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Superselection Structure at Small Scales in Algebraic Quantum Field Theory

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Contents

Introduction

The phenomenology of elementary particle physics is described on the theoretical side, to a high degree of accuracy, by the perturbative treatment of relativistic quantum field theories. On the mathematical and conceptual side, however, the understanding of these theories is far from being satisfactory, as illustrated, for instance, by the well known difficulties in the very problem of providing them with a mathematically sound definition in $d = 4$ spacetime dimensions. Another example, not unrelated to the previous one, even if more on the conceptual level, is given by the problem of *confinement*. This issue arises in the theoretical description of hadronic physics, which, according to the common belief, is provided by quantum chromodynamics (QCD), a non-abelian gauge theory coupled to fermion fields, and amounts to the fact that the asymptotic ultraviolet freedom of QCD, together with other experimental and theoretical results such as the scaling of deep inelastic scattering cross section, or hadron spectroscopy fitting with the quark model of Gell-Mann and Ne'eman, seem to speak in favour of the existence of particle-like costituents of hadronic matter, quarks and gluons, which do not appear as asymptotic states at large distances, and are hence permanently confined in hadrons, due to a force between them that grows with distance. The conceptual problem with this notion of confinement arises when one notes that it ascribes a physical reality to theoretical objects which are not observable, such as the gauge and Dirac fields out of which the QCD lagrangian is constructed, while the observables are the only elements of a quantum theory to which it is possible to attach a physical interpretation. From what we know at present, we cannot exclude at all the possibility that there exists some other lagrangian, with a totally different field content, and nevertheless such that the corresponding quantum field theory, provided it can be constructed, yields the same observables as QCD. On the contrary, several examples are known of such a situation. It is well known, for instance, that a given S-matrix can be obtained from different systems of Wightman fields, relatively local with respect to each other (a so-called *Borchers class* of Wightman fields [Bor60]). Another classical, more relevant example is given by the Schwinger model, i.e. QED in $d = 2$ spacetime dimensions with massless fermions, which has been considered as a simple illustration of QCD phenomenology, since in $d = 2$ the electric potential is linearly rising with distance, so electrically charged fermions are expected to be confined in this model, and only composite neutral objects can appear as asymptotic states. This model is exactly solvable, and what one indeed finds, is that the algebra of observables is isomorphic to the one generated by a massive free scalar field [LS71], so that actually the model has no charged states. But the same set of observables is obtained starting from a free field lagrangian, so it is unclear why one should speak of "confined electrons" described by the theory. More recently, the discovery of a web of dualities between pairs of (supersymmetric) Yang-Mills theories [Sei96] has given support to the possibility of existence, even in 4 dimensions, of theories defined by different lagrangians, with different gauge groups and matter content, but having the same infrared behaviour, and hence the same charges, particles etc.

In view of the above considerations, D. Buchholz has advocated the following point of view [Buc96b]: in order to decide if a given theory intrinsically describes at small scales objects corresponding to the physical idea of confined particles and charges, one has to look at the observables of the theory alone.

As a matter of fact, a conclusion of this kind should not have come as a surprise, since the principle that a theory is fixed by the assignment of its algebras of local observables has been at the heart of the algebraic approach to quantum field theory [Haa96] for more than forty years. This axiomatic framework has been considerably successful in analysing structural aspects of quantum field theory such as collision theory, gauge symmetry and superselection structure in physical and low-dimensional spacetimes, quantum field theory on curved spacetime, and thermal states. In particular superselection theory gives a completely general procedure to recover, from the knowledge of the net of local observables, the set of charges (also called superselection sectors) described by the theory, together with their composition rule, permutation statistic and charge-anticharge symmetry [DHR71, DHR74, BF82], as well as a canonical system of charge carrying fields and a global gauge group selecting the observables as the gauge invariant combinations of fields, and labelling charges by its irreducible representations [DR90]. We can expect then that this analysis should play a relevant role in an intrinsic understanding of confined charges in the spirit put forward above. The other essential ingredient in this task is necessarily a framework allowing a canonical analysis of the structure of local observables at small spatio-temporal scales. This framework has been provided in [BV95], where an algebraic version of the conventional renormalization group methods is established, and it is then used to show that the small scales behaviour of the observables of a given theory is canonically described by a new theory (or, more generally, by a family of theories), itself defined by a net of local algebras, which is then regarded as the *scaling (ultraviolet) limit* of the given theory. It is then possible to apply the superselection analysis to this new theory, and the resulting sectors, which, in view of their construction, are canonically determined by the observables of the underlying (i.e., finite scales) theory, can be naturally considered as the charges described by the underlying theory at small scales. This has to be contrasted with the unphysical degrees of freedom usually associated with the small scales behaviour of gauge theories considered above. The superselection structure of the underlying theory and of its scaling limit are in principle different. Suppose then that there is a canonical way to recognize some charges of the scaling limit as suitable small scales limits of charges of the underlying theory. Then, since the physical idea of confined charge is that of a charge that cannot be created by operations at finite scales, one would get a natural and intrinsic definition of confined charge by declaring that a confined charge of a given theory is a charge of its scaling limit that it is not obtained as a limit of charges of the underlying theory.

In this thesis we study the relations between the scaling limit and the underlying theory's superselection structures, with the aim of establishing such a notion of charge scaling limit. As the analysis of the phase space properties of renormalization group orbits carried out in [BV95] makes clear (see also chapter 2), one cannot expect on general grounds this limit to exist for an arbitrary sector of the underlying theory, since it may happen that localizing a charge in a sequence of regions shrinking to a point requires energies growing too fast with the inverse radius of those regions, and the resulting charges of the scaling limit theory, if they existed, would then require an infinite amount of energy to be created in finite regions. What we will do then is to single out a class of sectors of the underlying theory with "good" phase space properties, and show that the limit of these sectors can be defined in a natural way. Also, we will consider both the cases of sectors which are finitely localizable, the so called DHR charges, after [DHR71, DHR74], and of sectors which are localizable in arbitrary spacelike cones, first studied in [BF82], the reason for this being that the latters are expected to appear in non-abelian gauge theories, the cone representing a roughened version of the Mandelstam string emanating from charged particles observed in lattice approximations, and that, on the other hand, non-abelian gauge theories are precisely the ones expected to exhibit the confinement phenomenon, so that a physically interesting intrinsic confinement criterion should necessarily encompass cone-like localizable charges in its range of application.

The organization of the thesis is as follows. In chapter 1, after having stated explicitly the general assumptions of the algebraic approach to quantum field theory, we recall the main results of the theory of superselection sectors both for localizable and for topological (i.e. cone-like localizable) charges, including the results of [DR90] about the reconstrucion of the field net and the compact gauge group. In chapter 2 we briefly review the algebraic version of the renormalization group and the construction of the scaling limit, together with some illustrative examples, taken from [BV98], among which there is the above mentioned Schwinger model, which is the simplest case in which confined charges can be intrinsically identified. New results on the scaling limit of charges are exposed in chapter 3. Here, we first extend the scaling limit construction to the case of a net of field algebras with normal (i.e. Bose-Fermi) commutation relations, and carrying an action of a compact gauge group G, obtaining as a scaling limit a new net of field algebras, again with normal commutation relations and an action of G. We show then that the DHR sectors of the underlying theory which satisfy a condition of "ultraviolet stability" – physically motivated by the above phase space considerations, and which is expressed in terms of the Hilbert spaces in the field net implementing the considered sector – admit a natural notion of scaling limit, in the precise sense that for any of such sectors it is possible to construct Hilbert spaces in the scaling limit of the field net, which carry the gauge group representation associated to the given ultraviolet stable sector, and inducing a DHR sector on the scaling limit theory.[a)] We then expose the results on the extension of this analysis to quantum topological charges, as cone-like localizable sectors are also known. In this case there remain some conceptual and technical difficulties that will be discussed, but we feel that there are promising partial results, and work is in progress in order to obtain a clear physical picture. In particular, we

[a)] Work of C. D'Antoni and R. Verch in this direction is also in progress [DV] consider sectors satisfying a condition of ultraviolet stability similar to the one employed for localizable charges, together with some natural conditions of asymptotic localizability (in bounded regions), which are suggested by the observed behaviour of these charges in models. To these sectors we are able to associate in the scaling limit a normal net of field algebras on *bounded* regions, and Hilbert spaces in these algebras carrying the relevant representations of G. Adding a technical assumption on this net, the status of which still needs to be clarified, we are also able to show that these Hilbert spaces induce DHR sectors on the scaling limit theory. We remark that this is precisely what is expected to happen in asymptotically free theories, where, due to vanishing of interactions at small scales, the strings disappear in the scaling limit, leaving only finitely localized excitations. Finally, in appendix A we collect some geometrical results on spacelike cones which are needed in the above analysis, and in appendix B we exhibit an example of a theory (the Majorana free field with $\mathbb{Z}_2$ gauge group) whose localizable sectors comply with the ultraviolet stability assumption referred to above.

CHAPTER 1

Superselection sectors and the reconstruction of fields

The existence of a restriction to the superposition principle for pure states, represented by vectors in the physical Hilbert space of quantum field theory, was discovered in [WWW52], where it was shown that this Hilbert space is the direct sum of *coherent subspaces*, or *superselection sectors*, for instance labelled by the electric charge, or by univalence, in such a way that the phase relations between vectors belonging to different sectors are unobservable. In [HK64] this was recognized as an aspect of the representation problem in quantum field theory, i.e. the existence of several inequivalent irreducibile representations of the algebra of observables for systems with an infinite number of degrees of freedom (in contrast to the situation prevailing for non-relativistic finite systems): the algebra of observables of quantum field theory is faithfully represented on each superselection sector, such representations being inequivalent, and the role of unobservable fields is that of transferring some “superselection quantum number” from one sector to another. This shifts the attention from the Hilbert space formulation of quantum field theory, central in the Wightman approach, to the abstract net of algebras of local observables and its representations, which are the object of study in the algebraic approach. In this framework it is taken as a fundamental postulate that all the information is encoded in the net of algebras of local observables, which then characterizes a given theory completely; superselection theory is then the study of the structure of the set of irreducible representations of such a net (or better of the subset of them which are “relevant for particle physics”). Remarkably enough, one finds, as we will see below, that it is possible to endow this set with structures, such as composition or conjugation of sectors, which reflect the corresponding physical operations with charges, and also that sectors are in one-to-one correspondence with unitary equivalence classes of irreducible representations of a global gauge group. This culminates in the Doplicher-Roberts reconstruction theorem, also discussed below, which embeds the observable net as the gauge invariant part of a canonical field net with Bose-Fermi commutation relations.

In this chapter we will state explicitly the main postulates of the algebraic approach to quantum field theory, briefly discussing also some basic consequences needed in the following. Then we will review in section 1.2 the main results of superselection theory, both for localizable and topological charges, and finally, in section 1.3, we will discuss the above mentioned reconstruction of fields and global gauge group.

1.1 Basic assumptions of algebraic quantum field theory

For this work to be reasonably self-contained, and to fix a notation, in this section we will briefly discuss the fundamental assumptions of the algebraic approach to quantum field theory. We refer to the monograph [Haa96] for further details and references.

The arena of relativistic quantum field theory is Minkowski space, i.e. the (affine) space $\mathbb{R}^4$ endowed with the pseudo-euclidean structure, called *Minkowski metric*, induced by the symmetric matrix $g = (g_{\mu\nu})_{\mu,\nu=0,\dots,4} = \mathrm{diag}(1,-1,-1,-1)$. For $x, y \in \mathbb{R}^4$, we will write $x \cdot y = g_{\mu\nu}x^\mu y^\nu$ for this bilinear form (summation over repeated indices is understood), and $x^2 := x \cdot x$. The symmetry group of Minkowski space is the Poincaré group $\mathscr{P} := O(1,3) \ltimes \mathbb{R}^4$, with $O(1,3)$ the group of matrices $\Lambda \in M_4(\mathbb{R})$ leaving g invariant, $\Lambda^t g \Lambda = g$, acting in the natural way on $\mathbb{R}^4$. We will only make use of the connected subgroup $SO^\uparrow(1,3)$ of those Λ with $\det \Lambda = 1$, ${\Lambda^0}_0 > 0$, and correspondingly we get a subgroup $\mathscr{P}_+^\uparrow$ of $\mathscr{P}$. We will also need to consider the universal covering $SL(2,\mathbb{C}) = \{A \in M_2(\mathbb{C}) \,:\, \det A = 1\}$ of $SO^\uparrow(1,3)$, with covering homomorphism denoted by $A \to \Lambda(A)$ (for its definition we refer to [BLOT90, 3.1.C]). Correspondingly $\tilde{\mathscr{P}}_+^\uparrow := SL(2,\mathbb{C}) \ltimes \mathbb{R}^4$ is the universal covering of $\mathscr{P}_+^\uparrow$, and $\eta : \tilde{\mathscr{P}}_+^\uparrow \to \mathscr{P}_+^\uparrow$ will denote the covering homomorphism. Generic elements of $\mathscr{P}_+^\uparrow$ and $\tilde{\mathscr{P}}_+^\uparrow$ will be denoted by $s, t, \dots$ and their action on $x \in \mathbb{R}^4$ by $s \cdot x$. The metric g defines the standard causal structure, to which the terms timelike, lightlike etc. will be referred. The open forward (backward) lightcone $V_\pm$ is the set of points in $\mathbb{R}^4$ which are future (past) timelike to the origin. We set $V := V_+ \cup V_-$. For any set $\mathscr{S} \subseteq \mathbb{R}^4$, its spacelike complement $\mathscr{S}'$ is the set of points in $\mathbb{R}^4$ which are spacelike separated form all points in $\mathscr{S}$.

Throughout all this thesis, we shall use the symbol $\mathscr{O}$ to denote the generic element of the family of open bounded subsets of $\mathbb{R}^4$, which form an upward directed net under inclusion. A frequently used subnet of causally complete regions is that of open double cones $\mathscr{O}_{a,b} := (a + V_-) \cap (b + V_+)$, with $a \in b + V_+$. This family is clearly upward directed under inclusion, and the union of all its elements is all $\mathbb{R}^4$. Moreover, any double cone is Poincaré equivalent to a double cone of the form $\mathscr{O}_r := \mathscr{O}_{re_0,-re_0}$, with $e_0 = (1,\mathbf{0})$, and they form a basis for the standard topology of $\mathbb{R}^4$.

Definition 1.1. A *net of C*-algebras over Minkowski space* is a net

$$\mathscr{O} \to \mathfrak{A}(\mathscr{O}) \tag{1.1}$$

of unital C*-algebras over the directed set of open bounded regions in $\mathbb{R}^4$, all the algebras having the same unit.

We can embed all the local algebras $\mathfrak{A}(\mathscr{O})$ of a net as above in a *quasi-local* C*-algebra $\mathfrak{A}$, defined as the C*-inductive limit of the inductive system $\mathfrak{A}(\mathscr{O})$ of C*-algebras, i.e. as the completion of the $*$-algebra $\mathfrak{A}_{\mathrm{loc}} = \bigcup_{\mathscr{O}} \mathfrak{A}(\mathscr{O})$ in its unique C*-norm (as $\mathfrak{A}(\mathscr{O}_1) \subseteq \mathfrak{A}(\mathscr{O}_2)$ if $\mathscr{O}_1 \subseteq \mathscr{O}_2$, the C*-norms of $\mathfrak{A}(\mathscr{O}_1)$ and $\mathfrak{A}(\mathscr{O}_2)$ agree on $\mathfrak{A}(\mathscr{O}_1)$). We will use the symbol $\mathfrak{A}$ to denote both the net and its quasi-local algebra. To unbounded regions $\mathscr{S} \subseteq \mathbb{R}^4$, we can then associate the C*-subalgebra of $\mathfrak{A}$ generated by all the $\mathfrak{A}(\mathscr{O})$, $\mathscr{O} \subset \mathscr{S}$. Given two nets $\mathfrak{A}_i$, $i = 1,2$, a *net homomorphism* is an homomorphism $\phi : \mathfrak{A}_1 \to \mathfrak{A}_2$ of the quasi-local algebras, such that $\phi(\mathfrak{A}_1(\mathscr{O})) \subseteq \mathfrak{A}_2(\mathscr{O})$ for each $\mathscr{O}$. A *net isomorphism* is a net homomorphism which is invertible as such. For a subset $\mathfrak{S}$ of an algebra $\mathfrak{A}$, we denote its commutant as $\mathfrak{S}'$.

Definition 1.2. A net of C*-algebras is said to be *local* if $\mathscr{O}_1 \subseteq \mathscr{O}_2'$ implies $\mathfrak{A}(\mathscr{O}_1) \subseteq \mathfrak{A}(\mathscr{O}_2)'$ in $\mathfrak{A}$.

We recall that for a unitary strongly continuous representation U of $\mathbb{R}^4$ on a Hilbert space $\mathscr{H}$, the *spectrum* of U, $\mathrm{Sp}\,U$, is the support of the spectral measure determined by U, or, equivalently, the joint spectrum of its generators P_μ, $\mu = 0,\dots,3$.

Definition 1.3. A *Poincaré covariant net* of C*-algebras is a pair $(\mathfrak{A}, \alpha)$, where $\mathfrak{A}$ is a net of C*-algebras and $\alpha : \mathscr{P}_+^\uparrow \to \mathrm{Aut}\,\mathfrak{A}$ is a group homomorphism (also called an automorphic action of $\mathscr{P}_+^\uparrow$ on $\mathfrak{A}$) such that for each $\mathscr{O}$,

$$\alpha_s(\mathfrak{A}(\mathscr{O})) = \mathfrak{A}(s \cdot \mathscr{O}), \qquad s \in \mathscr{P}_+^\uparrow. \tag{1.2}$$

A *vacuum state*[a)] on $(\mathfrak{A}, \alpha)$ is a state ω on $\mathfrak{A}$ which is α-invariant, and such that, denoted by $(\pi, \mathscr{H}, \Omega)$ its GNS representation, and by U the associated unitary strongly continuous representation of $\mathscr{P}_+^\uparrow$, the translations $x \in \mathbb{R}^4 \to U(\mathbb{1}, x)$ satisfy the *spectrum condition* $\mathrm{Sp}\,U(\mathbb{1}, \cdot) \subseteq \overline{V}_+$.

An homomorphism of Poincaré covariant nets $(\mathfrak{A}_i, \alpha^i)$, $i = 1,2$, is a net homomorphism ϕ of the underlying nets $\mathfrak{A}_i$ which intertwines the actions of the Poincaré group

$$\phi \circ \alpha_s^1 = \alpha_s^2 \circ \phi, \qquad s \in \mathscr{P}_+^\uparrow. \tag{1.3}$$

Sometimes we will consider nets that are only translation covariant, and it is evident how to adapt the above definitions.

As already mentioned above, the fundamental postulate of the algebraic approach to quantum field theory is that all the physically relevant information on a given theory is encoded in its algebra of observables. Together with the trivial observation that all measurements on a physical system are performed in some bounded spacetime region, and with Einstein causality, which implies that observations localized in spacelike separated regions cannot interfere with each other, so that the associated quantum mechanical operators need to commute, this implies that the algebra of observables has the structure of a local net of C*-algebras, the algebras $\mathfrak{A}(\mathscr{O})$ being interpreted as generated by observables measurable in $\mathscr{O}$.

Taking also into account that we are interested in applications to particle physics, i.e. to describe localized excitations of the vacuum, we conclude that the data determining a theory are given by a triple $(\mathfrak{A}, \alpha, \omega)$, constituted by a Poincaré covariant local net with a pure vacuum state.

[a)] The traditional notation ω_0 for the vacuum state will be used in this thesis with a different meaning, so that we will here denote the vacuum state simply by ω. Correspondigly, its GNS representation will be denoted by $(\pi, \mathscr{H}, \Omega)$ instead of $(\pi_0, \mathscr{H}_0, \Omega)$.

Usually, as a consequence of the observation that all known superselection rules (including here not only charges in particle physics, but also thermodynamic quantities) are determined by global aspects of the states, the hypotesis is also made that all physically relevant states are *locally normal* states of the vacuum representation, i.e. for any such state φ on $\mathfrak{A}$, and any $\mathscr{O}$, $\varphi \upharpoonright \mathfrak{A}(\mathscr{O})$ is a normal state of the representation $\pi \upharpoonright \mathfrak{A}(\mathscr{O})$. Then we are locally in the same situation as in quantum mechanics of systems with a finite number of degrees of freedom, as we have only to deal with a single quasi-equivalence class of representations of each local algebra $\mathfrak{A}(\mathscr{O})$. Thus we can assume that the net is defined directly in its vacuum representation. By $B(\mathscr{H})$ we will denote the C*-algebra of bounded operators on a Hilbert space $\mathscr{H}$.

Definition 1.4. A local Poincaré covariant net of C*-algebras *in vacuum representation* is a quadruple $(\mathscr{H},\mathfrak{A},U,\Omega)$, with $\mathscr{H}$ a Hilbert space, $\mathfrak{A}$ a local net of C*-subalgebras of $B(\mathscr{H})$, U a strongly continuous unitary representation of $\mathscr{P}_+^\uparrow$ on $\mathscr{H}$ satisfying the spectrum condition, and $\Omega \in \mathscr{H}$ a unit vector cyclic for $\mathfrak{A}$, such that, with $\alpha_s := \mathrm{Ad}U(s)$, $(\mathfrak{A},\alpha)$ is a Poincaré covariant local net of C*-algebras, and Ω is the (up to a phase) unique unit vector invariant under translations $U(\mathbb{1},x)$.

The unicity condition on Ω implies that the vacuum state $\omega := (\Omega|(\cdot)\Omega)$ is pure on $\mathfrak{A}$, or, equivalently, $\mathfrak{A}$ is irreducible on $\mathscr{H}$, $\mathfrak{A}' = \mathbb{C}\mathbb{1}_{\mathscr{H}}$ [Haa96, thm 3.2.6]. In the situation described by the above definition, it is also customary to assume that the local algebras are actually von Neumann algebras, as the von Neumann algebra[b)] $\mathscr{A}(\mathscr{O}) := \mathfrak{A}(\mathscr{O})''$ has by definition the same normal states as $\mathfrak{A}(\mathscr{O})$. In this context, an assumption that is frequently made, but which we will use only occasionally, is *weak additivity* of the net $\mathscr{A}$, i.e. that

$$\bigvee_{x\in\mathbb{R}^4} \mathscr{A}(\mathscr{O}+x) = B(\mathscr{H}) \tag{1.4}$$

holds for each $\mathscr{O}$. This is clearly suggested by the idea that the algebras are generated by an underlying system of Wightman fields.

We now list a couple of basic results on the structure of local nets, essentially consequences of positivity of the energy, which will be needed in the following. For the proofs we refer to [D'A90].

Theorem 1.5 (Reeh-Schlieder). [RS61] *Let $(\mathscr{H},\mathfrak{A},U,\Omega)$ be a translation covariant net in vacuum representation. Let $\mathscr{S} \subseteq \mathbb{R}^4$ be such that there exists $\mathscr{S}_0 \subset \mathscr{S}$ and a neighbourhood $\mathscr{N}$ of zero in $\mathbb{R}^4$ for which $\mathscr{S}_0 + \mathscr{N} \subseteq \mathscr{S}$, and $\bigvee_{x\in\mathbb{R}^4} \mathfrak{A}(\mathscr{S}_0+x)'' = B(\mathscr{H})$. Then Ω is cyclic for $\mathfrak{A}(\mathscr{S})$.*

We remark explicitly that this result does not rely on locality of $\mathfrak{A}$. Examples of regions satisfying the hypothesis of the above theorem are given by the spacelike complements $\mathscr{O}'$ of any bounded $\mathscr{O}$, or by *wedges*, i.e. Poincaré transforms of $\mathscr{W}_{1,\pm} := \{x \in \mathbb{R}^4 \,:\, \pm x^1 > |x^0|\}$. Indeed, for any such region, one can find a translated region that contains any given bounded open set. If the net $\mathscr{A}$ satisfies weak additivity, another example is given by bounded regions, so that in this case, if the net is also local, the vacuum vector is cyclic and separating for the algebras $\mathscr{A}(\mathscr{O})$.

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b) Throughout the thesis, we will use script capital letters to denote nets of von Neumann algebras.

Theorem 1.6 (Borcher's property B). [Bor67] *Let $(\mathscr{H}, \mathscr{A}, U, \Omega)$ be a translation covariant local net of von Neumann algebras satisfying weak additivity. Then for any non-zero projection $E \in \mathscr{A}(\mathscr{O})$ and for any $\mathscr{O}_1 \supset \overline{\mathscr{O}}$, there exists an isometry $V \in \mathscr{A}(\mathscr{O}_1)$ whose final projection is E, $E = VV^*$.*

As we will see, this property finds an important application in superselection theory.

We will also have to deal with nets of algebras generated by unobservable fields, which then need not be local. For future reference, we collect here the relevant definitions for this case. As customary, we denote the commutator of operators by square brackets, and the anticommutator by curly brackets.

Definition 1.7. (i) A *normal net of C^*-algebras with gauge symmetry* is a triple $(\mathfrak{F}, \beta, k)$ where $\mathfrak{F}$ is a net of C^*-algebras, β is an action of a compact group G on $\mathfrak{F}$ (the *gauge group*) by net automorphisms, and $k \in G$ is a central element, such that $k^2 = e$ (the identity of G), and such that $\mathfrak{F}$ obeys *local $\mathbb{Z}_2$-graded commutativity* with respect to the grading defined by $\gamma := \beta_k$, i.e. if for $F \in \mathfrak{F}$ we define its *Bose and Fermi parts* as $F_\pm := \frac{1}{2}(F \pm \gamma(F))$, we have that for any pair $F_i \in \mathfrak{F}(\mathscr{O}_i)$, $i = 1, 2$, with $\mathscr{O}_1 \subset \mathscr{O}_2'$, there holds

$$[F_{1,+}, F_{2,+}] = [F_{1,+}, F_{2,-}] = [F_{1,-}, F_{2,+}] = \{F_{1,-}, F_{2,-}\} = 0. \tag{1.5}$$

(ii) A normal Poincaré covariant net with gauge symmetry will be a quadruple $(\mathfrak{F}, \beta, k, \alpha^{\mathfrak{F}})$, with $(\mathfrak{F}, \beta, k)$ a normal net with gauge symmetry, $(\mathfrak{F}, \alpha^{\mathfrak{F}})$ a $\tilde{\mathscr{P}}_+^\uparrow$-covariant net, and such that $\alpha^{\mathfrak{F}}_s$ and β_g commute for each $s \in \tilde{\mathscr{P}}_+^\uparrow$, $g \in G$.

(iii) Given a Poincaré covariant local net $(\mathfrak{A}, \alpha)$, a normal Poincaré covariant net with gauge symmetry *over* $(\mathfrak{A}, \alpha)$ is a quintuple $(\pi_{\mathfrak{F}}, \mathfrak{F}, \beta, k, \alpha^{\mathfrak{F}})$, with $(\mathfrak{F}, \beta, k, \alpha^{\mathfrak{F}})$ as in (ii), and $\pi_{\mathfrak{F}} : \mathfrak{A} \to \mathfrak{F}$ a net homomorphism, such that $\alpha^{\mathfrak{F}}_s \circ \pi_{\mathfrak{F}} = \pi_{\mathfrak{F}} \circ \alpha_{\eta(s)}$, and $\pi_{\mathfrak{F}}(\mathfrak{A}(\mathscr{O})) = \mathfrak{F}(\mathscr{O})^G$, the fixed points of $\mathfrak{F}(\mathscr{O})$ under the action β of G.

It is also clear what will be the definition of a vacuum state ω over $(\mathfrak{F}, \beta, k, \alpha^{\mathfrak{F}})$. As in the case of local nets, we have the following spatial version of the above definitions (actually, below we give only the spatial version of definition 1.7(iii), it is however clear how the spatial versions of the other two definitions should be formulated).

Definition 1.8. Let $(\mathscr{H}, \mathfrak{A}, U, \Omega)$ be a Poincaré covariant local net. A Poincaré covariant, normal net with gauge symmetry *in its physical representation* over $(\mathscr{H}, \mathfrak{A}, U, \Omega)$ is a quintuple $(\pi_{\mathfrak{F}}, \mathfrak{F}, V, k, U_{\mathfrak{F}})$ with $\pi_{\mathfrak{F}}$ a representation of $\mathfrak{A}$ on a Hilbert space $\mathscr{H}_{\mathfrak{F}}$ containing the vacuum representation, $\mathfrak{F}$ a net of C^*-subalgebras of $B(\mathscr{H}_{\mathfrak{F}})$, V a unitary strongly continuous representation of a compact group G on $\mathscr{H}_{\mathfrak{F}}$, $k \in G$ a central element with $k^2 = e$, and $U_{\mathfrak{F}}$ a unitary strongly continuous representation of $\tilde{\mathscr{P}}_+^\uparrow$ on $\mathscr{H}_{\mathfrak{F}}$ such that:

(i) with $\beta_g := \mathrm{Ad}\,V(g)$ and $\alpha^{\mathfrak{F}}_s := \mathrm{Ad}\,U_{\mathfrak{F}}(s)$, $(\pi_{\mathfrak{F}}, \mathfrak{F}, \beta, k, \alpha^{\mathfrak{F}})$ is a Poincaré covariant normal net over $(\mathfrak{A}, \mathrm{Ad}\,U)$;
(ii) the translations $x \to U_{\mathfrak{F}}(\mathbb{1}, x)$ satisfy the spectrum condition;
(iii) Ω is gauge invariant, $V(g)\Omega = \Omega$, $g \in G$, and is the unique translation invariant unit vector in $\mathscr{H}_{\mathfrak{F}}$.

Remark. If $(\pi_{\mathfrak{F}}, \mathfrak{F}, V, k, U_{\mathfrak{F}})$ is as in the above definition, Reeh-Schlieder theorem and normal commutation relations imply that Ω is separating for the local von Neumann algebras $\mathfrak{F}(\mathscr{O})^-$ (the bar denoting closure in the weak operator topology on $B(\mathscr{H}_{\mathfrak{F}})$): if $F \in \mathfrak{F}(\mathscr{O})^-$

is such that $F\Omega = 0$, then by gauge invariance of Ω, also $F_\pm\Omega = 0$, so that if $F' \in \mathfrak{F}(\mathscr{O}')$, by normality $FF'\Omega = F'F_+\Omega + (F'_+ - F'_-)F_-\Omega = 0$, and by the Reeh-Schlieder theorem applied to $\mathfrak{F}(\mathscr{O}')$, $F = 0$.

As we mentioned in the introduction, in the subsequent analysis we shall encounter charges which are localizable (in a sense made precise in section 1.2) only in certain unbounded regions, called spacelike cones (see appendix A for their definition). Corrispondingly, the fields carrying such charges will also only be localized in spacelike cones, so that they will generate, instead of a net $\mathscr{O} \to \mathfrak{F}(\mathscr{O})$ on bounded sets, a net $\mathscr{C} \to \mathfrak{F}(\mathscr{C})$ on spacelike cones.[c)] It is then clear how to modify the above definitions in order to deal with such situation, and we will call the resulting quadruple $(\mathfrak{F}, \beta, k, \alpha^{\mathfrak{F}})$, with $\mathfrak{F}$ a net on spacelike cones, an *extended* normal Poincaré covariant (field) net with gauge symmetry. The extended field net arising from topological sectors through the Doplicher-Roberts reconstruction theorem (see section 1.3 below) has some additional features, so that it deserves a formal definition.

Definition 1.9. Let $(\mathscr{H}, \mathscr{A}, U, \Omega)$ be a Poincaré covariant local net of von Neumann algebras. A *Poincaré covariant, normal extended net with gauge symmetry in its physical representation over* $(\mathscr{H}, \mathscr{A}, U, \Omega)$ is a quintuple $(\pi_{\mathscr{F}}, \mathscr{F}, V, k, U_{\mathscr{F}})$ with $\pi_{\mathscr{F}}$ a representation of $\mathscr{A}$ on a Hilbert space $\mathscr{H}_{\mathscr{F}}$ containing the vacuum representation, $\mathscr{F}$ an extended net of C*-subalgebras of $B(\mathscr{H}_{\mathscr{F}})$, V a unitary strongly continuous representation of a compact group G on $\mathscr{H}_{\mathscr{F}}$, $k \in G$ a central element with $k^2 = e$, and $U_{\mathfrak{F}}$ a unitary strongly continuous representation of $\tilde{\mathscr{P}}_+^\uparrow$ on $\mathscr{H}_{\mathscr{F}}$ such that:

(i) with $\beta_g := \mathrm{Ad}\,V(g)$ and $\alpha_s^{\mathscr{F}} := \mathrm{Ad}\,U_{\mathscr{F}}(s)$, $(\mathscr{F}, \beta, k, \alpha^{\mathscr{F}})$ is a Poincaré covariant normal extended net of von Neumann algebras;
(ii) $\alpha_s^{\mathscr{F}} \circ \pi_{\mathscr{F}} = \pi_{\mathscr{F}} \circ \alpha_{\eta(s)}$ and $\mathscr{F}(\mathscr{C})^G = \pi_{\mathscr{F}}(\mathscr{A}(\mathscr{C}))^-$;
(iii) for each $\mathscr{C}$, the union of all the algebras $\mathscr{F}(\mathscr{C} + x)$, $x \in \mathbb{R}^4$, is irreducible;
(iv) $\mathscr{H}$ is cyclic for each algebra $\mathscr{F}(\mathscr{C})$;
(v) the translations $x \to U_{\mathscr{F}}(\mathbb{1}, x)$ satisfy the spectrum condition;
(vi) Ω is gauge invariant and is the unique translation invariant unit vector in $\mathscr{H}_{\mathscr{F}}$.

1.2 Superselection theory

The states of interest in particle physics are characterized by the idealization that they describe a few localized excitations in empty space. The subject of superselection theory is to formulate precise criteria selecting, among all states on the quasi-local algebras, those that comply with this physical picture, and then to classify the (irreducible) representations of the quasi-local algebra induced by such states, and to study the structure of the resulting set of unitary equivalence classes, called *superselection sectors*. Here we will give a very brief account of the results obtained for the two main known classes of sectors: localizable – or DHR – sectors [DHR71, DHR74] (see also [Rob90] for a pedagogical overview), and topological sectors [BF82], which are essentially distinguished by the kind of localization

[c)] Here the term *net* is slightly abused, since the set of spacelike cones is not directed, and stands for the isotony property of the correspondence $\mathscr{C} \to \mathfrak{F}(\mathscr{C})$, i.e. $\mathscr{C}_1 \subseteq \mathscr{C}_2$ implies $\mathfrak{F}(\mathscr{C}_1) \subseteq \mathfrak{F}(\mathscr{C}_2)$.

regions of the corresponding charges, double cones for the former, and spacelike cones for the latter (a unified treatment is however possible, and has been given in [Kun01]).

1.2.1 Localizable sectors

Throughout this section, $\mathscr{O}$ will denote a double cone in Minkowski space. Let $(\mathscr{H}, \mathfrak{A}, U, \Omega)$ be a Poincaré covariant[d)] local net of C*-algebras in its vacuum representation, that we denote by ι. We also assume that $\mathfrak{A}$ satisfies property B, theorem 1.6, and *essential Haag duality*

$$\mathfrak{A}^d(\mathscr{O}) = \mathfrak{A}^d(\mathscr{O}')', \tag{1.6}$$

where $\mathscr{O} \to \mathfrak{A}^d(\mathscr{O}) := \mathfrak{A}(\mathscr{O}')'$ is the *dual net* of $\mathfrak{A}$. Equation (1.6) is equivalent to locality of $\mathfrak{A}^d$. If in particular $\mathfrak{A}^d(\mathscr{O}) = \mathfrak{A}(\mathscr{O})''$ (by locality of $\mathfrak{A}$, $\mathfrak{A}(\mathscr{O})'' \subseteq \mathfrak{A}^d(\mathscr{O})$, so this can be viewed as the requirement that $\mathfrak{A}$ is maximal with respect to locality) then $\mathfrak{A}$ is said to satisfy Haag duality *tout court*. This last property, in terms of which DHR theory was originally formulated, is known to hold in free field theories [Ara63, Ara64], but is violated in models with spontaneous symmetry breaking [Rob76]. Essential duality is however to be considered as a generic property of local nets, as it is satisfied whenever the net is generated, in any reasonable sense, by Wightman fields [BW75, BW76]. The class of states (or, equivalently, representations) which we will consider is specified by the following criterion, where we use $\cong$ to denote unitary equivalence, and $\restriction$ to denote restriction.

Definition 1.10 (DHR selection criterion). [DHR71] A representation π of the quasi-local algebra *satisfies the DHR selection criterion* (or is a *DHR representation*) if, for every double cone $\mathscr{O}$,

$$\pi \restriction \mathfrak{A}(\mathscr{O}') \cong \iota \restriction \mathfrak{A}(\mathscr{O}'), \tag{1.7}$$

The above criterion was suggested by the findings of [DHR69a], where an irreducible field net with gauge symmetry $\mathfrak{F}$ over $\mathfrak{A}$ is considered, and it is then shown that the irreducible representations of $\mathfrak{A}$ appearing in $\mathscr{H}_{\mathfrak{F}}$ are DHR.

For what concerns the physical interpretation of the DHR criterion, if φ is a normal state of a DHR representation, then

$$\lim_{\mathscr{O} \nearrow \mathbb{R}^4} \|(\varphi - \omega) \restriction \mathfrak{A}(\mathscr{O}')\| = 0,$$

and this statement also admits a partial converse under some not really restrictive hypotesis on the representation considered. Thus we see that the DHR criterion select states which are close to the vacuum at spacelike infinity in a rather strong sense, and therefore excludes states with nonvanishing total electric charge, since this can be measured, thanks to Gauss' law, in the spacelike complement of any bounded region. This is clearly due to the vanishing of the photon mass, implying that electromagnetic forces are long-range, but we will see below that topological charges, which are not DHR, arise also in purely massive theories.

[d)] Most of the results of DHR theory of superselection sectors are independent of covariance of the theory, which enters directly only when dealing with covariant sectors, but since in the application to the scaling limit theory covariance is an essential ingredient, we include this hypotesis from the beginning.

It follows from property B that the set $\mathrm{DHR}(\mathfrak{A})$ of DHR representations of $\mathfrak{A}$ is closed under direct sums and subrepresentations. Actually, it has a much richer structure, which can be uncovered by relating it to a set of endomorphisms of the quasi-local dual algebra $\mathfrak{A}^d$. This is accomplished as follows, using in an essential way notions from category theory, for which we refer to [McL98, DR89b]. The set $\mathrm{DHR}(\mathfrak{A})$ is, in a natural way, the set of objects of a C*-category, whose space of morphisms $(\pi_1 : \pi_2)$ between objects π_i, $i = 1,2$, is given by the *intertwiners* between π_1 and π_2, i.e. operators $T \in B(\mathscr{H}_{\pi_1}, \mathscr{H}_{\pi_2})$ such that $T\pi_1(A) = \pi_2(A)T$ for each $A \in \mathfrak{A}$. This C*-category is seen to be isomorphic to $\mathrm{DHR}(\mathfrak{A}^d)$ (with no risk of confusion, we denote by the same symbol a category and the set of its objects) by associating to each $\pi \in \mathrm{DHR}(\mathfrak{A})$ its unique extension $\tilde{\pi}$ to $\mathfrak{A}^d$, which is an element of $\mathrm{DHR}(\mathfrak{A}^d)$ [Rob90], and thanks to essential duality, this last category is in turn equivalent to the C*-category Δ of transportable localized endomorphisms of $\mathfrak{A}^d$, defined as follows.

Definition 1.11. Let $\mathfrak{A}$ be a net of C*-algebras. An endomorphism $\rho \in \mathrm{End}(\mathfrak{A})$ is *localized* in a double cone $\mathscr{O}$ if $\rho(A) = A$ for each $A \in \mathfrak{A}(\mathscr{O}')$, and is *transportable* if for any double cone $\mathscr{O}_1$ there exists ρ_1 localized in $\mathscr{O}_1$ and a unitary $U \in \mathfrak{A}$ which intertwines ρ and ρ_1. The space of morphisms between localized transportable endomorphisms ρ and σ, denoted by $(\rho : \sigma)$, is the subspace of intertwiners between ρ and σ which belong to $\mathfrak{A}$.

We denote by $\Delta(\mathscr{O})$ the full subcategory of Δ defined by endomorphisms localized in $\mathscr{O}$. The above mentioned equivalence is then the identity on morphisms, and is given by $\rho \in \Delta \to \tilde{\iota} \circ \rho \in \mathrm{DHR}(\mathfrak{A}^d)$ on objects. What one gains in considering endomorphisms, is that they allow a simple and natural definition of composition of sectors, since it is easy to show that $\rho\sigma \in \Delta$ (composition of endomorphisms) for $\rho, \sigma \in \Delta$, and that the semigroup structure thus defined on Δ, with the vacuum representation $\tilde{\iota}$ as a unit, passes to the quotient $\Delta/\cong$. One can then define a corresponding product $(T_1, T_2) \in (\rho_1 : \sigma_1) \times (\rho_2 : \sigma_2) \to T_1 \times T_2 \in (\rho_1\rho_2 : \sigma_1\sigma_2)$ on morphisms, in such a way as to equip Δ with the structure of a *tensor* C*-category.

The composition law of sectors is just the first example of a structure of physical charges which is encoded in the set of representations of the net $\mathfrak{A}$. Indeed, one can show [DHR71] that, thanks to locality and to the fact that in $d = 4$ spacetime dimensions the spacelike complement of a double cone is connected, exchange symmetry of identical charges is described by unitary representations $\varepsilon_\rho^{(n)}$, $n \in \mathbb{N}$, of the symmetric group of n objects S_n on $\rho^n(\mathfrak{A}^d)'$, whose possible irreducible components are classified, for all integers n, by a single $d_\xi \in \mathbb{N} \cup \{\infty\}$, the *statistical dimension*, and, if $d_\xi < \infty$, by a sign σ_ξ, both depending only on the class $\xi := [\rho]$, the case $1 < d_\xi < \infty$ being a generalization of ordinary Bose ($\sigma_\xi = +$) or Fermi ($\sigma_\xi = -$) statistics, called parastatistics of order d_ξ. Also, it is possible to show [DHR74], that if $d_\xi < \infty$, in which case ρ is said to have *finite statistics*, there exists a conjugate endomorphism $\overline{\rho}$ such that $\rho\overline{\rho}$ and $\overline{\rho}\rho$ both contain the vacuum representation of $\mathfrak{A}^d$, so that $\overline{\xi} := [\overline{\rho}]$ is interpreted as the anti-charge of ξ.

As we will not need directly these structures in the subsequent analysis, we will not go here into details, and we just mention that the resulting structure on the full subcategory Δ_f of Δ, determined by finite direct sums of irreducibles with finite statistics, is that of a tensor symmetric C*-category with subobjects, direct sums and conjugates and with $(\tilde{\iota} : \tilde{\iota}) = \mathbb{C}\mathbb{1}$ [DR89b], and that this is the central issue in the Doplicher-Roberts reconstruction theorem, discussed in the following section. In particular, since we will be concerned with *covariant endomorphisms* ρ, which are objects $\rho \in \Delta_f$ for which there exists a unitary strongly continuous representations U_ρ of $\tilde{\mathscr{P}}_+^\uparrow$ on $\mathscr{H}$, such that

$$U_\rho(s)\rho(A)U_\rho(s)^* = \rho\alpha_{\eta(s)}(A), \qquad A \in \mathfrak{A}^d, s \in \tilde{\mathscr{P}}_+^\uparrow, \tag{1.8}$$

i.e. such that (ρ, U_ρ) is a covariant representation of the C*-dynamical system $(\mathfrak{A}^d, \alpha_{\eta(\cdot)})$, we remark that if Δ_c is the full subcategory of Δ_f defined by covariant endomorphisms, then Δ_c is a category of the same kind as Δ_f, and that moreover all the representations U_ρ satisfy the spectrum condition [DHR74].

1.2.2 Topological sectors

Let $(\mathfrak{A}, \alpha)$ be a translation covariant local net. We say that a covariant representation (π_m, U_m) of $(\mathfrak{A}, \alpha)$, $m > 0$, is a *massive single particle representation* if π_m is a factorial representation of $\mathfrak{A}$, $U_m(x) \in \pi_m(\mathfrak{A})''$ for each $x \in \mathbb{R}^4$, the positive mass hyperboloid $\Omega_m^+ := \{p \in \mathbb{R}^4 \,:\, p^2 = m^2,\, p_0 > 0\}$ is contained in the singular spectrum of U_m, and $\mathrm{Sp}\,U_m \subseteq \Omega_m^+ \cup \{p \in \mathbb{R}^4 \,:\, p^2 \geq M^2,\, p_0 > 0\}$, for some $M > m$, i.e. the set of single particle states is separated from the continuum by a gap in the spectrum.

Theorem 1.12. [BF82, thm. 3.5] *Let $(\mathfrak{A}, \alpha)$ and (π_m, U_m) be as above. Then, there exists an irreducible vacuum representation π of $\mathfrak{A}$ such that, for any spacelike cone $\mathscr{C}$,*

$$\pi_m \restriction \mathfrak{A}(\mathscr{C}') \cong \pi \restriction \mathfrak{A}(\mathscr{C}'). \tag{1.9}$$

Then, even in theories without massless particles, as for instance in non-abelian gauge theories (according to the folklore), we may have superselection sectors not complying with the DHR selection criterion. That such sectors really should arise in this kind of theories is suggested by the fact that the cone can be viewed as a fattened version of the flux string joining opposite gauge charges in non-abelian gauge theories (see, for instance, [KS75]), once that a member of a charge-anticharge pair has been shifted to spacelike infinity to give a charged state.

Then, given a net $\mathfrak{A}$ in vacuum representation ι on $\mathscr{H}$, one is led to consider representations π of $\mathfrak{A}$ satisfying the weaker selection criterion obtained by replacing the double cone $\mathscr{O}$ by a spacelike cone $\mathscr{C}$ in (1.7),

$$\pi \restriction \mathfrak{A}(\mathscr{C}') \cong \iota \restriction \mathfrak{A}(\mathscr{C}'). \tag{1.10}$$

Assuming then also the analogous property of Haag duality with respect to spacelike cones,

$$\mathfrak{A}(\mathscr{C}')' = \mathfrak{A}(\mathscr{C})^-, \tag{1.11}$$

and *property B'*: for any non-zero projection $E \in \mathfrak{A}(\mathscr{C}')'$, and any spacelike cone $\mathscr{C}_1 \supset \overline{\mathscr{C}}$, there is an isometry $W \in \mathfrak{A}(\mathscr{C}_1')'$ with E as final projection – which is also a consequence of weak additivity and the spectrum condition –, it is possible to develop a superselection theory for topological sectors which is similar to the one for localizable sectors. In particular, in this case also it is convenient to shift from representations complying with (1.10) to the set Δ of transportable localized homomorphisms $\rho \in \mathrm{Hom}(\mathfrak{A}, B(\mathscr{H}))$, defined in the evident way. Here, one does not obtain endomorphisms of $\mathfrak{A}$, since by (1.11) a $\rho \in \Delta(\mathscr{C})$ is only such that

$\rho(\mathfrak{A}(\mathscr{C}_1)) \subseteq \mathfrak{A}(\mathscr{C}_1)^-$ for each $\mathscr{C}_1 \supseteq \mathscr{C}$. The composition law of sectors cannot therefore be directly defined in terms of composition of morphisms. Nevertheless it can be established in a natural way, and most of the results discussed for localizable charges hold in this case as well, as the classification of statistics, and the existence of conjugates [BF82, DR90]. In particular, if D is a double cone at spacelike infinity (see appendix A for the definition), and if $\Delta_f(D)$ is the set of all $\rho \in \Delta$ which are localized in a cone $\mathscr{C}$ with base $D(\mathscr{C}) \subseteq D$, then $\Delta_f(D)$ is the set of objects of a symmetric tensor C*-category with direct sums, subobjects and conjugates and with irreducible unit, and this allows the reconstruction of an extended field net in this case too.

1.3 Reconstruction of fields and gauge group

According to the above discussion, the main properties of charges appearing in quantum field theories are directly encoded in the algebraic structure of the observables of the theory, and in particular in the family of their representations. In view of the picture of superselection sectors as coherent subspaces of a "universal" Hilbert space on which unobservable fields act, recalled at the beginning of the present chapter, it is therefore tempting to conjecture that the algebraic structure of charge carrying fields itself, and in particular their commutation relations, with the Bose-Fermi alternative, can be directly read off the net of local observables and its superselection structure. This was indeed established already in [DHR69b] if the superselection sectors are all given by localizable *automorphisms* of $\mathfrak{A}$: in this case it is possible to construct a field net containing $\mathfrak{A}$ as the fixed point subnet under the action of an *abelian* gauge group, whose Pontrjagin dual is in 1-1 correspondence with the sectors. That this should hold as well in the general case is also suggested by the mathematical structure of the set of sectors, the most prominent example of a tensor C*-category as the ones arising in superselection theory being the category $\mathbf{U}(G)$ of finite-dimensional continuous unitary representations of a compact group G: the tensor structure is given by the tensor product of representations and intertwiners, the symmetry by the operator flipping the factors in a tensor product, and the conjugate by the conjugate representation. As was shown in [DR72] this is no accident: if $\mathfrak{F}$ is a normal field net with gauge group G over $\mathfrak{A}$, then the full subcategory of Δ_f determined by the sectors of $\mathfrak{A}$ appearing in $\mathscr{H}_{\mathfrak{F}}$ is equivalent to $\mathbf{U}(G)$, and the equivalence is given by a symmetric tensor functor. One is thus led to the conjecture, related to the one above about the existence of a field net describing superselection structure, that for any symmetric tensor C*-category with direct sums, subobjects, conjugates and irreducible unit, there exist a unique compact group G and a symmetric tensor equivalence from the given category to $\mathbf{U}(G)$. Were this the case, the gauge group of the theory would be uniquely determined by Δ_f.

Both these conjectures were solved affirmatively in [DR90] and [DR89b] respectively, the main technical tool used in these analyses being a crossed product construction of a C*-algebra $\mathfrak{A}$ by a subsemigroup $\Delta \subseteq \mathrm{End}(\mathfrak{A})$, satisfying certain hypoteses which are verified in the applications to quantum field theory [DR89a]. Here we will briefly review the main results on the reconstruction of the field net, which will be needed in the following.

We begin by recalling the notion of Hilbert space inside a unital C*-algebra $\mathfrak{B}$. This is a closed subspace $H \subset \mathfrak{B}$ such that, for each $\psi, \varphi \in H$, $\psi^*\varphi \in \mathbb{C}\mathbb{1}_{\mathfrak{B}}$, and then a scalar product

$(\cdot|\cdot)$ on H is defined by

$$\psi^*\varphi = (\psi|\varphi)\mathbb{1}_{\mathfrak{B}},$$

and the associated norm agrees on H with the norm of $\mathfrak{B}$. We will only consider finite-dimensional Hilbert spaces H. In this case, if ψ_j, $j = 1,\ldots,d$, is an orthonormal basis of H, the projection

$$\mathbb{1}_H := \sum_{j=1}^{d} \psi_j\psi_j^*, \tag{1.12}$$

is independent of the chosen orthonormal basis, and is called the *support* of H.

Theorem 1.13. [DR90] *Let $(\mathscr{H},\mathscr{A},U,\Omega)$ be a local, Poincaré covariant net of von Neumann algebras in its vacuum representation, and assume that $\mathscr{H}$ is separable, and that Haag duality and property B hold for $\mathscr{A}$. There exists then a unique (up to unitary equivalence) normal Poincaré covariant field net $(\pi_{\mathscr{F}},\mathscr{F},V,k,U_{\mathscr{F}})$ over $(\mathscr{H},\mathscr{A},U,\Omega)$ such that $\mathscr{H}$ is cyclic for each $\mathscr{F}(\mathscr{O})$, and any equivalence class of Poincaré covariant finite statistics DHR representations of $\mathscr{A}$ is realized as a subrepresentation of $\pi_{\mathscr{F}}$. Moreover*

(i) $\pi_{\mathscr{F}}(\mathscr{A})' \cap \mathscr{F} = \mathbb{C}\mathbb{1}$;

(ii) $\pi_{\mathscr{F}}(\mathscr{A})' = G''$ *and*

$$\pi_{\mathscr{F}} = \bigoplus_{\xi} d_\xi \pi_\xi, \tag{1.13}$$

where ξ runs over the set of localizable covariant sectors of $\mathscr{A}$ and π_ξ is an irreducible covariant finite statistics DHR representation of class ξ on $\mathscr{H}_\xi \subset \mathscr{H}$;

(iii) there is a 1-1 correspondence between covariant sectors ξ of $\mathscr{A}$ and classes of irreducible subrepresentations of V defined by the fact that, correspondingly to (1.13),

$$V = \bigoplus_{\xi} u_\xi \otimes \mathbb{1}_{\mathscr{H}_\xi}, \tag{1.14}$$

is the factorial decomposition of V, being then u_ξ an irreducible representation of G of dimension d_ξ;

(iv) the grading of $\mathscr{H}$ defined by $V(k)$ corresponds to the alternative between para-Bose and para-Fermi statistics, i.e. if $\Phi \in \mathscr{H}_\xi$, then $V(k)\Phi = \pm\Phi$ according as π_ξ has para-Bose or para-Fermi statistics;

(v) for each $\rho \in \Delta_c(\mathscr{O})$,

$$H_\rho := \{\psi \in \mathscr{F}(\mathscr{O}) : \psi\pi_{\mathscr{F}}(A) = \pi_{\mathscr{F}}\rho(A)\psi,\ A \in \mathscr{A}\} \tag{1.15}$$

is a $d_{[\rho]}$-dimensional β-invariant Hilbert space in $\mathscr{F}(\mathscr{O})$ with support $\mathbb{1}$, and $\mathscr{F}(\mathscr{O})$ is generated as a von Neumann algebra by H_ρ, $\rho \in \Delta_c(\mathscr{O})$;

(vi) for irreducible ρ, the representation u_ρ of G induced by β on H_ρ is equivalent to $u_{[\rho]}$.

It is evident that this is a remarkable result in several respects: in particular, we may note that the starting point is the algebra of local observables, which are gauge invariant by definition, and which contain no element that anticommutes with its spacelike translates, and we end up with a gauge group and with fields satisfying normal commutation and anticommutation relations.

Clearly, we have also the corresponding result for topological sectors, the main difference being that now the field algebras will be indexed by spacelike cones rather than double cones.

Theorem 1.14. [DR90] *Let $(\mathscr{H},\mathscr{A},U,\Omega)$ be a local, Poincaré covariant net of von Neumann algebras in its vacuum representation, and assume that $\mathscr{H}$ is separable, and that* (1.11) *and property B' hold for $\mathscr{A}$. There exists then a unique (up to unitary equivalence) normal Poincaré covariant extended field net $(\pi_{\mathscr{F}},\mathscr{F},V,k,U_{\mathscr{F}})$ over $(\mathscr{H},\mathscr{A},U,\Omega)$ such that any equivalence class of Poincaré covariant finite statistics representations of $\mathscr{A}$ satisfying* (1.10) *is realized as a subrepresentation of $\pi_{\mathscr{F}}$. Moreover*

(i) $\pi_{\mathscr{F}}(\mathscr{A})'\cap\mathscr{F}_D=\mathbb{C}\mathbb{1}$ *where D is a double cone at spacelike infinity, and $\mathscr{F}_D$ is the C^*-algebra generated by $\mathscr{F}(\mathscr{C})$, $D(\mathscr{C})\subseteq D$;*

(ii) $\pi_{\mathscr{F}}(\mathscr{A})'=G''$ *and*

$$\pi_{\mathscr{F}}=\bigoplus_{\xi}d_\xi\pi_\xi, \tag{1.16}$$

where ξ runs over the set of topological covariant sectors of $\mathscr{A}$ and π_ξ is an irreducible covariant finite statistics representation fulfilling (1.10) *of class ξ on $\mathscr{H}_\xi\subset\mathscr{H}$;*

(iii) there is a 1-1 correspondence between covariant topological sectors ξ of $\mathscr{A}$ and classes of irreducible subrepresentations of V defined by the fact that, correspondingly to (1.16),

$$V=\bigoplus_{\xi}u_\xi\otimes\mathbb{1}_{\mathscr{H}_\xi}, \tag{1.17}$$

is the factorial decomposition of V, being then u_ξ an irreducible representation of G of dimension d_ξ;

(iv) the grading of $\mathscr{H}$ defined by $V(k)$ corresponds to the alternative between para-Bose and para-Fermi statistics, i.e. if $\Phi\in\mathscr{H}_\xi$, then $V(k)\Phi=\pm\Phi$ according as π_ξ has para-Bose or para-Fermi statistics;

(v) for each $\rho\in\Delta_c(\mathscr{C})$,

$$H_\rho:=\{\psi\in\mathscr{F}(\mathscr{C}) : \psi\pi_{\mathscr{F}}(A)=\pi_{\mathscr{F}}\rho(A)\psi,\ A\in\mathscr{A}\} \tag{1.18}$$

is a $d_{[\rho]}$-dimensional β-invariant Hilbert space in $\mathscr{F}(\mathscr{C})$ with support $\mathbb{1}$, and $\mathscr{F}(\mathscr{C})$ is generated as a von Neumann algebra by H_ρ, $\rho\in\Delta_c(\mathscr{C})$;

(vi) for irreducible ρ, the representation u_ρ of G induced by β on H_ρ is equivalent to $u_{[\rho]}$.

CHAPTER 2

Scaling algebras and ultraviolet limit

In the conventional, lagrangian and perturbative, approach to quantum field theory, the ultraviolet (i.e. small scale or high energy) properties of a given model are uncovered, in favourable cases, with the help of renormalization group methods, which allow to control the short distance limit of correlation functions of fields. Since fields play such a central role in this kind of analysis, providing a set of operators with a fixed physical interpretation at all scales, it is not straightforward to translate these methods in the algebraic framework, in which the physical information is encoded only in the net structure of the observables. Such a translation has been however performed in [BV95], and is based, as we shall see below, on the observation that what really matters in the conventional framework are only the phase space properties of renormalization group orbits.

In this chapter we will review the work of Buchholz and Verch on this subject, on which our analysis of superselection sectors in the ultraviolet, which will be exposed in chapter 3, is based. In section 2.1 we will see how the analysis of the above mentioned phase space properties of renormalization group orbits leads to the introduction of the concept of *scaling algebras*, as a net of algebras subsuming the action of all possible renormalization group transformations on the given theory. In section 2.2 we will employ the net of scaling algebras to define, in a canonical, model independent fashion, the (ultraviolet) scaling limit of the theory under consideration, which turns out to be again a theory described by a net of local algebras, and we will classify the various possibilities for the structure of such a net. Finally, in section 2.3 we will discuss some simple examples of this construction, which illustrate some of these possibilities.

2.1 Scaling algebras as an algebraic version of the renormalization group

If we denote generically by $\varphi(x)$ an observable Wightman field, as for instance a component of a current or of a field strength, the conventional renormalization group $(R_\lambda)_{\lambda>0}$ is defined essentially by

$$R_\lambda : \varphi(x) \to \varphi_\lambda(x) := Z_\lambda \varphi(\lambda x) \tag{2.1}$$

where the renormalization constants Z_λ are fixed e.g. by requiring that for some fixed test-function f,

$$(\Omega|\varphi_\lambda(f)\varphi_\lambda(f)\Omega) = \text{const} \qquad \text{for } \lambda > 0, \tag{2.2}$$

or by some other condition of this kind, in such a way that the φ_λ correlation functions have the same order of magnitude at all scales. The transformation R_λ can then be considered to map the given theory at scale $\lambda = 1$ – from now on referred to as the *underlying theory* – to the theory at scale λ which is generated by the fields φ_λ, leaving fundamental constants, such as the speed of light c or Planck's $\hbar$, unaffected. In order to calculate the scaling (ultraviolet) limit of the underlying theory one has then to calculate the limit, for $\lambda \to 0$, of the φ_λ correlation functions. The main technical problem that has to be overcome in performing such a calculation, is that, due to the singular behaviour of Wightman fields at neighbouring spacetime points, in general Z_λ will go to zero in this limit, and quite precise information on the way it approaches zero will be needed in order to control the correlation functions' limit. Renormalization group equations provide this information in "good" cases, i.e. essentially only if the theory exhibits a perturbatively small ultraviolet fixed point, while leaving the problem open in all other cases.

Nevertheless, the following general properties of the transformations R_λ follow at once from their definition.

(i) R_λ maps observables localized in the bounded spacetime region $\mathscr{O}$ to observables localized in the scaled region $\lambda\mathscr{O}$,

$$R_\lambda : \mathfrak{A}(\mathscr{O}) \to \mathfrak{A}(\lambda\mathscr{O}), \tag{2.3}$$

and this reflects the fact that the value of c remains constant.

(ii) the fact that also the value of $\hbar$ has to be left fixed by R_λ implies that these transformations map observables which transfer to physical states 4-momentum contained in a compact region[a] $\Delta \subset \mathbb{R}^4$ to observables which transfer 4-momentum in $\lambda^{-1}\Delta$, in symbols

$$R_\lambda : \hat{\mathfrak{A}}(\Delta) \to \hat{\mathfrak{A}}(\lambda^{-1}\Delta) \tag{2.4}$$

$\hat{\mathfrak{A}}(\Delta)$ denoting the subspace of observabes which transfer 4-momentum in Δ.

(iii) When acting on bounded functions of the fields, due to the above definition of Z_λ, the transformations R_λ are bounded in norm, uniformly for $\lambda > 0$.

We will refer to the above properties (i)-(iii) as "phase space" properties of renormalization group transformations, since they state that renormalization group orbits $\lambda \to R_\lambda(A)$, $A \in \mathfrak{A}$, should occupy essentially the same phase space volume at all scales. All possible transformations R_λ complying with these conditions identify the same net $\mathfrak{A}_\lambda(\mathscr{O}) = \mathfrak{A}(\lambda\mathscr{O})$ at the scale λ, and then, if one sticks to the principle that all the physical information on the theory is given by the net structure of observables, they should be regarded as being all physically equivalent, and there is no need to use fields to single out a particular choice of these maps.

[a] See below for precise definitions.

We shall therefore construct the scaling algebra associated to a given underlying theory in such a way that it contains the orbits of local observables under all possible choices of renormalization group transformations as above. Before doing this, we shall state explicitly the assumptions under which such a construction can be performed, and we shall elaborate a little further on properties (i)-(iii).

Throughout the present chapter, $(\mathscr{H}, \mathfrak{A}, U, \Omega)$ will denote a Poincaré covariant local net of C*-algebras in vacuum representation, which satisfies the following

Hypothesis 2.1. For each $A \in \mathfrak{A}(\mathscr{O})$ the function $s \in \mathscr{P}_+^\uparrow \to \alpha_s(A)$ is continuous in the norm topology of $\mathfrak{A}$, and the local algebras $\mathfrak{A}(\mathscr{O})$ are maximal with respect to this property, i.e. if $A \in \mathfrak{A}(\mathscr{O})^-$ is such that $s \to \alpha_s(A)$ is norm continuous, then $A \in \mathfrak{A}(\mathscr{O})$.

This hypotesis is not really restrictive, as given any net $\mathfrak{B}$ and defined $\mathfrak{A}(\mathscr{O})$ as the C*-subalgebra of $\mathfrak{B}(\mathscr{O})^-$ of those $A \in \mathfrak{B}(\mathscr{O})^-$ for which $s \to \alpha_s(A)$ is norm continuous, the net $\mathfrak{A}$ complies with hypothesis 2.1, and for any $\mathscr{O}$ and $\mathscr{O}_1 \supset \overline{\mathscr{O}}$, $\mathfrak{A}(\mathscr{O}_1)^- \supseteq \mathfrak{B}(\mathscr{O})$,[b)] so that the two nets $\mathfrak{A}$ and $\mathfrak{B}$ have the same locally normal states, and they can be considered as physically equivalent.

Our conventions about Fourier transform on Minkowski space are as follows: the Fourier transform of f will be defined as

$$\hat{f}(p) := \int_{\mathbb{R}^4} d^4x \, e^{-ip\cdot x} f(x), \tag{2.5}$$

and correspondingly the inverse transform will be

$$\check{f}(x) := \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} \, e^{ip\cdot x} f(p). \tag{2.6}$$

The following elementary lemma, in which E denotes the spectral measure associated with translations $\alpha_x := \alpha_{(\mathbb{1},x)}$, is at the basis of the definition of the spaces of momentum transfer $\hat{\mathfrak{A}}(\Delta)$ considered above.

Lemma 2.2. *Let $\Phi_1, \Phi_2 \in \mathscr{H}$, and $\Delta_1, \Delta_2 \subset \mathbb{R}^4$ be compact sets. The function $x \in \mathbb{R}^4 \to (E(\Delta_2)\Phi_2 | \alpha_x(A) E(\Delta_1)\Phi_1)$ has (distributional) Fourier transform with support in $\Delta_2 - \Delta_1$.*

Proof. Let μ be the complex measure on $\mathbb{R}^4 \times \mathbb{R}^4$, determined by $\mu(\Gamma_2 \times \Gamma_1) := (E(\Gamma_2)E(\Delta_2)\Phi_2 | AE(\Gamma_1)E(\Delta_1)\Phi_1) = (E(\Gamma_2 \cap \Delta_2)\Phi_2 | AE(\Gamma_1 \cap \Delta_1)\Phi_1)$. Then μ has support in $\Delta_2 \times \Delta_1$. Thus since, for $f \in \mathscr{S}(\mathbb{R}^4)$,

$$\int_{\mathbb{R}^4} d^4x \, f(x)(E(\Delta_2)\Phi_2 | \alpha_x(A) E(\Delta_1)\Phi_1) = \int_{\Delta_2 \times \Delta_1} d\mu(p_2, p_1) \, \hat{f}(p_2 - p_1),$$

we get the statement. □

[b)] For the proof see also below, equation (3.13)

It follows from the above lemma that, if we define, for $A \in \mathfrak{A}$ and $f \in L^1(\mathbb{R}^4)$,

$$\alpha_f(A) := \int_{\mathbb{R}^4} d^4x\, f(x)\alpha_x(A), \tag{2.7}$$

where the integral is defined in weak sense, i.e. by its associated bounded sesquilinear form on $\mathscr{H}$, and if $\Phi_i \in \mathscr{H}$ has compact spectral support[c)] $\Delta_i \subset \mathbb{R}^4$, $i = 1, 2$, then $(\Phi_2|\alpha_f(A)\Phi_1) = 0$ if $(\Delta_1 + \operatorname{supp} \hat{f}) \cap \Delta_2 = \emptyset$, i.e. $\alpha_f(A)$ transfers to states 4-momentum contained in $\operatorname{supp} \hat{f}$. Taking into account that $A = \alpha_g(A)$ for each g with $\hat{g} = 1$ on Δ if and only if $\alpha_f(A) = 0$ whenever $\operatorname{supp} \hat{f} \cap \Delta = \emptyset$, this motivates the following

Definition 2.3. The *4-momentum support* of $A \in \mathfrak{A}$ is the smallest closed set $\Delta \subseteq \mathbb{R}^4$ such that $\alpha_f(A) = 0$ for each $f \in L^1(\mathbb{R}^4)$ with $\operatorname{supp} \hat{f} \subseteq \mathbb{R}^4 \setminus \Delta$. We shall denote by $\hat{\mathfrak{A}}(\Delta)$ the subspace of $\mathfrak{A}$ of the elements with 4-momentum support contained in Δ.

Let then $A \in \mathfrak{A}(\mathscr{O})$ and $(R_\lambda)_{\lambda>0}$ be a family of renormalization group transformations. Then by (i) above, $R_\lambda(A) \in \mathfrak{A}(\lambda\mathscr{O})$. Moreover, it can be shown [BV95, lemma 2.2] that, thanks to hypotesis 2.1, A can be approximated in norm by operators in $\hat{\mathfrak{A}}(\Delta)$ for Δ sufficiently big, so that, by conditions (ii) and (iii), we deduce that for each $\varepsilon > 0$ there exists a compact Δ such that

$$R_\lambda(A) \in \hat{\mathfrak{A}}(\lambda^{-1}\Delta) + \varepsilon B_{\mathfrak{A}}, \qquad \lambda > 0, \tag{2.8}$$

having denoted by $B_{\mathfrak{A}}$ the unit ball in $\mathfrak{A}$.

This last condition can be reformulated as a more manageable condition of continuity with respect to translations, uniform in λ, as in [BV95, lemma 3.1] it is shown that (2.8), together with boundedness in norm of $\lambda \to R_\lambda(A)$, which follows again from (iii), is equivalent to

$$\lim_{x\to 0} \sup_{\lambda>0} \left\| \alpha_{\lambda x}\big(R_\lambda(A)\big) - R_\lambda(A) \right\| = 0. \tag{2.9}$$

Similarly, since angular momentum has the dimensions of an action, and since Planck's constant is not rescaled by renormalization group transformations, it can be shown that the orbits $\lambda \to R_\lambda(A)$ have also to satisfy

$$\lim_{\Lambda\to \mathbb{1}} \sup_{\lambda>0} \left\| \alpha_{\Lambda}\big(R_\lambda(A)\big) - R_\lambda(A) \right\| = 0, \tag{2.10}$$

where $\alpha_\Lambda := \alpha_{(\Lambda,0)}$.

These remarks suggest the following definition of scaling algebra. We consider the C*-algebra $B(\mathbb{R}_+^\times, \mathfrak{A})$ of bounded $\mathfrak{A}$-valued functions on the positive reals, where algebraic operations are defined pointwise,

$$\begin{aligned} (a\underline{A} + b\underline{B})(\lambda) &:= a\underline{A}(\lambda) + b\underline{B}(\lambda), \\ (\underline{A}\underline{B})(\lambda) &:= \underline{A}(\lambda)\underline{B}(\lambda), \\ (\underline{A}^*)(\lambda) &:= \underline{A}(\lambda)^*, \end{aligned} \qquad \begin{aligned} &\underline{A}, \underline{B} \in B(\mathbb{R}_+^\times, \mathfrak{A}), \\ &a, b \in \mathbb{C}, \end{aligned} \tag{2.11}$$

and with C*-norm

$$\|\underline{A}\| := \sup_{\lambda>0} \|\underline{A}(\lambda)\|. \tag{2.12}$$

[c)] The spectral support of a $\Phi \in \mathscr{H}$ is the support of the vector valued Borel measure $\Delta \to E(\Delta)\Phi$.

We can lift the action of the Poincaré group to this algebra by defining

$$\underline{\alpha}_s(\underline{A})(\lambda) := \alpha_{s_\lambda}(\underline{A}(\lambda)), \qquad \lambda > 0,\ s \in \mathscr{P}_+^\uparrow \tag{2.13}$$

where $(\Lambda, x)_\lambda := (\Lambda, \lambda x)$, which is an endomorphism of $\mathscr{P}_+^\uparrow$, so that we obtain again an action of $\mathscr{P}_+^\uparrow$ by automorphisms of $B(\mathbb{R}_+^\times, \mathfrak{A})$.

Definition 2.4. Let $\mathscr{O} \subset \mathbb{R}^4$ be open and bounded. The *local scaling algebra* associated to $\mathscr{O}$ is the C*-subalgebra $\underline{\mathfrak{A}}(\mathscr{O})$ of $B(\mathbb{R}_+^\times, \mathfrak{A})$ of those $\underline{A}$ such that $\underline{A}(\lambda) \in \mathfrak{A}(\lambda\mathscr{O})$ for each $\lambda > 0$, and

$$\lim_{s \to e} \|\underline{\alpha}_s(\underline{A}) - \underline{A}\| = 0. \tag{2.14}$$

The *(quasi-local) scaling algebra* $\underline{\mathfrak{A}}$ is the C*-inductive limit of the algebras $\underline{\mathfrak{A}}(\mathscr{O})$.

It is clear that $(\underline{\mathfrak{A}}, \underline{\alpha})$ is a local Poincaré covariant net of C*-algebras, which moreover is non trivial, i.e. not reduced to scalar functions $\underline{A}(\lambda) = c_\lambda \mathbb{1}$, since functions $\underline{A}$ with the desired properties can be constructed quite easily using results in [BV95, sec. 2], where it is shown that, thanks to hypotesis 2.1, for each fixed pair of regions $\mathscr{O}_0, \Delta_0$ there exists a "large" set $\mathfrak{S}_\lambda \subset \mathfrak{A}(\lambda\mathscr{O}_0)$ such that each $A_\lambda \in \mathfrak{S}_\lambda$ has $\|A_\lambda\| = 1$, and for each $\varepsilon > 0$, $A_\lambda \in \hat{\mathfrak{A}}(\lambda^{-1}\Delta) + \varepsilon B_{\mathfrak{A}}$ with $\Delta = \varepsilon^{-1}\Delta_0$. Then if $\underline{A}(\lambda) := \int_{\mathscr{N}} d\Lambda\, \alpha_\Lambda(A_\lambda)$, where $\mathscr{N} \subseteq SO^\uparrow(1,3)$ is a neighbourhood of the identity, $\underline{A} \in \underline{\mathfrak{A}}(\mathscr{O})$ for $\mathscr{O} \supseteq \mathscr{N}\mathscr{O}_0$.

Renormalization group transformations are then implemented on the scaling algebra as geometrical symmetries, given by an automorphic action $\underline{\sigma}$ of the multiplicative group $\mathbb{R}_+^\times$ of positive reals on $\underline{\mathfrak{A}}$, defined by

$$\underline{\sigma}_\mu(\underline{A})(\lambda) := \underline{A}(\mu\lambda), \qquad \lambda, \mu > 0. \tag{2.15}$$

The geometrical character of this automorphisms group is seen by noting that

$$\underline{\sigma}_\mu(\underline{\mathfrak{A}}(\mathscr{O})) = \underline{\mathfrak{A}}(\mu\mathscr{O}), \tag{2.16}$$

$$\underline{\sigma}_\mu \circ \underline{\alpha}_s = \underline{\alpha}_{s_\mu} \circ \underline{\sigma}_\mu, \qquad s \in \mathscr{P}_+^\uparrow, \tag{2.17}$$

so that $\underline{\sigma}$ defines an action of the dilatations on the scaling algebra.

2.2 Construction of the scaling limit

Let φ be a locally normal state of $\mathfrak{A}$. We associate to it the family $(\underline{\varphi}_\lambda)_{\lambda>0}$ of states over the scaling algebra $\underline{\mathfrak{A}}$, defined by

$$\underline{\varphi}_\lambda(\underline{A}) := \varphi(\underline{A}(\lambda)), \qquad \underline{A} \in \underline{\mathfrak{A}},\ \lambda > 0. \tag{2.18}$$

Considering $(\underline{\varphi}_\lambda)_{\lambda>0}$ as a net for $\lambda \to 0$, the set of its weak* limit points will be non-empty, thanks to the Banach-Bourbaki-Alaoglu theorem.

Definition 2.5. Every weak* limit point of the net $(\underline{\varphi}_\lambda)_{\lambda>0}$ will be called a *scaling limit state* of the state φ. The set of all scaling limit states of φ will be denoted by $SL_{\mathfrak{A}}(\varphi)$.

Actually, $SL_{\mathfrak{A}}(\varphi)$ is independent of the locally normal state φ.

Proposition 2.6. *Let φ_1, φ_2 be locally normal states of $\mathfrak{A}$. Then, for each open bounded $\mathscr{O}$,*

$$\lim_{\lambda\to 0} \left\| (\underline{\varphi}_{1,\lambda} - \underline{\varphi}_{2,\lambda}) \upharpoonright \underline{\mathfrak{A}}(\mathscr{O}) \right\| = 0, \tag{2.19}$$

thus, in particular, $SL_{\mathfrak{A}}(\varphi_1) = SL_{\mathfrak{A}}(\varphi_2)$.

For the proof, we refer to [Rob74, BV95] (see also proposition 3.6, where the slightly more general case of normal, instead of local, commutation relations is considered). We will then talk simply of the *scaling limit states of* $\underline{\mathfrak{A}}$, without reference to any particular locally normal state on $\mathfrak{A}$, and we will denote this set as $SL_{\mathfrak{A}}$.

Let $\underline{\omega}_0 \in SL_{\mathfrak{A}}$ and denote by $(\pi_0, \mathscr{H}_0, \Omega_0)$ the corresponding GNS representation. On $\mathscr{H}_0$ we can then consider the local net of C*-algebras defined by

$$\mathscr{O} \to \mathfrak{A}_0(\mathscr{O}) := \pi_0(\underline{\mathfrak{A}}(\mathscr{O})), \tag{2.20}$$

with the state $\omega_0 := (\Omega_0|(\cdot)\Omega_0)$, and, since $\underline{\omega}_0$ can be obtained as a weak* limit of a net $(\omega_{\lambda_\iota})_{\iota\in I}$, ω being the vacuum state on $\mathfrak{A}$, it is $\underline{\alpha}$-invariant, and we get a corresponding unitary representation U_0 of $\mathscr{P}_+^\uparrow$ on $\mathscr{H}_0$. We set $\alpha^0_s := \mathrm{Ad}\, U_0(s)$, $s \in \mathscr{P}_+^\uparrow$.

Theorem 2.7. *With the above notations, $(\mathscr{H}_0, \mathfrak{A}_0, U_0, \Omega_0)$ is a Poincaré covariant, local net of C*-algebras in vacuum representation.*

Again, for the proof see [BV95, prop. 4.4] or theorem 3.7 below, where this is generalized to the scaling limit of a field net.

Remark. The above result holds for a number of spacetime dimensions $d \geq 3$. For $d = 2$ one still gets that $(\mathfrak{A}_0, \alpha^0)$ is a Poincaré covariant local net, and ω_0 is a vacuum state on it, but ω_0 needs not be pure in general.

Definition 2.8. Every net $(\mathfrak{A}_0, \alpha^0)$ obtained as above from a scaling limit state $\underline{\omega}_0 \in SL_{\mathfrak{A}}$ will be called a *scaling limit net* of the *underlying net* $(\mathfrak{A}, \alpha)$.

The fact that in general we get several in principle different scaling limit theories, one for each choice of a scaling limit state on the scaling algebra, can be traced back to the already mentioned fact that $\underline{\mathfrak{A}}$ is constructed from the orbits of all possible choices of renormalization group transformations complying with the very general phase space properties discussed in the previous section, so that, loosely speaking, any scaling limit net can be attributed to a particular such choice. But since the particular renormalization group transformation chosen should not matter for the physical interpretation of the theory at small scales, we can expect that, in generic cases, all the scaling limit nets should describe the same physics. The following definition formalizes this favourable scenario.

Definition 2.9. The underlying theory $(\mathfrak{A}, \alpha)$ is said to have a *unique scaling limit* if all the scaling limit nets $(\mathfrak{A}_0, \alpha^0)$ are isomorphic. If, moreover, there exist net isomorphisms that connect the respective vacuum states, the theory is said to have a *unique vacuum structure* in the scaling limit.

There is the possibility that the various isomorphic nets $\mathfrak{A}_0$ are all trivial, i.e. reduced to the multiples of the identity. In this case we speak of a *classical* scaling limit, due to the fact that correlations between observables vanish in every state on $\mathfrak{A}_0$. This situation may arise if observables in $\mathfrak{A}$ exhibit an exceptional quantum behaviour at small scales, the 4-momentum transfer of observables localized in $\mathfrak{A}(\lambda\mathscr{O})$ being of the order of λ^{-q} for some $q > 1$, so that do not exist renormalization group orbits occupying a finite volume of phase space at all scales, apart from multiples of the identity.[d)] We may expect a situation of this kind to be realized in non-renormalizable theories. The only alternative to this scenario is the much more interesting case in which the isomorphic nets $\mathfrak{A}_0$ are all infinite dimensional and non-abelian [Buc96a], and we say in this case that the unique scaling limit is a *quantum* one. This situation should correspond to theories that in the conventional setting have an ultraviolet fixed point of the renormalization group. We will see some simple examples of this case in the following section.

Of course, it could happen that not all the scaling limit nets are isomoprhic, since the structure of the theory at small scales is continually varying as $\lambda \to 0$, and we speak then of a *degenerate* scaling limit.

As can be expected, in the physically relevant case of unique (quantum) scaling limit, the dilatations are geometrical symmetries of the scaling limit theories.

Definition 2.10. A local Poincaré covariant net $(\mathfrak{A}, \alpha)$ is *dilatation covariant* if for each $\mu > 0$ there exists $\delta_\mu \in \mathrm{Aut}\,\mathfrak{A}$ such that

$$\delta_\mu(\mathfrak{A}(\mathscr{O})) = \mathfrak{A}(\mu\mathscr{O}), \qquad \mu > 0,\ \mathscr{O} \subset \mathbb{R}^4, \tag{2.21}$$

$$\delta_\mu \circ \alpha_s = \alpha_{s_\mu} \circ \delta_\mu, \qquad \mu > 0,\ s \in \mathscr{P}_+^\uparrow. \tag{2.22}$$

A vacuum state ω on $(\mathfrak{A}, \alpha)$ is *dilatation invariant* if $\omega \circ \delta_\mu = \omega$, $\mu > 0$.

Proposition 2.11. [BV95, prop. 4.4] *If the underlying theory has unique scaling limit, then each scaling limit theory is dilatation covariant. If, moreover, the underlying theory has a unique vacuum structure in the scaling limit, then the scaling limit vacuum states are dilatation invariant.*

The automorphisms $\delta^0_\mu \in \mathrm{Aut}\,\mathfrak{A}_0$ implementing dilatations in the scaling limit are induced by the scaling transformations $\underline{\sigma}_\mu$ through the fact that if $\underline{\omega}_0 \in SL_{\mathfrak{A}}$, then also $\underline{\omega}_0 \circ \underline{\sigma}_\mu \in SL_{\mathfrak{A}}$, and it induces a scaling limit net isomorphic to the one induced by $\underline{\omega}_0$.

We close this section by discussing results which allow us to analyse the superselection structure in the scaling limit.

We denote by $\Lambda_{\mathscr{W}_{1,+}}(t) \in SO^\uparrow(1,3)$, $t \in \mathbb{R}$, the one-parameter group of Lorentz boosts in the x^1 direction, with speed $\beta = \tanh t$, explicitly

$$\Lambda_{\mathscr{W}_{1,+}}(t) = \begin{pmatrix} \cosh t & \sinh t & \multicolumn{2}{c}{0} \\ \sinh t & \cosh t & & \\ & & 1 & \\ \multicolumn{2}{c}{0} & & 1 \end{pmatrix}. \tag{2.23}$$

These transformations leave the wedge $\mathscr{W}_{1,+}$ (and hence also $\mathscr{W}_{1,-} = \overline{\mathscr{W}}'_{1,+}$) invariant, since if $e_\pm := e_1 \pm e_0$, e_μ, $\mu = 0, \dots, 3$, being the canonical basis in $\mathbb{R}^4$, then $x \in \mathscr{W}_{1,+}$

d) Such a connection between phase space properties of the underlying theory and the structure of its scaling limit is further clarified in [Buc96a].

if and only if $x \cdot e_\pm < 0$, and $\Lambda_{\mathscr{W}_{1,+}}(t)e_\pm = e^{\pm t}e_\pm$. For any other wedge $\mathscr{W} = s \cdot \mathscr{W}_{1,+}$, $s \in \mathscr{P}_+^\uparrow$, the corresponding one-parameter group in $\mathscr{P}_+^\uparrow$ leaving $\mathscr{W}$ invariant is given by $\Lambda_{\mathscr{W}}(t) := s\Lambda_{\mathscr{W}_{1,+}}(t)s^{-1}$, $t \in \mathbb{R}$. Also, we consider the non-orthocronous Lorentz transformation $j := \mathrm{diag}(-1,-1,1,1)$, which is such that $j^2 = \mathbb{1}$ and $j\mathscr{W}_{1,+} = \mathscr{W}_{1,-}$. In generic cases, these Lorentz transformations represent the geometric action of the modular objects associated through Tomita-Takesaki theory (cfr. [BR79a, KR86]) to the algebra of $\mathscr{W}_{1,+}$, with the vacuum as cyclic and separating vector.

Definition 2.12. Let $(\mathscr{H},\mathfrak{A},U,\Omega)$ be a Poincaré covariant local net in vacuum representation, and let (Δ,J) be the modular objects associated to $(\mathfrak{A}(\mathscr{W}_{1,+})^-,\Omega)$. We say that $(\mathscr{H},\mathfrak{A},U,\Omega)$ satisfies the *condition of geometric modular action* if

$$J\mathfrak{A}(\mathscr{O})^-J = \mathfrak{A}(j\mathscr{O})^- \qquad \mathscr{O} \subset \mathbb{R}^4, \tag{2.24}$$

$$JU(\Lambda,x)J = U(j\Lambda j, jx), \qquad (\Lambda,x) \in \mathscr{P}_+^\uparrow, \tag{2.25}$$

$$\Delta^{it} = U(\Lambda_{\mathscr{W}_{1,+}}(2\pi t)), \qquad t \in \mathbb{R}. \tag{2.26}$$

This condition holds for nets generated by underlying Wightman fields [BW76], and it has been proven at a purely algebraic level in two spacetime dimensions [Bor92], and in physical dimension under fairly general assumptions [BGL93, Mun01], so that it can be considered as holding in generic cases. An immediate consequence is *wedge duality*:

$$\mathfrak{A}(\mathscr{W})' = \mathfrak{A}(\overline{\mathscr{W}}')^-, \tag{2.27}$$

which follows from (2.24) through $\mathfrak{A}(\mathscr{W}_{1,-})^- = \mathfrak{A}(j\mathscr{W}_{1,+})^- = J\mathfrak{A}(\mathscr{W}_{1,+})^-J = \mathfrak{A}(\mathscr{W}_{1,+})'$ and Poincaré covariance, and which implies essential duality, thanks to the fact that, as is easy to see, $\mathfrak{A}^d(\mathscr{O}) = \bigcap_{\mathscr{W}\supset\mathscr{O}}\mathfrak{A}(\mathscr{W})^-$ for each double cone $\mathscr{O}$. As this last property is, as we have seen, a basic hypotesis needed to apply superselection theory to a given net, the following result is quite welcome.

Proposition 2.13. [BV95, prop. 6.3] *If the underlying theory complies with the condition of geometric modular action, then any scaling limit theory does the same.*

2.3 Examples of scaling limit calculation

As can be expected from the geometrical significance of the renormalization group, a class of theories for which the scaling limit theory should be identified rather easily is that of dilatation covariant theories. This is indeed true, at least for theories which satisfy the following mild phase space condition.

Definition 2.14. [HS65] Let $(\mathscr{H},\mathfrak{A},U,\Omega)$ be a translation covariant local net, and define, for each $\beta > 0$ and $\mathscr{O} \subset \mathbb{R}^4$, the operator $\Theta_{\beta,\mathscr{O}} : \mathfrak{A}(\mathscr{O}) \to \mathscr{H}$ by

$$\Theta_{\beta,\mathscr{O}}(A) := e^{-\beta H}A\Omega, \qquad A \in \mathfrak{A}(\mathscr{O}), \tag{2.28}$$

where H is the generator of time translations. Then the theory is said to satisfy the *Haag-Swieca compactness condition* if all the operators $\Theta_{\beta,\mathscr{O}}$ are compact.

This condition was initially proposed in [HS65] (to which we refer for its motivations) in order to characterize theories with a complete asymptotic particle interpretation. It has been verified in several models, such as the free massive [HS65] and massless [BJ87] scalar field, or in locally Fock interacting theories in two dimensions, as the $P(\varphi)_2$ [Dri79] and Y_2 [Sum82] models.

Proposition 2.15. [BV95, prop. 5.1] *If the underlying net* $(\mathscr{H}, \mathfrak{A}, U, \Omega)$ *is dilatation covariant, satisfies the Haag-Swieca compactness condition, and the vacuum* ω *is dilatation invariant, then the theory has unique scaling limit and vacuum structure in this limit. In particular, each scaling limit net* $(\mathfrak{A}_0, \alpha^0)$ *is isomorphic to* $(\mathfrak{A}, \alpha)$, *and the isomorphism connects the respective vacuum states* ω_0 *and* ω.

In [Buc96a] it is also shown that if the underlying theory complies with a strengthened version of the compacteness condition, then all scaling limit theories satisfy the compactness condition, so that, in view also of proposition 2.11, the scaling limit of such theories, if it's unique, can be considered as a fixed point of the renormalization group.

Another example in which the scaling limit theory turns out to be unique, and can be explicitly calculated, is provided by the free scalar field in $d = 3, 4$ spacetime dimensions.

Theorem 2.16. [BV98, thm. 3.1] *The theory of the free scalar field of mass* $m \geq 0$ *in* $d = 3, 4$ *dimensional Minkowski space has unique quantum scaling limit and unique vacuum structure in this limit, as each of its scaling limit theories is isomorphic with the theory of the free scalar field of mass* $m = 0$ *in the same spacetime dimension, and the isomorphisms connect the respective vacuum states.*

This example, though rather simple, illustrates several aspects of the algebraic approach to renormalization group reviewed here. In particular, it shows that, in contrast to the conventional approach, it is not necessary to single out a particular choice of renormalization group transformations, with a corresponding gain in flexibility and generality, but still enabling one to perform explicit calculations. Also, it shows that the apparent ambiguities inherent to the existence of a multiplicity of scaling limit theories disappear after identifying isomorphic nets.

It is however in the example of the massive free scalar field in $d = 2$ spacetime dimensions, where the application of the conventional methods is complicated by infrared divergences, that the use of local algebras appears to be more effective. Though the explicit calculation of the scaling limit theories remains an open problem in this case, it can be shown that each of them contain a central extension of the net generated by the massless free field in Weyl form as a subnet, and this is sufficient to show that, in contrast to the situation at finite scales, the scaling limit theories describe charged states.

Theorem 2.17. [BV98, thm. 4.1] *Let* ω_0 *be any scaling limit state of the theory of the free scalar field of mass* $m > 0$ *in* $d = 2$ *dimensional Minkowski space, and let* $(\mathfrak{A}_0, \alpha^0)$ *the corresponding scaling limit theory. Then* $\mathfrak{A}_0$ *has a non-trivial centre, and there exist states* ω_q, $q \in \mathbb{R}$, *on* $\mathfrak{A}_0$, *locally normal to* ω_0, *such that*

(i) for each sufficiently large double cone $\mathscr{O}$, $\omega_q \restriction \mathfrak{A}_0(\mathscr{O}^\pm) = \omega_0 \restriction \mathfrak{A}_0(\mathscr{O}^\pm)$, *where* $\mathscr{O}^{+(-)}$ *is the right (left) component of* $\mathscr{O}'$;

(ii) for all double cones $\mathscr{O}$, $\omega_q \restriction \mathfrak{A}_0(\mathscr{O}')$ *is disjoint from* $\omega_0 \restriction \mathfrak{A}_0(\mathscr{O}')$ *if* $q \neq 0$;

(iii) in the GNS-representation induced by ω_q *the translations* $x \in \mathbb{R}^2 \to \alpha^0_x$ *are implemented by a unitary strongly continuous representation satisfying the spectrum condition.*

Then the states ω_q describe topological charges, and, in view of the fact that the massive scalar field has no non-trivial superselection sector, this illustrates the fact that the superselection structures of the underlying theory and of its scaling limit are in general different. Taking into account the fact that the net generated by the free massive scalar field in two dimensions coincides with the net of observables defined by the Schwinger model, as recalled in the introduction, the existence of such charged states in the scaling limit can be interpreted as the appearence of "confined" charges intrinsically described by the Schwinger model, in agreement with the folklore, and this feature is established without the need to attach a physical interpretation to unobservable fields. The dynamical justification usually given of this phenomenon remains however questionable, as it can be interpreted as a purely quantum feature of a free theory [Buc96b, sec. 4]. Also, if we denote by G and G_0 the canonical gauge groups determined respectively by the underlying theory and by a fixed scaling limit theory according to the Doplicher-Roberts reconstruction theorem, we have in this case $G \subset G_0$ (G is trivial in this example). This is the situation expected in asymptotically free theories, where symmetries possibly concealed by interactions at finite scales show up in the scaling limit. The opposite situation, $G_0 \subset G$, is also possible. An example of theory with classical scaling limit has been constructed [Lut97]: it is a theory which satisfies standard conditions, such as essential duality and compactness, but which contains only operators with a very singular short distance behaviour, as the ones expected to appear in non-renormalizable theories (provided they can be defined at all). For theories with a classical scaling limit G_0 is clearly trivial, thus the charges they describe disappear in the scaling limit. In the general case both phenomena – underlying charges disappearing and new ones appearing – may happen, so that G and G_0 have at best some subgroup in common.

CHAPTER 3

Ultraviolet stability and scaling limit of charges

The intuitive physical picture of a confined charge is that of a charge which appears only in the limit of small spatio-temporal scales, but which cannot be created by physical operations performed at finite scales. In order to obtain from this vague statement an intrinsic understanding of the confinement phenomenon, in the framework we have discussed up to now, the following idea presents almost by itself to mind: *confined charges are described by superselection sectors of the scaling limit theory, which do not appear as sectors of the underlying theory*. Since, as we have seen, both the scaling limit construction and the superselection structure are canonically determined by the assignement of the (underlying) net of algebras of local observables, such a notion of confinement does not present the drawbacks of the conventional one, recalled in the introduction, and then it is possible to regard at confined charges as theoretical objects intrinsically described by the theory, and not as artifacts of the particular way chosen to represent it. When applied to the already considered example of the Schwinger model (sect. 2.3), the above confinement criterion yields the existence of (at least) a continuum of confined charges, carried by the states ω_q: these charges appear in the scaling limit theory, but they cannot be described by the underlying theory, which has no non-trivial superselection sectors (it is the theory of the massive free scalar field). However it is apparent that, in order for the criterion to be applicable to more complicated models, in which both the underlying theory and the scaling limit one have non-trivial superselection structures, a way to compare the two superselection structures is needed. In particular, it is sufficient to have a natural notion of charges of the underlying theory "surviving" the scaling limit, giving an identification of (a subset of) sectors of the underlying theory with a subset of the ones in the scaling limit, regarded as scaling limits of the formers. Then these sectors can be interpreted, in a natural way, as described by the theory both at finite scales and in the scaling limit, and the scaling limit sectors which do not arise in this way from those of the underlying theory will be the confined ones.

In this chapter we will discuss such a notion of scaling limit of charges, which we call *ultraviolet stability*, after a suggestion of Detlev Buchholz. Recalling the remarks about the phase space properties of renormalization group orbits made in sect. 2.1, and the ensuing definition of scaling algebras, as well as the comments in section 2.3 about the possible disappearing of charges in the scaling limit, for instance in the case of classical scaling limit, it is to be expected that not all underlying charges will survive the limit, and that the "good" charges will be singled out by some phase space property of the fields carrying them: if the fields carrying a charge ξ require too much energy to be localized in smaller and smaller regions, then ξ will not appear in the scaling limit. It is also to be remarked that, to obtain a physically meaningful criterion, it is necessary in this setting to consider the scaling limit of both double cone and cone-like localizable charges, since, as already recalled, in non-abelian gauge theories, which are the candidate ones to exhibit confinement, the latter charges are naturally expected to appear, cfr. the introduction of [BF82] and references quoted there. This adds some conceptual (as well as technical) difficulties, since it is not a priori clear in what sense cone-like fields can be localized in vanishingly small regions, as space-like cones are unaffected by scaling trasformations. For this reason, we shall consider first the simpler case of double cone localizable charges. In the first section we shall generalize the scaling limit construction to the case of a normal field net with compact gauge group describing the superselection structure of the underlying theory. Then, in section 3.2, we shall show that double cone localizable charges satisfying an energetic condition of the above stated kind actually survive the scaling limit, in the sense that there is a Hilbert space of isometries in any field net scaling limit, carrying the associated gauge group representation and inducing a localizable sector on the corresponding scaling limit theory. In section 3.3 finally, we shall extend the discussion to quantum topological charges. In this case the results obtained up to now are not yet of a completely general character, as the above mentioned identification of topological charges surviving the scaling limit can be achieved only under some technical assumptions on the structure of the scaling limit, whose status we plan to clarify in our future work. However, we feel it worthwile to present here our investigations on this subject, as they seem promising.

3.1 Scaling limit for field nets

To begin with, in this section we extend the scaling limit construction of chapter 2 to the case of a field net with normal commutation relations on which a compact gauge group acts. Throughout this section, unless otherwise stated, $(\mathscr{H}, \mathfrak{A}, U, \Omega)$ will denote a Poincaré covariant local net of C*-algebras in vacuum representation, and $(\pi_{\mathfrak{F}}, \mathfrak{F}, V, k, U_{\mathfrak{F}})$ a corresponding Poincaré covariant normal field net, acting on a universal Hilbert space $\mathscr{H}_{\mathfrak{F}}$ (definitions 1.4 and 1.8). As usual, we shall indicate by α and $\alpha^{\mathfrak{F}}$ the action of the (universal covering of the) Poincaré group induced by U and $U_{\mathfrak{F}}$ on the nets $\mathfrak{A}$ and $\mathfrak{F}$ respectively, by β the action of the gauge group G induced by V on $\mathfrak{F}$, and by $\gamma := \beta_k$ the automorphism inducing the Bose-Fermi grading of $\mathfrak{F}$. In order to perform the scaling limit construction, in analogy with the case of observable nets, we shall add the following hypotesis on the field net.

Hypothesis 3.1. (i) For any $F \in \mathfrak{F}(\mathscr{O})$, the functions $s \in \tilde{\mathscr{P}}_+^\uparrow \to \alpha_s^{\mathfrak{F}}(F)$, $g \in G \to \beta_g(F)$ are continuous in the norm topology of $\mathfrak{F}$, and $\mathfrak{F}(\mathscr{O})$ is maximal with respect to this property, i.e. any $F \in \mathfrak{F}(\mathscr{O})^-$ for which these functions are norm continuous is already contained in $\mathfrak{F}(\mathscr{O})$.

(ii) For each $\lambda > 0$ there exists a unitary, strongly continuous representation $V^{(\lambda)}$ of the gauge group G on $\mathscr{H}_{\mathfrak{F}}$ leaving the vacuum invariant, still inducing a representation $\beta^{(\lambda)}$ of G by automorphisms of the underlying net $\mathfrak{F}$ which commute with Poincaré transformations, and such that $\gamma^{(\lambda)} := \beta_k^{(\lambda)}$ defines the $\mathbb{Z}_2$-grading of $\mathfrak{F}$, and $\pi_{\mathfrak{F}}(\mathfrak{A}(\mathscr{O})) = \mathfrak{F}(\mathscr{O})^{\beta^{(\lambda)}}$ for every $\mathscr{O}$ and every $\lambda > 0$.

The first of the above hypoteses is a natural extension to the field net of the analogous assumption of continuity of the observable net with respect to Poincaré transformations, hypotesis 2.1 (and in fact implies it, as we will see shortly, proposition 3.2), supplemented by a similar requirement concerning the action of the gauge group. The second hypotesis is not restrictive at all: a simple example of a family of representations $(V^{(\lambda)})_{\lambda>0}$ complying with such assumption is given by $V^{(\lambda)}(g) := V(g_\lambda)$ for some family of continuous homomorphisms $g \in G \to g_\lambda \in G$ (this is also essentially the unique possibility [DR90, thm 3.6]). In particular, we could take $g_\lambda = g$ for each $\lambda > 0$, so any field net with gauge symmetry complies with this assumption. However, we add this statement here because we consider the family of representations $(V^{(\lambda)})_{\lambda>0}$ as part of the data determining the field net scaling limit construction, the choice of different families giving rise to a priori different field net scaling limits. The reason for this being that in general charges may carry a dimension, as for instance the electric charge in the Schwinger model, which has dimensions of a mass, and then exhibit a non trivial running under renormalization group. So, allowing for a λ dependence of the gauge group representation, and restricting accordingly in an appropriate way the set of renormalization group orbits under consideraton, one may expect that the resulting scaling limit describes those charges whose scaling behaviour is given by the λ dependence of the spectra of $V^{(\lambda)}$.

Proposition 3.2. *The observable net $(\mathfrak{A}, U, \Omega)$ satisfies hypotesis 2.1.*

Before proving the proposition, we state and prove a preliminary result.

Lemma 3.3. *The representation $\pi_{\mathfrak{F}} \restriction \mathfrak{A}(\mathscr{O})$ extends to a strongly continuous isomorphism $\overline{\pi}_{\mathfrak{F}} : \mathfrak{A}(\mathscr{O})^- \to \pi_{\mathfrak{F}}(\mathfrak{A}(\mathscr{O}))^-$.*

Proof. Identifying $\mathscr{H}$ with a subspace of $\mathscr{H}_{\mathfrak{F}}$ as customary, define $\overline{\pi}_{\mathfrak{F}}$ by

$$\overline{\pi}_{\mathfrak{F}}(A)F\Omega := FA\Omega, \qquad A \in \mathfrak{A}(\mathscr{O})^-, F \in \mathfrak{F}(\mathscr{O}').$$

This defines $\overline{\pi}_{\mathfrak{F}}(A)$ on the dense set $\mathfrak{F}(\mathscr{O}')\Omega$. We show that $\overline{\pi}_{\mathfrak{F}}(A)$ is bounded and belongs to $\pi_{\mathfrak{F}}(\mathfrak{A}(\mathscr{O}))^-$: by Kaplansky's theorem, there is a net $(A_\iota)_{\iota\in I} \subseteq \mathfrak{A}(\mathscr{O})$, such that $\|\pi_{\mathfrak{F}}(A_\iota)\| = \|A_\iota\| \le \|A\|$, and $s\text{–}\lim_{\iota\in I} A_\iota = A$; then, using normal commutation relations,

$$\overline{\pi}_{\mathfrak{F}}(A)F\Omega = FA\Omega = \lim_{\iota\in I} FA_\iota\Omega = \lim_{\iota\in I} \pi_{\mathfrak{F}}(A_\iota)F\Omega,$$

so that, being $(\pi_{\mathfrak{F}}(A_\iota))_{\iota\in I}$ boundend in norm, $\overline{\pi}_{\mathfrak{F}}(A) = s\text{–}\lim_{\iota\in I} \pi_{\mathfrak{F}}(A_\iota)$ belongs to $\pi_{\mathfrak{F}}(\mathfrak{A}(\mathscr{O}))^-$, and $\overline{\pi}_{\mathfrak{F}}$ extends $\pi_{\mathfrak{F}}$ and is strongly continuous. This also implies that $\overline{\pi}_{\mathfrak{F}}(A) \restriction \mathscr{H} = A$, so that $\overline{\pi}_{\mathfrak{F}}$ is injective. To show that it is also surjective, we note that from $\pi_{\mathfrak{F}}(\mathfrak{A}(\mathscr{O})) \restriction \mathscr{H} = \mathfrak{A}(\mathscr{O})$, $\pi_{\mathfrak{F}}(\mathfrak{A}(\mathscr{O}))^- \restriction \mathscr{H} \subseteq \mathfrak{A}(\mathscr{O})^-$ follows, and if $A \in \pi_{\mathfrak{F}}(\mathfrak{A}(\mathscr{O}))^-$, by definition $\overline{\pi}_{\mathfrak{F}}(A \restriction \mathscr{H}) = A$. □

Remark. We should use the more precise notation $\overline{\pi}_{\mathfrak{F},\mathscr{O}}$ for the isomorphisms just defined, but since, as evident, $\overline{\pi}_{\mathfrak{F},\mathscr{O}_2} \restriction \mathfrak{A}(\mathscr{O}_1) = \overline{\pi}_{\mathfrak{F},\mathscr{O}_1}$ if $\mathscr{O}_1 \subseteq \mathscr{O}_2$, $\overline{\pi}_{\mathfrak{F},\mathscr{O}}(A)$ is independent of the region of localization of A, so that the simpler notation used above will not cause any confusion.

Proof of proposition 3.2. Since $(\pi_{\mathfrak{F}}, U_{\mathfrak{F}})$ is a covariant representation of $(\mathfrak{A}, \alpha \circ \eta)$ ($\eta : \tilde{\mathscr{P}}_+^\uparrow \to \mathscr{P}_+^\uparrow$ the covering homomorpishm), it follows, by strong continuity of $\overline{\pi}_{\mathfrak{F}}$, $\alpha_s^{\mathfrak{F}}$, α_s,

$$\alpha_s^{\mathfrak{F}}(\overline{\pi}_{\mathfrak{F}}(A)) = \overline{\pi}_{\mathfrak{F}}(\alpha_{\eta(s)}(A)), \qquad A \in \mathfrak{A}(\mathscr{O})^-, s \in \tilde{\mathscr{P}}_+^\uparrow,$$

and then if $s \to \alpha_s(A)$, $A \in \mathfrak{A}(\mathscr{O})^-$ is norm continuous, so is $s \to \alpha_s^{\mathfrak{F}}(\overline{\pi}_{\mathfrak{F}}(A))$, which, being $\overline{\pi}_{\mathfrak{F}}(A) \in \mathfrak{F}(\mathscr{O})^-$ and gauge invariant (still by strong continuity), implies, by hypotesis (i) above, that $\overline{\pi}_{\mathfrak{F}}(A)$ belongs to $\mathfrak{F}(\mathscr{O})$ and then, again by gauge invariance, to $\pi_{\mathfrak{F}}(\mathfrak{A}(\mathscr{O}))$, so that $A \in \mathfrak{A}(\mathscr{O})$. On the other hand, if $A \in \mathfrak{A}(\mathscr{O})$, $t \to \alpha_t^{\mathfrak{F}}(\pi_{\mathfrak{F}}(A))$ is norm continuous, and

$$\|\alpha_s(A) - A\| = \|\alpha_t^{\mathfrak{F}}(\pi_{\mathfrak{F}}(A)) - \pi_{\mathfrak{F}}(A)\|, \qquad s = \eta(t),$$

so that, as $\eta(\mathscr{N})$ is a neighbourhood of the identity in $\mathscr{P}_+^\uparrow$ for any such neighbourhood $\mathscr{N}$ in $\tilde{\mathscr{P}}_+^\uparrow$, $s \to \alpha_s(A)$ is continuous, and hypotesis 2.1 is satisfied. □

It is then possible to apply the construction of the scaling algebra and of the scaling limit to the observable net $\mathfrak{A}$. We now want to extend this construction to the field net. We will proceed along the lines of the discussion in chapter 2, taken from [BV95].

We recall then that the starting point for the definition of the scaling observable algebra is the usual, field theoretical approach to renormalization group, based on the choice of a family $(R_\lambda)_{\lambda>0}$ of (uniformly bounded) transformations of $\mathfrak{A}$ into itself, which have specific phase-space properties: namely, they have to rescale space-time coordinates by a factor λ, and momentum coordinates by a factor λ^{-1}, so to leave the velocity of light and Planck's constant unaffected. However, in the usual approach to renormalization group, the action of the transformations R_λ is by no means restricted to the observables: on the contrary, usually one considers the scaling of all the correlation functions of the theory, which in general involve unobservable fields, as Dirac or gauge fields, as well. This implies that we can assume that the R_λ's are transformations of $\mathfrak{F}$ in itself, retaining the above mentioned phase space properties. Then, all the considerations about these transformations in 2.1 apply verbatim to the present setting, if we replace the observable net $\mathfrak{A}$ by the field net $\mathfrak{F}$, and we conclude that RG orbits $\lambda \to R_\lambda(F)$ of fields $F \in \mathfrak{F}$ will enjoy properties of continuity with respect to the action of the Poincaré group analogous to the ones found to hold for orbits of observables. In addition, a similar spectral analysis of RG orbits with respect to the action of the gauge group shows that the functions complying with the physically prescribed running of charges, encoded in the λ-dependence of $\mathrm{Sp}\,V^{(\lambda)}$, are those for which

$$\lim_{g \to e} \sup_{\lambda>0} \|\beta_g^{(\lambda)}(R_\lambda(F)) - R_\lambda(F)\| = 0$$

holds.

We will therefore consider the C*-algebra $B(\mathbb{R}_+^\times, \mathfrak{F})$ of functions $\underline{F} : \mathbb{R}_+^\times \to \mathfrak{F}$, such that

$$\|\underline{F}\| := \sup_{\lambda>0} \|\underline{F}(\lambda)\| < +\infty, \tag{3.1}$$

and where the algebraic structure is defined pointwise:

$$\begin{aligned}(a\underline{F}+b\underline{G})(\lambda) &:= a\underline{F}(\lambda)+b\underline{G}(\lambda),\\ (\underline{F}\,\underline{G})(\lambda) &:= \underline{F}(\lambda)\underline{G}(\lambda),\\ (\underline{F}^*)(\lambda) &:= \underline{F}(\lambda)^*,\end{aligned} \tag{3.2}$$

where $\underline{F},\underline{G} \in B(\mathbb{R}_+^\times,\mathfrak{F})$ and $a,b\in\mathbb{C}$. On this algebra we can define actions $\underline{\alpha}^{\mathfrak{F}}$ of $\tilde{\mathscr{P}}_+^\uparrow$ and $\underline{\beta}$ of G by automorphisms as

$$\begin{aligned}\underline{\alpha}^{\mathfrak{F}}_s(\underline{F})(\lambda) &:= \alpha^{\mathfrak{F}}_{s_\lambda}(\underline{F}(\lambda)), \qquad s\in\tilde{\mathscr{P}}_+^\uparrow,\\ \underline{\beta}_g(\underline{F})(\lambda) &:= \beta_g^{(\lambda)}(\underline{F}(\lambda)), \qquad g\in G,\end{aligned} \tag{3.3}$$

where as usual $(a,x)_\lambda := (a,\lambda x)$ for any $(a,x)\in\tilde{\mathscr{P}}_+^\uparrow$.

Definition 3.4. The *local field scaling algebra* associated to the bounded open region $\mathscr{O}$ of Minkowskij space-time is the C*-subalgebra $\underline{\mathfrak{F}}(\mathscr{O})$ of $B(\mathbb{R}_+^\times,\mathfrak{F})$ of those $\underline{F}$ such that $\underline{F}(\lambda)\in\mathfrak{F}(\lambda\mathscr{O})$ and

$$\lim_{s\to e}\|\underline{\alpha}^{\mathfrak{F}}_s(\underline{F})-\underline{F}\| = 0, \qquad \lim_{g\to e}\|\underline{\beta}_g(\underline{F})-\underline{F}\| = 0. \tag{3.4}$$

The *(quasi-local) field scaling algebra* $\underline{\mathfrak{F}}$ is the C*-inductive limit of the algebras $\underline{\mathfrak{F}}(\mathscr{O})$.

It is then evident that $\mathscr{O}\to\underline{\mathfrak{F}}(\mathscr{O})$ is a net of C*-algebras over Minkowski space – which we will denote, as usual, still by $\underline{\mathfrak{F}}$ – and since the actions $\underline{\alpha}^{\mathfrak{F}}$ and $\underline{\beta}$ clearly commute, they restrict to actions of $\tilde{\mathscr{P}}_+^\uparrow$ and G on $\underline{\mathfrak{F}}$, with respect to which the net is covariant. Likewise, it is evident that $\underline{\mathfrak{F}}$ satisfies local $\mathbb{Z}_2$-graded commutativity with respect to the grading defined by $\underline{\gamma} := \underline{\beta}_k$, as follows immediately from

$$\underline{F}_\pm(\lambda) = \frac{1}{2}\Big(\underline{F}(\lambda)\pm\gamma^{(\lambda)}(\underline{F}(\lambda))\Big) = \underline{F}(\lambda)_\pm.$$

Finally, it is clear that

$$\underline{\pi_{\mathfrak{F}}}(\underline{A})(\lambda) := \pi_{\mathfrak{F}}(\underline{A}(\lambda)), \qquad \underline{A}\in\underline{\mathfrak{A}}(\mathscr{O}) \tag{3.5}$$

defines, by continuity, an homomorphism of nets $\underline{\pi_{\mathfrak{F}}}$ of $\underline{\mathfrak{A}}$ in $\underline{\mathfrak{F}}$, such that $\underline{\alpha}^{\mathfrak{F}}_s\circ\underline{\pi_{\mathfrak{F}}} = \underline{\pi_{\mathfrak{F}}}\circ\underline{\alpha}_{\eta(s)}$, and $\underline{\pi_{\mathfrak{F}}}(\underline{\mathfrak{A}}(\mathscr{O}))$ is contained in $\underline{\mathfrak{F}}(\mathscr{O})^G$, fixed-point subalgebra of $\underline{\mathfrak{F}}(\mathscr{O})$ under the action of $\underline{\beta}$. On the converse, if $\underline{F}\in\underline{\mathfrak{F}}(\mathscr{O})^G$, then $\beta_g^{(\lambda)}(\underline{F}(\lambda)) = \underline{F}(\lambda)$ for every $\lambda>0$, i.e. $\underline{F}(\lambda)\in\mathfrak{F}(\lambda\mathscr{O})^{\beta^{(\lambda)}} = \pi_{\mathfrak{F}}(\mathfrak{A}(\lambda\mathscr{O}))$, and then $\underline{F}\in\underline{\pi_{\mathfrak{F}}}(\underline{\mathfrak{A}}(\mathscr{O}))$ (the continuity of $s\to\underline{\alpha}_s(\underline{A})$ follows easily from that of $s\to\underline{\alpha}^{\mathfrak{F}}_s(\underline{\pi_{\mathfrak{F}}}(\underline{A}))$, as in the proof of proposition 3.2), so that $\underline{\mathfrak{A}}(\mathscr{O})$ and $\underline{\mathfrak{F}}(\mathscr{O})^G$ really coincide. In summary, we have proven:

Proposition 3.5. *The quintuple* $(\underline{\pi_{\mathfrak{F}}},\underline{\mathfrak{F}},\underline{\beta},k,\underline{\alpha}^{\mathfrak{F}})$ *is a Poincaré covariant normal field net over* $(\underline{\mathfrak{A}},\underline{\alpha})$.

Given a locally normal state φ on $\mathfrak{F}$ we can define its lift to $\underline{\mathfrak{F}}$ as the family of states $(\underline{\varphi}_\lambda)_{\lambda>0}$ such that

$$\underline{\varphi}_\lambda(\underline{F}) := \varphi(\underline{F}(\lambda)), \qquad \underline{F}\in\underline{\mathfrak{F}},\lambda>0, \tag{3.6}$$

and we can consider the set $SL_{\mathfrak{F}}(\varphi)$ of its weak* limit points, which is non-void by Banach-Bourbaki-Alaoglu theorem. As for the case of the observable scaling algebra, we have the following.

Proposition 3.6. *$SL_{\mathfrak{F}}(\varphi)$ is independent of the locally normal state φ on $\mathfrak{F}$.*

Proof. Arguing as in [BV95, corollary 4.2], it follows that it is sufficient to show that, for any couple of locally normal states φ_1, φ_2 on $\mathfrak{F}$, it holds

$$\lim_{\lambda\to 0^+} \|(\varphi_1-\varphi_2)\restriction \mathfrak{F}(\lambda\mathscr{O})\| = 0.$$

Furthermore, since the second half of the proof of [BV95, lemma 4.1] can be repeated verbatim in the present case (only the net structure of $\mathfrak{F}$ is involved, and not commutation relations), we need only to show that

$$\bigcap_{\mathscr{O}\ni 0} \mathfrak{F}(\mathscr{O})^- = \mathbb{C}\mathbb{1}.$$

To show this, we follow the first half of the proof of [BV95, lemma 4.1]. Since any $Z\in\bigcap_{\mathscr{O}\ni 0}\mathfrak{F}(\mathscr{O})^-$ is a sum of a bosonic and a fermionic part, we can assume that Z is purely bosonic or purely fermionic. Then Z^* has the same Bose-Fermi parity as Z and is also "localized at 0". Thus, by locality, for $x^2>0$ we have $[Z^*,\alpha^{\mathfrak{F}}_x(Z)]=0$ (resp. $\{Z^*,\alpha^{\mathfrak{F}}_x(Z)\}=0$) if Z is bosonic (resp. fermionic) and, being $x\to\alpha^{\mathfrak{F}}_x(Z)$ weakly continuous, this holds also for $x^2=0$. Then, if e is a lightlike vector, we have

$$\begin{aligned}(Z\Omega|U_{\mathfrak{F}}(te)Z\Omega) &= (\Omega|Z^*\alpha^{\mathfrak{F}}_{te}(Z)\Omega)\\ &= \pm(\Omega|\alpha^{\mathfrak{F}}_{te}(Z)Z^*\Omega) = \pm(Z^*\Omega|U_{\mathfrak{F}}(-te)Z^*\Omega),\end{aligned}$$

thus, by the spectrum condition, the function $t\to(Z\Omega|U_{\mathfrak{F}}(te)Z\Omega)$ has Fourier transform whose support is $\{0\}$ and hence, being bounded, is constant. This implies

$$\|U_{\mathfrak{F}}(te)Z\Omega - Z\Omega\|^2 = 2-2\,\mathrm{Re}(Z\Omega|U_{\mathfrak{F}}(te)Z\Omega) = 0$$

for any lightlike vector e, and then $U_{\mathfrak{F}}(x)Z\Omega = Z\Omega$ for any $x\in\mathbb{R}^4$, and finally, by uniqueness of the vacuum, and its separating property with respect to the algebras $\mathfrak{F}(\mathscr{O})^-$, $Z=(\Omega|Z\Omega)\,\mathbb{1}$. □

It is then meaningful to talk about the *set of scaling limit states of* $\underline{\mathfrak{F}}$, $SL_{\mathfrak{F}}$, without reference to any particular (locally normal) state of $\mathfrak{F}$. We remark also that $SL_{\mathfrak{F}}\circ\pi_{\underline{\mathfrak{F}}} = SL_{\mathfrak{A}}$ (scaling limit states of $\underline{\mathfrak{A}}$), since both can be calculated through the lifting of the vacuum states ω and $\omega^{\mathfrak{A}} := \omega\circ\pi_{\mathfrak{F}}$: if $\underline{\omega}_0\in SL_{\mathfrak{F}}(\omega)$ it is clear that $\underline{\omega}_0\circ\pi_{\underline{\mathfrak{F}}}\in SL_{\mathfrak{A}}(\omega^{\mathfrak{A}})$, and on the converse, if $\underline{\omega}^{\mathfrak{A}}_0\in SL_{\mathfrak{A}}(\omega^{\mathfrak{A}})$ is a weak* limit of a net $(\underline{\omega}^{\mathfrak{A}}_{\lambda_\iota})_{\iota\in I}$, by compactness one can find a subnet of $(\underline{\omega}_{\lambda_\iota})_{\iota\in I}$ convergent to a state $\underline{\omega}_0\in SL_{\mathfrak{F}}(\omega)$, and $\underline{\omega}_0\circ\pi_{\underline{\mathfrak{F}}} = \underline{\omega}^{\mathfrak{A}}_0$.

Now, fix a scaling limit state $\underline{\omega}_0\in SL_{\mathfrak{F}}$, and denote by $(\pi^0,\mathscr{H}^0,\Omega_0)$ its GNS representation. One can then define a net of C*-algebras over Minkowskij space, acting on $\mathscr{H}^0$, by $\mathfrak{F}^0(\mathscr{O}) := \pi^0(\underline{\mathfrak{F}}(\mathscr{O}))$. Since $\underline{\omega}_0$ can be obtained as a weak* limit point of the lifting of the vacuum state ω on $\mathfrak{F}$, we have $\underline{\omega}_0\circ\underline{\alpha}^{\mathfrak{F}}_s = \underline{\omega}_0$ and $\underline{\omega}_0\circ\underline{\beta}_g = \underline{\omega}_0$, so that we get unitary representations $V_{\mathfrak{F}^0}$ and $U_{\mathfrak{F}^0}$, of G and $\tilde{\mathscr{P}}^{\uparrow}_+$ respectively, on $\mathscr{H}^0$, leaving Ω_0 invariant. Let also $(\pi_0,\mathscr{H}_0,\Omega_0)$ be the GNS representation associated to $\underline{\omega}_0\circ\pi_{\underline{\mathfrak{F}}}\in SL_{\mathfrak{A}}(\omega)$ (we drop the notational distinction between the vacuum states on $\mathfrak{A}$ and $\mathfrak{F}$), and $\mathfrak{A}_0$ the associated scaling limit net. By unicity of the GNS representation, we have that $\mathscr{H}_0$ is identified, through a unitary equivalence, with a subspace of $\mathscr{H}^0$, in such a way that the respective cyclic vectors agree (and that is why we have used the same symbol Ω_0 for them from the beginning), and that π_0 is the restriction to $\mathscr{H}_0$ of $\pi^0\circ\pi_{\underline{\mathfrak{F}}}$.

Theorem 3.7. *The representation $\pi_{\mathfrak{F}^0}$ of $\mathfrak{A}_0$ on $\mathscr{H}^0$ given by*

$$\pi_{\mathfrak{F}^0}(\pi_0(\underline{A})) := \pi^0 \circ \pi_{\underline{\mathfrak{F}}}(\underline{A}) \qquad \underline{A} \in \underline{\mathfrak{A}}, \tag{3.7}$$

is well defined, and the quintuple $(\pi_{\mathfrak{F}^0}, \mathfrak{F}^0, V_{\mathfrak{F}^0}, k, U_{\mathfrak{F}^0})$ is a Poincaré covariant, normal field net with gauge symmetry over $(\mathscr{H}_0, \mathfrak{A}_0, U_0, \Omega_0)$.

We introduce the following notation: for a tempered distribution $f \in \mathscr{S}(\mathbb{R}^4)'$, such that $\hat{f}$ is a measure with $\text{supp}\, \hat{f} \subseteq \overline{V}_+ \cup \overline{V}_-$, its positive (negative) frequency part is $f_\pm := (\chi_{\overline{V}_\pm} \hat{f})\check{}$. Then the following lemma, useful for the proof of the theorem, can be extracted from the calculations in [AHR62].

Lemma 3.8. *Let $h \in C_b^\infty(\mathbb{R}^4)$ be such that*
(i) $h(x) = 0$ for $x \in \mathscr{O}_r'$ ($\mathscr{O}_r$ double cone generated by the 3-sphere of radius r centered at 0 in the $x^0 = 0$ plane);
(ii) $p^\alpha d\hat{h}(p)$ is a bounded complex measure for every multiindex $\alpha \in \mathbb{N}_0^4$;
(iii) $\text{supp}\, \hat{h} \subseteq \overline{V}_+ \cup \overline{V}_-$.
Then there exists a constant $A > 0$, independent of h, such that, for $|\boldsymbol{x}| > r$,

$$|h_+(0, \boldsymbol{x})| \leq \frac{Ar^3}{(|\boldsymbol{x}| - r)^2} \left| \int_{\mathbb{R}^4} p^0 d\hat{h}(p) \right|. \tag{3.8}$$

The proof is based on an application of the Jost-Lehmann-Dyson representation.

Proof of theorem 3.7. Since the function $g \to \underline{\omega}_0(\underline{G}\underline{\beta}_g(\underline{F}))$ is continuous for every $\underline{F}, \underline{G} \in \underline{\mathfrak{F}}$, the representation $V_{\mathfrak{F}^0}$ is weakly, and hence strongly, continuous. Analogously for $U_{\mathfrak{F}^0}$. By the definition of $V_{\mathfrak{F}^0}$, it is immediate to verify that $\beta_g^{\mathfrak{F}^0} := \mathrm{Ad} V_{\mathfrak{F}^0}(g)$, $g \in G$, is such that, for $\underline{F} \in \underline{\mathfrak{F}}$, $\beta_g^{\mathfrak{F}^0}(\pi^0(\underline{F})) = \pi^0(\underline{\beta}_g(\underline{F}))$, so that $\beta^{\mathfrak{F}^0}$ defines an action of G on $\mathfrak{F}^0$ by net automorphisms, and $\gamma^{\mathfrak{F}^0} := \beta_k^{\mathfrak{F}^0}$ defines a $\mathbb{Z}_2$-grading of $\mathfrak{F}^0$ such that $\pi^0(\underline{F})_\pm = \pi^0(\underline{F}_\pm)$, and then $\mathfrak{F}^0$ satisfies local $\mathbb{Z}_2$-graded commutativity with respect to $\gamma^{\mathfrak{F}^0}$.

We show that $\pi_{\mathfrak{F}^0}(\mathfrak{A}_0)$ is the fixed point subnet of $\mathfrak{F}^0$ under the action of G (we will prove below that $\pi_{\mathfrak{F}^0}$ is well defined). To this end, consider the conditional expectation $\underline{m}$ on $\underline{\mathfrak{F}}$ defined by

$$\underline{m}(\underline{F}) := \int_G dg\, \underline{\beta}_g(\underline{F}),$$

where dg is the left invariant Haar measure on G, and the integral exists in Bochner sense [Yos68] since the integrand is a bounded, norm continuous function on a compact space, and hence its range, being metrizable, is separable. Let also m be the analogous mean on $\mathfrak{F}^0$. We have then $\underline{m}(\underline{\mathfrak{F}}(\mathscr{O})) = \underline{\mathfrak{F}}(\mathscr{O})^G$: the inclusion $\underline{\mathfrak{F}}(\mathscr{O})^G \subseteq \underline{m}(\underline{\mathfrak{F}}(\mathscr{O}))$ is evident, as $\underline{m}(\underline{F}) = \underline{F}$ for $\underline{F} \in \underline{\mathfrak{F}}(\mathscr{O})^G$, the reverse one also follows immediately from invariance of dg, which implies G-invariance of $\underline{m}$. For the same reasons, $m(\mathfrak{F}^0(\mathscr{O})) = \mathfrak{F}^0(\mathscr{O})^G$. By continuity of π^0, we have $m \circ \pi^0 = \pi^0 \circ \underline{m}$, and then

$$\begin{aligned}\pi_{\mathfrak{F}^0}(\mathfrak{A}_0(\mathscr{O})) = \pi^0 \circ \pi_{\underline{\mathfrak{F}}}(\underline{\mathfrak{A}}(\mathscr{O})) = \pi^0(\underline{\mathfrak{F}}(\mathscr{O})^G) \\ = \pi^0(\underline{m}(\underline{\mathfrak{F}}(\mathscr{O}))) = m(\mathfrak{F}^0(\mathscr{O})) = \mathfrak{F}^0(\mathscr{O})^G.\end{aligned}$$

For what concerns covariance, if $\alpha_s^{\mathfrak{F}^0} := \mathrm{Ad}U_{\mathfrak{F}^0}(s)$, we have $\alpha_s^{\mathfrak{F}^0} \circ \pi^0 = \pi^0 \circ \underline{\alpha}_s^{\mathfrak{F}}$, so that $\mathfrak{F}^0$ is covariant with respect to the action of $\tilde{\mathscr{P}}_+^\uparrow$ defined by $\alpha^{\mathfrak{F}^0}$, and $s \to \alpha_s^{\mathfrak{F}^0}(F)$ is norm continuous for every $F \in \mathfrak{F}^0(\mathscr{O})$. Furthermore, it is clear that $\alpha^{\mathfrak{F}^0}$ and $\beta^{\mathfrak{F}^0}$ commute.

Let us show that $U_{\mathfrak{F}^0}$ fulfills the spectrum condition. For this, it is sufficient to show that, for every $f \in L^1(\mathbb{R}^4)$ whose Fourier transform has support in $\mathbb{R}^4 \setminus \overline{V}_+$, and for every $\underline{F}, \underline{G} \in \underline{\mathfrak{F}}$,

$$\begin{aligned}\int_{\mathbb{R}^4} dx\, f(x)(\pi^0(\underline{G})\Omega_0|U_{\mathfrak{F}^0}(x)\pi^0(\underline{F})\Omega_0) &= \int_{\mathbb{R}^4} dx\, f(x)\underline{\omega}_0(\underline{G}^*\underline{\alpha}_x^{\mathfrak{F}}(\underline{F}))\\ &= \underline{\omega}_0(\underline{G}^*\underline{\alpha}_f^{\mathfrak{F}}(\underline{F})) = 0,\end{aligned}$$

where we have also used the continuity of the action of $\underline{\alpha}^{\mathfrak{F}}$ on $\underline{\mathfrak{F}}$. But, making again use of this fact, it is easy to see that

$$\underline{\omega}_\lambda(\underline{G}^*\underline{\alpha}_f^{\mathfrak{F}}(\underline{F})) = \omega\big(\underline{G}(\lambda)^*\alpha_{f_\lambda}^{\mathfrak{F}}(\underline{F}(\lambda))\big),$$

where $f_\lambda(x) := \lambda^{-4} f(\lambda^{-1}x)$, so that $\mathrm{supp}\,\hat{f}_\lambda = \lambda^{-1}\,\mathrm{supp}\,\hat{f} \subseteq \mathbb{R}^4 \setminus \overline{V}_+$, and $\underline{\omega}_\lambda(\underline{G}^*\underline{\alpha}_f^{\mathfrak{F}}(\underline{F})) = 0$ for every $\lambda > 0$. Furthermore, by definition Ω_0 is cyclic for $\mathfrak{F}^0$, and is Poincaré and gauge invariant.

To complete the proof, it remains then only to show that Ω_0 is the unique translation invariant unit vector. In fact, if this is true, $\mathfrak{F}^0$ is irreducible, and Ω_0 is separating for the local von Neumann algebras $\mathfrak{F}^0(\mathscr{O})^-$ (Reeh-Schlieder theorem 1.5 and remark after definition 1.8), and this implies that $\pi_{\mathfrak{F}^0}$ is well defined: if $\pi_0(\underline{A}) = 0$, $\underline{A} \in \underline{\mathfrak{A}}(\mathscr{O})$, then

$$\|\pi^0 \circ \pi_{\underline{\mathfrak{F}}}(\underline{A})\Omega_0\| = \lim_{n\to+\infty} \|\pi_{\mathfrak{F}}(\underline{A}(\lambda_n))\Omega\| = \|\pi_0(\underline{A})\Omega_0\| = 0,$$

for a suitable sequence $(\lambda_n)_{n\in\mathbb{N}}$ converging to zero, and then $\pi^0 \circ \pi_{\underline{\mathfrak{F}}}(\underline{A}) = 0$, so that $\pi_{\mathfrak{F}^0}$ is a well defined representation of $\mathfrak{A}_{0,\mathrm{loc}}$, which then extends to the inductive limit, giving a net homomorphism from $\mathfrak{A}_0$ to $\mathfrak{F}^0$.

To show that Ω_0 is the unique translation invariant unit vector, it is sufficient [Haa96, lemma 3.2.5] to verify that the state $\omega_0 := (\Omega_0|(\cdot)\Omega_0)$ is clustering, i.e. that for every $F, G \in \mathfrak{F}^0$,

$$\lim_{|\boldsymbol{x}|\to+\infty} \omega_0(F\alpha_{\boldsymbol{x}}^{\mathfrak{F}^0}(G)) = \omega_0(F)\omega_0(G). \tag{3.9}$$

Furthermore, since the operators $F_\pm\alpha_{\boldsymbol{x}}^{\mathfrak{F}^0}(G_\mp)$ are odd under $\gamma^{\mathfrak{F}^0}$, and therefore $\omega_0(F_\pm\alpha_{\boldsymbol{x}}^{\mathfrak{F}^0}(G_\mp)) = 0 = \omega_0(F_\pm)\omega_0(G_\mp)$, it suffices to show that (3.9) holds for F, G both purely bosonic or purely fermionic. Any purely bosonic (resp. fermionic) $F \in \mathfrak{F}^0$ is of the form $F = \pi^0(\underline{F})$ with $\underline{F} \in \underline{\mathfrak{F}}$ purely bosonic (resp. fermionic), so we pick, to begin with, $\underline{F}, \underline{G} \in \underline{\mathfrak{F}}(\mathscr{O}_r)$ purely bosonic, and such that $x \to \underline{\alpha}_x^{\mathfrak{F}}(\underline{F})$, $x \to \underline{\alpha}_x^{\mathfrak{F}}(\underline{G})$ are infinitely continuously differentiable in norm. The set of such operators, for all $r > 0$ is norm dense in $\underline{\mathfrak{F}}_+$, as can be seen considering operators of the form $\underline{\alpha}_f^{\mathfrak{F}}(\underline{F})$, with $f \in C_c^\infty(\mathbb{R}^4)$ and $\underline{F} \in \underline{\mathfrak{F}}(\mathscr{O})_+$ (it is easy to see that $x \to \underline{\alpha}_x^{\mathfrak{F}}(\underline{\alpha}_f^{\mathfrak{F}}(\underline{F}))$ is differentiable in norm with $\partial_\mu\underline{\alpha}_x^{\mathfrak{F}}(\underline{\alpha}_f^{\mathfrak{F}}(\underline{F})) = -\underline{\alpha}_x^{\mathfrak{F}}(\underline{\alpha}_{\partial_\mu f}^{\mathfrak{F}}(\underline{F}))$, and, if $(\delta_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^4)$, is an approximate identity, $\underline{\alpha}_{\delta_n}^{\mathfrak{F}}(\underline{F}) \to \underline{F}$ in norm, thanks to norm continuity of $x \to \underline{\alpha}_x^{\mathfrak{F}}(\underline{F})$). Consider, for $0 < \lambda \leq 1$, the functions

$$\begin{aligned}h_{\underline{F},\underline{G}}^{(\lambda)}(x) :&= (\Omega|\underline{F}(\lambda)\alpha_{\lambda x}^{\mathfrak{F}}(\underline{G}(\lambda))\Omega) - (\Omega|\underline{F}(\lambda)\Omega)(\Omega|\underline{G}(\lambda)\Omega)\\ &= \underline{\omega}_\lambda(\underline{F}\underline{\alpha}_x^{\mathfrak{F}}(\underline{G})) - \underline{\omega}_\lambda(\underline{F})\underline{\omega}_\lambda(\underline{G}),\end{aligned}$$

and

$$h^{(\lambda)}(x) := h^{(\lambda)}_{\underline{F},\underline{G}}(x) - h^{(\lambda)}_{\underline{G},\underline{F}}(-x) = (\Omega|[\underline{F}(\lambda), \alpha^{\mathfrak{F}}_{\lambda x}(\underline{G}(\lambda))]\Omega).$$

We have $h^{(\lambda)} \in C^\infty_b(\mathbb{R}^4)$ and, due to the fact that $\underline{F}(\lambda)$, $\underline{G}(\lambda)$ are purely bosonic for every λ, $h^{(\lambda)}(x) = 0$ for $x \in \mathscr{O}'_{2r}$. If we denote by E the spectral measure determined by the translation group, we have

$$d\hat{h}^{(\lambda)}(p) = (2\pi)^4 \left[d(\Omega|\underline{F}(\lambda)E(\lambda^{-1}p)\underline{G}(\lambda)\Omega) - d(\Omega|\underline{G}(\lambda)E(-\lambda^{-1}p)\underline{F}(\lambda)\Omega) \right],$$

so that, by the spectrum condition, $\operatorname{supp} \hat{h}^{(\lambda)} \subseteq \overline{V}_+ \cup \overline{V}_-$. From the fact that $x \to \alpha^{\mathfrak{F}}_x(\underline{F}(\lambda))$, $x \to \alpha^{\mathfrak{F}}_x(\underline{G}(\lambda))$ are C^∞, it follows that $\underline{F}(\lambda)\Omega$, $\underline{G}(\lambda)\Omega$ are in the domain of all monomials in the generators of the translations P^μ, and then, from the above formula, we have that, for any $\alpha \in \mathbb{N}_0^4$,

$$\begin{aligned} p^\alpha d\hat{h}^{(\lambda)}(p) = (2\pi)^4 \Big[& \lambda^{|\alpha|} d(\Omega|\underline{F}(\lambda)E(\lambda^{-1}p)P^\alpha \underline{G}(\lambda)\Omega) \\ & - (-\lambda)^{|\alpha|} d(\Omega|\underline{G}(\lambda)E(-\lambda^{-1}p)P^\alpha \underline{F}(\lambda)\Omega) \Big] \end{aligned} \tag{3.10}$$

is a bounded measure. So, if we observe that, due to the fact that Ω is the unique translation invariant vector,

$$\begin{aligned} h^{(\lambda)}_+(x) &= \frac{1}{(2\pi)^4} \int_{\overline{V}_+} e^{ipx} d\hat{h}^{(\lambda)}(p) \\ &= (\Omega|\underline{F}(\lambda)U_{\mathfrak{F}}(\lambda x)\underline{G}(\lambda)\Omega) - (\Omega|\underline{G}(\lambda)E(\{0\})\underline{F}(\lambda)\Omega) = h^{(\lambda)}_{\underline{F},\underline{G}}(x), \end{aligned}$$

applying lemma 3.8 and (3.10), we have, for some (universal) constant $C > 0$, independent of λ, $\underline{F}$ and $\underline{G}$, and for $|\boldsymbol{x}| > 2r$,

$$\begin{aligned} \left| \underline{\omega}_\lambda(\underline{F}\underline{\alpha}^{\mathfrak{F}}_x(\underline{G})) - \underline{\omega}_\lambda(\underline{F})\underline{\omega}_\lambda(\underline{G}) \right| &\leq \\ &\leq \frac{Cr^3}{(|\boldsymbol{x}| - 2r)^2} \left| \lambda(\Omega|\underline{F}(\lambda)P^0\underline{G}(\lambda)\Omega) + \lambda(\Omega|\underline{G}(\lambda)P^0\underline{F}(\lambda)\Omega) \right| \\ &\leq \frac{Cr^3}{(|\boldsymbol{x}| - 2r)^2} \left(\|\underline{F}(\lambda)\| \|\dot{\underline{G}}(\lambda)\| + \|\underline{G}(\lambda)\| \|\dot{\underline{F}}(\lambda)\| \right) \\ &\leq \frac{Cr^3}{(|\boldsymbol{x}| - 2r)^2} \left(\|\underline{F}\| \|\dot{\underline{G}}\| + \|\underline{G}\| \|\dot{\underline{F}}\| \right) \end{aligned}$$

where $\dot{\underline{F}}$ is the derivative of $t \to \underline{\alpha}^{\mathfrak{F}}_t(\underline{F})$ at $t = 0$, and we have used the fact that, due to norm differentiability, $\dot{\underline{F}}(\lambda) = \frac{d}{dt}\alpha^{\mathfrak{F}}_{\lambda t}(\underline{F}(\lambda))|_{t=0}$ (and the same for $\underline{G}$). Since the above estimate is uniform in λ, we conclude that (3.9) holds for any pair $\pi^0(\underline{F})$, $\pi^0(\underline{G})$ of the form considered, and since such operators are norm dense in the bosonic part of $\mathfrak{F}^0$, we have that ω_0 is clustering for any bosonic F and G. For F and G purely fermionic, one has $\underline{\omega}_\lambda(\underline{F}) = 0 = \underline{\omega}_\lambda(\underline{G})$, and the result is obtained by applying the above argument to the function $h^{(\lambda)}_{\underline{F},\underline{G}}(x) + h^{(\lambda)}_{\underline{G},\underline{F}}(-x) = (\Omega|\{\underline{F}(\lambda), \alpha^{\mathfrak{F}}_{\lambda x}(\underline{G}(\lambda))\}\Omega)$. □

This result shows therefore that the scaling limit construction can then be applied, without essential modifications, to a net of localized fields.

Definition 3.9. Every net $(\pi_{\mathfrak{F}^0}, \mathfrak{F}^0, V_{\mathfrak{F}^0}, k, U_{\mathfrak{F}^0})$ arising, as above, from a scaling limit state $\underline{\omega}_0 \in SL_{\mathfrak{F}}$, will be called a *scaling limit field net* over the associated scaling limit observable net $(\alpha^{\mathfrak{F}^0}, U_0, \Omega_0)$.

The various possibilities for the structure of the scaling limit, considered for the case of the observable net in section 2.2, arise also in the case at hand, and the analysis made there of their different physical meanings applies as well. We remark, however, that it can happen that different, non isomorphic scaling limit field nets are associated to the same scaling limit observable net, or to isomorphic ones. For instance, it may happen that for two different scaling limit states $\underline{\omega}_0$ on $\underline{\mathfrak{F}}$, the corresponding states $\underline{\omega}_0 \circ \pi_{\underline{\mathfrak{F}}}$, determining $\mathfrak{A}_0$, coincide. In particular, the situation can be realized in which there is a unique quantum scaling limit in the sense of definition 2.9, but the various scaling limit field net are not isomorphic to each other, and then describe different set of charges of the scaling limit theory. Sticking to the general principles of the algebraic approach to quantum field theory, according to which the theory is identified by its net of local observables, in such a case we will still talk of a unique scaling limit.

3.2 Ultraviolet stable localizable charges

The superselection structure of a theory is described by a net of charge carrying fields with a compact gauge group acting on it, and then it is natural to try and define a scaling limit procedure for sectors of the underlying theory through the scaling limit field net defined in the previous section. However, as already remarked at the beginning of this chapter, one cannot expect that, in general, all (localizable) sectors of the underlying theory possess a sensibile scaling limit, giving rise to corresponding (localizable) sectors of the scaling limit theory, and this is due to the specific phase space properties of renormalization group orbits, encoded in the (field or observable) scaling algebra: if a charge sistematically requires, in order to be created from the vacuum in a region of diameter λ, energy of order λ^{-q}, with $q > 1$, then the fields carrying it cannot be expected to give rise to elements of the field scaling algebra possesing a non trivial scaling limit, and the corresponding sector will vanish in the limit, or, in more physical terms, it cannot appear in the scaling limit since its creation would require an infinite amount of energy[a)]. For this reason, we shall now single out a subclass of sectors which are energetically "well behaved" in the above sense, and then we shall apply the scaling limit procedure of the previous section to the subnet of the canonical field net of a given theory constructed out of fields carrying such charges.

We shall then assume to be given a Poincaré covariant observable net $(\mathscr{H}, \mathfrak{A}, U, \Omega)$ in vacuum representation satisfying hypotesis 2.1, and such that the corresponding net of von Neumann algebras $\mathscr{A}(\mathscr{O}) := \mathfrak{A}(\mathscr{O})^-$ satisfies Haag duality and property B. Let then $(\pi_{\mathscr{F}}, \mathscr{F}, V, k, U_{\mathscr{F}})$ be the corresponding unique complete normal Poincaré covariant field net with gauge symmetry determined by the superselection structure of $\mathscr{A}$, theorem 1.13. To simplify the notation, from now on, unless where confusion can arise, we shall drop

[a)]On the other hand, it may be possible that such charges give rise to some kind of *non-localizable* sector in the scaling limit, since, if one insists in not spending energies bigger than λ^{-1}, then the charge can only be localized in regions of diameter much bigger than λ, which in the limit become the whole space-time the superscripts from the action of the Poincaré and gauge groups on the various nets, underlying or scaling limit ones, we shall consider, and we shall only distinguish the actions in the scaling limit by denoting them by α^0 and β^0. As usual, we shall indicate by Δ the semigroup of transportable localized endomorphisms of $\mathscr{A}$, $\Delta(\mathscr{O})$ being the subsemigroup of those localized in the double cone $\mathscr{O}$, and H_ρ will be the finite dimensional Hilbert space in $\mathscr{F}(\mathscr{O})$ implementing a $\rho \in \Delta(\mathscr{O})$.

Definition 3.10. A localizable covariant sector ξ of the underlying theory will be called *ultraviolet stable* if for every double cone $\mathscr{O}$ there exists a family $(\rho_\lambda)_{\lambda>0}$ of covariant transportable localized endomorphisms of $\mathscr{A}$ of class ξ, ρ_λ localized in $\lambda\mathscr{O}$, such that for every bounded function $\lambda \to \psi(\lambda) \in H_{\rho_\lambda}$ there holds

$$\lim_{s\to e} \sup_{\lambda\in(0,1]} \|[\alpha_{s_\lambda}(\psi(\lambda)) - \psi(\lambda)]\Omega\| = 0 = \lim_{s\to e} \sup_{\lambda\in(0,1]} \|[\alpha_{s_\lambda}(\psi(\lambda)^*) - \psi(\lambda)^*]\Omega\|. \tag{3.11}$$

Any such function will be called a *(renormalization group) quasi-orbit*. By a slight abuse, we shall also say that an endomorphism $\rho \in \xi$ is ultraviolet stable.

Remarks. (i) That these charges do not exhibit the pathological phase space behaviour discussed above can be easily seen using methods completely analogous to those employed in the analysis of chapter 2. For instance, a particular case of (3.11) is

$$\lim_{x\to 0} \sup_{\lambda\in(0,1]} \|[\alpha_{\lambda x}(\psi(\lambda)) - \psi(\lambda)]\Omega\| = 0,$$

and, repeating the proof of lemma 3.1 in [BV95] with trivial modifications, we get that this is equivalent to demanding that for any $\varepsilon > 0$, there exists a compact set $\Delta \subseteq \mathbb{R}^4$ such that

$$\sup_{\lambda\in(0,1]} \|[E(\lambda^{-1}\Delta) - \mathbb{1}]\psi(\lambda)\Omega\| < \varepsilon,$$

and then, as required, $\psi(\lambda)\Omega$ is (essentially) localized in a region of radius λ, and has 4-momentum scaling as λ^{-1}. As we consider a Poincaré covariant scaling limit, we have to require also similar continuity with respect to Lorentz transformations, which is equivalent to the fact that the charged states $\psi(\lambda)\Omega$ carry angular momentum independent of λ.

(ii) The choice of the interval $(0,1]$ in (3.11) is clearly arbitrary, and, for what concerns the analysis of the short distance behaviour of charges, it could have been replaced by any interval $(0,\delta]$ with $\delta > 0$. It may be too strong a requirement, however, to ask (3.11) to hold with such interval replaced by $(0,+\infty)$, as it is conceivable that charges exist which cannot be localized in a region of radius, say, $\lambda > 1$ without being localized also in a region of radius $\lambda = 1$, and it would then be unreasonable to require that arbitrarily small energies are needed to create such a charge in larger regions (cfr. the previous remark).

(iii) In appendix B it is shown there is at least a free field model in which all sectors are ultraviolet stable.

We are going to show in this section that ultraviolet stable sectors survive the scaling limit.

From field operators carrying ultraviolet stable charges we construct a field net, to which we will then apply the scaling limit procedure, in the following way. Let $\mathscr{F}_s(\mathscr{O})$ be the von Neumann subalgebra of $\mathscr{F}(\mathscr{O})$, generated by $\pi_{\mathscr{F}}(\mathscr{A}(\mathscr{O}))$ and by the Hilbert spaces

H_ρ, where the sector determined by ρ is ultraviolet stable. Then we define $\mathfrak{F}(\mathscr{O})$ as the C*-subalgebra of $\mathscr{F}_s(\mathscr{O})$ consisting of those elements $F \in \mathscr{F}_s(\mathscr{O})$ such that the functions $s \to \alpha_s(F)$, $g \to \beta_g(F)$ are norm continuous, and denote by $\mathscr{H}_\mathfrak{F}$ the closure of $\mathfrak{F}\Omega$. Using the fact that Ω is separating for $\mathscr{F}(\mathscr{O})$, it is then easy to see that $\mathfrak{F}$ is net isomorphic with its restriction to $\mathscr{H}_\mathfrak{F}$, and we will then identify the two nets.

Proposition 3.11. *With the notations above $\mathscr{H}_\mathfrak{F}$ is Poincaré and gauge invariant, and with $\pi_\mathfrak{F} := \pi_\mathscr{F}(\cdot) \restriction \mathscr{H}_\mathfrak{F}$, $U_\mathfrak{F} := U_\mathscr{F}(\cdot) \restriction \mathscr{H}_\mathfrak{F}$, and still denoting by V the restriction to $\mathscr{H}_\mathfrak{F}$ of the gauge group representation, $(\pi_\mathfrak{F}, \mathfrak{F}, V, k, U_\mathfrak{F})$ is a Poincaré covariant, normal field net with gauge symmetry satisfying hypotesis 3.1(i) over $(\mathfrak{A}, U, \Omega)$.*

Proof. The auxiliary net $\mathscr{F}_s$ is Poincaré and gauge covariant: $\alpha_s\big(\pi_\mathscr{F}(\mathscr{A}(\mathscr{O}))\big) = \pi_\mathscr{F}(\mathscr{A}(s\cdot\mathscr{O}))$ and $\alpha_s(H_\rho) = H_{\alpha_s\rho\alpha_{s^{-1}}}$ and, being ρ covariant, $\alpha_s\rho\alpha_{s^{-1}}$ is in its same ultraviolet stable sector, so that $\alpha_s(\mathscr{F}_s(\mathscr{O})) = \mathscr{F}_s(s\cdot\mathscr{O})$. Similarly $\beta_g(\pi_\mathscr{F}(\mathscr{A}(\mathscr{O}))) = \pi_\mathscr{F}(\mathscr{A}(\mathscr{O}))$ and $\beta_g(H_\rho) = H_\rho$, giving $\beta_g(\mathscr{F}_s(\mathscr{O})) = \mathscr{F}_s(\mathscr{O})$. It follows then immediately, using commutativity of α and β, that $\mathfrak{F}$ is also Poincaré and gauge covariant, and that $\mathscr{H}_\mathfrak{F}$ is globally Poincaré and gauge invariant. Normality of commutation relations is also immediate, as $\mathfrak{F}(\mathscr{O}) \subseteq \mathscr{F}(\mathscr{O})$ and the action of k is the same on the two nets. That the vacuum is cyclic for $\mathfrak{F}$, thought as a net over $\mathscr{H}_\mathfrak{F}$, and that it is the unique translation invariant vector is true by definition, as the fact that hypotesis 3.1(i) holds. Finally as $\pi_\mathscr{F}(\mathfrak{A}) \subseteq \mathfrak{F}$, $\mathscr{H}$, cyclically generated by $\mathfrak{A}$ on Ω, is a subspace of $\mathscr{H}_\mathfrak{F}$, and $\pi_\mathfrak{F}$ restricted to it is the vacuum representation of $\mathfrak{A}$, and if $F \in \mathfrak{F}(\mathscr{O})^G \subseteq \mathscr{F}(\mathscr{O})^G$ then $F = \pi_\mathscr{F}(A)$, $A \in \mathscr{A}(\mathscr{O})$, but then, by the usual argument, $s \to \alpha_s(A)$ is norm continuous, and, by maximality, $A \in \mathfrak{A}(\mathscr{O})$ so that, restricting to $\mathscr{H}_\mathfrak{F}$, $\mathfrak{F}(\mathscr{O})^G = \pi_\mathfrak{F}(\mathfrak{A}(\mathscr{O}))$. ☐

Hypotesis 3.1(ii) is trivially satisfied if we assume $V^{(\lambda)} = V$, $\lambda > 0$, which is the scaling behaviour expected for conserved (Nöther) charges from perturbative quantum field theory [IZ80, chp. 13].

Consider then the scaling field net $\underline{\mathfrak{F}}$, and fix a scaling limit state $\underline{\omega}_0 \in SL_\mathfrak{F}$, with corresponding scaling limit field net $(\pi_{\mathfrak{F}^0}, \mathfrak{F}^0, V_0, k, U_0)$ and scaling limit observable net $(\mathscr{H}_0, \mathfrak{A}_0, U_0, \Omega_0)$. In general, a quasi-orbit $\lambda \to \psi(\lambda)$ will not be an element of $\underline{\mathfrak{F}}(\mathscr{O})$, as (3.11) does not imply norm continuity, uniform in λ, of $s \to \alpha_{s_\lambda}(\psi(\lambda))$. We proceed then to a *smeared quasi-orbit*[b)]

$$\underline{\alpha}_h\psi(\lambda) := \int_{\tilde{\mathscr{P}}_+^\uparrow} ds\, h(s)\alpha_{s_\lambda}(\psi(\lambda)), \qquad \lambda > 0, \tag{3.12}$$

where $h \in L^1(\tilde{\mathscr{P}}_+^\uparrow)$, ds denotes left Haar measure on $\tilde{\mathscr{P}}_+^\uparrow$, and the integral is understood in the weak sense.

Lemma 3.12. *If $\lambda \to \psi(\lambda) \in H_{\rho_\lambda}$ is bounded, $\rho_\lambda \in \Delta_c(\lambda\mathscr{O}_1)$ ultraviolet stable, and $h \in C_c(\tilde{\mathscr{P}}_+^\uparrow)$ (continuous functions of compact support), then $\underline{\alpha}_h\psi$ is in $\underline{\mathfrak{F}}(\mathscr{O})$ for any $\mathscr{O}$ containing* $\operatorname{supp} h\cdot\mathscr{O}_1 := \bigcup_{s\in\operatorname{supp} h} s\cdot\mathscr{O}_1$.

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b) The relevance of these objects was pointed out to me by R. Verch. See also [DV], where they are called *lifted scaled multiplets*.

Proof. It is clear that $\underline{\alpha}_h\psi(\lambda)$ is a well defined bounded operator, with $\|\underline{\alpha}_h\psi(\lambda)\| \leq \|h\|_1\|\psi(\lambda)\|$, so that, being $\lambda \to \psi(\lambda)$ bounded, the same is true for $\lambda \to \underline{\alpha}_h\psi(\lambda)$. Let $B \in \mathscr{F}_s(\lambda\mathscr{O})'$ and $\Phi \in \mathscr{H}_\mathscr{F}$. Then

$$\begin{aligned}(\Phi|\underline{\alpha}_h\psi(\lambda)B\Phi) &= \int ds\, h(s)(\Phi|\alpha_{s_\lambda}(\psi(\lambda))B\Phi) \\ &= \int ds\, h(s)(\Phi|B\alpha_{s_\lambda}(\psi(\lambda))\Phi) = (\Phi|B\underline{\alpha}_h\psi(\lambda)\Phi),\end{aligned}$$

where the one but to last equality follows from the fact that $\alpha_{s_\lambda}(\psi(\lambda)) \in \mathscr{F}_s\big(\lambda(s\cdot\mathscr{O}_1)\big)$, and $s\cdot\mathscr{O}_1 \subseteq \mathscr{O}$. Then $\underline{\alpha}_h\psi(\lambda) \in \mathscr{F}_s(\lambda\mathscr{O})$. With ${}_sh(t) := h(s^{-1}t)$, we have

$$\|\alpha_{s_\lambda}(\underline{\alpha}_h\psi(\lambda)) - \underline{\alpha}_h\psi(\lambda)\| = \|\underline{\alpha}_{{}_sh}\psi(\lambda) - \underline{\alpha}_h\psi(\lambda)\| \leq \|{}_sh - h\|_1\|\psi(\lambda)\|,$$

which, as left translations are continuous on L^1 [Loo53] and $\lambda \to \psi(\lambda)$ is bounded, implies $\|\underline{\alpha}_s(\underline{\alpha}_h\psi) - \underline{\alpha}_h\psi\| \to 0$ for $s \to e$. Finally, let u_{ρ_λ} be the representation of G induced by β on H_{ρ_λ}. Then, since

$$\beta_g(\underline{\alpha}_h\psi(\lambda)) = \int ds\, h(s)\alpha_{s_\lambda}\big(\beta_g(\psi(\lambda))\big) = \int ds\, h(s)\alpha_{s_\lambda}\big(u_{\rho_\lambda}(g)\psi(\lambda)\big),$$

we have

$$\begin{aligned}\|\underline{\beta}_g(\underline{\alpha}_h\psi)(\lambda) - \underline{\alpha}_h\psi(\lambda)\| &\leq \|h\|_1\|u_{\rho_\lambda}(g) - 1\|_{B(H_{\rho_\lambda})}\|\psi(\lambda)\| \\ &= \|h\|_1\|u_{\rho_1}(g) - 1\|_{B(H_{\rho_1})}\|\psi(\lambda)\|\end{aligned}$$

the last equality holding by unitary equivalence of u_{ρ_λ} for different λ. Being these representations continuous, we have continuity of $g \to \underline{\beta}_g(\underline{\alpha}_h\psi)$ and $\underline{\alpha}_h\psi \in \underline{\mathfrak{F}}(\mathscr{O})$. □

We are going to see that actually the smearing function h can be removed in the scaling limit. We think of the space I_δ of non-negative functions $h \in C_c(\tilde{\mathscr{P}}_+^\uparrow)$ such that $\int_{\tilde{\mathscr{P}}_+^\uparrow} h = 1$ and $e \in \operatorname{supp} h$, as a net ordered by $h_1 \succeq h_2$ if $\operatorname{supp} h_1 \subseteq \operatorname{supp} h_2$, and we will write $h \to \delta$ to denote limit along this net. By [Loo53, thm 31E] it follows that, for any $f \in L^1(\tilde{\mathscr{P}}_+^\uparrow)$, $f * h \to f$ and $h * f \to f$ in L^1 as $h \to \delta$. We have also, for any $\Phi \in \mathscr{H}_\mathscr{F}$,

$$\begin{aligned}\|\underline{\alpha}_h\psi(\lambda)\Phi - \psi(\lambda)\Phi\| &\leq \int_{\tilde{\mathscr{P}}_+^\uparrow} ds\, h(s)\|\alpha_{s_\lambda}(\psi(\lambda))\Phi - \psi(\lambda)\Phi\| \\ &\leq \sup_{s\in\operatorname{supp} h} \|\alpha_{s_\lambda}(\psi(\lambda))\Phi - \psi(\lambda)\Phi\|,\end{aligned} \tag{3.13}$$

and the analogous estimate with $\psi(\lambda)^*$ replacing $\psi(\lambda)$, from which, by strong continuity of $s \to \alpha_s(F)$, $F \in \mathscr{F}$, we get $\underline{\alpha}_h\psi(\lambda) \xrightarrow{*s} \psi(\lambda)$ for any fixed $\lambda > 0$. Also note that, since $\tilde{\mathscr{P}}_+^\uparrow$ is a Lie group, and hence in particular a metric space, there exist sequences $(h_n)_{n\in\mathbb{N}}$ which are subnets of this net, since it is sufficient to take $\operatorname{supp} h_n \subseteq \mathscr{N}_n$, with $(\mathscr{N}_n)_{n\in\mathbb{N}}$ a monotonically decreasing basis of neighbourhoods of e in $\tilde{\mathscr{P}}_+^\uparrow$. We will write $h_n \to \delta$ in this case.

Lemma 3.13. *With the notations above, there exists the *strong limit*

$$\psi^0 := {}^*s\text{-}\lim_{h\to\delta} \pi^0(\underline{\alpha}_h\psi) \in \mathfrak{F}^0(\mathscr{O})^-$$

Proof. As there exist sequences $(h_n)_{n\in\mathbb{N}}$ such that $h_n \to \delta$, Ω_0 is separating for $\mathfrak{F}^0(\mathscr{O})^-$ and the norms $\|\pi^0(\underline{\alpha}_h\psi)\|$ are uniformly bounded for $h \in I_\delta$, it is sufficient to show that for any $\varepsilon > 0$ there exists $h_\varepsilon \in I_\delta$ such that for any $h, g \succeq h_\varepsilon$,

$$\|\pi^0(\underline{\alpha}_h\psi)\Omega_0 - \pi^0(\underline{\alpha}_g\psi)\Omega_0\| < \varepsilon, \qquad \|\pi^0(\underline{\alpha}_h\psi)^*\Omega_0 - \pi^0(\underline{\alpha}_g\psi)^*\Omega_0\| < \varepsilon.$$

Clearly, it is enough to prove the first inequality, the second one being proven in a completely analogus way. For any $\lambda \in (0,1]$ we have, using $\int_{\tilde{\mathscr{P}}_+^\uparrow} h = \int_{\tilde{\mathscr{P}}_+^\uparrow} g = 1$,

$$\begin{aligned}
&\|[\underline{\alpha}_h\psi(\lambda) - \underline{\alpha}_g\psi(\lambda)]\Omega\| \leq \\
&\qquad \leq \int_{\tilde{\mathscr{P}}_+^\uparrow} ds\, h(s)\|[\alpha_{s_\lambda}(\psi(\lambda)) - \psi(\lambda)]\Omega\| + \int_{\tilde{\mathscr{P}}_+^\uparrow} ds\, g(s)\|[\alpha_{s_\lambda}(\psi(\lambda)) - \psi(\lambda)]\Omega\| \\
&\qquad \leq \sup_{s\in\operatorname{supp} h}\ \sup_{\lambda\in(0,1]} \|[\alpha_{s_\lambda}(\psi(\lambda)) - \psi(\lambda)]\Omega\| + \sup_{s\in\operatorname{supp} g}\ \sup_{\lambda\in(0,1]} \|[\alpha_{s_\lambda}(\psi(\lambda)) - \psi(\lambda)]\Omega\|,
\end{aligned}$$

and then, for a suitable sequence $(\lambda_k)_{k\in\mathbb{N}}$ converging to zero,

$$\begin{aligned}
&\|\pi^0(\underline{\alpha}_h\psi)\Omega_0 - \pi^0(\underline{\alpha}_g\psi)\Omega_0\| = \lim_{k\to+\infty} \|[\underline{\alpha}_h\psi(\lambda_k) - \underline{\alpha}_g\psi(\lambda_k)]\Omega\| \\
&\qquad \leq \sup_{s\in\operatorname{supp} h}\ \sup_{\lambda\in(0,1]} \|[\alpha_{s_\lambda}(\psi(\lambda)) - \psi(\lambda)]\Omega\| + \sup_{s\in\operatorname{supp} g}\ \sup_{\lambda\in(0,1]} \|[\alpha_{s_\lambda}(\psi(\lambda)) - \psi(\lambda)]\Omega\|,
\end{aligned}$$

and since, thanks to (3.11), we can find a neighbourhood $\mathscr{N}_\varepsilon$ of the identity in $\tilde{\mathscr{P}}_+^\uparrow$ such that $\sup_{\lambda\in(0,1]}\|[\alpha_{s_\lambda}(\psi(\lambda)) - \psi(\lambda)]\Omega\| < \frac{\varepsilon}{2}$ for any $s \in \mathscr{N}_\varepsilon$, we conclude by taking $\operatorname{supp} h_\varepsilon \subseteq \mathscr{N}_\varepsilon$. □

We shall use the notations $\mathscr{A}_0(\mathscr{O}) := \mathfrak{A}_0(\mathscr{O})^-$, $\mathscr{F}^0(\mathscr{O}) := \mathfrak{F}^0(\mathscr{O})^-$ and $\pi_{\mathscr{F}^0} := \overline{\pi}_{\mathfrak{F}^0}$ will be the extension of $\pi_{\mathfrak{F}^0}$ to $\mathscr{A}_0$ given by lemma 3.3. It is easy to verify that $(\pi_{\mathscr{F}^0}, \mathscr{F}^0, V_0, k, U_0)$ is still a Poincaré covariant normal field net over $(\mathscr{H}_0, \mathscr{A}_0, U_0, \Omega_0)$: the only thing that really needs a check is that $\mathscr{F}^0(\mathscr{O})^G \subseteq \pi_{\mathscr{F}^0}(\mathscr{A}_0(\mathscr{O}))$ (the reverse inclusion being trivial); let $F \in \mathscr{F}^0(\mathscr{O})^G$ and $(F_\iota)_{\iota\in I}$ be a norm bounded net in $\mathfrak{F}^0(\mathscr{O})$ converging strongly to F, which exists by Kaplanski theorem. Then by

$$\|m(F_\iota)\Omega_0 - F\Omega_0\| = \left\| \int_G dg\, \beta_g^0(F_\iota - F)\Omega_0 \right\| \leq \|(F_\iota - F)\Omega_0\|,$$

$m(F_\iota) \in \pi_{\mathfrak{F}^0}(\mathfrak{A}_0(\mathscr{O}))$, being bounded, converges strongly to F, and then $F \in \pi_{\mathscr{F}^0}(\mathscr{A}_0(\mathscr{O}))$.

Theorem 3.14. *Let ξ be an ultraviolet stable covariant sector. For any double cone $\mathscr{O}$ there is a finite dimensional Hilbert space H_ρ in $\mathscr{F}^0(\mathscr{O})$ of support $\mathbb{1}$ carrying a G representation of class ξ, and implementing a transportable irreducible endomorphism ρ of $\mathscr{A}_0$ localized in $\mathscr{O}$, which is covariant with positive energy.*

Proof. Fix a double cone $\mathscr{O}_1$ whose closure is contained in $\mathscr{O}$, let $\rho_\lambda \in \Delta_c(\lambda\mathscr{O}_1)$, $\lambda > 0$, be as in definition 3.10, and choose, for each $\lambda > 0$, an orthonormal basis $\psi_j(\lambda)$, $j = 1, \dots, d := d(\xi)$, of H_{ρ_λ}, transforming under a fixed, λ-independent, irreducible matrix representation u_ξ of G of class ξ,

$$\psi_j(\lambda)^*\psi_i(\lambda) = \delta_{ji}\mathbb{1}, \qquad \sum_{j=1}^d \psi_j(\lambda)\psi_j(\lambda)^* = \mathbb{1}, \qquad \beta_g(\psi_j(\lambda)) = \sum_{i=1}^d u_\xi(g)_{ij}\psi_i(\lambda)$$

(e.g., it could be $\psi_j(\lambda) = \pi_{\mathscr{F}}(V_\lambda)\psi(1)$, for a unitary $V_\lambda \in (\rho_\lambda : \rho_1)$). If ψ_j^0 is as in lemma 3.13, we show that ψ_j^0, $j = 1,\dots,d$ is a multiplet of class ξ of orthogonal isometries of support $\mathbb{1}$ in $\mathscr{F}^0(\mathscr{O})$ (and then in particular ψ_j^0 is not a multiple of $\mathbb{1}$). To this end, it is clearly sufficient, by cyclicity of the vacuum for $\mathfrak{F}^0(\mathscr{O}')$, to show that, for any $\underline{F} \in \underline{\mathfrak{F}}(\mathscr{O}')$,

$$\omega_0(\pi^0(\underline{F})\psi_j^{0*}\psi_i^0) = \delta_{ji}\omega_0(\pi^0(\underline{F})), \tag{3.14}$$

$$\sum_{j=1}^{d} \omega_0(\pi^0(\underline{F})\psi_j^0\psi_j^{0*}) = \omega_0(\pi^0(\underline{F})). \tag{3.15}$$

We begin by proving (3.14). We can assume that $\underline{F}$ is a bosonic element, for if it is fermionic, since ψ_j^0 has a defined Bose-Fermi parity (it has the same parity as $\psi_j(\lambda)$, and then as ξ, as is easily verified), and ω_0 is even, both sides of (3.14) are zero, and the equality is trivially satisfied. Then we have, for $h_n \to \delta$, and for sufficiently big n, using normal commutation relations,

$$\begin{aligned}
&\left|\big(\Omega\big|\underline{F}(\lambda)\underline{\alpha}_{h_n}\psi_j^*(\lambda)\underline{\alpha}_{h_n}\psi_i(\lambda)\Omega\big) - \delta_{ji}(\Omega|\underline{F}(\lambda)\Omega)\right| \\
&\quad = \Big|\int_{\tilde{\mathscr{P}}_+^\uparrow} ds \int_{\tilde{\mathscr{P}}_+^\uparrow} dt\, h_n(s)h_n(t)\, \big[\big(\Omega\big|\alpha_{s_\lambda}(\psi_j(\lambda)^*)\underline{F}(\lambda)\alpha_{t_\lambda}(\psi_i(\lambda))\Omega\big) \\
&\qquad - \big(\Omega\big|\psi_j(\lambda)^*\underline{F}(\lambda)\psi_i(\lambda)\Omega\big)\big]\Big| \\
&\quad \le \sup_{s,t\in\operatorname{supp} h_n} \left|\big(\alpha_{s_\lambda}(\psi_j(\lambda))\Omega\big|\underline{F}(\lambda)\alpha_{t_\lambda}(\psi_i(\lambda))\Omega\big) - \big(\psi_j(\lambda)\Omega\big|\underline{F}(\lambda)\psi_i(\lambda)\Omega\big)\right| \\
&\quad \le \|\underline{F}\| \sup_{s\in\operatorname{supp} h_n} \sup_{\lambda\in(0,1]} \left\{\|\alpha_{s_\lambda}(\psi_i(\lambda))\Omega - \psi_i(\lambda)\Omega\| + \|\alpha_{s_\lambda}(\psi_j(\lambda))\Omega - \psi_j(\lambda)\Omega\|\right\},
\end{aligned}$$

so that, in view of (3.11), if $(\lambda_\iota)_{\iota\in I}$ is a net such that $\underline{\omega}_{\lambda_\iota} \to \underline{\omega}_0$, the limit

$$\lim_{n\to+\infty} \big(\Omega\big|\underline{F}(\lambda_\iota)\underline{\alpha}_{h_n}\psi_j^*(\lambda_\iota)\underline{\alpha}_{h_n}\psi_i(\lambda_\iota)\Omega\big) = \delta_{ji}(\Omega|\underline{F}(\lambda_\iota)\Omega),$$

following from $\underline{\alpha}_{h_n}\psi_j(\lambda_\iota) \xrightarrow{*s} \psi_j(\lambda_\iota)$, is uniform in ι, and then it is possible to interchange the limits and get

$$\begin{aligned}
\omega_0(\pi_0(\underline{F})\psi_j^{0*}\psi_i^0) &= \lim_{n\to+\infty}\lim_{\iota\in I} \big(\Omega\big|\underline{F}(\lambda_\iota)\underline{\alpha}_{h_n}\psi_j^*(\lambda_\iota)\underline{\alpha}_{h_n}\psi_i(\lambda_\iota)\Omega\big) \\
&= \lim_{\iota\in I}\lim_{n\to+\infty} \big(\Omega\big|\underline{F}(\lambda_\iota)\underline{\alpha}_{h_n}\psi_j^*(\lambda_\iota)\underline{\alpha}_{h_n}\psi_i(\lambda_\iota)\Omega\big) \\
&= \lim_{\iota\in I}\delta_{ji}(\Omega|\underline{F}(\lambda_\iota)\Omega) = \delta_{ji}\omega_0(\pi_0(\underline{F})).
\end{aligned}$$

The proof of (3.15) is completely analogous.

Then the linear span of ψ_j^0, $j = 1,\dots,d$ is a d-dimensional Hilbert space H_ρ of support $\mathbb{1}$ in $\mathscr{F}^0(\mathscr{O})$, and by

$$\beta_g^0(\psi_j^0) = {}^*s\text{-}\lim_{n\to+\infty}\pi^0\big(\underline{\beta}_g(\underline{\alpha}_{h_n}\psi_j)\big) = {}^*s\text{-}\lim_{n\to+\infty}\sum_{i=1}^{d} u_\xi(g)_{ij}\pi^0(\underline{\alpha}_{h_n}\psi_i) = \sum_{i=1}^{d} u_\xi(g)_{ij}\psi_i^0,$$

it carries a unitary representation of G of class ξ. The endomorphism ρ of $\mathscr{A}_0$ implemented by H_ρ is given, as usual, by

$$\pi_{\mathscr{F}^0}\rho(A) = \sum_{j=1}^{d} \psi_j^0\pi_{\mathscr{F}^0}(A)\psi_j^{0*}, \qquad A \in \mathscr{A}_0$$

which is well defined, since the left hand side is gauge invariant (ψ_j^{0*} transforms with $\bar{u}_\xi$) and $\pi_{\mathscr{F}^0}$ is an isomorphism. Clearly ρ is localized in $\mathscr{O}$. If $\mathscr{K}$ is another double cone, let $H_\sigma \subseteq \mathscr{F}^0(\mathscr{K})$ and $\sigma \in \mathrm{End}(\mathscr{A}_0)$ be constructed as H_ρ and ρ above, and φ_j^0, $j = 1,\dots,d$, be an orthonormal basis of H_σ. Then the operator $W \in \mathscr{A}_0$ defined by $\pi_{\mathscr{F}^0}(W) = \sum_{j=1}^d \varphi_j^0 \psi_j^{0*}$ is isometric,

$$\pi_{\mathscr{F}^0}(W^*W) = \sum_{i,j}^{1,d} \psi_i^0 \varphi_i^{0*} \varphi_j^0 \psi_j^{0*} = \sum_{j=1}^d \psi_j^0 \psi_j^{0*} = \mathbb{1},$$

and hence, interchanging the roles of the ψ_j^0's and φ_j^0's, unitary, and it intertwines ρ and σ

$$\begin{aligned}\pi_{\mathscr{F}^0}(W\rho(A)W^*) &= \sum_{i,j,k}^{1,d} \varphi_i^0 \psi_i^{0*} \psi_j^0 \pi_{\mathscr{F}^0}(A) \psi_j^{0*} \psi_k^0 \varphi_k^{0*} \\ &= \sum_{j=1}^d \varphi_j^0 \pi_{\mathscr{F}^0}(A) \varphi_j^{0*} = \pi_{\mathscr{F}^0}\sigma(A),\end{aligned}$$

so that ρ is transportable. Covariance of ρ is proved as follows. Define $W_\rho(s) \in \mathscr{A}_0$, $s \in \tilde{\mathscr{P}}_+^\uparrow$, by $\pi_{\mathscr{F}^0}(W_\rho(s)) = \sum_{j=1}^d \alpha_s^0(\psi_j^0)\psi_j^{0*}$; then since, as is easily seen, $\alpha_s^0(H_\rho)$ is a Hilbert space implementing $\alpha_s^0 \rho \alpha_{s^{-1}}^0$, $W_\rho(s) \in (\rho : \alpha_s^0 \rho \alpha_{s^{-1}}^0)$ is unitary, as above. Furthermore, W_ρ is a strongly continuous α^0-cocycle:

$$\begin{aligned}\pi_{\mathscr{F}^0}\left(\alpha_{s_1}^0\left(W_\rho(s_2)\right)W_\rho(s_1)\right) &= \alpha_{s_1}^0\left(\sum_{j=1}^d \alpha_{s_2}^0(\psi_j^0)\psi_j^{0*}\right)\sum_{i=1}^d \alpha_{s_1}^0(\psi_i^0)\psi_i^{0*} \\ &= \sum_{i=1}^d \alpha_{s_1}^0\left(\sum_{j=1}^d \alpha_{s_2}^0(\psi_j^0)\psi_j^{0*}\psi_i^0\right)\psi_i^{0*} \\ &= \sum_{i=1}^d \alpha_{s_1 s_2}^0(\psi_i^0)\psi_i^{0*} = \pi_{\mathscr{F}^0}(W_\rho(s_1 s_2)).\end{aligned}$$

Define then $U_\rho(s) := W_\rho(s)^* U_0(s)$. That U_ρ is a strongly continuous representation of $\tilde{\mathscr{P}}_+^\uparrow$ follows easily from strong continuity and the cocycle property of W_ρ, while the intertwining property of W_ρ implies

$$U_\rho(s)\rho(A)U_\rho(s)^* = W_\rho(s)^*\alpha_s^0\rho(A)W_\rho(s) = \rho\alpha_s^0(A),$$

and ρ is covariant. Finally translations $x \to U_\rho(x)$ satisfy the spectrum condition thanks to

$$U_\rho(x) = W_\rho(x)^* U_0(x) = \sum_{j=1}^d \psi_j^0 U_0(x) \psi_j^{0*} \restriction \mathscr{H}_0.$$

It remains only to show that ρ is irreducible. We adapt standard arguments from [DHR69a, section 3]. These imply that, thanks to norm and σ-weak continuity of m, from $\pi_{\mathscr{F}^0}(\mathscr{A}_0(\mathscr{O})) = m(\mathscr{F}^0(\mathscr{O})) = \mathscr{F}^0(\mathscr{O}) \cap V_0(G)'$ it follows $\pi_{\mathscr{F}^0}(\mathscr{A}_0)^- = \mathscr{F}^{0-} \cap V_0(G)'$ $= V_0(G)'$, where the last equality uses irreducibility of $\mathscr{F}^0$, theorem 3.7. Consider now the closed subspace $\mathscr{H}_{\bar{\xi}}$ of $\mathscr{H}^0$ generated by the vectors $\psi_j^{0*}\Phi$, $j = 1,\dots,d$, $\Phi \in \mathscr{H}_0$. This subspace is isomorphic to $\mathbb{C}^d \otimes \mathscr{H}_0$, for

$$(\psi_j^{0*}\Phi_1|\psi_i^{0*}\Phi_2) = (\Phi_1|\beta_g^0(\psi_j^0\psi_i^{0*})\Phi_2) = \sum_{h,k}^{1,d} \bar{u}_\xi(g)_{jk}(\psi_k^{0*}\Phi_1|\psi_h^{0*}\Phi_2)u_\xi(g)_{hi}$$

implies, by irreducibility of u_ξ and Schur's lemma, $(\psi_j^{0*}\Phi_1|\psi_i^{0*}\Phi_2) = \frac{\delta_{ji}}{d}(\Phi_1|\Phi_2)$, so that the above isomorphism is defined by sending $\psi_j^{0*}\Phi \in \mathscr{H}_{\bar\xi}$ to $\frac{e_j}{\sqrt{d}} \otimes \Phi \in \mathbb{C}^d \otimes \mathscr{H}_0$, e_j, $j = 1,\dots,d$, being the canonical basis. We have also $V_0(g) \restriction \mathscr{H}_{\bar\xi} \cong \bar{u}_\xi(g) \otimes \mathbb{1}_{\mathscr{H}_0}$, i.e. $\bar{u}_\xi$ is a subrepresentation of V_0 with multiplicity at least $\dim \mathscr{H}_0$, and let then $E_{\bar\xi}$ be the central support, in $V_0(G)''$, of the projector on the subspace of one of the subrepresentations $\bar{u}_\xi$ of V_0. As in [DHR69a], from $\pi_{\mathscr{F}^0}(\mathscr{A}_0)'' = V_0(G)'$ we have $V_0(g) \restriction E_{\bar\xi}\mathscr{H}^0 \cong \bar{u}_\xi(g) \otimes \mathbb{1}_{\mathscr{H}'_{\bar\xi}}$, $\pi_{\mathscr{F}^0}(A) \restriction E_{\bar\xi}\mathscr{H}^0 \cong \mathbb{1}_{\mathbb{C}^d} \otimes \pi_\xi(A)$, for some multiplicity space $\mathscr{H}'_{\bar\xi}$ containing $\mathscr{H}_0$ as a subspace and carrying an irreducibile representation π_ξ of $\mathscr{A}_0$. But for $\Phi \in \mathscr{H}_0$,

$$\frac{e_j}{\sqrt{d}} \otimes \pi_\xi(A)\Phi \cong \pi_{\mathscr{F}^0}(A)\psi_j^{0*}\Phi = \psi_j^{0*}\rho(A)\Phi \cong \frac{e_j}{\sqrt{d}} \otimes \rho(A)\Phi,$$

and ρ is a subrepresentation of π_ξ, but this last one is irreducible, so that it coincides with ρ, which is irreducible as well. ☐

From the proof of the above theorem, we see that the unitary equivalence class of the endomorphism ρ depends only on the considered sector ξ, and not on the choices made, as, e.g., the family ρ_λ of endomorphisms of $\mathscr{A}$. We have then a well defined mapping from ultraviolet stable sectors to sectors in the scaling limit theory fixed by the chosen scaling limit state $\underline{\omega}_0 \in SL_{\mathfrak{F}}$, and, as discussed at the beginning of this chapter, it is then natural to regard as non-confined the sectors of the scaling limit theory which are obtained in this way. We obtain thus, for a theory having only localizable sectors, an intrinsic notion of confinement, by declaring a sector of the scaling limit theory confined if it is not in the range of the map just defined.

3.3 Ultraviolet stable topological charges

According to the understanding gained through the perturbatrive treatment of quantum field models, the main class of theories which are expected to exhibit the confinement phenomenon, at least in certain regimes, is the class of asymptotically free theories, for which the coupling constant is seen to raise with energy in the perturbative region. Therefore a confinement criterion, in order to be really useful as mean to decide if a given model describes confined charges or not, has to encompass this kind of theories. On the other hand, as was discussed in section 1.2.2, the charges described by non abelian gauge theories, which are the only ones that can be asymptotically free, will not in general be localizable in bounded spacelike regions. Rather, if these theories are purely massive (which is expected to be the case precisely if the massless coloured glouns are confined) such charges can be localized in spacelike cones. Because of these facts, a confinement criterion based on a notion of stability of localizable charges as the one established in the previous section, is certainly not suitable for application to these theories: it can well be that some charge, which is cone-like localizable at finite scales, becomes finitely localizable in the scaling limit (and this is exactly what is expected to happen, see below). Thus one should not be allowed to call confined such a charge, but this is what the mentioned criterion would suggest, since this charge could not be created by *localizable* field operator at finite scales. We see then that, in order to have a sufficiently general confinement criterion, a notion of ultraviolet stability is needed for cone-like localizable charges as well.

A moment's thought shows, however, that one cannot trivially generalize the constructions of the previous section, since those where based essentially on the phase space properties of renormalization group orbits of field operators, and in particular on the requirement that the considered charged states occupy a fixed volume of phase space in the limit of small scales. Thus, since spacelike cones have infinite volume, and, what's worse, are invariant under scaling trasformations (at least the ones with apex at the origin), it is not immediately clear how to implement the above mentioned phase space requirements on orbits of cone-like localizable fields. In this respect, the key observation that allows to generalize, at least to a certain extent, the previous discussion, is the following one: what emerges from the analysis of models, in particular in the lattice approximation (cfr., for instance, [FM83] and reference quoted there), is that, at least in asymptotically free theories, the Mandelstam string attached to a gauge charges becomes weaker and weaker at smaller scales, leaving a compactly localizable charge in the scaling limit. Then, a notion of phase space occupation can be recovered, at least in an asymptotic sense.

In the following, we will therefore consider a class of topological sectors whose behaviour at small scales is, in a sense that we will make precise, of the kind just discussed, and we will show that these sectors give rise, in a natural way, to a net of *localizable* fields in the scaling limit. At the present stage of our work, we are then able to prove that this net induces localizable charges of the scaling limit theory $\mathfrak{A}_0$ only if a technical assumption, which will be discussed below, is added. Work is in progress, however, to extend this result to the general setting.

As in the previous section, we consider a Poincaré covariant observable net $(\mathscr{H}, \mathfrak{A}, U, \Omega)$, satisfying hypotesis 2.1 and such that the associated net of von Neumann algebras $\mathscr{A}$ satisfies (1.11) and property B'. Correspondingly, we consider the unique complete normal Poincaré covariant extended field system with gauge symmetry $(\pi_{\mathscr{F}}, \mathscr{F}, V, k, U_{\mathscr{F}})$ determined by the cone-like localizable sectors of $\mathscr{A}$, theorem 1.14. We will also make essential use of the assumption that the field net $\mathscr{F}$ satisfies the condition of geometric modular action,[c)] where, as usual, for every wedge $\mathscr{W}$, $\mathscr{F}(\mathscr{W})$ is the C*-algebra generated by $\mathscr{F}(\mathscr{C})$, $\mathscr{C} \subset \mathscr{W}$. That the vacuum Ω is cyclic and separating for $\mathscr{F}(\mathscr{W})^-$, so that this last assumption is meaningful, comes as usual from the Reeh-Schlieder theorem, thanks to irreducibility of $\bigvee_{x \in \mathbb{R}^4} \mathscr{F}(\mathscr{C} + x)$, and from locality. The notations introduced in section 1.3 will be employed throughout.

We will say that a bounded function $\lambda \in \mathbb{R}_+^\times \to F(\lambda) \in \mathscr{F}$ is *asymptotically localized* in a bounded open region $\mathscr{O}$ if

$$\lim_{\lambda \to 0} \sup_{\underline{A} \in \underline{\mathfrak{A}}(\mathscr{O}')_1} \| [F(\lambda), \pi_{\mathscr{F}}(\underline{A}(\lambda))] \| = 0. \tag{3.16}$$

Definition 3.15. A covariant sector ξ of the underlying theory will be called *ultraviolet stable* if

(i) for every spacelike cone $\mathscr{C}$ and every double cone $\mathscr{O} \subset \mathscr{C}$ there exists a family $\rho_\lambda \in \Delta_c(\lambda\mathscr{C})$, ρ_λ of class ξ, $\lambda > 0$, such that every bounded function $\lambda \to \psi(\lambda) \in H_{\rho_\lambda}$

[c)] It is clear how to adapt definition 2.12 to the present context, but actually we will only make use of the adaptation of relation (2.26), in which the lift of $\Lambda_{\mathscr{W}_{1,+}}$ to $\tilde{\mathscr{P}}_+^\uparrow$ (which exists, cfr. [BW76]) appears.

is asymptotically localized in $\mathscr{O}$ and

$$\lim_{s\to e}\sup_{\lambda\in(0,1]}\|[\alpha_{s_\lambda}(\psi(\lambda))-\psi(\lambda)]\Omega\| = 0 = \lim_{s\to e}\sup_{\lambda\in(0,1]}\|[\alpha_{s_\lambda}(\psi(\lambda)^*)-\psi(\lambda)^*]\Omega\|;$$

(ii) for every choice of $\mathscr{C}$, $\mathscr{O}$ and $\lambda\to\psi(\lambda)\in H_{\rho_\lambda}$ as in (i), and for every other spacelike cone $\tilde{\mathscr{C}}$ containing $\mathscr{O}$ there is a choice of $\sigma_\lambda\in\Delta_c(\lambda\tilde{\mathscr{C}})$ of class ξ as in (i) and a quasi orbit $\lambda\to\varphi(\lambda)\in H_{\sigma_\lambda}$ asymptotically localized in $\mathscr{O}$ such that

$$\lim_{\lambda\to 0}\|[\psi(\lambda)-\varphi(\lambda)]\Omega\| = 0. \tag{3.17}$$

We will say that any function $\lambda\to\psi(\lambda)\in H_{\rho_\lambda}$ as in (i) is a renormalization group *topological quasi-orbit* localized in $\mathscr{C}$ and asymptotically localized in $\mathscr{O}$.

Some remarks about this definition are in order. Condition (i) expresses two requirements on the considered class of sectors. The first is a formalization of the physical behavoiur of the gauge strings in the scaling limit pointed out before: if the string becomes weak at small scales, its effect on measurements performed in the spacelike complement of a bounded region roughly around the tip of the cone should vanish in the limit, so that field operators carrying such charges should asymptotically commute with observables localized in the part of the cone outside some bounded region. These charges can then be associated, in this weak sense, to a family of bounded region shrinking to a point. The second requirement is then the phase space condition on the states created by the selected fields, which is familiar from the discussion in the previous section. For what concerns condition (ii), it formalizes the fact that the direction in which the string emanates is irrelevant, and then that for any choice of an orbit of fields with strings in a fixed direction, one can find an equivalent orbit with its string in any other fixed direction, such that the two create form the vacuum the same state in the scaling limit.

We note that if $\lambda\to\psi(\lambda)$ is asymptotically localized in $\mathscr{O}_1$ and if $h\in L^1(\tilde{\mathscr{P}}_+^\uparrow)$ has compact support, so that $\operatorname{supp} h\cdot\mathscr{O}_1$ is bounded, we have, for $\mathscr{O}\supseteq\operatorname{supp} h\cdot\mathscr{O}_1$ and for any $\underline{A}\in\underline{\mathfrak{A}}(\mathscr{O}')_1$,

$$\begin{aligned}\|[\underline{\alpha}_h\psi(\lambda),\pi_{\mathscr{F}}(\underline{A}(\lambda))]\| &\le \int_{\tilde{\mathscr{P}}_+^\uparrow} ds\,|h(s)|\,\|[\psi(\lambda),\pi_{\mathscr{F}}(\underline{\alpha}_{s^{-1}}(\underline{A})(\lambda))]\|\\ &\le \|h\|_1 \sup_{\underline{B}\in\underline{\mathfrak{A}}(\mathscr{O}_1')_1}\|[\psi(\lambda),\pi_{\mathscr{F}}(\underline{B}(\lambda))]\|,\end{aligned}$$

so that $\underline{\alpha}_h\psi$ is asymptotically localized in $\mathscr{O}$, and if $\psi(\lambda)\in\mathscr{F}(\lambda\mathscr{C}_1)$ for some spacelike cone $\mathscr{C}_1$ containing $\mathscr{O}_1$, $\operatorname{supp} h$ may be chosen so small that there is a spacelike cone $\mathscr{C}\supset\mathscr{O}$ such that $\underline{\alpha}_h\psi(\lambda)\in\mathscr{F}(\lambda\mathscr{C})$, proposition A.6 in appendix A. Furthermore it is immediate that if (3.17) is satisfied, then for any $h\in L^1(\tilde{\mathscr{P}}_+^\uparrow)$,

$$\lim_{\lambda\to 0}\|[\underline{\alpha}_h\psi(\lambda)-\underline{\alpha}_h\varphi(\lambda)]\Omega\| = 0.$$

Analogously to section 3.1, we can consider on the C*-algebra $B(\mathbb{R}_+^\times,\mathscr{F})$ of bounded functions of $\mathbb{R}_+^\times$ in $\mathscr{F}$, and automorphic actions $\underline{\alpha}$ and $\underline{\beta}$ of $\tilde{\mathscr{P}}_+^\uparrow$ and G respectively, defined as in (3.3), where we again assume $\beta^{(\lambda)}=\beta$ for each $\lambda>0$. Furthermore we get an homomorphism $\pi_{\underline{\tilde{\mathfrak{F}}}}:\underline{\mathfrak{A}}\to B(\mathbb{R}_+^\times,\mathscr{F})$ by $\pi_{\underline{\tilde{\mathfrak{F}}}}(\underline{A})(\lambda) := \pi_{\mathscr{F}}(\underline{A}(\lambda))$, $\underline{A}\in\underline{\mathfrak{A}}$, $\lambda>0$. Then we start by considering an auxiliary scaling algebra of conelike localized fields, associated to ultraviolet stable charges.

Definition 3.16. The *extended field scaling algebra* associated to the spacelike cone $\mathscr{C}$ is the C*-subalgebra $\underline{\tilde{\mathfrak{F}}}(\mathscr{C})$ of $B(\mathbb{R}_+^\times, \mathscr{F})$ generated by the algebras $\pi_{\underline{\tilde{\mathfrak{F}}}}(\underline{\mathfrak{A}}(\mathscr{O}))$ with $\mathscr{O} \subset \mathscr{C}$ and by all the functions $\underline{\alpha}_h\psi$ such that

(i) $\lambda \to \psi(\lambda)$ is a topological quasi-orbit localized in some $\mathscr{C}_1$ and asymptotically localized in a $\mathscr{O}_1 \subset \mathscr{C}_1$, associated to some ultraviolet stable sector ξ of $\mathscr{A}$;

(ii) $h \in L^1(\tilde{\mathscr{P}}_+^\uparrow)$;

(iii) $\underline{\alpha}_h\psi(\lambda) \in \mathscr{F}(\lambda\mathscr{C})$ for each $\lambda > 0$ and $\underline{\alpha}_h\psi$ is asymptotically localized in some $\mathscr{O} \subset \mathscr{C}$.

Every $\underline{\alpha}_h\psi$ satisfying conditions (i)-(iii) will be called a *(topological) lifted quasi-orbit* localized in $\mathscr{C}$ and asymptotically localized in $\mathscr{O}$.

Proposition 3.17. *The actions $\underline{\alpha}$ and $\underline{\beta}$ of $\tilde{\mathscr{P}}_+^\uparrow$ and G restrict to $\bigcup_{\mathscr{C}} \underline{\tilde{\mathfrak{F}}}(\mathscr{C})$, and $(\underline{\tilde{\mathfrak{F}}}, \underline{\beta}, k, \underline{\alpha})$ is a normal, Poincaré covariant extended field net, such that for any $\underline{F} \in \underline{\tilde{\mathfrak{F}}}(\mathscr{C})$, the functions $s \to \underline{\alpha}_s(\underline{F})$, $g \to \underline{\beta}_g(\underline{F})$ are norm continuous.*

Proof. It's an easy verification. As usual $\underline{\alpha}_s\pi_{\underline{\tilde{\mathfrak{F}}}} = \pi_{\underline{\tilde{\mathfrak{F}}}}\underline{\alpha}_{\eta(s)}$, so that $\underline{\alpha}_s(\pi_{\underline{\tilde{\mathfrak{F}}}}(\underline{\mathfrak{A}}(\mathscr{O}))) = \pi_{\underline{\tilde{\mathfrak{F}}}}(\underline{\mathfrak{A}}(s\cdot\mathscr{O}))$, and, if $\underline{\alpha}_h\psi$ is a lifted quasi-orbit localized in $\mathscr{C}$, $\underline{\alpha}_s(\underline{\alpha}_h\psi) = \underline{\alpha}_{_sh}\psi$ is a lifted quasi-orbit localized in $s\cdot\mathscr{C}$, so that $\underline{\alpha}_s(\underline{\tilde{\mathfrak{F}}}(\mathscr{C})) = \underline{\tilde{\mathfrak{F}}}(s\cdot\mathscr{C})$, $s \in \tilde{\mathscr{P}}_+^\uparrow$. Likewise, $\underline{\beta}_g(\underline{\alpha}_h\psi) = \underline{\alpha}_h\psi_g$, where $\psi_g(\lambda) := u_{\rho_\lambda}(g)\psi(\lambda) \in H_{\rho_\lambda}$ is still a topological quasi-orbit, and $\beta_g(\underline{\alpha}_h\psi(\lambda)) \in \mathscr{F}(\lambda\mathscr{C})$, $\|[\beta_g(\underline{\alpha}_h\psi(\lambda)), \pi_{\mathscr{F}}(\underline{A}(\lambda))]\| = \|[\underline{\alpha}_h\psi(\lambda), \pi_{\mathscr{F}}(\underline{A}(\lambda))]\|$, so that $\underline{\beta}_g(\underline{\alpha}_h\psi)$ is a lifted quasi-orbit localized in $\mathscr{C}$ and $\underline{\beta}_g(\underline{\tilde{\mathfrak{F}}}(\mathscr{C})) = \underline{\tilde{\mathfrak{F}}}(\mathscr{C})$. Furthermore, the arguments in the proof of lemma 3.12 apply here as well to show that $s \to \underline{\alpha}_s(\underline{\alpha}_h\psi)$, $g \to \underline{\beta}_g(\underline{\alpha}_h\psi)$ are norm continuous functions, and then this extends to any $\underline{F} \in \underline{\tilde{\mathfrak{F}}}(\mathscr{C})$. It remains to show that normal commutation relations hold. To this end, let $\underline{\tilde{\mathfrak{F}}}(\mathscr{C})_0$ denote the dense $*$-subalgebra of $\underline{\tilde{\mathfrak{F}}}(\mathscr{C})$ generated by $\pi_{\underline{\tilde{\mathfrak{F}}}}(\underline{\mathfrak{A}}(\mathscr{O}))$, $\mathscr{O} \subset \mathscr{C}$, and by lifted quasi-orbits localized in $\mathscr{C}$. It is immediate to verify that $\underline{\tilde{\mathfrak{F}}}(\mathscr{C})_\pm$ is the norm closure of $\underline{\tilde{\mathfrak{F}}}(\mathscr{C})_{0,\pm}$ (grading defined by $\underline{\gamma}$), and it is then sufficient to show that normal commutation relations hold for $\mathscr{C} \to \underline{\tilde{\mathfrak{F}}}(\mathscr{C})_0$. Since all generators have a definite parity (elements of $\pi_{\underline{\tilde{\mathfrak{F}}}}(\underline{\mathfrak{A}}(\mathscr{O}))$ are bosonic, and $\underline{\alpha}_h\psi$ and its adjoint have the same parity as the associated sector), all elements of $\underline{\tilde{\mathfrak{F}}}(\mathscr{C})_0$ are finite sums of monomials in the generators, each of which has a definite parity, the product of the parities of the factors, and furthermore any $\underline{F} \in \underline{\tilde{\mathfrak{F}}}(\mathscr{C})_{0,\pm}$ can be written as a sum of monomials with the same parity as $\underline{F}$ itself, for if $\underline{F} = \sum_i \underline{M}_i$, $\underline{\gamma}(\underline{M}_i) = (-1)^{\sigma_i}\underline{M}_i$, then also

$$\underline{F} = \frac{1}{2}\big(\underline{F} \pm \underline{\gamma}(\underline{F})\big) = \sum_i \frac{1}{2}(1 \pm (-1)^{\sigma_i})\underline{M}_i = \sum_{i\,:\,(-1)^{\sigma_i} = \pm 1} \underline{M}_i.$$

In order to conclude it is then sufficient to show that any two such monomials $\underline{M}_i \in \underline{\tilde{\mathfrak{F}}}(\mathscr{C}_i)$, $i = 1, 2$ with $\mathscr{C}_1$ and $\mathscr{C}_2$ spacelike separated, obey normal commutation relations, $\underline{M}_1\underline{M}_2 = (-1)^{\sigma_1\sigma_2}\underline{M}_2\underline{M}_1$, but this is easily established by direct computation of the required permutations of generators. □

Remark. In general, it is not true that $\underline{\tilde{\mathfrak{F}}}(\mathscr{C})^G$ coincides with the C*-algebra generated by $\pi_{\underline{\tilde{\mathfrak{F}}}}(\underline{\mathfrak{A}}(\mathscr{O}))$, $\mathscr{O} \subset \mathscr{C}$. We will comment later on a possible physical interpretation of this fact.

We now consider the lift $(\underline{\omega}_\lambda)_{\lambda>0}$ of the vacuum state ω to the C*-algebra $\underline{\tilde{\mathfrak{F}}}$ generated by all the $\underline{\tilde{\mathfrak{F}}}(\mathscr{C})$, and denote as usual by $SL_{\tilde{\mathfrak{F}}}$ the set of its weak* limit points. As in section 3.1, we have $SL_{\tilde{\mathfrak{F}}} \circ \pi_{\underline{\tilde{\mathfrak{F}}}} = SL_{\mathfrak{A}}$. Let then $(\pi^0, \mathscr{H}^0, \Omega_0)$ be the GNS representation determined by a fixed scaling limit state $\underline{\omega}_0 \in SL_{\tilde{\mathfrak{F}}}$, and define $\tilde{\mathfrak{F}}^0(\mathscr{C}) := \pi^0(\underline{\tilde{\mathfrak{F}}}(\mathscr{C}))$ for any spacelike cone

$\mathscr{C}$. As usual, by Poincaré and gauge invariance of $\underline{\omega}_0$ we get unitary strongly continuous representations U_0, V_0 of $\tilde{\mathscr{P}}_+^\uparrow$ and G on $\mathscr{H}^0$. Then, the relevant parts of the proof of theorem 3.7 can be easily adapted to the present context to show that $(\tilde{\mathfrak{F}}^0, V_0, k, U_0, \Omega_0)$ is a Poincaré covariant, normal extended field net with gauge symmetry. In particular, Poincaré invariance of Ω_0 and the spectrum condition are satisfied, though in general Ω_0 is not the only translation invariant unit vector, i.e. $\tilde{\mathfrak{F}}^0$ need not be irreducible. For a given wedge $\mathscr{W}$, we will denote by $\underline{\tilde{\mathfrak{F}}}(\mathscr{W})$, $\tilde{\mathfrak{F}}^0(\mathscr{W})$ the C*-algebras generated by the respective algebras $\underline{\tilde{\mathfrak{F}}}(\mathscr{C})$, $\tilde{\mathfrak{F}}^0(\mathscr{C})$, with $\mathscr{C} \subset \mathscr{W}$. It is evident that $\tilde{\mathfrak{F}}^0(\mathscr{W}) = \pi^0(\underline{\tilde{\mathfrak{F}}}(\mathscr{W}))$.

The following weak version of the Reeh-Schlieder theorem, which uses in an essential way analyticity of both translations and Lorentz boosts, consequence of geometric modular action and Tomita-Takesaki theory, will be central in the following. Similar results can be found in [BB99, DSW86].

Theorem 3.18. *The vacuum Ω_0 is a cyclic and separating vector for the algebras $\tilde{\mathfrak{F}}^0(\mathscr{W})^-$.*

We need some preparations before proving the theorem. We recall that we denote by $\Lambda_\mathscr{W}(t) \in \mathscr{P}_+^\uparrow$, $t \in \mathbb{R}$, the one parameter group of Poincaré transformations leaving the wedge $\mathscr{W}$ invariant, $\Lambda_\mathscr{W}(t) = s\Lambda_{\mathscr{W}_{1,+}}(t)s^{-1}$ if $\mathscr{W} = s \cdot \mathscr{W}_{1,+}$. By abuse of notation, we will identify $\Lambda_\mathscr{W}(t)$ with its unique smooth lift to $\tilde{\mathscr{P}}_+^\uparrow$ which is the identity for $t = 0$. Also, we will identify $\Lambda \in SL(2,\mathbb{C})$ with $(\Lambda, 0) \in \tilde{\mathscr{P}}_+^\uparrow$.

Lemma 3.19. *Let U_0 be a strongly continuous unitary representation of $\tilde{\mathscr{P}}_+^\uparrow$, and $\mathscr{N} \subseteq \tilde{\mathscr{P}}_+^\uparrow$ an open neighbourhood of the identity. Then $U_0(\tilde{\mathscr{P}}_+^\uparrow)$ is the strong closure of the group $U_\mathscr{N}$ generated by the elements $U_0(s\Lambda_\mathscr{W}(t)s^{-1})$, $t \in \mathbb{R}$, $s \in \mathscr{N}$.*

Proof. Thanks to Poincaré invariance of the problem, we can assume that $\mathscr{W} = \mathscr{W}_{1,+}$. Also, we can assume that $\mathscr{N} = \mathscr{N}_1 \times \mathscr{N}_2 \subseteq SL(2,\mathbb{C}) \times \mathbb{R}^4$ is a rectangular open neighbourhood of the identity. Then by [BB99, lemma 2.1] $U_0(SL(2,\mathbb{C}))$ is the strong closure of the subgroup of $U_\mathscr{N}$ generated by $U_0(\Lambda\Lambda_{\mathscr{W}_{1,+}}(t)\Lambda^{-1})$, $t \in \mathbb{R}$, $\Lambda \in \mathscr{N}_1$,[d)] so that, since $U_0(\Lambda, x) = U_0(x)U_0(\Lambda)$, $\Lambda \in SL(2,\mathbb{C})$, $x \in \mathbb{R}^4$, it is sufficient to show that $U_0(x) \in U_\mathscr{N}^-$. Furthermore, as $U_0((\Lambda,x)\Lambda_{\mathscr{W}_{1,+}}(t)(\Lambda,x)^{-1})U_0(\Lambda\Lambda_{\mathscr{W}_{1,+}}(t)\Lambda^{-1})^* = U_0(\Lambda(\mathbb{1} - \Lambda_{\mathscr{W}_{1,+}}(t))\Lambda^{-1}x)$, and $U_0(x) = U_0(x/n)^n$, $x \in \mathbb{R}^4$, $n \in \mathbb{N}$, we reduce the problem to showing that the set $\mathscr{U} := \{\sum_i \Lambda_i(\mathbb{1} - \Lambda_{\mathscr{W}_{1,+}}(t_i))\Lambda_i^{-1}x_i \,:\, (\Lambda_i, x_i) \in \mathscr{N},\, t_i \in \mathbb{R}\}$ is a neighbourhood of zero in $\mathbb{R}^4$. Then with e_μ, $\mu = 0, \dots, 3$ the canonical basis of $\mathbb{R}^4$ and $e_\pm := e_1 \pm e_0$, $(\mathbb{1} - \Lambda_{\mathscr{W}_{1,+}}(t))e_\pm = (1 - e^{\pm t})e_\pm$ so that there is an $\varepsilon > 0$ such that $se_\pm \in \mathscr{U}$ for $|s| < \varepsilon$. Also, if R_φ denotes the rotation around the e_3 axis of an angle φ, and $\xi := (1 - e^t)R_\varphi e_- + (1 - e^{-t})R_\varphi e_+$, then

$$R_\varphi(\mathbb{1} - \Lambda_{\mathscr{W}_{1,+}}(t))R_\varphi^{-1}\xi = 4(1 - \cosh t)(\cos\varphi\, e_1 - \sin\varphi\, e_2),$$

so that, for $u \in \mathbb{R}$,

$$R_\varphi(\mathbb{1} - \Lambda_{\mathscr{W}_{1,+}}(t))R_\varphi^{-1}(u\xi) + R_{-\varphi}(\mathbb{1} - \Lambda_{\mathscr{W}_{1,+}}(t))R_{-\varphi}^{-1}(-u\xi) = -8u(1 - \cosh t)\sin\varphi\, e_2$$

and then, since $(R_\varphi, u\xi) \in \mathscr{N}$ for $|u|$, $|\varphi|$ sufficiently small, $se_2 \in \mathscr{U}$ for $|s| < \varepsilon$. Analogously we show that $se_3 \in \mathscr{U}$, $|s| < \varepsilon$, and since (e_+, e_-, e_2, e_3) is a basis of $\mathbb{R}^4$, $\{\sum_{\alpha=\pm,2,3} s_\alpha e_\alpha \,:\, |s_\alpha| < \varepsilon\}$ is a neighbourhood of 0 contained in $\mathscr{U}$. □

[d)] the cited results refers actually to representations of $SO^\uparrow(1,3)$, but since the proof uses only properties of its Lie algebra, it can be also applied to the present case

Lemma 3.20. *The state* $\omega_0 = (\Omega_0|(\cdot)\Omega_0)$ *is a* 2π*-KMS state for the* C^**-dynamical system* $(\tilde{\mathfrak{F}}^0(\mathscr{W}), \alpha^0_{\Lambda_{\mathscr{W}}})$.

Proof. As above, we may assume $\mathscr{W} = \mathscr{W}_{1,+}$. By geometric modular action and Tomita-Takesaki theory, the underlying vacuum state ω is 2π-KMS for the C^*-dynamical system $(\mathscr{F}(\mathscr{W}), \alpha_{\Lambda_{\mathscr{W}}})$. This implies that any of the scaled vacuum states $\underline{\omega}_\lambda$, $\lambda > 0$, is 2π-KMS for $(\underline{\tilde{\mathfrak{F}}}(\mathscr{W}), \underline{\alpha}_{\Lambda_{\mathscr{W}}})$: for $\underline{F}, \underline{G} \in \underline{\tilde{\mathfrak{F}}}(\mathscr{W})$ let $F_\lambda(t) := \underline{\omega}_\lambda(\underline{F\alpha}_{\Lambda_{\mathscr{W}}(t)}(\underline{G})) = \omega\big(\underline{F}(\lambda)\alpha_{\Lambda_{\mathscr{W}}(t)}(\underline{G}(\lambda))\big)$, $t \in \mathbb{R}$, then, being $\underline{F}(\lambda), \underline{G}(\lambda) \in \mathscr{F}(\mathscr{W})$, F_λ is analytic in the open strip $\{0 < \operatorname{Im} z < 2\pi\}$ and continuous and bounded in the closed strip, and $F_\lambda(t + 2\pi i) = \omega(\alpha_{\Lambda_{\mathscr{W}}(t)}(\underline{G}(\lambda))\underline{F}(\lambda)) = \underline{\omega}_\lambda(\underline{\alpha}_{\Lambda_{\mathscr{W}}(t)}(\underline{G})\underline{F})$, so that $\underline{\omega}_\lambda$ is KMS. Then being the set of KMS states *weakly closed [BR79b, thm. 5.3.30], $\underline{\omega}_0$ is 2π-KMS for $(\underline{\tilde{\mathfrak{F}}}(\mathscr{W}), \underline{\alpha}_{\Lambda_{\mathscr{W}}})$, and finally, considering the function $F_0(t) := \omega_0\big(\pi^0(\underline{F})\alpha^0_{\Lambda_{\mathscr{W}}(t)}(\pi^0(\underline{G}))\big) = \underline{\omega}_0(\underline{F\alpha}_{\Lambda_{\mathscr{W}}(t)}(\underline{G}))$, we conclude the proof. □

For any two spacetime regions $\mathscr{R}_1$, $\mathscr{R}_2$, we will use the notation $\mathscr{R}_1 \Subset \mathscr{R}_2$ to mean that there exists a neighbourhood $\mathscr{N}$ of the identity in $\mathscr{P}^\uparrow_+$ such that $\mathscr{N} \cdot \mathscr{R}_1 \subseteq \mathscr{R}_2$. For any finite set of spacelike cones $\mathscr{C}_1, \ldots, \mathscr{C}_n$, we introduce the C^*-algebra $\tilde{\mathfrak{F}}^0(\mathscr{C}_1, \ldots, \mathscr{C}_n)$ as the one generated by the algebras $\tilde{\mathfrak{F}}^0(\mathscr{C}_1), \ldots, \tilde{\mathfrak{F}}^0(\mathscr{C}_n)$. We also define $\mathfrak{G}^0(\mathscr{C}_1, \ldots, \mathscr{C}_n)$ to be the set of operators $G \in \tilde{\mathfrak{F}}^0(\mathscr{C}_1, \ldots, \mathscr{C}_n)$ for which there exists a neighbourhood $\mathscr{N}$ of the identity in $\tilde{\mathscr{P}}^\uparrow_+$ such that $\alpha^0_s(G) \in \tilde{\mathfrak{F}}^0(\mathscr{C}_1, \ldots, \mathscr{C}_n)$ for any $s \in \mathscr{N}$. It is clear that $\mathfrak{G}^0(\mathscr{C}_1, \ldots, \mathscr{C}_n)$ is a $*$-algebra and that for any n-tuple $\tilde{\mathscr{C}}_1, \ldots, \tilde{\mathscr{C}}_n$ with $\tilde{\mathscr{C}}_i \Subset \mathscr{C}_i$, $i = 1, \ldots, n$, $\tilde{\mathfrak{F}}^0(\tilde{\mathscr{C}}_1, \ldots, \tilde{\mathscr{C}}_n) \subseteq \mathfrak{G}^0(\mathscr{C}_1, \ldots, \mathscr{C}_n)$.

Lemma 3.21. *Let* $\mathscr{W}$ *be a wedge in Minkowski space and let* $\mathscr{C}_i \Subset \mathscr{W}$, $i = 1, \ldots, n$ *be spacelike cones. If* $\Phi \in (\mathfrak{G}^0(\mathscr{C}_1, \ldots, \mathscr{C}_n)\Omega_0)^\perp$*, then*

$$(\Phi|\alpha^0_{s_1}(G_1)\ldots\alpha^0_{s_m}(G_m)\Omega_0) = 0 \tag{3.18}$$

for any $s_i \in \tilde{\mathscr{P}}^\uparrow_+$, $G_i \in \mathfrak{G}^0(\mathscr{C}_1, \ldots, \mathscr{C}_n)$, $i = 1, \ldots, m$.

Proof. We begin by showing that $\Phi \in (\mathfrak{G}^0(\mathscr{C}_1, \ldots, \mathscr{C}_n)\Omega_0)^\perp$ implies $U_0(s)\Phi \in (\mathfrak{G}^0(\mathscr{C}_1, \ldots, \mathscr{C}_n)\Omega_0)^\perp$ for any $s \in \tilde{\mathscr{P}}^\uparrow_+$. Let $\mathscr{N}$ be a neighbourhood of the identity in $\tilde{\mathscr{P}}^\uparrow_+$ such that $\mathscr{N}^{-1} \cdot \mathscr{C}_i \subset \mathscr{W}$, $i = 1, \ldots, n$, and let $G \in \mathfrak{G}^0(\mathscr{C}_1, \ldots, \mathscr{C}_n)$. Then $G \in \tilde{\mathfrak{F}}^0(s \cdot \mathscr{W})$ for any $s \in \mathscr{N}$. By continuity of $t \to s\Lambda_{\mathscr{W}}(t)s^{-1}$ there exists $\varepsilon > 0$, depending on $s \in \mathscr{N}$, such that $\alpha^0_{s\Lambda_{\mathscr{W}}(t)s^{-1}}(G) \in \mathfrak{G}^0(\mathscr{C}_1, \ldots, \mathscr{C}_n)$ for $|t| < \varepsilon$, and then $(\Phi|\alpha^0_{s\Lambda_{\mathscr{W}}(t)s^{-1}}(G)\Omega_0) = 0$ for $|t| < \varepsilon$. But $s\Lambda_{\mathscr{W}}(t)s^{-1} = \Lambda_{s\mathscr{W}}(t)$, and the fact that $\omega_0 = (\Omega_0|(\cdot)\Omega_0)$ is a 2π-KMS state for $(\tilde{\mathfrak{F}}^0(s \cdot \mathscr{W}), \alpha^0_{\Lambda_{s\mathscr{W}}})$ implies that [KR86]

$$\alpha^0_{\Lambda_{s\mathscr{W}}(t)}(G)\Omega_0 = \sigma^{\omega_0}_{-t/2\pi}(G)\Omega_0 = \Delta_{\omega_0}^{-\frac{it}{2\pi}} G\Omega_0,$$

where Δ_{ω_0} is the modular operator associated by Tomita-Takesaki theory to the restriction of $\tilde{\mathfrak{F}}^0(s \cdot \mathscr{W})^-$ to the cyclic subspace generated by its action on Ω_0. From this last equation it follows that $t \to \alpha^0_{s\Lambda_{\mathscr{W}}(t)s^{-1}}(G)\Omega_0$ has an analytic continuation to a function on the strip $\{0 < \operatorname{Im} z < \pi\}$, and then

$$(U_0(s\Lambda_{\mathscr{W}}(t)s^{-1})\Phi|G\Omega_0) = (\Phi|\alpha^0_{s\Lambda_{\mathscr{W}}(-t)s^{-1}}(G)\Omega_0) = 0, \qquad t \in \mathbb{R},\ s \in \mathscr{N},$$

i.e. $U_0(s\Lambda_{\mathscr{W}}(t)s^{-1})\Phi \in (\mathfrak{G}^0(\mathscr{C}_1,\dots,\mathscr{C}_n)\Omega_0)^\perp$ for any $t \in \mathbb{R}$, $s \in \mathscr{N}$. Then, iterating the argument,

$$U_0(s_1\Lambda_{\mathscr{W}}(t_1)s_1^{-1})\dots U_0(s_m\Lambda_{\mathscr{W}}(t_m)s_m^{-1})\Phi \in (\mathfrak{G}^0(\mathscr{C}_1,\dots,\mathscr{C}_n)\Omega_0)^\perp$$

for every choice of $s_i \in \mathscr{N}$, $t_i \in \mathbb{R}$, $i = 1,\dots,m$, and since, by lemma 3.19, $U_0(s)\Phi$, $s \in \tilde{\mathscr{P}}_+^\uparrow$, is a limit of vectors as the one in the left hand side of the last equation and $(\mathfrak{G}^0(\mathscr{C}_1,\dots,\mathscr{C}_n)\Omega_0)^\perp$ is closed, $U_0(s)\Phi \in (\mathfrak{G}^0(\mathscr{C}_1,\dots,\mathscr{C}_n)\Omega_0)^\perp$.

If we now show that, for any $G_i \in \mathfrak{G}^0(\mathscr{C}_1,\dots,\mathscr{C}_n)$, $s_i \in \tilde{\mathscr{P}}_+^\uparrow$, $i = 1,\dots,m$, $\alpha^0_{s_1}(G_1)\dots\alpha^0_{s_m}(G_m)\Phi \in (\mathfrak{G}^0(\mathscr{C}_1,\dots,\mathscr{C}_n)\Omega_0)^\perp$, since $\mathfrak{G}^0(\mathscr{C}_1,\dots,\mathscr{C}_n)$ is a $*$-algebra containing the identity operator, the conclusion of the lemma will follow. We prove this by induction on m. For $m = 1$ we have, by what we have just seen and by the fact that $\mathfrak{G}^0(\mathscr{C}_1,\dots,\mathscr{C}_n)$ is a $*$-algebra, $G_1U_0(s_1)^*\Phi \in (\mathfrak{G}^0(\mathscr{C}_1,\dots,\mathscr{C}_n)\Omega_0)^\perp$ and then, applying again the first part of the proof, $\alpha^0_{s_1}(G_1)\Phi \in (\mathfrak{G}^0(\mathscr{C}_1,\dots,\mathscr{C}_n)\Omega_0)^\perp$. Then, if $\alpha^0_{s_2}(G_1)\dots\alpha^0_{s_m}(G_m)\Phi \in (\mathfrak{G}^0(\mathscr{C}_1,\dots,\mathscr{C}_n)\Omega_0)^\perp$ the argument just made leads to the conclusion. □

Proof of theorem 3.18. It is sufficient to show that Ω_0 is cyclic for $\tilde{\mathfrak{F}}^0(\mathscr{W})$, i.e. $(\tilde{\mathfrak{F}}^0(\mathscr{W})\Omega_0)^\perp = \{0\}$: if this is true, the fact that Ω_0 is separating for $\tilde{\mathfrak{F}}^0(\mathscr{W})^-$ follows from the fact that the interior of $\mathscr{W}'$ is again a wedge and by normal commutation relations, as in the usual Reeh-Schlieder theorem. Let then $\Phi \in (\tilde{\mathfrak{F}}^0(\mathscr{W})\Omega_0)^\perp$ and $F_i \in \tilde{\mathfrak{F}}^0(\tilde{\mathscr{C}}_i)$, $i = 1,\dots,n$, be arbitrary operators. For any $i = 1,\dots,n$ there exists $s_i \in \tilde{\mathscr{P}}_+^\uparrow$ and a spacelike cone $\mathscr{C}_i$ such that $s_i^{-1}\cdot\tilde{\mathscr{C}}_i \Subset \mathscr{C}_i \Subset \mathscr{W}$. Then $\alpha^0_{s_i^{-1}}(F_i) \in \tilde{\mathfrak{F}}^0(s_i^{-1}\cdot\tilde{\mathscr{C}}_i) \subseteq \mathfrak{G}^0(\mathscr{C}_1,\dots,\mathscr{C}_n)$ and, being $(\tilde{\mathfrak{F}}^0(\mathscr{W})\Omega_0)^\perp \subseteq (\mathfrak{G}^0(\mathscr{C}_1,\dots,\mathscr{C}_n)\Omega_0)^\perp$,

$$(\Phi|F_1\dots F_n\Omega_0) = (\Phi|\alpha^0_{s_1}(\alpha^0_{s_1^{-1}}(F_1))\dots\alpha^0_{s_n}(\alpha^0_{s_n^{-1}}(F_n))\Omega_0) = 0$$

by lemma 3.21, thus Φ is orthogonal to a total set of vectors in $\mathscr{H}^0$, and then vanishes. □

We now introduce a new net of C*-algebras on the scaling limit Hilbert space $\mathscr{H}^0$, which will be associated to bounded regions instead of cones, and which we are willing to regard as the "true" field net determined by the scaling limit of ultraviolet stable topological charges.

Definition 3.22. The *localized scaling limit field algebra* associated to the double cone $\mathscr{O} \subset \mathbb{R}^4$ is the C*-algebra $\mathfrak{F}_0(\mathscr{O})$ generated by $\pi^0 \circ \pi_{\underline{\mathfrak{F}}}(\underline{\mathfrak{A}}(\mathscr{O}))$ and by all elements $\pi^0(\underline{\alpha}_h\psi)$ such that $\underline{\alpha}_h\psi$ is a lifted quasi-orbit aysmptotically localized in $\mathscr{O}$ and localized in some spacelike cone $\mathscr{C} \supset \mathscr{O}$.

At first sight, the adjective "localized" used in the above definition may seem inadequate, since a priori $\mathfrak{F}_0(\mathscr{O}) \subseteq \bigcup_{\mathscr{C}\supset\mathscr{O}}\tilde{\mathfrak{F}}^0(\mathscr{C})$, which is highly non-local, as the union of all such cones covers a complete spacelike hyperplane. We are going to see however, that, thanks to the physically motivated requirements imposed on the considered class of topological charges, the net $\mathfrak{F}_0$ enjoys much better localization properties.

Lemma 3.23. *With the above notations, we have*

$$\mathfrak{F}_0(\mathscr{O}) \subseteq \bigcap_{\mathscr{C}\supset\mathscr{O}}\tilde{\mathfrak{F}}^0(\mathscr{C}).$$

Proof. It is evident that $\pi^0(\underline{\mathfrak{A}}(\mathscr{O}))$ is contained in the intersection of the algebras $\tilde{\mathfrak{F}}^0(\mathscr{C})$, $\mathscr{C} \supset \mathscr{O}$. It is then sufficient to show that this is the case for any $\pi^0(\underline{\alpha}_h\psi)$ with $\underline{\alpha}_h\psi$ asymptotically localized in $\mathscr{O}$, irrespective of its spacelike cone of localization. Let then $\underline{\alpha}_h\psi$ be localized in some spacelike cone $\mathscr{C}$ containing $\mathscr{O}$ and of class ξ, and pick another spacelike cone $\tilde{\mathscr{C}} \supset \mathscr{O}$ such that there exist a wedge $\mathscr{W}$ which is spacelike to both $\mathscr{C}$ and $\tilde{\mathscr{C}}$. Correspondingly we can find, thanks to definition 3.15(ii) and the following remarks, another lifted quasi-orbit $\pi^0(\underline{\alpha}_h\varphi)$ associated to the same charge, asymptotically localized in $\mathscr{O}$ and localized in $\tilde{\mathscr{C}}$. Thus, for a suitable sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \mathbb{R}_+^\times$,

$$
\begin{aligned}
\|[\pi^0(\underline{\alpha}_h\psi) - \pi^0(\underline{\alpha}_h\varphi)]\Omega_0\|^2 &= \omega_0\big((\pi^0(\underline{\alpha}_h\psi) - \pi^0(\underline{\alpha}_h\varphi))^*(\pi^0(\underline{\alpha}_h\psi) - \pi^0(\underline{\alpha}_h\varphi))\big)\\
&= \lim_{k\to+\infty} \omega\big((\underline{\alpha}_h\psi(\lambda_k) - \underline{\alpha}_h\varphi(\lambda_k))^*(\underline{\alpha}_h\psi(\lambda_k) - \underline{\alpha}_h\varphi(\lambda_k))\big)\\
&= \lim_{k\to+\infty} \|[\underline{\alpha}_h\psi(\lambda_k) - \underline{\alpha}_h\varphi(\lambda_k)]\Omega\|^2 = 0,
\end{aligned}
$$

and then, for any $F \in \tilde{\mathfrak{F}}^0(\mathscr{W})$ of Bose-Fermi parity $(-1)^{\sigma(F)}$,

$$
\begin{aligned}
\pi^0(\underline{\alpha}_h\psi)F\Omega_0 &= (-1)^{\sigma(F)\sigma(\xi)}F\pi^0(\underline{\alpha}_h\psi)\Omega_0\\
&= (-1)^{\sigma(F)\sigma(\xi)}F\pi^0(\underline{\alpha}_h\varphi)\Omega_0 = \pi^0(\underline{\alpha}_h\varphi)F\Omega_0,
\end{aligned}
$$

where $\sigma(\xi)$ is the parity of the sector ξ. This implies, by theorem 3.18, that $\pi^0(\underline{\alpha}_h\psi) = \pi^0(\underline{\alpha}_h\varphi) \in \tilde{\mathfrak{F}}^0(\tilde{\mathscr{C}})$ for any spacelike cone $\tilde{\mathscr{C}}$ satisfying the above conditions. For a general spacelike cone $\tilde{\mathscr{C}}$ containing $\mathscr{O}$, we can find cones $\mathscr{C}_1, \dots, \mathscr{C}_n$ containing $\mathscr{O}$ and such that $\mathscr{C}_1 = \mathscr{C}$, $\mathscr{C}_n = \tilde{\mathscr{C}}$, and cones $\hat{\mathscr{C}}_1, \dots, \hat{\mathscr{C}}_{n-1}$ such that $\mathscr{C}_j \cup \mathscr{C}_{j+1} \subseteq \hat{\mathscr{C}}_j$, $j = 1, \dots, n-1$ (proposition A.9), and for any $j = 1, \dots, n-1$ there exists a wedge $\mathscr{W}_j \subseteq \hat{\mathscr{C}}_j'$, so that, iterating the argument above, $\pi^0(\underline{\alpha}_h\psi) \in \tilde{\mathfrak{F}}^0(\tilde{\mathscr{C}})$. □

We denote by $(\mathscr{H}_0, \mathfrak{A}_0, U_0, \Omega_0)$ the scaling limit observable net determined by $\underline{\omega}_0 \circ \pi_{\underline{\mathfrak{F}}} \in SL_{\mathfrak{A}}$.

Theorem 3.24. *The quadruple $(\mathfrak{F}_0, V_0, k, U_0)$ is a normal, Poincaré covariant field net with gauge symmetry, Ω_0 is a cyclic vacuum for it, and the formula*

$$
\pi_{\mathfrak{F}_0}(\pi_0(\underline{A})) := \pi^0 \circ \pi_{\underline{\mathfrak{F}}}(\underline{A}), \qquad \underline{A} \in \underline{\mathfrak{A}}, \tag{3.19}
$$

defines a representation $\pi_{\mathfrak{F}_0}$ of $\mathfrak{A}_0$ on $\mathscr{H}^0$ containing the identical representation of $\mathfrak{A}_0$ and such that $\pi_{\mathfrak{F}_0}(\mathfrak{A}_0(\mathscr{O})) \subseteq \mathfrak{F}_0(\mathscr{O})^G$.

Proof. Poincaré and gauge covariance of the net $\mathfrak{F}_0$ are easily established, using arguments analogous to the ones in the proof of proposition 3.17. Normality of commutation relations for $\mathfrak{F}_0$ follows at once from the same property for the extended net $\tilde{\mathfrak{F}}^0$, the previous lemma and the fact that for any two spacelike separated double cones $\mathscr{O}_i$, $i = 1, 2$, there exist spacelike separated spacelike cones $\mathscr{C}_i$, such that $\mathscr{O}_i \subset \mathscr{C}_i$, $i = 1, 2$, proposition A.10 in appendix A. We have already seen that translations satisfy the spectrum condition, and Ω_0 is Poincaré invariant. Cyclicity of $\mathfrak{F}_0$ on the vacuum follows from the fact that $\mathfrak{F}_0 = \tilde{\mathfrak{F}}^0$ (equality of the respective quasi-local algebras): since every generator of $\mathfrak{F}_0(\mathscr{O})$ is in some $\tilde{\mathfrak{F}}^0(\mathscr{C})$, we have $\mathfrak{F}_0(\mathscr{O}) \subseteq \tilde{\mathfrak{F}}^0$ for any double cone $\mathscr{O}$, and then $\mathfrak{F}_0 \subseteq \tilde{\mathfrak{F}}^0$; the reverse inclusion is analogously proven. We can then apply Reeh-Schlieder theorem to conclude that Ω_0 is cyclic also for the C*-algebras $\mathfrak{F}_0(\mathscr{O}')$ and then separating for local von Neumann algebras $\mathfrak{F}_0(\mathscr{O})^-$. Then, by the argument employed in the proof of theorem 3.7, $\pi_{\mathfrak{F}_0}$ is a well defined representation of $\mathfrak{A}_0$ and, by GNS unicity, it contains the defining representation of $\mathfrak{A}_0$ on the subspace $\overline{\pi^0 \circ \pi_{\underline{\mathfrak{F}}}(\underline{\mathfrak{A}})\Omega_0} \cong \mathscr{H}_0$ of $\mathscr{H}^0$. Finally, it is evident that $\pi_{\mathfrak{F}_0}(A)$ is gauge invariant for any $A \in \mathfrak{A}_0$. □

We see then that ultraviolet stable topological charges give rise, in the scaling limit, to a net of finitely localizable charge carrying fields. We would like then to show that among the charges carried by these fields, there are charges which we can regard as scaling limits of the cone-like localizable charges which we started with, in a sense similar to the one that was empolyed in the localizable case, i.e. we would like to construct Hilbert spaces in $\mathfrak{F}_0$ carrying appropriate representations of G and implementing localizable endomorphisms of $\mathfrak{A}_0$. However, at the present stage of our work, this can be achieved only at the price of some additional technical assumption on the net $\mathfrak{F}_0$ (see theorem 3.27 below), the status of which has yet to be clarified.

Before stating explicitly these assumptions, and therefore sticking, for the time being, to the level of generality used up to now, we can at any rate show that the fields in $\mathfrak{F}_0$ give rise, in a sense made precise in the following theorem, to positive energy representations of $\mathfrak{A}_0$, so that we get at least charges which are localizable in this weak sense. As in the previous section, we let $\mathscr{F}_0(\mathscr{O}) := \mathfrak{F}_0(\mathscr{O})^-$.

Theorem 3.25. *Let ξ be an ultraviolet stable topological sector, and let $\mathscr{O}$ be a double cone and $\rho_\lambda \in \Delta_c(\lambda\mathscr{C}_1)$, $[\rho_\lambda] = \xi$, $\lambda > 0$, a family as in definition 3.15(i), where $\mathscr{C}_1$ is a spacelike cone such that $\mathscr{C}_1 \Subset \mathscr{C}$ for some $\mathscr{C} \supset \mathscr{O}$. There is then a finite dimensional Hilbert space H_ρ in $\mathscr{F}_0(\mathscr{O})$ of support $\mathbb{1}$, carrying a G representation of class ξ, and for any such Hilbert space the state ω_ρ on $\mathfrak{A}_0$ defined by*

$$\omega_\rho(A) := \sum_{j=1}^{d(\xi)} (\Omega_0|\psi_j^0 \pi_{\mathfrak{F}_0}(A)\psi_j^{0*}\Omega_0), \qquad A \in \mathfrak{A}_0, \tag{3.20}$$

where ψ_j^0, $j = 1, \dots, d(\xi)$, is an orthonormal basis of H_ρ, is such that $\omega_\rho \upharpoonright \mathfrak{A}_0(\mathscr{O}') = \omega_0 \upharpoonright \mathfrak{A}_0(\mathscr{O}')$, and induces, via the GNS construction, a representation π_ρ of $\mathfrak{A}_0$ which is translation covariant with positive energy.

During the proof of the above theorem, we shall need a result on the existence of covariant representations of C*-dynamical systems due to Borchers [Bor96, thm. II.6.6], which we state here (without proof) for the reader convenience and for later reference.

Theorem 3.26. *Let $\mathfrak{A}$ be a C*-algebra, $\alpha : \mathbb{R}^4 \to \mathrm{Aut}(\mathfrak{A})$ a group homomorphism, and let $\mathfrak{A}^*(V_+)$ be the norm closure of the linear space of those $\phi \in \mathfrak{A}^*$ such that*

(i) for any $A, B \in \mathfrak{A}$, $x \to \phi(A\alpha_x(B))$ is a continuous function on $\mathbb{R}^4$;

(ii) $x \to \phi(A\alpha_x(B))$ is the boundary value of a function $z \to W(z)$ analytic in the future tube $\mathscr{T}(V_+) := \{z \in \mathbb{C}^4 : \mathrm{Im}\, z \in V_+\}$;

(iii) there exists a constant $m > 0$ such that, for $z \in \mathscr{T}(V_+)$,

$$|W(z)| \le \|\phi\|\|A\|\|B\|e^{m|\mathrm{Im}\, z|}.$$

Let π be a representation of $\mathfrak{A}$. Then there exists a unitary, strongly continuous representation U of $\mathbb{R}^4$ on $\mathscr{H}$ with spectrum in $\overline{V}_+$ and such that (π, U) is a covariant representation of $(\mathfrak{A}, \alpha)$ if and only if every vector state of π belongs to $\mathfrak{A}^*(V_+)$.

Proof of theorem 3.25. Let $\mathscr{O}$, $\mathscr{C}$, $\mathscr{C}_1$ and ρ_λ be as in the statement. There is then a double cone $\mathscr{O}_1 \subset \mathscr{C}_1$ such that if $\psi_j(\lambda)$, $j = 1, \ldots, d := d(\xi)$, is an orthonormal basis of H_{ρ_λ}, then any function $\lambda \to \psi_j(\lambda)$ is a topological quasi-orbit asymptotically localized in $\mathscr{O}_1$. We can also assume that the $\psi_j(\lambda)$'s transform according to a λ-independent G representation u_ξ. We can then repeat, *mutatis mutandis*, the proof of lemma 3.13 and the first part of the proof of theorem 3.14 to conclude that, as in the case of localizable charges, the limit $\psi_j^0 := {}^*s\text{–}\lim_{h\to\delta} \underline{\alpha}_h \psi_j$ exists and defines a multiplet of class ξ of orthogonal isometries with support $\mathbb{1}$ in $\mathscr{F}_0(\mathscr{O})$, and the linear span of these operators is therefore a d dimensional Hilbert space H_ρ of support $\mathbb{1}$ in $\mathscr{F}_0(\mathscr{O})$ carrying a unitary matrix representation of G of class ξ. This Hilbert space is independent of the choice of the basis $\psi_j(\lambda)$: let $\varphi_j(\lambda) \in H_{\rho_\lambda}$ be another such choice, then we have

$$\underline{\alpha}_h \varphi_j(\lambda) = \sum_{k=1}^{d} c_{kj}(\lambda) \underline{\alpha}_h \psi_k(\lambda), \qquad \lambda > 0, j = 1, \ldots, d,$$

where $c_{kj}(\lambda) = \psi_k(\lambda)^* \varphi_j(\lambda)$ is the unitary basis change matrix. The functions $\lambda \to c_{kj}(\lambda)$ are therefore bounded, and then, again as in the proof of theorem 3.14, we have that for any $\underline{F} \in \underline{\tilde{\mathfrak{F}}}(\mathscr{W})$, $\mathscr{W} \subset \mathscr{C}'$, the limit

$$\lim_{h\to\delta} \sum_{k=1}^{d} c_{kj}(\lambda_\iota)(\underline{F}(\lambda_\iota)\Omega | \underline{\alpha}_h \psi_l(\lambda_\iota)^* \underline{\alpha}_h \psi_k(\lambda_\iota)\Omega) = c_{lj}(\lambda_\iota)(\underline{F}(\lambda_\iota)\Omega|\Omega)$$

is uniform in ι, so that

$$\begin{aligned}\lim_{\iota\in I} c_{lj}(\lambda_\iota)(\underline{F}(\lambda_\iota)\Omega|\Omega) &= \lim_{h\to\delta} \lim_{\iota\in I} \sum_{k=1}^{d} c_{kj}(\lambda_\iota)(\underline{F}(\lambda_\iota)\Omega | \underline{\alpha}_h \psi_l(\lambda_\iota)^* \underline{\alpha}_h \psi_k(\lambda_\iota)\Omega) \\ &= \lim_{h\to\delta} (\pi^0(\underline{F})\Omega_0 | \pi^0(\underline{\alpha}_h \psi_l)^* \pi^0(\underline{\alpha}_h \varphi_j)\Omega_0) = (\pi^0(\underline{F})\Omega_0 | \psi_l^{0*} \varphi_j^0 \Omega_0).\end{aligned}$$

Putting $\underline{F} = \underline{\mathbb{1}}$ in this last equation, we get that there exists the limit $c_{lj}^0 = \lim_\iota c_{lj}(\lambda_\iota) = (\Omega_0 | \psi_l^{0*} \varphi_j^0 \Omega_0)$, which is again a unitary matrix, and then

$$c_{lj}^0 (\pi^0(\underline{F})\Omega_0|\Omega_0) = (\pi^0(\underline{F})\Omega_0 | \psi_l^{0*} \varphi_j^0 \Omega_0),$$

so that we conclude, by cyclicity of Ω_0 for $\mathfrak{F}_0(\mathscr{W})$, that $\psi_l^{0*}\varphi_j^0 = c_{lj}^0 \mathbb{1}$, and the linear span of φ_j^0, $j = 1, \ldots, d$, coincides with H_ρ.

Let then $(\pi_\rho, \mathscr{H}_\rho, \Omega_\rho)$ be the GNS representation induced by the state ω_ρ in (3.20). To show that this is translation covariant with positive energy, take $\Delta_n \subseteq \mathbb{R}^4$ to be the closed double cone in momentum space with vertices 0 and $(n, \mathbf{0})$, and let $C \in \hat{\mathfrak{A}}_0(\Delta)$ for some compact set Δ (e.g. $C = \alpha_f^0(C_1)$ with $\operatorname{supp} \hat{f} \subseteq \Delta$). Then the vectors $\Phi_j^{C,n} := \pi_{\mathfrak{F}_0}(C) E(\Delta_n) \psi_j^{0*} \Omega_0$, $j = 1, \ldots, d$, have momentum support in $\Delta + \Delta_n$, and if we define $\phi_{C,n} \in \mathfrak{A}_0^*$ by

$$\phi_{C,n}(A) := \sum_{j=1}^{d} (\Omega_0 | \psi_j^0 E(\Delta_n) \pi_{\mathfrak{F}_0}(C^* A C) E(\Delta_n) \psi_j^{0*} \Omega_0),$$

we get that

$$\phi_{C,n}(A\alpha^0_x(B)) = \sum_{j=1}^d (\pi_{\mathfrak{F}_0}(A)^*\Phi_j^{C,n}|U_0(x)\pi_{\mathfrak{F}_0}(B)U_0(x)^*E(\Delta+\Delta_n)\Phi_j^{C,n})$$

is obviously continuous in x, and has (distributional) Fourier transform with support in $-(\Delta+\Delta_n)+\overline{V}_+$, lemma 2.2, so that, if $p=(-m,\mathbf{0})$ is such that $-(\Delta+\Delta_n)+\overline{V}_+\subset p+\overline{V}_+$, according to [Bor96, thm. II.1.7], $x\to\phi_{C,n}(A\alpha^0_x(B))$ is the boundary value of a function $z\to W(z)$ analytic in $\mathscr{T}(V_+)$ and satisfying, for suitable constants $M,N>0$, the bound

$$|W(z)|\leq k(1+|x|)^N(1+\mathrm{dist}(y,\partial V_+)^{-1})^M e^{m|y|}, \qquad z=x+iy\in\mathscr{T}(V_+),$$

and since $|W(x)|\leq\|\phi_{C,n}\|\|A\|\|B\|$ for real x, a Phragmén-Lindelöf type argument (see the proof of lemma II.3.4 in [Bor96]) gives the desired estimate $|W(z)|\leq\|\phi_{C,n}\|\|A\|\|B\|e^{m|\mathrm{Im}\,z|}$, $z\in\mathscr{T}(V_+)$, and $\phi_{C,n}\in\mathfrak{A}_0^*(V_+)$. If $\phi_C(A):=\omega_\rho(C^*AC)$, $A\in\mathfrak{A}_0$, we have

$$|\phi_C(A)-\phi_{C,n}(A)|\leq\sum_{j=1}^d 2\|C\|\|A\|\|[E(\Delta_n)-\mathbb{1}]\psi_j^{0*}\Omega_0\|,$$

so that $\phi_{C,n}\to\phi_C$ in norm as $n\to+\infty$ and, being $\mathfrak{A}_0^*(V_+)$ norm closed, $\phi_C\in\mathfrak{A}_0^*(V_+)$. Finally, since the Fourier transforms of continuous functions of compact support are dense in $L^1(\mathbb{R}^4)^{e)}$ and translations act norm continuously on $\mathfrak{A}_0$, the operators of the form $C=\alpha^0_f(C_1)$ with compact supp $\hat{f}$ lie norm dense in $\mathfrak{A}_0$, which implies that the set of vector states of π_ρ is contained in the norm closure of the set of corresponding functionals ϕ_C, so that any such state belongs to $\mathfrak{A}_0^*(V_+)$, and we can apply theorem 3.26 to get a unitary, strongly continuous representation U_ρ of the translations group on $\mathscr{H}_\rho$ with spectrum in $\overline{V}_+$ and such that (π_ρ,U_ρ) is a covariant representation of $(\mathfrak{A}_0,\alpha^0)$. □

The representation π_ρ thus constructed, will not be, in general, a DHR representation of $\mathfrak{A}_0$, since the fact that $\omega_\rho\restriction\mathfrak{A}_0(\mathscr{O}')=\omega_0\restriction\mathfrak{A}_0(\mathscr{O}')$, together with translation covariance of π_ρ, and the fact that, as is easily verified, $\omega_\rho\circ\alpha^0_x$ is the state determined as in theorem 3.25 by the family $\alpha_{\lambda x}\rho_\lambda\alpha_{-\lambda x}$, $\lambda>0$, implies only that $\pi_\rho\restriction\mathfrak{A}_0(\mathscr{O}'+x)$ has a subrepresentation that is equivalent to $\iota\restriction\mathfrak{A}_0(\mathscr{O}'+x)$, ι being the defining representation of $\mathfrak{A}_0$. Thus, in order to have DHR property at least for the class of translates of the given double cone $\mathscr{O}$ (which is sufficient to perform the superselection analysis), it would be sufficient to know that property B holds in the representation π_ρ (cfr. the appendix of [DHR71]), which in turn would follow from irreducibility and local normality of π_ρ, as well as weak additivity of the net $\mathfrak{A}_0$, which is quite natural to expect to hold in relevant cases. However, a proof of this properties is lacking at present.

At the technical level, the main obstruction is represented by the fact that, in general, $\pi_{\mathscr{F}_0}(\mathscr{A}_0(\mathscr{O}))\subsetneq\mathscr{F}_0(\mathscr{O})^G$, as it is easy to construct gauge invariant combinations of the $\underline{\alpha}_h\psi$'s, which need not belong to some scaling algebra $\underline{\mathfrak{A}}(\mathscr{O})$, as they are only localized

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$^{e)}C_c^\infty(\mathbb{R}^4)$ is dense in the Schwartz topology in $\mathscr{S}(\mathbb{R}^4)$, which in turn is norm dense in $L^1(\mathbb{R}^4)$, and being the $\mathscr{S}$ topology stonger than the L^1 topology, $(C_c^\infty(\mathbb{R}^4)\hat{\ })^{-\|\cdot\|_1}\supseteq(C_c^\infty(\mathbb{R}^4)\hat{\ })^{-\mathscr{S}}=(C_c^\infty(\mathbb{R}^4)^{-\mathscr{S}})\hat{\ }=\mathscr{S}(\mathbb{R}^4)$.

in spacelike cones. However, thanks to the fact that these functions are asymptotically localizable in $\mathscr{O}$, it may well happen that, at least in some models, their scaling limits do belong to $\mathfrak{A}_0(\mathscr{O})$. Adding the simple hypotesis that this is indeed the case, yields a quite satisfactory picture of the scaling limit of ultraviolet stable topological charges.

Theorem 3.27. *Assume that* $\pi_{\mathscr{F}_0}(\mathscr{A}_0(\mathscr{O})) = \mathscr{F}_0(\mathscr{O})^G$*, and that* $\mathscr{F}_0$ *acts irreducibly on* $\mathscr{H}^0$*. Then for any covariant, ultraviolet stable, topological sector* ξ*, any double cone* $\mathscr{O}$*, and any family* $\rho_\lambda \in \Delta_c(\lambda\mathscr{C}_1)$*,* $[\rho_\lambda] = \xi$*,* $\lambda > 0$*, as in the previous theorem, there is a finite dimensional Hilbert space* H_ρ *in* $\mathfrak{F}_0(\mathscr{O})$ *of support* $\mathbb{1}$*, carrying a G representation of class* ξ*, and implementing an irreducibile transportable endomorphism* ρ *of* $\mathfrak{A}_0$ *localized in* $\mathscr{O}$*, covariant with positive energy. Moreover, any two such endomorphisms* ρ *and* σ*, obtained as above from familes* ρ_λ*,* σ_λ*,* $\lambda > 0$*,* $\rho_\lambda \cong \sigma_\lambda$*, are unitarily equivalent.*

Proof. It's an easy adaptation of the proof of theorem 3.14 to the present setting. □

Thus we obtain, as in the case of localizable charges, a well defined mapping from the subset of ultraviolet stable charges to the set of charges of the fixed scaling limit theory.

As a final comment, we would like to remark that the condition $\pi_{\mathscr{F}_0}(\mathscr{A}_0(\mathscr{O})) = \mathscr{F}_0(\mathscr{O})^G$, introduced here as a technical assumption in order to get a well defined scaling limit of topological charges, may turn out to have a sensibile physical interpretation. By the above remarks, we see that $\mathscr{F}_0(\mathscr{O})^G$ contains, apart from the scaling limit observables localized in $\mathscr{O}$, the scaling limit of functions $\lambda \to \underline{A}(\lambda) \in \mathfrak{A}(\lambda\mathscr{C})$, for every spacelike cone $\mathscr{C} \supset \mathscr{O}$, i.e. there are gauge invariant families of operators, with localization regions extending to spacelike infinity, which give rise to objects in the scaling limit which are charged with respect to the intrinsic gauge group of $\mathfrak{A}_0$, so that new charges appear at small scales. This situation, which does not have to be confused with the confinement one, in which the fields carrying the new charges cannot be approximated at all at finite scales, is instead reminiscent of the phenomenon of *charge screening*,[f)] much discussed in the physical literature (cfr. for instance [Swi76, RRS79] and references quoted). In this scenario, a charge which is described by an asymptotically free theory at small scales, disappears at finite scales because, due to nonvanishing interactions, it is always accompanied by a cloud, extending to spacelike infinity, of charge-anticharge pairs, so that one can expect that the corresponding "charge carrying fields" are neutral and non-compactly localized at finite scales, and become instead charged and localized in the scaling limit. Then the condition $\pi_{\mathscr{F}_0}(\mathscr{A}_0(\mathscr{O})) = \mathscr{F}_0(\mathscr{O})^G$ could be interpreted as the requirement that in the theory under consideration, no charges are screened. Work is currently in progress in order to clarify further these matters.

[f)] This connection was pointed out to me by Detlev Buchholz.

Conclusions and outlook

In this thesis we addressed the conceptual problem of formulating a notion of confined charge in quantum field theory free of the ambiguities of the one generally adopted, which relies heavily on the description of the theory in terms of unobservable gauge degrees of freedom. This was done in the algebraic framework of quantum field theory, which is the most suitable one to this task, since it disregards completely the existence of unobservable fields – which, as we saw, comes out as a consequence of the structure of the observable net and its representations, rather than being assumed from the outset – and focuses on the information encoded in the net of local observables. In particular, superselection theory on one hand, and scaling algebras on the other, allow a natural and intrinsic identification of the charges described by a theory at small spatio-temporal scales with the superselection sectors of the scaling limit theory (supposed for simplicity to be unique) canonically defined by the given theory, so that the required definition of confined charge is obtained through a comparison of the superselection structure of the given theory with that of its scaling limit.

In order to establish such a comparison, we studied the scaling behaviour of sectors, and singled out a class of sectors, both finitely and cone-like localizable, for which a natural notion of scaling limit exists, so that they can be identified with sectors in the scaling limit theory, which are then naturally regarded as non-confined. In the DHR case, such an ultraviolet stable sector ξ has been defined by requiring essentially that for fields $\psi(\lambda)$, carrying charge ξ and localized in scaled regions $\lambda\mathscr{O}$, the charged states $\psi(\lambda)\Omega$ have energy and momentum growing not faster than λ^{-1} for $\lambda \to 0$, which, in view of the construction of the scaling limit, is a natural phase space restriction. Then we showed that to any such sector we can associate a system of Hilbert spaces in the scaling limit field net $\mathscr{F}^0$, which are generated essentially by the limits, for $\lambda \to 0$, of the just mentioned operators $\psi(\lambda)$, and which carry a gauge group representation of class ξ. Finally, these Hilbert spaces implement localized endomorphisms on the scaling limit observable net $\mathscr{A}_0$, whose sector is then identified with the (scaling limit of the) starting sector ξ. Then, we tried to generalize these results to topological sectors. In this case, an ultraviolet stable sector ξ is defined first of all by requiring that, according to the physical picture of this kind of charges emerging from non-abelian gauge theories, the effect in the scaling limit of the operators $\psi(\lambda)$ outside some bounded region $\lambda\mathscr{O}$ becomes negligible, and that, still in the limit, the states $\psi(\lambda)\Omega$

become independent on the direction of the string emanating to spacelike infinity, so that it is meaningful to impose on these states a phase space requirement analogous to the one for DHR charges. A first non trivial result is then that these operators, though being localized in spacelike cones at each scale $\lambda > 0$, generate, in the scaling limit, a net $\mathscr{O} \to \mathscr{F}_0(\mathscr{O})$ of field algebras associated to bounded regions $\mathscr{O}$ and satisfying normal commutation relations, and that they also generate, as above, Hilbert spaces with gauge group representations in this net. Adding then the technical hypotesis that the fixed point net of $\mathscr{F}_0$ under the action of the gauge group coincide with $\mathscr{A}_0$, together with the irreducibility of $\mathscr{F}_0$, gives scaling limit sectors as in the DHR case. We have also seen, in appendix B, that the charged sector of the free Majorana field with $\mathbb{Z}_2$ gauge group is ultraviolet stable.

There are several directions along which this work could be improved and extended. First of all, clearly, we are trying to get a better understanding of the conditions under which ultraviolet stable topological sectors admit a scaling limit along the lines discussed in the thesis. In particular, as suggested at the end of section 3.3, the condition $\mathscr{F}_0(\mathscr{O})^G = \mathscr{A}_0(\mathscr{O})$ may be replaced with a physically more transparent condition formulated in terms of the underlying theory and expressing the absence of screening, and, eventually, discarded altogether, as there are indications that ultraviolet stability alone is a sufficient condition for the existence of the charge scaling limit.

Another natural issue to be investigated is the structure of the set of ultraviolet stable sectors with respect to the standard operations of composition, direct sums and conjugation defined by superselection theory. In particular, it seems likely, on physical grounds, that the irreducible components appearing in the direct sum decomposition of a product of ultraviolet stable sectors are again ultraviolet stable, as, due to additivity of the spectrum [DHR74], the energy momentum transfer of fields carrying the product charge cannot be substantially higher at small scales than that of the component fields.

To complete the analysis of scaling properties of charges, it would be desirable to understand better the fate, in the scaling limit, of non-ultraviolet stable charges, and in particular if there is a natural way to associate with them some class of non-localizable states on $\mathfrak{A}_0$. Also, it would be interesting to construct examples of this kind of charges. In connection to this we just recall that examples of theories having classical scaling limit have been constructed in [Lut97], and similar ideas and techniques could be useful in this task.

Still on the examples side, it would be interesting to have at one's disposal other models exhibiting ultraviolet stable sectors, apart from the very simple one treated in appendix B. The particular features of the latter, namely the existence of a single charged sector, with the basic field itself interpolating between this sector and the vacuum, made the establishing of the ultraviolet stability condition straightforward in this case. In more complicated examples, possibly with non-abelian gauge groups, the charge multiplets are in general non-linear combinations of the fields, (or, rather, of the isometries appearing in their polar decompositions if the fields are unbounded), which, moreover, are not explicitly known, except that for G abelian. The direct method used in appendix B is thus unlikely to be useful for non-abelian examples.

Finally, some more insight on the status of the asymptotic localizability conditions employed to treat the scaling limit of topological charge could come from the rigorous analysis of lattice models of gauge theories, as the essentially unique continuum example of such charges is provided by the already considered theory of the free massless scalar field in $d = 2$ spacetime dimensions.

APPENDIX A

Some geometrical results about spacelike cones

In this appendix we shall collect several geometrical definitions and results, mostly concerning spacelike cones in 4-dimensional Minkowski space, which are needed for the analysis in section 3.3. The basic definitions will be taken from the appendix of [DR90].

The spacelike hyperboloid $\mathscr{D} := \{n \in \mathbb{R}^4 : n^2 = -1\}$ will be called *spacelike infinity*, since we identify $n \in \mathscr{D}$ with the "point at infinity" in the spacelike direction λn, $\lambda \in \mathbb{R}_+$. We endow $\mathscr{D}$ with the causal structure induced by the one in Minkowksi space, i.e. two points $n, n' \in \mathscr{D}$ will be called timelike (resp. lightlike, spacelike) if they are timelike (resp. lightlike, spacelike) when considered as points in $\mathbb{R}^4$, and in the first two cases, n will be said to be future (resp. past) to n' if $n^0 - n'^0 > 0$ (resp. < 0).

Given $n_+, n_- \in \mathscr{D}$, n_+ future timelike to n_-, the (open) *double cone* in $\mathscr{D}$ with vertices n_+, n_- will be the set D_{n_+,n_-} of points $n \in \mathscr{D}$ which are past timelike to n_+ and future timelike to n_-, i.e. $D_{n_+,n_-} = \mathscr{O}_{n_+,n_-} \cap \mathscr{D}$, where $\mathscr{O}_{n_+,n_-}$ is the double cone in Minkowski space with vertices n_+, n_-, which also shows that D_{n_+,n_-} is open in the relative topology of $\mathscr{D}$. We remark explicitly that $\mathscr{O}_{n_+,n_-}$ is spacelike to the origin, for if there would be an $x \in \mathscr{O}_{n_+,n_-}$ such that, for instance, $x \in \overline{V}_+$, then also $n_+ = (n_+ - x) + x \in \overline{V}_+$, which is not true.

Definition A.1. Let D be a double cone in $\mathscr{D}$. The *spacelike cone* $\mathscr{C} = \mathscr{C}_{a,D}$ with apex $a \in \mathbb{R}^4$ and base D is the set

$$\mathscr{C}_{a,D} := \{a + \lambda n : n \in D, \lambda > 0\}.$$

If $\mathscr{C}$ is a spacelike cone, we shall denote by $D(\mathscr{C})$ the double cone in $\mathscr{D}$ which is the base of $\mathscr{C}$.

We have also the following expressions for $\mathscr{C}_{a,D}$:

$$\mathscr{C}_{a,D} = \left\{a + x : x^2 < 0, \frac{x}{\sqrt{-x^2}} \in D\right\} = a + \bigcup_{\lambda>0} \lambda \mathscr{O}_D, \tag{A.1}$$

where $\mathscr{O}_D$ is the double cone in $\mathbb{R}^4$ with the same vertices as D. The first of these equalities is evident by putting $x = \lambda n$ in the definition of $\mathscr{C}_{a,D}$. For what concerns the second, it is sufficient to prove that for any $x \in \mathscr{O}_D$, we have $x/\sqrt{-x^2} \in D$ ($x^2 < 0$ by the remark above), the inclusion $\mathscr{C}_{a,D} \subseteq a + \bigcup_{\lambda>0} \lambda \mathscr{O}_D$ being trivial. It is clear that $x/\sqrt{-x^2} \in \mathscr{D}$, and we can then assume, by an appropriate choice of the Lorentz frame, that $x^0 = 0$. Then, if $n_\pm$ are the vertices of D, from $(n_+ - x)^2 > 0$ it follows that $2\boldsymbol{n}_+ \cdot \boldsymbol{x} > |\boldsymbol{x}|^2 + 1$, and then

$$\left(n_+ - \frac{x}{\sqrt{-x^2}}\right)^2 = -2 + \frac{2\boldsymbol{n}_+ \cdot \boldsymbol{x}}{|\boldsymbol{x}|} > -2 + \frac{|\boldsymbol{x}|^2 + 1}{|\boldsymbol{x}|} \geq 0,$$

being 2 the minimum of the fuction $t \to (t^2+1)/t$ for $t > 0$. Analogously one sees that $x/\sqrt{-x^2}$ is future timelike to n_-.

From the last expression, it follows immediately that any spacelike cone $\mathscr{C}_D := \mathscr{C}_{0,D}$ (i.e. with apex at the origin of Minkowski space) is a convex cone, in the sense that $x, y \in \mathscr{C}_D$ and $\lambda > 0$ imply λx, $x + y \in \mathscr{C}_D$: $\lambda x \in \mathscr{C}_D$ is evident, while if $x = \lambda\xi$, $y = \mu\eta$ with $\xi, \eta \in \mathscr{O}_D$, being $\mathscr{O}_D$ convex,

$$x + y = (\lambda + \mu)\left(\frac{\lambda}{\lambda+\mu}\xi + \frac{\mu}{\lambda+\mu}\eta\right) \in \mathscr{C}_D.$$

The following three results are also taken from the appendix of [DR90], and we include for completeness the easy proofs.

Lemma A.2. *Let $\mathscr{C}$ be a spacelike cone, and $x \in \mathbb{R}^4$, $n \in \mathscr{D}$. Then*
(i) if $n \in D(\mathscr{C})$ then $x + \lambda n \in \mathscr{C}$ for $\lambda > 0$ sufficiently large;
(ii) if $x + \lambda n \in \mathscr{C}$ for $\lambda > 0$ sufficiently large, then $n \in \overline{D(\mathscr{C})}$ (closure in the relative topology of $\mathscr{D}$).

Proof. (i) Since x is arbitrary, we can clearly assume that the apex a of $\mathscr{C}$ is the origin. We have $(x + \lambda n)^2 < 0$ for λ sufficiently big and since

$$\lim_{\lambda\to+\infty} \frac{x + \lambda n}{\sqrt{-(x+\lambda n)^2}} = n \in D(\mathscr{C}), \tag{A.2}$$

and $D(\mathscr{C})$ is open in $\mathscr{D}$, $(x+\lambda n)/\sqrt{-(x+\lambda n)^2} \in D(\mathscr{C})$ for λ sufficiently big, i.e., being $a = 0$, $x + \lambda n \in \mathscr{C}$.

(ii) Again we assume $a = 0$. Then by hypotesis $(x+\lambda n)/\sqrt{-(x+\lambda n)^2} \in D(\mathscr{C})$ for λ sufficiently big and by (A.2), $n \in \overline{D(\mathscr{C})}$. ☐

Corollary A.3. *If $\mathscr{C}_1 \subseteq \mathscr{C}_2$ then $D(\mathscr{C}_1) \subseteq D(\mathscr{C}_2)$.*

Proof. Let a_1 be the apex of $\mathscr{C}_1$. If $n \in D(\mathscr{C}_1)$ then $a_1 + \lambda n \in \mathscr{C}_2$ for each $\lambda > 0$, and by the above lemma $n \in \overline{D(\mathscr{C}_2)}$, and we conclude by noting that the two sets are open. ☐

Corollary A.4. *If D is a double cone in $\mathscr{D}$, and $\mathscr{C}_1, \dots, \mathscr{C}_r$ are spacelike cones such that $D(\mathscr{C}_i) \subseteq D$, $i = 1, \dots, r$, then there exists a spacelike cone $\mathscr{C}$ with $D(\mathscr{C}) = D$, and $\mathscr{C}_i \subseteq \mathscr{C}$, $i = 1, \dots, r$.*

Proof. Let a_i be the apex of $\mathscr{C}_i$, $i = 1, \dots, r$, and let $n \in D$. Then for $\lambda > 0$ sufficiently large, $a_i + \lambda n \in \mathscr{C}_D$ for any $i = 1, \dots, r$. Then, being $\mathscr{C}_D$ a convex cone, $\mathscr{C} := \mathscr{C}_{-\lambda n, D}$. ☐

The following lemma is also completely straightforward.

Lemma A.5. *If $\mathscr{O}$ is a double cone in $\mathbb{R}^4$ and D is a double cone in $\mathscr{D}$, there exists a spacelike cone $\mathscr{C}$, with $D(\mathscr{C}) = D$ and $\mathscr{O} \subseteq \mathscr{C}$.*

Proof. There are $c \in \mathbb{R}^4$, $b \in V_+$ such that $\mathscr{O}_D = c + \mathscr{O}_{b,-b}$, and since $\bigcup_{\lambda>0} \lambda\mathscr{O}_{b,-b} = \mathbb{R}^4$, by compactness there is $\lambda > 0$ such that $\mathscr{O} \subseteq \lambda\mathscr{O}_{b,-b}$, and then $\mathscr{O} \subseteq \mathscr{C}_{-\lambda c,D}$. □

The following easy result is used in the construction of the lifted topological quasi-orbits of section 3.3.

Proposition A.6. *For any spacelike cone $\mathscr{C}_1$ there exists a neighbourhood of the identity $\mathscr{N} \subseteq \mathscr{P}_+^\uparrow$ and a spacelike cone $\mathscr{C}_2$ such that $\mathscr{N} \cdot \mathscr{C}_1 \subseteq \mathscr{C}_2$.*

We shall give the proof after having proven two elementary lemmas, which we single out for reference's sake.

Lemma A.7. *Let $\mathscr{C}_1 \subseteq \mathscr{C}_2$ be spacelike cones such that, for the apex a_1 of $\mathscr{C}_1$ there holds $a_1 + B_\varepsilon \subseteq \mathscr{C}_2$, B_ε being the open ball of radius ε around the origin. Then $\mathscr{C}_1 + B_\varepsilon \subseteq \mathscr{C}_2$.*

Proof. We can clearly assume that the apex of $\mathscr{C}_2$ is the origin. Then $\mathscr{C}_2$ is a convex cone. Furthermore, by corollary A.3, $\mathscr{C}_{D(\mathscr{C}_1)} \subseteq \mathscr{C}_2$, and then $\mathscr{C}_1 + B_\varepsilon = (a_1 + B_\varepsilon) + \mathscr{C}_{D(\mathscr{C}_1)} \subseteq \mathscr{C}_2$. □

Lemma A.8. *Let $\mathscr{C}_1 \subseteq \mathscr{C}_2$ be spacelike cones with the same apex and such that $\overline{D(\mathscr{C}_1)} \subset D(\mathscr{C}_2)$. Then there exists a neighbourhood of the identity $\mathscr{N} \subseteq \mathscr{P}_+^\uparrow$ such that $\mathscr{N} \cdot \mathscr{C}_1 \subseteq \mathscr{C}_2$.*

Proof. Assume first that the apex of the two cones coincide with the origin. Thanks to the continuity of the function $\Lambda \in SO^\uparrow(1,3) \to \Lambda n \in \mathscr{D}$, we can find, for any $n \in \overline{D(\mathscr{C}_1)}$, a neighbourhood of the identity $\tilde{\mathscr{N}_n} \subseteq SO^\uparrow(1,3)$ such that $\tilde{\mathscr{N}_n} n \subseteq D(\mathscr{C}_2)$, and, being $SO^\uparrow(1,3)$ a topological group, we can also find neighbourhoods of the identity $\mathscr{N}_n \subseteq \tilde{\mathscr{N}_n}$ such that $\mathscr{N}_n^2 \subseteq \tilde{\mathscr{N}_n}$. By compactness of $\overline{D(\mathscr{C}_1)}$ there exist then $n_1, \dots, n_r \in \overline{D(\mathscr{C}_1)}$ such that $\mathscr{N}_{n_i} n_i$, $i = 1, \dots, r$, is an open covering of $\overline{D(\mathscr{C}_1)}$. Then if $\mathscr{N} := \bigcap_{i=1}^r \mathscr{N}_{n_i}$, for any $n \in D(\mathscr{C}_1)$ there is i such that $\mathscr{N} n \subseteq \tilde{\mathscr{N}_{n_i}} n_i \subseteq D(\mathscr{C}_2)$, i.e. $\mathscr{N} \cdot D(\mathscr{C}_1) \subseteq D(\mathscr{C}_2)$, which immediately implies $\mathscr{N} \cdot \mathscr{C}_1 \subseteq \mathscr{C}_2$. For the general case, if a is the common apex of $\mathscr{C}_1, \mathscr{C}_2$, by what we have just seen there exists $\mathscr{N} \subseteq \mathscr{P}_+^\uparrow$ such that $\mathscr{N} \cdot (\mathscr{C}_1 - a) \subseteq \mathscr{C}_2 - a$ and if $\mathscr{N}_a := (\mathbb{1}, a)\mathscr{N}(\mathbb{1}, -a)$ then $\mathscr{N}_a \cdot \mathscr{C}_1 \subseteq \mathscr{C}_2$. □

Proof of proposition A.6. Given a double cone $D \subseteq \mathscr{D}$ containing $\overline{D(\mathscr{C}_1)}$ and $n \in D(\mathscr{C}_1)$, the spacelike cone $\mathscr{C}_2 := \mathscr{C}_{a_1 - \delta n, D}$, where a_1 is the apex of $\mathscr{C}_1$ and $\delta > 0$, is such that $a_1 \in \mathscr{C}_2$, and then $a_1 + B_\varepsilon \subseteq \mathscr{C}_2$ for some $\varepsilon > 0$. Then if $\tilde{\mathscr{C}_1} := \mathscr{C}_{a_1, D}$, by the last lemma there exists a neighbourhood of the identity $\tilde{\mathscr{N}}$ in $\mathscr{P}_+^\uparrow$ such that $\tilde{\mathscr{N}} \cdot \mathscr{C}_1 \subseteq \tilde{\mathscr{C}_1}$, and by lemma A.7, $\tilde{\mathscr{C}_1} + B_\varepsilon \subseteq \mathscr{C}_2$, so that the proposition is proven with $\mathscr{N} := (\{\mathbb{1}\} \times B_\varepsilon) \cdot \tilde{\mathscr{N}}$. □

The next proposition gives the justification of the homotopy argument used at the end of the proof of lemma 3.23.

Proposition A.9. *If $\mathscr{O}$ is a double cone, and $\mathscr{C}, \tilde{\mathscr{C}}$ are spacelike cones containing $\mathscr{O}$, there exist spacelike cones $\mathscr{C}_1, \dots, \mathscr{C}_r$ and $\hat{\mathscr{C}}_1, \dots, \hat{\mathscr{C}}_{r-1}$, such that $\mathscr{C}_1 = \mathscr{C}$, $\mathscr{C}_r = \tilde{\mathscr{C}}$, and $\mathscr{O} \subset \mathscr{C}_i$, $\mathscr{C}_i \cup \mathscr{C}_{i+1} \subseteq \hat{\mathscr{C}}_i$, $i = 1, \dots, r-1$.*

Proof. By combining lemma A.5 and corollary A.4, it is clear that it is sufficient to show the existence of double cones $D_1,\dots,D_r$ and $\hat{D}_1,\dots,\hat{D}_{r-1}$ in $\mathscr{D}$, such that $D_1 = D(\mathscr{C})$, $D_r = D(\tilde{\mathscr{C}})$ and $D_i \cup D_{i+1} \subseteq \hat{D}_i$, $i = 1,\dots,r-1$. To this end we fix a Lorentz frame e_μ, $\mu = 0,\dots,3$, and we endow $\mathscr{D}$ with the metric induced by the metric d on $\mathbb{R}^4$ given by $d(x,y) := |x^0 - y^0| + |\boldsymbol{x} - \boldsymbol{y}|$. We denote by $K_r(n) \subseteq \mathscr{D}$ the open ball of radius $r > 0$ and centered at $n \in \mathscr{D}$ defined by this metric. Clearly there holds $K_r(n) = \mathscr{K}_r(n) \cap \mathscr{D}$, where $\mathscr{K}_r(n)$ is the corresponding ball in $\mathbb{R}^4$, i.e. the "upright" double cone $\mathscr{O}_{n+re_0,n-re_0}$, but in general $K_r(n)$ is not a double cone in $\mathscr{D}$. However any $K_r(n)$ contains a double cone $D \subseteq \mathscr{D}$ (it is sufficient to take $n_\pm \in K_r(n)$ with n_+ future to n_-, then $\mathscr{O}_{n_+,n_-} \subseteq \mathscr{K}_r(n)$ and $D_{n_+,n_-} \subseteq K_r(n)$), and if $r < |\boldsymbol{n}| - |n^0|$, i.e. if the vertices $(n^0 \pm r, \boldsymbol{n})$ of $\mathscr{K}_r(n)$ are spacelike to the origin, then $K_r(n)$ is contained in some double cone in $\mathscr{D}$: it is in fact easy to check that the points

$$n_\pm := (n^0 \pm r, \boldsymbol{n}) + s_\pm \left(\omega_\pm, \frac{\boldsymbol{n}}{|\boldsymbol{n}|}\right), \qquad s_\pm := \pm \frac{r(2n^0 \pm r)}{2\big(|\boldsymbol{n}| - \omega_\pm(n^0 \pm r)\big)},$$

where $\omega_\pm = \operatorname{sgn}(2n^0 \pm r)$, are such that $n_\pm \in \mathscr{D}$ and $n_\pm \in n \pm re_0 + \partial V_\pm$ (since $r < |\boldsymbol{n}| - |n^0|$, the denominator of $s_\pm$ is positive, and then $s_\pm\omega_\pm \gtrless 0$), so that $K_r(n) \subseteq D_{n_+,n_-}$.

Having established that, we fix $n \in D(\mathscr{C})$, $\tilde{n} \in D(\tilde{\mathscr{C}})$ and a continuous curve $t \in [0,1] \to z(t) \in \mathscr{D}$ joining n to $\tilde{n}$. Given $\varepsilon < \min_{t\in[0,1]}\{|\boldsymbol{z}(t)| - |z^0(t)|\}$, $\varepsilon > 0$, we can find, by uniform continuity of z, a $\delta > 0$ such that if $|t - t'| < \delta$, then $d(z(t), z(t')) < \varepsilon$. Fix then $0 = \hat{t}_0 < \hat{t}_1 < \dots < \hat{t}_r = 1$ such that $|\hat{t}_i - \hat{t}_{i-1}| < \delta$, and $t_i \in (\hat{t}_{i-1}, \hat{t}_i)$, $i = 1,\dots,r$. This implies $z(t_i), z(t_{i+1}) \in K_\varepsilon(z(\hat{t}_i))$ and then, by what we have just seen, there exist double cones $D_i \ni z(t_i)$ and $\hat{D}_i$ such that $D_i \cup D_{i+1} \subseteq K_\varepsilon(z(\hat{t}_i)) \subseteq \hat{D}_i$, which concludes the proof. □

Finally we turn to the proof of a result concerning the existence of "sufficiently many" spacelike separated spacelike cones, which is needed to prove that the scaling limit field net constructed in section 3.3 has normal commutation relations.

We introduce the following notation: given a double cone $\mathscr{O}_{a,b}$ we denote by $M_{a,b}$ the spacelike (affine) hyperplane of those $x \in \mathbb{R}^4$ such that $(x-c)\cdot(a-b) = 0$, where $c \in M_{a,b}$ is the midpoint between a and b (the *centre* of $\mathscr{O}_{a,b}$). We also denote by $\tilde{M}_{a,b}$ the subset of $M_{a,b}$ which is spacelike to $\mathscr{O}_{a,b}$. As an example, chosen a Lorentz frame e_μ, $M_{re_0,-re_0}$ is the time zero hyperplane $\{0\} \times \mathbb{R}^3$ and $\tilde{M}_{re_0,-re_0} = \{0\} \times \{\boldsymbol{x} : |\boldsymbol{x}| > r\}$. It will be important in the following that for $a \in \tilde{M}_{a_+,a_-}$, $(a - a_+)^2 = (a - a_-)^2$, so that, if $n_\pm := (a_\pm - a)/\sqrt{-(a_\pm - a)^2} \in \mathscr{D}$, then $\mathscr{O}_{a_+,a_-} - a = \lambda D_{n_+,n_-}$ with $\lambda := \sqrt{-(a_\pm - a)^2}$, and then $\mathscr{O}_{a_+,a_-} \subset \mathscr{C}_{a,D_{n_+,n_-}}$.

Proposition A.10. *Given spacelike separated double cones $\mathscr{O}_1$, $\mathscr{O}_2$, there exist spacelike separated spacelike cones $\mathscr{C}_1$, $\mathscr{C}_2$, such that $\mathscr{O}_i \subset \mathscr{C}_i$, $i = 1,2$.*

Proof. Let $\mathscr{O}_i = \mathscr{O}_{a_{i,+},a_{i,-}}$, $c_i = (a_{i,+} + a_{i,-})/2$, $i = 1,2$. Assume first that $a_{1,+} - a_{1,-}$ and $a_{2,+} - a_{2,-}$ are not proportional. Then $M_{a_{1,+},a_{1,-}} \cap M_{a_{2,+},a_{2,-}}$ is a 2-dimensional spacelike affine subspace of $\mathbb{R}^4$. Furthermore, assume that $a_{1,+} - a_{2,+}$ is not a linear combination of $a_{1,+} - a_{1,-}$ and $a_{2,+} - a_{2,-}$. Then if N is the hyperplane

$$N = \left\{ a \in \mathbb{R}^4 \,:\, a \cdot (a_{1,+} - a_{2,+}) = \frac{1}{2}(a_{1,+}^2 - a_{2,+}^2) \right\},$$

$M_{a_{1,+},a_{1,-}} \cap M_{a_{2,+},a_{2,-}} \cap N$ is a spacelike line. There exists then an a on this line which is spacelike to both $\mathscr{O}_1$ and $\mathscr{O}_2$, and since $a \in N$ if and only if $(a - a_{1,+})^2 = (a - a_{2,+})^2$, we put $\lambda := \sqrt{-(a_{1,\pm} - a)^2} = \sqrt{-(a_{2,\pm} - a)^2}$ and $n_{i,\pm} := (a_{i,\pm} - a)/\lambda \in \mathscr{D}$. It is then clear that the double cones $D_i := D_{n_{i,+},n_{i,-}}$, $i = 1,2$ are spacelike separated, because such are their vertices (for instance, $(n_{1,+} - n_{2,-})^2 = \lambda^{-2}(a_{1,+} - a_{2,-})^2 < 0$), and then so are also the spacelike cones $\mathscr{C}_i := \mathscr{C}_{a,D_i}$: if $(n_1 - n_2)^2 = -2(1 + n_1 \cdot n_2) < 0$ then $n_1 \cdot n_2 > -1$, so that, for any $\lambda_1, \lambda_2 > 0$, $(\lambda_1 n_1 - \lambda_2 n_2)^2 < -\lambda_1^2 - \lambda_2^2 + 2\lambda_1\lambda_2 = -(\lambda_1 - \lambda_2)^2 < 0$. But, by the remark above, $\mathscr{O}_i \subseteq \mathscr{C}_i$ and we have the statement in this case.

If $a_{1,+} - a_{1,-}$ and $a_{2,+} - a_{2,-}$ are proportional, we can find a Lorentz frame in which $a_{i,+} - a_{i,-} = 2r_i e_0$, $r_i > 0$, and $c_1 = c^0 e_0 - (r_1 + \delta)e_1$, $c_2 = -c^0 e_0 + (r_2 + \delta)e_1$, and, if the two double cones are not tangent, i.e. if $\delta > 0$, it is clear that $\mathscr{C}_{a_i, D_{n_{i,+},n_{i,-}}}$, where $a_1 = c_1 + (r_1 + \varepsilon)e_1 \in \tilde{M}_{a_{1,+},a_{1,-}}$, $a_2 = c_2 - (r_2 + \varepsilon)e_1 \in \tilde{M}_{a_{2,+},a_{2,-}}$, $n_{i,\pm} := (a_{i,\pm} - a_i)/\sqrt{-(a_{i,\pm} - a_i)^2}$, are spacelike separated for $\varepsilon > 0$ sufficiently small, and contain $\mathscr{O}_{a_{i,+},a_{i,-}}$. If the two double cones are tangent, $c^0 = 0$ and the hyperplanes $M_{a_{1,+},a_{1,-}}$ and $M_{a_{2,+},a_{2,-}}$ really coincide, so that we can still apply the argument of the first part of the proof, because $a_{1,+} - a_{2,+}$, being lightlike, cannot be proportional to the timelike vector $a_{1,+} - a_{1,-}$.

Finally, the case in which $a_{1,+} - a_{2,+}$ is a linear combination of $a_{1,+} - a_{1,-}$ and $a_{2,+} - a_{2,-}$ (not proportional to each other) can again be reduced to a 2 dimensional situation (in the plane defined by $a_{1,+} - a_{1,-}$ and $a_{2,+} - a_{2,-}$) and it is then similar to the last one. $\square$

APPENDIX B

An example of ultraviolet stable charge

In this appendix we shall consider the simple free field model defined by the Majorana field in $d=1+3$ spacetime dimensions with $\mathbb{Z}_2$ gauge group, and after having discussed at some extent its superselection structure, we shall show that the localizable $\mathbb{Z}_2$ charge described by this model indeed satisfies the condition of ultraviolet stability, definition 3.10.

The free spin $1/2$ field and its associated local algebras are discussed in many references (see, for instance, [BLOT90], [Del68]). However, since the conventions may vary considerably form one source to another, here, for the convenience of the reader, we will give a brief outline of the construction. We will mainly follow [Fre], where also a discussion of the superselection structure is given.

We begin with some notational conventions. Let $\gamma^\mu \in M_4(\mathbb{C})$, $\mu = 0,\dots,3$, be the Dirac matrices, satisfying the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{B.1}$$

g being as usual the Minkowski metric. A consequence is $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, $A^\dagger$ being the adjoint matrix of A. A possible solution to these relations, to which we will stick in the following, is the so called *chiral representation*:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \qquad j = 1,2,3, \tag{B.2}$$

where σ_j are the Pauli matrices. A vector $u \in \mathbb{C}^4$ (also called a spinor) will be thought as a column matrix

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix},$$

and correspondingly its adjoint $u^\dagger = \begin{pmatrix} \overline{u}_1 & \overline{u}_2 & \overline{u}_3 & \overline{u}_4 \end{pmatrix}$ will be a row matrix, so that the standard scalar product on $\mathbb{C}^4$ is given by $(u,v) \to u^\dagger v$ (rows by columns product of matrices). A very useful notation is $\not{v} := v_\mu \gamma^\mu$ for any (covariant) vector $v \in \mathbb{R}^4$, and one has

$\not{v}^2 = v_\mu v_\nu \gamma^\mu \gamma^\nu = 1/2 v_\mu v_\nu \{\gamma^\mu, \gamma^\nu\} = v^2$, where in the one but to last equality the symmetry of the tensor $v_\mu v_\nu$ was used. By $\Omega_m^\pm$ we shall indicate the upper and lower mass $m > 0$ hyperboloid, $\Omega_m^\pm := \{p \in \mathbb{R}^4 : p^2 = m^2, \pm p_0 > 0\}$.

For a given mass $m > 0$, the Dirac operator is $D := \gamma^\mu \partial_\mu + im$, and, denoting as usual by $\mathscr{D}(\mathbb{R}^4; \mathbb{C}^4)$ the space of spinor valued, compactly supported smooth functions on Minkowski space, we endow the space $H_{0,m} := \mathscr{D}(\mathbb{R}^4; \mathbb{C}^4) / \operatorname{Im} D$ with the scalar product

$$\langle f, g \rangle_m := \int_{\mathbb{R}^3} d^3\boldsymbol{p} \sum_\pm \hat{f}(\pm\omega_m(\boldsymbol{p}), \boldsymbol{p})^\dagger P_\pm(\boldsymbol{p}) \hat{g}(\pm\omega_m(\boldsymbol{p}), \boldsymbol{p}), \tag{B.3}$$

where

$$P_\pm(\boldsymbol{p}) = \pm \frac{\gamma^0(\not{p} + m)}{2\omega_m(\boldsymbol{p})}\bigg|_{p_0 = \pm\omega_m(\boldsymbol{p})}, \qquad \omega_m(\boldsymbol{p}) = \sqrt{|\boldsymbol{p}|^2 + m^2}, \tag{B.4}$$

and where we made no notational distinction between elements in $H_{0,m}$ and their representatives in $\mathscr{D}(\mathbb{R}^4; \mathbb{C}^4)$.

We have, for $p_0 = \pm\omega_m(\boldsymbol{p})$, $\not{p}\gamma^0\not{p} = \{\not{p}, \gamma^0\}\not{p} - \gamma^0\not{p}^2 = \pm 2\omega_m(\boldsymbol{p})\not{p} - m^2\gamma^0$, and then

$$\begin{aligned} P_\pm(\boldsymbol{p})^2 &= \frac{1}{4\omega_m(\boldsymbol{p})^2}\gamma^0(\not{p} + m)\gamma^0(\not{p} + m) \\ &= \frac{1}{4\omega_m(\boldsymbol{p})^2}\gamma^0\big(\pm 2\omega_m(\boldsymbol{p})\big)(\not{p} + m) = P_\pm(\boldsymbol{p}), \end{aligned}$$

and moreover, it is clear that $P_\pm(\boldsymbol{p})^\dagger = P_\pm(\boldsymbol{p})$, i.e. $P_\pm(\boldsymbol{p})$ are orthogonal projections on $\mathbb{C}^4$, for which it also holds $P_+(\boldsymbol{p}) + P_-(\boldsymbol{p}) = \mathbb{1}$, $P_+(\boldsymbol{p})P_-(\boldsymbol{p}) = 0$. Furthermore $(\not{p} + m)(\not{p} - m) = p^2 - m^2$, so that $P_\pm(\boldsymbol{p})(\not{p} - m)|_{p_0 = \pm\omega_m(\boldsymbol{p})} = 0$ and then, taking into account that $\widehat{Df}(p) = -i(\not{p} - m)\hat{f}(p)$, we find that $\langle\cdot,\cdot\rangle_m$ is well defined and positive semidefinite on $H_{0,m}$. To see that it is really strictly positive, and hence a scalar product, note that if $(\not{p} + m)\hat{f}(p) = 0$ for each $p \in \Omega_m^\pm$, then for $q \in \Omega_m^\pm$, $\partial^\mu\big((\not{p} + m)\hat{f}(p)\big)|_{p=q}$ is normal in q to $\Omega_m^\pm$ (with respect to Minkowski metric), i.e. is proportional to q, so that

$$\begin{aligned} (\not{p} + m)\hat{f}(p) &= \alpha(2q)\cdot(p - q) + O\big(|p - q|^2\big) \\ &= \alpha\big((q + p) + (q - p)\big)\cdot(p - q) + O\big(|p - q|^2\big) = \alpha(p^2 - m^2) + O\big(|p - q|^2\big), \end{aligned}$$

and hence, by a straightforward application of the Paley-Wiener theorem (cfr. [RS75], theorem IX.11), the function

$$\hat{g}(p) := \frac{i(\not{p} + m)\hat{f}(p)}{p^2 - m^2}$$

is the Fourier transform of a function $g \in \mathscr{D}(\mathbb{R}^4; \mathbb{C}^4)$ and $f = Dg$, so that $f = 0$ in $H_{0,m}$. We will then denote by H_m the completion of $H_{0,m}$ in this scalar product.

We shall need to consider the action of the universal covering of the Poincaré group on H_m, defined, for $(A, a) \in \tilde{\mathscr{P}}_+^\uparrow$, by

$$\big(u(A,a)f\big)(x) := S(A)f\big(\Lambda(A)^{-1}(x - a)\big), \qquad S(A) := \begin{pmatrix} A & 0 \\ 0 & (A^\dagger)^{-1} \end{pmatrix}, \tag{B.5}$$

$A \in SL(2,\mathbb{C}) \to \Lambda(A) \in SO^\uparrow(1,3)$ being the covering homomorphism. The basic identity satisfied by $S(A)$ is

$$S(A)\gamma^\mu S(A)^{-1} = (\Lambda(A)^{-1})^\mu{}_\nu\gamma^\nu, \tag{B.6}$$

and using this it is easy to verify that $u(A,a)D = Du(A,a)$, so that $u(A,a)$ is well defined as an operator on $H_{0,m}$. Equation (B.6) also implies $\not{p}\, S(A) = S(A)\,\not{p}'$, with $p' = \Lambda(A)^{-1}p$ (recall that p, being covariant, transforms under Λ as $(\Lambda p)_\mu = p_\nu(\Lambda^{-1})^\nu{}_\mu = \big((\Lambda^{-1})^t\big)_\mu{}^\nu p_\nu$), which, together with $\big(u(A,a)f\big)\hat{}(p) = e^{ip\cdot a}S(A)\hat{f}(\Lambda(A)^{-1}p)$, the Lorentz invariance of the measure $d^3\boldsymbol{p}/2\omega_m(\boldsymbol{p})$ [RS75, appendix to IX.8], and $S(A)^\dagger\gamma^0 S(A) = \gamma^0$, entails unitarity of $u(A,a)$ on H_m. The strong continuity of the unitary representation u of $\tilde{\mathscr{P}}_+^\uparrow$ thus defined, is a consequence of an argument, based on the dominated convergence theorem, that is essentially a particular case of the argument used below to show that the charge carried by the Majorana field is ultraviolet stable, so we don't repeat it here and refer the reader to the proof of proposition B.5.

The last piece of structure that we have to introduce on the single particle space H_m is *charge conjugation*: define the antilinear operator C on $\mathbb{C}^4$ by $Cu := i\gamma^2\overline{u}$, where the bar denotes complex conjugation.[a)] Since $i\gamma^2 = (i\gamma^2)^\dagger = (i\gamma^2)^t$, it is readily verfied that $C^2 = \mathbb{1}$ and $C^\dagger = C$, where, being C antilinear, $C^\dagger$ is defined by $\langle C^\dagger u, v\rangle = \langle Cv, u\rangle$, and, using Dirac matrices anticommutation relations, $C\gamma^\mu C = -\gamma^\mu$. We then define an antilinear involution Γ on $H_{0,m}$ by $(\Gamma f)(x) := Cf(x)$, which is well defined, since it anticommutes with D. Furthermore, using some γ-gymnastic, one verifies that $i\gamma^2 P_\pm(\boldsymbol{p}) i\gamma^2 = P_\mp(-\boldsymbol{p})^t$, which implies that Γ is antiunitary, $\langle \Gamma f, \Gamma g\rangle_m = \langle g, f\rangle_m$, and it extends then to H_m. One also has $i\gamma^2 S(A) i\gamma^2 = \overline{S(A)}$, so that Γ commutes with the action of the Poincaré group on H_m.

We consider then the CAR algebra $\mathfrak{A}(H_m)$ over H_m [BR79b], generated as a C*-algebra by elements $a(f)$, $f \in H_m$, such that $f \to a(f)$ is antilinear, and

$$\{a(f), a(g)^*\} = \langle f, g\rangle_m \mathbb{1}, \qquad \{a(f), a(g)\} = 0 = \{a(f)^*, a(g)^*\}. \tag{B.7}$$

By unicity of the CAR algebra, the representation u of $\tilde{\mathscr{P}}_+^\uparrow$ on H_m induces an automorphic action α of $\tilde{\mathscr{P}}_+^\uparrow$ on $\mathfrak{A}(H_m)$, defined by

$$\alpha_{(A,a)}(a(f)) := a(u(A,a)f), \qquad (A,a) \in \tilde{\mathscr{P}}_+^\uparrow, f \in H_m,$$

and, by the fact that $\|a(f)\| = \|f\|_m$ and strong continuity of u, it follows that this action is strongly continuous, i.e. $(A,a) \to \alpha_{(A,a)}(B)$ is norm continuous for each $B \in \mathfrak{A}(H_m)$.

We restrict our attention to the elements $B(f) \in \mathfrak{A}(H_m)$ given by

$$B(f) := \frac{1}{\sqrt{2}}[a(\Gamma f) + a(f)^*], \qquad f \in H_m, \tag{B.8}$$

and to the sub-C*-algebra $\mathfrak{B}(H_m)$ of $\mathfrak{A}(H_m)$ they generate, and we consider on this algebra the state ω defined by

$$\begin{aligned} &\omega\big(B(f_1)\dots B(f_{2n+1})\big) := 0,\\ &\omega\big(B(f_1)\dots B(f_{2n})\big) := (-1)^{\frac{n(n+1)}{2}} \sum_{\sigma\in P_{2n}} \operatorname{sgn}\sigma \prod_{i=1}^n \langle \Gamma f_{\sigma(i)}, f_{\sigma(i+n)}\rangle_{m,+}, \end{aligned} \tag{B.9}$$

[a)] The appearing of γ^2 in the definition of C depends on the chosen representation of the Dirac matrices.

where $P_{2n} \subset S_{2n}$ is the set of *pairings* of $\{1,\dots,2n\}$, i.e. permutations $\sigma \in S_{2n}$ such that $\sigma(1) < \cdots < \sigma(n)$, and $\sigma(i) < \sigma(i+n)$, $i = 1,\dots,n$, and where $\langle\cdot,\cdot\rangle_{m,+}$ is the positive energy part of $\langle\cdot,\cdot\rangle_m$, obtained from it by dropping the term containing P_-.

Since $\alpha_{(A,a)}(B(f)) = B(u(A,a)f)$, the action α of $\tilde{\mathscr{P}}_+^\uparrow$ restricts to $\mathfrak{B}(H_m)$, and ω is left invariant by α, because, by the same calculation showing that $u(A,a)$ is unitary, $\langle \Gamma u(A,a) f, u(A,a)g\rangle_{m,+} = \langle \Gamma f, g\rangle_{m,+}$. If we then consider the GNS representation $(\pi,\mathscr{H},\Omega)$ induced by ω, we get on $\mathscr{H}$ a unitary strongly continuous representation U of $\tilde{\mathscr{P}}_+^\uparrow$ leaving Ω invariant and such that (π,U) is a covariant representation of $(\mathfrak{B}(H_m),\alpha)$.

Definition B.1. The *free Majorana field of mass* $m > 0$ is the operator valued distribution $f \in \mathscr{D}(\mathbb{R}^4;\mathbb{C}^4) \to \psi(f) \in \mathscr{B}(\mathscr{H})$ given by $\psi(f) := \pi\big(B(f)\big)$, $f \in \mathscr{D}(\mathbb{R}^4;\mathbb{C}^4)$, where on the left hand side f is identified with its image in H_m.

If we introduce a formal column matrix of fields of operators $\psi(x)$, $x \in \mathbb{R}^4$, such that

$$\psi(f) = \int_{\mathbb{R}^4} d^4x\, (\psi(x)^*)^t \gamma^0 f(x),$$

from $\psi(\Gamma f) = \psi(f)^*$ we find $C\psi(x) = i\gamma^2\psi(x)^* = -\psi(x)$, i.e. $\psi(x)$ is neutral (the overall phase -1 being irrelevant).

As customary, $\Delta_m := (2\pi)^4 i(d\Omega_m^+ - d\Omega_m^-)^\vee$, will be the commutator distribution for the Klein-Gordon field of mass m, where $d\Omega_m^\pm$ are the positive Radon measures on $\mathbb{R}^4$ defined by

$$\int_{\mathbb{R}^4} d\Omega_m^\pm(p)\; f(p) = \int_{\mathbb{R}^3} \frac{d^3\boldsymbol{p}}{2\omega_m(\boldsymbol{p})}\; f(\pm\omega_m(\boldsymbol{p}),\boldsymbol{p}), \qquad f \in C_c(\mathbb{R}^4).$$

We shall indicate by $\langle\cdot,\cdot\rangle$ the canonical pairing between $\mathscr{S}(\mathbb{R}^4)$ and $\mathscr{S}'(\mathbb{R}^4)$. We then summarize in the following proposition the main properties of the Majorana field.

Proposition B.2. *With the above notations, there holds:*

(i) *ψ is covariant with respect to U,*

$$U(A,a)\psi(f)U(A,a)^* = \psi(u(A,a)f),$$

the spectrum of the representation of the translation group $a \to U(\mathbb{1},a)$ is contained in the forward light cone, and Ω is the unique (up to a phase) translation invariant unit vector in $\mathscr{H}$;

(ii)

$$\{\psi(f),\psi(g)^*\} = \langle f_\beta * \big[\big(\gamma^0(i\partial\!\!\!/ + m)\big)_{\alpha\beta}\Delta_m\big], \overline{g}_\alpha\rangle\,\mathbb{1},$$

and in particular $\{\psi(f),\psi(g)^*\} = 0$ if the supports of f and g are spacelike separated;

(iii) *for any $f \in \mathscr{D}(\mathbb{R}^4,\mathbb{C}^4)$, $\psi((\partial\!\!\!/ + im)f) = 0$, i.e. ψ is a distributional solution of the Dirac equation.*

Introducing the 4×4 matrix of tempered distributions $S_m := (i\partial\!\!\!/ + m)\Delta_m$, the anticommutation relations in (ii) can be formally written as

$$\{\psi(f),\psi(g)^*\} = \int_{\mathbb{R}^4} d^4x \int_{\mathbb{R}^4} d^4y\; g(x)^\dagger \gamma^0 S_m(x-y) f(y)$$

which is frequently found in physics texts.

Proof of proposition B.2. It is standard, so we can be brief. (i) Covariance of ψ is immediate from the definitions. To show that the translations satisfy the spectrum condition, we first note that for any choice of $f_1,\dots,f_n$, $g_1,\dots,g_n$ in $\mathscr{D}(\mathbb{R}^4,\mathbb{C}^4)$ and h in $L^1(\mathbb{R}^4)$ with $\operatorname{supp}\hat{h}$ disjoint from $\overline{V}_+$, we have, using the shorthand notation $p_+ = (\omega_m(\boldsymbol{p}),\boldsymbol{p})$,

$$\int_{\mathbb{R}^4} d^4a\, h(a) \prod_{i=1}^n \langle f_i, u(\mathbb{1},a) g_i\rangle_{m,+} = \int_{\mathbb{R}^{3n}} \prod_{i=1}^n d^3\boldsymbol{p}_i\ \hat{h}(p_{1+}+\dots+p_{n+}) \prod_{i=1}^n \hat{f}_i(p_{i+})^\dagger P_+(\boldsymbol{p}_i)\hat{g}_i(p_{i+}) = 0,$$

since, being $p_{i+}\in\Omega_m^+$, $p_{1+}+\dots+p_{n+}\in\overline{V}_+$. Then, by the defintion of ω this implies

$$\int_{\mathbb{R}^4} d^4a\, h(a) \big(\psi(f_1)\dots\psi(f_n)\Omega\big|U(\mathbb{1},a)\psi(g_1)\dots\psi(g_m)\Omega\big) = \int_{\mathbb{R}^4} d^4a\, h(a)\omega\big(B(\Gamma f_1)\dots B(\Gamma f_n)B(u(\mathbb{1},a)g_1)\dots B(u(\mathbb{1},a)g_m)\big) = 0,$$

for any choice of the functions f_i, g_j and h as above, and, as Ω is cyclic for $\pi(\mathfrak{B}(H_m))$, this is sufficent to conclude that $\operatorname{Sp}U(\mathbb{1},\cdot)\subseteq\overline{V}_+$. To show that Ω is the unique translation invariant vector, it is sufficient to show that clustering holds, and in particular that

$$\lim_{|\boldsymbol{a}|\to+\infty} \big(\Omega\big|\psi(f_1)\dots\psi(f_n)\alpha_{\boldsymbol{a}}\big(\psi(g_1)\dots\psi(g_m)\big)\Omega\big) = \big(\Omega\big|\psi(f_1)\dots\psi(f_n)\Omega\big)\big(\Omega\big|\psi(g_1)\dots\psi(g_m)\Omega\big),$$

for any choice of the functions f_i, g_j in $\mathscr{D}(\mathbb{R}^4,\mathbb{C}^4)$, but this is a consequence of the vanishing of the truncated n-point functions and of the fact that, due to the smoothness of f and g,

$$\lim_{|\boldsymbol{a}|\to+\infty}\langle f, u(\mathbb{1},\boldsymbol{a})g\rangle_{m,+} = \lim_{|\boldsymbol{a}|\to+\infty}\int_{\mathbb{R}^3} d^3\boldsymbol{p}\, e^{i\boldsymbol{p}\cdot\boldsymbol{a}}\hat{f}(p_+)^\dagger P_+(\boldsymbol{p})\hat{g}(p_+) = 0.$$

(ii) From the definitions, we have $\{\psi(f),\psi(g)^*\} = \pi(\{B(f),B(\Gamma g)\}) = \langle g,f\rangle_m\mathbb{1}$ and a calculation using (B.3) shows that the formula in the statement holds. In particular, since $\operatorname{supp}\Delta_m\subseteq\overline{V}$ [RS75, thm. IX.48], $\{\psi(f),\psi(g)^*\}=0$ if $\operatorname{supp} f$ is spacelike separated from $\operatorname{supp} g$.

(iii) Immediate. □

We now turn to the consideration of the net of local von Neumann algebras associated to the free Majorana field, and defined by

$$\mathscr{F}(\mathscr{O}) := \{\psi(f) : \operatorname{supp} f\subseteq\mathscr{O}\}'', \tag{B.10}$$

for $\mathscr{O}\subset\mathbb{R}^4$ open and bounded. On this net the group $\mathbb{Z}_2$ acts by an automorphism β_k induced by the automorphism of $\mathfrak{B}(H_m)$ defined by $B(f)\to -B(f)$ (which is in turn the restriction to $\mathfrak{B}(H_m)$ of the automorphism of $\mathfrak{A}(H_m)$ defined, through CAR unicity, by $a(f)\to -a(f)$), which leaves the vacuum state ω invariant. This also implies that β_k is implemented by a unitary operator $V(k)$ on $\mathscr{H}$ such that $V(k)^2 = V(k^2) = \mathbb{1}$. $V(k)$ then induces a direct sum decompostion $\mathscr{H} = \mathscr{H}_+\oplus\mathscr{H}_-$ according to its eigenspaces, i.e. $V(k)\restriction\mathscr{H}_\pm = \pm\mathbb{1}_{\mathscr{H}_\pm}$, which is clearly preserved by the operators $F \in \mathscr{F}(\mathscr{O})$ which are $\mathbb{Z}_2$-invariant, $\beta_k(F) = F$, as well as by $U(A,a)$ for each $(A,a) \in \tilde{\mathscr{P}}_+^\uparrow$, since α and β clearly commute, so that we can define $U_+(A,a) := U(A,a) \restriction \mathscr{H}_+$, and the net of observable von Neumann algebras associated to the free Majorana field as

$$\mathscr{A}(\mathscr{O}) := \mathscr{F}(\mathscr{O})^{\mathbb{Z}_2} \restriction \mathscr{H}_+. \tag{B.11}$$

That in this way we get an example satisfying the assumptions made in section 3.2 is the content of the following proposition.

Proposition B.3. *With the above notations, let $\pi_{\mathscr{F}}$ be the representation of the quasi-local algebra $\mathscr{A}$ defined by $\pi_{\mathscr{F}}(A \restriction \mathscr{H}_+) := A$. Then $\pi_{\mathscr{F}}$ is well defined, $(\mathscr{H}_+, \mathscr{A}, U_+, \Omega)$ is a Poincaré covariant observable net, and $(\pi_{\mathscr{F}}, \mathscr{F}, V, k, U)$ is a Poincaré covariant, normal field net with gauge symmetry over $(\mathscr{H}_+, \mathscr{A}, U_+, \Omega)$. Furthermore*

$$\pi_{\mathscr{F}}(\mathscr{A}(\mathscr{O})) = \{\psi(f)\psi(g) : \operatorname{supp} f, \operatorname{supp} g \subseteq \mathscr{O}\}'', \tag{B.12}$$

and if we denote by $\mathfrak{A}(\mathscr{O})$ the sub-C^*-algebra of $\mathscr{A}(\mathscr{O})$ of those $A \in \mathscr{A}(\mathscr{O})$ such that $s \in \tilde{\mathscr{P}}_+^\uparrow \to \alpha_s(A)$ is norm continuous, then $\mathscr{A}(\mathscr{O}) = \mathfrak{A}(\mathscr{O})^-$.

Proof. As usual, the fact that $\pi_{\mathscr{F}}$ is well defined will be a consequence of the Reeh-Schlieder theorem, once we will have shown that $\mathscr{F}$ is a covariant normal field net. Isotony and Poincaré covariance of $\mathscr{O} \to \mathscr{F}(\mathscr{O})$ and of $\mathscr{O} \to \mathscr{A}(\mathscr{O})$ are evident from the definitions, since $\operatorname{supp} u(A,a) f = \Lambda(A) \operatorname{supp} f + a$. We also note that U_+ factors through $\mathscr{P}_+^\uparrow$, since from $u(-\mathbb{1},0) f = -f$ $(S(-\mathbb{1}) = -\mathbb{1})$, $U(-\mathbb{1},0) = V(k)$ follows. In order to prove normality, we first show that

$$\mathscr{F}(\mathscr{O})_+ = \mathscr{F}(\mathscr{O})^{\mathbb{Z}_2} = \{\psi(f)\psi(g) : \operatorname{supp} f, \operatorname{supp} g \subseteq \mathscr{O}\}''.$$

Provided that $\pi_{\mathscr{F}}$ is well defined, we have $\pi_{\mathscr{F}}(\mathscr{A}(\mathscr{O})) = \mathscr{F}(\mathscr{O})^{\mathbb{Z}_2}$, so that from the above formula we will get (B.12). The inclusion $\{\psi(f)\psi(g) : \operatorname{supp} f, \operatorname{supp} g \subseteq \mathscr{O}\}'' \subseteq \mathscr{F}(\mathscr{O})^{\mathbb{Z}_2}$ is evident. To prove the converse one, note that $\mathscr{F}(\mathscr{O})^{\mathbb{Z}_2} = m(\mathscr{F}(\mathscr{O}))$, with $m = (\iota + \beta_k)/2$ the invariant mean over $\mathbb{Z}_2$ (ι being the identity automorphism of $\mathscr{B}(\mathscr{H})$). Then since $\mathscr{F}(\mathscr{O})$ is the weak closure of the span of elements $\psi(f_1)\dots\psi(f_n)$, $\operatorname{supp} f_i \subseteq \mathscr{O}$, and $m\big(\psi(f_1)\dots\psi(f_{2n})\big) = \psi(f_1)\dots\psi(f_{2n})$, $m\big(\psi(f_1)\dots\psi(f_{2n+1})\big) = 0$, we get that $\mathscr{F}(\mathscr{O})^{\mathbb{Z}_2}$ is the weak closure of the span of elements $\psi(f_1)\dots\psi(f_{2n})$, $\operatorname{supp} f_i \subseteq \mathscr{O}$, which coincides with the von Neumann algebra generated by $\psi(f)\psi(g)$, $\operatorname{supp} f$, $\operatorname{supp} g \subseteq \mathscr{O}$. Analogously one finds that $\mathscr{F}(\mathscr{O})_-$ is the weakly closed span of products of an odd number of $\psi(f)$, $\operatorname{supp} f \subseteq \mathscr{O}$. Then if $\operatorname{supp} f_i \subseteq \mathscr{O}_1$, $i = 1,\dots,n$, $\operatorname{supp} g_j \subseteq \mathscr{O}_2$, $j = 1,\dots,m$, and $\mathscr{O}_1 \subseteq \mathscr{O}_2'$, we have, from proposition B.2(ii), $\psi(f_i)\psi(g_j) = -\psi(g_j)\psi(f_i)$, and then

$$\psi(f_1)\dots\psi(f_n)\psi(g_1)\dots\psi(g_m) = (-1)^{nm}\psi(g_1)\dots\psi(g_m)\psi(f_1)\dots\psi(f_n),$$

which implies normality of commutation relations for $\mathscr{F}$, and spacelike commutativity for $\mathscr{A}$. Positivity of the energy for U, and hence for U_+, as well as uniqueness of the vacuum, have been proven in proposition B.2, thus the quasi-local algebras $\mathscr{F}$, $\mathscr{A}$ are irreducible on the respective Hilbert spaces. It remains only to show that $\mathscr{A}(\mathscr{O}) = \mathfrak{A}(\mathscr{O})^-$, but this is an immediate consequence of the strong continuity of the action of $\tilde{\mathscr{P}}_+^\uparrow$ on $\pi(\mathfrak{B}(H_m))$, pointed out before, and of formula (B.12). □

In the next proposition, the very simple superselection structure of $\mathscr{A}$ described by the field net $\mathscr{F}$ is analysed.

Proposition B.4. *The representation π_- of $\mathscr{A}$ given by $\pi_- := \pi_{\mathscr{F}}(\cdot) \restriction \mathscr{H}_-$ is irreducibile and satisfies the DHR criterion, and any irreducible representation of $\mathscr{A}$ appearing in $\mathscr{H}$ is equivalent either to ι, the identity representation, or to π_-. Moreover, if ρ_f is the automorphism of $\mathscr{A}$ induced by the 1-dimensional Hilbert space $H_f = \mathbb{C}\psi(f)$, with $\operatorname{supp} f \subseteq \mathscr{O}$, $\Gamma f = f$ and $\|f\|_m = \sqrt{2}$, then ρ_f is localized in $\mathscr{O}$ and transportable, and $\rho_f \cong \pi_-$.*

Proof. The irreducibility of π_- follows from the arguments in [DHR69a], since from $\pi_{\mathscr{F}}(\mathscr{A}(\mathscr{O})) = \mathscr{F}(\mathscr{O})^{\mathbb{Z}_2}$ and irreducibility of $\mathscr{F}$, $\pi_{\mathscr{F}}(\mathscr{A})'' = V(\mathbb{Z}_2)'$ follows, and $\mathscr{H}_-$ is the subspace associated to the irreducible representation $k \to -1$ of $\mathbb{Z}_2$ in the factorial decomposition of V. This also implies that any other irreducible representation of $\mathscr{A}$ in $\mathscr{H}$ is equivalent to ι or π_-. To show that π_- satisfies the DHR criterion, fix a double cone $\mathscr{O}$ and an $f \in \mathscr{D}(\mathscr{O};\mathbb{C}^4)$ with $\Gamma f = f$ and $\|f\|_m = \sqrt{2}$. Then

$$\psi(f)\psi(f)^* = \psi(f)^*\psi(f) = \psi(f)^2 = \frac{1}{2}\{\psi(f),\psi(f)\} = \frac{1}{2}\|f\|_m^2 \mathbb{1} = \mathbb{1},$$

i.e. $\psi(f)$ is unitary, and, since $\mathscr{H}_\pm$ is spanned by products of an even (odd) number of field operators applied to the vacuum, $\psi(f)\mathscr{H}_\pm = \mathscr{H}_\mp$. Let then $V_f := \psi(f) \restriction \mathscr{H}_-$, and $g, h \in \mathscr{D}(\mathscr{O}', \mathbb{C}^4)$. We have

$$V_f \pi_-\big(\psi(g)\psi(h)\big) = \psi(f)\psi(g)\psi(h) \restriction \mathscr{H}_- = \psi(g)\psi(h)\psi(f) \restriction \mathscr{H}_- = \psi(g)\psi(h)V_f,$$

so that V_f intertwines between $\pi_- \restriction \mathscr{A}(\mathscr{O}')$ and $\iota \restriction \mathscr{A}(\mathscr{O}')$. Finally this also shows that if $\pi_{\mathscr{F}}\rho_f(A) = \psi(f)\pi_{\mathscr{F}}(A)\psi(f)^*$ then ρ_f is localized in $\mathscr{O}$, and V_f intertwines between π_- and ρ_f, and if $\operatorname{supp} f_1 \subseteq \mathscr{O}_1$, $V := \psi(f)\psi(f_1)^* \restriction \mathscr{H}_+$ intertwines between ρ_{f_1} and ρ_f, which is therefore transportable. □

Finally we come to the proof of the fact that the charge $\xi := [\pi_-]$ in the above proposition is ultraviolet stable, in the sense of definition 3.10. To this end, since the Hilbert spaces inducing the various ρ_f are 1-dimensional, it is sufficient to find, for every double cone $\mathscr{O}$, a family $(f_\lambda)_{\lambda\in(0,1]}$ of functions such that $\operatorname{supp} f_\lambda \subseteq \lambda\mathscr{O}$, $\|f_\lambda\|_m = \sqrt{2}$, and such that condition (3.11) is satisfied with $\psi(\lambda) = \psi(f_\lambda)$.

Proposition B.5. *For every double cone $\mathscr{O}$, there exists $f \in \mathscr{D}(\mathscr{O};\mathbb{C}^4)$ such that, if $f_\lambda \in \mathscr{D}(\lambda\mathscr{O};\mathbb{C}^4)$ is defined by*

$$f_\lambda(x) := \lambda^{\frac{3}{2}-4} f(\lambda^{-1}x), \qquad \lambda \in (0,1],$$

then $\|f_\lambda\|_m = \sqrt{2}$ and $\Gamma f_\lambda = f_\lambda$ for $\lambda \in (0,1]$, and

$$\lim_{(A,a)\to(\mathbb{1},0)} \sup_{\lambda\in(0,1]} \big\| \big[\alpha_{(A,\lambda a)}\big(\psi(f_\lambda)\big) - \psi(f_\lambda)\big]\Omega \big\| = 0. \tag{B.13}$$

In the course of the proof of this proposition, we will need the following simple result concerning the action of the Lorentz group on Minkowski space.

Lemma B.6. *Fix a mass $m > 0$. For any sufficiently large $R > 0$, there exists a neighbourhood of the identity $\mathscr{N}$ in $SO^\uparrow(1,3)$ such that, for any $p \in \overline{V}_+$ with $0 \le p^2 \le m^2$ and $|\boldsymbol{p}| > R$, and for any $\Lambda \in \mathscr{N}$, it holds, for $p' := \Lambda p$, $|\boldsymbol{p}'| > |\boldsymbol{p}|/\sqrt{2}$.*

Proof. To simplify the notation, we will write $\Lambda\cdot\boldsymbol{p}$ for the spatial part, in a given Lorentz frame, of the 4-vector Λp, $\Lambda\in SO^{\uparrow}(1,3)$, $p\in\mathbb{R}^4$. One should always keep in mind, however, that $\Lambda\cdot\boldsymbol{p}$ depends on the time component p_0 of p as well. Let $\Lambda_1(s)$, $s\in\mathbb{R}$, denote the 1-parameter group of boosts in the p_1 direction.

If $p\in\mathbb{R}^4$ is such that $|p_1|^2\le|p_2|^2+|p_3|^2$, since $\Lambda_1(s)$ leaves the components p_2, p_3 unaffected, we have, for any $s\in\mathbb{R}$, $|\Lambda_1(s)\cdot\boldsymbol{p}|^2\ge|p_2|^2+|p_3|^2\ge|\boldsymbol{p}|^2/2$.

Assume now that $|p_1|^2>|p_2|^2+|p_3|^2$. This implies $|p_1|\ge|\boldsymbol{p}|/\sqrt{2}$ and, since for any sufficiently large $R>0$,

$$\inf_{\substack{0\le p^2\le m^2\\ |\boldsymbol{p}|>R}}\frac{|\boldsymbol{p}|}{|p_0|}\ge\inf_{|\boldsymbol{p}|>R}\frac{|\boldsymbol{p}|}{\sqrt{|\boldsymbol{p}|^2+m^2}}>0,$$

we can find a $\delta>0$ such that, if $|s|<\delta$,

$$|(\Lambda_1(s)p)_1|=|\sinh s\,p_0+\cosh s\,p_1|\ge|p_1|-|\sinh s|p_0\ge\frac{|p_1|}{\sqrt{2}},$$

for any $p\in\overline{V}_+$ with $0\le p^2\le m^2$ and $|\boldsymbol{p}|>R$, so that

$$|\Lambda_1(s)\cdot\boldsymbol{p}|^2\ge\frac{p_1^2}{2}+p_2^2+p_3^2\ge\frac{|\boldsymbol{p}|^2}{2}.$$

Then, if we identify in the canonical way $SO(3)$ with a subgroup of $SO^{\uparrow}(1,3)$, since $|R\cdot\boldsymbol{p}|=|\boldsymbol{p}|$ for any $R\in SO(3)$, we conclude with $\mathscr{N}:=\{R_1\Lambda_1(s)R_2\,:\,|s|<\delta,\,R_1,R_2\in SO(3)\}$. □

Proof of proposition B.5. In order to shorten formulae, we will use the notation $p_{\lambda,\pm}:=(\pm\omega_{\lambda m}(\boldsymbol{p}),\boldsymbol{p})$, as well as the notation $\Lambda\cdot\boldsymbol{p}$ introduced in the proof of the above lemma. Also, $|\cdot|$ will denote the norm of a vector both in $\mathbb{R}^3$ and in $\mathbb{C}^4$. A calculation shows

$$\|f_\lambda\|_m^2=\|f\|_{\lambda m}^2=\int_{\mathbb{R}^3}\frac{d^3\boldsymbol{p}}{4\omega_{\lambda m}(\boldsymbol{p})^2}\sum_{\pm}\left|\gamma^0(\not{p}_{\lambda,\pm}+\lambda m)\hat{f}(p_{\lambda,\pm})\right|^2,$$

and then, in order to show that there is an $f\in\mathscr{D}(\mathscr{O},\mathbb{C}^4)$ such that $\|f_\lambda\|_m=\sqrt{2}$ and $\Gamma f_\lambda=f_\lambda$ for each $\lambda\in(0,1]$, it is sufficient to exhibit an $f\in\mathscr{D}(\mathscr{O},\mathbb{C}^4)$ such that $\Gamma f=f$, and for which $(\not{p}+\mu)\hat{f}(p)$ is not identically zero on each hyperboloid $\Omega_\mu:=\Omega_\mu^+\cup\Omega_\mu^-$, $\mu>0$. A direct check shows that these conditions are met by $f(x):=g(x)(\mathbb{1}+i\gamma^2)\begin{pmatrix}1&0&0&0\end{pmatrix}^t$ where $g\in\mathscr{D}(\mathscr{O};\mathbb{R})$.

We show then that for any f satisfying the stated conditions, (B.13) holds. Since $\|\gamma^\mu\|^2=\|(\gamma^\mu)^\dagger\gamma^\mu\|=\|\gamma^0\gamma^\mu\gamma^0\gamma^\mu\|=1$ (norm in $M_4(\mathbb{C})$), we have $\|\gamma^0(\not{p}_{\lambda,\pm}+\lambda m)\|\le\omega_{\lambda m}(\boldsymbol{p})+3|\boldsymbol{p}|+\lambda m\le5\omega_{\lambda m}(\boldsymbol{p})$, and then, being $u(A,\lambda a)f_\lambda=\big(u(A,a)f\big)_\lambda$, we have, writing $p_\lambda:=p_{\lambda,+}$,

$$\begin{aligned}\left\|\left[\alpha_{(A,\lambda a)}\big(\psi(f_\lambda)\big)-\psi(f_\lambda)\right]\Omega\right\|^2&=\|u(A,a)f-f\|_{\lambda m,+}^2\\&=\int_{\mathbb{R}^3}\frac{d^3\boldsymbol{p}}{4\omega_{\lambda m}(\boldsymbol{p})^2}\left|\gamma^0(\not{p}_\lambda+\lambda m)\Big(e^{ip_\lambda\cdot a}S(A)\hat{f}(\Lambda(A)^{-1}p_\lambda)-\hat{f}(p_\lambda)\Big)\right|^2\\&\le\frac{5}{4}\int_{\mathbb{R}^3}\frac{d^3\boldsymbol{p}}{|\boldsymbol{p}|}\left|e^{ip_\lambda\cdot a}S(A)\hat{f}(\Lambda(A)^{-1}p_\lambda)-\hat{f}(p_\lambda)\right|^2,\end{aligned}$$

where we also used $\omega_{\lambda m}(\boldsymbol{p}) \geq |\boldsymbol{p}|$. Then denoting $\|\cdot\|_2$ the standard norm in $L^2(\mathbb{R}^3, d^3\boldsymbol{p}/|\boldsymbol{p}|)\otimes\mathbb{C}^4$, and using repeatedly the triangle inequality,

$$\begin{aligned}\|u(A,a)f - f\|_{\lambda m,+} \leq \sqrt{\frac{5}{4}}\Big\{\|S(A)\|\Big[\big\|\hat{f}(\Lambda(A)^{-1}p_\lambda) - \hat{f}(p_\lambda)\big\|_2 \\ + \big\|(e^{ip_\lambda\cdot a}-1)\hat{f}(p_\lambda)\big\|_2\Big] \\ + \|S(A)-\mathbb{1}\|\|\hat{f}(p_\lambda)\|_2\Big\},\end{aligned} \tag{B.14}$$

where, for more clarity, we indicated explicitly the variable of integration inside the norms. The last term in this equation can be estimated uniformly in λ by the fact that, being $\hat{f}\in\mathscr{S}(\mathbb{R}^4;\mathbb{C}^4)$, there are constants $C>0$, $n>1$, such that

$$\int_{\mathbb{R}^3}\frac{d^3\boldsymbol{p}}{|\boldsymbol{p}|}\big|\hat{f}(p_\lambda)\big|^2 \leq C\int_{\mathbb{R}^3}\frac{d^3\boldsymbol{p}}{|\boldsymbol{p}|}\frac{1}{(1+\omega_{\lambda m}(\boldsymbol{p})^2+|\boldsymbol{p}|^2)^n} \leq C\int_{\mathbb{R}^3}\frac{d^3\boldsymbol{p}}{|\boldsymbol{p}|}\frac{1}{(1+2|\boldsymbol{p}|^2)^n},$$

so that it can be made arbitrarily small, as $A\to\mathbb{1}$, uniformly in $\lambda\in(0,1]$. For the second term in square brackets in (B.14), we have, by an application of Lagrange theorem to the exponential, and using $\omega_{\lambda m}(\boldsymbol{p})\leq\omega_m(\boldsymbol{p})$ for $\lambda\in(0,1]$,

$$\int_{\mathbb{R}^3}\frac{d^3\boldsymbol{p}}{|\boldsymbol{p}|}\big|(e^{ip_\lambda\cdot a}-1)\hat{f}(p_\lambda)\big|^2 \leq C(|a^0|^2+|\boldsymbol{a}|^2)\int_{\mathbb{R}^3}\frac{d^3\boldsymbol{p}}{|\boldsymbol{p}|}\frac{|\omega_m(\boldsymbol{p})|^2+|\boldsymbol{p}|^2}{(1+2|\boldsymbol{p}|^2)^n},$$

and then, if $n>2$, this term is also uniformly small in the relevant limit. Finally, we use the above lemma to estimate the first term in square bracket in (B.14). For each sufficiently large $R>0$ let $\mathscr{N}_R$ be a neighbourhood of the identity in $SL(2,\mathbb{C})$, such that $\Lambda(\mathscr{N}_R)\subseteq SO^\uparrow(1,3)$ is as in the lemma. Then, for $|\boldsymbol{p}|>R$, $A\in\mathscr{N}_R$,

$$\begin{aligned}\big|\hat{f}(\Lambda(A)^{-1}p_\lambda)-\hat{f}(p_\lambda)\big| &\leq C\left[\frac{1}{(1+2|\Lambda(A)^{-1}\cdot\boldsymbol{p}_\lambda|^2)^n}+\frac{1}{(1+2|\boldsymbol{p}_\lambda|^2)^n}\right] \\ &\leq C\left[\frac{1}{(1+|\boldsymbol{p}|^2)^n}+\frac{1}{(1+2|\boldsymbol{p}|^2)^n}\right].\end{aligned}$$

Thus, again by Lagrange theorem, we have, for $A\in\mathscr{N}_R$,

$$\begin{aligned}\int_{\mathbb{R}^3}\frac{d^3\boldsymbol{p}}{|\boldsymbol{p}|}\big|\hat{f}(\Lambda(A)^{-1}p_\lambda)-\hat{f}(p_\lambda)\big|^2 &\leq C\int_{|\boldsymbol{p}|>R}\frac{d^3\boldsymbol{p}}{|\boldsymbol{p}|}\left[\frac{1}{(1+|\boldsymbol{p}|^2)^n}+\frac{1}{(1+2|\boldsymbol{p}|^2)^n}\right]^2 \\ &\quad+\|\partial\hat{f}\|_\infty^2\|\Lambda(A)^{-1}-\mathbb{1}\|^2\int_{|\boldsymbol{p}|<R}\frac{d^3\boldsymbol{p}}{|\boldsymbol{p}|}(|\omega_m(\boldsymbol{p})|^2+|\boldsymbol{p}|^2).\end{aligned}$$

and the λ independent right hand side can be made arbitrarily small by taking R sufficiently large, and A in a corresponding neighbourhood $\tilde{\mathscr{N}}_R\subseteq\mathscr{N}_R$. □

Bibliography

[AHR62] H. Araki, K. Hepp and D. Ruelle, *On the asymptotic behaviour of Wightman functions in spacelike directions*, Helv. Phys. Acta **35** (1962), 164–174.

[Ara63] H. Araki, *A lattice of von Neumann algebras associated with the quantum theory of a free boson field*, J. Math. Phys. **4** (1963), 1343–1362.

[Ara64] H. Araki, *Von Neumann algebras of local observables for free scalar field*, J. Math. Phys. **5** (1964), 1–13.

[BB99] H. J. Borchers and D. Buchholz, *Global properties of vacuum states in de Sitter space*, Ann. Inst. Henri Poincaré **70** (1999), 23–40.

[BF82] D. Buchholz and K. Fredenhagen, *Locality and the structure of particle states*, Commun. Math. Phys. **84** (1982), 1–54.

[BGL93] R. Brunetti, D. Guido and R. Longo, *Modular structure and duality in conformal quantum field theory*, Commun. Math. Phys. **156** (1993), 201–220.

[BJ87] D. Buchholz and P. Jacobi, *On the nuclearity condition for massless fields*, Lett. Math. Phys. **13** (1987), 313–323.

[BLOT90] N. N. Bogolubov, A. A. Logunov, A. I. Oksak and I. T. Todorov, *General principles of quantum field theory*, Kluwer Academic Publishers, 1990.

[Bor60] H. J. Borchers, *Über die Mannigfaltigkeit der interpolierenden Felder zu einer kausalen S-Matrix*, Nuovo Cimento **15** (1960), 784–794.

[Bor67] H. J. Borchers, *A remark on a theorem of B. Misra*, Commun. Math. Phys. **4** (1967), 315–323.

[Bor92] H. J. Borchers, *The CPT theorem in two-dimensional theories of local observables*, Commun. Math. Phys. **143** (1992), 315–332.

[Bor96] H. J. Borchers, *Translation group and particle representations in quantum field theory*, Lecture Notes in Physics, no. m40, Springer, 1996.

[BR79a] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics*, vol. 1, Springer-Verlag, 1979.

[BR79b] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics*, vol. II, Springer-Verlag, 1979.

[Buc96a] D. Buchholz, *Phase space properties of local observables and structure of scaling limits*, Ann. Inst. Henri Poincaré **64** (1996), 433–460.

[Buc96b] D. Buchholz, *Quarks, gluons, colour: Facts or fiction?*, Nucl. Phys. **B469** (1996), 333–356.

[BV95] D. Buchholz and R. Verch, *Scaling algebras and renormalization group in algebraic quantum field theory*, Rev. Math. Phys. **7** (1995), 1195–1239.

[BV98] D. Buchholz and R. Verch, *Scaling algebras and renormalization group in algebraic quantum field theory. II: Instructive examples*, Rev. Math. Phys. **10** (1998), 775–800.

[BW75] J. J. Bisognano and E. H. Wichmann, *On the duality condition for a hermitian scalar field*, J. Math. Phys. **16** (1975), 985–1007.

[BW76] J. J. Bisognano and E. H. Wichmann, *On the duality condition for quantum fields*, J. Math. Phys. **17** (1976), 303–321.

[D'A90] C. D'Antoni, *Technical properties of the quasi-local algebra* in *The algebraic theory of superselection sectors. Introduction and recent results* (Palermo, 1989), D. Kastler, ed., World Scientific, 1990, pp. 248–258.

[Del68] G. F. Dell'Antonio, *Structure of the algebras of some free systems*, Commun. Math. Phys. **9** (1968), 81–117.

[DHR69a] S. Doplicher, R. Haag and J. E. Roberts, *Fields, observables and gauge transformations I*, Commun. Math. Phys. **13** (1969), 1–23.

[DHR69b] S. Doplicher, R. Haag and J. E. Roberts, *Fields, observables and gauge transformations II*, Commun. Math. Phys. **15** (1969), 173–200.

[DHR71] S. Doplicher, R. Haag and J. E. Roberts, *Local observables and particle statistics I*, Commun. Math. Phys. **23** (1971), 199–230.

[DHR74] S. Doplicher, R. Haag and J. E. Roberts, *Local observables and particle statistics II*, Commun. Math. Phys. **35** (1974), 49–85.

[DR72] S. Doplicher and J. E. Roberts, *Fields, statistics and non-abelian gauge groups*, Commun. Math. Phys. **28** (1972), 331–348.

[DR89a] S. Doplicher and J. E. Roberts, *Endomorphisms of C^*-algebras, cross products and duality for compact groups*, Ann. Math. **130** (1989), 75–119.

[DR89b] S. Doplicher and J. E. Roberts, *A new duality theory for compact groups*, Invent. Math. **98** (1989), 157–218.

[DR90] S. Doplicher and J. E. Roberts, *Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics*, Commun. Math. Phys. **131** (1990), 51–107.

[Dri79] W. Driessler, *Duality and absence of locally generated superselection sectors for CCR type algebras*, Commun. Math. Phys. **70** (1979), 213–220.

[DSW86] W. Driessler, S. J. Summers and E. H. Wichmann, *On the connection between quantum fields and von Neumann algebras of local operators*, Commun. Math. Phys. **105** (1986), 49–84.

[DV] C. D'Antoni and R. Verch, in preparation.

[FM83] K. Fredenhagen and M. Marcu, *Charged states in $\mathbb{Z}_2$ gauge theories*, Commun. Math. Phys. **92** (1983), 81–119.

[Fre] K. Fredenhagen, *Superselection sectors*, notes from lectures held at Hamburg University in the Winter Term 1994/95.

[Haa96] R. Haag, *Local quantum physics*, IInd ed., Springer, 1996.

[HK64] R. Haag and D. Kastler, *An algebraic approach to quantum field theory*, J. Math. Phys. **5** (1964), 848–861.

[HS65] R. Haag and J. A. Swieca, *When does a quantum field theory describes particles?*, Commun. Math. Phys. **1** (1965), 308–320.

[IZ80] C. Itzykson and J.-B. Zuber, *Quantum field theory*, McGraw-Hill, 1980.

[KR86] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras*, vol. II, Academic Press, 1986.

[KS75] J. B. Kogut and L. Susskind, *Hamiltonian formulation of Wilson's lattice gauge theory*, Phys. Rev. **D11** (1975), 395–408.

[Kun01] W. Kunhardt, *On infravacua and the superselection structure of theories with massless particles*, Ph.D. thesis, Göttingen University, 2001.

[Loo53] L. H. Loomis, *An introduction to abstract harmonic analysis*, Van Nostrand, 1953.

[LS71] J. H. Lowenstein and J. A. Swieca, *Quantum electrodynamics in two dimensions*, Ann. Phys. **68** (1971), 172–195.

[Lut97] M. Lutz, *Ein lokales Netz ohne Ultraviolettfixpunkte der Renormierungsgruppe*, Diploma thesis, Göttingen University, 1997.

[McL98] S. McLane, *Categories for the working mathematician*, IInd ed., Springer, 1998.

[Mun01] J. Mund, *The Bisognano-Wichmann theorem for massive theories*, Ann. Inst. Henri Poincaré **2** (2001), 907–926.

[Rob74] J. E. Roberts, *Some applications of dilatation invariance to structural questions in the theory of local observables*, Commun. Math. Phys. **37** (1974), 273–286.

[Rob76] J. E. Roberts, *Spontaneously broken gauge symmetries and superselection rules* in *Proceedings of the international school of mathematical physics*, G. Gallavotti, ed., Università di Camerino, 1976, pp. 423–437.

[Rob90] J. E. Roberts, *Lectures on algebraic quantum field theory* in *The algebraic theory of superselection sectors. Introduction and recent results* (Palermo, 1989), D. Kastler, ed., World Scientific, 1990, pp. 1–112.

[RRS79] H. J. Rothe, K. D. Rothe and J. A. Swieca, *Screening versus confinement*, Phys. Rev. **D19** (1979), 3020–3023.

[RS61] H. Reeh and S. Schlieder, *Bemerkungen zur Unitäräquivalenz von Lorentzinvarianten Feldern*, Nuovo Cimento **22** (1961), 1051–1068.

[RS75] M. Reed and B. Simon, *Methods of modern mathematical physics*, vol. II: Fourier analysis, self-adjointess, Academic Press, 1975.

[Sei96] N. Seiberg, *The power of duality – Exact results in 4D SUSY field theory* in *Particles, strings and cosmology. 19th Johns Hopkins Workshop on Current Problems in Particle Theory – PASCOS Interdisciplinary Symposium* (Baltimore, 1995), J. Bagger, G. Domokos, A. Falk and S. Kovesi-Domokos, eds., World Scientific, 1996, pp. 183–197.

[Sum82] S. J. Summers, *Normal product states for fermions and twisted duality for CCR and CAR type algebras with application to the Yukawa-2 quantum field model*, Commun. Math. Phys. **86** (1982), 111–141.

[Swi76] J. A. Swieca, *Charge screening and mass spectrum*, Phys. Rev. **D13** (1976), 312–314.

[WWW52] G. C. Wick, A. S. Wightman and E. P. Wigner, *The intrinsic parity of elementary particles*, Phys. Rev. **88** (1952), 101–105.

[Yos68] K. Yosida, *Functional analysis*, IInd ed., Springer-Verlag, 1968.

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