(\tilde{u}_0)_y\right]$$

that is hypotesis (3.9).
This proves that (S_u) is a special case of system (3.25) and it verifies hypoteses of Theorem 3. So Theorem 6 is a generalization of Theorem 3.

3.2.3 Proof of Theorem 6

First of all we give the following definition of a weak solution for (3.25).

Definition 2 *A function u defined in* $[\eta_1, \eta_2] \times [0, T]$ *is a weak solution for (3.25), if:*

1. *u is real, non negative and continuous;*
2. $$\begin{cases} u(y,0) = \tilde{u}_0(y) & x \in [\eta_1, \eta_2] \\ u(\eta_1, t) = \psi_1(t) & t \in [0, T] \\ u(\eta_2, t) = \psi_2(t) & t \in [0, T] \end{cases}$$
3. $G_2(u) = \left(\int_0^u g_2(s)ds\right)$ *is in* $C^{2,1}$ *and its derivative with respect to y is a square integrable function;*
4. *u verifies the following identity:*
$$\int_0^T \int_{\eta_1}^{\eta_2} [u_t\varphi - g_1(t)(g_2(u)u)_y\varphi_y - (b(y,t)\varphi)_y u]dydt = -\int_{\eta_1}^{\eta_2} \tilde{u}_0(y)\varphi(y,0)dydt \tag{3.32}$$
for all $\varphi \in C^2$ *such that:*
$$\varphi(\eta_1, t) = \varphi(\eta_2, t) = 0 \quad \forall t \in (0, T]$$
and
$$\varphi(y, T) = 0 \quad y \in (\eta_1, \eta_2)$$

In the literature about first boundary problems for degenerate parabolic equations, this special case is not treated in detail. Among others we point out the article of Gilding [13] and the one of M. Bertsch and S. Kamin [4]. In particular, in [13] the first boundary problem for equations of the following type are treated:

$$u_t = (a(u)u_y)_y + b(u)u_y \tag{3.33}$$

while in [4]:
$$u_t = (a(t,u)u_y)_y + b(y,t,u) \tag{3.34}$$

Our equation doesn't belong neither to (3.33) nor to (3.34).
The indipendence of the coefficients from y, t and u_y, makes the study of equation (3.33) and (3.34) easier then the ours; also the study of (3.34) is easier than the ours, since $b(y,t,u)$ does not depend on u_y. However, since $a(t,u) = a_1(t)a_2(u)$, for our problem we can follow the proof of the existence for (3.33).
The idea of the proof of Theorem 6 is to show that if, for all $y \in [0, r_0]$, $\tilde{u}(y) > 0$, then there exists a sequence of functions $\{u_{0,k}\}$ which uniformly converges to $\tilde{u}_0$ for

$k \to \infty$. Then we show that for all k, the system (3.25) with initial datum $u_{0,k}$ has a unique solution, u_k. Finally we prove that setting

$$u(y,t) = \lim_{k\to\infty} u_k(y,t)$$

then u is a weak solution for system 3.25. In the next section we will show that this solution is also unique.

For the proof we need the following four lemma.

Lemma 3 *Let $f \in C^1(0,\infty)$.*
Then given any M positive constant there exists a function $\vartheta \in C^2[0,M]$ and a positive constant C such that for $s \in (0,M]$:

1. $C \geq |\vartheta(s)| \geq \dfrac{1}{C}$;
2. $\vartheta''(s)\vartheta(s) < 0$;
3. $|f'(s)\vartheta(s) + 2f(s)\vartheta'(s)| \leq -C\vartheta''(s)\vartheta(s)$;
4. $f^2(s) \leq -C\vartheta''\vartheta$.

if and only if

$$F(s) = s|f'(s)| \in L^1(0,M)$$

The proof of this lemma follows from Lemma 3 of [13], simply replacing $b(s)$ with a constant.

Lemma 4 *Let $\varepsilon, \alpha \in (0,1]$ and $M > 0$ be fixed arbitrary constants.*
Let Q_T be the rectangulus $Q_T = (\eta_1,\eta_2) \times (0,T]$, with $-\infty < \eta_1 < \eta_2 < \infty$.
Suppose that $\tilde{u}_0(y)$ is a $C^{2+\alpha}[0,r_0]$ function and that $\psi_1(t)$, $\psi_2(t)$ are $C^{1+\alpha}[0,T]$ functions such that:

$$\begin{cases} \varepsilon \leq \tilde{u}_0(y) \leq M & y \in [\eta_1,\eta_2] \\ \varepsilon \leq \psi_1(t), \psi_2(t) \leq M & t \in [0,T] \\ \psi_i(0) = \tilde{u}_0(\eta_i) & i = 1,2 \\ \psi_i'(0) = \Big(g(0,\tilde{u}_0)(\tilde{u}_0)_y\Big)_y + b(y,0)\tilde{u}_0(y)_y & i = 1,2 \end{cases} \tag{3.35}$$

Then, if $g_1(t), g_2(u)$ and $b(y,t)$ verify respectively conditions (3.26), (3.27) and (3.28) there exists a unique function $u(y,t)$ such that:

1. $u(y,t) \in C^{2,1}(\overline{Q}_T)$;
2. $G_2(u) = \displaystyle\int_0^{u(y,t)} g_2(r)dr$ *is such that* $(G_2(u))_y \in C^{2,1}(Q)$ *and its derivative with respect to y is a square integrable function;*
3. $\varepsilon \leq u \leq M$, $\forall (y,t) \in Q_T$;
4. $u_t = (g(t,u)u_y)_y + b(y,t)u_y$, $\forall (y,t) \in Q_T$;

5.

$$\begin{cases} u(y,0) = \tilde{u}_0(y) \\ u(\eta_i, t) = \psi_i(t) \quad \forall t \in [0,T], i = 1,2 \end{cases}$$

The proof of this Lemma uses Lemma 3 and some properties of the non degenerate parabolic equations. In particular, since $g(t,u)$ has a continuous derivative with respect to u and verifies conditions (3.26) and (3.27), then there exists $h(u)$ such that:

$$\begin{cases} h(s) = g_2(s) & \varepsilon \leq u \leq M \\ h'(s) = 0 & \text{otherwise} \end{cases}$$

Then there exist $\gamma \in [0,1]$ and a function $u : [0, r_0] \times [0,T] \to \mathbb{R}$ in $C^{2+\gamma,1}$ such that:

$$u_t - g_1(t)(h(u)u_y)_y - b(y,t)u_y = 0 \quad \text{for all} \quad (y,t) \in Q_T \tag{3.36}$$

with boundary conditions (B_F) has a unique solution.
In fact, $g_1(t)h(u)$ and $b(y,t)$ verifies the hypoteses of the existence Theorem 6.1 (pag.452 of [17]), for parabolic non degenerate equation. In particular, writing equation (3.36) as follows:

$$u_t - (g_1(t)h(u))u_{yy} - b(y,t)u_y = 0 \quad \text{for all} \quad (y,t) \in Q_T$$

we have that:

$$g_1(t)h(u)\xi^2 = E\frac{r_0^2}{(r(t)-l(t))^2}u\xi^2 \geq \lambda\varepsilon\xi^2 > 0$$

and

$$(g_1(t)h(u) + b(y,t))u_y\Big|_{u_y=0} = 0 > -c_1u^2 - c_2$$

for all c_1 and c_2 positive constants.
Then to prove the Lemma, we have only to show that $G_2(u)$ has a generalized square integrable derivative. This can be shown setting $v = G_2(u)$ and noting that v verifies:

$$v_t = g_1(t)h(u)v_{yy} + b(y,t)v_y \quad \text{in} \quad Q_T$$

The required regularity is then obtained noting that $u \in C^{2,1}(Q_T)$ and using standard theory on parabolic equations, for further detail see [11].

Lemma 5 *Let the assumptions of Lemma 4 hold and let $u(y,t)$ be the function exhibited in the same Lemma. Suppose that*

$$\left|G_2'(\tilde{u}_0)\right| \leq K_0 \quad \forall y \in \left[\eta_1 + \frac{\delta}{2}, \eta_2 - \frac{\delta}{2}\right]$$

with K_0 and δ positive constants.
Then if

$$sg_2'(s) \in L^1(0,1)$$

there exists a constant $K = K(K_0, \delta, M)$ such that:

$$|G_2(u(y_1,t_1)) - G_2(u(y_2,t_2))| \leq K\left[|y_1 - y_2|^2 + |t_1 - t_2|\right]^{\frac{1}{2}}$$

$\forall (y_1,t_1),(y_2,t_2) \in \overline{Q_\delta} = \left[\eta_1 + \frac{1}{2}\delta, \eta_2 - \frac{1}{2}\delta\right] \times [0,T].$

Proof. $sg_2'(s) \in L^1(0,1)$, so

$$F(s) = s|g_2'(s)| \in L^1(0,M).$$

Then, for Lemma 3, there exists a function $\vartheta(s) \in C^2[0,M]$ such that:

1. $C \geq |\vartheta(s)| \geq \dfrac{1}{C}$;
2. $\vartheta''(s)\vartheta(s) < 0$;
3. $|g_2'(s)\vartheta(s) + 2g_2(s)\vartheta'(s)| \leq -C\vartheta''(s)\vartheta(s)$;
4. $g_2^2(s) \leq -C\vartheta''\vartheta$.

Set

$$w(y,t) = \int_0^{u(y,t)} g_2(s)\vartheta^{-1}(s)ds$$

The proof of the lemma needs some properties of $w(y,t)$.

$$w_t = \frac{g_2(u)}{\vartheta(u)}u_t \quad \text{and} \quad w_y = \frac{g_2(u)}{\vartheta(u)}u_y$$

and

$$w_{yy} = \frac{g_2(u)}{\vartheta(u)}u_{yy} + \left[\frac{g_2'(u)}{\vartheta(u)} - g_2(u)\frac{\vartheta'(u)}{\vartheta^2(u)}\right]u_y^2 \tag{3.37}$$

Since $u_t = (g(t,u)u_y)_y + b(y,t)u_y$, then:

$$\begin{aligned}
w_t &= \frac{g_2(u)}{\vartheta(u)}\left[g_1(t)g_2(u)u_{yy} + g_1(t)g_2'(u)u_y^2 + b(y,t)u_y\right] \\
&= g_1(t)g_2(u)\left[w_{yy} + \left(g_2(u)\frac{\vartheta'(u)}{\vartheta^2(u)} - \frac{g_2'(u)}{\vartheta(u)}\right)u_y^2\right] + g_1(t)g_2(u)\frac{g_2'(u)}{\vartheta(u)}u_y^2 + b(y,t)w_y = \\
&= g(t,u)w_{yy} + g_1(t)\vartheta'(u)w_y^2 + b(y,t)w_y
\end{aligned}$$

We differentiate this equation with respect to y and multiplicate it for w_y. So, we obtain:

$$\begin{aligned}
w_y w_{yt} &= g_1(t)g_2(u)w_y w_{yyy} + \left[\frac{g_1(t)}{g_2(u)}g_2'(u)\vartheta(u) + 2g_1(t)\vartheta'(u)\right]w_y^2 w_{yy} + \\
&+ \frac{g_1(t)}{g_2(u)}\vartheta(u)\vartheta''(u)(w_y)^4 + b_y(y,t)w_y^2 + b(y,t)w_y w_{yy}
\end{aligned}$$

Setting $p = w_y$ the previous equation becomes:

$$\begin{aligned}
\frac{1}{2}(p^2)_t &= g_1(t)g_2(u)pp_{yy} + \left[\frac{g_1(t)}{g_2(u)}g_2'(u)\vartheta(u) + 2g_1(t)\vartheta'(u)\right]p^2 p_y \\
&+ \frac{g_1(t)}{g_2(u)}\vartheta(u)\vartheta''(u)p^4 + b_y(y,t)p^2 + b(y,t)pp_y
\end{aligned} \tag{3.38}$$

Let's define $z(y,t)$ in the following way:

$$z(y,t) = \zeta^2(y)p^2(y,t) \tag{3.39}$$

where $\zeta \in C^2[\eta_1, \eta_2]$ is a *cut-off* function such that:

$$\zeta(y) = \begin{cases} 1 & y \in \left[\eta_1 + \frac{3}{4}\delta, \eta_2 - \frac{3}{4}\delta\right] \\ 0 & y \in \left[\eta_1, \eta_1 + \frac{1}{2}\delta\right] \cup \left[\eta_2 - \frac{1}{2}\delta, \eta_2\right] \end{cases} \tag{3.40}$$

and $\forall y \in [\eta_1, \eta_2] \quad 0 \leq \zeta(y) \leq 1$.
If $z(y, t)$ has a maximum point in Q_T, then at this point:

$$z_y = z_t = 0 \text{ and } z_{yy} < 0$$

or, since $g(t, u)$ is positive,

$$z_y = 0 \text{ and } g(t, u)z_{yy} - z_t = 0$$

From (3.39):

$$z_t = 2\zeta^2(y)pp_t \tag{3.41}$$

$$z_y = 2\zeta p(\zeta' p + \zeta p_y) \tag{3.42}$$

and

$$z_{yy} = 2[\zeta' p + \zeta p_y]^2 + 2\zeta p[\zeta'' p + 2\zeta' p_y + \zeta p_{yy}] \tag{3.43}$$

$z_y = 0$ implies:

$$\zeta' p = -\zeta p_y \tag{3.44}$$

and at this point $g(t, u)z_{yy} - z_t$ becomes:

$$\begin{aligned} g(t, u)z_{yy} - z_t &= 2g(t, u)\zeta p[\zeta'' p + 2\zeta' p_y + \zeta p_{yy}] - 2\zeta^2 pp_t = \\ &= 2[\zeta^2(g(t, u)pp_{yy} - pp_t) + g(t, u)\zeta\zeta'' p^2 - 2g(t, u)p^2(\zeta')^2] \end{aligned}$$

and so $g(t, u)z_{yy} - z_t < 0$ if and only if:

$$\zeta^2 \left(\frac{1}{2}(p^2)_t - g(t, u)pp_{yy}\right) \geq g(t, u)p^2(\zeta\zeta'' - 2(\zeta')^2) \tag{3.45}$$

and using equation (3.38):

$$\begin{aligned} \zeta^2 \left(\frac{1}{2}(p^2)_t - g(t, u)pp_{yy}\right) &= \zeta^2 \left[\frac{g_1(t)}{g_2(u)}\vartheta\vartheta'' p^4 + \right. \\ &+ \left. \left(\frac{g_1(t)}{g_2(u)}g_2'\vartheta + 2g_1(t)\vartheta'\right) p^2 p_y + b_y p^2 + bpp_y\right] = \\ &= \zeta^2 p^2 \left[\frac{g_1(t)}{g_2(u)}\vartheta\vartheta'' p^2 + \left(\frac{g_1(t)}{g_2(u)}g_2'\vartheta + 2g_1(t)\vartheta'\right) p_y + b_y\right] + \\ &- b\zeta\zeta' p^2 \end{aligned}$$

where in the last equation we have used (3.44). So equation (3.45) is equivalent to:

$$\zeta^2 \left[\frac{g_1(t)}{g_2(u)}\vartheta\vartheta'' p^2 + \left(\frac{g_1(t)}{g_2(u)}g_2'\vartheta + 2g_1(t)\vartheta'\right) p_y + b_y\right] - b\zeta\zeta' \geq g(t, u)(\zeta\zeta'' - 2(\zeta')^2)$$

That is:

$$-\frac{g_1(t)}{g_2(u)}\vartheta\vartheta''\zeta^2 p^2 \leq \zeta^2 p_y\left(\frac{g_1}{g_2}g_2'\vartheta + 2g_1\vartheta'\right) + b_y\zeta^2 - b\zeta\zeta'' + g(t,u)(2\zeta' - \zeta\zeta'') =$$

$$= -\zeta\zeta' p\left(\frac{g_1}{g_2}g_2'\vartheta + 2g_1\vartheta'\right) + b_y\zeta^2 - b\zeta\zeta'' + g(t,u)(2\zeta' - \zeta\zeta'')$$

So, since for Lemma 3, $\vartheta\vartheta'' < 0$ and $g_1(t), g_2(u) > 0$:

$$\zeta^2 p^2 \leq \frac{\zeta\zeta'}{\vartheta\vartheta''}p(g_2'\vartheta + 2g_2\vartheta) + \frac{g_2^2}{\vartheta\vartheta''}(\zeta\zeta'' - 2\zeta') - \frac{g_2}{g_1}\frac{b_y}{\vartheta\vartheta''}\zeta^2 + \frac{g_2}{g_1}\frac{b}{\vartheta\vartheta''}\zeta\zeta' \leq$$

$$\leq \left|\frac{\zeta\zeta'}{\vartheta\vartheta''}p(g_2'\vartheta + 2g_2\vartheta)\right| + \left|\frac{g_2^2}{\vartheta\vartheta''}(\zeta\zeta'' - 2\zeta')\right| + \left|\frac{g_2}{g_1}\frac{b_y}{\vartheta\vartheta''}\zeta^2\right| + \left|\frac{g_2}{g_1}\frac{b}{\vartheta\vartheta''}\zeta\zeta'\right|$$

and again from Lemma 3:

$$\left|\frac{g_2'\vartheta + 2g_2\vartheta'}{\vartheta\vartheta''}\right| \leq C$$

and

$$\left|\frac{g_2^2}{\vartheta\vartheta''}\right| \leq C$$

Moreover, since $g_2' \geq 0$ in $[0, M]$ and $\vartheta > 0$:

$$\left|\frac{2g_2\vartheta'}{\vartheta\vartheta''}\right| \leq \left|\frac{g_2'\vartheta + 2g_2\vartheta'}{\vartheta\vartheta''}\right| \leq C$$

and then:

$$\left|\frac{g_2}{\vartheta\vartheta''}\right| \leq \frac{1}{2}\frac{C}{\vartheta'}$$

From Lemma 3, $\vartheta'' < 0$ and so ϑ' is a decreasing function. Then:

$$\left|\frac{g_2}{\vartheta\vartheta''}\right| \leq \frac{1}{2}\frac{C}{\vartheta'(M)} = C'$$

Using these inequalities we obtain the following upper bound for $\zeta^2 p^2$:

$$\zeta^2 p^2 \leq C\left[|\zeta\zeta' p| + |\zeta\zeta'' - 2\zeta'| + \frac{1}{|\vartheta'(M)|}\left|\frac{b}{g_1}\zeta\zeta'\right| + \frac{1}{|\vartheta'(M)|}\left|\frac{b_y}{g_1}\zeta^2\right|\right] \tag{3.46}$$

Since b and b_y are bounded functions and g_1 is positive and continuous in Q_T, b/g_1 and b_y/g_1 are also bounded. Let C_b and C_{b_y} their respectively upper bounds.
Set

$$C' = \frac{1}{|\vartheta'(M)|}\max\{C_b, C_{b_y}\}$$

then equation (3.46) becomes:

$$\zeta^2 p^2 \leq C\left[|\zeta\zeta' p| + |\zeta\zeta'' - 2\zeta'| + C'|\zeta\zeta'| + C'|\zeta^2|\right]$$

Using Young's inequality:

$$C\zeta\zeta' p \le \frac{\zeta^2 p^2}{2} + \frac{C^2(\zeta')^2}{2} \le \frac{\zeta^2 p^2}{2} + C^2(\zeta')^2$$

and so:

$$\frac{1}{2}z = \frac{1}{2}\zeta^2 p^2 \le C\left[C(\zeta')^2 + |\zeta\zeta'' - 2\zeta'| + C'|\zeta\zeta'| + C'|\zeta^2|\right] \tag{3.47}$$

On the other hand if the maximum value of z is not an interior point of Q_T it has to be on the lower bound of $\overline{Q}_T$ and by definition:

$$\begin{aligned} z(y,0) &= \zeta^2(y)p^2(y,0) = \zeta^2(y)(w_y(y,0))^2 = \\ &= \zeta^2(y)\left[\left(\int_0^u (y,0)g_2(s)\vartheta^{-1}(s)ds\right)_y\right]^2 = \\ &= \zeta^2(y)(u_y(y,0))^2 g_2^2(u(y,0))\vartheta^{-2}(u(y,0)) = \\ &= \zeta^2(y)\left((G_2)'(u_0)\right)^2 \vartheta^{-2} \end{aligned}$$

$|\zeta| \le 1$ for its definition, (3.40), and since for Lemma 3:

$$\vartheta^{-1}(s) \ge \frac{1}{C}$$

and for the hypotesis of the lemma:

$$|(G_2)'(\tilde{u}_0)| \le K_0 \quad \text{for} \quad y \in \left[\eta_1 + \frac{1}{2}\delta, \eta_2 - \frac{1}{2}\delta\right]$$

then:

$$z(y,0) \le C^2 K_0^2 \tag{3.48}$$

From equations (3.47) and (3.48) we have that:

$$\sup_{\overline{Q}_\delta} |w_y| \le C_1 = C_1(K_0, M, \delta)$$

and since $w_y = \frac{1}{\vartheta}(G_2(u))_y$:

$$\sup_{\overline{Q}_\delta} |(G_2(u))_y| \le CC_1$$

Now setting $v(y,t) = G_2(y,t)$, then:

$$v_t = g(t,u)v_{yy} + b(y,t)v_y$$

and

$$|v(y_1,t) - v(y_2,t)| \le CC_1|y_1 - y_2| \quad \forall (y_1,t),(y_2,t) \in \overline{Q}_\delta \tag{3.49}$$

Moreover v has a bound which depends only on M.
Then there exists C_2 such that:

$$|v(y,t_1) - v(y,t_2)| \le C_2|t_1 - t_2|^{\frac{1}{2}} \tag{3.50}$$

From equations (3.49) and (3.50) the theorem is proved. In fact:

$$\begin{aligned}|v(y_1,t_1)-v(y_2,t_2)| &\leq |v(y_1,t_1)-v(y_1,t_2)|+|v(y_1,t_2)-v(y_2,t_2)|\\ &\leq C_2|t_1-t_2|^{\frac{1}{2}}+CC_1|y_1-y_2|\\ &\leq K\left(|y_1-y_2|^2+|t_1-t_2|\right)^{\frac{1}{2}}\end{aligned}$$

Then:

$$|G_2(y_1,t_1)-G_2(y_2,t_2)|\leq K(|y_1-y_2|^2+|t_1-t_2|)^{\frac{1}{2}}\quad\forall (y_1,t_1),(y_2,t_2)\in\overline{Q}_\delta$$

Lemma 6 *Let the assumptions of Lemma 4 hold and let $u(y,t)$ the function exhibited in the same Lemma. Suppose that there exist two positive constants, K_0 and K_0' such that:*

$$|(G_2(\tilde{u}_0))'|\leq K_0\quad\forall y\in\left[\eta_1+\frac{1}{2}\delta,\eta_2-\frac{1}{2}\delta\right]$$

and

$$\int_0^T|(G_2(\psi_i))'|dt\leq K_0'\quad i=1,2$$

Then, if $sg'(s)\in L^1(0,M)$, there exists a positive constant $L=L(K_0,K_0',M,\delta,T)$ such that:

$$\int\int_{Q\setminus Q_\delta}\left((G_2(u)_y)\right)^2dydt\leq L$$

Proof. We will only prove that

$$\int_0^T\int_{\eta_1}^{\eta_1+\frac{1}{2}\delta}(G_2(u)_y)^2dydt\leq\frac{1}{2}L$$

The proof is the same for

$$\int_0^T\int_{\eta_2-\frac{1}{2}\delta}^{\eta_2}(G_2(u)_y)^2dydt\leq\frac{1}{2}L$$

Set:

$$\chi(y,t)=\frac{1}{g_1(t)}[G_2(u(y,t))-G_2(\psi_1(t))]$$

then, since $u_t-(g(t,u)u_y)_y-b(y,t)u_y=0$:

$$\int_0^T\int_{\eta_1}^{\eta_1+\frac{1}{2}\delta}[u_t-(g_1(t)g_2(u)u_y)_y-b(y,t)u_y]\chi(y,t)dydt=0\tag{3.51}$$

Noting that:

$$(g_2(u)u_y)_y=(G_2(u))_{yy}$$

Equation (3.51) is equivalent to:

$$
\begin{aligned}
0 &= \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} [u_t - (g_1(t)g_2(u)u_y)_y - b(y,t)u_y]\chi(y,t)dydt = \\
&= \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} u_t \frac{G_2(u(y,t))}{g_1(t)} dydt - \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} u_t \frac{G_2(\psi_1(t))}{g_1(t)} dydt + \\
&- \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} (G_2(u))_{yy}[G_2(u) - G_2(\psi_1)]dydt - \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} b(y,t)u\chi(y,t)dydt
\end{aligned}
$$

But

$$
\begin{aligned}
&\int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} (G_2(u))_{yy}[G_2(u) - G_2(\psi_1)]dydt = \\
&= \int_0^T [G_2(u)_y(G_2(u) - G_2(\psi_1))]_{\eta_1}^{\eta_1+\frac{1}{2}\delta} dt - \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} \left(G_2(u)_y\right)^2 dydt = \\
&= \int_0^T \left[G_2\left(u\left(\eta_1 + \frac{1}{2}\delta, t\right)\right)_y (G_2(u) - G_2(\psi_1))\right]\left(\eta_1 + \frac{1}{2}\delta, t\right) dt + \\
&- \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} \left(G_2(u)_y\right)^2 dydt
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} b(y,t)u_y\chi(y,t)dydt = \\
&= \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} [b(y,t)u]_y\chi(y,t) - \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} b_y(y,t)u\chi(y,t)dydt
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} [b(y,t)u]_y\chi(y,t)dydt = \\
&\int_0^T [b(y,t)u(y,t)\chi(y,t)]_{\eta_1}^{\eta_1+\frac{1}{2}\delta} dt - \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} b(y,t)u(y,t)\frac{G_2(u)_x}{g_1(t)} dydt
\end{aligned}
$$

So, equation (3.51) implies:

$$
\begin{aligned}
\int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} \left(G_2(u)_y\right)^2 dydt = &- \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} u_t \frac{G_2(u) - G_2(\psi_1)}{g_1(t)} dydt + \\
&- \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} b(y,t)u(y,t)\frac{G_2(u)_y}{g_1(t)} dydt + \\
&- \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} b_y(y,t)u\chi(y,t)dydt + \\
&+ \int_0^T \left[\left(G_2(u)_y - \frac{b(y,t)}{g_1(t)}u(y,t)\right)(G_2(u) - G_2(\psi_1))\right](\eta_1 + \frac{1}{2}\delta, t)dt
\end{aligned}
\tag{3.52}
$$

Denote the four integrals on the right hand side by I_1, I_2, I_3, I_4 respectively. We estimate them in turn.
Let C_1 and C_{g_1} be positive constants such that:

$$0 < C_{g_1} \leq \inf_{t\in[0,T]} g_1(t)$$

and

$$C_1 \geq \sup_{s\in(0,M]} G_2(s), \quad \sup_{s\in(0,M]} \int_0^s G_2(r)dr$$

$$C_1 \geq \sup_{y,t\in[\eta_1,\eta_1+\frac{1}{2}\delta]\times[0,T]} \left|\frac{b(y,t)}{g_1(t)}u(y,t)\right|, \quad \sup_{y,t\in[\eta_1,\eta_1+\frac{1}{2}\delta]\times[0,T]} \left|\frac{b_y(y,t)}{g_1(t)}u(y,t)\right|$$

Since for Lemma 5 there exists $K = K(K_0, M, \delta)$ such that $\forall t \in (0, T]$ and $\forall u \in (0, M]$:

$$\left|G_2(u)_y\left(\eta_1 + \frac{1}{2}\delta, t\right)\right| \leq K \quad \forall t \in [0, T]$$

then:

$$I_1 = \int_0^T \left[\left(G_2(u)_y - \frac{b(y,t)}{g_1(t)}u(y,t)\right)(G_2(u) - G_2(\psi_1))\right](\eta_1 + \frac{1}{2}\delta, t)dt \leq$$
$$\leq \int_0^T (|(G_2(u(\eta_1 + \frac{1}{2}\delta), t))_y| + C_1)2C_1dt = 2C_1\left[\int_0^T |(G_2(u(\eta_1 + \frac{1}{2}\delta), t))_y| + C_1T\right]$$

and so $I_1 \leq\leq 2C_1T(K + C_1)$

$$I_2 = \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} b(y,t)u(y,t)\frac{G_2(u)_y}{g_1(t)}dydt \leq$$
$$\leq C_1 \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} G_2(u)_ydydt = C_1 \int_0^T [|G_2(u)|]_{\eta_1}^{\eta_1+\frac{1}{2}\delta}dt \leq$$
$$\leq 2C_1^2T$$

$$I_3 = \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} b_y(y,t)u\chi(y,t)dydt \leq$$
$$\leq \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} C_1|G_2(u) - G_2(\psi_1)|dydt \leq$$
$$\leq 2C_1^2T\delta$$

$$
\begin{aligned}
I_4 &= \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} u_t \frac{G_2(u) - G_2(\psi_1)}{g_1(t)} dydt = \\
&= \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} u_t \frac{G_2(u)}{g_1(t)} dydt - \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} u_t \frac{G_2(\psi_1)}{g_1(t)} dydt \leq \\
&\leq \frac{1}{C_{g_1}} \left[\left| \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} u_t G_2(u) dydt \right| + \left| \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} u_t G_2(\psi_1) dydt \right| \right] = \\
&= \frac{1}{C_{g_1}} \left[\left| \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} \frac{\partial}{\partial t} \left(\int_0^u G_2(s) ds \right) dydt \right| + \right. \\
&+ \left. \left| \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} [uG_2(\psi_1)]_0^T \, dx - \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} u(G_2(\psi_1(t))')dydt \right| \right]
\end{aligned}
$$

then from the hypoteses of the lemma $\int_0^T |G(\psi_1(t))|dt \leq K_0'$

$$
\begin{aligned}
I_4 &\leq \frac{1}{C_{g_1}} \left[\int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} \left| \left(\int_0^u G_2(s)ds \right)_0^T \right| + \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} 2MC_1 + \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} MK_0' \right] \leq \\
&\leq \frac{1}{C_{g_1}} \left[2C_1\delta + 2MC_1\delta + MK_0'\delta \right]
\end{aligned}
$$

These estimates of I_1, I_2, I_3 and I_4 make $\int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} (A_2(u)_x)^2 dxdt$ bounded.
In fact, setting:

$$
\frac{1}{2}L = 2C_1T(K + C_1) + 2C_1^2T + 2C_1^2T\delta + \frac{\delta}{C_{a_1}}[2C_1 + 2MC_1 + MK_0']
$$

and substituting this estimate in equation (3.52), we obtain:

$$
\int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} \left(G_2(u)_y \right)^2 dydt \leq \frac{1}{2}L \tag{3.53}
$$

and $L = L(K, K_0', M, \delta)$ but it doesn't depend on ε.
Proof of Theorem 6. As in [13], we can choose positive constants M and K_0', sequences of positive constants $\{\varepsilon_k\}_{k=1}^{\infty}$, $\{\alpha_k\}_{k=1}^{\infty}$ and sequences of functions $\{u_{0,k}\}_{k=0}^{\infty}$, $\{\psi_{1,k}\}_{k=1}^{\infty}$ and $\{\psi_{2,k}\}_{k=1}^{\infty}$ such that:

1. $\varepsilon_k, \alpha_k \in (0, 1] \quad \forall k$;
2. $u_{0,k} \in C^{2+\alpha_k}[\eta_1, \eta_2] \quad \forall k$;
3. $\psi_{1,k}, \psi_{2,k} \in C^{1+\alpha_k}[0, T] \quad \forall k$;
4.

$$
\varepsilon_k \leq u_{0,k}(y) \leq M \qquad \forall y \in [\eta_1, \eta_2], \forall k \tag{3.54}
$$

$$
\varepsilon_k \leq \psi_{1,k}(t), \psi_{2,k}(t) \leq M \qquad \forall t \in [0, T], \forall k \tag{3.55}
$$

5.

$$u_{0,k+1}(y) \leq u_{0,k}(y) \qquad \forall x \in [\eta_1, \eta_2], \forall k \tag{3.56}$$

$$\psi_{1,k+1}(t) \leq \psi_{1,k}(t) \qquad \forall t \in [0, T], \forall k \tag{3.57}$$

$$\psi_{2,k+1}(t) \leq \psi_{2,k}(t) \qquad \forall t \in [0, T], \forall k \tag{3.58}$$

$$\tag{3.59}$$

6.

$$\psi_{1,k}(0) = u_{0,k}(\eta_1) \text{ and } \psi_{2,k}(0) = u_{0,k}(\eta_2)$$

Moreover:

7.

$$(\psi_{i,k})'(0) = (g(t, u_{0,k})(u_{0,k})_y)_y(\eta_i) + b(y, t)(u_{0,k})_y(\eta_i) \quad i = 1, 2$$

8. $\forall \delta \in (0, 1)$ there exists a constant $K_0(\delta)$ such that:

$$|(G_2(u_{0,k}))'(y)| \leq K_0(\delta) \quad \forall y \in (\eta_1 + \delta, \eta_2 - \delta) \quad \forall k$$

9. $\int_0^T G_2(\psi_{1,k}(t))dt, \int_0^T G_2(\psi_{2,k}(t))dt \leq K_0' \quad \forall k$

10. $u_{0,k} \to u_0(y)$ for $k \to \infty$ uniformly $\forall y \in [\eta_1, \eta_2]$

11. $\psi_{1,k} \to \psi_1$ and $\psi_{2,k} \to \psi_2$ for $k \to \infty$ uniformly $\forall t \in [0, T]$

12. $\varepsilon_k \to 0$ as $k \to \infty$

So, for Lemma 4 there exists a unique function $u_k(y, t)$ such that:

1. $(A_2(u_y))_y \in C^{2,1}(R)$
2. $\varepsilon_k \leq u_k(y, t) \leq M \quad \forall (y, t) \in \overline{Q}_T$
3. $(u_k)_t - (g(t, u_k)(u_k)_y)_y - b(y, t)(u_k)_y = 0 \quad \forall (y, t) \in \overline{Q}_T$
4. $u_k(y, 0) = \tilde{u}_0(y) \quad \forall y \in [\eta_1, \eta_2]$
5. $u_k(\eta_1, t) = \psi_{1,k} \quad \forall t \in [0, T]$
6. $u_k(\eta_2, t) = \psi_{2,k} \quad \forall t \in [0, T]$

In view of the monotonicity conditions on $\{u_{0,k}\}_{k=1}^{\infty}, \{\psi_{1,k}\}_{k=1}^{\infty}$ and on $\{\psi_{2,k}\}_{k=1}^{\infty}$ we can define a real non negative bounded function

$$u(y, t) = \lim_{k \to \infty} u_k(y, t)$$

As in [13] we can prove that this is a weak solution for system (3.25).

Moreover, as in [13], thanks to Lemma 5 and Lemmma 6, we can prove the continuity of u in the interior of Q and that the operator $G_2(u)$ has a generalized square integrable derivative.

So, we have only to prove the continuity of $u(y,t)$ for $y = \eta_1, \eta_2$. We will show the continuity only in $y = \eta_1$. For $y = \eta_2$ it can be shown in a similar way.
In order to prove the continuity of $u(y,t)$ in $y = \eta_1$, it's enough to prove that $\forall t_0 \in [0,T]$:

$$\limsup_{(y,t)\to(\eta_1,t_0)} u(y,t) \leq \psi_1(t_0) \tag{3.60}$$

and

$$\liminf_{(y,t)\to(\eta_1,t_0)} u(y,t) \geq \psi_1(t_0) \tag{3.61}$$

Equation (3.60) can be proven as follows:

$$u(y,t) \leq u_k(y,t) \quad \forall (y,t) \in \overline{Q}_T \quad \forall k$$

Then:

$$\limsup_{(y,t)\to(\eta_1,t_0)} u(y,t) \leq \limsup_{(y,t)\to(\eta_1,t_0)} u_k(y,t) = \psi_{1.k}(t_0)$$

and equation (3.60) is obtained letting $k \to \infty$.
In order to prove equation (3.61) we will show that for any $\varepsilon \in (0, \psi_1(t_0))$, we can define a function $w(y,t)$ such that:

$$\liminf_{(y,t)\to(\eta_1,t_0)} w(y,t) = \psi_1(t_0) - \varepsilon$$

and

$$u_k(y,t) \geq w(y,t) \text{ for sufficiently large } k \text{ and } \forall (y,t) \in \overline{Q}_T \tag{3.62}$$

If $\psi_1(t_0) = 0$ then trivially equation (3.61) is verified. In fact:

$$\liminf_{(y,t)\to(\eta_1,t_0)} u(y,t) \geq 0$$

then we focused on $\psi(t_0) > 0$.
Let ε be a constant such that $\varepsilon \in (0, \psi_1(t_0))$. Set

$$\beta = 1 + M sup_{(y,t)\in\overline{Q}_T} b(y,t)$$

Then we can define the following functions:

$$\rho(c) = \int_0^M a(t,r)(cr + b(y,t)r + \beta)^{-1} dr \tag{3.63}$$

and

$$\lambda(c) = t_0 - \frac{1}{c} \int_0^{\psi_1(t_0)-\varepsilon} a(t,r)(cr + b(y,t)r + \beta)^{-1} dr \tag{3.64}$$

We note that:

$$\lim_{c\to\infty} c(t_0 - \lambda(c)) = 0^+$$

If $t_0 > 0$ then we choose and fix c so large that:

$$\begin{cases} \lambda(c) > 0 \\ c(t_0 - \lambda(c)) \leq \eta_2 - \eta_1 \\ \psi_1(t) \geq \psi_1(t_0) - \dfrac{\varepsilon}{2} \qquad \forall t \in [\lambda(c), t_0] \end{cases} \tag{3.65}$$

and set $t_1 = \lambda(c)$.
Else we choose and fix c so large that:

$$\begin{cases} -c\lambda(c) = c(t_0 - \lambda(c)) < \eta_2 - \eta_1 \\ u_0(y) \geq u_0(0) - \dfrac{\varepsilon}{2} = \psi_1(t_0) - \dfrac{\varepsilon}{2} \quad \forall y \in [\eta_1, \eta_1 - c\lambda(c)] \end{cases} \tag{3.66}$$

and set $t_1 = t_0$.
Now we define an increasing function $h : [0, \rho(c)] \to [0, M]$ as follows:

$$\eta = \int_0^{h(\eta)} g(t,r)(cr + b(y,t)r + \beta)^{-1} dr$$

This expression identify a bijection between $[0, \rho(c)]$ and $[0, M]$.
In particular, if we define $G(t,u)$ as follows:

$$G(t,u) = \int_0^u g(t,r)dr$$

we find that:

$$(G(t,h(\eta)))_\eta = g(t,h)h' = ch + b(y,t)h + \beta \text{ on } [0,\rho(c)]$$

and

$$(G(t,h(\eta)))_{\eta\eta} = h'(c + b(y,t)) \text{ on } [0,\rho(c)] \tag{3.67}$$

We remark that, by definition:

$$h(c(t_0 - \lambda(c))) = \psi_1(t_0) - \varepsilon \tag{3.68}$$

If $t_0 < T$ then from (3.65), (3.66) and (3.68) we can choose t_2 such that:

$$\begin{array}{ll} t_0 < t_2 \leq T & \\ c(t_2 - \lambda(c)) < \eta_2 - \eta_1 & \\ h(c(t_2 - \lambda(c))) < \psi_1(t_0) - \dfrac{\varepsilon}{2} & \\ \psi_1(t) \geq \psi_1(t_0) - \dfrac{\varepsilon}{2} & \forall t \in [t_0, t_2] \end{array}$$

else we set $t_2 = T = t_0$.
Let m so large that $\varepsilon_k < \psi_1(t_0) - \varepsilon$ for all $k \geq m$ and for each m we define a point σ_k such that:

$$h(\sigma_k) = \varepsilon_k$$

We set:

$$\Omega_k = \{(y,t) : t_1 < t \leq t_2, \eta_1 < y < \eta_1 - \sigma_k - c(t - \lambda(c))\}$$

and

$$\Gamma_k = \{(y,t) : t_1 < t \leq t_2, y = \eta_1 - \sigma_k - c(t - \lambda(c))\}$$

Set $\Omega = \cup_k \Omega_k$. Since

$$\sigma_k \to 0 \text{ for } \varepsilon_k \to 0$$

then

$$\Omega = \{(y,t) : t_1 < t \le t_2, \eta_1 < y < \eta_1 + c(t - \lambda(c))\}$$

and since, by definition $c(t-\lambda(c)) < \eta_2 - \eta_1$, then $\Omega \subset Q$. Now, we define the function $w(y,t)$ in the following way:

$$w(y,t) = \begin{cases} h(\eta_1 - y + c(t - \lambda(c))) & (y,t) \in \overline{\Omega} \\ 0 & (y,t) \in \overline{Q} \setminus \overline{\Omega} \end{cases}$$

In particular:

$$\liminf_{(y,t)\to(\eta_1,t_0),(y,t)\in R} w(y,t) = \liminf_{(y,t)\to(\eta_1,t_0),(y,t)\in\Omega} w(y,t) = w(\eta_1,t_0) = \psi_1(t_0) - \varepsilon$$

So, in order to prove the theorem we had only to show that equation (3.62) holds. That is
$\forall (y,t) \in \overline{\Omega}_k \setminus \Omega_k \quad \forall k \ge m \quad u_k(y,t) > w(y,t)$.
In fact, if $t \in [t_1, t_2]$ then:

$$\begin{aligned} u_k(\eta_1,t) = \psi_{1,k}(t) \ge \psi_1(t) \ge \psi_1(t) - \frac{\varepsilon}{2} > \\ > h(c(t_2 - \lambda(c))) > h(c(t - \lambda(c))) = w(\eta_1, t) \end{aligned}$$

and if $(y,t) \in \Gamma_k$ then:

$$u_k(y,t) \ge \varepsilon_k = h(\sigma_k) = w(y,t)$$

Moreover, if $t_1 = 0$ then for $y \in [\eta_1, \eta_1 - c\lambda(c)]$:

$$\begin{aligned} u_k(y,t) = u_{0,k}(y) \ge u_0(y) \ge u_0(0) - \frac{\varepsilon}{2} = \psi_1(t_0) - \frac{\varepsilon}{2} > \\ > h(c(t_2 - \lambda(c))) > h(-c\lambda(c)) > h(\eta_1 - x - c\lambda(c)) = w(y,0) \end{aligned}$$

Now, we use the maximum principle to prove that inequality (3.62) holds in $\overline{\Omega}_k$.
From (3.67), we observe that $w(y,t)$ is a classical solution of (3.25) in Ω_k. So, $w(y,t)$ is bounded away from 0 in $\overline{\Omega}_k$, by ε_k.
Then, $\forall (y,t) \in \overline{\Omega}_k$:

$$u_k(y,t) \ge w(y,t)$$

Moreover, $\forall (y,t) \in \overline{Q} \setminus \overline{\Omega}_k$:

$$u_k(y,t) \ge \varepsilon_k = h(\sigma_k) \ge w(y,t)$$

and so

$$u_k(y,t) \ge w(y,t) \quad \forall (y,t) \in \overline{Q}$$

and this proves the theorem.

3.2.4 Uniqueness of the solution of (3.25)

In order to prove the uniqueness of the solution of (3.25), we follow the proof of a similar result in [31].

Theorem 7 *Assume $u_0(y) \in L^\infty(\eta_1, \eta_2)$ and $g_1(t)$, $g_2(u)$ and $b(y,t)$ verify conditions (3.26), (3.27) and (3.28). Then problem (3.25) has at most one weak solution.*

Proof. Suppose that there exists two separate weak solutions of (3.25), $u_1(y,t)$ and $u_2(y,t)$.
By the definition of weak solution, u_1 and u_2 verify:

$$\begin{aligned}\int_0^T \int_{\eta_1}^{\eta_2} [(u_1 - u_2)\varphi_t - g_1(t)(G_2(u_1) - G_2(u_2))_y\varphi_y]dydt+ \\ - \int_0^T \int_{\eta_1}^{\eta_2} [b(y,t)(u_1 - u_2)\varphi_y - b_y(y,t)(u_1 - u_2)\varphi]dydt = 0\end{aligned} \tag{3.69}$$

$\forall \varphi \in C^\infty$ such that:

$$\varphi(\eta_1, t) = \varphi(\eta_2, t) = 0 \quad \forall t \in [0, T]$$

and

$$\varphi(y, T) = 0 \quad \forall y \in [\eta_1, \eta_2]$$

Equation (3.69) can be written as follows:

$$\begin{aligned}\int_0^T \int_{\eta_1}^{\eta_2} (u_1 - u_2)[\varphi_t - b(y,t)\varphi_y - b_y(y,t)\varphi]dydt = \\ = \int_0^T \int_{\eta_1}^{\eta_2} g_1(t)(G_2(u_1) - G_2(u_2))_y\varphi_y dydt\end{aligned} \tag{3.70}$$

and integrating by parts with respect to y, the second integral of the previous equation, we obtain:

$$\int_0^T \int_{\eta_1}^{\eta_2} g_1(t)(G_2(u_1) - G_2(u_2))_g\varphi_g dydt = - \int_0^T \int_{\eta_1}^{\eta_2} g_1(t)(G_2(u_1) - G_2(u_2))\varphi_{yy} dydt$$

Set:

$$\overline{G}(u_1, u_2) = \int_0^1 g_1(t)G_2(\vartheta u_1 + (1 - \vartheta)u_2)d\vartheta$$

then

$$G_2(u_1) - G_2(u_2) = (u_1 - u_2)\overline{G}(u_1, u_2)$$

So equation (3.70) becomes:

$$\int_0^T \int_{\eta_1}^{\eta_2} (u_1 - u_2)[\varphi_t + \overline{G}\varphi_{yy} - b(y,t)\varphi_y - b_y(y,t)\varphi]dydt = 0$$

So, if we show that $\forall f \in C_0^\infty$ there exists a solution of the following problem

$$\begin{cases} \varphi_t + \overline{G}\varphi_y - b(y,t)\varphi_y - b_y(y,t)\varphi = f & y \in [\eta_1, \eta_2], t \in (0, T] \\ \varphi(\eta_1, t) = \varphi(\eta_2, t) = 0 & \forall t \in (0, T] \\ \varphi(y, T) = 0 & \forall y \in [\eta_1, \eta_2] \end{cases} \tag{3.71}$$

then $u_1 - u_2 = 0$ and the theorem is shown.
Since $\overline{G}$ is merely bounded, it's not easy to study the solvability of (3.71). So, we approximate the operator $\overline{G}$ as follows:
for sufficiently small $\eta, \delta > 0$ we define:

$$\lambda_\eta^\delta = \begin{cases} \dfrac{b(y,t)}{\eta + \overline{G}} & |u_1 - u_2| > \delta \\ 0 & \text{otherwise} \end{cases}$$

Since $G_2(s)$ is a strictly increasing function and $u_1, u_2 \in L^\infty(Q)$ there exists $L = L(\delta, T)$ and $K = K(\delta, T)$ such that:

$$\overline{G} = g_1(t)\frac{G_2(u_1) - G_2(u_2)}{u_1 - u_2} \geq L(\delta) \quad \text{if } |u_1 - u_2| > \delta$$

$$|\lambda_\eta^\delta| \leq K(\delta) \quad \text{if } |u_1 - u_2| > \delta$$

Then there exists a sequence $\{\overline{G}_\varepsilon\}$ such that:

$$\lim_{\varepsilon\to 0} \overline{G}_\varepsilon = \overline{G}$$

and

$$|G_\varepsilon| \leq C$$

where C is a positive constant.
Then for given $f \in C_0^\infty(Q)$ the approximated system:

$$\begin{cases} \varphi_t + (\eta + \overline{G}_\varepsilon)\varphi_{yy} - b(y,t)\varphi_y - b_y(y,t)\varphi = f & y \in [\eta_1, \eta_2], t \in (0,T] \\ \varphi(\eta_1, t) = \varphi(\eta_2, t) = 0 & \forall t \in (0,T] \\ \varphi(y, T) = 0 & \forall y \in [\eta_1, \eta_2] \end{cases} \tag{3.72}$$

has a unique solution following the standard theory of parabolic linear equations.
From [31], Lemma 13.3.1, the solution of system (3.72) satisfies the following inequalities:

$$\sup_Q |\varphi(y,t)| \leq C$$

$$\int\int_Q (\eta + \overline{G}_\varepsilon)\left(\frac{\partial^2\varphi}{\partial y^2}\right)^2 dydt \leq \frac{K(\delta)}{\eta}$$

and

$$\int\int_Q \left(\frac{\partial\varphi}{\partial y}\right)^2 dydt \leq \frac{K(\delta)}{\eta}$$

From Theorem 13.3.1 of [31] we know that:

$$\lim_{\varepsilon\to 0}\int\int_Q \left(\frac{\partial\varphi}{\partial y}\right)^2 dydt \leq \frac{K(\delta)}{\eta} = 0$$

and it's enough to prove our theorem.

3.2.5 Proof of Theorem 3

The proof of Theorem 3, uses Lemma 4 and Theorem 5. More in detail, we are looking for a solution of system (M) with (I), (B), (LR) and (1.15)-(1.17), with positive initial data.

In order to apply Lemma 4 at (S_u), we have to verify that $\psi_1(t)$ and $\psi_2(t)$ are C^1 functions. This is trivial for $\psi_1(t) = u_0$, since it is a constant function. While, since $\psi_2(t) = u_0 + \frac{\mu}{bE} l'(t)$, then it is in C^1 if and only if $l'(t) \in C^1$. From (LR), the regularity of l' is the same of $u_y(0,t)$. So, $\psi_2(t) \in C^1$ if and only if $u_y(0,t)$ is in C^1 and this condition is equivalent to the second order compatibility condition at $y = 0$. That is:

$$\psi_{1tt} = ((a(t,u)u_y)_y + b(y,t)u_y)_t$$

But, since ψ_1 is a constant, we can write the previous equation as:

$$0 = ((a(t,u)u_y)_y + b(y,t)u_y)_t \quad \text{for} y = 0, t = 0 \tag{3.73}$$

$$\begin{aligned}
((a(t,u)u_y)_y + b(y,t)u_y)_t = &-2Er_0^2 \frac{r'(t) - l'(t)}{(r(t) - l(t))^3}(uu_y)_y + \\
&+ E\frac{r_0^2}{(r(t) - l(t))^2}(uu_y)_{yt} + \frac{l'(t)(r_0 - y) + r'(t)y}{r(t) - l(t)}u_{yt} + \\
&+ \frac{(l''(t)(r_0 - y) + r''(t)y)(r(t) - l(t))}{(r(t) - l(t))^2}u_{yt} + \\
&- \frac{(r'(t) - l'(t))(l'(t)(r_0 - y) + r'(t)y)}{(r(t) - l(t))^2}u_{yt}
\end{aligned}$$

and

$$\begin{aligned}
u_{yt} &= ((a(t,u)u_y)_y + b(y,t)u_y)_y = \\
&= E\frac{r_0^2}{(r(t) - l(t))^2}(uu_y)_{yy} + \frac{l'(t)(r_0 - y) + r'(t)y}{r(t) - l(t)}u_{yy} + \\
&+ \frac{r'(t) - l'(t)}{r(t) - l(t)}u_y
\end{aligned}$$

while

$$\begin{aligned}
(uu_y)_{yt} = &E\frac{r_0^2}{(r(t) - l(t))^2}(u(uu_y)_y)_{yy} + \\
&+ \frac{l'(t)(r_0 - y) + r'(t)y}{r(t) - l(t)}(uu_y)_{yy} + \frac{r'(t) + 2l'(t)}{r(t) - l(t)}(uu_y)_y
\end{aligned}$$

Equation (3.73) becomes:

$$0 = E^2(\tilde{u}_0(\tilde{u}_0(\tilde{u}_0)_y)_y)_{yy} + 2El'(0)(\tilde{u}_0(\tilde{u}_0)_y)_{yy} + l'(0)^2(\tilde{u}_0)_{yy} + l''(0)(\tilde{u}_0)_y$$

that is hypotesis (3.10) of the Theorem. So, the hypoteses of the Theorem guarantee that ψ_1 and ψ_2 are both C^1 functions and as a consequence Lemma 4 holds. So, (S_u) has a solution that is unique, positive and in $C^{2,1}$.

Moreover, from Theorem 5, there exists ρ which solves (S_ρ) and since $u \in C^{2,1}$ then it is a continuous and positive function.

In this way Theorem 3 is proved.

4 Numerical results

In this section we summarize the numerical results of this work. In particular we will describe the two numerical schemes we have implemented for the approximation of the travelling wave and of the general solution of system (M) with (I), (B), (LR) and (1.15)-(1.17). In the last part of this section we will show some tests about these schemes.

Note that, since (M_o) is a special case of system (M), the proposed algorithms can be used for it as well.

4.1 A numerical scheme for the travelling wave solution

From Section 2, a travelling wave, (W, r_0, u, ρ), is a solution of the two Cauchy problems (C_u), (C_ρ), respectively for the evolution of u and ρ, together with an explicit equation for r_0, (Eq_{r_0}), and an implicit one for W, (Eq_W).

From Theorem 2 and Lemma 2, under suitable choices of the parameters (2.8)-(2.18), for every fixed $\overline{W} \in I_W = (0, \beta/(h(\mu - \alpha)))$, there exists a unique $(\overline{r}_0, \overline{u}, \overline{\rho})$, solution of (C_u), (C_ρ) and (Eq_{r_0}).

$(\overline{W}, \overline{r}_0, \overline{u}, \overline{\rho})$ is a travelling wave solution if it satisfies equation (Eq_W) and its existence is guaranteed from Theorem 1. Let $f(W) = 0$ be the implicit expression for W, induced by (Eq_W); that is:

$$f(W) = W - \frac{1}{\mu - 2\alpha}\left[\beta - \frac{\nu\alpha}{\rho(r_0) - du(r_0)^{\frac{2}{3}}}\frac{u(r_0)}{u_0}\right]$$

Since it is not a monotone function with respect to W, $f(W)$ is not invertible and so the standard root finding algorithms are not useful to approximate W. Our approach is to scan the interval I_W searching for a $\overline{W}$ such that $(\overline{W}, \overline{r}_0, \overline{u}, \overline{\rho})$ verifies (Eq_W). More in detail, we have implemented the following algorithm:

Step 1: fix $\overline{\Delta z}$, Err_{max} and $Err_{u_{max}} > 0$;

Step 2: fix $N_W \in \mathbb{N}$ and define $\Delta W = \dfrac{1}{N_W + 1}\dfrac{\beta}{h(\mu - \alpha)}$;

Step 3: $\forall i \in \{1, \cdots, N_W\}$:

- compute (r_{0i}, u_i, ρ_i), an approximation of the solution of (C_u), (C_ρ) and (Eq_{r_0}) for $W = W_i = i\Delta W$;
- compute

$$Err_{W_i} = \left| W_i - \frac{1}{\mu - 2\alpha}\left(\beta - \frac{\nu\alpha}{\rho_i(r_{0i} - du_i(r_{0i})^{\frac{2}{3}})}\frac{u_i(r_{0i})}{u_0}\right)\right| \tag{4.1}$$

Step 4: choose $\bar{i}$ which minimizes Err_{W_i}; that is, $\bar{i}$ such that:

$$Err_{W_{\bar{i}}} \leq Err_{W_i} \qquad \text{for all } i \in \{1, \cdots, N_W\}$$

Step 5: if $Err_{W_{\bar{i}}} < Err_{max}$ then the approximated travelling wave solution is $(W_{\bar{i}} r_{0\bar{i}}, u_{\bar{i}}, \rho_{\bar{i}})$. Else set $N_W = 2N_W$ and come back to Step 2

Now, let's turn to a detailed description of the numerical scheme for the computation of (r_{0i}, u_i, ρ_i). For the sake of simplicity in the notation, we neglect the subscript i. First of all the algorithm computes r_0 from (Eq_{r_0}) and determines $N_z \in \mathbb{N}$ such that $(N_z - 1)\overline{\Delta z} < r_0 \leq N_z\overline{\Delta z}$. Then the interval $[0, r_0]$ is diveded in N_z subintervals of same length

$$\Delta z = \frac{r_0}{N_z}$$

In this way $r_0 = N_z\Delta z$ and

$$\frac{N_z - 1}{N_z}\overline{\Delta z} \leq \Delta z \leq \overline{\Delta z}$$

Then the algorithm computes u_j and ρ_j for all $j \in \{1, \cdots, N_z\}$.
From the boundary conditions:

$$\begin{cases} u(0) = u_0 \\ u(r_0) = u_0 - \frac{\mu\varepsilon}{b}W \end{cases}$$

so, we need the approximation of u_j for all $j \in \{1, \cdots, N_z - 1\}$.
From (2.28), $\forall j \in \{1, \cdots, N_z - 1\}$:

$$u_j - u_0 + \frac{C}{\varepsilon W}\log\left|\frac{\varepsilon W u_j - C}{\varepsilon W u_0 - C}\right| = \varepsilon W z_j$$

then if we define $F(u_j)$ as in Theorem 2, that is:

$$F(u_j) = u_j - u_0 + \frac{C}{\varepsilon W}\log\left|\frac{\varepsilon W u_j - C}{\varepsilon W u_0 - C}\right|,$$

we have that $\varepsilon W z_j = F(u_j)$. Assumptions of Theorem 1 guarantee the invertibility of $F(u_j)$. In fact, as we have shown in the proof of that theorem, $F(u_j)$ is a monotone decreasing function. So:

$$u_j = F^{-1}(\varepsilon W z_j)$$

and we can approximate u_j using a classical numerical scheme for the solution of non-linear equations.
In particular, since $\forall j \in \{1, \cdots, N - 1\}$:

$$F(u_0) < \varepsilon W z_j < F(u(r_0))$$

and

$$F(u(z_j)) \leq F(u(z_l)) \quad \forall j > l$$

we can approximate u_j through the bisection method and using the monotonicity of u in the following way:

u_1 and u_{N-1} are approximated using the bisection method in $I_1 = [u(r_0), u_0]$

u_2 and u_{N-2} are approximated using the bisection method in $I_2 = [u_{N-1}, u_1]$

$$\vdots$$

u_j and u_{N-j} are approximated using the bisection method in $I_j = [u_{N-(j+1)}, u_{j-1}]$

This is a great advantage from the computational point of view; in fact, denoting the maximum acceptable error in the approximation of $u(z_j)$ by $E_{u_{max}}$ and the length of I_j by l_{I_j}, we will get, using the bisection method, that:

$$\frac{l_{I_j}}{2^{n_j}} \leq E_{u_{max}}$$

whit a decreasing numbers of iterations, n_j, of the bisection method.

Finally the algorithm approximates ρ_j for all $j \in \{1, \cdots, N_z - 1\}$ and $\rho(r_0)$.

From (2.33), $\forall j \in \{1, \cdots, N_z - 1\}$:

$$\rho_j = u_j \left(\frac{\rho_0}{u_0} - \frac{\varepsilon}{C} \int_0^{z_j} K(\xi) u(\xi) d\xi \right)$$

Let $\tilde{u}_j$ be the above computed aproximation of u_j. We can approximate ρ_j using trapezoidal rule and replacing u_j with $\tilde{u}_j$. If $\overline{\rho}_j$ denotes such an approximation, then:

$$\overline{\rho}_j = \overline{u}_j \left(\frac{\rho_0}{u_0} - \frac{\varepsilon}{C} \frac{\Delta z}{2} \sum_{l=1}^{j} (K_l \overline{u}_l + K_{l-1} \overline{u}_{l-1}) \right) \tag{4.2}$$

4.2 Remarks on the consistency of the numerical method for the travelling wave solution

From a computational point of view, the algorithm described above is slow, since it computes at least N_W solutions of (C_u), (C_ρ) and (Eq_{r_0}), corresponding at $W_i = i\Delta W$ for $i \in \{1, \cdots, N_W\}$, but it is efficient and consistent.

In order to prove its consistency we first that for every fixed W_i, with $i \in \{1, \cdots, N_W\}$, the algorithm for the approximation of (r_{0i}, u_i, ρ_i) is convergent. Again we neglect the subscrits i; we denote by (r_0, u, ρ) the exact solution of (C_u), (C_ρ) and (Eq_{r_0}) for fixed W and by $(\overline{r}_0, \overline{u}, \overline{\rho})$ the approximated one. We set:

$$\begin{aligned}
Err_{r_0,W} &= |\overline{r}_0 - r_0| \\
Err_{u,W} &= \max_{j \in \{1,\cdots,N_z\}} |\overline{u}(z_j) - u(z_j)| \\
Err_{\rho,W} &= \max_{j \in \{1,\cdots,N_z\}} |\overline{\rho}(z_j) - \rho(z_j)|
\end{aligned}$$

Since $\overline{r}_0$ is computed exactly from (Eq_{r_0}), then $\overline{r}_0 = r_0$ and so $Err_{r_0,W} = 0$.

As far as the approximation of $u(z_j)$ is concerned, $\overline{u}(0)$ and $\overline{u}(r_0)$ are computed exactly from the boundary conditions.

Moreover, for all $j \in \{1, \cdots, N_z - 1\}$, $\overline{u}(z_j)$ is computed applying the bisection method to an interval, I_j, of length

$$L_j \leq u_0 - u(r_0) = \frac{\mu\varepsilon}{b} W < \frac{\mu\varepsilon}{b} \frac{\beta}{h(\mu - \alpha)}$$

So, from classical theory:

$$|\overline{u}(z_j) - u(z_j)| \leq \frac{\mu\varepsilon}{b} \frac{\beta}{h(\mu - \alpha)} \frac{1}{2^n}$$

where n is the number of iterations in the bisection method. Then $Err_{u,W}$ can be small as we want.
With regard to $Err_{\rho,W}$, from (2.33) and (4.2):

$$|\overline{\rho}_j - \rho_j| = \left|\frac{\rho_0}{u_0}(\overline{u}_j - u_j) + \frac{\varepsilon}{C}\left(\overline{u}_j I_T(K\overline{u}, z_j) - u_j \int_0^{z_j} K(\xi)u(\xi)d\xi\right)\right|$$

where $I_T(K\overline{u}, z_j) = \frac{\Delta z}{2}\sum_{l=1}^{j}(K(z_l)\overline{u}(z_l) + K(z_{l-1})\overline{u}(z_{l-1}))$ is the approximation of the integral with the trapezoidal rule.
So:

$$|\overline{\rho}_j - \rho_j| \leq \frac{\rho_0}{u_0}Err_{u,W} + \frac{\varepsilon}{C}\left|\overline{u}_j I_T(K\overline{u}, z_j) - u_j \int_0^{z_j} K(\xi)u(\xi)d\xi\right|$$

and:

$$\begin{aligned}
&\left|\overline{u}_j I_T(K\overline{u}, z_j) - u_j \int_0^{z_j} K(\xi)u(\xi)d\xi\right| \leq \\
&\qquad \leq |\overline{u}_j I_T(K\overline{u}, z_j) - \overline{u}_j I_T(Ku, z_j)| + |\overline{u}_j I_T(Ku, z_j) - u_j I_T(Ku, z_j)| + \\
&\qquad\qquad + \left|u_j I_T(Ku, z_j) - u_j \int_0^{z_j} K(\xi)u(\xi)d\xi\right|
\end{aligned}$$

and since $\overline{u}_j, u_j < u_0$, then:

$$\begin{aligned}
&\left|\overline{u}_j I_T(K\overline{u}, z_j) - u_j \int_0^{z_j} K(\xi)u(\xi)d\xi\right| \leq \\
&\qquad \leq u_0|I_T(K\overline{u}, z_j) - I_T(Ku, z_j)| + |\overline{u}_j - u_j||I_T(Ku, z_j)| + \\
&\qquad\qquad + u_0\left|I_T(Ku, z_j) - \int_0^{z_j} K(\xi)u(\xi)d\xi\right|
\end{aligned}$$

Let's estimate the three terms on the right of the previous inequality. Recalling that $K(z)$, defined in $K(x,t)$ for $t = 0$, is a not increasing C^2 function such that $K_2 \leq K(z) \leq K_2$, for the first one:

$$\begin{aligned}
|I_T(K\overline{u}, z_j) - I_T(Ku, z_j)| &= \frac{\Delta z}{2}\left|\sum_{l=1}^{j}(K(z_l)(\overline{u}_l - u_l) + K(z_{l-1})(\overline{u}_{l-1} - u_{l-1}))\right| = \\
&= \Delta z\left|\sum_{l=1}^{j-1} K(z_l)(\overline{u}_l - u_l) + \frac{1}{2}K(z_j)(\overline{u}_j - u_j)\right| \leq \\
&\leq \Delta z \sum_{l=1}^{j} |K(z_l)||\overline{u}_l - u_l| \leq \\
&\leq j\Delta z \max\{K_1, K_2\}Err_{u,W} = z_j \max\{K_1, K_2\}Err_{u,W}
\end{aligned}$$

So:

$$|I_T(K\overline{u}, z_j) - I_T(Ku, z_j)| \leq r_0 \max\{K_1, K_2\}Err_{u,W}$$

For the second term of the inequality:

$$\begin{aligned}|\overline{u}_j - u_j||I_T(Ku, z_j)| \le Err_{u,W}|I_T(Ku, z_j)| &= \\ &\le Err_{u,W}\Delta z \sum_{l=1}^{j} |K_l u_l| \le \\ &\le Err_{u,W}\, j\Delta z u_0 \max\{K_1, K_2\}\end{aligned}$$

and so

$$|\overline{u}_j - u_j||I_T(Ku, z_j)| \le r_0 u_0 \max\{K_1, K_2\} Err_{u,W}$$

Finally, by the known error estimate for the trapezoidal rule:

$$\begin{aligned}\left|I_T(Ku, z_j) - \int_0^{z_j} K(\xi)u(\xi)d\xi\right| &\le \frac{z_j}{12}\Delta z^2 |(K(\xi)u(\xi))''| \le \\ &\le r_0 \Delta z^2 |(K(\xi)u(\xi))''|\end{aligned}$$

With regard to the term $|(K(\xi)u(\xi))''|$, it is bounded since both $K(\xi)$ and $u(\xi)$ are C^2 functions.
So:

$$Err_{\rho,W} \le \frac{\rho_0}{u_0} Err_{u,W} + \frac{\varepsilon}{C}\left[2u_0 r_0 \max\{K_1, K_2\} Err_{u,W} + u_0 r_0 \Delta z^2 |(K(\xi)u(\xi))''|\right]$$

And

$$\frac{\varepsilon}{C} = \frac{\alpha}{(\beta - W(\mu - 2\alpha))u_0} \le 2\frac{\mu - \alpha}{\beta}$$

where we have used the hypotesis on u_0 (2.10) and that $W \in (0, \beta/(h(\mu - \alpha)))$. Then,

$$Err_{\rho,W} \le \frac{\rho_0}{u_0} Err_{u,W} + 2\frac{\mu - \alpha}{\beta}\left[2r_0 \max\{K_1, K_2\} Err_{u,W} + r_0 \Delta z^2 |(K(\xi)u(\xi))''|\right]$$

And since

$$\lim_{n\to\infty} Err_{u,W} = 0$$

then

$$\lim_{\substack{n \to \infty \\ \Delta z \to 0}} Err_{\rho,W} = 0$$

and so for $n \to \infty$ and $\Delta z \to 0$, $(\overline{r}_0, \overline{u}, \overline{\rho}) \to (r_0, u, \rho)$. Note that since

$$\overline{\Delta z}((N_z - 1)/N_z) \le \Delta z \le \overline{\Delta z}$$

then the previous limit holds also for $\overline{\Delta z} \to 0$. Then, the numerical method for the approximation of the solution of (C_u), (C_ρ) and (Eq_{r_0}) for a fixed W is convergent.

In order to prove the consistency of the numerical scheme for the travelling wave solution, we note that, from (4.1):

$$\begin{aligned}
Err_{W_i} &= \left| W_i - \frac{1}{\mu - 2\alpha}\left(\beta - \frac{\nu\alpha}{\rho_i(r_{0i}) - du_i(r_{0i})^{\frac{2}{3}}}\right)\frac{u_i(r_{0i})}{u_0}\right| \leq \\
&\leq |W_i - W| + \left| W - \frac{1}{\mu - 2\alpha}\left(\beta - \frac{\nu\alpha}{\rho(r_0) - du(r_0)^{\frac{2}{3}}}\right)\frac{u(r_0)}{u_0}\right| + \\
&\quad + \frac{\nu\alpha}{(\mu - 2\alpha)u_0}\left|\frac{u(r_0)}{\rho(r_0) - du(r_0)^{\frac{2}{3}}} - \frac{u_i(r_{0i})}{\rho_i(r_{0i}) - du_i(r_{0i})^{\frac{2}{3}}}\right| = \\
&= |W_i - W| + \frac{\nu\alpha}{(\mu - 2\alpha)u_0}\left|\frac{u(r_0)}{\rho(r_0) - du(r_0)^{\frac{2}{3}}} - \frac{u_i(r_{0i})}{\rho_i(r_{0i}) - du_i(r_{0i})^{\frac{2}{3}}}\right|
\end{aligned}$$

where in last equation we have considered that the exact solution verifies (Eq_W).

Since the method scans the interval $\left(0, \dfrac{\beta}{h(\mu - \alpha)}\right)$, then there exists $\bar{i} \in \{1, \cdots, N_W\}$ such that $W_{\bar{i}} \leq W < W_{\bar{i}+1}$ and so $|W_{\bar{i}} - W| < \Delta W$. Then:

$$Err_{W_{\bar{i}}} \leq \Delta W + \frac{\nu\alpha}{(\mu - 2\alpha)u_0}\left|\frac{u(r_0)}{\rho(r_0) - du(r_0)^{\frac{2}{3}}} - \frac{u_{\bar{i}}(r_{0\bar{i}})}{\rho_{\bar{i}}(r_{0\bar{i}}) - du_{\bar{i}}(r_{0\bar{i}})^{\frac{2}{3}}}\right|$$

Since the scheme for the approximation of the solution of (C_u), (C_ρ) and (Eq_{r_0}) is convergent, if $\Delta W \to 0$, that is $W_{\bar{i}} \to W$, we have that:

$$\begin{aligned}
&r_{0\bar{i}} \to r_0 \\
&u_{\bar{i}}(r_{0\bar{i}}) \to u(r_0) \\
&\rho_{\bar{i}}(r_{0\bar{i}}) \to \rho(r_0)
\end{aligned}$$

and consequently $Err_{W_{\bar{i}}} \to 0$. So, Err_{W_i} can be interpreted as the measure of the quality of the approximation of the solution (W, r_0, u, ρ) with $(W_i, r_{0i}, u_i, \rho_i)$. Then, the choice of the approximated solution as the one which minimize Err_W makes the method consistent. Moreover, from the tests on the travelling wave solution Err_W decreases as ΔW decreases. Then it is possible to fix Err_{Wmax} and ΔW in such a way that $Err_W \leq Err_{Wmax}$.

4.3 A numerical scheme for the problem

From the literature about free and moving boundaries equations, see for example [7], it is known that we can use two different approaches to this kind of problems.

Let $\Omega(t) = (l(t), r(t))$ denote the moving domain of definition for the problem. One approach is to study the system in the moving domain $\Omega(t)$, the other is to study it in a fixed domain. Numerical methods of the first type are the so called *front-tracking* methods. In these schemes, at every time $\bar{t}$ the positions of the two extrema are approximated. Then the method approximates the solution in the new domain $\Omega(\bar{t})$.

Numerical schemes of the second type are the so called *front-fixing* methods. They use a spatio-temporal transformation and study the problem in a fixed domain. Once they have approximated the solution in the fixed domain they apply the inverse transfomation to obtain the approximate solution in the original variables.

Front-tracking methods are very useful for numerical characterization of the solution of problems with vanishing boundary data or for free boundary problems. The reason will be clear through the following example.

Let t_0, t_1 be such that $0 \leq t_0 < t_1$ and let $\Omega(t_0)$, $\Omega(t_1)$ be the definition domain of the problem respectively at $t = t_0$ and $t = t_1$. Let's suppose we want to approximate the solution, $w(x,t)$, of a partial differential equation involving its time derivative w_t. Applying forward Euler method for w_t we get:

$$w_t(x, t_1) = \frac{w(x, t_1) - w(x, t_0)}{t_1 - t_0}$$

and if we want to approximate $w(x, t_1)$ we need to know $w(x, t_0)$. Assume, now, that $\Omega(t_1) \nsubseteq \Omega(t_0)$, so there exists $\overline{x}$ such that $\overline{x} \in \Omega(t_1) \setminus \Omega(t_0)$. Since $\overline{x} \notin \Omega(t_0)$, $w(\overline{x}, t_0)$ is not defined. So we need to extend the solution, $w(x, t_0)$ out of the domain $\Omega(t_0)$. This can be easily done setting $w(x,t) \equiv 0 \in \mathbb{R} \setminus \Omega(t_0)$, if the boundary data are such that $w(l(t), t) = w(r(t), t) = 0$. Otherwise this extension is not trivial and could increase the approximation error.

Since the boundary conditions for our system, (B), are:

$$\begin{cases} u(l(t), t) = u_0 = \tilde{u}_0(0) > 0 & t > 0 \\ u(r(t), t) = u_0 - \dfrac{\mu}{bE}\dfrac{dl}{dt} > 0 & t > 0 \\ \rho(l(t), t) = \rho_0 = \tilde{\rho}_0(0) > 0 & t > 0 \end{cases}$$

then it seems better to use a *front-fixing* method. In this way we avoid the difficulty of the extension of the solution but, due to the transformation of the variables, we have to study a more complex system. Note, also, that a *front-fixing* method allows us to approximate the solution using a fixed grid.

We transform the moving domain in a fixed one using the transformation we introduced in Section 3, equation (3.13):

$$y = r_0 \frac{x - l(t)}{r(t) - l(t)}$$

and so we study (M). The existence of a unique solution for the problem is guaranteed by Theorem 6.

Let's now describe the numerical method we have implemented to approximate the solution of system (M).

Fix $\Delta t > 0$ and $N \in \mathbb{N} > 0$.

Let $\Delta y > 0$ defined as $\Delta y = r_0/N$, so that $r_0 = N\Delta y$.

For all $j \in \{1, \cdots, N\}$ set $y_j = j\Delta y$, for all $n > 0$ set $t^n = n\Delta t$ and for a generic function $f = f(x,t)$, set $f_j^n = f(y_j, t^n)$.

Assume that we know an approximated solution $(\overline{u}(y,t), \overline{\rho}(y,t))$ for all $y \in [0, r_0]$ at $t = t^{n-1}$, and we want an approximated solution at time $t = t^n$. For the sake of simplicity in the notation in what follows we will neglect the overline.

The numerical algorithm executes the following steps:

Step 1: it computes l^n and r^n. These two variables, in the fixed domain, lose their physical sense of boundary positions. However their computation, as those of

$l'(t^n)$ and $r'(t^n)$ is necessary to compute the coefficients of the partial differential equation for u and for ρ;

Step 2: it approximates u_j^n for all $j \in \{0, \cdots, N-1\}$;

Step 3: it computes ρ_j^n for all $j \in \{0, \cdots, N-1\}$

More in detail, from (LR):

$$\begin{cases} l^n = l^{n-1} - \Delta t \left[E \dfrac{\alpha}{\mu - \alpha} \dfrac{r_0}{r^{n-1} - l^{n-1}} u_y(0, t^{n-1}) - \dfrac{\beta}{\mu - \alpha} \right] \\ r^n = r^{n-1} - \Delta t \left[E \dfrac{r_0}{r^{n-1} - l^{n-1}} u_y(r_0, t^{n-1}) - \dfrac{\nu}{\rho_N^{n-1} - d(u_N^{n-1})^{\frac{2}{3}}} \right] \end{cases}$$

where we approximate $u_y(0, t^{n-1})$ e $u_y(r_0, t^{n-1})$ using a second order scheme, that is:

$$\begin{cases} u_y(0, t^{n-1}) = \dfrac{1}{2\Delta y}[-3u_0^{n-1} + 4u_1^{n-1} - u_2^{n-1}] \\ u_y(r_0, t^{n-1}) = \dfrac{1}{2\Delta y}[3u_N^{n-1} - 4u_{N-1}^{n-1} + u_{N-2}^{n-1}] \end{cases} \tag{4.3}$$

Once the algorithm has computed l^n and r^n it turns to the approximation of u_j^n and ρ_j^n. For the approximation of u, we rearrange a method introduced by E. A. Socolovsky in [28], for the porous media equation $w_t = (w^2)_{xx}$ with a compact supported initial data on the whole space $\mathbb{R}$. In analogy with the second order central scheme for the second derivative, he suggests the following scheme for $(w^2)_{xx}$:

$$(w^2(x,t))_{xx} = \frac{w^2(x + \Delta x, t) - 2w^2(x,t) + w^2(x - \Delta x, t)}{\Delta x^2} \tag{4.4}$$

We adapt this finite difference scheme to approximate the solution of the Cauchy problem for u, (S_u), introduced in Section 3, that we rewrite as:

$$\begin{cases} u_t - \dfrac{1}{2} a_1(t)(u^2)_{yy} - b(y,t)u_y = 0 \quad y \in (0, r_0), t > 0 \\ u(0,t) = u_0 \\ u(r_0, t) = u_0 + \dfrac{\mu}{bE} \dfrac{dl}{dt} \end{cases}$$

where $a_1(t)$ and $b(y,t)$ are defined respectively in (3.18) and (3.19).
For all $j \in \{1, \cdots, N-1\}$ we define the following implicit scheme where we have used the forward Euler method for $(u_t)_j^n$, the central scheme for $(u_y)_j^n$ and the scheme described in (4.4) for $((u^2)_{yy})_j^n$.

$$\frac{u_j^n - u_j^{n-1}}{\Delta t} - a_1^n \frac{1}{2\Delta y^2} \left[(u_{j-1}^n)^2 - 2(u_j^n)^2 + (u_{j+1}^n)^2 \right] - b_j^n \frac{1}{2\Delta y}[u_{j+1}^n - u_{j-1}^n] = 0$$

Taking care of the boundary conditions, then we have to solve the following nonlinear system:

$$\begin{cases} u_0^n = u_0 \\ u_j^n - a_1^n \dfrac{\Delta t}{\Delta y^2}\left[(u_{j-1}^n)^2 - 2(u_j^n)^2 + (u_{j+1}^n)^2\right] - b_j^n \dfrac{\Delta t}{2\Delta y}[u_{j+1}^n - u_{j-1}^n] \quad = u_j^{n-1} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j \in \{1, \cdots, N-1\} \\ u_N^n = u_0 + \dfrac{\mu}{bE} l'(t^n) \end{cases}$$

and we solve it using Newton method with u^{n-1} as initial data. In our C++ program we have used the algorithm for the Newton method described in [26], adapting it to our case.

As far as the error order for this scheme is concerned, in [28] it was proved that it is a first order scheme in Δy.

Let's now describe the approximation scheme for ρ. The partial differential equation for ρ can be written as:

$$\rho_t - (a(t,u)\rho)_y - b(y,t)\rho_y = Ku \tag{4.5}$$

As we said previously this is an equation of hyperbolic type. From the literature, see for example [27], it is known that explicit schemes are commonly used for the approximation of the solution of this kind of problems. But an explicit schemes has to satisfy CFL condition to be stable. For a generic hyperbolic problem:

$$\frac{df}{dt} + a\frac{df}{dy} + bf + g = 0$$

the CFL condition can be written as:

$$\left| a\frac{\Delta t}{\Delta y} \right| < 1$$

This condition for (4.5) is:

$$\left| \frac{\Delta t}{\Delta y}\left[E\frac{r_0^2}{(r(t)-l(t))^2}u_y + \frac{l'(t)(r_0-y)+r'(t)y}{r(t)-l(t)} \right] \right| < 1 \tag{4.6}$$

but the quantity on the left hand side is not easy to estimate.

In fact, an upper bound for it strictly depends on an estimate of $r(t)-l(t)$. One approach is to choose Δy and Δt in order to strictly verify the CFL condition at the initial time, i.e.

$$\left| \frac{\Delta t}{\Delta y}\left[E\frac{r_0^2}{(r(t)-l(t))^2}u_y + \frac{l'(t)(r_0-y)+r'(t)y}{r(t)-l(t)} \right] \right| << 1,$$

Then the continuity of $l(t)$, $r(t)$, $l'(t)$ and $r'(t)$ guarantee that there exists $T > 0$ such that the CFL condition is verified for all $t \in [0, T]$. But we can't estimate T, since we have not *a priori* estimate for these quantities.

Another approach should be to approximate the solution on a grid with variable temporal step. In particular at $t = t^n$ we should choose Δt^n such that:

$$\left| \frac{\Delta t^n}{\Delta y}\left[E\frac{r_0^2}{(r(t^n)-l(t^n))^2}u_y + \frac{l'(t)(r_0-y)+r'(t)y}{r(t^n)-l(t^n)} \right] \right| < 1 \quad \forall y \in (0, r_0)$$

But, if $r(t) - l(t)$ are small or l', r' are large, this process could affects the computational time step for the approximation. For this reason we preferred to use an implicit scheme for the approximation of the solution of (4.5).

Another difficulty in the choice of the numerical scheme to use, is that the coefficients of ρ and ρ_y depend on u_y and u_{yy}. In order to avoid the computation of the second derivative for u, we generalize the idea for the approximation of the solution of conservation laws, introduced in [18]. In particular, suppose we want to approximate the solution of the following equation:

$$w_t + (fw)_x = 0$$

then we define $g = fw$ and we use the method we prefer for the approximation of g.

We generalize this scheme to our equation (4.5). We will use a fixed grid with spatial step Δy and temporal step Δt. Suppose that we have approximated the solution for $t = t^{n-1}$ and we would like to estimate the solution at time $t = t^n$. We use forward Euler scheme for the temporal derivative:

$$(\rho_t)_j^n = \frac{\rho_j^n - \rho_j^{n-1}}{\Delta t}$$

the central scheme for the spatial derivatives:

$$(\rho_y)_j^n = \frac{\rho_{j+1}^n - \rho_{j-1}^n}{2\Delta y}$$

and from what we said previously about conservative laws:

$$(u_y\rho)_y = \frac{(u_y)_{j+1}^n \rho_{j+1}^n - (u_y)_{j-1}^n \rho_{j-1}^n}{2\Delta y}$$

where $(u_y)_{j+1}^n$ and $(u_y)_{j+1}^n$ are the derivatives of u respectively in y_{j+1} and y_{j-1}.

These derivatives are approximated again with the central scheme.

Moreover since u is defined only for $y \in (0, r_0)$, the previous equation is not defined in $j = 0$ and $j = N$, where $N \in \mathbb{N}$ is such that $N\Delta y = r_0$.

So in those points we used the non symmetric second order scheme introduced for the computation of l^n and r^n, equation (4.3). Summing up:

$$(u_y)_j^n = \begin{cases} \frac{1}{2\Delta y}(-u_{j+2}^n + 4u_{j+1}^n - 3u_j^n) & j = 0 \\ \frac{1}{2\Delta y}(u_{j+1}^n - u_{j-1}^n) & j \in \{1, \cdots, N-1\} \\ \frac{1}{2\Delta y}(u_{j-2}^n - 4u_{j-1}^n + 3u_j^n) & j = N \end{cases}$$

So, we have to solve the following system:

$$\begin{cases} \rho_0^n = \rho_0 \\ \frac{\rho_j^n - \rho_j^{n-1}}{\Delta t} = a^n \frac{1}{2\Delta y}(u_{j+1}^n \rho_{j+1}^n - u_{j-1}^n \rho_{j-1}^n) b_j^n \frac{1}{2\Delta y}(\rho_{j+1}^n - \rho_{j-1}^n) \quad + K_j^n u_j^n \\ \qquad j \in \{1, \cdots, N-1\} \end{cases}$$

We need also an equation for ρ_N^n; in particular we set ρ_N^n on the line thorugh points (y_{N-2}, ρ_{N-2}^n) and (y_{N-1}, ρ_{N-1}^n). In this way:

$$\rho_N^n = 2\rho_{N-1}^n - \rho_{N-2}^n$$

Then we have to solve the following tridiagonal system:

$$\begin{cases} \rho_0^n = \rho_0 \\ \dfrac{\Delta t}{2\Delta y}(a^n(u_y)_{j-1}^n + b_j^n)\rho_{j-1}^n + \rho_j^n - \dfrac{\Delta t}{2\Delta y}(a^n(u_y)_{j+1}^n + b_j^n)\rho_{j+1}^n & = \rho_j^{n-1} + \Delta t K_j^n u_j^n \\ & j \in \{1, \cdots, N-1\} \\ \rho_N^n = 2\rho_{N-1}^n - \rho_{N-2}^n \end{cases}$$

An error estimate for this method is not very easy to prove. In fact it is a system of partial differential equations with time dependent coefficients. The estimates of those coefficients involves the estimate of $r(t) - l(t)$ and we are not able to give a lower or an upper bound for that quantity. So, we test its efficiency using an approximate travelling wave solution as initial data and compare it with the approximate solution computed with the method described above at every time $t = t^n$.

4.4 Tests for the travelling wave solution

One of the difficulties in testing the numerical methods described in previous sections is the choice of the parameters of the problem. From the model we only know that $\mu > \alpha$.

For the travelling wave solution, Theorem 1 gives us other conditions on the parameters. In particular it fixes in which interval u_0, h and v have to be and it also gives a lower bound for ρ_0, but it's not enough. Moreover we have not informations about the values they should assume from a biological point of view. Since we are specially interested on the qualitative behaviour of the solution, without loss of generality we set the parameters in order to have u_0 and ρ_0 approximately of the same size.

Let's now come back to choose the parameters of the problem. First of all, we remark that in the proof of Theorem 1:

$$u_0 \in \left(2\varepsilon\frac{\mu\beta}{b(\mu-\alpha)}, \varepsilon\frac{\mu\beta}{2b(h(\mu-\alpha)-(\mu-2\alpha))}\right)$$

We choose b and β in order to have ε as lower bound for u_0, setting:

$$\begin{aligned} b &= 2\mu \\ \beta &= \mu - \alpha \end{aligned} \tag{4.7}$$

Then:

$$u_0 \in \left(\varepsilon, \varepsilon\frac{(\mu-\alpha)}{4(h(\mu-\alpha)-(\mu-2\alpha))}\right) \tag{4.8}$$

Now we turn to study how to choose K_1, K_2, δ_l, δ_r and d to have $\rho_0 > \rho_{0_{min}} \cong u_0$. From Theorem 1

$$\rho_{0_{min}} = \frac{bu_0h(\mu-\alpha) - \mu\varepsilon\beta}{bu_0(\mu-2\alpha) - \mu\varepsilon\beta}\rho_{min}$$

where

$$\rho_{min} = u_0 \left[(K_2(R_0 - \delta_r) - K_1\delta_l) \frac{\alpha h(\mu - \alpha)}{\beta(h(\mu - \alpha) - (\mu - 2\alpha))} + d\left(u_0 - \frac{\mu\varepsilon\beta}{bh(\mu - \alpha)}\right)^{-\frac{1}{3}} \right]$$

and using (4.7):

$$R_0 = \frac{\mu}{b}\frac{h}{h-1} = \frac{h}{2(h-1)} \tag{4.9}$$

We would like $\rho_{0_{min}} \cong u_0$.
We set

$$\rho_{01} = [(K_2(R_0 - \delta_r) - K_1\delta_l) \frac{bu_0h(\mu - \alpha) - \mu\varepsilon\beta}{bu_0(\mu - 2\alpha) - \mu\varepsilon\beta} \frac{\alpha h(\mu - \alpha)}{\beta(h(\mu - \alpha) - (\mu - 2\alpha))} \tag{4.10}$$

and

$$\rho_{02} = d\frac{bu_0h(\mu - \alpha) - \mu\varepsilon\beta}{bu_0(\mu - 2\alpha) - \mu\varepsilon\beta}\left(u_0 - \frac{\mu\varepsilon\beta}{bh(\mu - \alpha)}\right)^{-\frac{1}{3}} \tag{4.11}$$

We will choose the parameters such that

$$\frac{1}{4} \leq \rho_{01}, \rho_{02} \leq \frac{3}{4}$$

so that:

$$\rho_{0_{min}} = u_0(\rho_{01} + \rho_{02}) \in \left(\frac{1}{2}u_0, \frac{3}{2}u_0\right)$$

Since one of our purposes is to compare the solution of (M_o) with the one of (M), we fix K_2 in order to have $\rho_{0_{min}} \subseteq \left(\frac{1}{2}u_0, \frac{3}{2}u_0\right)$ both for the case $\delta_l = \delta_r = 0$ than for $0 \leq \delta_l < \delta_r$. Then we choose K_1, δ_l and δ_r in order to verify $K_1\delta_l < K_2(R_0 - \delta_r)$ and $0 \leq \delta_l \leq \delta_r$. Since from the hypoteses of Theorem 1:

$$h \in \left(1, \frac{5\mu - 9\alpha}{4(\mu - \alpha)}\right)$$

then:

$$\frac{bu_0h(\mu - \alpha) - \mu\varepsilon\beta}{bu_0(\mu - 2\alpha) - \mu\varepsilon\beta} > 1$$

Using (4.7), (4.8) and $\mu > 5\alpha$:

$$\begin{aligned}
\frac{bu_0h(\mu - \alpha) - \mu\varepsilon\beta}{bu_0(\mu - 2\alpha) - \mu\varepsilon\beta} &< (\mu - \alpha)\frac{2u_0h - \varepsilon}{\varepsilon(\mu - 3\alpha)} < \\
&< \frac{\mu - \alpha}{2\varepsilon\alpha}(2u_0h - \varepsilon) < \\
&< \frac{\mu - \alpha}{\varepsilon\alpha}u_0h < \\
&< \frac{\mu - \alpha}{\alpha}\frac{h(\mu - \alpha)}{4(h(\mu - \alpha) - (\mu - 2\alpha))}
\end{aligned}$$

and then:

$$\frac{h(\mu - \alpha)}{4(h(\mu - \alpha) - (\mu - 2\alpha))} < 1$$

So

$$1 < \frac{bu_0h(\mu-\alpha)-\mu\varepsilon\beta}{bu_0(\mu-2\alpha)-\mu\varepsilon\beta} < \frac{\mu-\alpha}{\alpha} \tag{4.12}$$

Since $h \in \left(1, \frac{5\mu-9\alpha}{4(\mu-\alpha)}\right)$ then:

$$1 > \frac{\alpha h}{h(\mu-\alpha)-(\mu-2\alpha)} > \frac{4\alpha}{\mu-\alpha} \tag{4.13}$$

We choose K_2 as follows:

$$K_2 \in \left(\frac{1}{16}\frac{1}{R_0}\frac{\mu-\alpha}{\alpha}, \frac{3}{4}\frac{1}{R_0}\frac{\alpha}{\mu-\alpha}\right) \tag{4.14}$$

In this way, $\rho_{01} \in \left(\frac{1}{4}, \frac{3}{4}\right)$. In fact, replacing (4.12), (4.13) and (4.14) in (4.10):

$$K_2R_0\frac{4\alpha}{\mu-\alpha} < \rho_{01} < K_2R_0\frac{\mu-\alpha}{\alpha}$$

$$\begin{aligned}\rho_{01} &> K_2R_0\frac{4\alpha}{\mu-\alpha}\\ &> \frac{1}{16}\frac{\mu-\alpha}{\alpha}\frac{4\alpha}{\mu-\alpha} =\\ &= \frac{1}{4}\end{aligned}$$

and:

$$\begin{aligned}\rho_{01} &< K_2R_0\frac{\mu-\alpha}{\alpha} <\\ &= \frac{3}{4}\frac{\alpha}{\mu-\alpha}\frac{\mu-\alpha}{\alpha} <\\ &= \frac{3}{4}\end{aligned}$$

Similarly choosing d such that:

$$d \in \left(\frac{1}{4}\left(\frac{\alpha}{\mu-\alpha}\right)^{\frac{1}{3}}\left(\frac{\varepsilon}{2}\right)^{\frac{1}{3}}, \frac{3}{4}\frac{\alpha}{\mu-\alpha}\left(\frac{\varepsilon}{2}\right)^{\frac{1}{3}}\right) \tag{4.15}$$

we have $\rho_{02} \in \left(\frac{1}{4}, \frac{3}{4}\right)$.

In fact:

$$u_0 - \frac{\mu\varepsilon\beta}{bh(\mu-\alpha)} = u_0 - \frac{\varepsilon}{2h}$$

Moreover, since:

$$u_0 - \frac{\varepsilon}{2h} < u_0 < \varepsilon\frac{(\mu-\alpha)}{4(h(\mu-\alpha)-(\mu-2\alpha))} < \frac{\varepsilon}{2}\frac{\mu-\alpha}{2\alpha}$$

and

$$u_0 - \frac{\varepsilon}{2h} > u_0 - \frac{\varepsilon}{2} > \frac{\varepsilon}{2}$$

then:

$$\frac{\varepsilon}{2} < u_0 - \frac{\varepsilon}{2h} < \varepsilon \frac{(\mu - \alpha)}{4(h(\mu - \alpha) - (\mu - 2\alpha))} \tag{4.16}$$

So, from (4.12) and (4.16):

$$d\left(\frac{\varepsilon}{2}\right)^{-\frac{1}{3}} \left(\frac{\mu - \alpha}{\alpha}\right)^{-\frac{1}{3}} < \rho_{02} < d\left(\frac{\varepsilon}{2}\right)^{-\frac{1}{3}} \frac{\mu - \alpha}{\alpha}$$

and (4.15) implies:

$$\frac{1}{4} < \rho_{02} < \frac{3}{4}$$

In the following examples we show the results of some tests. In the first one the consistency error for the equation (Eq_W), Err_W defined in (4.1), is shown to be decreasing as ΔW decreases. In the second one we show how a change in K_1, δ_l and δ_r affects ρ. We don't focus our attention on u, because its qualitative behaviour has not appreciable changes. The last example is a comparison between a travelling wave solutions of our model and of the one introduced in [3].

Example 1

Following the idea of this section we choose:

$$\begin{aligned} \alpha &= 1 \\ \mu &= 6 \\ \beta &= \mu - \alpha = 5 \\ b &= 2\mu = 12 \\ \varepsilon &= 1 \\ h &= 1.025 \end{aligned} \tag{4.17}$$

So that:

$$R_0 = \frac{h}{2(h-1)} = 20.5$$

$$K_2 = 0.00762195 \in \left(\frac{1}{16}\frac{1}{R_0}\frac{\mu - \alpha}{4\alpha}, \frac{15}{16}\frac{1}{R_0}\frac{\alpha}{\mu - \alpha}\right) = (0.003811, 0.009146) \tag{4.18}$$

$$d = 0.1 \in \left(\frac{1}{4}\left(\frac{\alpha}{\mu - \alpha}\right)^{\frac{1}{3}}\left(\frac{\varepsilon}{2}\right)^{\frac{1}{3}}, \frac{3}{4}\frac{\alpha}{\mu - \alpha}\left(\frac{\varepsilon}{2}\right)^{\frac{1}{3}}\right) = (0.02901, 0.11955) \tag{4.19}$$

$$u_0 = 1.05556 \in \left(\varepsilon, \varepsilon\frac{(\mu - \alpha)}{4(h(\mu - \alpha) - (\mu - 2\alpha))}\right) = (1, 1.11111) \tag{4.20}$$

$$\rho_0 = 2.11111 = 2u_0 > \rho_{0_{min}} \tag{4.21}$$

and

$$K_1 = 100$$
$$\delta_l = 0.002$$
$$\delta_r = 0.02$$

We fix $\overline{\Delta z} = 0.001$, $Err_{W_{max}} = 10^{-6}$ and since $W \in \left(0, \dfrac{\beta}{h(\mu - alpha)}\right)$, we search for $W < 0.97561$. We obtain the results summarized in the following table:

ΔW	$\overline{W}$	$\bar{r}_0$	err_W
0.1	0.8	0.286097	0.0620885
0.01	0.79	0.272001	0.0155516
0.001	0.786	0.26667	0.00153417
0.0001	0.7859	0.266539	$5.73937 \cdot 10^{-4}$
0.00001	0.78581	0.266421	$3.390741 \cdot 10^{-5}$
0.000001	0.785753	0.266346	$4.07957 \cdot 10^{-6}$
0.0000001	0.785982	0.266646	$2.74301 \cdot 10^{-7}$

This table shows that Err_W has the order of ΔW.
We obtain, for $\Delta W = 10^{-7}$ the following graphs for u and ρ, respectively figure 1 and 2:

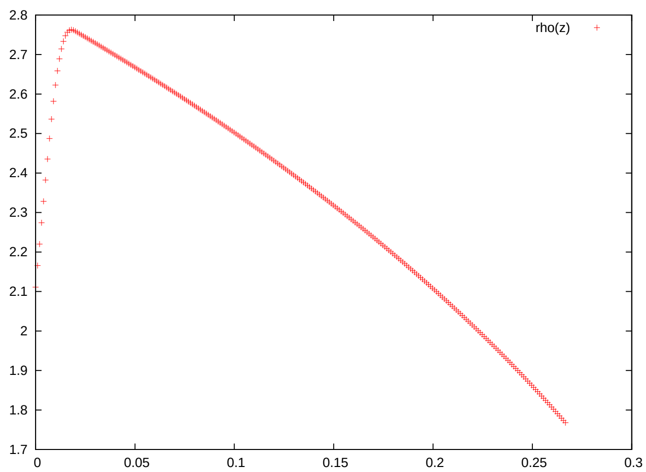

Figure 1: $\rho(z)$

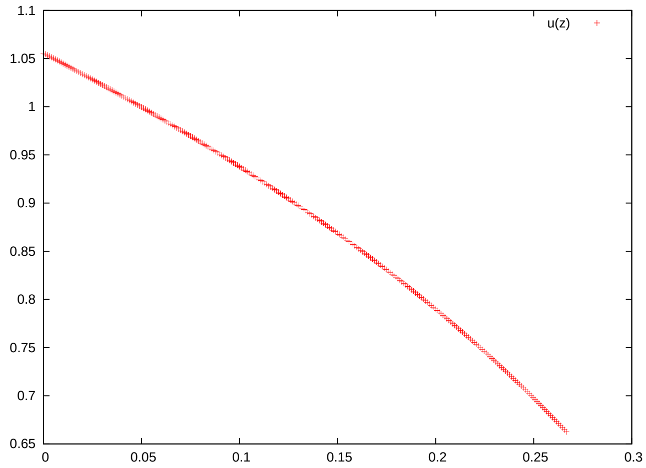

Figure 2: $u(z)$

Example 2

Suppose that ρ assumes its maximum value, ρ_M, in z_M. Then, as we show in the following tables, these values are affected by changes in K_1, δ_l and δ_r.
We choose Δz, α, β, μ, ν, h, b, d, u_0, ρ_0, K_2 as in the previous example.
Fixing $K_1 = 100$ and $\delta_r = 0.02$ we obtain:

δ_l	z_M	ρ_M	r_0	W
0.001	0.0169347	2.61687	0.23609	0.760388
0.002	0.0169624	2.6508	0.242463	0.766134
0.005	0.0179761	2.76284	0.266646	0.785982
0.01	0.0189793	2.97298	0.316655	0.819172

Fixing $K_1 = 100$ and $\delta_l = 0.005$ we obtain:

δ_r	z_M	ρ_M	r_0	W
0.01	0.008988905	2.45127	0.198758	0.721368
0.015	0.0139468	2.59417	0.22813	0.752871
0.02	0.0179761	2.76284	0.266646	0.785982
0.03	0.026931	3.19844	0.384071	0.852238

Fixing $\delta_l = 0.005$ and $\delta_r = 0.02$ we obtain:

K_1	z_M	ρ_M	r_0	W
0	0	2.11111	0.134445	0.622341
50	0.0159303	2.34976	0.18519	0.704442
100	0.0179761	2.76284	0.266646	0.785982
150	0.0189781	3.35948	0.42551	0.868053

From these tables we note that z_M, ρ_M, r_0 and W increase as one parameters among δ_l, δ_r or K_1 increases. In particular, last table show us that z_M strictly depends on K_1. In fact for $K_1 = 0$ for every values of δ_l and δ_r $z_M = 0$ and so, $\rho(z)$ is a decreasing function over the whole interval $[0, r_0]$. Moreover, from the first two tables, we observe that δ_r affects the values of z_M, ρ_M, r_0 and W more than δ_l.
These tests show that our model could represent several configuration for $\rho(z)$. Since we have not experimental data to compare with the approximated solution of the problem, this seems to be an improvement in the model.

Example 3

We compare the travelling wave solution for the original system (M_o) with some travelling wave solutions for system (M) computed for several values of K_1 (with fixed $\delta_l = 0.005$ and $\delta_r = 0.02$).
We denote by *BBF* the travelling wave solution for the original model, (M_o); in this case $r_0 = 0.134684$ and $W = 0.622816$.

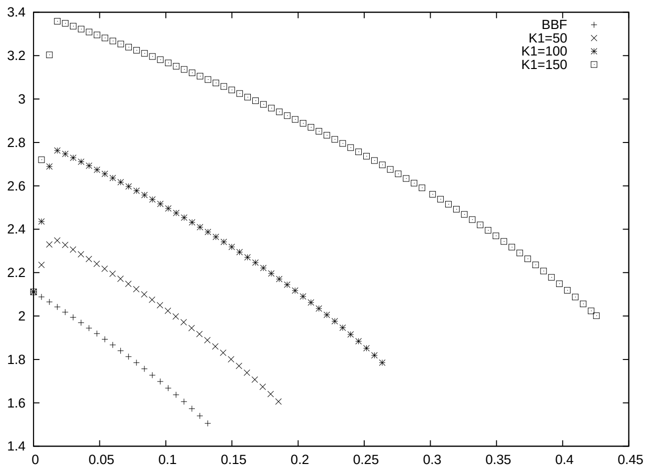

Figure 3: Comparison between ρ of travelling wave solution for (M_o) and for (M)

This graph shows how the positive value of $K(z)$ affected the configuration of ρ. In particular, as we said in the previous examples $K(z)$ affected the position and the value of the maximum for ρ. Moreover, passing from the original model to the one introduced by us, ρ loses its monocity.

Note that in the previous graph we have not included the case for $K = 0$ and $\delta_l, \delta_r \neq 0$, since it's roughly the same of the travelling wave solution for the original model, (M_o).

4.5 Tests for the general case

In this section we show some tests about the method described in Section 4.3 for the approximation of the solution of problem (M) with (I), (B), (LR) and (1.15)-(1.17).

In particular we would like to show the efficiency of that method using the travelling wave solution as initial datum. The idea is that of comparing, at every time, the approximated solution with the travelling wave.

We choose α, β, μ, ν, E, b as in (4.17); K_2 as in (4.18), d as in (4.19), u_0 and ρ_0 respectively as in (4.20) and (4.21).

We fix $\Delta t = \Delta x = 0.001$ and we set

$$\begin{aligned} K_1 &= 100 \\ \delta_l &= 0.005 \\ \delta_r &= 0.02 \end{aligned}$$

We set the initial datum of the problem as the travelling wave solution fo system (M) subject to (I) and (B). Then we approximate the solution at several times and we compare it with the evolving travelling wave.

Let $(W_T, r_{0_T}, u_T, \rho_T)$ be the approximated travelling wave solution that we use as initial datum, and (l, r, u, ρ) the approximated solution of the problem.

First of all we have to check if the travelling wave solution verifies the hypoteses of Theorem 6, conditions (3.2)-(3.10).

From Theorem 1:

$$r_{0_T} > 0$$

$$u_T(x), \rho_T(x) > 0 \quad \forall x \in [0, r_{0_T}]$$

and

$$\rho_T(r_{0_T}) - d u_T(r_{0_T})^{\frac{2}{3}} > 0$$

Moreover, from Theorem 2, u_T is a decreasing function. Since $W \in (0, \beta/(h(\mu - \alpha)))$:

$$u_T'(0) = \frac{\varepsilon}{\alpha}(\beta - W(\mu - \alpha)) > -\beta \frac{\varepsilon}{\alpha}$$

$$u_T'(r_{0_T}) = \varepsilon W - \frac{\varepsilon}{\alpha} \frac{u_0}{u_T(r_{0_T})}(\beta - W(\mu - 2\alpha))$$

and since from (2.3):

$$\varepsilon W = u_T'(r_0) + \frac{\nu \varepsilon}{\rho_T(r_{0_T}) - d u_T(r_{0_T})^{\frac{2}{3}}}$$

then:

$$u'_T(r_{0_T}) = \frac{\varepsilon}{\mu - 2\alpha}\left[\beta - \frac{\nu\alpha}{\rho_T(r_{0_T}) - du_T(r_{0_T})^{\frac{2}{3}}}\frac{u_T(r_{0_T})}{u_0)}\right] - \frac{\nu\varepsilon}{\rho_T(r_{0_T}) - du_T(r_{0_T})^{\frac{2}{3}}}$$

From Theorem 1:

$$\left[\beta - \frac{\nu\alpha}{\rho_T(r_{0_T}) - du_T(r_{0_T})^{\frac{2}{3}}}\frac{u_T(r_{0_T})}{u_0)}\right] > 0$$

then

$$u'_T(r_{0_T}) > -\frac{\nu\varepsilon}{\rho(r_0) - du(r_0)^{\frac{2}{3}}}$$

So hypoteses (3.3)-(3.5) of Theorem 6 are satisfied.
We have also to prove that the compatibility conditions (3.6)-(3.10) are verified.
From the boundary condition (2.4):

$$\begin{cases} u_T(0,t) = u_0 & t > 0 \\ u_T(r_0,t) = u_0 + \dfrac{\mu\varepsilon}{b} l'(0) = u_0 - \dfrac{\mu\varepsilon}{b} W & t > 0 \end{cases}$$

that are the zero order compatibility conditions for the initial data.
In order to prove the first order compatibility conditions we note that:

$$\begin{cases} u'(x) = \varepsilon W - \dfrac{C}{u} \\ u''(x) = \dfrac{C}{u^2} u_x \\ u'''(x) = -2\dfrac{C}{u^3}(u_x)^2 + \dfrac{C}{u^2} u_{xx} \end{cases}$$

where, for the sake of simplicity, we have neglect the subscript T.
Then for all $x \in (0, r_0)$:

$$\begin{aligned} E(uu_x)_x + l'(0)u_x &= E(uu_x)_x - Wu_x = \\ &= E[(uu_x)_x - \varepsilon W u_x] = \\ &= E[u_x^2 + uu_{xx} - \varepsilon W u_x] = \\ &= Eu_x[u_x + \frac{C}{u} - \varepsilon W] = 0 \end{aligned}$$

So condition (3.8) is verified. Moreover:

$$\begin{aligned} E(uu_x)_{xx} - Wu_{xx} &= E[(uu_x)_{xx} - \varepsilon W u_{xx}] = \\ &= E[u_{xx}(3u_x - \varepsilon W) + uu_{xxx}] = \\ &= E\left[u_{xx}\left(2\varepsilon W - 3\frac{C}{u}\right) - 2\frac{C}{u^2}u_x^2 + \frac{C}{u}u_{xx}\right] = \\ &= E\left[2u_x u_{xx} - 2\frac{C}{u^2}u_x^2\right] = 0 \end{aligned}$$

and as a consequence of this equation (3.9) holds.

With a similar argument it can be proved that the travelling wave solution verifies also (3.10).
In order to study the efficiency of the method, we have to estimate the error between $(W_T, r_{0_T}, u_T, \rho_T)$ and (l, r, u, ρ).
So we compare:

- W with l' and r':

$$E_{l'} = |W_T - l'|$$

$$E_{r'} = |W_T - r'|$$

  This comparison is justified since W is the travelling wave speed, while l' and r' are respectively the left and the right extremum velocity. If the solution is a travelling wave, as we said in Section 2, l' and r' have to be equal and their value represents the velocity of the wave.

- u_T and u:

$$||E_u||_1 = \Delta x \sum_{j=0}^{N} |u_j - u_{T_j}|$$

$$||E_u||_2 = \left(\Delta x \sum_{j=0}^{N} |u_j - u_{T_j}|^2 \right)^{\frac{1}{2}}$$

- ρ_T and ρ:

$$||E_\rho||_1 = \Delta x \sum_{j=0}^{N} |\rho_j - \rho_{T_j}|$$

$$||E_\rho||_2 = \left(\Delta x \sum_{j=0}^{N} |\rho_j - \rho_{T_j}|^2 \right)^{\frac{1}{2}}$$

We summarize the errors for some fixed times in the following tables.

t	$\|\|E_u\|\|_1$	$\|\|E_u\|\|_2$	$\|\|E_\rho\|\|_1$	$\|\|E_\rho\|\|_2$
1.	0.0164197	0.000444808	0.138456	0.0201135
2.	0.018036	0.000534637	0.14184	0.0211304
3.	0.0180724	0.000536747	0.141912	0.0211525
4.	0.0180731	0.000536791	0.141913	0.0211529
5.	0.0180731	0.000536791	0.141913	0.0211529

t	$E_{l'}$	$E_{r'}$	l'	r'
1.	0.0855267	0.0639523	−0.700455	−0.72203
2.	0.0934786	0.0929874	−0.692503	−0.692995
3.	0.0936562	0.093646	−0.692326	−0.692336
4.	0.09366	0.0936597	−0.692322	−0.692322
5.	0.09366	0.09366	−0.692322	−0.692322

From last table we note that l' and r' in a first period are increasing and then, from $t = 4.$ they become constant and in particular:

$$l'(t) = r'(t)$$

At the same time also the error functions become constant. This means that for $t > 4$ the distance between the approximated solution and the approximated travelling wave is constant. This result can be interpreted as the achievement of a steady state for the approximated solution. Moreover, in the moving domain it corresponds to a solution that is traslated in the negative x direction and setting $\tilde{W} = l' = r'$, this steady state is a travelling wave solution with speed $\tilde{W}$. This solution is different from the one we have used as initial datum. This suggests us that the travelling wave solution is a stable solution. In fact, starting from a perturbation of it, due to the numerical approximation, the system reaches the configuration of a travelling wave in a finite time.

In the following graphs we show the solutions ρ and u in the moving domain for several times.

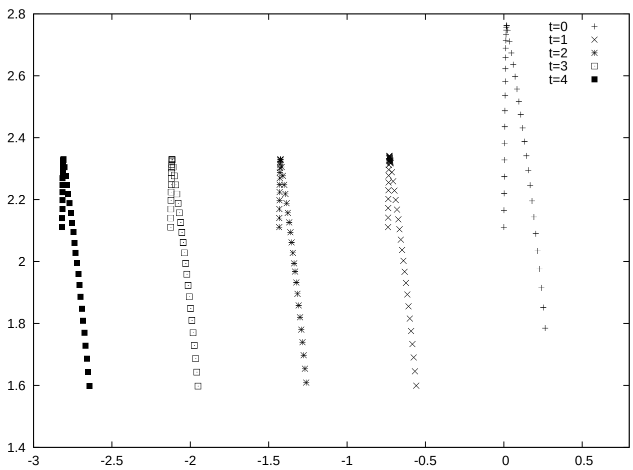

Figure 4: ρ at several times

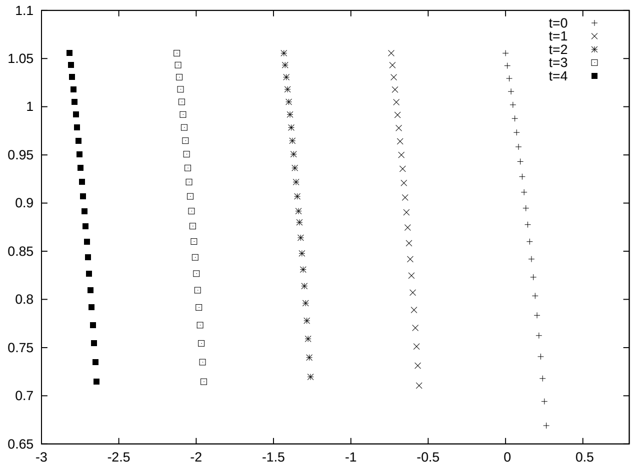

Figure 5: u at several times

We have tested the method also with different initial data. Since from Theorem 3, the existence of the solution is only guaranteed for an initial datum which verifies conditions (3.2)-(3.10), we have chosen it as the travelling wave solution for system M_o. So, as we have shown at the beginning of this section, the hypoteses of Theorem 3 are satisfied.We have obtained, for some fixed times, the following results:

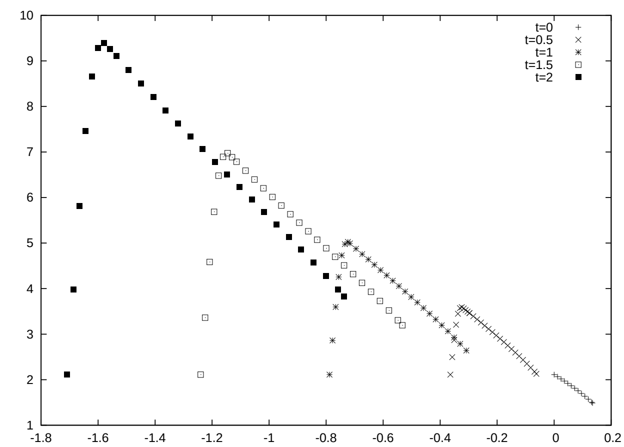

Figure 6: ρ for $t \in [0, 2]$

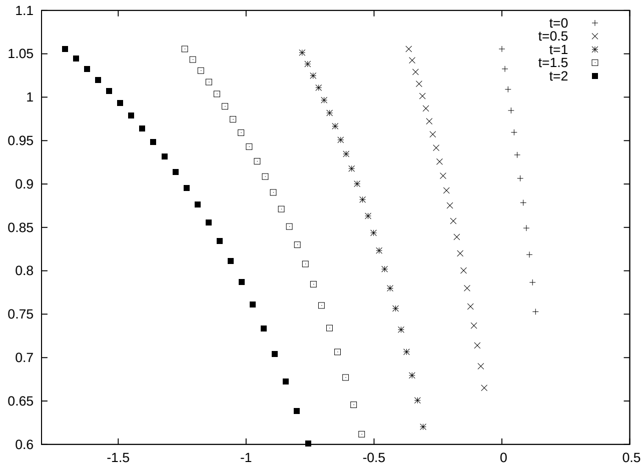

Figure 7: u for $t \in [0, 2]$

This example shows how the solution is affected by $K(x, t)$. In fact, the strictly decreasing configuration of the initial datum for ρ is rapidly transformed in a function that is increasing in a small interval close to $x = l(t)$ and then it decreases. Moreover, the maximum value for ρ is increasing with respect to t.

Note that also u changes; its characteristic monotonicity still holds but its second order derivative decreases as we can observe from Figure 7.

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