Università degli Studi di Roma "La Sapienza"
Dottorato di Ricerca in Matematica - XIII ciclo
Dipartimento di Matematica Istituto "Guido Castelnuovo"

Tesi

"Functional equations and Lie algebras"

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ANNO ACCADEMICO 2001/2002

Introduction

We consider a commutative superring $\mathbb{K} = \mathbb{K}_0 + \mathbb{K}_1$, and for all Lie superalgebra $\mathfrak{g}$ over $\mathbb{K}$ we study the representations of $\mathfrak{g}$ on its symmetric algebra $S(\mathfrak{g})$ which are *by coderivations and universal.* We stress that many of our results are new also for ordinary Lie algebras over a field of characteristic zero.

The symmetric algebra $S(\mathfrak{g})$ has a natural structure of coalgebra, so we have a notion of coderivation of $S(\mathfrak{g})$. A representation ρ of $\mathfrak{g}$ in $S(\mathfrak{g})$ is called *by coderivations* if $\rho(a)$ is a coderivation of $S(\mathfrak{g})$ for all $a \in \mathfrak{g}$. We focus on representations ρ by coderivations which are *universal.* This means informally that ρ is given by a formula independent of $\mathfrak{g}$. We will explain in section 2.4 why it is natural to consider this kind of representations.

To each formal power series

$$\varphi = c_0 + c_1 t + \cdots \in \mathbb{K}_0[[t]]$$

we associate a family $\Phi(a) \equiv \Phi^a$ of coderivations of $S(\mathfrak{g})$ depending linearly of $a \in \mathfrak{g}$.

We show that Φ is a universal representation by coderivations if and only if

$$\varphi(t)\frac{\varphi(t+u)-\varphi(u)}{t} + \varphi(u)\frac{\varphi(t+u)-\varphi(t)}{u} + \varphi(t+u) = 0 \qquad (1)$$

in $\mathbb{K}_0[[t, u]]$.

The most interesting case is when the constant term c_0 is equal to 1, In this case, it is a simple matter to solve the functional equation, because we show (lemma 2.2.2) that it is equivalent to the functional equation for the exponential function. This last equation has a non-trivial solution exactly when $\mathbb{K}$ contains $\mathbb{Q}$. In this case the unique solution of (1) is

$$\varphi = \frac{t}{e^t - 1}.$$

We think that it is an interesting fact that this famous function, the generation function for Bernoulli numbers, occurs in such an elementary manner.

We explain the relation of our results with classical subjects like Poincaré-Birkhoff-Witt theorem, Lie third theorem (chapter 3), Maurer-Cartan equation, etc. For example, let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$ and assume that $\mathbb{K} \supseteq \mathbb{Q}$. We use the representation obtained using the function $\frac{t}{e^t-1}$ to define a symbol map $\sigma : U(\mathfrak{g}) \to S(\mathfrak{g})$. We show that σ is an inverse for the symmetrization $\beta : S(\mathfrak{g}) \to U(\mathfrak{g})$, which gives a natural and direct proof of the fact that β is an isomorphism. This *strong* form of the Poincaré-Birkhoff-Witt theorem, with no assumption on $\mathfrak{g}$ as a $\mathbb{K}$-module, is due to P.M. Cohn (**[Coh]**) in the case of Lie algebras and to D. Quillen (**[Qui]**) in the case of super Lie algebras with $\mathbb{K} = \mathbb{K}_0$.

Let $N \geq 2$ be an integer. We consider a N-nilpotent Lie superalgebras $\mathfrak{g}$ over $\mathbb{K}$, and a truncated power series $\varphi \in \mathbb{K}_0[[t]]/t^N$. To φ is associated a family of coderivations Φ^a depending linearly of $a \in \mathfrak{g}$. We show that Φ is an universal representation by coderivations if and only if φ verifies equation (1) in $\mathbb{K}_0[[t,u]]/I_N$, where I_N is the ideal generated by $\{t^i u^{N-1-i} | 0 \leq i \leq N-1\}$. There exists a solution with $\varphi(0) = 1$ exactly when

$$\frac{1}{2}, \frac{1}{3}, ..., \frac{1}{N} \in \mathbb{K},$$

and in this case it is unique. As a consequence, we show that for a N-nilpotent Lie superalgebra over a commutative superring containing $\frac{1}{2}, \frac{1}{3}, ..., \frac{1}{N}$, there is a canonical symbol map $\sigma : U(\mathfrak{g}) \to S(\mathfrak{g})$ which is an isomorphism. When $\mathbb{K} \supseteq \mathbb{Q}$, σ^{-1} is (as seen above) the symmetrization, but in general we do not know a formula for σ^{-1}. For $N = 2$ this is due to M. El-Agawany and A. Micali (**[ElM]**). The case $N \geq 3$ is new.

The method used to get the universal representations by coderivations can be used to study the Maurer-Cartan equations over $\mathfrak{g}$. For simplicity we consider only the case of Lie algebras over a field $\mathbb{K}$ of characteristic zero. We show that each solution $f \in \mathbb{K}[[t]]$ of

$$\frac{f(u+t) - f(t)}{u} + \frac{f(u+t) - f(u)}{t} = f(u)f(t)$$

provides a solution of Maurer-Cartan equation. This functional equation has an unique solution with $f(0) = 1$, it is $f(t) = \frac{e^t-1}{t}$.

Let $(\mathfrak{g}, \mathfrak{h}, \mathfrak{q})$, with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ and $\mathfrak{h}$ a Lie $\mathbb{K}$-subsuperalgebra, be a $\mathbb{K}$-super symmetric space. In this situation we get similar results. In particular, if $\mathbb{K} \supseteq \mathbb{Q}$, we get representations $\Pi : \mathfrak{g} \to Hom(S(\mathfrak{q}), S(\mathfrak{q}))$ which are universal and by coderivations. Our proof uses functional equations of type

$$\varphi(t)\frac{\psi(t+u)-\psi(u)}{t}+\frac{\varphi(t+u)-\varphi(t)}{u}\psi(u)=h(t+u) \tag{2}$$

where φ, ψ, h are formal power series with coefficients in $\mathbb{K}_0$. In section 2.3 we classify all triples verifying (2), when $\mathbb{K}_0$ is a field of characteristic zero.

As an application of the representations Π, we study the image in $U(\mathfrak{g})$ of $S(\mathfrak{q})$ by the symmetrization map β. The super vector space $\beta(S(\mathfrak{q}))$ is stable under the so-called twisted adjoint action ad' of $\mathfrak{g}$. This induces an action of $\mathfrak{g}$ on $S(\mathfrak{q})$ which is one of the representations Π.

Let $\mathbb{K}$ be a field of characteristic zero. M. Gorelik proved in **[Gor]** that if $\mathfrak{q}$ a finite-dimensional, odd, $\mathbb{K}$-super vector space such that the trace of $\mathrm{ad}(a)|_{\mathfrak{q}} : \mathfrak{q} \to \mathfrak{q}$ is zero for all $a \in \mathfrak{h}$, then $\beta(S(\mathfrak{q}))$ has an unique line of ad'-invariant vectors. By solving

$$\begin{cases} t\coth\left(\frac{t}{2}\right)h'(t+u)-\frac{(t+u)\coth\left(\frac{t+u}{2}\right)-t\coth\left(\frac{t}{2}\right)}{u}=0 \\ h'(u)=h'(-u) \end{cases}$$

we get, in proposition 4.3.1, an explicit formula for Gorelik 's line. This formula is new and it provides a new proof of Gorelik 's result.

For each Lie superalgebra over $\mathbb{K}$ equipped with an invariant, even, symmetric bilinear form (we do not suppose that it is not-degenerate), we consider in chapter 5.2 a variant of the Classical Dynamical Yang-Baxter Equation (vCDYBE) with coupling constant $\varepsilon \in \mathbb{K}_0$. We show that to each odd series $f \in \mathbb{K}_0[[t]]$ verifying

$$\begin{aligned} &\tfrac{f(t+u)-f(u)}{t}+\tfrac{f(u+v)-f(v)}{u}+\tfrac{f(v+t)-f(t)}{v}= \\ &\qquad f(t)f(u)+f(u)f(v)+f(t)f(v)+\epsilon \mod u+t+v \end{aligned} \tag{3}$$

we can associate a solution of vCDYBE. Considering $v=-t-u$, from (3) we obtain a functional equation in two variables which was independently found by L. Feher and B. G. Pusztai in **[FeP]**, for a quadratic Lie algebra over $\mathbb{R}$ or $\mathbb{C}$.

The interesting case is when $\mathbb{K} \supseteq \mathbb{Q}$ and $\varepsilon = \frac{1}{4}$. In this case the unique solution of (3) is $f(z)=\frac{1}{z}-\frac{1}{2}\coth(\frac{z}{2})$, which gives the so-called *universal solution* of vCDYBE.

Let $\mathfrak{g}$ be a quadratic Lie algebra (over $\mathbb{C}$ or $\mathbb{R}$). It means that $\mathfrak{g}$ is finite-dimensional, provided with a not-degenerate invariant symmetric bilinear form. For each quadratic sub-algebra $\mathfrak{h}$ of $\mathfrak{g}$ there is a Classical Dynamical Yang-Baxter Equation (CDYBE) (see **[EtV]**).

A. Alekseev and E. Meinrenken considered the case $\mathfrak{h} = \mathfrak{g}$ and in this case the CDYBE is equivalent to the vCDYBE. They solve the vCDYBE in **[AlM]** for $\mathfrak{g}$ a compact Lie algebra. Our method of proof is much simpler, and it provides a solution of the vCDYBE for arbitrary quadratic Lie superalgebras. This was in fact the origin of the present work. For other proofs in the case of quadratic Lie algebras, see **[FeP]** and **[AlM1]**.

A. Alekseev and E. Meinrenken used the solution of CDYBE to prove some properties of their non-commutative Weil algebra. As the universal solution is valid also for any quadratic Lie superalgebra, we can extend a result of Alekseev and Meinrenken to any quadratic Lie superalgebra over a field of zero characteristic (see chapter 6).

Desidero innanzitutto ringraziare il Prof. Michel Duflo per avermi suggerito questo argomento di tesi e avermi poi seguito con disponibilità e pazienza. Molto del tempo dedicato a questa ricerca è stato trascorso a Parigi. Ringrazio il Collegio dei Docenti del Dipartimento di Matematica dell'Università di Roma "La Sapienza" per aver autorizzato i miei soggiorni a Parigi. Un ringraziamento speciale al Prof. Alessandro Silva. Tali soggiorni sono stati facilitati dal fatto che ho potuto usufruire dei mezzi e delle infrastrutture messe a disposizione dall'Ecole Normale Superieure e dall'Università "Paris - VII". Anche di questo sono grata a Michel Duflo.

Infine, un ringraziamento a quegli amici che, direttamente o indirettamente, hanno dato il loro sostegno durante gli anni passati in tesi.

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Chapter 1

Preliminaries

1.1 Lie superalgebras over a superring

In this section we recall the basic definitions and examples used in the text, they are from super linear algebra (**[Lei]**).

We say that $\mathbb{K}$ is a *superring* if it is an unitary ring graded over $\mathbb{Z}/2\mathbb{Z}$. We note $\mathbb{K}_0$ and $\mathbb{K}_1$ the subgroups of elements with even and odd degree, for each non-zero homogeneous element $a \in \mathbb{K}$ we note $p(a)$ its degree. We have $1 \in \mathbb{K}_0$.

The superring $\mathbb{K}$ is called *commutative* if $ab = (-1)^{p(a)p(b)}ba$ for all homogeneous and not-zero $a, b \in \mathbb{K}$ and $a^2 = 0$ for $a \in \mathbb{K}_1$.

Convention 1.1.1. *Each time we use the symbol $p(a)$ for an element a of a graded group occurring in a linear expression, it is implicitly assumed that it is not zero and homogeneous. Moreover the expression is extended by linearity. For example, the expression above will be written as $ab = (-1)^{p(b)p(a)}ba$ for any $a, b \in \mathbb{K}$.*

From now to the end of this section, $\mathbb{K}$ will be a fixed commutative superring.
We denote by $\mathbb{K}_0^\times \subseteq \mathbb{K}$ the subgroup of invertible elements of $\mathbb{K}_0$.

Definition. 1.1.1. *A commutative group $(M, +)$ graded over $\mathbb{Z}/2\mathbb{Z}$ is a $\mathbb{K}$-module if it is equipped with a bilinear application $M \times \mathbb{K} \to M$ such that, for all $\alpha, \beta \in \mathbb{K}$ and $m, n \in M$ we have*

$$(m\alpha)\beta = m(\alpha\beta)$$
$$p(m\alpha) = p(m) + p(\alpha).$$

We note M_0 and M_1 the $\mathbb{K}_0$-submodules composed of even and odd elements.
If $\mathbb{K}$ is a field, M is also called a $\mathbb{K}$-supervector space.

In a $\mathbb{K}$-module M we use the notation $\alpha m := m\alpha(-1)^{p(\alpha)p(m)}$, for any $m \in M$ and $\alpha \in \mathbb{K}$. Let N be another $\mathbb{K}$-module. A map $f : M \to N$ is a *morphism of* $\mathbb{K}$*-modules* if $f(m\alpha) = f(m)\alpha$ for any $m \in M$ and $\alpha \in \mathbb{K}$.

Definition. 1.1.2. *We say that A is a $\mathbb{K}$-superalgebra if it a $\mathbb{K}$-module equipped of a distributive application $A \times A \xrightarrow{\cdot} A$ such that*

$$p(a \cdot b) = p(a) + p(b)$$
$$(a \cdot b)\alpha = a \cdot (b\alpha) = (-1)^{p(b)p(\alpha)}(a\alpha) \cdot b$$

for all $a, b \in A$ *and* $\alpha \in \mathbb{K}$.
We say that A *is commutative if* $a \cdot b = (-1)^{p(a)p(b)} b \cdot a$ *for* $a, b \in A$, *and* $c^2 = 0$ *for* $c \in A_1$.

Let A and B be two $\mathbb{K}$-superalgebras. A map $f : A \to B$ is said a *morphism of* $\mathbb{K}$-*superalgebras* if it is a morphism of $\mathbb{K}$-modules such that

$$p(f(a)) = p(a), \quad f(a \cdot b) = f(a) \cdot f(b)$$

for any $a, b \in A$ and $\alpha \in \mathbb{K}$.

Notation 1.1.1. *Let* A *be a* $\mathbb{K}$*-superalgebra and* $a \in A$. *We denote by* $a^L : A \to A$ *the left multiplication by* a, *and by* $a^R : A \to A$ *the right multiplication:* $a^R(b) = (-1)^{p(a)p(b)} b \cdot a$, *for any* $b \in A$.

The following is our definition of Lie superalgebra.

Definition. 1.1.3. *Let* $\mathfrak{g}$ *be a* $\mathbb{K}$*-superalgebra such that its product* $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *verifies*

$$[X, Y] = -(-1)^{p(X)p(Y)}[Y, X], \quad \forall X, Y \in \mathfrak{g} \tag{1.1}$$

$$[X, X] = 0, \quad \forall X \in \mathfrak{g}_0 \tag{1.2}$$

$$[[X, Y], Z] = [X, [Y, Z]] - (-1)^{p(Y)p(X)}[Y, [X, Z]], \quad \forall X, Y, Z \in \mathfrak{g} \tag{1.3}$$

$$[Y, [Y, Y]] = 0, \quad \forall Y \in \mathfrak{g}_1. \tag{1.4}$$

Such $\mathfrak{g}$ *is called a Lie* $\mathbb{K}$*-superalgebra.*

The product in a Lie superalgebra is called Lie product or Lie bracket, and (1.3) is the *Jacobi identity.*

Remark 1.1.1. *If* $2 \in \mathbb{K}$ *is invertible (1.2) follows from (1.1). If* $3 \in \mathbb{K}$ *is invertible (1.4) follows from (1.1) and (1.3).*

As explained in **[BMP]**, if $\mathfrak{g}_1 \neq \{0\}$ and $2 \in \mathbb{K}$ is not invertible, definition 1.1.3 is not the right one, but it is sufficient for the purpose of this text.
We end this section with some useful examples.

Example 1.1.1. *If* A *is a commutative superring,* $A[[z]]$ *denote the set formal series in* s, *with coefficients in* A. *It inherits the graduation* $(A[[z]])_0 = A_0[[z]]$, $(A[[z]])_1 = A_1[[z]]$ *and a natural structure of commutative superring.*

Example 1.1.2. *Let* M, N *be two* $\mathbb{K}$*-modules.*

a) $Hom(M, N)$ *is the group of functions* $F : M \to N$ *which are morphisms of* $\mathbb{K}$*-modules. It is graded in the following way:* F *is even if* $F(M_0) \subseteq N_0$ *and* $F(M_1) \subseteq N_1$, F *is odd if* $F(M_0) \subseteq N_1$ *and* $F(M_1) \subseteq N_0$. *More over,* $Hom(M, N)$ *is a* $\mathbb{K}$*-module by* $F\alpha : v \mapsto (-1)^{p(v)p(\alpha)} F(v)\alpha$, *for all* $\alpha \in \mathbb{K}$, $v \in M$.

b) $M \otimes N$ *is the* $\mathbb{K}$*-module generated by* $\{v \otimes w; v \in M, w \in N, \}$ *with relations*

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$$
$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$$
$$(v \otimes w)\alpha = v \otimes w\alpha = (-1)^{p(w)p(\alpha)} v\alpha \otimes w, \quad \forall \alpha \in \mathbb{K}$$

and graduation $p(v \otimes w) = p(v) + p(w)$.

c) The tensor algebra of M *is* $T(M) := \mathbb{K} + (M \otimes M) + (M \otimes M \otimes M) + \cdots$ *with product* $(v_1 \otimes \cdots \otimes v_i) \cdot (v_{i+1} \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_n$, *for all* $i, n \in \mathbb{N}$. *It is an associative* $\mathbb{K}$*-superalgebra.*

Example 1.1.3. *Let* M *a* $\mathbb{K}$*-module,* $Hom(M, M)$ *is a Lie* $\mathbb{K}$*-superalgebra by* $[F, G] = F \circ G - (-1)^{p(F)p(G)} G \circ F$, $\forall F, G \in Hom(M, M)$.

1.2 Symmetric algebras

Let $\mathbb{K}$ be a commutative superring and M a $\mathbb{K}$-module, we recall the definition of its symmetric algebra $S(M)$.
The tensor algebra $T(M)$ contains the ideal I generated by

$$\left\{ v \otimes w - (-1)^{p(v)p(w)} w \otimes v, u \otimes u |\ v, w \in M, u \in M_1 \right\},$$

and we define $S(M) := T(M)/I$. It is a commutative and associative $\mathbb{K}$-superalgebra.

We have $S(M) = \mathbb{K} \oplus \bigoplus_{n=1}^{\infty} S^n(M)$, where $S^n(M)$ is the $\mathbb{K}$-module generated by products of n elements of M.

1.2.1 Formal functions

We recall that $S(M)$ has a natural structure of cocommutative Hopf superalgebra, and in particular it is a coalgebra. This means that $S(M)$ is equipped of three morphisms of superalgebras $\Delta : S(M) \to S(M) \otimes S(M)$, $\epsilon : S(M) \to \mathbb{K}$, $\delta : S(M) \to S(M)$, such that

$$(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta \tag{1.5}$$

$$\text{Mult} \circ (id \otimes \delta) \circ \Delta = \text{Mult} \circ (\delta \otimes id) \circ \Delta = \epsilon \tag{1.6}$$

$$\text{Mult} \circ (id \otimes \epsilon) \circ \Delta = \text{Mult} \circ (\epsilon \otimes id) \circ \Delta = id \tag{1.7}$$

$$\Delta = \sigma \circ \Delta \tag{1.8}$$

where $\text{Mult} : S(M) \otimes S(M) \to S(M)$ is the multiplication of $S(M)$ and $\sigma : S(M) \otimes S(M) \ni W \otimes Z \mapsto (-1)^{p(W)p(Z)} Z \otimes W$ is the exchange operator. We call δ an antipode, and each even morphism of $\mathbb{K}$-modules verifying (1.5) is called an *associative comultiplication.* We refer to (1.8) saying that Δ is *cocommutative.*
For all $X \in M$ we have

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \epsilon(X) = 0, \quad \delta(X) = -X.$$

To give formulas for Δ we introduce the following notation. Let $n \in \mathbb{N}$ and Σ_n be the group of permutations of n elements. For $s \in \Sigma_n$ and $X_1, .., X_n \in M$, let $\alpha(X_{s(1)}, ..., X_{s(n)}) \in \{1, -1\}$ be the sign such that $\alpha(X_{s(1)}, ..., X_{s(n)}) X_{s(1)} \cdots X_{s(n)} = X_1 \cdots X_n$ in $S(M)$.
If $X \in M_0$

$$\Delta\left(X^n\right) = \sum_{j=0}^{n} \binom{n}{j} X^j \otimes X^{n-j}, \ \forall n \geq 0 \tag{1.9}$$

and, if $X_1, ..., X_n \in M$

$$\Delta\left(X_1 \cdots X_n\right) = \sum_{j=0}^{n} \sum_{1 \leq p_1 < \cdots < p_j \leq n} \alpha(\vec{X_{\vec{p}}}) X_{p_1} \cdots X_{p_j} \otimes X_1 \cdots \widehat{X_{p_1}} \cdots \widehat{X_{p_j}} \cdots X_n$$

where $\alpha(\vec{X_{\vec{p}}}) := \alpha(X_{p_1}, ..., X_{p_j}, X_1, ..., \widehat{X_{p_1}}, ..., \widehat{X_{p_j}}, ..., X_n)$.

We note $S(M)^* := Hom(S(M), \mathbb{K})$. Because of $S(M)$ is a coalgebra, $S(M)^*$ is a commutative superalgebra and it is called the algebra of formal power series over M. More generally, if N is a $\mathbb{K}$-module, $Hom(S(M), N)$ is called the space of *formal functions* on M with values in N.

Each $X \in N$ defines a "constant function" of $Hom(S(M), N)$: it is the function such that $1 \mapsto X$ and $S^n(M) \mapsto \{0\}$ for $n \neq 0$. We have the following structure of $S(M)^*$-module: $F\varphi := (F \otimes \varphi) \circ \Delta$ for $\varphi \in S(M)^*$ and $F \in Hom(S(M), N)$. Let $Y \in M$, we define $\partial(Y) : Hom(S(M), N) \to Hom(S(M), N)$ as $f \mapsto (-1)^{p(f)p(Y)} f \circ Y^L$ for any $f \in Hom(S(M), N)$. It is called the derivative in the direction Y.

Remark 1.2.1. *By definition, $\partial(Y)(X) = 0$ for any $X \in N$.*

When $N = M$, $Hom(S(M), M)$ is called the space of *formal vectors field* over M. The identity of M extends to a morphism of $\mathbb{K}$-modules $x_M : S(M) \to M$ by $S^n(M) \mapsto \{0\}$ for $n \neq 1$. It is called the *generic point of* M, and it will be denoted by x when there is no risk of confusion.

Remark 1.2.2. *We have $\partial(Y)(x) = Y$ for all $Y \in M$.*

Let A be a $\mathbb{K}$-superalgebra. In $Hom(S(M), A)$ we have the following structure of $S(M)^*$-algebra: $F \cdot G := \mathrm{Mult} \circ (F \otimes G) \circ \Delta$, for any $F, G \in Hom(S(M), A)$.

Remark 1.2.3. *For all $Y \in M$, $\partial(Y)$ is a derivation of $Hom(S(M), A)$.*

We have seen that $A \subseteq Hom(S(M), A)$, moreover A is a $\mathbb{K}$-subsuperalgebra of $Hom(S(M), A)$. If A is associative $Hom(S(M), A)$ is associative, because Δ verifies (1.5). If A is unitary $Hom(S(M), A)$ is unitary, with unit given by $\epsilon : S(M) \ni W \mapsto 1\epsilon(W) \in A$. If A is commutative $Hom(S(M), A)$ is commutative, because Δ is a cocommutative comultiplication.

In the particular case $A = S(M)$ it is traditional to note $*$ the product of $Hom(S(M), S(M))$. In this case $\delta \in Hom(S(M), S(M))$ and identities (1.6), (1.7) give

$$\delta * id = id * \delta = \epsilon. \tag{1.10}$$

Lemma 1.2.1. *If $\mathfrak{g}$ is a Lie $\mathbb{K}$-superalgebra, $Hom(S(M), \mathfrak{g})$ is a Lie $S(M)^*$-superalgebra.*

1.2.2 Coderivations of a symmetric algebra

Definition. 1.2.1. *Let A be a $\mathbb{K}$-module equipped of a comultiplication Δ. A coderivation of A is a morphism of $\mathbb{K}$-modules $\Phi : A \to A$ such that $\Delta \circ \Phi = (\Phi \otimes id + id \otimes \Phi) \circ \Delta$.*

To describe the coderivations of $S(M)$, we introduce the $\mathbb{K}$-module $P(S(M) = \{W \in S(M) | \Delta(W) = 1 \otimes W + W \otimes 1\}$. Its elements are called the *primitive elements of* $S(M)$. By definition of Δ, $M \subseteq P(S(M))$.

Let $\varphi : S(M) \to P(S(M))$ be a morphism of $\mathbb{K}$-modules, we define $\Phi := id * \varphi : S(M) \to S(M)$.

Theorem 1.2.1. ([Rad])
The map Φ is a coderivation of $S(M)$ and $\delta * \Phi = \varphi$.

Proof. As a first step we show that Φ is a coderivation. By definition

$$(id \otimes \Phi) \circ \Delta = (id \otimes \mathrm{Mult}) \circ (id \otimes id \otimes \varphi) \circ (id \otimes \Delta) \circ \Delta$$
$$(\Phi \otimes id) \circ \Delta = (\mathrm{Mult} \otimes id) \circ (id \otimes \varphi \otimes id) \circ (\Delta \otimes id) \circ \Delta$$
$$\Delta \circ \Phi = \Big((\mathrm{Mult} \otimes id) \circ (id \otimes \varphi \otimes id) \circ (id \otimes \sigma) + (id \otimes \mathrm{Mult}) \circ (id \otimes id \otimes \varphi) \Big) \circ (\Delta \otimes id) \circ \Delta$$

so identities (1.5) and (1.8) give that Φ is a coderivation.

The second part of the theorem follows from identities (1.10). In fact we get $\varphi = \delta * id * \varphi = \delta * \Phi$. ▮

Remark 1.2.4. *The previous theorem is valid if $S(M)$ is replaced by any cocommutative Hopf superalgebra.*

1.3 Generic point of a Lie superalgebra

Let $\mathbb{K}$ be a commutative superring and $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie $\mathbb{K}$-superalgebra. For any $X \in \mathfrak{g}$, we note $\mathrm{ad}X$ the application $[X, \cdot] : \mathfrak{g} \to \mathfrak{g}$.

Let t and u be two even commuting variables. For any $r, q \in \mathbb{N}$ we introduce the notation

$$(t^r u^q : [Y, Z])_X := [(\mathrm{ad}X)^r(Y), (\mathrm{ad}X)^q(Z)], \quad \forall X \in \mathfrak{g}_0,\ \forall Y, Z \in \mathfrak{g}. \qquad (1.11)$$

By linearity it is extended to all polynomials in $\mathbb{K}[t, u]$.

Lemma 1.3.1. *For any polynomial $q \in \mathbb{K}[z]$, $X \in \mathfrak{g}_0$ and $Y, Z \in \mathfrak{g}$,*

$$q(\mathrm{ad}X)([Y, Z]) = (q(t + u) : [Y, Z])_X \,.$$

Proof. It is sufficient to consider $q(z) = z^k$ with $k \geq 1$. It means that it is sufficient to show that

$$(\mathrm{ad}X)^k([Y, Z]) = \sum_{p=0}^{k} \binom{k}{p} [(\mathrm{ad}X)^p(Y), (\mathrm{ad}X)^{k-p}(Z)],\ \ k \geq 1.$$

This identity means that $\mathrm{ad}X$ is an even derivation, which follows from the Jacobi identity. ▮

We introduce $\mathfrak{g}_x := Hom(S(\mathfrak{g}), \mathfrak{g})$. Each $X \in \mathfrak{g}$ is identified to its image in $\mathfrak{g}_x$. As seen above (section 1.2.1), the comultiplication Δ of $S(\mathfrak{g})$ and the bracket for $\mathfrak{g}$ allow to define the bilinear application $[F, G] := [\cdot, \cdot] \circ (F \otimes G) \circ \Delta$, for all $F, G \in \mathfrak{g}_x$. We have seen also that $\mathfrak{g}_x$ is a Lie $S(\mathfrak{g})^*$-superalgebra and $\mathfrak{g} \subseteq \mathfrak{g}_x$ is a $\mathbb{K}$-Lie subsuperalgebra. Let x be the generic point of $\mathfrak{g}$.

Remark 1.3.1. *Let $n \in \mathbb{N} \setminus \{0\}$, $(\mathrm{ad}x)^n : \mathfrak{g}_x \to \mathfrak{g}_x$ is a $S(\mathfrak{g})^*$-morphism of $\mathfrak{g}_x$. In particular, if $Y \in \mathfrak{g}$, $(\mathrm{ad}x)^n(Y) : S(\mathfrak{g}) \to \mathfrak{g}$ is the map*

$$X_1 \cdots X_p \mapsto \begin{cases} 0, & p \neq n \\ (-1)^{p(Y)p(X_1+\cdots+X_n)} \sum_{s \in \Sigma_n} \alpha(X_{s(1)}, ..., X_{s(n)}) \mathrm{ad}X_{s(1)} \circ \cdots \circ \mathrm{ad}X_{s(n)}(Y), & p = n \end{cases}$$

for any $p \geq 0$ and $X_1, ..., X_p \in \mathfrak{g}$. If $n = 0$, $(\mathrm{ad}x)^0(Y) := Y \in \mathfrak{g}_x$.

Let $q = c_0 + c_1 t + c_2 t^2 + \cdots \in \mathbb{K}[[t]]$ and $Y \in \mathfrak{g}$. As a consequence of the last remark, we can define

$$q(\mathrm{ad}x)(Y) := c_0 Y + c_1(\mathrm{ad}x)(Y) + c_2(\mathrm{ad}x)^2(Y) + \cdots.$$

It is the morphism of $\mathbb{K}$-modules from $S(\mathfrak{g})$ to $\mathfrak{g}$ such that, for any $n \in \mathbb{N}$, its restriction to $S^n(\mathfrak{g})$ is $c_n(\mathrm{ad}x)^n(Y)$.

Remark 1.3.2. *In the Lie superalgebra $\mathfrak{g}_x$, we consider the formula (1.11) with $X = x$. This gives the formula of a morphism of $\mathbb{K}$-modules from $S(\mathfrak{g})$ to $\mathfrak{g}$. Let $Y, Z \in \mathfrak{g}$, for any $n \in \mathbb{N}$ and $X_1, ..., X_n \in \mathfrak{g}$, this morphism is given by*

$$(t^r u^q : [Y, Z])_x (X_1 \cdots X_n) =$$
$$= \begin{cases} 0, & n \neq r + q \\ \sum_{s \in \Sigma_n} \alpha(X_s)[\mathrm{ad}X_{s(1)} \circ \cdots \circ \mathrm{ad}X_{s(r)}(Y), \mathrm{ad}X_{s(r+1)} \circ \cdots \circ \mathrm{ad}X_{s(n)}(Z)], & n = q + r \end{cases}$$

where the coefficients $\alpha(X_s)$ are given by

$$\alpha(X_s) := (-1)^{p(X_1 + \cdots + X_n)p(Z) + p(Y)p(X_{s(1)} + \cdots + X_{s(r)})} \alpha(X_{s(1)}, ..., X_{s(r+q)}).$$

As above this allows to define $(p(t, u) : [Y, Z])_x$ for any formal power series $p \in \mathbb{K}[[t, u]]$. The following theorem plays a basic role in this text.

Theorem 1.3.1. *Let $Y, Z \in \mathfrak{g}$ and $q(z) \in \mathbb{K}[[z]]$. In $\mathfrak{g}_x$ we have*

$$\partial(Y)\big(q(\mathrm{ad}x)(Z)\big) = (-1)^{p(q)p(Y)} \left(\frac{q(t+u) - q(u)}{t} : [Y, Z] \right).$$

PROOF. We only need to consider the case $q(z) = z^k$, with $k \geq 0$. For $k = 0$ the statement follows from remark 1.2.1. We recall that $\partial(Y)$ is a derivation. By induction over k and by the Jacobi identity in $\mathfrak{g}_x$, we get

$$\begin{aligned} & \partial(Y)((\mathrm{ad}x)^{k+1}(Z)) = \partial(Y)([x, \mathrm{ad}x)^k(Z)]) = \\ = \;& [Y, (\mathrm{ad}x)^k(Z)] + [x, \partial(Y)((\mathrm{ad}x)^k)(Z)] \\ = \;& (u^k : [Y, Z]) + \mathrm{ad}x \left(\left(\frac{(u+t)^k - u^k}{t} : [Y, Z] \right)_x \right) \\ = \;& (u^k : [Y, Z])_x + \left((t+u) \frac{(u+t)^k - u^k}{t} : [Y, Z] \right)_x \\ = \;& \left(\frac{(u+t)^{k+1} - u^{k+1}}{t} : [Y, Z] \right)_x . \blacksquare \end{aligned}$$

Chapter 2

Universal representations by coderivations

2.1 Functional equations associated to coderivations

Let $\mathbb{K}$ be a commutative superring and $\varphi(z) = \sum_j z^j c_j \in \mathbb{K}[[z]]$. For any Lie $\mathbb{K}$-superalgebra $\mathfrak{g}$ and $a \in \mathfrak{g}$, we define $\varphi^a := \varphi(\mathrm{ad}x)(a) \in \mathfrak{g}_x$. We recall from remark (1.3.1) that

$$\varphi^a(X_1 \cdots X_n) = (-1)^{p(a)p(X_1+\cdots+X_n)} \sum_{\sigma \in \Sigma_n} c_n \alpha(X_{\sigma(1)}, ..., X_{\sigma(n)}) \mathrm{ad}X_{\sigma(1)} \circ \cdots \circ \mathrm{ad}X_{\sigma(n)}(a) \quad (2.1)$$

for any $n \in \mathbb{N}$ and $X_1, ..., X_n \in \mathfrak{g}$. In particular, if $X \in \mathfrak{g}_0$ we get

$$\begin{aligned} &\varphi^a\,(1) = c_0 a \\ &\varphi^a\,(X^n) = n! c_n (\mathrm{ad}X)^n (a), \quad \forall n \geq 1. \end{aligned} \quad (2.2)$$

Remark 2.1.1. (Functorial property)
Let $\mathfrak{h}$ be a Lie $\mathbb{K}$-superalgebra and $f : \mathfrak{g} \to \mathfrak{h}$ be a morphism of Lie $\mathbb{K}$-superalgebras. The formula (2.1) shows that $f \circ \varphi^a(X_1 \cdots X_n) = \varphi^{f(a)}(f(X_1) \cdots f(X_n))$, for any $n \in \mathbb{N}$ and $X_1, ..., X_n \in \mathfrak{g}$.

Let $\rho(t,u) = \sum_{j \geq 0} \sum_{i=0}^{j} t^i u^{j-i} d_{i,j-i} \in \mathbb{K}[[t,u]]$. To $a,\ b \in \mathfrak{g}$ we associate the map $(\rho(t,u) : [a,b])_x \in \mathfrak{g}_x$. We recall that (remark 1.3.2) for $X \in \mathfrak{g}_0$ and $n \in \mathbb{N}$ we get $(\rho(t,u) : [a,b])_x(X^n) = n! \sum_{i=0}^{n} (t^i u^{n-i} d_{i,n-i} : [a,b])_X$.

Remark 2.1.2. *By lemma 1.3.1 we have $\varphi^{[a,b]} = (\varphi(t+u) : [a,b])_x$.*

By theorem 1.2.1, to the map φ^a we associate the coderivation $\Phi^a := id * \varphi^a \equiv \mathrm{Mult} \circ (1 \otimes \varphi^a) \circ \Delta$. We have

$$\Phi^a\,(X_1 \cdots X_n) = \sum_{j=0}^{n} \sum_{1 \leq p_1 < \cdots < p_j \leq n} a(\vec{X}, \vec{p}) X_{p_1} \cdots X_{p_j} \cdot \varphi^a \left(X_1 \cdots \widehat{X_{p_1}} \cdots \widehat{X_{p_j}} \cdots X_n \right) \quad (2.3)$$

where $a(\vec{X}, \vec{p}) := (-1)^{p(\varphi^a)p(X_{p_1}+\cdots+X_{p_j})} \alpha(X_{p_1}, ..., X_{p_j}, X_1, ..., \widehat{X_{p_1}}, ..., \widehat{X_{p_j}}, ..., X_n)$. In particular, if $X \in \mathfrak{g}_0$ we get $\Phi^a\,(X^n) = \sum_{j=0}^{n} \binom{n}{j} X^j \cdot \varphi^a(X^{n-j})$.
Let $\psi \in \mathbb{K}[[t]]$, $b \in \mathfrak{g}$, and let $\Psi^b : S(\mathfrak{g}) \to S(\mathfrak{g})$ be the associated coderivation.

Remark 2.1.3. *By definition, for all $Y \in \mathfrak{g}$ we have $id * Y = Y^L$.*

Lemma 2.1.1. *For any $Y \in \mathfrak{g}$ we have*
i) $\Phi^a \circ Y^L = id * \left(\varphi^a * Y - \left(\frac{\varphi(t+u)-\varphi(t)}{u} : [a, Y]\right)_x\right)$

ii) $\Phi^a \circ \Psi^b = id * \left(\varphi^a * \psi^b - (-1)^{p(a)p(\psi)} \left(\frac{\varphi(t+u)-\varphi(t)}{u}\psi(u) : [a, b]\right)_x\right).$

Proof. i) From the fact that Δ is, in particular, a morphism of algebras, and from remark 2.1.3 we have

$$\Phi^a \circ Y^L \equiv \left(id * \varphi^a\right) \circ Y^L = id * \left\{\varphi^a \circ Y^L + (-1)^{p(\varphi^a)p(Y)} Y * \varphi^a\right\}.$$

As $*$ is commutative, this shows that

$$\Phi^a \circ Y^L = id * \left\{\varphi^a \circ Y^L\right\} + id * \varphi^a * Y.$$

By definition $\varphi^a \circ Y^L = (-1)^{p(Y)(p(\varphi)+p(a))}\partial(Y)(\varphi^a)$, so the theorem 1.3.1 gives the desired formula.

ii) Let us consider the Lie superalgebra $\mathfrak{g}_x$ and its generic point $y \in Hom(S(\mathfrak{g}_x), \mathfrak{g}_x)$. By definition and by remark 2.1.3 we have

$$\Phi^a_{\mathfrak{g}} \circ \Psi^b_{\mathfrak{g}} = \Phi^a_{(\mathfrak{g}_x)_y} \circ (id * \psi(\mathrm{ad}y)(b))\,|_{S(\mathfrak{g})} = \Phi^a_{(\mathfrak{g}_x)_y} \circ \psi(\mathrm{ad}y)(b)^L|_{S(\mathfrak{g})}.$$

As $\psi(\mathrm{ad}y)(b) \in (\mathfrak{g}_x)_y$, from case i we get

$$\Phi^a_{\mathfrak{g}} \circ \Psi^b_{\mathfrak{g}} \;\; = \;\; id * \left\{\varphi(\mathrm{ad}y)(a) * \psi(\mathrm{ad}y)(b) - \left(\frac{\varphi(t+u)-\varphi(t)}{u} : [a, \psi(\mathrm{ad}y)(b)]\right)_y\right\}|_{S(\mathfrak{g})}.$$

By definition $\left(\frac{\varphi(t+u)-\varphi(t)}{u} : [a, \psi(\mathrm{ad}y)(b)]\right)_y = (-1)^{p(a)p(\psi)} \left(\frac{\varphi(t+u)-\varphi(t)}{u}\psi(t) : [a, b]\right)_y$ so the proof is finished. ∎

Theorem 2.1.1.

$$[\Phi^a, \Psi^b] = id * (-1)^{p(\psi)p(a)} \left(-\frac{\varphi(t+u)-\varphi(t)}{u}\psi(u) - \varphi(t)\frac{\psi(t+u)-\psi(u)}{t} : [a, b]\right)_x.$$

Proof. Let $\omega(t, u) := -\frac{\varphi(t+u)-\varphi(t)}{u}\psi(u) - \varphi(t)\frac{\psi(t+u)-\psi(u)}{t}$, we denote by $\Omega^{[a,b]}$ the coderivation corresponding to $(-1)^{p(\psi)p(a)}\left(\omega(t, u) : [a, b]\right)_x$. By theorem 1.2.1 we want to show that $[\Phi^a, \Psi^b] = \Omega^{[a,b]}$. From lemma 2.1.1 we get

$$\begin{aligned}
&[\Phi^a, \Psi^b] = \\
= \;& id* \left(-(-1)^{p(a)p(\psi)} \left(\frac{\varphi(t+u)-\varphi(t)}{u}\psi(u) : [a, b]\right)_x + \varphi^a * \psi^b\right) + \\
& id* \left((-1)^{p(b)p(\varphi)+p(\Phi^a)p(\Psi^b)} \left(\frac{\psi(t+u)-\psi(t)}{u}\varphi(u) : [b, a]\right)_x - (-1)^{p(\Phi^a)p(\Psi^b)}\psi^b * \varphi^a\right) \\
= \;& -(-1)^{p(a)p(\psi)} id * \left((-1)^{p(\varphi)p(\psi)}\frac{\psi(t+u)-\psi(u)}{t}\varphi(t) + \frac{\varphi(t+u)-\varphi(t)}{u}\psi(u) : [a, b]\right)_x \\
= \;& id * (-1)^{p(\psi)p(a)} \left(\omega(t, u) : [a, b]\right)_x. \;\blacksquare
\end{aligned}$$

To prove the next theorem we need some preliminaries, which we state in a form that will be useful later.

Definition. 2.1.1. *Let $N \geq 1$ be an integer. A Lie superalgebra $\mathfrak{g}$ is said to be N-nilpotent if we have*

$$\mathrm{ad}X_1 \circ \cdots \circ \mathrm{ad}X_N = 0, \ \forall X_1, ..., X_N \in \mathfrak{g}.$$

Remark 2.1.4. *For $N = 1$ we have a commutative Lie superalgebra, for $N = 2$ we have a Lie superalgebra of Heisenberg type.*

Lemma 2.1.2. *For any $N \geq 2$, there exists a Lie $\mathbb{K}$-superalgebra $\mathfrak{g}_N$, N-nilpotent, equipped of an infinite family of even elements $\{\alpha, \beta, X_1, X_2, ...\}$ such that*

$$\bigcup_{0 \leq r+s \leq N-2} \{[\mathrm{ad}X_{i(1)} \circ \cdots \circ \mathrm{ad}X_{i(r)}(\alpha), \mathrm{ad}X_{i(r+1)} \circ \cdots \circ \mathrm{ad}X_{i(r+s)}(\beta)]; i(1), ..., i(r+s) \in \mathbb{N}\}$$

is a free family.

Proof. We start by considering the free Lie algebras $\mathfrak{h}$ over $\mathbb{Z}$, with an infinite family of generators α, β, $X_1, X_2, ...$ By properties of free Lie algebras (**[Bou]** prop. 10, page 26) we know that

$$\bigcup_{r,s \geq 0} \{[\mathrm{ad}X_{i(1)} \circ \cdots \circ \mathrm{ad}X_{i(r)}(\alpha), \mathrm{ad}X_{i(r+1)} \circ \cdots \circ \mathrm{ad}X_{i(r+s)}(\beta)]; i(1), ..., i(r+s) \in \mathbb{N}\}$$

is contained in a basis of $\mathfrak{h}$.
Let I_N be the ideal of $\mathfrak{h}$ generated by $\{\mathrm{ad}x_1 \circ \cdots \circ \mathrm{ad}x_N(Y) | x_1, ..., x_N, Y \in \mathfrak{h}\}$. The quotient $\mathfrak{h}_N := \mathfrak{h}/I_N$ is a N-nilpotent Lie superalgebra over $\mathbb{Z}$ and the family $f_N := \bigcup_{0 \leq r+s \leq N-2} \{[\mathrm{ad}X_{i(1)} \circ \cdots \circ \mathrm{ad}X_{i(r)}(\alpha), \mathrm{ad}X_{i(r+1)} \circ \cdots \circ \mathrm{ad}X_{i(r+s)}(\beta)]; i(1), ..., i(r+s) \in \mathbb{N}\}$ is contained in a basis of $\mathfrak{h}_N$.
We define $\mathfrak{g}_N := \mathfrak{h}_N \otimes \mathbb{K}$, it is a N-nilpotent Lie superalgebra over $\mathbb{K}$ and $\mathfrak{h}_N$ is a Lie subsuperalgebra. As the tensor product of modules is distributive, $\mathfrak{g}_N$ inherits from $\mathfrak{h}_N$ the property that f_N is contained in a basis of $\mathfrak{g}_N$. ∎

Lemma 2.1.3. *Let $\omega(t, u) = \sum_{i,j=0}^{\infty} c_{ij} t^i u^j \in \mathbb{K}[[t, u]]$ and $N \geq 2$. If for any N-nilpotent Lie $\mathbb{K}$-superalgebra $\mathfrak{g}$*

$$(\omega(t, u) : [a, b])_x = 0, \ \forall a, b \in \mathfrak{g},$$

then $c_{ij} = 0$ for any $0 \leq i + j \leq N - 2$.

Proof. We consider the case $\mathfrak{g} = \mathfrak{g}_N$, where $\mathfrak{g}_N$ is the N-nilpotent Lie superalgebra of lemma 2.1.2. Choosing $a = \alpha$ and $b = \beta$ we get $(\omega(t, u), [\alpha, \beta])_x = \sum_{i+j=0}^{N-2} c_{i,j} (t^i u^j : [\alpha, \beta])_x$.
Let $0 \leq p \leq N - 2$, remark 1.3.2 gives

$$(\omega(t, u), [\alpha, \beta])_x (X_1 \cdots X_p) = \sum_{i=0}^{p} c_{i,p-i} \sum_{s \in \Sigma_p} [\mathrm{ad}X_{s(1)} \circ \cdots \circ \mathrm{ad}X_{s(i)}(\alpha), , \mathrm{ad}X_{s(i+1)} \circ \cdots \circ \mathrm{ad}X_{s(p)}(\beta)].$$

As $(\omega(t, u), [\alpha, \beta])_x (X_1 \cdots X_p)$ is zero, lemma 2.1.2 gives that $c_{i,p-i} = 0$ for any $i = 0, ..., p$. As $0 \leq p \leq N - 2$, the proof is finished. ∎

Let $\lambda \in \mathbb{K}[[z]]$, we consider the coderivation $\Lambda^a : S(\mathfrak{g}) \to S(\mathfrak{g})$ for any $a \in \mathfrak{g}$.

Theorem 2.1.2. *Let $\varphi, \psi, \lambda \in \mathbb{K}_0[[t]]$. For any Lie $\mathbb{K}$-superalgebra $\mathfrak{g}$ we have*

$$[\Phi^a, \Psi^b] = \Lambda^{[a,b]}, \quad \forall a, b \in \mathfrak{g} \tag{2.4}$$

if and only if φ, ψ, λ verify

$$\left(-\frac{\varphi(t+u)-\varphi(t)}{u}\psi(u) - \varphi(t)\frac{\psi(t+u)-\psi(u)}{t}\right) = \lambda(t+u)$$

in $\mathbb{K}_0[[t,u]]$.

PROOF. Let $\omega(t,u) := -\frac{\varphi(t+u)-\varphi(t)}{u}\psi(u) - \varphi(t)\frac{\psi(t+u)-\psi(u)}{t} - \lambda(t+u)$. Using theorem 2.1.1 and remark 2.1.2, we see that (2.4) is equivalent to

$$id * (\omega(t,u) : [a,b])_x = 0, \forall a, b \in \mathfrak{g}.$$

By theorem 1.2.1, this identity is equivalent to

$$(\omega(t,u) : [a,b])_x = 0, \forall a, b \in \mathfrak{g}.$$

We get immediately that the functional equation is sufficient. To show the converse, it is sufficient to apply the lemma 2.1.3 to any N-nilpotent Lie superalgebra $\mathfrak{g}_N$, with $N \geq 2$. We get that in $\omega(t,u)$ the coefficients of degree $N-2$ are zero, for any $N \geq 2$. In particular $\omega(t,u) = 0$. ▮

Theorem 2.1.3. *Let $\varphi \in \mathbb{K}[[t]]$. For any Lie $\mathbb{K}$-superalgebra $\mathfrak{g}$, we have*

$$[\Phi^a, \Phi^b] = \Phi^{[a,b]}, \quad \forall a, b \in \mathfrak{g} \tag{2.5}$$

if and only if φ has even coefficients ($\varphi \in \mathbb{K}_0[[t]]$) and verifies

$$\left(-\frac{\varphi(t+u)-\varphi(t)}{u}\varphi(u) - \varphi(t)\frac{\varphi(t+u)-\varphi(u)}{t}\right) = \varphi(t+u). \tag{2.6}$$

PROOF. As $p(\Phi^a) \equiv p(\varphi) + p(a)$, identity (2.5) needs $p(\varphi) = 0$. The theorem 2.1.3 follows from theorem 2.1.2. ▮

2.2 Universal representations

Let $\mathbb{K}$ be a commutative superring, $\varphi(t) = \sum_j t^j c_j \in \mathbb{K}_0[[t]]$, $\mathfrak{g}$ a Lie $\mathbb{K}$-superalgebra. We consider the map $\Phi : \mathfrak{g} \ni a \mapsto \Phi^a \in Hom(S(\mathfrak{g}), S(\mathfrak{g}))$ defined in section 2.1 by formulas (2.1) and (2.3). From theorem 2.1.3 we know that Φ is a representation for all Lie $\mathbb{K}$-superalgebras $\mathfrak{g}$, if and only if $\varphi \in \mathbb{K}_0[[t]]$ verifies the functional equation (2.6). We look for solutions of this functional equation.

Definition. 2.2.1. (universal representation)
Let $M(\mathfrak{g})$ be the symmetric algebra or the enveloping algebra of $\mathfrak{g}$. We call a representation $\Phi : \mathfrak{g} \to End(M(\mathfrak{g}))$ universal, if for any $a \in \mathfrak{g}$ and any morphism of Lie $\mathbb{K}$-superalgebras $f : \mathfrak{g} \to \mathfrak{h}$ the following diagram, with $\tilde{f}$ the morphism of superalgebras induced by f, is commutative:

$$\begin{array}{ccc} M(\mathfrak{g}) & \xrightarrow{\Phi^a} & M(\mathfrak{g}) \\ \downarrow_{\tilde{f}} & & \downarrow_{\tilde{f}} \\ M(\mathfrak{h}) & \xrightarrow{\Phi^{f(a)}} & M(\mathfrak{h}) \end{array}.$$

For any commutative superring $\mathbb{K}$ we introduce

$$\varphi_0(t) := -t \in \mathbb{K}_0[[t]]. \tag{2.7}$$

For any $c \in \mathbb{K}_0^\times$, if $\mathbb{K} \supseteq \mathbb{Q}$, we introduce also

$$\varphi_c(t) = \frac{t}{e^{\frac{t}{c}} - 1} \in \mathbb{K}_0[[t]]. \tag{2.8}$$

All these series verify $\varphi_c(0) = c$.

Lemma 2.2.1. *Let $\mathbb{K}_0$ be a commutative field. The solutions of equation (2.6) which lie in $\mathbb{K}_0[[t]]$ and such that $\varphi(0) = 0$, are $\varphi = 0$ and $\varphi = \varphi_0$.*

Proof. If $\varphi(0) = 0$, the limit $\lim_{u \to 0}$ applied to the equation (2.6) gives $\varphi(t)\left(1 + \frac{\varphi(t)}{t}\right) = 0$, so $\varphi(t)$ is zero or $\varphi(t) = -t$ because $\mathbb{K}_0[[t]]$ is a domain. ∎

Lemma 2.2.2. *i) Let $\varphi \in \mathbb{K}_0[[t]]$ be a solution of equation (2.6). If the constant term $\varphi(0) =: c$ is invertible, $f(t) := \frac{\varphi(t)+t}{\varphi(t)}$ satisfies*

$$\begin{cases} f(t) \cdot f(u) = f(t+u) \\ f(0) = 1 \\ f'(0) = \frac{1}{c} \end{cases} . \tag{2.9}$$

ii) System (2.9) has solutions if and only if $\mathbb{K}_0$ contains $\mathbb{Q}$. In this case the unique solution is $e^{\frac{t}{c}} \in \mathbb{K}_0[[t]]$.
iii) Let $\mathbb{K} \supseteq \mathbb{Q}$ and $c \in \mathbb{K}_0^\times$. The unique solution of (2.6) in $\mathbb{K}_0[[t]]$ verifying $\varphi(0) = c$ is $\varphi_c(t)$.

Proof. *i*) We recall that c is invertible if and only if the series φ is invertible, so we write the equation (2.6) as

$$\frac{\varphi(t)+t}{\varphi(t)} \cdot \frac{\varphi(u)+u}{\varphi(u)} = \frac{\varphi(t+u)+t+u}{\varphi(t+u)}.$$

We have $\frac{\varphi(t)+t}{\varphi(t)} = 1 + \frac{1}{c}t + \cdots$.
ii) Let $f = 1 + \frac{1}{c}t + \sum_{k=2}^{\infty} f_k t^k$. The system (2.9) gives $f'(t) = \frac{1}{c}f(t)$, so $2f_2 = \frac{1}{c^2}$ and $kf_k = \frac{1}{c}f_{k-1}$ for any $k \geq 3$. By induction we get that k is invertible and $f_k = \frac{1}{k!c^k}$ for any $k \geq 2$.
iii) When $f(t) = e^{\frac{1}{c}t}$, we get $\varphi(t) = \varphi_c(t)$. ∎

We have shown the following theorems

Theorem 2.2.1. *The map $\Phi_0 : \mathfrak{g} \to Hom(S(\mathfrak{g}), S(\mathfrak{g}))$ associated to φ_0 is a representation by coderivations.*

Remark 2.2.1. *Let $a \in \mathfrak{g}$. The map Φ_0^a is in the same time a derivation and a coderivation of $S(\mathfrak{g})$: for any $X_1, ..., X_n \in \mathfrak{g}$ we have*

$$\Phi_0^a(X_1 \cdots X_n) = \sum_{j=1}^{n} (-1)^{p(a)p(X_1+\cdots+X_{j-1})} X_1 \cdots \Phi_0^a(X_j) \cdots X_n.$$

It is the only derivation of $S(\mathfrak{g})$ such that $\Phi_0^a(X) = [a, X]$ for $X \in \mathfrak{g}$, so Φ_0 is the adjoint representation of $\mathfrak{g}$ in $S(\mathfrak{g})$.

Theorem 2.2.2. *Let $\mathbb{K} \supseteq \mathbb{Q}$. For any $c \in \mathbb{K}_0^\times$, the series $\frac{z}{e^{c^{-1}z}-1} \in \mathbb{K}_0[[z]]$ gives a representation by coderivations $\Phi_c : \mathfrak{g} \to Hom(S(\mathfrak{g}), S(\mathfrak{g}))$.*

Theorem 2.2.3. *Let $\varphi \in \mathbb{Q}[[t]]$. For any commutative superring $\mathbb{K} \supseteq \mathbb{Q}$ and any Lie $\mathbb{K}$-superalgebra $\mathfrak{g}$, the map $\Phi : \mathfrak{g} \to Hom(S(\mathfrak{g}), S(\mathfrak{g}))$ is a representation by coderivations, if and only if φ is zero, or φ_c, $c \in \mathbb{Q}$.*

Remark 2.2.2. *The Bernoulli numbers $\{b_k \in \mathbb{Q}, k \in \mathbb{N}\}$ are defined by the generating series*

$$\varphi_1(z) \equiv \frac{1}{e^z - 1} = \sum_{k \geq 0}^{\infty} \frac{b_k}{k!} z^k.$$

For example $b_0 = 1$, $b_1 = -\frac{1}{2}$, $b_2 = \frac{1}{6}$. Let $c \in \mathbb{K}_0^\times$, the fact that $\varphi_c(t) \in \mathbb{K}_0[[t]]$ verifies the identity (2.6) can be written in the following way:

$$\forall k \geq 0, \quad 0 = b_k + \sum_{p=0}^{k-l} \binom{k-l}{p} \frac{b_p b_{k+1-p}}{(l+1)} + \sum_{p=0}^{l} \binom{l}{p} \frac{b_p b_{k+1-p}}{(k+1-l)}, \qquad l = 0, ..., k.$$

2.2.1 The case of nilpotent Lie superalgebras

We are going to give an analogue of theorem 2.2.2 for $\mathbb{K}$ not necessarily containing $\mathbb{Q}$.

Let $N \geq 2$, $\mathfrak{g}$ a N-nilpotent Lie superalgebra over $\mathbb{K}$, a and $b \in \mathfrak{g}$.

Remark 2.2.3. *i) The notation $\varphi(\mathrm{ad}x)(a) \in \mathfrak{g}_x$ is well-defined for $\varphi \in \mathbb{K}[t]/t^N$ a truncated polynomial with coefficients in $\mathbb{K}$.*
ii) The notation $(\rho(t, u) : [a, b])_x \in \mathfrak{g}_x$ is well-defined if $\rho(t, u) \in \mathbb{K}[t, u]/I_N$, where I_N is the ideal generated by $\{t^i u^j, i + j \geq N - 1\}$.

To a truncated polynomial $\varphi(z) \in \mathbb{K}_0[t]/t^N$ we associate a family of coderivations still denoted by $\Phi^a : S(\mathfrak{g}) \to S(\mathfrak{g})$, $a \in \mathfrak{g}$. The direct part of the following theorem is a particular case of theorem 2.1.2.

Theorem 2.2.4. *For any $\mathfrak{g}$ a N-nilpotent Lie superalgebra over $\mathbb{K}$ the map $\Phi : \mathfrak{g} \ni a \mapsto \Phi^a \in Hom(S(\mathfrak{g}), S(\mathfrak{g}))$ is a representation by coderivations, if and only if φ verifies*

$$\varphi(u)\frac{\varphi(t+u) - \varphi(t)}{u} + \varphi(t)\frac{\varphi(t+u) - \varphi(u)}{t} = -\varphi(t+u) \tag{2.10}$$

in $\mathbb{K}_0[t, u]/I_N$.

PROOF. Let $\omega(,u) := -\varphi(u)\frac{\varphi(t+u)-\varphi(t)}{u} - \varphi(t)\frac{\varphi(t+u)-\varphi(u)}{t} - \varphi(t+u)$. Proceeding as in the proof of theorem 2.1.2 we get that equation (2.10) is equivalent to

$$(\omega(t,u) : [a,b])_x = 0, \ \forall a,b \in \mathfrak{g}.$$

Moreover, for a N-nilpotent Lie superalgebra, this reduces to

$$(\omega(t,u) \mod I_N : [a,b])_x = 0, \ \forall a,b \in \mathfrak{g}.$$

We get immediately that the functional equation (2.10) is sufficient to get a representation. The converse follows from lemma 2.1.3. ∎

Example 2.2.1. *Let $\mathbb{K}_0$ be a field.*
a) Let $N = 2$ and $\frac{1}{2} \in \mathbb{K}_0$. We look for $\varphi(t) = c_0 + c_1 t \mod t^2$ solution of

$$2c_0c_1 = -c_0.$$

We have $\varphi(t) = c_1 t$ or $\varphi(t) = c_0 - \frac{1}{2}t$.
b) Let $N = 3$ and $\frac{1}{2}, \frac{1}{3} \in \mathbb{K}_0$. We look for $\varphi(t) = c_0 + c_1 t + c_2 t^2 \mod t^3$ solution of

$$2c_0c_1 + (3c_0c_2 + c_1^2)(u+t) = -c_0 - c_1(t+u).$$

We get $\varphi(t) = c_2 t^2$, or $\varphi(t) = -t + c_2 t^2$, or $\varphi(t) = c_0 - \frac{1}{2}t + \frac{1}{12c_0}t^2$ with $c_0 \neq 0$.

Lemma 2.2.3. *Let $N \geq 2$. The equation (2.10) has solutions in $\mathbb{K}[t]/t^N$ with $\varphi(0) \in \mathbb{K}_0^\times$, if and only if $\frac{1}{2}, ..., \frac{1}{N} \in \mathbb{K}$. In this case the unique solution such that $\varphi(0) =: c \in \mathbb{K}_0^\times$ is*

$$\varphi_c(t) \text{ mod } t^N.$$

PROOF. We look for $\varphi \in \mathbb{K}_0[t]/t^N$ such that $1 + \frac{t}{\varphi(t)} \in \mathbb{K}_0[t]/t^{N+1}$ solves the system (2.9) in $\mathbb{K}[t,u]/I_{N+2}$.

The system (2.9) has solutions in $\mathbb{K}_0[t,u]/I_{N+2}$ exactly when $2, ..., N$ are invertible in $\mathbb{K}$. In this case the unique solution is $e^{\frac{1}{c}t} \mod t^{N+1}$ with $c \in \mathbb{K}_0^\times$, it means that $\varphi(t) = \varphi_c(t)$. ∎

We have shown that

Theorem 2.2.5. *Let $\mathbb{K} \supseteq \{\frac{1}{2}, ..., \frac{1}{N}\}$. For any $c \in \mathbb{K}_0^\times$, the truncated polynomial $\varphi_c(t) \in \mathbb{K}_0[t]/t^n$ gives a representation by coderivations $\Phi_c : \mathfrak{g} \to Hom(S(\mathfrak{g}), S(\mathfrak{g}))$.*

Theorem 2.2.6. *Let $\mathbb{K}$ be a field, p its characteristic, $\varphi \in \mathbb{K}[t]/t^N$. We assume that $2 \leq N < p$. For all N-nilpotent Lie $\mathbb{K}$-superalgebra $\mathfrak{g}$ the map $\Phi : \mathfrak{g} \to Hom(S(\mathfrak{g}), S(\mathfrak{g}))$ is a representation by coderivations, if and only if $\varphi = 0$ or $\varphi = \varphi_c \mod t^N$, $c \in \mathbb{K}$.*

2.2.2 Some properties of the representations Φ_c

This section applies to the cases $\mathbb{K} \supseteq \mathbb{Q}$ and $\mathfrak{g}$ any Lie $\mathbb{K}$-superalgebra, and to the case $\mathbb{K} \ni \frac{1}{2}, ..., \frac{1}{N}$ with $N \geq 2$ and $\mathfrak{g}$ a N-nilpotent Lie $\mathbb{K}$-superalgebra.

Remark 2.2.4. *If $c \in \mathbb{K}_0^\times$, the representation $\mathfrak{g} \ni a \mapsto \Phi_c^a \in Hom(S(\mathfrak{g}), S(\mathfrak{g}))$ is faithful because $\Phi_c^a(1) = c \cdot a$.*

Remark 2.2.5. *From theorem 2.1.2 we get $[\Phi_0^a, \Phi_c^b] = \Phi_c^{[a,b]}$, for all $a, b \in \mathfrak{g}$ and $c \in \mathbb{K}_0^\times \cup \{0\}$.*

Theorem 2.2.7. *Each representation Φ_c, $c \in \mathbb{K}_0^\times$, is equivalent to Φ_1.*

PROOF. Let $c \in \mathbb{K}_0^\times$. We consider the map $f_c : S(\mathfrak{g}) \to S(\mathfrak{g})$ such that $f_c(X_1 \cdots X_n) = c^n X_1 \cdots X_n$ for all $X_1, ..., X_n \in \mathfrak{g}$. We have $f_c^{-1} \circ \Phi_c^a \circ f_c = \Phi_1^a$, $\forall a \in \mathfrak{g}$. ∎

Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie $\mathbb{K}$-superalgebras, $f : \mathfrak{g} \to \mathfrak{h}$ be a morphism of Lie $\mathbb{K}$-superalgebras. It extends to a morphism of algebras $\tilde{f} : S(\mathfrak{g}) \to S(\mathfrak{h})$.

Remark 2.2.6. (Functorial property)
By remark 2.1.1, for any $a \in \mathfrak{g}$ and $c \in \mathbb{K}_0^\times$, the following diagram commutes

$$\begin{array}{ccc} S(\mathfrak{g}) & \xrightarrow{\Phi_c^a} & S(\mathfrak{g}) \\ \downarrow_{\tilde{f}} & & \downarrow_{\tilde{f}} \\ S(\mathfrak{h}) & \xrightarrow{\Phi_c^{f(a)}} & S(\mathfrak{h}) \end{array}.$$

2.3 A more general equation

Let $\mathbb{K}$ be a field of characteristic zero, t and u be two commutating variables. We classify the triples of formal series (φ, ψ, ρ) such that $\varphi, \psi, \rho \in \mathbb{K}[[t]]$ and

$$\varphi(t)\frac{\psi(t+u) - \psi(u)}{t} + \frac{\varphi(t+u) - \varphi(t)}{u}\psi(u) = \rho(t+u). \tag{2.11}$$

This is motivated by theorem 2.1.2, it is clear that equation (2.6) is a particular case of equation (2.11). The classification is contained in theorems 2.3.2 et 2.3.3.

Remark 2.3.1. *Applying the limit $t \to 0$ we get*

$$\varphi(0)\psi'(u) + \frac{\varphi(u) - \varphi(0)}{u}\psi(u) = \rho(u). \tag{2.12}$$

Remark 2.3.2. *Applying limits $t \to 0$ and $u \to 0$ to equation (2.11) we get*

$$\varphi(0)\psi'(u) - \psi(0)\varphi'(u) = \frac{\varphi(0)\psi(u) - \psi(0)\varphi(u)}{u}.$$

This differential equation gives the existence of $a \in \mathbb{K}$ such that $\varphi(0)\psi(u) - \psi(0)\varphi(u) = au$.

As the series ρ is determined by (2.12), it is natural to ask if equation (2.11) can be reduced to an equation for the couple (φ, ψ). To get this equation we introduce $p(t), q(t) \in \frac{\mathbb{K}}{t} + \mathbb{K}[[t]]$ such that

$$\varphi(t) = tp(t), \quad \psi(t) = tq(t).$$

Theorem 2.3.1. *The pair $(p(t), q(t))$ gives a solution of (2.11) if and only if*

$$q^{'}(u)\{p(t+u) - p(t)\} = p^{'}(t)\{q(t+u) - q(u)\}. \tag{2.13}$$

PROOF. Equation (2.11) becomes

$$p(t)\{(t+u)q(t+u) - uq(u)\} + q(u)\{(t+u)p(t+u) - tp(t)\} = \rho(t+u).$$

We recall that a function $f(t,u)$ is a function of $t+u$ if and only if $\frac{\partial}{\partial t}f(t,u) - \frac{\partial}{\partial u}f(t,u) = 0$. We apply this fact to $f(t,u) := p(t)\{(t+u)q(t+u) - uq(u)\} + q(u)\{(t+u)p(t+u) - tp(t)\}$ and we get equation (2.13). ∎

Formula (2.13) is very elegant. However, we will use it only trough the following remark.

Remark 2.3.3. *If the pair (p, q) is a solution without poles of (2.13) then $q^{'}(u)(p(u) - p(0)) = 0$, so p or q is constant.*

Theorem 2.3.2. *All triples of series (φ, ψ, ρ) verifying (2.11) and $\varphi(0)\psi(0) = 0$, are given by the following list*
i) $(\varphi(t), \psi(t), \rho(t)) = (\varphi(t), ct, c\varphi(t)), \qquad c \in \mathbb{K},\ \varphi \in \mathbb{K}[[t]],$
ii) $(\varphi(t), \psi(t), \rho(t)) = (ct, \psi(t), c\psi(t)), \qquad c \in \mathbb{K},\ \psi \in \mathbb{K}[[t]].$

PROOF. It is sufficient to consider the case $\psi(0) = 0$. From remark 2.3.2 we get $\varphi(0)\psi(u) = au$, with $a \in \mathbb{K}$. Let $\varphi(0) \neq 0$, we get $\psi(u) = cu$ with $c \in \mathbb{K}$. Equation (2.11) is verified with $c\varphi(t) = \rho(t)$, so we have a triple of type i.
Let $\psi(0) = \varphi(0) = 0$. From remark 2.3.3 we get $q(u)$ or $p(u)$ constant, that means we get a triple of type i or ii. ∎

Now we treat the case $\varphi(0) \cdot \psi(0) \neq 0$.

Remark 2.3.4. *Let $a, b \in \mathbb{K}$. If $(\varphi(t), \psi(t), \rho(t))$ verifies the functional equation (2.11), also the triple $(a\varphi(t), b\psi(t), a \cdot b\rho(t))$ verifies (2.11).*

It is sufficient to look for series such that $\psi(0) = \varphi(0) = 1$.

Remark 2.3.5. *Let $a, b \in \mathbb{K}$. If (φ, ψ, ρ) verifies (2.11), the triple $(\varphi(t) + at, \psi + bt, \rho(t) - a \cdot bt - a\psi(t) - b\varphi(t))$ verifies it.*

By this remark, we can restrict ourself to look for triples with $\varphi = \psi$ and $\varphi(0) = 1$, $\varphi'(0) = 0$.

Remark 2.3.6. *Let $a \in \mathbb{K}$. If $(\varphi(t), \psi(t), \rho(t))$ verifies the functional equation (2.11), then $(\varphi(at), \psi(at), a\rho(at))$ verifies (2.11).*

For any $c \in \mathbb{K}$ we introduce the notation

$$\theta_c(t) = \sqrt{c}\coth(\sqrt{c}t) = 1 + \frac{c}{3}t^2 - \frac{c^2}{45}t^4 + \cdots \in \mathbb{K}[[t]].$$

In particular $\theta_0(t) = 1$.

Lemma 2.3.1. *Let $c \in \mathbb{K}$. There exist exactly one triple (ψ_c, ψ_c, ρ_c) such that $\psi_c(t) = 1 + \frac{c}{3}t^2 + o(t^2)$. Moreover $\psi_c(t) = \psi_c(-t)$,*

$$(\psi_c(t), \rho_c(t)) = (\sqrt{c}t)\coth(\sqrt{c}t), ct).$$

Proof. We consider the left hand side of (2.11). As its derivatives by t must be equal to its derivative by u (see the proof of theorem 2.3.1), if $\varphi = \psi$ we get

$$\psi'(t)\frac{\psi(t+u)-\psi(u)}{t} + \psi(t)\frac{\partial}{\partial t}\left(\frac{\psi(t+u)-\psi(u)}{t}\right) + \psi(u)\frac{\psi'(t+u)-\psi'(t)}{u}$$
$$= \psi(t)\frac{\psi'(t+u)-\psi'(u)}{t} + \psi'(u)\frac{\psi(t+u)-\psi(t)}{u} + \psi(u)\frac{\partial}{\partial u}\left(\frac{\psi(t+u)-\psi(t)}{u}\right).$$

As $\psi(0) = 1$ and $\psi'(0) = 0$, the limit $t \to 0$ gives

$$-\frac{1}{2}\psi''(u) = \frac{\psi'(u)}{u}(\psi(u)-1) - \psi(u)\frac{\psi(u)-1}{u^2}. \tag{2.14}$$

Substituting $\psi(t) = 1 + \frac{c}{3}t^2 + \sum_{k=3}^{\infty} c_k t^k$ we get

$$c_{k+2} = \frac{-2}{k(k+3)}\left(\frac{kc_kc}{3} + \sum_{p=3}^{k-1} c_p c_{k-p+2}(k-p+1)\right), \quad k \geq 1.$$

This formula gives $c_3 = 0$. By induction, all coefficients c_{2j+1}, $j \geq 2$ are zero. The series $t\coth(t) = 1 + \frac{1}{3}t^2 + \cdots$ is a solution of equation (2.14). By remark 2.3.6 also $ct\coth(ct) = 1 + \frac{c}{3}t + \cdots$ is a solution of equation (2.14). ∎

Theorem 2.3.3. *All solutions of (2.11) verifying $\psi(0)\varphi(0) \neq 0$ are in the following list:*

$$\begin{cases} \varphi(t) = a\theta_c(ft) + bft \\ \psi(t) = d\theta_c(ft) + eft \\ \rho(t) = f(ae+bd)\theta_c(ft) + (adc+be)ft \end{cases}$$

with $a, b, c, d, e, f \in \mathbb{K}$ and $a \cdot d \neq 0$.

Remark 2.3.7. *We have*

$$\varphi_d + \varphi_{-d} = \varphi_0, \quad \forall d \in \mathbb{K}^\times \tag{2.15}$$

and $\theta_c(t) \equiv \sqrt{c}\left(\varphi_{\frac{1}{2\sqrt{c}}}(t) - \varphi_{-\frac{1}{2\sqrt{c}}}(t)\right) = 2\sqrt{c}\left(\varphi_{\frac{1}{2\sqrt{c}}}(t) - \frac{1}{2}\varphi_0(t)\right)$.

2.4 Motivations

In this paragraph we consider the case of Lie algebras over a $\mathbb{Q}$-algebra. So we assume that $\mathbb{K} = \mathbb{K}_0$ is a commutative ring, and $\mathfrak{g}$ a Lie $\mathbb{K}$-algebra.

Let $\mathbb{K} \supseteq \mathbb{Q}$ and $\mathfrak{g}$ be a Lie $\mathbb{K}$-algebra. In this chapter we have considered coderivations associated to vector fields on $\mathfrak{g}$ of type

$$\varphi^a = \varphi(\mathrm{ad}x)(a)$$

with $a \in \mathfrak{g}$, $\varphi \in \mathbb{K}[[t]]$, $x \in \mathfrak{g}_x$ the generic point of $\mathfrak{g}$. We have seen in remarks 2.1.1 and 2.2.6 that φ^a and the corresponding coderivation Φ^a satisfy a functorial property. We are going to prove the converse.

We look for any morphism of $\mathbb{K}$-modules $F_\mathfrak{g} : S(\mathfrak{g}) \otimes \mathfrak{g} \to \mathfrak{g}$ such that, for any morphism of Lie $\mathbb{K}$-algebras $f : \mathfrak{g} \to \mathfrak{h}$ the diagram

$$\begin{array}{ccc} S(\mathfrak{g}) \otimes \mathfrak{g} & \xrightarrow{F_\mathfrak{g}} & \mathfrak{g} \\ \downarrow^{\tilde{f}\otimes f} & & \downarrow^{f} \\ S(\mathfrak{h}) \otimes \mathfrak{h} & \xrightarrow{F_\mathfrak{g}} & \mathfrak{h} \end{array} , \tag{2.16}$$

where $\tilde{f} : S(\mathfrak{g}) \to S(\mathfrak{h})$ is the algebra-morphism induced by f, commutes.

Theorem 2.4.1. *For each $n \in \mathbb{N}$, there exists $c_n \in \mathbb{K}$ such that*

$$F_\mathfrak{g}(X_1 \cdots X_n \otimes a) = (c_n(\mathrm{ad}x)^n(a))(X_1 \cdots X_n)$$

for any $X_1, ..., X_n, a \in \mathfrak{g}$.

Proof. We consider the free Lie $\mathbb{K}$-algebra $\mathfrak{h}$ with generators $x_1, ...x_{n+1}$. Let $Y := F_\mathfrak{g}(x_1 \cdot \cdot \cdot x_n \otimes x_{n+1})$. Let $t \in \mathbb{Q}$. We fix $i \in \{1, ..., n+1\}$. By the universal property of free Lie algebras, the map

$$f_{t,i} : X_j \mapsto \left\{ \begin{array}{ll} X_j, & j \neq i \\ X_i t, & j = i \end{array} \right. , \quad \forall j = 1, ..., n+1$$

extends to a morphism of Lie $\mathbb{K}$-algebras $\tilde{f}_{t,i} : \mathfrak{h} \to \mathfrak{h}$. As the diagram (2.16) associated to this map commutes, we get $Yt = \tilde{f}_{t,i}(Y)$. We write $Y = \sum_n Y_{n,i}$ where $Y_{n,i}$ is a bracket containing n times x_i, so $f_{t,i}(Y) \equiv \sum_n Y_{n,i} t^n$. To get $Yt = \sum_n Y_{n,i} t^n$, we need $\sum_{n \neq 1} Y_{n,i}(t - t^n) = 0$ for any $t \in \mathbb{Q}$. As the family $\{Y_{n,i} | Y_{n,i} \neq 0, n \geq 0\}$ is free (see **[Bou]**, prop. 10, page 26), we get that $Y = \sum_{i=1}^{n+1} Y_{1,i}$. This is true for any $i \in \{i, ..., n+1\}$, so Y is a linear combination of brackets of n elements, exactly elements $x_1, ..., x_{n+1}$.
Using the Jacobi identity and the fact that the bracket of a Lie algebra is antisymmetric, we show that Y is a linear combination of $\mathrm{ad}x_{s(1)} \circ \cdots \circ \mathrm{ad}x_{s(n)}(x_{n+1})$, with $s \in \Sigma_n$. Let $Y = \sum_{s \in \Sigma_n} c_s \mathrm{ad}x_{s(1)} \circ \cdots \circ \mathrm{ad}x_{s(n)}(x_{n+1})$, with $c_s \in \mathbb{K}$. As a permutation u of $\{x_1, ..., x_n\}$ extends to a morphism g_u of Lie $\mathbb{K}$-algebras, from the commutative diagrams (2.16) for g_u we get $\sum_{s \in \Sigma_n} (c_s - c_{sou}) \mathrm{ad}x_{s(1)} \circ \cdots \circ \mathrm{ad}x_{s(n)}(x_{n+1}) = 0$. By properties of free Lie algebras (see **[Bou]**, prop. 10 page 26) we get that the family $\{\mathrm{ad}x_{s(1)} \circ \cdots \circ \mathrm{ad}x_{s(n)}(x_{n+1}) | s \in \Sigma_n\}$ is free. In particular $c_s - c_{sou} = 0$ for any $s \in \Sigma_n$. As it is true for any permutation s, we get $c_s = c_{id}$ for any $s \in \Sigma_n$. We note c_{id} as c_n, so $Y = c_n \sum_{j \in \Sigma_n} \mathrm{ad}x_{j(1)} \circ \cdots \circ \mathrm{ad}x_{j(n)}(x_{n+1})$.

Let f be a map $\{x_1, ..., x_n, x_{n+1}\} \to \mathfrak{g}$ in a Lie $\mathbb{K}$-algebra. From the universal property of free Lie algebras, f extends to a morphism of Lie $\mathbb{K}$-superalgebras still noted f. Let $a := f(x_{n+1}), f(x_i) =: X_i$. The commutative diagram for f gives

$$F_{\mathfrak{g}}(X_1 \cdots X_n \otimes a) = c_n \sum_{j \in \Sigma_n} \mathrm{ad}X_{j(1)} \circ \cdots \circ \mathrm{ad}X_{j(n)}(a). \blacksquare$$

Remark 2.4.1. *The previous theorem is not valid for a Lie superalgebra $\mathfrak{g}$. For example, if $\theta_{\mathfrak{g}} = \theta$ is the map such that $\theta|_{\mathfrak{g}_0} = id$ and $\theta|_{\mathfrak{g}_1} = -id$ then $\mathfrak{g} \ni a \mapsto (\mathrm{ad}x)^n(\theta(a))$ has the functorial property expressed in diagram (2.16).*

2.5 The Poincaré-Birkhoff-Witt theorem

Let $\mathbb{K}$ be a commutative superring and $\mathfrak{g}$ be a Lie $\mathbb{K}$-superalgebra. We assume that $\frac{1}{2} \in \mathbb{K}$ or that $\mathbb{K} = \mathbb{K}_0$ and $\mathfrak{g} = \mathfrak{g}_0$.

We recall that the enveloping algebra $U(\mathfrak{g})$ is defined as the quotient of the tensor algebra $T(\mathfrak{g})$ by the ideal J generated by $\{a \otimes b - (-1)^{p(a)p(b)} b \otimes a - [a, b] | a, b \in \mathfrak{g}\}$. The inclusion of $\mathfrak{g}$ in $T(\mathfrak{g})$ gives a map $j : \mathfrak{g} \to U(\mathfrak{g})$. Let $gr(U(\mathfrak{g}))$ be the graded module of $U(\mathfrak{g})$ associated to the filtration $\{U_i\}_{i \geq 0}$ with $U_0 = \mathbb{K}$ and U_i the $\mathbb{K}$-module generated by $\{j(X_1) \cdots j(X_l) | l \leq i, X_1, ..., X_l \in \mathfrak{g}\}$. The hypothesis give that it is a commutative superalgebra.

Remark 2.5.1. *If our assumptions are not verified, $gr(U(\mathfrak{g}))$ is not commutative. For example let us consider $\mathfrak{g} = \mathbb{Z}e$ with odd e and $[e, e] = 0$. As $j(e)^2 \notin U_1(\mathfrak{g})$, $gr(U(\mathfrak{g}))$ is not commutative.*

By the universal property of symmetric algebras, j extends to the algebra-morphism

$$\tilde{j} : S(\mathfrak{g}) \to gr(U(\mathfrak{g}))$$

such that $X_1 \cdots X_n \mapsto j(X_1) \cdots j(X_n) \mod U_{n-1}$, for any $X_1, ..., X_n$. This map is onto.

Definition. 2.5.1. *We say that $\mathfrak{g}$ verifies the weak Poincaré-Birkhoff-Witt theorem if $\tilde{j}$ is bijective.*

Before giving the next definition we recall that a map $f : \mathfrak{g} \to \mathfrak{g}$ is said to be an automorphism if it is an invertible morphism of Lie $\mathbb{K}$-superalgebras. Such a map induces two isomorphisms of algebras: $\tilde{f} : S(\mathfrak{g}) \to S(\mathfrak{g})$ and $\overline{f} : U(\mathfrak{g}) \to U(\mathfrak{g})$.

Remark 2.5.2. *A derivation $g : \mathfrak{g} \to \mathfrak{g}$ extends to derivations $g_1 : U(\mathfrak{g}) \to U(\mathfrak{g})$ and $g_2 : S(\mathfrak{g}) \to S(\mathfrak{g})$. Moreover, g_2 is also a coderivation.*

Definition. 2.5.2. *We say that $\mathfrak{g}$ verify the strong Poincaré-Birkhoff-Witt theorem if it does exist an isomorphism $\rho \in Hom(S(\mathfrak{g}), U(\mathfrak{g}))$ such that*
i) $\rho(S^n(\mathfrak{g})) \subseteq U_n(\mathfrak{g})$ for any $n \in \mathbb{N}$,
ii) the associated graded map $gr(\rho)$ is $\tilde{j}$,
iii) ρ commutes with any derivation of $\mathfrak{g}$ and any automorphism of $\mathfrak{g}$.

Remark 2.5.3. *Let $n \geq 1$. From i and ii we have $\rho(S^n(\mathfrak{g})) \oplus U_{n-1} = U_n$. From ii we get that $\rho(S^n(\mathfrak{g}))$ is stable by any derivation or automorphism of $\mathfrak{g}$. In particular $S(\mathfrak{g})$ and $U(\mathfrak{g})$ are isomorphic for the adjoint representation.*

Now we suppose also that

Hypothesis 2.5.1. $\mathbb{K} \supseteq \mathbb{Q}$ *or* $\mathfrak{g}$ *is N-nilpotent with $N \geq 2$ and $2, ..., N \in \mathbb{K}_0^\times$.*

From theorems 2.2.3 and 2.2.5 we have a representation $\Phi_1 : \mathfrak{g} \to End(S(\mathfrak{g}))$. By the universal property of enveloping algebras, it extends to an algebra-morphism $\Phi : U(\mathfrak{g}) \to End(S(\mathfrak{g}))$ such that $\Phi_1 = \Phi \circ j$. From Φ we construct the map $\sigma : U(\mathfrak{g}) \to S(\mathfrak{g})$, called the *symbol map* and defined by

$$\sigma(u) := \Phi(u)(1), \ \forall u \in U(\mathfrak{g}).$$

For example, for all $a_1, a_2, a_3 \in \mathfrak{g}$ we have

$$\begin{aligned}
&\sigma(1) = 1\\
&\sigma(j(a_1)) = a_1\\
&\sigma(j(a_1)j(a_2)) = a_1 \cdot a_2 + \frac{1}{2}[a_1, a_2]\\
&\sigma(j(a_1)j(a_2)j(a_3)) = a_1 \cdot a_2 \cdot a_3 + \frac{1}{2}\left\{a_1 \cdot [a_2, a_3] + [a_1, a_2] \cdot a_3 + (-1)^{p(a_1)p(a_2)} a_2 \cdot [a_1, a_3]\right\} +\\
&\qquad + \frac{1}{12}\left\{-(-1)^{p(a_2)p(a_1)}[a_2, [a_1, a_3]] + [[a_1, a_2], a_3]\right\} + \frac{1}{4}[a_1, [a_2, a_3]].
\end{aligned}$$

Lemma 2.5.1. *Let $\mathbb{K}$ be any commutative superring and $\mathfrak{g}$ any Lie $\mathbb{K}$-superalgebra. If $\lambda \in \mathbb{K}[[z]]$ with $\lambda(0) = 1$, the coderivations corresponding to λ have the property*

$$\Lambda^{a_n} \circ \cdots \circ \Lambda^{a_1}(1) - a_n \cdots a_1 \in \oplus_{j=0}^{n-1} S^j(\mathfrak{g}), \quad \forall a_1, ..., a_n \in \mathfrak{g}.$$

PROOF. If $n = 1$ the theorem is evident. As $\Lambda^{a_n} \circ \cdots \circ \Lambda^{a_1}(1) = \Lambda^{a_n}(\Lambda^{a_{n-1}} \circ \cdots \circ \Lambda^{a_1}(1))$, by induction there exists $p_n \in \oplus_{j=0}^{n-2} S^j(\mathfrak{g})$ such that $\Lambda^{a_{n-1}} \circ \cdots \circ \Lambda^{a_1}(1) = a_{n-1} \cdots a_1 + p_n$. As for all $p \geq 0$

$$\Delta(S^p(\mathfrak{g})) \subseteq \bigoplus_{j=0}^{p} S^j(\mathfrak{g}) \otimes S^{p-j}(\mathfrak{g}),$$

we have $\Lambda^{a_n}(p_n) \subseteq \bigoplus_{j=1}^{n-1} S^j(\mathfrak{g})$, it is sufficient to show that $\Lambda^{a_n}(a_{n-1} \cdots a_1) - a_n \cdots a_1 \in \oplus_{j=0}^{n-1} S^j(\mathfrak{g})$. This identity follows using $\Delta(a_{n-1} \cdots a_1) - a_{n-1} \cdots a_1 \otimes 1 \in \oplus_{j=0}^{n-2} S^j(\mathfrak{g}) \otimes S^{n-1-j}(\mathfrak{g})$, using the definition of Λ, using (2.2). ∎

Theorem 2.5.1. *Assume hypothesis 2.5.1. Then*
i) the map σ is invertible,
ii) $\mathfrak{g}$ verifies the strong Poincaré-Birkhoff-Witt theorem with $\rho = \sigma^{-1}$.

PROOF. *i*) By lemma 2.5.1 the graded map $gr(\sigma) : gr(U(\mathfrak{g})) \to S(\mathfrak{g})$ is well-defined and onto: for any $n \in \mathbb{N}$ we have

$$gr(\sigma)(a_1 \cdots a_n + U_{n-1}) = \sigma(a_1) \cdots \sigma(a_n), \quad \forall a_1, ..., a_n \in \mathfrak{g}.$$

The inverse of $gr(\sigma)|_{U_n/U_{n-1}}$ is $\tilde{j}|_{S^n(\mathfrak{g})}$, so $gr(\sigma)$ is one-to-one.
ii) In i we have seen, in particular, that $\tilde{j} = gr(\sigma^{-1})$. Let $f : \mathfrak{g} \to \mathfrak{g}$ be an automorphism of $\mathfrak{g}$ we want to show that $\sigma \circ \overline{f} = \tilde{f} \circ \sigma$. It follows from remark 2.2.6.

Let g be a derivation of $\mathfrak{g}$ and $a_1, ..., a_n \in \mathfrak{g}$. We want to show that $g_2 \circ \sigma = \sigma \circ g_1$, which means

$$(g_2 \circ \Phi^{a_1} \circ \cdots \circ \Phi^{a_n})(1) = \sum_{j=1}^{n} (-1)^{p(g)p(a_1+\cdots+a_{j-1})} \left(\Phi^{a_1} \circ \Phi^{g(a_j)} \circ \Phi^{a_n}\right)(1), \quad \forall a_1, ..., a_n \in \mathfrak{g}.$$

By induction, it is sufficient to show that $[g_2, \Phi_1^a] = \Phi^{g(a)}$ for any $a \in \mathfrak{g}$. By definitions $g_2 \circ \varphi_1^a = \varphi_1^a \circ g_2 + \varphi^{g(a)}$, it gives $[g_2, \Phi_1^a] = 1 \otimes \varphi_1^a \circ (g_2 \otimes 1 + 1 \otimes g_2 - \Delta \circ g_2) + \Phi^{g(a)}$. The fact that g_2 is a coderivation ends the proof. ∎

Let $\beta := \sigma^{-1}$.

Remark 2.5.4. (Functorial property)
Let $f : \mathfrak{g} \to \mathfrak{h}$ be a morphism of Lie $\mathbb{K}$-superalgebras. By remark 2.2.6 we get a commuting diagram

$$\begin{array}{ccc} S(\mathfrak{g}) & \xrightarrow{\beta} & U(\mathfrak{g}) \\ \downarrow_{\tilde{f}} & & \downarrow_{\overline{f}} \\ S(\mathfrak{h}) & \xrightarrow{\beta} & U(\mathfrak{h}) \end{array} .$$

To get formulas for β we use the following lemma.

Lemma 2.5.2. *For all $n \in \mathbb{N}$ and $a \in \mathfrak{g}_0$ we have $(\Phi_1^a)^n(1) = a^n$.*

PROOF. If $n = 1$ the statement is obvious. By induction

$$(\Phi_1^a)^{n+1}(1) \equiv \Phi_1^a \circ (\Phi_1^a)^n(1) = \Phi_1^a(a^n) = \sum_{j=0}^{n} \binom{n}{j} a^j \cdot \varphi^a(a^{n-j}).$$

From identity (1.2) we get $\varphi^a(a^j) = 0$ for $j \geq 1$, so $(\Phi_1^a)^{n+1}(1) = a^n \cdot \varphi^a(1) = a^{n+1}$. ∎

Corollary 2.5.1. *i) For all $a \in \mathfrak{g}_0$ we have*

$$\beta(a^n) = \beta(a)^n = j(a)^n, \ \forall n \in \mathbb{N}.$$

ii) For each $n \in \mathbb{N}$ and $a_1, ..., a_n \in \mathfrak{g}$

$$n!\beta(a_1 \cdots a_n) = \sum_{s \in \Sigma_n} \alpha(a_{s(1)}, ..., a_{s(n)})\beta(a_{s(1)}) \cdots \beta(a_{s(n)}).$$

From now on β will be called the *symmetrization map*. If $\mathbb{K}$ contains $\mathbb{Q}$, β is the usual symmetrization map. If $\mathbb{K}$ does not contains $\mathbb{Q}$, the previous corollary does not give an explicit formula for the symmetrization map. However, we can compute $\beta(X_1 \cdots X_n)$ (as in the following example) but we do not know a nice formula.

Example 2.5.1. *Let $\mathbb{K} = \mathbb{Z}/3\mathbb{Z}$, it contains $\frac{1}{2}$. Let $\mathfrak{g}$ be a 2-nilpotent Lie superalgebra over $\mathbb{K}$. For each $a \in \mathfrak{g}$, $\Phi^a = a^L + \frac{1}{2}\mathrm{ad}a$. Let $a_1, a_2, a_3 \in \mathfrak{g}$, we have*

$$\sigma(1) = 1$$
$$\sigma(j(a_1)) = a_1$$
$$\sigma(j(a_1)j(a_2)) = a_1 \cdot a_2 + \frac{1}{2}[a_1, a_2]$$
$$\sigma(j(a_1)j(a_2)j(a_3)) = a_1 \cdot a_2 \cdot a_3 + \frac{1}{2}\left\{a_1 \cdot [a_2, a_3] + [a_1, a_2] \cdot a_3 + (-1)^{p(a_2)p(a_1)} a_2 \cdot [a_1, a_3]\right\}.$$

By inversion we get

$$\beta(a_1 \cdot a_2) = j(a_1)j(a_2) - \frac{1}{2}\beta([a_1, a_2]) = j(a_1)j(a_2) - \frac{1}{2}j([a_1, a_2])$$
$$\beta(a_1 \cdot a_2 \cdot a_3) = j(a_1)j(a_2)j(a_3) - \frac{1}{2}\beta\left(a_1 \cdot [a_2, a_3] + [a_1, a_2] \cdot a_3 + (-1)^{p(a_2)p(a_1)} a_2 \cdot [a_1, a_3]\right) =$$
$$= j(a_1)\left(j(a_2)j(a_3) - \frac{1}{2}j([a_2, a_3])\right) - \frac{1}{2}\left(j([a_1, a_2])j(a_3) + (-1)^{p(a_2)p(a_1)} j(a_2)j([a_1, a_3])\right).$$

Remark 2.5.5. (Historical note)
*In the literature you can find proofs of the fact that β is an isomorphism of $\mathbb{K}$-modules for $\mathbb{K} = \mathbb{K}_0 \supseteq \mathbb{Q}$, $\mathfrak{g}$ a Lie $\mathbb{K}$-algebra (***[Coh]***,* **[Bou]** *exercise 16, page 78) or a Lie superalgebra (appendix of* **[Qui]***). All these proofs are reduced to the case of free Lie algebras.*
The case of N-nilpotent Lie superalgebras was known only for $N = 2$. It was proved by M. El-Agawany and A. Micali (see **[ElM]***).*

Before **[Coh]** *the theorem was known for some class of Lie algebras. For example P. Cartier showed the Poincaré-Birkhoff-Witt theorem for a Lie algebra over a Dedeking ring (see* **[Car]***). In the same paper there is an example of Lie algebra not verifying weak Poincaré-Birkhoff-Witt property. An older example is contained in* **[Sir]**.

Cohn uses that the theorem is true for $\mathfrak{g}$ a free Lie algebra, to prove that β is injective if $\mathbb{K}$ is torsion-free and $\mathfrak{g}$ a Lie $\mathbb{K}$-algebra. He is the first to prove the strong Poincaré-Birkhoff-Witt theorem with no assumption on $\mathfrak{g}$ as a $\mathbb{K}$-module. The same paper contains a family of examples of Lie algebras over a ring of prime characteristic p, not verifying weak Poincaré-Birkhoff-Witt theorem, one for each p prime.

In **[Bou]***) it is shown that β is one-to-one for a Lie algebra over $\mathbb{Q}$. The proof is reduced to the case of free Lie algebras.*

In the appendix of **[Qui]***, to prove that β is one-to-one, the proof is also reduced to the case of free Lie superalgebras.*

2.5.1 Universal representations over the enveloping algebra

We still assume hypothesis (H 2.5.1). By theorem 2.5.1 we can transport each coderivations Φ_c, $c \in \mathbb{K}_0^\times \cup \{0\}$, on $U(\mathfrak{g})$.

We recall that $U(\mathfrak{g})$ is equipped of a natural comultiplication Δ', such that for $a \in \mathfrak{g}$ we have $\Delta'(j(a)) = 1 \otimes j(a) + j(a) \otimes 1$.

Lemma 2.5.3. *The symmetrization map verify $\Delta' \circ \beta = (\beta \otimes \beta) \circ \Delta$. In particular for all $a \in \mathfrak{g}$ and $c \in \mathbb{K}_0^\times \cup \{0\}$, $\beta \circ \Phi_c^a \circ \beta^{-1}$ is a coderivation of $U(\mathfrak{g})$.*

PROOF. We consider the map $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ such that $X \mapsto (X, X)$. It induces the comultiplication over $S(\mathfrak{g})$ and $U(\mathfrak{g})$, so remark (2.5.4) ends the proof. ∎

Let $a \in \mathfrak{g}$. In $U(\mathfrak{g})$ we have $\mathrm{ad}j(a) = j(a)^L - j(a)^R$.

Theorem 2.5.2. *For all $a \in \mathfrak{g}$ we have*
i) $\beta^{-1} \circ \mathrm{ad}j(a) \circ \beta = \Phi_0^a$
ii) $\beta^{-1} \circ j(a)^L \circ \beta = \Phi_1^a$
iii) $\beta^{-1} \circ j(a)^R \circ \beta = -\Phi_{-1}^a$.

PROOF. i) The map $\mathfrak{g} \ni X \mapsto [a, X]$ is a derivation of $\mathfrak{g}$, it extend to derivations $\mathrm{ad}j(a)$ and Φ_0^a. The theorem 2.5.1 gives the identity i.
ii) Let $W \in S(\mathfrak{g})$, to show that $\sigma\left(j(a) \cdot \beta(W)\right) = \Phi_1^a(W)$ we only need to recall that by definitions we have $\sigma\left(j(a) \cdot \beta(W)\right) = \Phi_1^a \circ \sigma\left(\beta(W)\right) \equiv \Phi_1^a(W)$.
iii) As $\mathrm{ad}j(a) = j(a)^L - j(a)^R$ in $U(\mathfrak{g})$, the previous cases give

$$\beta^{-1} \circ j(a)^R \circ \beta = \Phi_1^a - \Phi_0^a.$$

From (2.15), the coderivation $\Phi_1^a - \Phi_0^a$ is equal to $-\Phi_{-1}^a$. ∎

Remark 2.5.6. *The map $a \mapsto \beta \circ \Phi_c^a \circ \beta^{-1}$ interpolates the regular left representation $a \mapsto j(a)^L$ ($c = 1$) and the regular right representation $a \mapsto -j(a)^R$ ($c = -1$).*

Theorem 2.5.3. *Let $\mathbb{K} = \mathbb{K}_0$ be a field of characteristic zero and $\mathfrak{g} = \mathfrak{g}_0$. All universal representations $\mathfrak{g} \to Hom(U(\mathfrak{g}), U(\mathfrak{g}))$ by coderivations are equivalent to the zero representation, or to the adjoint representation, or to the regular left representation.*

PROOF. Let $F : \mathfrak{g} \to Hom(U(\mathfrak{g}), U(\mathfrak{g}))$ be a representation by coderivations. We assume that F is not the zero representation. By lemma 2.5.3, for any $a \in \mathfrak{g}$, $G(a) := \beta^{-1} \circ F(a) \circ \beta$ is a coderivation of $S(\mathfrak{g})$. In particular G is a representation by coderivations of $\mathfrak{g}$ in $S(\mathfrak{g})$. Using that $\mathbb{K}$ is a field and using theorem 2.4.1, we get that G is one of the representations given in theorems 2.2.1 and 2.2.2. From theorem 2.2.7 we get that G is equivalent to Φ_1 or Φ_0. By theorem 2.5.2, $\mathfrak{g} \ni a \mapsto G(a)$ is equivalent to $\mathfrak{g} \ni a \mapsto \mathrm{ad}j(a)$ or $\mathfrak{g} \ni a \mapsto j(a)^L$. ∎

Remark 2.5.7. *Let $\mathbb{K}$ a field of characteristic zero. We note $\pi_1 : S(\mathfrak{g}) \to \mathfrak{g}$ the projection over $\mathfrak{g}$ and we put $P = \pi_1 \circ \beta^{-1}$. By theorem 1.2.1 the part ii and iii of theorem 2.5.2 is equivalent to $P \circ a^L = \varphi_1^a \circ \beta^{-1}$, $P \circ a^R = -\varphi_{-1}^a \circ \beta^{-1}$. In* **[Sol]** *and* **[Hel]** *we can find a formula for P.*

Remark 2.5.8. *In* **[Ber]** *and* **[Ras]** *we can find the formula for $\beta^{-1} \circ a^L \circ \beta$ which is in theorem 2.5.2.*

Chapter 3

Lie groups

3.1 Dual representations

In this section we suppose that $\mathbb{K}$ is a field of characteristic zero and that $\mathfrak{g}$ is a Lie $\mathbb{K}$-superalgebra.

We recall that, if A is a $\mathbb{K}$-algebra equipped with a comultiplication Δ, its dual $A^* := Hom(A, \mathbb{K})$ is a $\mathbb{K}$-algebra.

Lemma 3.1.1. *The dual map of a coderivation is a derivation.*

Proof. Let $\Phi : A \to A$ be a coderivation and let Φ^T be its transposed map. By definitions we have

$$\Phi^T \circ \mathrm{Mult} = \mathrm{Mult} \circ (1 \otimes \Phi^T + \Phi^T \otimes 1).$$

In particular if $a, b \in A$ we have $\Phi^T(a \cdot b) = \Phi^T(a) \cdot b + (-1)^{p(\Phi)p(a)} a \cdot \Phi^T(b)$. ∎

We note $\mathfrak{g}^* = Hom(\mathfrak{g}, \mathbb{K})$ the dual supervector space of $\mathfrak{g}$. Let be $X \in \mathfrak{g}$ and let $\partial(X)$ be the derivation of $S(\mathfrak{g}^*)$, with parity $p(X)$ such that for $f \in \mathfrak{g}^*$ we have $\partial(X)(f) = f(X)(-1)^{p(f)p(X)}$. Between the superalgebras $S(\mathfrak{g})$ and $S(\mathfrak{g}^*)$ we have a natural pairing noted $< \cdot, \cdot >$ and defined by

$$< X_1 \cdots X_n, f >= \partial(X_1) \circ \cdots \circ \partial(X_n)|_0(f) \tag{3.1}$$

for all $X_1, ..., X_n \in \mathfrak{g}$ and $f \in S(\mathfrak{g}^*)$.

Let $c \in \mathbb{K}$. The map $\mathfrak{g} \ni a \mapsto (\Phi_c^a)^T$ gives a representation of $\mathfrak{g}$ in $S(\mathfrak{g})^*$, by derivations. We note ξ_c this representation, for all $a \in \mathfrak{g}$ we have $\xi_c^a : S(\mathfrak{g})^* \to S(\mathfrak{g})^*$.

Theorem 3.1.1. *For all $a \in \mathfrak{g}$ and $c \in \mathbb{K}_0^\times$, the restriction of ξ_c^a to $\mathfrak{g}^*$ is $-\varphi_c(\mathrm{ad}x)(a)$.*

Proof. Let $f \in \mathfrak{g}^*$ and $X_1, ..., X_n \in \mathfrak{g}$. By definitions

$$\begin{aligned} -(-1)^{p(X_1 \cdots X_n)p(a)} < X_1 \cdots X_n, \xi_c^a(f) > &= - < \Phi_c^a(X_1 \cdots X_n), f > \\ &= - < \varphi_c^a(X_1 \cdots X_n), f > . \end{aligned}$$

Remark 3.1.1. *Let $\xi_c^a|_X : S(\mathfrak{g})^* \to \mathbb{K}$ be the evaluation of ξ_c^a in $X \in \mathfrak{g}_0$. For all $X \in \mathfrak{g}_0$, the previous theorem gives $\xi_c^a|_X = -\varphi_c(\mathrm{ad}X)(a)$.*

Remark 3.1.2. *Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. The vector fields $\{\xi_c^a, c \in \mathbb{K}_0^\times\}_{a\in\mathfrak{g}}$ are analytic. As explained for instance in* **[DuK]**, *the existence of such a family of analytic vector fields can be used to show that $\mathfrak{g}$ contains an open neighborhood U of the origin, equipped with an associative map $u : U\times U \to \mathfrak{g}$. This is the theorem, due to Sophus Lie, which states that we can choose U in such a way that u verifies also $u(X,-X)=0$ and $u(X,0)=u(0,X)=X$, for any $X\in U$.*

3.2 Lie groups

Let be $\mathbb{K}=\mathbb{K}_0=\mathbb{R}$, $\mathfrak{g}=\mathfrak{g}_0$ a finite-dimensional Lie $\mathbb{K}$-algebra, G a Lie group with Lie algebra $\mathfrak{g}$.

For any $c\in\mathbb{K}$, we consider the open set $\mathfrak{g}_c$ composed of $X\in\mathfrak{g}$ such that $\mathrm{ad}X$ has no eigenvalues in $(2\pi ci)\mathbb{Z}\setminus\{0\}\in\mathbb{C}$. For any $a\in\mathfrak{g}$ and $X\in\mathfrak{g}_c$, the series $\varphi_c(\mathrm{ad}X)(a)$ is convergent, and ξ_c^a is a vector field over $\mathfrak{g}_c$. Moreover for any $X\in\mathfrak{g}_c$, $\xi_c^a|_X$ is the corresponding tangent vector in the point $\mathfrak{g}$.

We note $C^\infty_{\mathfrak{g}}$ the sheaf of regular functions over $\mathfrak{g}$ and C^∞_G the sheaf of regular functions over G. If $exp:\mathfrak{g}\to G$ the exponential map of G, it induces $exp^*: C^\infty_G \to C^\infty_{\mathfrak{g}}$.
Let $V_0\subset\mathfrak{g}$ be an open set such that $exp|_{V_0}$ is a invertible, then $exp^*: C^\infty_G(expV_0)\to C^\infty_{\mathfrak{g}}(V_0)$ is invertible. In particular $T^{a,c}:=(exp^*)^{-1}\circ\xi_c^a\circ exp^*$ is a vector field over $exp(V_0\cap\mathfrak{g}_c^a)$.

Remark 3.2.1. *For any $c\in\mathbb{K}$, the map $\mathfrak{g}\ni a\mapsto T^{a,c}$ is a representation of $\mathfrak{g}$.*

Let $g\in G$, we note T_gG the tangent space in g, $l_g: G\to G$ is the left multiplication by $g\in G$, $dl_g:\mathfrak{g}\to T_gG$ is its differential.

Theorem 3.2.1. *Let $a\in\mathfrak{g}$, $c\in\mathbb{K}$. For all $X\in V_0\cap\mathfrak{g}_c^a$ the tangent vector $T^{a,c}_{exp(X)}\in T_{exp(X)}G$ is given by*

$$T^{a,c}_{exp(X)}=\begin{cases} dl_{exp(X)}\left(\frac{1-e^{-\mathrm{ad}X}}{1-e^{\frac{\mathrm{ad}X}{c}}}(a)\right), & c\neq 0\\ \\ dl_{exp(X)}\left((e^{-\mathrm{ad}X}-1)(a)\right), & c=0\end{cases}$$

Proof. We have $T^{a,c}_{exp(X)}=(\xi_c^a\circ exp^*)|_X=d(exp)_X\circ\xi_c^a|_X$. It is well known (see for example **[Var]**) that $d(exp)_X=dl_{exp(X)}\circ\frac{1-e^{-\mathrm{ad}X}}{\mathrm{ad}X}$. ∎

Remark 3.2.2. *We have*

$$T^{a,1}_{exp(X)}=-dl_{exp(X)}\left(e^{-\mathrm{ad}X}(a)\right),\quad T^{a,-1}_{exp(X)}=dl_{exp(X)}(a)$$

so $a\mapsto T^{a,1}$ and $a\mapsto T^{a,-1}$ are respectively the right and left regular representations.

Remark 3.2.3. *The family $\{\mathfrak{g}\ni a\mapsto T^{a,c}\}_c$ interpolates the left and the right regular representations.*

3.3 Differential forms over a module

Let $\mathbb{K}$ be a commutative superring and M a $\mathbb{K}$-module. To introduce the notion of differential form we need the following definition.

Definition. 3.3.1. *We denote by ΠM the $\mathbb{K}$-module with graduation $(\Pi M)_0 = M_1$ and $(\Pi M)_1 = M_0$. The identity over M gives an odd map $\pi \in Hom(M, \Pi M)$. The structure of $\mathbb{K}$-module for ΠM is given by*

$$(\pi m)\alpha = \pi(m\alpha), \forall m \in M, \forall \alpha \in \mathbb{K}.$$

Let N be a $\mathbb{K}$-superalgebra.

Definition. 3.3.2. *The space of differential forms over M and with values in N is*

$$A(M,N) := \bigoplus_{n=0}^{\infty} Hom(S^n(\Pi M) \otimes S(M), N).$$

Let $n \in \mathbb{N}$. A differential n-form is an element of $A^n(M,N) := Hom(S^n(\Pi M) \otimes S(M), N)$.

As $Hom(S(M), N)$ is $S(M)^*$-superalgebra, we have a structure of $S(M)^*$-superalgebra over the space of differential forms.

Let V be a $\mathbb{K}$-module. For any $\varphi \in Hom(S(V), V)$ and $F \in Hom(S(V), N)$, we introduce

$$\partial(\varphi)(F) = (-1)^{p(\varphi)p(F)} F \circ \Phi \in Hom(S(V), N)$$

where $\Phi \in Hom(S(V), S(V))$ is the coderivation corresponding to φ. Let $\alpha \in A^0(M,N)$ and $b \in M$, $\partial(b)(\alpha) = (-1)^{p(b)p(\alpha)} \alpha \circ b^L$ (it is equal to the derivative in the direction b introduced in section 1.2.1). Let $\alpha \in A(M,N)$ and $\varphi \in Hom(S(M), M)$. We introduce

$$i(\varphi)(\alpha) := \partial(\pi \circ \varphi)(\alpha) \in A^0(M,N).$$

In particular, if $\alpha \in A^n(M,N)$, with $n \geq 2$, and $a_1, ..., a_n \in \mathfrak{g}$, we have $i(a_1)\cdots i(a_n)(\alpha) := (-1)^{p(\alpha)p(\pi a_1 + \cdots + a_n)} \alpha(\pi a_1 \cdots \pi a_n \otimes id) \in A^0(M,N)$. We introduce the map $\delta \in Hom(S(M \oplus \Pi M), M \oplus \Pi M)$ such that

$$\begin{aligned}
&\delta(\pi m) = m, \quad \forall m \in M \\
&\delta(M) = \{0\} \\
&\delta(S^n(M \oplus \Pi M)) = \{0\}, \quad \forall n \neq 1.
\end{aligned}$$

The *de Rham differential* of $\alpha \in A(M,N)$ is the differential form $d\alpha := -\partial(\delta)(\alpha)$. We have the following commutation rules

$$[d, i(a)] = \partial(a),\ [i(a), \partial(b)] = 0,\ [i(a), i(b)] = 0,\ \forall a, b \in M.$$

Remark 3.3.1. *Let $\alpha_0 \in A^0(M,\mathbb{K})$, $\alpha_1 \in A^1(M,\mathbb{K})$ and $\alpha_2 \in A^2(M,\mathbb{K})$. For any $X, Y, Z \in M$ we have*

$$\begin{aligned}
i(X)(d\alpha_0) &= (-1)^{p(X)}\partial(X)(\alpha_0) \\
i(X)i(Y)(d\alpha_1) &= -\partial(X)(i(Y)(\alpha_1)) + (-1)^{p(X)p(Y)}\partial(Y)(i(X)(\alpha_1)) \\
i(X)i(Y)i(Z)(d\alpha_2) &= \partial(X)\big(i(Y)i(Z)(\alpha_2)\big) + (-1)^{p(\pi Z)p(X+Y)}\partial(Z)\big(i(X)i(Y)(\alpha_2)\big) + \\
&\quad +(-1)^{p(\pi X)p(Y+Z)}\partial(Y)\big(i(Z)i(X)(\alpha_2)\big).
\end{aligned}$$

3.4 Maurer-Cartan equations

Let $\mathbb{K}$ be a commutative superring and $\mathfrak{g}$ a Lie $\mathbb{K}$-superalgebra. We call a *left invariant Maurer-Cartan form* over $\mathfrak{g}$, each 1-differential form $\overline{\alpha} \in Hom(\Pi\mathfrak{g} \otimes S(\mathfrak{g}), \mathfrak{g})$ such that, $i(a)(\alpha) : S(\mathfrak{g}) \ni 1 \mapsto a$ for any $a \in \mathfrak{g}$, its de Rham differential $d(\overline{\alpha})$ verifies the Maurer-Cartan equation

$$d(\overline{\alpha}) = -\frac{1}{2}[\overline{\alpha}, \overline{\alpha}]. \tag{3.2}$$

In an analog way we have a notion of *right invariant Maurer-Cartan* form over $\mathfrak{g}$. We note $\widetilde{\alpha}$ such a 1-form. The difference with $\overline{\alpha}$ it that the differential $d\widetilde{\alpha}$ verifies another equation of Maurer-Cartan:

$$d(\widetilde{\alpha}) = \frac{1}{2}[\widetilde{\alpha}, \widetilde{\alpha}]. \tag{3.3}$$

For any $a \in \mathfrak{g}$, the contraction $i(a)(\overline{\alpha})$ belongs to $Hom(S(\mathfrak{g}), \mathfrak{g})$. We look for differential forms $\overline{\alpha}$ and $\widetilde{\alpha}$ described in the following way: $i(a)(\overline{\alpha}) = f(\mathrm{ad}x)(a)$ where $f \in \mathbb{K}_0[[t]]$ is a formal series such that $f(0) = 1$.
Let $\beta \in \{1, -1\}$, the two Maurer-Cartan equations (3.2) and (3.3) can written as

$$-\partial(a)(f(\mathrm{ad}x)(b)) + (-1)^{p(a)p(b)}\partial(b)(f(\mathrm{ad}x)(a)) = \beta[f(\mathrm{ad}x)(a), f(\mathrm{ad}x)(b)],\ \forall a, b \in \mathfrak{g} \tag{3.4}$$

where $\beta = 1$ corresponds to $\overline{\alpha}$ and $\beta = -1$ corresponds to $\widetilde{\alpha}$.

Lemma 3.4.1. *Equation (3.4) is equivalent to*

$$\left(\frac{f(u+t) - f(t)}{u} + \frac{f(u+t) - f(u)}{t} + \beta f(u)f(t) : [a, b]\right)_x = 0,\ \forall a, b \in \mathfrak{g}. \tag{3.5}$$

PROOF. From theorem 1.3.1 we have

$$\partial(a)(f(\mathrm{ad}x)(b)) = \left(\frac{f(t+u) - f(u)}{t} : [a, b]\right)_x,$$

so $\partial(b)(f(\mathrm{ad}x)(a)) = -(-1)^{p(a)p(b)} \left(\frac{f(t+u)-f(t)}{u} : [a, b]\right)_x$. ∎

Theorem 3.4.1. *i) Let $\beta \in \{1, -1\}$. The equation (3.5) is verified for all Lie $\mathbb{K}$-superalgebras if and only if*

$$\frac{f(u+t) - f(t)}{u} + \frac{f(u+t) - f(u)}{t} + \beta f(u)f(t) = 0 \tag{3.6}$$

in $\mathbb{K}_0[[t, u]]$.
ii) Let $\mathbb{K} \supseteq \mathbb{Q}$, the unique solution in $\mathbb{K}_0[[t]]$ with $f(0) = 1$ is $f(t) := \frac{e^{-\beta t}-1}{-\beta t}$.

PROOF. The first part follows from lemma 3.4.1. As seen in the proof of theorem 2.1.2, an equation of type (3.5) is verified for any $\mathbb{K}$-Lie superalgebra if and only if the functional equation (3.6) is verified.
ii) The functional equation (3.6) gives $f'(t) + \frac{f(t)-f(0)}{t} = -\beta f(0)f(t)$. If we fix $f(0) := c$, this equation has only one solution which is a formal series. For $c = 1$ and we have the unique solution $f(t) = \frac{e^{-\beta t}-1}{-\beta t}$. ∎

Corollary 3.4.1. *Let $\mathbb{K} \supseteq \mathbb{Q}$. Each Maurer-Cartan equations has one analytic solution independent of the Lie $\mathbb{K}$-superalgebra:*

$$\overline{\alpha} = \frac{1 - e^{-\mathrm{ad}x}}{\mathrm{ad}x}, \qquad \widetilde{\alpha} = \frac{e^{\mathrm{ad}x} - 1}{\mathrm{ad}x}.$$

Remark 3.4.1. *Let $\mathbb{K} = \mathbb{R}$. An analytic solution of the Maurer-Cartan equation can be used to prove the third Lie theorem on the existence of local Lie groups (see* **[Sha]***).*

Remark 3.4.2. *For any $a \in \mathfrak{g}$, there exist $\overline{a}$, $\widetilde{a}$ analytic vector fields over $\mathfrak{g}$ such that $i(\overline{a})(\overline{\alpha}) = i(\widetilde{a})(\widetilde{\alpha}) = a$. More over $\overline{a} = -\varphi_{-1}(\mathrm{ad}x)(a)$, $\widetilde{a} = \varphi_1(\mathrm{ad}x)(a)$.*

The vector field $\overline{a}$ is called the left invariant vector field of a, $\widetilde{a}$ is the right invariant one.

Chapter 4

Symmetric spaces

Let $\mathbb{K}$ be commutative superring. We recall the definition of a $\mathbb{K}$-supersymmetric space.

Definition. 4.0.1. *Let $\mathfrak{g}$ be a Lie $\mathbb{K}$-superalgebra, $\mathfrak{h}$ and $\mathfrak{q}$ two $\mathbb{K}$-modules such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. We say that $(\mathfrak{h}, \mathfrak{q})$ is a supersymmetric space if*

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}, \quad [\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{q}] \subseteq \mathfrak{q}.$$

Example 4.0.1. *Let $\mathfrak{g}$ be a Lie $\mathbb{K}$-superalgebra. As examples of symmetric spaces we have*
i) $\mathfrak{g}$ with $\mathfrak{h} = \mathfrak{g}_0$ and $\mathfrak{q} = \mathfrak{g}_1$, if $\mathfrak{g} = \mathfrak{g}_0$ or $\frac{1}{2} \in \mathbb{K}$.
ii) The direct product of Lie superalgebras $\mathfrak{g} \times \mathfrak{g}$ becomes a symmetric space if $\mathfrak{h}$ is the diagonal and $\mathfrak{q} = \{(X, -X), X \in \mathfrak{g}\}$, when $\frac{1}{2} \in \mathbb{K}$.

Let $\mathfrak{g}$ be a Lie $\mathbb{K}$-superalgebra. We consider the family $\{\Phi_c\}_c$ of representations introduced in section 2.2. We start with some properties of Φ_c related to $\mathfrak{g} \times \mathfrak{g}$.

Lemma 4.0.2. *Assume $\mathbb{K} \supseteq \mathbb{Q}$. The couple (Φ_c, Φ_d), with c and d in $\mathbb{K}_0^\times$, is composed of commuting representations for any Lie $\mathbb{K}$-superalgebra $\mathfrak{g}$, if and only if $c = -d$.*

Proof. Let us consider the couple Φ_c et Φ_d. By theorem 2.1.2 and by formula (2.12) they commute if

$$0 = d \cdot \varphi_c'(z) + \varphi_c(z) \cdot \frac{\varphi_d(z) - d}{z}.$$

If we derive and we set $z = 0$ we get $0 = c + d$.

If $c = d = 0$, we get Φ_0, which commutes with itself only for $\mathfrak{g}$ commutative. To end the proof, we need to show that $c = -d \neq 0$ is also a sufficient condition. This means that

$$-\varphi_{-c}(y)\frac{\varphi_c(x+y) - \varphi_c(x)}{y} - \varphi_c(x)\frac{\varphi_{-c}(x+y) - \varphi_{-c}(y)}{x} = 0.$$

this equation is equivalent to $\varphi_c(-y)\frac{\varphi_c(x+y)-\varphi_c(x)}{y} = -\varphi_c(x)\frac{\varphi_c(-x-y)-\varphi_c(-y)}{x}$ who is true. ∎

Let $\rho : \mathfrak{g} \times \mathfrak{g} \to Hom(S(\mathfrak{g}), S(\mathfrak{g}))$ be a representation, it decomposes into the sum of two commuting representations $\rho_1, \rho_2 : \mathfrak{g} \to Hom(S(\mathfrak{g}), S(\mathfrak{g}))$ such that $\rho(a_1, a_2) = \rho_1(a_1) + \rho_2(a_2)$ for each $(a_1, a_2) \in \mathfrak{g} \times \mathfrak{g}$. We write $\rho = (\rho_1, \rho_2)$.
In the following theorem we consider only Lie $\mathbb{K}$-algebras $\mathfrak{g}$ such that one of the following hypothesis is verified:
a_1) $\mathbb{K} \supseteq \mathbb{Q}$,
a_N) $\mathfrak{g}$ is N-nilpotent and $\frac{1}{2}, ..., \frac{1}{N} \in \mathbb{K}$.

Theorem 4.0.2. *Let $N \geq 1$, $\mathbb{K} = \mathbb{K}_0$ be a field, $\mathfrak{g} = \mathfrak{g}_0$ be a Lie $\mathbb{K}$-algebra. A representations by coderivations of $\mathfrak{g} \times \mathfrak{g}$ over $S(\mathfrak{g})$ is universal in the family of Lie algebras $\mathfrak{g}$ verifying a_N if and only if it is the zero representation, or $(\Phi_c, 0)$, or $(0, \Phi_c)$ with $c \in \mathbb{K}_0^\times \cup \{0\}$, (Φ_d, Φ_{-d}) with $d \in \mathbb{K}_0^\times$.*

Proof. If ρ is a representation by coderivations for any Lie algebra verifying a_N, also ρ_1 and ρ_2 have the same property. Theorems 2.2.3, 2.2.5 and lemma 4.0.2 end the proof. ▮

Using the previous theorem, corollary 2.2.7, theorem 2.5.2 we get

Corollary 4.0.2. *Let $N \geq 1$, $\mathbb{K} = \mathbb{K}_0$ be a field, $\mathfrak{g} = \mathfrak{g}_0$ be a Lie algebra. We have 5 classes of equivalence for non-zero representations by coderivations of $\mathfrak{g} \times \mathfrak{g} \to Hom(U(\mathfrak{g}), U(\mathfrak{g}))$ which are universal in the family of Lie algebras $\mathfrak{g}$ verifying a_N:*

$$\begin{aligned} &\mathfrak{g} \times \mathfrak{g} \ni (a,b) \mapsto \alpha \mathrm{ad}a + (1-\alpha)\mathrm{ad}b, && \alpha \in \{0,1\} \\ &\mathfrak{g} \times \mathfrak{g} \ni (a,b) \mapsto \alpha a^L - (1-\alpha) b^R, && \alpha \in \{0,1\} \\ &\mathfrak{g} \times \mathfrak{g} \ni (a,b) \mapsto a^L - b^R. \end{aligned}$$

Let $m(t) \in \mathbb{K}[[t]]$. To any $a \in \mathfrak{g}$ we associate the coderivation $M^a : S(\mathfrak{g}) \to S(\mathfrak{g})$. By theorem 2.1.2 we get

Lemma 4.0.3. *Assume that $\frac{1}{2} \in \mathbb{K}$. For any Lie $\mathbb{K}$-superalgebra $\mathfrak{g}$ the identity*

$$[M^a, M^b] = \frac{1}{4}\Phi_0^{[a,b]}, \forall a, b \in \mathfrak{g} \tag{4.1}$$

is verified, if and only if $m \in \mathbb{K}_0[[t]]$ verifies

$$m(u)\frac{m(t+u) - m(t)}{u} + m(t)\frac{m(t+u) - m(u)}{t} = \frac{1}{4}(t+u). \tag{4.2}$$

We consider the identity (4.1) because it occurs in the paper **[AlM]**.

Theorem 4.0.3. *i) Let $\mathbb{K} \supseteq \mathbb{Q}$. The identity (4.1) is verified for all $\mathbb{K}$-Lie superalgebras $\mathfrak{g}$, if $m(t) = \frac{1}{2}\varphi_0(t)$, or $m(t) = -\frac{1}{2}\varphi_0(t)$, or $m(t) = \frac{1}{2}t\coth(ct)$ with $c \in \mathbb{K}_0^\times$.*

ii) Let $N \geq 2$ and $\mathbb{K} \supseteq \{\frac{1}{2}, ..., \frac{1}{N}\}$. The identity (4.1) is verified for all N-nilpotent Lie superalgebra $\mathfrak{g}$, if $m(t) = \frac{1}{2}\varphi_0(t)$, or $m(t) = -\frac{1}{2}\varphi_0(t)$, or $m(t) = \frac{1}{2}t\coth(ct) \mod t^N$ with $c \in \mathbb{K}_0^\times$.

Proof. *i)* By the previous lemma we are reduced to look for all triples $(\varphi(t), \varphi(t), \frac{1}{4}t)$ verifying equation 2.11. Theorem 2.3.2 gives $\varphi = \pm\frac{1}{2}\varphi_0$ and theorem 2.3.3 gives $\varphi = \pm\frac{1}{2}\left(\varphi_{\frac{1}{2c}} - \varphi_{\frac{-1}{2c}}\right)$.
ii) Let I_N be the ideal of $\mathbb{K}_0[[t,u]]$ generated by $\{t^i u^j | i + j \geq N - 1\}$. In analogy with section 2.2.1, we want to solve equation (4.2) in $\mathbb{K}_0[[t,u]]/I_N$. Any solution of (4.2) in $\mathbb{K}_0[[t]]$, truncated to degree N, gives a solution of (4.2) in $\mathbb{K}_0[[t,u]]/I_N$. ▮

4.1 The symmetrization map for super symmetric spaces

Let $\mathbb{K} \supseteq \mathbb{Q}$ be a commutative superring. Let $\mathfrak{g} = (\mathfrak{h}, \mathfrak{q})$ be a supersymmetric space over $\mathbb{K}$. We consider $d(t), p(t) \in \mathbb{K}[[t]]$ such that $p(t) = p(-t)$, $d(t) = -d(-t)$. To any $a \in \mathfrak{g}$ we associate the coderivation $\Pi^a : S(\mathfrak{q}) \to S(\mathfrak{q})$ such that

$$\Pi^a = \begin{cases} id * p(\mathrm{ad}x)(a), a \in \mathfrak{q} \\ id * d(\mathrm{ad}x)(a), a \in \mathfrak{h} \end{cases}.$$

In particular

$$\Pi^a(1) = 0, \quad \forall a \in \mathfrak{h} \tag{4.3}$$

We note $\Pi : \mathfrak{g} \to Hom(S(\mathfrak{q}), S(\mathfrak{q}))$ the map such that $\mathfrak{g} \ni a \mapsto \Pi^a$.

Theorem 4.1.1. *The map* Π *is a representation for all* $\mathbb{K}$*-supersymmetric space, if* $(p(t), d(t)) = (0, 0)$ *or* $(d(t), p(t)) = (-t, t\coth(ct))$, $c \in \mathbb{K}_0^\times$.

Proof. By theorem 2.1.2 we solve the following system of equations

$$d(y)\frac{d(x+y) - d(x)}{y} + d(x)\frac{d(x+y) - d(y)}{x} = -d(x+y) \tag{4.4}$$

$$p(y)\frac{p(x+y) - p(x)}{y} + p(x)\frac{p(x+y) - p(y)}{x} = -d(x+y) \tag{4.5}$$

$$p(y)\frac{d(x+y) - d(x)}{y} + d(x)\frac{p(x+y) - p(y)}{x} = -p(x+y). \tag{4.6}$$

Equation (4.4) give $d(x)\left(\frac{d(x)}{x} + 1\right) = 0$. We consider only $d(x) = 0$ and $d(x) = -x$.

If $d = 0$, equation (4.6) give $p = 0$ and equation (4.5) is verified.

Let be $d(x) = -x$, equation (4.6) is verified and equation (4.5) becomes

$$p(y)\frac{p(x+y) - p(x)}{y} + p(x)\frac{p(x+y) - p(y)}{x} = x + y.$$

If we compare with equation (4.2) we get $p(z) = 2m(z)$. As $p(z) = p(-z)$, we get $p(t) = t\coth(ct)$ with $c \in \mathbb{K}_0^\times$. ∎

For any $c \in \mathbb{K}_0^\times$ we denote by $\Pi_c : \mathfrak{g} \to Hom(S(\mathfrak{q}), S(\mathfrak{q}))$ the representation corresponding to the couple of series

$$(p, d) = (t\coth\left(\frac{t}{c}\right), -t).$$

In the case of a supersymmetric space, the symmetrization map $\beta : S(\mathfrak{g}) \to U(\mathfrak{g})$ induces

$$\beta|_{S(\mathfrak{q})} : S(\mathfrak{q}) \to U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathfrak{h}.$$

Considering the case $c = 1$ we have the representation $\Pi_1 : \mathfrak{g} \to Hom(S(\mathfrak{q}), S(\mathfrak{q}))$. By universal property of the enveloping algebras, it extends to a morphism of algebras $\Pi : U(\mathfrak{g}) \to Hom(S(\mathfrak{q}), S(\mathfrak{q}))$. It gives a map $U(\mathfrak{g}) \ni u \mapsto \Pi^u(1) \in S(\mathfrak{q})$. By (4.3), this map induces a symbol map σ:

$$\sigma : U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathfrak{h} \to S(\mathfrak{q}).$$

Theorem 4.1.2. *The map $\beta|_{S(\mathfrak{q})} : S(\mathfrak{q}) \to U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathfrak{h}$ is a module-isomorphism with inverse map σ.*

PROOF. Let $a \in \mathfrak{q}_0$. By $\Pi^a(1) = a$ we get

$$\sigma \circ \beta(a^n) = a^n, \ \forall n \in \mathbb{N}. \tag{4.7}$$

By looking at the generic point of $\mathfrak{q}$, this implies the assertion: in the following we give the details.

We consider the Lie superalgebra $\mathfrak{g}_y := Hom(S(\mathfrak{q}), \mathfrak{g})$ over $S(\mathfrak{q})^*$. It has a natural structure of symmetric space $\mathfrak{g}_y = \mathfrak{q}_y \oplus \mathfrak{h}_y$, where $\mathfrak{h}_y = Hom(S(\mathfrak{q}), \mathfrak{h})$ and $\mathfrak{q}_y = Hom(S(\mathfrak{q}), \mathfrak{q})$. We denote by $y \in \mathfrak{q}_y$ the generic point of $\mathfrak{q}$. From identity (4.7) we get

$$\sigma_{\mathfrak{g}_y} \circ \beta_{\mathfrak{g}_y}|_{S(\mathfrak{q}_y)}(y^n) = y^n, \ \forall n \in \mathbb{N}. \tag{4.8}$$

Let $X_1, ..., X_n \in \mathfrak{q}$. By definitions we have $y^n(X_1 \cdots X_n) = n! X_1 \cdots X_n$, and

$$\left(\sigma_{\mathfrak{g}_y} \circ \beta_{\mathfrak{g}_y}(y^n)\right)(X_1 \cdots X_n) = n! \sigma \circ \beta(X_1 \cdots X_n).$$

In particular $X_1 \cdots X_n = \sigma \circ \beta(X_1 \cdots X_n)$ for any $X_1, ..., X_n \in \mathfrak{q}$, so $\sigma \circ \beta = id$. As β is onto, this shows that σ is the inverse of β. ∎

We denote by $\theta : \mathfrak{g} \to \mathfrak{g}$ the map such that $\theta|_{\mathfrak{q}} = -id$ and $\theta|_{\mathfrak{h}} = id$. Let $ad' : \mathfrak{g} \times U(\mathfrak{g}) \to U(\mathfrak{g})$ be defined as

$$ad'(a)(X) = a \cdot X - (-1)^{p(a)p(X)} X \cdot \theta(a), \quad \forall X \in U(\mathfrak{g}).$$

We call ad' the *twisted adjoint action* of $\mathfrak{g}$.

Theorem 4.1.3. *For all $a \in \mathfrak{g}$ we have $ad'(a) \circ \beta = \beta \circ \Pi_2^a$.*

PROOF. We remark that $ad'(a) \circ \beta = a^L \circ \beta - \theta(a)^R \circ \beta$. If we note $\Psi_a = \beta^{-1} \circ a^L \circ \beta - \beta^{-1} \circ \theta(a)^R \circ \beta$, from theorem 2.5.2 we get

$$\Psi_a = \Phi_1^a + \Phi_{-1}^{\theta(a)} = \begin{cases} \Phi_0^a, \ a \in \mathfrak{h} \\ \Phi_1^a + \Phi_{-1}^{-a}, \ a \in \mathfrak{q} \end{cases} = \Pi_2^a.$$

This means that $\beta^{-1} \circ ad'(a) \circ \beta = \Pi_2^a$. ∎

Corollary 4.1.1. *The set $\beta(S(\mathfrak{q}))$ is invariant by the* ad'*-action of* $\mathfrak{g}$.

4.2 Supertrace and Berezinian determinant

Let $\mathbb{K}$ be a commutative superring. The algebra of square matrices with coefficients in $\mathbb{K}$ and size $(n+m)$ has the following graduation: let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a matrix such that A, B, C, D have size n, (n, m), (m, n), m. We say that M is even if A, D have coefficients in

$\mathbb{K}_0$ and B, C have coefficients in $\mathbb{K}_1$. We say that M is odd if A, D have coefficients in $\mathbb{K}_1$ and B, C have coefficients in $\mathbb{K}_0$. The *supertrace* of the graded matrix M is defined as

$$str(M) := tr(A) - (-1)^{p(M)} tr(D).$$

Let us consider an even matrix M with D an invertible matrix. The berezinian determinant is defined as

$$Ber(M) := det(A - B \cdot D^{-1} \cdot C) det(D^{-1}) \in \mathbb{K}.$$

Proposition. 4.2.1. ([Ber1])
i) $str(M \cdot N) = (-1)^{p(M)p(N)} str(N \cdot M)$.
ii) When the $Ber(M)$ *and* $Ber(N)$ *are defined,* $Ber(M \cdot N) = Ber(M)Ber(N)$.
iii) Let $\mathbb{K} \supseteq \mathbb{Q}$. *When the exponential matrix* e^M *is defined*[1], $Ber(e^M) = e^{str(M)}$.

Let consider a free $\mathbb{K}$-module V of finite rang. Let $\{e_i\}_i$ be a basis of V and $\{e_1^*, ..., e_{n+m}^*\} \subset V^*$ be the dual basis. If $f : V \to V$ is a $\mathbb{K}$-linear map, it is represented by the matrix $M_{ij} := e_i^*(f(e_j))$. We define $str(f) := M$ and $Ber(f) := Ber(M)$.

4.3 A formula for Gorelik 's line

Let $\mathbb{K}$ be a field of zero characteristic.

In the paper **[Gor]**, M. Gorelik assumes that $\mathfrak{q} = \mathfrak{q}_1$ is a finite-dimensional vector space and $str_{\mathfrak{q}}(\mathrm{ad}a) = 0, \forall a \in \mathfrak{h}$. She shows that $U(\mathfrak{g})$ contains a line of ad'-invariant vectors . By theorem 4.1.3, this is equivalent to say that $S(\mathfrak{q})$ contains a line belonging to the kernel of Π_2.

In this section we give a formula for the Gorelik 's line in $S(\mathfrak{q})$. Before giving the proof we give a an example the formula when $\mathfrak{q}$ is of dimension two:

Lemma 4.3.1. *Let* $(\mathfrak{h}, \mathfrak{q})$ *be a supersymmetric space over a field of characteristic zero, with* $str_{\mathfrak{q}}(\mathrm{ad}a) = 0$ *for all* $a \in \mathfrak{h}$, $\mathfrak{q} = \mathfrak{q}_1$, $\{v_1, v_2\}$ *a basis for* $\mathfrak{q}$. *Gorelik 's line is generated by*

$$v_1 \cdot v_2 + \frac{1}{12} str_{\mathfrak{q}}(\mathrm{ad}v_1 \circ \mathrm{ad}v_2).$$

PROOF. Let $p(t) = t\coth(\frac{t}{2}) = p_1 + p_2 t^2 + \cdots = 1 + \frac{1}{6}t^2 + \cdots$. By definitions

$$\Pi_2^{v_1}(v_1 \cdot v_2) = p_2(\mathrm{ad}x)^2(v_1 \cdot v_2) + p_1 v_1 \cdot v_1 \cdot v_2 = p_2(-[[v_2, v_1], v_1] + (-1)^{p(v_1)}[v_2, [v_1, v_1]]).$$

Using the Jacobi identity we get

$$\Pi_2^{v_1}(v_1 \cdot v_2) = 3p_2[v_1, [v_1, v_2]] = \frac{1}{2}[v_1, [v_1, v_2]].$$

Let $\{v_1^*, v_2^*\}$ be the dual basis of $\{v_1, v_2\}$. As $str_{\mathfrak{q}}(\mathrm{ad}[v_1, v_1]) = 0$, identity (1.4) and the Jacobi identity give $v_2^*([[v_1, v_1], v_2]) = 0$. In particular $[v_1, [v_1, v_2]] \in span\{v_1\}$ and

$$\Pi_2^{v_1}(v_1 \cdot v_2) = \frac{1}{2} 1 v_1^*([v_1, [v_1, v_2]]) = \frac{1}{4}\Pi_2^{v_1}(1) v_1^*([v_1, [v_1, v_2]]).$$

[1]For instance if M is nilpotent matrix

From the Jacobi identity we get $[[v_1,v_1],v_2]=2[v_1,[v_2,v_1]])$. In particular

$$str_{\mathfrak{q}}(\mathrm{ad}v_1\circ\mathrm{ad}v_2)=-v_1^*([v_1,[v_2,v_1]])+2v_2^*([v_2,[v_2,v_1]]).$$

We get that

$$str_{\mathfrak{q}}(\mathrm{ad}v_1\circ\mathrm{ad}v_2)=3v_1^*([[v_2,v_1],v_1])+2str_{\mathfrak{q}}(\mathrm{ad}[v_1,v_2])=3v_1^*([[v_2,v_1],v_1])=-3v_1^*([v_1,[v_1,v_2]])$$

so

$$\Pi_2^{v_1}(v_1\cdot v_2)=-\frac{1}{12}\Pi_2^{v_1}\big(str_{\mathfrak{q}}(\mathrm{ad}v_1\circ\mathrm{ad}v_2)\big).$$

In particular $\Pi_2^{v_2}\left(v_1\cdot v_2-\frac{1}{12}str_{\mathfrak{q}}(\mathrm{ad}v_2\circ\mathrm{ad}v_1)\right)=0$. In a similar way we show that $\Pi_2^{v_2}(v_1\cdot v_2)+\frac{1}{12}\Pi_2^{v_2}(str_{\mathfrak{q}}(\mathrm{ad}v_1\circ\mathrm{ad}v_2))=0$. If $a\in\mathfrak{h}$ we have $\Pi_2^a(1)=0$ and

$$\Pi_2^a(v_1\cdot v_2)=[a,v_1]\cdot v_2+v_1\cdot[a,v_2]=-str_{\mathfrak{q}}(\mathrm{ad}a)v_1\cdot v_2$$

By hypotheses we get $\Pi_2^a(v_1\cdot v_2)=0$. ∎

Let $n:=dim(\mathfrak{q})$ be the dimension of $\mathfrak{q}$ as $\mathbb{K}$-vector space. If $n\geq 3$, it is not easy to generalize the previous proof. We are going to reduce the research of a formula for the Gorelik 's line to the resolution of a functional equation.

By the natural pairing (3.1) each $T\in S(\mathfrak{q})$ gives the morphism of $\mathbb{K}$-modules $<T,\cdot>$: $S(\mathfrak{q}^*)\to\mathbb{K}$. Let $v_1,...,v_n$ be a basis of $\mathfrak{q}=\mathfrak{q}_1$, to each $f\in S(\mathfrak{q}^*)$ we can associate the morphism of $\mathbb{K}$-modules $m_f:S(\mathfrak{q}^*)\to\mathbb{K}$ defined by

$$m_f(g)=(-1)^{np(f)}<v_1\cdots v_n,f\cdot g>,\quad\forall g\in S(\mathfrak{q}^*).$$

We denote by $\partial_f:S(\mathfrak{q})\to S(\mathfrak{q})$ the derivation such that $\partial_f(v_i):=f(v_i)$. If $g_1,...,g_n\in S(\mathfrak{q}^*)$ we note $\partial_{g_1\cdots g_n}:=\partial_{g_1}\circ\cdots\circ\partial_{g_n}$.

Remark 4.3.1. *We have* $<v_1\cdots v_n,f\cdot g>=(-1)^{np(f)}<\partial_f(v_1\cdots v_n),g>$.

Let $v_0^{(n)}=v_1\cdots v_n+\cdots\in S(\mathfrak{q})$ be in the Gorelik 's line, we can write $v_0^{(n)}=\partial_f(v_1\cdots v_n)$ with $f\in S(\mathfrak{q}^*)$ such that its component over $\mathbb{K}=S^0(\mathfrak{q}^*)$ is 1. We note this component with $f(0)$. In the following we find a formula for f.

Let $\{\pi^a,a\in\mathfrak{g}\}$ be the family of vector fields over $\mathfrak{q}$ defined by the representation Π_2. If y is the generic point of $\mathfrak{q}$ we have $\pi^a=g^a(\mathrm{ad}y)(a)$ with

$$g^a(z)=\begin{cases}-z, & a\in\mathfrak{h}\\ z\cdot\coth(\frac{z}{2}), & a\in\mathfrak{q}\end{cases}.$$

Remark 4.3.2. *By duality* $(-1)^{p(a)p(T)}<T,\pi^a(\cdot)>=<\Pi_2^a(T),\cdot>$ *for all* $a\in\mathfrak{g}$ *and* $T\in S(\mathfrak{q}^*)$.

By this remark, $v_0^{(n)}$ verifies $<v_0^{(n)},\pi^a(\cdot)>\equiv 0$ for any $a\in\mathfrak{g}$.
Let $x_1,...,x_n$ be the dual base of v_i, in this section we note $\frac{\partial}{\partial x_i}:=\partial(v_i)$. As $y=v_i\otimes x_i\in\mathfrak{q}\otimes S(\mathfrak{q}^*)$, for $a\in\mathfrak{g}$ we note

$$\pi_i^a:=(-1)^{p(v_i)p(a)}x_i(g^a(\mathrm{ad}y)(a))\in S(\mathfrak{q}^*),\ i=1,...,n$$

so $\pi^a=\sum_{i=1}^n\pi_i^a\cdot\frac{\partial}{\partial x_i}\in\mathfrak{q}\otimes S(\mathfrak{q}^*)$.

Lemma 4.3.2. *For any $f \in S(\mathfrak{q}^*)_0$ and $a \in \mathfrak{g}$, we have*

$$< \partial_f(v_1 \cdots v_n), \pi^a(g) >= - < v_1 \cdots v_n, (\pi^a(f) + f \cdot div(\pi^a)) \cdot g >, \quad \forall g \in \ S(\mathfrak{q}_1^*).$$

Proof. By definitions we get

$$f \cdot \pi^a(g) = \sum_{i=1}^{n} (-1)^{p(v_i)p(a+v_i)} \frac{\partial}{\partial x_i}(f \cdot \pi_i^a \cdot g) - \pi^a(f) \cdot g - f \cdot div(\pi^a) \cdot g.$$

As $\mathfrak{q}_1$ is finite-dimensional, for each i we have $< v_1 \cdots v_n, \frac{\partial}{\partial x_i}(g) >= 0$. ∎

We have shown that

Theorem 4.3.1. *For each $n \geq 1$, $v_0^{(n)} = \partial_f(v_0 \cdots v_n) = v_0 \cdots v_n + \cdots \in S(\mathfrak{q})$ is in Gorelik 's line if and only if the element $f \in S(\mathfrak{q}^*)$ verifies*

$$\begin{cases} \pi^a(f) + f \cdot div(\pi^a) = 0, & \forall a \in \mathfrak{g} \\ f(0) = 1 \end{cases} . \tag{4.9}$$

To solve (4.9) we use a preliminary lemma, it uses the following notation. Let be $p(z), q(z) \in \mathbb{K}[z]$ and $a \in \mathfrak{g}$,

$$(p(t)q(u) : \text{ad}a)_y := [p(\text{ad}y)(a), q(\text{ad}y)(\cdot)] \in Hom(\mathfrak{g}, \mathfrak{g})$$

which is extended by linearity to polynomial in $\mathbb{K}[t, u]$.

Lemma 4.3.3. *Let be $(\mathfrak{h}, \mathfrak{q})$ a super symmetric space over a field $\mathbb{K}$, such that $\mathfrak{q}$ is finite-dimensional. For each polynomial $q(z) \in \mathbb{K}[z]$ such that $q(z) = -q(-z)$ we have*

$$str_{\mathfrak{q}}\left(\left(q(u+t)t : \text{ad}a\right)_y\right) = 0, \quad \forall a \in \mathfrak{h}, .$$

Proof. It is sufficient to consider $q(z) = q_k(z) := z^{2k+1}$ with $k \geq 0$. By definitions

$$str_{\mathfrak{q}}\left(\left(q_k(u+t)t : \text{ad}a\right)_y\right) = \sum_{j=0}^{2k+1} \binom{2k+1}{j} str_{\mathfrak{q}}\left(\text{ad}\left((\text{ad}y)^j(a)\right) \circ (\text{ad}y)^{2k+2-j}\right).$$

This supertrace is zero because bracket rules in a symmetric space give

$$[\text{ad}^j y(\mathfrak{h}), \text{ad}^{2k+2-j} y(\mathfrak{q})] \subseteq \mathfrak{h} \otimes S(\mathfrak{q}_1^*), \ \forall j. \ ∎$$

Proposition. 4.3.1. *Let $(\mathfrak{h}, \mathfrak{q})$ be a supersymmetric space over a field of characteristic zero, with $\mathfrak{q}$ is finite-dimensional and odd supervector space and $str_{\mathfrak{q}}(\text{ad}a) = 0$ for $a \in \mathfrak{h}$. For all $(\mathfrak{h}, q)$, $f = Ber_{\mathfrak{q}}\left(\frac{\sinh(\frac{\text{ad}y}{2})}{\frac{\text{ad}y}{2}}\right) \in S(\mathfrak{q}^*)_0$ is a solution of (4.9).*

PROOF. Using definitions and theorem 1.3.1 we get

$$div(\pi^a) = (-1)^{p(v_i)}\frac{\partial}{\partial x_i}x_i(g^a(\mathrm{ad}y)(a)) = x_i\left(\frac{\partial}{\partial x_i}g^a(\mathrm{ad}y)(a)\right) =$$

$$= \quad (-1)^{p(v_i)}x_i\left(\frac{g^a(t+u)-g^a(u)}{t} : [v_i,a]\right)_y$$

$$=: \quad -str_{\mathfrak{q}}\left(\left(\frac{g^a(t+u)-g^a(u)}{t} : \mathrm{ad}a\right)_y\right).$$

Choosing $f = e^{str_{\mathfrak{q}}(h(\mathrm{ad}y))}$ with $h(t)\in\mathbb{Q}[t]$ we have

$$\frac{\partial f}{\partial x_i} = f\cdot\frac{\partial}{\partial x_i}(str_{\mathfrak{q}}(h(\mathrm{ad}y))) = f\cdot str_{\mathfrak{q}}\left(\frac{\partial h(\mathrm{ad}y)}{\partial x_i}\right) = f\cdot str_{\mathfrak{q}}\left(h'(\mathrm{ad}y)\frac{\partial \mathrm{ad}y}{\partial x_i}\right)$$

and by fundamentals properties of the supertrace we get

$$\frac{\partial f}{\partial x_i} = (-1)^{p(v_i)}f\cdot str_{\mathfrak{q}}\left(h'(\mathrm{ad}y)\cdot \mathrm{ad}v_i\right).$$

From lemma 1.3.1 we get $\frac{\partial f}{\partial x_i} = (-1)^{p(v_i)}f\cdot str_{\mathfrak{q}}\left((h'(t+u):\mathrm{ad}v_i)_y\right)$, so we have

$$\pi^a(f) = (-1)^{p(v_i)p(a)}x_i(g^a(\mathrm{ad}y)(a))\frac{\partial f}{\partial x_i} =$$

$$= \quad f\cdot str_{\mathfrak{q}}\left((h'(t+u):\mathrm{ad}g^a(\mathrm{ad}y)(a))_y\right)$$

$$= \quad f\cdot str_{\mathfrak{q}}\left((h'(t+u)g^a(t):\mathrm{ad}a)_y\right).$$

Equation (4.9) becomes

$$\begin{cases} str_{\mathfrak{q}}\left(\left(h'(t+u)g^a(t) - \frac{g^a(t+u)-g^a(t)}{u} : \mathrm{ad}a\right)_y\right) = 0, \forall a\in\mathfrak{g} \\ h(0)=0 \end{cases}$$

which means

$$\begin{cases} str_{\mathfrak{q}}\left((-h'(t+u)t+1:\mathrm{ad}a)_y\right) = 0, & a\in\mathfrak{h} \\ str_{\mathfrak{q}}\left(\left(t\coth(\frac{t}{2})h'(t+u) - \frac{(t+u)\coth(\frac{t+u}{2})-t\coth(\frac{t}{2})}{u} : \mathrm{ad}a\right)_y\right) = 0, & a\in\mathfrak{q} \\ h(0)=0 \end{cases} \cdot \tag{4.10}$$

Hypothesis over the supertrace can be written as $str_{\mathfrak{q}}((1:\mathrm{ad}a)_y)=0$ for $a\in\mathfrak{h}$. By lemma 4.3.3 we put

$$\begin{cases} h'(u) = -h'(-u) \\ t\coth(\frac{t}{2})h'(t+u) - \frac{(t+u)\coth(\frac{t+u}{2})-t\coth(\frac{t}{2})}{u} = 0 \\ h(0)=0 \end{cases}$$

who can be reduced to

$$\begin{cases} h'(u) = \frac{1}{2}\coth(\frac{u}{2}) - \frac{1}{u} \\ h(0)=0 \end{cases}$$

so $h(u) = \lg\left(\frac{\sinh(\frac{u}{2})}{\frac{u}{2}}\right)$. ∎

Remark 4.3.3. *In the previous proof we use the formal series* $\frac{\sinh(\frac{z}{2})}{\frac{z}{2}}$. *Its logarithmic derivative is equal to the series giving the universal solution of the cyclotomic equation in theorem 5.2.2.*

Remark 4.3.4. *By definition* $str_{\mathfrak{q}}(\mathrm{ad}v_i \circ \mathrm{ad}v_j + \mathrm{ad}v_j \circ \mathrm{ad}v_i) = str_{\mathfrak{q}}(\mathrm{ad}[v_i, v_j])$ *for all* $i, j = 1, ..., n$, *so*

$$str_{\mathfrak{q}}(\mathrm{ad}v_i \circ \mathrm{ad}v_j) = -str_{\mathfrak{q}}(\mathrm{ad}v_j \circ \mathrm{ad}v_i).$$

For any permutation $s \in \Sigma_n$ we denote by $|s|$ its signature.

Corollary 4.3.1.

$$\begin{aligned}
v_0^{(2)} &= v_1 \cdot v_2 + \frac{1}{12} str_{\mathfrak{q}}(\mathrm{ad}v_1 \circ \mathrm{ad}v_2) \\
v_0^{(3)} &= v_1 \cdot v_2 \cdot v_3 + \sum_{s\in\Sigma_3, s(1)<s(2)} (-1)^{|s|} \frac{1}{12} str_{\mathfrak{q}}(\mathrm{ad}v_{s(1)} \circ \mathrm{ad}v_{s(2)}) v_{s(3)} \\
v_0^{(4)} &= v_1 \cdot v_2 \cdot v_3 \cdot v_4 + \frac{1}{12} \sum_{s\in\Sigma_4, s(1)<s(2)} (-1)^{|s|} str_{\mathfrak{q}}(\mathrm{ad}v_{s(1)} \circ \mathrm{ad}v_{s(2)}) v_{s(3)} \cdot v_{s(4)} + \\
& \frac{1}{278} \sum_{s\in\Sigma_4, s(1)<s(2), s(3)<s(4)} (-1)^{|s|} str_{\mathfrak{q}}(\mathrm{ad}v_{s(1)} \circ \mathrm{ad}v_{s(2)}) str_{\mathfrak{q}}(\mathrm{ad}v_{s(3)} \circ \mathrm{ad}v_{s(4)}) + \\
& -\frac{1}{2880} \sum_{s\in\Sigma_4} (-1)^{|s|} str_{\mathfrak{q}}(\mathrm{ad}v_{s(1)} \circ \mathrm{ad}v_{s(2)} \circ \mathrm{ad}v_{s(3)} \circ \mathrm{ad}v_{s(4)}).
\end{aligned}$$

PROOF. Let $M := \mathrm{ad}y$, if $\dim(\mathfrak{q}) \leq 4$ we have $M^5 = 0$ so

$$\begin{aligned}
f = Ber_{\mathfrak{q}} & \left(1 + \frac{1}{24} M^2 + \frac{1}{1920} M^4\right) =: Ber_{\mathfrak{q}} \left(1 + aM^2 + bM^4\right) = \\
&= e^{str_{\mathfrak{q}} lg\left(1 + aM^2 + bM^4\right)} \\
&= e^{str_{\mathfrak{q}} \; aM^2 + bM^4 - \frac{a^2}{2} M^4} \\
&= 1 + \left(b - \frac{a^2}{2}\right) str_{\mathfrak{q}}(M^4) + a \cdot str_{\mathfrak{q}}(M^2) + \frac{a^2}{2} \cdot str_{\mathfrak{q}}(M^2)^2 \\
&= 1 - \frac{1}{2880} str_{\mathfrak{q}}(M^4) + \frac{1}{24} str_{\mathfrak{q}}(M^2) + \frac{1}{1152} str_{\mathfrak{q}}(M^2)^2.
\end{aligned}$$

If $n = 2$ we have $M^2 = (-\mathrm{ad}v_1 \circ \mathrm{ad}v_2 + \mathrm{ad}v_2 \circ \mathrm{ad}v_1) x_1 \cdot x_2$ so

$$\begin{aligned}
f &= 1 - \frac{1}{12} str_{\mathfrak{q}}(\mathrm{ad}v_1 \circ \mathrm{ad}v_2) x_1 \cdot x_2 \\
\partial_f(v_1 \cdot v_2) &= v_1 \cdot v_2 + \frac{1}{12} str_{\mathfrak{q}}(\mathrm{ad}v_1 \circ \mathrm{ad}v_2) \partial_{x_1} \circ \partial_{x_2}(v_2 \cdot v_1)
\end{aligned}$$

If $n = 3$, $M^2 = \sum_{s\in\Sigma_3} (-\mathrm{ad}v_{\sigma(1)} \circ \mathrm{ad}v_{\sigma(2)} + \mathrm{ad}v_{\sigma(2)} \circ \mathrm{ad}v_{\sigma(1)}) x_{\sigma(1)} \cdot x_{\sigma(2)}$ and

$$f = 1 - \frac{1}{12} \sum_{s\in\Sigma_3, s(1)<s(3)} str_{\mathfrak{q}}(\mathrm{ad}v_{s(1)} \circ \mathrm{ad}v_{s(2)}) x_{s(1)} \cdot x_{s(2)}$$

$$\partial_f(v_1 \cdot v_2 \cdot v_3) = v_1 \cdot v_2 \cdot v_3 - \frac{1}{12} \sum_{s\in\Sigma_3, s(1)<s(2)} (-1)^{|s|} str_{\mathfrak{q}}(\mathrm{ad}v_{s(1)} \circ \mathrm{ad}v_{s(2)}) \partial_{x_{\sigma(1)}} \circ \partial_{x_{\sigma(2)}}(v_{s(1)} v_{s(2)} v_{s(3)}).$$

If $n = 4$ we have $M^2 = \sum_{1\le i<j\le 4} -(\mathrm{ad}v_i \circ \mathrm{ad}v_j - \mathrm{ad}v_j \circ \mathrm{ad}v_i)x_i \cdot x_j$ and

$$M^4 = \sum_{s\in\Sigma_4} (\mathrm{ad}v_{s(1)}\circ\mathrm{ad}v_{s(2)} - \mathrm{ad}v_{s(2)}\circ\mathrm{ad}v_{s(1)})(\mathrm{ad}v_{s(3)}\circ\mathrm{ad}v_{s(4)} - \mathrm{ad}v_{s(4)}\circ\mathrm{ad}v_{s(3)})x_{s(1)}\cdot x_{s(2)}\cdot x_{s(3)}\cdot x_{s(4)}$$

so

$$\begin{aligned} f = 1 + \frac{1}{12} \sum_{s\in\Sigma_4, s(1)<s(2)} & str_{\mathfrak{q}}(\mathrm{ad}v_{s(1)} \circ \mathrm{ad}v_{s(2)})v_{s(2)} \cdot v_{s(1)} + \\ + \frac{1}{278} \sum_{s\in\Sigma_4, s(1)<s(2), s(3)<s(4)} & (-1)^{|s|} str_{\mathfrak{q}}(\mathrm{ad}v_{s(1)} \circ \mathrm{ad}v_{s(2)}) str_{\mathfrak{q}}(\mathrm{ad}v_{s(3)} \circ \mathrm{ad}v_{s(4)}) x_1 \cdot x_2 \cdot x_3 \cdot x_4 + \\ - \frac{1}{2880} \sum_{s\in\Sigma_4} & (-1)^{|s|} str_{\mathfrak{q}}(\mathrm{ad}v_{s(1)} \circ \mathrm{ad}v_{s(2)} \circ \mathrm{ad}v_{s(3)} \circ \mathrm{ad}v_{s(4)}) x_1 \cdot x_2 \cdot x_3 \cdot x_4. \blacksquare \end{aligned}$$

Example 4.3.1. ([ArB])
As a first example we consider the Lie superalgebra

$$\mathfrak{g} = osp(1,2) = \left\{ \begin{pmatrix} 0 & a & b \\ -b & c & d \\ a & e & -c \end{pmatrix} ; a, b, c, d, e \in \mathbb{K} \right\}$$

with $\mathfrak{q} = \mathfrak{g}_1$ equipped of the basis

$$v_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in this basis $v_0^{(2)} = v_1 \cdot v_2 + \frac{1}{4}$.

Example 4.3.2. [ArB]
As second example we have

$$\mathfrak{g} = osp(1,4) = \left\{ \begin{pmatrix} 0 & v^T & w^T \\ -w^T & C & D \\ v & E & -C^T \end{pmatrix} ; v, w \in \mathbb{K}^2, D = D^T, E = E^T \right\}$$

with C, D, E 2-dimensional matrices. As a basis for $\mathfrak{q}_1$ we take

$$v_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In this basis

$$v_0^{(4)} = v_1 \cdot v_2 \cdot v_3 \cdot v_4 - \frac{5}{12}(v_2 \cdot v_4 + v_1 \cdot v_3) - \frac{3}{16}.$$

In **[ArB]** *are given formulas for the image of the Gorelik 's element in* $U(\mathfrak{g})$*, for all* $\mathfrak{g} = osp(1, 2n)$ *with* $n \geq 1$*. From these formulas it is easy to get the corresponding formulas for* $S(\mathfrak{q})$*. For* $n = 1$ *and* $n = 2$ *we have formulas which look slightly different because we use a different basis.*

Chapter 5

A special classical dynamical Yang-Baxter equation

Let $\mathbb{K}$ be a commutative superring.

5.1 Quadratic Lie superalgebras

Let M, N be two $\mathbb{K}$-modules. A linear application $\alpha : M \otimes M \to N$ is called a *bilinear form* and we write $\alpha(X, Y) = \alpha(X \otimes Y)$ for any $X, Y \in M$. We say that α is *symmetric* if $\alpha(X, Y) = (-1)^{p(X)p(Y)}\alpha(Y, X)$ for any $X, Y \in \mathfrak{g}$. We say that α it is *not-degenerate* when $\alpha' : \mathfrak{g} \ni X \mapsto \alpha(X, \cdot) \in \mathfrak{g}^*$ is one-to-one.

Let $\mathfrak{g}$ be a Lie $\mathbb{K}$-superalgebra, we say that $\alpha : \mathfrak{g} \otimes \mathfrak{g} \to N$ is *invariant* if $\alpha(X, [Y, Z]) = \alpha([X, Y], Z)$ for any $X, Y, Z \in \mathfrak{g}$.

Lemma 5.1.1. *Let $\mathfrak{g}$ be equipped with an invariant bilinear form $\alpha : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{K}$. For all $X \in \mathfrak{g}_0$ and $Y, Z \in \mathfrak{g}$ we have*

$$\alpha((\mathrm{ad}X)^j(Y), Z) = (-1)^j \alpha(Y, (\mathrm{ad}X)^j(Z)),\ \forall j \in \mathbb{N}.$$

PROOF. The statement follows by induction over j. ▮

We extend α to a bilinear form $\mathfrak{g}_x \otimes \mathfrak{g}_x \to S(\mathfrak{g})^*$ still noted α, in the following way:

$$\alpha(F, G) := \alpha \circ (F \otimes G) \circ \Delta, \quad \forall F, G \in \mathfrak{g}_x.$$

Remark 5.1.1. *If $\alpha : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{K}$ is invariant, the extension $\alpha : \mathfrak{g}_x \otimes \mathfrak{g}_x \to S(\mathfrak{g})^*$ is invariant.*

Definition. 5.1.1. *Let $\mathbb{K}$ be a field, $\mathfrak{g}_0$ and $\mathfrak{g}_1$ be finite-dimensional vector spaces. If α is even, symmetric, invariant and non-degenerate, we say that $(\mathfrak{g}, \alpha)$ is a quadratic Lie $\mathbb{K}$-superalgebra.*

5.2 A solution of vCDYBE

Let $\mathfrak{g}$ be a Lie $\mathbb{K}$-superalgebra equipped of an invariant, symmetric, even bilinear form $\gamma : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{K}$.

Let $f(t) \in \mathbb{K}_0[[t]]$. For any $Y \in \mathfrak{g}$ we consider the map $f^Y = f(\mathrm{ad}x)(Y) \in Hom(S(\mathfrak{g}), \mathfrak{g})$.

Remark 5.2.1. *Let be $Y, Z \in \mathfrak{g}$. As γ is invariant, $\gamma(f(\mathrm{ad}x)(Y), Z) = \gamma(Y, f(-\mathrm{ad}x)(Z))$. As γ is symmetric $\gamma(f(\mathrm{ad}x)(Y), Z) = (-1)^{p(Z)p(Y)}\gamma(Z, f(-\mathrm{ad}x)(Y))$.*

We suppose also that $f(t) = -f(-t)$. The previous remark allows to introduce the differential 2-form $\omega \in Hom(S^2(\Pi\mathfrak{g}) \otimes S(\mathfrak{g}), \mathbb{K})$ such that

$$i(X)i(Y)(\omega) = (-1)^{p(X)}\gamma(f^X, Y), \quad \forall X, Y \in \mathfrak{g}$$

As γ is invariant, we introduce the 3-differential form $\alpha \in Hom(S^3(\Pi\mathfrak{g}) \otimes S(\mathfrak{g}), \mathbb{K})$ such that

$$i(X)i(Y)i(Z)(\gamma) = (-1)^{p(Y)}\gamma([X, Y], Z), \ \ \forall X, Y, Z \in \mathfrak{g}.$$

If l, q, r are three power series with coefficients in $\mathbb{K}_0$, we introduce the notation

$$\left(\left(l(t)q(u)r(v) : \alpha(X, Y, Z)\right)_x\right) := (-1)^{p(Y)}\gamma([l^X, q^Y], r^Z) \in S(\mathfrak{g})^*$$

for any $X, Y, Z \in \mathfrak{g}$. This notation is extended to all formal power series in $\mathbb{K}_0[[t, u, v]]$.

Remark 5.2.2. *For all $X, Y, Z \in \mathfrak{g}$ we have*

$$\begin{aligned}\left(\left(l(t,u,v) : \alpha(X,Y,Z)\right)_x\right) &= (-1)^{p(\pi X)p(\pi Y)}\left(\left(l(u,t,v) : \alpha(Y,X,Z)\right)_x\right)\\ \left(\left(l(t,u,v) : \alpha(X,Y,Z)\right)_x\right) &= (-1)^{p(\pi Z)p(\pi Y)}\left(\left(l(t,v,u) : \alpha(X,Z,Y)\right)_x\right).\end{aligned}$$

In particular, if $l(t,v,u) = l(t,u,v) = l(u,v,t)$ we get a 3-differential form over $\mathfrak{g}$. In this case we write $(l(t,u,v) : \alpha(X,Y,Z))_x =: i(X)i(Y)i(Z)((l(t,u,v) : \alpha)_x)$.

Let $\epsilon \in \mathbb{K}_0$, by remark 5.2.2, the equation

$$(f(t)f(v) + f(u)f(v) + f(t)f(u) + \epsilon : \alpha)_x = d(\omega) \tag{5.1}$$

is well defined. This equation is (see below) a variant of the Classical Dynamical Yang-Baxter Equation (vCDYBE). If equation (5.1) holds, we say that the formal series $f(t) \in \mathbb{K}[[t]]$ is a *solution of (5.1)*.

Lemma 5.2.1. *We have the identity*

$$d(\omega) = \left(\frac{f(t+u) - f(u)}{t} + \frac{f(v+t) - f(t)}{v} + \frac{f(v+u) - f(v)}{u} : \alpha\right)_x.$$

PROOF. Let $X, Y, Z \in \mathfrak{g}$. From remark 3.3.1 we get

$$\begin{aligned}i(X)i(Y)i(Z)(d\omega) = (-1)^{p(Y)}\partial(X)(\gamma(f^Y, Z)) + (-1)^{p(X)+p(\pi Z)p(X+Y)}\partial(Z)(\gamma(f^X, Y)) +\\ +(-1)^{p(Z)+p(\pi X)p(Y+Z)}\partial(Y)(\gamma(t^Z, X)).\end{aligned}$$

From theorem 1.3.1 and from definitions we get

$$\begin{aligned}&\partial(X)(\gamma(f^Y, Z)) = \gamma(\partial(X)(f^Y), Z)) =\\ =\ &\gamma\left(\left(\frac{f(t+u) - f(u)}{t} : [X, Y]\right)_x, Z\right) =\\ =\ &(-1)^{p(Y)}i(X)i(Y)i(Z)\left(\left(\frac{f(t+u) - f(u)}{t} : \alpha\right)_x\right).\end{aligned}$$

In particular $\partial(Z)(\gamma(f^X,Y)) = (-1)^{p(X)} i(Z)i(X)i(Y)\left(\left(\frac{f(t+u)-f(u)}{t}:\alpha\right)_x\right)$. From remark 5.2.2 we get

$$\begin{aligned}\partial(Z)(\gamma(f^X,Y)) &= (-1)^{p(X)+p(\pi Z)p(\pi X)} i(X)i(Z)i(Y)\left(\left(\frac{f(t+u)-f(t)}{u}:\alpha\right)_x\right) = \\ &= (-1)^{p(X)+p(\pi Z)p(Y+X)} i(X)i(Y)i(Z)\left(\left(\frac{f(t+v)-f(t)}{v}:\alpha\right)_x\right).\end{aligned}$$

In the same way we get

$$\partial(Y)(\gamma(t^Z,X)) = (-1)^{p(Z)+p(\pi X)p(Y+Z)} i(X)i(Y)i(Z)\left(\left(\frac{f(u+v)-f(u)}{v}:\alpha\right)_x\right). \blacksquare$$

By this lemma, equation (5.1) is equivalent to the equation

$$\begin{aligned} &\left(\frac{f(t+u)-f(u)}{t}+\frac{f(v+t)-f(t)}{v}+\frac{f(v+u)-f(v)}{u}:\alpha\right)_x = \\ = \quad &(f(t)f(v)+f(u)f(v)+f(t)f(u)+\epsilon:\alpha)_x .\end{aligned}$$

Theorem 5.2.1. *The previous equation is verified for all Lie $\mathbb{K}$-superalgebras equipped of an invariant bilinear form, if*

$$\begin{aligned}&\frac{f(t+u)-f(u)}{t}+\frac{f(u+v)-f(v)}{u}+\frac{f(v+t)-f(t)}{v} = \\ &= f(t)f(u)+f(u)f(v)+f(v)f(t)+\epsilon \mod t+u+v\end{aligned} \tag{5.2}$$

in $\mathbb{K}_0[[t,u,v]]$.

Proof. We only need to remark that the invariance of $\gamma(\cdot,\cdot)$ gives $(t+u+v:\alpha)_x \equiv 0$ so, if $g(t,u,v)\in\mathbb{K}_0[[t,u,v]]$ we have $((t+u+v)g(t,u,v):\alpha)_x\equiv 0$. $\blacksquare$

If $t+u+v=0$ (5.2) becomes

$$\frac{f(t+u)-f(u)}{t}+\frac{f(u+t)-f(t)}{u}+\frac{f(u)+f(t)}{u+t} = f(t)f(u)-f(u)f(t+u)-f(u+t)f(t)+\epsilon. \tag{5.3}$$

Lemma 5.2.2. *Equation (5.2) has only one solution $f\in\mathbb{Q}[[t]]$. If $\epsilon=\frac{1}{4}$ we get $f(t) = -\frac{1}{t}+\frac{1}{2}\coth\left(\frac{t}{2}\right)=\frac{1}{2}+\frac{\varphi_1(t)-1}{t}$.*

Proof. We apply $u\to 0$ to equation (5.3) and we get $f'(t) = -2\frac{f(t)}{t}-f(t)^2+\epsilon$. This equation has only one odd formal power series in $\mathbb{Q}[[t]]$ as solution. $\blacksquare$

Theorem 5.2.2. *Let $\mathbb{K}\supseteq\mathbb{Q}$ be a commutative superring. For any Lie $\mathbb{K}$-superalgebra equipped of a invariant bilinear form, the odd series $\rho(z)=-\frac{1}{z}+\frac{1}{2}\coth\left(\frac{z}{2}\right)\in\mathbb{Q}[[z]]$ gives a solution of equation (5.1) with $\epsilon=\frac{1}{4}$.*

The series $\rho(z)$ is called the *universal solution* of the cyclotomic equation (5.1).

Remark 5.2.3. (The classical dynamical Yang-Baxter equation)
Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$, $\mathfrak{g}$ be a finite-dimensional Lie algebras, $\mathfrak{h}$ a Lie subalgebra. We consider a map $r:\mathfrak{h}^*\to\mathfrak{g}\otimes\mathfrak{g}$ analytic in an open set containing zero. For any $1\le i,j\le 3$ with $i\neq j$, we use the standard notation $r_{ij}:\mathfrak{h}^*\to\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$ (for example $r_{12}=r\otimes 1$ and $r_{23}=1\otimes r$). Let

$$CYBE(r):=[r_{12},r_{13}]+[r_{12},r_{23}]+[r_{13},r_{23}]\in\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}.$$

For any $X\in\mathfrak{g}$ we use the standard notations $X_1=X\otimes 1\otimes 1$, $X_2=1\otimes X\otimes 1$, $X_3=1\otimes 1\otimes X$. Let $\{e_j\}_j$ be a basis of $\mathfrak{h}$, we introduce (see **[EtV]***)*

$$CDYBE(r):=CYBE(r)+\sum_j (e_j)_1\frac{\partial r_{23}}{\partial e_j}-(e_j)_2\frac{\partial r_{13}}{\partial e_j}+(e_j)_3\frac{\partial r_{12}}{\partial e_j}\in\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}.$$

We say that r verifies the classical dynamical Yang-Baxter equation if $CDYBE(r)=0$.

Let us suppose that $\mathfrak{g}$ is equipped of a bilinear form γ such that $(\mathfrak{g},\gamma)$ is a quadratic Lie algebra. We identify $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. The bilinear form γ defines an element $c\in\mathfrak{g}\otimes\mathfrak{g}$. By looking for solutions for which $r-r_{21}$ is a constant multiple of c, one obtains a modified Classical Dynamical Yang-Baxter Equation (vCDYBE) for the antisymmetric part of r. We denote by $\{e^i\}_i$ the basis for $\mathfrak{g}$ such that $\gamma(e^i,e_j)=\delta_{i,j}$ for any i,j. Let $\varepsilon\in\mathbb{K}$ and $\varphi:=e^j\otimes e^k\otimes[e_j,e_k]\in\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$. We consider $\mathfrak{h}=\mathfrak{g}$. The modified equation vCDYBE with coupling constant ε is the equation $CDYBE(r)=\varepsilon\varphi$, where $r:\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$, is a function with values in the antisymmetric part of $\mathfrak{g}\otimes\mathfrak{g}$.

Let $f(t)\in\mathbb{K}[[t]]$. We consider $r_\gamma=\sum_{j,k}\gamma(e_j,f^{e^k})e^j\otimes e^k$. Using properties of γ, we get that the modified equation vCDYBE for r_β is equivalent to

$$\begin{aligned}\gamma\left([f^Z,Y],f^X\right)+\partial(Z)\left(\gamma(f^Y,X)\right)+\gamma\left([f^Y,X],f^Z\right)+\\ +\partial(Y)\left(\gamma(f^X,Z)\right)+\gamma\left([f^X,Z],f^Y\right)+\partial(X)\left(\gamma(f^Z,Y)\right)=\\ =\epsilon\gamma([X,Y],Z),\qquad\qquad \forall X,Y,Z\in\mathfrak{g}.\end{aligned}$$

This cyclotomic equation is equation (5.1) for $\mathfrak{g}=\mathfrak{g}_0$.

Remark 5.2.4. *P. Etingof and A. Varchenko give in* **[EtV]** *a classification of solutions of the classical dynamical Yang Baxter equation when $\mathfrak{g}$ is a simple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra.*

A. Alekseev and E. Meinrenken in **[AlM]** *consider the modified equation vCDYBE with $\epsilon=\frac{1}{4}$ and $\mathfrak{g}$ a simple Lie algebra. They show that $f(t)=-\frac{1}{t}+\frac{1}{2}\coth(\frac{z}{2})$ gives a solution of vCDYBE (they deduce it using* **[EtV]***).*

The functional equation (5.3) was found independently by L. Feher and B.G. Pusztai in **[FeP]***. They give another direct proof of the fact that r_β is a solution of vCDYBE when $f(t)=-\frac{1}{t}+\frac{1}{2}\coth(\frac{z}{2})$. Their proof uses the theory of holomorphic functional calculus of linear operators.*

Chapter 6

A remarkable application

6.1 Some properties of quadratic Lie superalgebras

Let $\mathbb{K} = \mathbb{K}_0$ be a field, $\mathfrak{g}$ be a Lie finite-dimensional $\mathbb{K}$-superalgebra equipped of a bilinear form γ such that $(\mathfrak{g}, \gamma)$ is quadratic (see definition 5.1.1). We fix a basis $\{e_i\}_i$ and the corresponding dual basis $\{e_i^*\}_i \subseteq \mathfrak{g}^*$. We consider also the basis $\{e^i\}_i$ for $\mathfrak{g}$ such that $\gamma(e^i, e_j) = \delta_{i,j}$ for any i, j.

Remark 6.1.1. *For any $X \in \mathfrak{g}$ we have $X = \sum_i e_i \gamma(e^i, X) = \sum_i e^i \gamma(X, e_i)$.*

Corollary 6.1.1. *For each $X \in \mathfrak{g}_0$ we have*

$$str\Big((\mathrm{ad}X)^{2k+1}\Big) = 0, \ \forall k \geq 0.$$

PROOF. By definition $str((\mathrm{ad}X)^k) = \sum_i (-1)^{p(e_i)} \gamma(e^i, (\mathrm{ad}X)^k(e_i))$. We have $str((\mathrm{ad}X)^k) = \sum_i (-1)^{p(e_i)} \gamma(e^i, e^j) \gamma((\mathrm{ad}X)^k(e_i), e_j)$. As γ is invariant and symmetric we get

$$\gamma((\mathrm{ad}X)^k e_i, e_j) = (-1)^k \gamma(e_i, (\mathrm{ad}X)^k e_j) = (-1)^{k+p(e_i)p(e_j)} \gamma((\mathrm{ad}X)^k e_j, e_i)$$

so $str((\mathrm{ad}X)^k) = (-1)^k str((\mathrm{ad}X)^k)$. ∎

The bilinear form $\gamma : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{K}$ extends to a bilinear form $\gamma : \mathfrak{g}_x \otimes \mathfrak{g}_x \to S(\mathfrak{g}^*)$ (see section 5.1). This last one preserve properties such that the degree, the property of being symmetric. Let x be the generic point of $\mathfrak{g}$. As $\mathfrak{g}$ is finite-dimensional, $x = \sum_i e_i \otimes e_i^* \in \mathfrak{g} \otimes S(\mathfrak{g}^*)$. Let $J := Ber\left(\frac{\sinh\left(\frac{\mathrm{ad}x}{2}\right)}{\frac{\mathrm{ad}x}{2}}\right) \in S(\mathfrak{g})^*$ (it has been introduced in section 4.3).

Lemma 6.1.1. *i) It does exists $F(z) \in \mathbb{K}[[t]]$ such that $F(z)^2 = \frac{\sinh(\frac{z}{2})}{\frac{z}{2}}$, $F(z) = F(-z)$ and $F(0) = 1$.*
ii) It does exist $J^{\frac{1}{2}} \in S(\mathfrak{g})^*$, such that $J^{\frac{1}{2}} \cdot J^{\frac{1}{2}} = J$ and $J^{\frac{1}{2}}(1_{S(\mathfrak{g})}) = 1$. More over $J^{\frac{1}{2}} = Ber\,(F\,(\mathrm{ad}x))$.

We consider

$$t := \rho(\mathrm{ad}x) \equiv \frac{1}{\mathrm{ad}x} - \frac{1}{2}\coth\left(\frac{\mathrm{ad}x}{2}\right) \in Hom(S(\mathfrak{g}), End(\mathfrak{g}))$$

and we introduce the matrix S with coefficients in $S(\mathfrak{g})^*$ given by

$$S_{ij} := -(-1)^{p(e_j)} \gamma\left(t(e^j), e^i\right).$$

Lemma 6.1.2. *For any couple of index i, j we have $S_{ij} = (-1)^{p(\pi e_i)p(\pi e_j)} S_{ji}$.*

PROOF. Let ω be the differential form introduced in section 5.2, with $f = \rho$. The statement follows from $S_{ij} = -i(e^j)i(e^i)(\omega)$. ∎

6.2 The quantization map

The paper **[AlM]** contains an application of the universal solution ρ in theorem 5.2.2.

Let $\mathfrak{h}$ be the Lie algebra of a compact Lie group H. Using the fact that the universal solution verifies the cyclotomic equation (5.1), A. Alekseev and E. Meinrenken get a quantization map from the usual Weil algebra to their non-commutative Weil algebra.

Here we generalize this result of A. Alekseev and E. Meinrenken to any quadratic, finite-dimensional Lie superalgebra over a field $\mathbb{K} = \mathbb{K}_0$ of characteristic zero. From now on $\mathfrak{g}$ will be such a Lie superalgebra. We start with the following definition

Definition. 6.2.1. (H. Cartan)
Let A be a superalgebra over a ring. We say that A is a $\mathfrak{g}$-differential algebra if it is equipped with derivations d, $\{L(a), a \in \mathfrak{g}\}$, $\{i(a), a \in \mathfrak{g}\}$ such that d has degree 1, $L(a)$ degree $p(a)$, $i(a)$ degree $p(a)+1$ and

$$\begin{array}{lll} [i(a), i(b)] = 0, & [L(a), L(b)] = L([a,b]), & d \circ d = 0 \\ [d, i(a)] = L(a), & [i(a), L(b)] = i([a,b]), & \end{array} \quad \forall a, b \in \mathfrak{g}. \tag{6.1}$$

Each element of $A_{bas} := \{w \in A; L(a)(w) = i(a)(w) = 0, \forall a \in \mathfrak{g}\}$ is called a basic element.

We introduce the superalgebra $S(\mathfrak{g}^*) \otimes S(\Pi\mathfrak{g}^*)$ called the *Weil algebra.* It is equipped with the following structure of $\mathfrak{g}$-differential algebra. Let $\{e_i\}_i$ be an homogeneous basis of $\mathfrak{g}$ and $\{e_i^*\}_i \subset \mathfrak{g}^*$ its dual basis. For any $a \in \mathfrak{g}$, we have

$$\begin{array}{ll} i^W(a)(\pi\psi) = \psi(a), & L^W(a)(\pi\psi) = -(-1)^{p(a)p(\pi\psi)}\pi(\psi \circ \mathrm{ad}a) \\ i^W(a)(\psi) = 0, & L^W(a)(\psi) = -(-1)^{p(a)p(\psi)}\psi \circ \mathrm{ad}a, \qquad \forall \psi \in \mathfrak{g}^* \\ d^W(\pi\psi) = \psi + \frac{1}{2}\sum_i (-1)^{p(e_i)}\pi e_i^* \cdot L^W(e_i)(\pi\psi) & \\ d^W(\psi) = \sum_i -(-1)^{p(e_i)} e_i^* \cdot L^W(e_i)(\pi\psi). & \end{array}$$

Definition. 6.2.2. *Let (B, d^B, i^B, L^B) be a $\mathfrak{g}$-differential algebra. The equivariant cohomology $H_{\mathfrak{g}}(B)$ is the cohomology of $((S(\mathfrak{g}^*) \otimes S(\Pi\mathfrak{g}^*) \otimes B)_{bas}, d^W \otimes id + id \otimes id \otimes d^B)$.*

Example 6.2.1. *Let $B = \mathbb{K}$. It is equipped of the trivial structure of $\mathfrak{g}$-differential algebra. The usual equivariant cohomology $H_{\mathfrak{g}}(B)$ is the set of invariants of $S(\mathfrak{g}^*)$.*

Let $(\mathfrak{g}, \gamma)$ be a quadratic Lie superalgebra. We introduce the bilinear form $\gamma^\pi : \pi\mathfrak{g} \otimes \pi\mathfrak{g} \to \mathbb{K}$ such that $\gamma^\pi(\pi X, \pi Y) = (-1)^{p(X)}\gamma(X, Y)$ for any $X, Y \in \mathfrak{g}$. Let I be the ideal of $T(\Pi\mathfrak{g})$ generated by $\{\pi X \otimes \pi Y - (-1)^{p(\pi Y)p(\pi X)}\pi Y \otimes \pi X - \gamma^\pi(\pi X, \pi Y) | X, Y \in \mathfrak{g}\}$. We define the Clifford algebra

$$Clif(\Pi\mathfrak{g}, \gamma^\pi) := T(\Pi\mathfrak{g})/I.$$

A. Alekseev and E. Meinrenken have introduced a new equivariant cohomology. This one is defined by a $\mathfrak{g}$-differential algebra structure over $U(\mathfrak{g})\otimes Clif(\Pi\mathfrak{g},\gamma^\pi)$ which A. Alekseev and E. Meinrenken called the *non-commutative Weil algebra.* This $\mathfrak{g}$-differential structure is

$$\begin{array}{cc} i^{\mathcal{W}}(a)(\pi Z)=\gamma(Z,a), & i^{\mathcal{W}}(a)\big(j(Z)\big)=0 \\ L^{\mathcal{W}}(a)(\pi Z)=(-1)^{p(a)}\pi[a,Z], & L^{\mathcal{W}}(a)\big(j(Z)\big)=j([a,Z]) \\ \multicolumn{2}{c}{d^{\mathcal{W}}(\pi Z)=j(Z)+\frac{1}{2}\sum_i(-1)^{p(e_i)}\pi e^i\cdot L^{\mathcal{W}}(e_i)(\pi Z)} \\ \multicolumn{2}{c}{d^{\mathcal{W}}(j(Z))=\sum_i \pi e_i\cdot j([e^i,Z])} \end{array} \tag{6.2}$$

where $a,X,Z\in\mathfrak{g}$ and j is the inclusion of $\mathfrak{g}$ in the enveloping algebra $U(\mathfrak{g})$.

Definition. 6.2.3. ([AlM])
Let (B,d^b,i^b,L^B) be a $\mathfrak{g}$-differential algebra. The new equivariant cohomology $\hat{H}_{\mathfrak{g}}(B)$ is the cohomology of $((U(\mathfrak{g})\otimes Clif(\Pi\mathfrak{g},\gamma^\pi)\otimes B)_{bas}, d^{\mathcal{W}}\otimes id+id\otimes id\otimes d^B)$.

Example 6.2.2. *Let $B=\mathbb{K}$ be equipped of the trivial structure of $\mathfrak{g}$-differential algebra. Then $\hat{H}_{\mathfrak{g}}(B)$ is the center of the enveloping algebra $U(\mathfrak{g})$.*

As $\mathfrak{g}$ is quadratic, we have the natural isomorphism $\mathfrak{g}\ni X\to\gamma(X,\cdot)\in\mathfrak{g}^*$. In particular (d^W,i^W,L^W) induces a structure of $\mathfrak{g}$-differential algebra over $S(\mathfrak{g})\otimes S(\Pi\mathfrak{g})$: for any $a\in\mathfrak{g}$

$$\begin{array}{ccc} i'(a)(\pi Z)=\gamma(Z,a), & i'(a)(Z)=0 & \\ L'(a)(\pi Z)=(-1)^{p(a)}\pi[a,Z], & L'(a)(Z)=[a,Z], & \forall Z\in\mathfrak{g}. \\ \multicolumn{2}{c}{d'(\pi Z)=Z-\frac{1}{2}\sum_i(-1)^{p(e_i)}\pi e^i\cdot L'(e_i)(\pi Z)} & \\ \multicolumn{2}{c}{d'(Z)=-\sum_i e^i\cdot\pi[e_i,Z]} & \end{array} \tag{6.3}$$

A. Alekseev and E. Meinrenken have introduced a map

$$Q: S(\mathfrak{g})\otimes S(\Pi\mathfrak{g})\to U(\mathfrak{g})\otimes Clif(\Pi\mathfrak{g},\gamma^\pi)$$

exchanging differentials d' and $d^{\mathcal{W}}$. Its explicit formula uses two supervector space isomorphisms.

One is

$$\begin{array}{rcl} \sigma_\gamma : Clif(\Pi\mathfrak{g},\gamma^\pi) & \to & S(\Pi\mathfrak{g}) \\ 1 & \mapsto & 1. \end{array} \tag{6.4}$$

Let $X_1,...,X_n\in\mathfrak{g}$. We denote by $s_\gamma(X_1)$ the sum of the left multiplication $(\pi X_1)^L$ and of the derivation $\frac{1}{2}i'(X_1)$. We define $\sigma_\gamma(X_1\cdots X_n)=s_\gamma(X_1)\circ\cdots\circ s_\gamma(X_n)(1)$.

Notation 6.2.1. *Let $f\in S(\mathfrak{g})^*$. It defines the operator $\partial_f:S(\mathfrak{g})\to S(\mathfrak{g})$ such that*

$$\begin{array}{rcl} \partial_f(1) & = & f(1) \\ \partial_f(X\cdot W) & = & \partial_{f\circ X^L}(W)+(-1)^{p(f)p(X)}X\cdot\partial_f(W), \forall X\in\mathfrak{g},\ \forall W\in S(\mathfrak{g}). \end{array}$$

To get simpler notations, it will be denoted also by f.

Example 6.2.3. *Let $X, Y \in \mathfrak{g}$. We have*

$$\begin{aligned} \partial_f(X) &= f(X) + f(1)X \\ \partial_f(X \cdot Y) &= f(X \cdot Y) + f(X)Y + (-1)^{p(f)p(X)} X f(Y) + f(1) X \cdot Y \end{aligned}$$

Remark 6.2.1. *For any $f, g \in S(\mathfrak{g})^*$ we have $\partial_{f \cdot g} = \partial_f \circ \partial_g$.*

The other vector space isomorphism is $Duf := \beta \circ \partial_{J^{\frac{1}{2}}} : S(\mathfrak{g}) \to U(\mathfrak{g})$ ($J^{\frac{1}{2}}$ was introduced in lemma 6.1.1). It allowed M. Duflo to show (**[Duf]**), in the case of a Lie algebra, that the center of $U(\mathfrak{g})$ is isomorph, as an algebra, to $\{W \in S(\mathfrak{g}); L^W(a)(W) = 0, \forall a \in \mathfrak{g}\}$.

The formula of A. Alekseev and E. Meinrenken is

$$Q = (Duf \otimes \sigma_\gamma^{-1}) \circ e^{\sum_{a,b} S_{ab} \otimes i'(e_a) \circ i'(e_b)}.$$

By lemma 6.1.2, the map Q is well-defined.

Remark 6.2.2. *For any $W \in S(\mathfrak{g})$, $Q(W \otimes 1) = Duf(W) \otimes 1$.*

Theorem 6.2.1. *For each $a \in \mathfrak{g}$ we have*

$$Q \circ i'(a) = i^{\mathcal{W}}(a) \circ Q, \quad Q \circ L'(a) = L^{\mathcal{W}}(a) \circ Q.$$

PROOF. See appendix A.

Theorem 6.2.2. *The map Q intertwines the differentials d' and $d^{\mathcal{W}}$ if $\rho(z)$ is solution of the cyclotomic equation (5.1) with $\epsilon = \frac{1}{4}$.*

A. Alekseev and E. Meinrenken give two proofs of the fact that Q exchanges differentials. One is a direct calculation, we can find in appendix A its generalization to quadratic finite-dimensional Lie superalgebras. This calculation uses commutation rules (6.1), the properties of $\gamma(\cdot,\cdot)$. It shows that exchanging differentials is a consequence of the fact that ρ verify the cyclotomic equation (5.1). The hypothesis on the topology of H and the results of **[EtV]**, allow A. Alekseev and E. Meinrenken to use this property of ρ.

Theorem 6.2.3. *Application Q induces a one-to-one map in cohomology.*

PROOF. As Q is a morphism of $\mathfrak{g}$-differentials algebras and $Q(0) = 0$, Q induces a map in cohomology. The map Q is one-to-one because Duf and σ_γ are one-to-one. In particular also the map induced by Q in cohomology is one-to-one. ∎

Appendix A

Intertwining property

Let $\mathbb{K} = \mathbb{K}_0$ be a field of characteristic zero and $\mathfrak{g}$ be a quadratic Lie $\mathbb{K}$-superalgebra. This appendix is devoted to show theorems 6.2.1 and 6.2.2.

Let $\{e_j\}$ be a basis of $\mathfrak{g}$ and $\{x_j\} \subseteq \mathfrak{g}^*$ its dual basis.

A.1 Preliminary properties

For any $X \in \mathfrak{g}$ we introduce $d^\Lambda(\pi X) := \frac{1}{2}\sum_j \pi e^j \cdot \pi[j, X] \in S(\Pi\mathfrak{g})$. In the following lemma we give formulas for the structure of $\mathfrak{g}$-differential algebra over $S(\mathfrak{g}) \otimes S(\Pi\mathfrak{g})$ introduced in (6.3).

Lemma A.1.1. *For any $a \in \mathfrak{g}$ we have*

$$L^{'}(a) = \Phi_0^a + (-1)^{p(a)} \sum_j (\pi[a, e^j])^L \circ i^{'}(e_j) \tag{A.1}$$

$$d^{'} = \sum_j \left((e^j)^L + d^\Lambda(\pi e^j)^L\right) \circ i^{'}(e_j) + (-1)^{p(e_j)}(\pi e^j)^L \circ \Phi_0^{e_j}. \tag{A.2}$$

Proof. The formula (A.1) follows from definitions and from the remark 6.1.1. As $d^{'}$ is a derivation of $S(\mathfrak{g} \otimes \Pi\mathfrak{g})$, it is sufficient to show that the formula is true over $\Pi\mathfrak{g}$ and $\mathfrak{g}$. We remark that the formula (A.2) is verified over $\Pi\mathfrak{g}$. From $d^{'} \circ d^{'} = 0$ we get $d^{'}(X) = \sum_j (-1)^{p(e_j)} \pi e^j \cdot [e_j, X]$ for any $X \in \mathfrak{g}$. In particular the formula (A.2) is verified over $\mathfrak{g}$. ∎

Remark A.1.1. *For any $a \in \mathfrak{g}$ and $f \in S(\mathfrak{g})^*$ we have $[a^L, \partial_f] = -\partial_{\partial(a)(f)}$.*

The identification between $\mathfrak{g}$ and $\mathfrak{g}^*$ allows to extend $L^{'}(a)$ to $End(\mathfrak{g}_x)$.

Lemma A.1.2. *For any $a \in \mathfrak{g}$ we have*
i) $L^{'}(a)(\mathrm{ad}Y) = \mathrm{ad}[a, Y]$, $\forall Y \in \mathfrak{g}$
ii) $L^{'}(a)(\mathrm{ad}x) = 0$.

Proof. *i)* We have $\mathrm{ad}Y = \sum_i [Y, e_i] \otimes x_i = \sum_i [Y, e_i] \otimes e^i$, so

$$\begin{aligned} L^{'}(a)(\mathrm{ad}Y) &= \sum_i L^{'}(a)([Y, e_i]) \otimes x_i + (-1)^{p(Y+e_i)p(X)}[Y, e_i] \otimes L^{'}(a)(e^i) = \\ &= \sum_i [a, [Y, e_i]] \otimes x_i - (-1)^{p(Y)p(X)} \sum_{i,k} [Y, e_i] \otimes e_i^*([k, e_k])x_k \\ &= \mathrm{ad}a \circ \mathrm{ad}Y - (-1)^{p(Y)p(a)} \mathrm{ad}Y \circ \mathrm{ad}a. \end{aligned}$$

ii) From i we get

$$L^{'}(a)(\mathrm{ad}x) = \sum_i L^{'}(a)(\mathrm{ad}e_i \otimes x_i) = \sum_i \mathrm{ad}[a, e_i] \otimes x_i - \sum_k \mathrm{ad}[a, e_k] \otimes x_k. \blacksquare$$

The family $\{L^{'}(a), a \in \mathfrak{g}\}$ induces the dual action over $S(\mathfrak{g})^*$. It is still noted $L^{'}$.

Lemma A.1.3. *For any $a \in \mathfrak{g}$ and $f \in S(\mathfrak{g})^*$ we have $[L^{'}(a), \partial_f] = \partial_{L^{'}(a)(f)}$.*

PROOF. We remark that $[L^{'}(a), \partial_f] = [\Phi_0^a, \partial_f]$. By definitions $[L^{'}(a), \partial_f](1) = 0$. Let $b \in \mathfrak{g}$ and $W \in S(\mathfrak{g})$. By notation 6.2.1 and by definition of Φ_0 we get

$$\begin{aligned} & [\Phi_0^a, \partial_f](b \cdot W) = \\ = \quad & [\Phi_0^a, \partial_{f \circ b^L}](W) + (-1)^{(p(f)+p(a))p(b)} b^L \circ [\Phi_0^a, \partial_f](W) - (-1)^{(p(f)p(a)} \partial_{f \circ [a,b]^L}(W). \end{aligned}$$

We suppose by induction that $[\Phi_0^a, \partial_g](\cdot W) = \partial_{L^{'}(a)(g)}(W)$ for any $g \in S(\mathfrak{g})^*$, so

$$\begin{aligned} & [\Phi_0^a, \partial_f](b \cdot W) = \\ = \quad & \left(\partial_{L^{'}(a)(f \circ b^L)} + (-1)^{(p(f)+p(a))p(b)} b^L \circ \partial_{L^{'}(a)(f)} - (-1)^{p(f)p(a)} \partial_{f \circ [a,b]^L}\right)(W) \\ = \quad & \left(-(-1)^{p(a)(p(f)+p(b))} \partial_{f \circ b^L \circ \Phi_0^a} + (-1)^{(p(f)+p(a))p(b)} b^L \circ \partial_{L^{'}(a)(f)} - (-1)^{p(f)p(a)} \partial_{f \circ [a,b]^L}\right)(W) \\ = \quad & \left(-(-1)^{p(f)p(a)} \partial_{f \circ \Phi_0^a \circ b^L} + (-1)^{(p(f)+p(a))p(b)} b^L \circ \partial_{L^{'}(a)(f)}\right)(W) \\ = \quad & -(-1)^{p(f)p(a)} \left(\partial_{f \circ \Phi_0^a \circ b^L} + (-1)^{(p(f)+p(a))p(b)} b^L \circ \partial_{f \circ \Phi_0^a}\right)(W) \\ \equiv \quad & -(-1)^{p(f)p(a)} \left(\partial_{f \circ \Phi_0^a}\right)(b \cdot W) \\ =: \quad & \partial_{L^{'}(a)(f)}(b \cdot W). \blacksquare \end{aligned}$$

Theorem A.1.1. *For any $a \in \mathfrak{g}$, $L^{'}(a)(J) = 0$.*

PROOF. We remark that $J = e^{str\ \ln \frac{\sinh(\frac{\mathrm{ad}x}{2})}{\frac{\mathrm{ad}x}{2}}}$, so $L^{'}(a)(J) = J \cdot str\left(L^{'}(a)\left(\ln \frac{\sinh(\frac{\mathrm{ad}x}{2})}{\frac{\mathrm{ad}x}{2}}\right)\right)$. Let $G(z) := \ln \frac{\sinh(\frac{z}{2})}{\frac{z}{2}}$. We have $str\left(L^{'}(a)\left(G(\mathrm{ad}x)\right)\right) = str\left(G'(\mathrm{ad}x) \cdot L^{'}(a)(\mathrm{ad}x)\right)$, and by lemma A.1.2 we get $L^{'}(a)(\mathrm{ad}x) = 0$. $\blacksquare$

Lemma A.1.4. *Let $\gamma : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{K}_0$ be an invariant, even, bilinear form. For any $a \in \mathfrak{g}$,*
i) $L^{'}(a)(\gamma(X, t(Y))) = \gamma(a, [X, t(Y)] + [t(X), Y])$,
ii) $\sum_{i,j} \frac{1}{2} L^{'}(a)(S_{ij}) \circ i^{'}(e_i) \circ i^{'}(e_j) = \sum_{i,j} S_{ij} \circ i^{'}(e_i) \circ i^{'}([e_j, a])$.

PROOF. i) From lemma A.1.2 part ii we get $L^{'}(a)(t) = 0$. In particular

$$\begin{aligned} 0 \quad = \quad & \sum_{i,j} L^{'}(a)(e_j \otimes t_{ji} \otimes x_i)) = \\ = \quad & \sum_i [a, t(e_i)] \otimes x_i + (-1)^{p(a)p(e_j)} e_j \otimes L^{'}(a)(e_j^*(t(e_i)) \otimes x_i - \sum_k t([a, e_k]) \otimes x_k. \end{aligned}$$

Using that γ is invariant we get

$$0 = -\gamma([a, e^j], t(Y)) - L^{'}(a)(\gamma(t(e^j), Y)) - \gamma([a, t(e^j)], Y), \quad \forall j.$$

In particular,

$$
\begin{aligned}
0 &= -\gamma([a,X],t(Y)) - L^{'}(a)(\gamma(t(X),Y)) - \gamma([a,t(X)],Y) \\
&= -\gamma(a,[X,t(Y)]) + L^{'}(a)(\gamma(X,t(Y))) - \gamma(a,[t(X),Y]).
\end{aligned}
$$

ii) This part follows from part i, using the definition of S_{ij} and the commutation rules for $i^{'}(\cdot)$. ∎

A.1.1 Formulas for the non-commutative Weil algebra

In this section we give formulas for the structure of $\mathfrak{g}$-differential algebra over $U(\mathfrak{g}) \otimes Clif(\Pi\mathfrak{g}, \beta^{\pi})$ introduced in (6.2). The most important formulas are given in theorem (A.1.3).

For any $a \in \mathfrak{g}$ we introduce $\gamma^a := \frac{1}{2}\sum_j \pi e^j \cdot \pi[e_j, a] \equiv \frac{1}{2}(-1)^{p(a)}\sum_j \pi[a, e^j] \cdot \pi e_j \in Clif(\pi\mathfrak{g}, \gamma^{\pi})$.

Lemma A.1.5. *Let $\gamma : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{K}_0$ be an invariant, even, not-degenerate bilinear form. For any $a, b \in \mathfrak{g}$ we have*
i) $[\pi b, \gamma^a] = \pi[b, a]$, $\quad [\gamma^a, \pi b] = (-1)^{p(a)}\pi[a, b]$
ii) $\left[\gamma^a, \gamma^b\right] = \gamma^{[a,b]}$.

Proof. *i)* For the first part we have

$$2[\pi b, \gamma^a] = (-1)^{p(b)}\sum_j \gamma(b, e^j)\pi[e_j, a] + (-1)^{p(b)+p(\pi b)p(\pi e_j)}\pi e^j \gamma(b, [e_j, a]) = 2\pi[b, a].$$

ii) From the first part we get

$$
\begin{aligned}
2[\gamma^a, \gamma^b] &= (-1)^{p(a)}\sum_j \pi[e^j, a] \cdot \pi[e_j, b] - (-1)^{p(a)p(b)}\pi e^j \cdot [[e_j, b], a] = \\
&= \sum_k \pi e^k \cdot \pi([[a, e_k], b] - (-1)^{p(a)p(b)}[[e_k, b], a])
\end{aligned}
$$

so the Jacobi identity gives $2[\gamma^a, \gamma^b] = 2\gamma^{[a,b]}$. ∎

From the previous lemma we get

Theorem A.1.2. *Let $\gamma : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{K}_0$ be an invariant, even, not-degenerate bilinear form. For any $a \in \mathfrak{g}$ we have*

$$
\begin{aligned}
&L^{\mathcal{W}}(a) = \mathrm{ad}j(a) + \mathrm{ad}\gamma^a, \qquad i^{\mathcal{W}}(a) = \mathrm{ad}\pi a \\
&d^{\mathcal{W}} = \mathrm{ad}\sum_i \pi e_i \cdot j(e^i) + \mathrm{ad}\sum_i \frac{1}{3}(-1)^{p(e_i)}\pi e^i \cdot \gamma^{e_i}.
\end{aligned}
$$

We identify $Clif(\pi\mathfrak{g}, \gamma^{\pi})$ and $S(\Pi\mathfrak{g})$ by σ_γ (see (6.4)), so we get a structure of $\mathfrak{g}$-differential algebra over $U(\mathfrak{g}) \otimes S(\Pi\mathfrak{g})$. In theorem A.3 we give formulas for this structure, they will be used in this section A.3.3. As a preliminary remark we have

$$
\begin{cases}
\sigma_\gamma \circ (\pi a)^L \circ \sigma_\gamma^{-1} = (\pi a)^L + \frac{1}{2}i'(a) \\
\sigma_\gamma \circ (\pi a)^R \circ \sigma_\gamma^{-1} = (\pi a)^L - \frac{1}{2}i'(a)
\end{cases}
\qquad \forall a \in \mathfrak{g}. \tag{A.3}
$$

Lemma A.1.6. *Let $\gamma : \mathfrak{g}\otimes\mathfrak{g}\to\mathbb{K}$ be an even, symmetric, not-degenerate bilinear form. For any $a\in\mathfrak{g}$ we have*

$$\sigma_\gamma\circ(\gamma^a)^R\circ\sigma_\gamma^{-1}=(-1)^{p(a)}\sum_i\frac{1}{2}\pi[a,e^i]^L\circ\pi e_i^L-\frac{1}{2}(\pi[a,e^i])^L\circ i'(e_i)+\frac{1}{8}i'([a,e^i])\circ i'(e_i)$$

$$\sigma_\gamma\circ(\gamma^a)^L\circ\sigma_\gamma^{-1}=(-1)^{p(a)}\sum_i\frac{1}{2}\pi[a,e^i]^L\circ\pi e_i^L+\frac{1}{2}(\pi[a,e^i])^L\circ i'(e_i)+\frac{1}{8}i'([a,e^i])\circ i'(e_i).$$

PROOF. Using the formulas (A.3) we get

$$\begin{aligned}
&2(-1)^{p(a)}\sigma_\gamma\circ(\gamma^a)^R\circ\sigma_\gamma^{-1}=\\
=\;&\sum_i(-1)^{p(\pi e_i)p(\pi a)}\left((\pi e_i)^L-\frac{1}{2}i'(e_i)\right)\circ\left((\pi[a,e^i])^L-\frac{1}{2}i'([a,e^i])\right)\\
=\;&\sum_i(-1)^{p(\pi e_i)p(\pi a)}(\pi e_i)^L\circ(\pi[a,e^i])^L-\frac{1}{2}str(\mathrm{ad}a)-\frac{1}{2}\pi[a,e^i]\circ i'(e_i)+\\
&-\sum_i\frac{1}{2}(-1)^{p(\pi e_i)p(\pi a)}\pi e_i\circ i'([a,e^i])+\frac{1}{4}(-1)^{p(\pi e_i)p(\pi a)}i'(e_i)\circ i'([a,e^i])\\
=\;&\sum_i\pi[a,e^i]^L\circ\pi e_i^L-(\pi[a,e^i])^L\circ i'(e_i)+\frac{1}{4}i'([a,e^i])\circ i'(e_i).
\end{aligned}$$

In the same way we get

$$\begin{aligned}
&2(-1)^{p(a)}\sigma_\gamma\circ(\gamma^a)^L\circ\sigma_\gamma^{-1}=\\
=\;&\sum_i\left((\pi[a,e^i])^L+\frac{1}{2}i'([a,e^i])\right)\circ\left((\pi e_i)^L+\frac{1}{2}i'(e_i)\right)\\
=\;&\sum_i(\pi[a,e^i]\cdot\pi e_i)^L+\frac{1}{2}str(\mathrm{ad}a)+\frac{1}{2}(-1)^{p(\pi e_i)p(\pi a)}(\pi e_i)^L\circ i'([a,e^i])+\frac{1}{2}(\pi[a,e^i])^L\circ i'(e_i)+\\
&+\sum_i\frac{1}{4}i'([a,e^i])\circ i'(e_i)\\
=\;&\sum_i(\pi[a,e^i]\cdot\pi e_i)^L+(\pi[a,e^i])^L\circ i'(e_i)+\frac{1}{4}i'([a,e_i])\circ i'(e_i)).\ \blacksquare
\end{aligned}$$

Theorem A.1.3. *The structure of $\mathfrak{g}$-differential algebra induced over $U(\mathfrak{g})\otimes S(\Pi\mathfrak{g})$ is*

$$\begin{aligned}
\sigma_\gamma\circ i^{\mathcal{W}}(a)\circ\sigma_\gamma^{-1}&=i'(a)\\
\sigma_\gamma\circ L^{\mathcal{W}}(a)\circ\sigma_\gamma^{-1}&=L'(a)\\
\sigma_\gamma\circ d^{\mathcal{W}}\circ\sigma_\gamma^{-1}&=\sum_i d^\Lambda(\pi e^i)^L\circ i'(e_i)+(-1)^{p(e_i)}(\pi e^i)^L\circ\mathrm{ad}j(e_i)+\frac{j(e^i)^L+j(e^i)^R}{2}\circ i'(e_i)+\\
&+\frac{1}{24}\sum_{i,j,k}i'(e_i)\circ i'(e_j)\circ i'(e_k)\gamma([e^k,e^j(-1)^{p(e_j)}],e^i)
\end{aligned}$$

for any $a\in\mathfrak{g}$.

PROOF. The formula for $\sigma_\gamma \circ i^{\mathcal{W}}(a) \circ \sigma_\gamma^{-1}$ follows from formulas (A.3), the formula for $\sigma_\gamma \circ L^{\mathcal{W}}(a) \circ \sigma_\gamma^{-1}$ follows from lemma A.1.6. From theorem A.1.2 we get $3d^{\mathcal{W}}|_{Clif(\pi\mathfrak{g},\gamma^\pi)} =: 3d^{\mathcal{W}}|_{Clif} = \sum_j (-1)^{p(e_j)} \left((\pi e^j)^L \circ (\gamma^{e_j})^L - (\gamma^{e_j})^R \circ (\pi e^j)^R \right)$. Using formulas (A.3), lemma A.1.6 and definitions, we get

$$\begin{aligned} & 3\sigma_\gamma \circ d^{\mathcal{W}}|_{Clif} \circ \sigma_\gamma^{-1} = \\ = & \sum_{i,j} \frac{3}{2} (\pi[e_j, e^i] \cdot \pi e_i)^L \circ i'(e^j) + \frac{1}{8} i'([e_j, e^i]) \circ i'(e_i) \circ i'(e^j) + \sum_i \frac{1}{2} str(\mathrm{ad} e^i) i'(e_i). \end{aligned}$$

By (A.3) we have

$$\begin{aligned} & \sigma_\gamma \circ \mathrm{ad} \left(\sum_i \pi e_i \cdot j(e^i) \right) \circ \sigma_\gamma^{-1} = \\ = & \sum_i \sigma_\gamma \circ \left((\pi e_i \cdot j(e^i))^L - (j(e^i) \cdot \pi e_i)^R \right) \circ \sigma_\gamma^{-1} \\ = & \sum_i \sigma_\gamma \circ (\pi e_i)^L \circ \sigma_\gamma^{-1} \circ j(e^i)^L - \sigma_\gamma \circ (\pi e_i)^R \circ \sigma_\gamma^{-1} \circ j(e^i)^R \\ = & \sum_i (\pi e_i)^L \circ \mathrm{ad} j(e^i) + \frac{1}{2} \left(j(e^i)^L + j(e^i)^R \right) \circ i'(e_i). \ \blacksquare \end{aligned}$$

A.2 Proof of theorem 6.2.1

Let $a \in \mathfrak{g}$. The first identity follows from from $\sigma_\gamma^{-1} \circ i'(a) \circ \sigma_\gamma = i^{\mathcal{W}}(a)$ (theorem A.1.3) and from that fact that $[i'(a), i'(b)] = 0$ for any $a, b \in \mathfrak{g}$.

To show the second identity we use the following lemma. Let $W = \sum_{i,l} S_{il} \circ i'(e_i) \circ i'(e_l)$.

Lemma A.2.1. *For any $a \in \mathfrak{g}$ we have $W \circ L'(a) = L'(a) \circ W$.*

PROOF. By commutation rules in a $\mathfrak{g}$-differential algebra we have

$$\begin{aligned} & W \circ L'(a) = \sum_{i,j} S_{ij} \circ i'(e_i) \circ i'(e_j) \circ L'(a) = \\ = & \sum_{i,j} (S_{ij} + (-1)^{p(\pi e_i) p(\pi e_i)} S_{ji}) \circ i'(e_i) \circ i'([e_j, a]) - [L'(a), S_{ij}] \circ i'(e_i) \circ i'(e_j) + \\ & + L'(a) \circ S_{ij} \circ i'(e_i) \circ i'(e_j). \end{aligned}$$

By lemmas 6.1.2 and A.1.3 we get

$$W \circ L'(a) = \sum_{i,j} \left(2S_{ij} \circ i'(e_i) \circ i'([e_j, a]) - L'(a)(S_{ij}) \circ i'(e_i) \circ i'(e_j) \right) + L'(a) \circ W,$$

so it is sufficient to show that $0 = \sum_{i,j} 2S_{ij} \circ i'(e_i) \circ i'([e_j, a]) - L'(a)(S_{ij}) \circ i'(e_i) \circ i'(e_j)$. It follows from lemma A.1.4, part *ii*. $\blacksquare$

From the previous lemma we get $Q \circ L'(a) = \sigma_\gamma^{-1} \circ L'(a) \circ \sigma_\gamma \circ Q$. The theorem A.1.3 gives $\sigma_\gamma^{-1} \circ L'(a) \circ \sigma_\gamma = L^{\mathcal{W}}(a)$, so the proof is finished.

A.3 Proof of theorem 6.2.2

Let $W=\sum_{i,k}\frac{1}{2}S_{ik}\circ i'(e_i))\circ i'(e_k)$, in this section we show theorem 6.2.2. It is equivalent to show the identity $\beta\circ J^{\frac{1}{2}}\circ e^{-W}\circ d'\circ e^{W}\circ J^{-\frac{1}{2}}=\sigma_\gamma^{-1}\circ d^{\mathcal{W}}\circ\sigma_\gamma\circ\beta$. If Ad denote the conjugation, this identity becomes

$$Ad(J^{\frac{1}{2}})\circ Ad(e^{-W})(d')=\beta^{-1}\circ\sigma_\gamma^{-1}\circ d^{\mathcal{W}}\circ\sigma_\gamma\circ\beta. \tag{A.4}$$

Lemma A.3.1.

$$Ad\Big(e^{-W}\Big)(d')=d'-\sum_{i,j}\frac{1}{2}\gamma(e^j,t(e^i))\circ i'([e_i,e_j])+\gamma(e^i,t(e^j))\circ i'(e_j)\circ\Phi_0^{e_i}+$$
$$-\frac{1}{2}\sum_{i,j,l}\Bigg(\partial(e^l)\left((-1)^{p(e^j)}\gamma\left(e^j,t(e^i)\right)\right)+\frac{1}{2}(-1)^{p(e_j)}\gamma\left(t(e^l),[e^j,t(e^i)]\right)\Bigg)\circ i'(e_i)\circ i'(e_j)\circ i'(e_l).$$

The proof of this identity is a long calculation so we give it in a different paragraph.

A.3.1 Proof of lemma A.3.1

We have $Ad\big(e^{-W}\big)(d')=e^{ad(-W)}(d')=d'-[W,d']+\frac{1}{2}[W,[W,d']]+\cdots$. The proof of lemma A.3.1 is the computation of the terms of this series.

Lemma A.3.2.

$$[W,d']=\sum_{i,j}\frac{1}{2}\gamma(e^j,t(e^i))\circ i'([e_i,e_j])+\gamma(e^i,t(e^j))\circ i'(e_j)\circ\Phi_0^{e_i}+$$
$$+\frac{1}{2}\sum_{i,j,l}\partial(e^l)\left((-1)^{p(e_j)}\gamma(e^j,t(e^i))\right)\circ i'(e_i)\circ i'(e_j)\circ i'(e_l).$$

Proof. By commutation rules in a $\mathfrak{g}$-differential algebra (see (6.1)) we get

$$2[W,d']=\sum_{i,j}S_{ij}\circ[i'(e_i)\circ i'(e_j),d']-[d',S_{ij}]\circ i'(i)\circ i'(j)=$$
$$=\sum_{i,j}(-1)^{p(e_j)}S_{ij}\circ i'([e_i,e_j])+(-1)^{p(e_j)p(e_i)}(S_{ij}(-1)^{p(\pi e_i)p(\pi e_j)}+S_{ji})\circ L'(e_i)\circ i'(e_j)+$$
$$-\sum_{i,j}[d',S_{ij}]\circ i'(e_i)\circ i'(e_j).$$

From (A.2) we have a formula for d'. We compute

$$[d',S_{ij}]=\sum_l-[S_{ij},(e^l)^L]\circ i'(e_l)(-1)^{p(e_i)+p(e_j)}+(\pi e^l)^L\circ[L'(e_l),S_{ij}](-1)^{p(e_l)}=$$
$$=\sum_l[(e^l)^L,S_{ij}]\circ i'(e_l)(-1)^{p(\pi e_l)p(e_i+e_j)}+(\pi e^l)^L\circ L'(e_l)(S_{ij})(-1)^{p(e_l)}.$$

For any $a \in \mathfrak{g}$ we have $L^{'}(a) = \Phi_0^a + L^{'}(a)|_{S(\Pi\mathfrak{g})}$ (see formula A.1). From lemmas A.1.4 and 6.1.2 we get

$$\sum_{i,j,l}(-1)^{p(e_l)}(\pi e^l)^L \circ L^{'}(e_l)(S_{ij}) \circ i^{'}(e_i) \circ i^{'}(e_j) = 2\sum_{i,j,l}(-1)^{p(e_j)p(e_i)}S_{ji} \circ L^{'}(e_i)|_{S(\Pi\mathfrak{g})} \circ i^{'}(e_j).$$

The lemma 6.1.2 and the remark A.1.1 give

$$2[W, d^{'}] = \sum_{i,j}(-1)^{p(e_j)}S_{ij} \circ i^{'}([e_i, e_j]) + 2(-1)^{p(e_j)p(e_i)}S_{ji} \circ \Phi_0^{e_i} \circ i^{'}(e_j) +$$
$$+ \sum_{i,j,l} -\partial(e^l)(S_{ij}) \circ i^{'}(e_i) \circ i^{'}(e_j) \circ i^{'}(e_l). \blacksquare$$

Lemma A.3.3. $\frac{1}{2}[W, [W, d^{'}]] = \frac{1}{4}\sum_{i,j,l} -(-1)^{p(e_j)}\gamma\left(t(e^l), [e^j, t(e^i)]\right) \circ i^{'}(e_i) \circ i^{'}(e_j) \circ i^{'}(e_l).$

PROOF. By the commutation rules in a $\mathfrak{g}$-differential algebra and by the fact that $S(\mathfrak{g})^*$ is a commutative superalgebra we get

$$4[W, [W, d^{'}]] = [W, 2\sum_{l,m}(-1)^{p(e_l)p(e_m)}S_{lm} \circ L^{'}(e_m) \circ i^{'}(e_l)].$$

We compute

$$[W, \sum_{l,m}(-1)^{p(e_l)p(e_m)}S_{lm} \circ L^{'}(e_m) \circ i^{'}(e_l)] =$$
$$= \sum_{i,j,l,m}(-1)^{p(e_l)p(e_m)}S_{lm} \circ \left(L^{'}(e_m)(S_{ij}) \circ i^{'}(e_l) \circ i^{'}(e_i) \circ i^{'}(e_j) + S_{ij} \circ [i^{'}(e_i) \circ i^{'}(e_j), L^{'}(e_m)] \circ i^{'}(e_l)\right)$$
$$= \sum_{i,j,l,m}(-1)^{p(e_l)p(e_m)}S_{lm} \circ \left(-L^{'}(e_m)(S_{ij}) \circ i^{'}(e_i) \circ i^{'}(e_j) \circ i^{'}(e_l) + S_{ji} \circ i^{'}(e_j) \circ i^{'}([e_i, e_j]) \circ i^{'}(e_l)\right)$$
$$= \sum_{i,j,l,m}(-1)^{p(e_l)p(e_m)}S_{lm} \circ \left(-L^{'}(e_m)(S_{ij}) \circ i^{'}(e_i) \circ i^{'}(e_j) \circ i^{'}(e_l)\right)$$
$$= \sum_{i,j,l,m}(-1)^{p(e_l)+p(e_m)}S_{ml} \circ \left(L^{'}(e_m)(S_{ij}) \circ i^{'}(e_i) \circ i^{'}(e_j) \circ i^{'}(e_l)\right).$$

The lemma A.1.4 and the remark 6.1.1 complete the proof. $\blacksquare$

Lemma A.3.4. $[W, [W, [W, d^{'}]]] = 0.$

PROOF. The proof follows from the commutation rules for $i^{'}(\cdot)$ and from the fact that $[\partial_f, \partial_g] = 0$ for any $f, g \in S(\mathfrak{g})^*$. $\blacksquare$

A.3.2 Some others properties

Theorem A.3.1. *Let $X \in \mathfrak{g}$. We have* $2J^{\frac{-1}{2}} \cdot \partial(X)(J^{\frac{1}{2}}) = \sum_i(-1)^{p(e_i)}\gamma(e^i, t([e_i, X]))$.

PROOF. As seen in lemma 6.1.1, $J^{\frac{1}{2}} = Ber(F(\mathrm{ad}x))$ where the formal series $F(z)$ is invertible. We have $\partial(X)(J^{\frac{1}{2}}) = J^{\frac{1}{2}} \cdot str\left(\frac{F'(\mathrm{ad}x)}{F(\mathrm{ad}x)} \cdot \partial(X)(\mathrm{ad}x)\right)$ and $\partial(X)(\mathrm{ad}x) = \mathrm{ad}X$, so

$$str\left(\frac{F'(\mathrm{ad}x)}{F(\mathrm{ad}x)} \cdot \partial(X)(\mathrm{ad}x)\right) = -\sum_i (-1)^{p(e_i)}\gamma\left(e^i, \frac{F'(\mathrm{ad}x)}{F(\mathrm{ad}x)}([e_i, X])\right).$$

As $F(z) = F(-z)$ we have $2\frac{F'(z)}{F(z)} = \frac{F'(z)}{F(z)} - \frac{F'(-z)}{F(-z)}$. By $F(z)F(-z) = \frac{\sinh(\frac{z}{2})}{\frac{z}{2}}$ (lemma 6.1.1) we get $\frac{F'(z)}{F(z)} - \frac{F'(-z)}{F(-z)} = -\rho(z)$. In particular $2\partial(X)(J^{\frac{1}{2}}) \cdot J^{-\frac{1}{2}} = \sum_i (-1)^{p(e_i)}\gamma\left(e^i, t([e_i, X])\right)$. ∎

Corollary A.3.1.

$$Ad(J^{\frac{1}{2}} \circ e^{-W})(d') = Ad(e^{-W})(d') + \frac{1}{2}\sum_{i,j} \gamma(e^i, t(e^j)) \circ i'([e_j, e_i]) \qquad \text{(A.5)}$$

PROOF. The operators in the formula of $Ad(e^{-W})(d')$ (lemma A.3.1) not commuting with $J^{\frac{1}{2}}$ are $L'(X)$, X^L with $X \in \mathfrak{g}$.

Let $f \in S(\mathfrak{g})_0^*$ be invertible. From notation 6.2.1 and lemma A.1.3 we get

$$\begin{aligned} f^{-1} \circ X^L \circ f &= X^L + f \circ \partial(X)(f^{-1}) \\ L'(X) \circ f &= L'(X)(f) + f \circ L'(X). \end{aligned}$$

If $L'(X)(f) = 0$ for any $X \in \mathfrak{g}$, these formulas give

$$Ad(f^{-1} \circ e^{-W})(d') \;=\; Ad(e^{-W})(d') + \sum_i f \circ \partial_{e^i}(f^{-1}) \circ i'(e_i).$$

To end the proof we only need to consider the case $f = J^{-\frac{1}{2}}$, in fact

$$L'(X)(J^{-\frac{1}{2}}) = -\frac{1}{2}J^{-1} \cdot J^{-\frac{1}{2}} \cdot L'(X)(J)$$

which is zero by theorem A.1.1. The theorem A.3.1 and the remark 6.1.1 give

$$\sum_j f \circ \partial_{e^j}(f^{-1}) \circ i'(e_j) = \frac{1}{2}\sum_{j,i} \gamma(e^i, t(e^j)) \circ i'([e_j, e_i]). \;∎$$

Let $e(X, Y, Z)$ be an expression depending of three elements of $\mathfrak{g}$. We introduce the notation

$$Cycl(e(X,Y,Z)) := e(X,Y,Z) + (-1)^{p(\pi Z)p(X+Y)}e(Z,X,Y) + (-1)^{p(\pi X)p(Y+Z)}e(Y,Z,X).$$

Let

$$L_{i,l,k} := -\partial(e^k)(\gamma(t(e^l(-1)^{p(e_l)}), e^i)) + \gamma([t(e^k), e^l(-1)^{p(e_l)}], t(e^i)).$$

We have shown in corollary A.3.1 and lemma A.3.1 that

$$Ad(J^{\frac{1}{2}} \circ e^{-W})(d') = d' + \sum_{i,k} \gamma(e^i, t(e^k)) \circ i'(e_k) \circ \Phi_0^{e_i} - \frac{1}{6}\sum_{i,l,k} Cycl(L_{i,l,k}) \circ i'(e_i) \circ i'(e_l) \circ i'(e_k).$$

A.3.3 The role of vCDYBE

We recall that $Hom(S(\mathfrak{g}), S(\mathfrak{g}))$ has the following structure of $S(\mathfrak{g})^*$-module: $Ff = \text{Mult} \circ (F \otimes f) \circ \Delta$ for any $F \in Hom(S(\mathfrak{g}), S(\mathfrak{g}))$ and $f \in S(\mathfrak{g})^*$.

Theorem A.3.2. *Let $f \in S(\mathfrak{g})^*$.*
i) For any $\varphi \in Hom(S(\mathfrak{g}), \mathfrak{g})$ we have $id * (\varphi f) = (id * \varphi) f$,
ii) for any derivation $F : S(\mathfrak{g}) \to S(\mathfrak{g})$ we have $\partial_f \circ F = fF$.

Proof. $i)$ This part follows from definitions.
$ii)$ We remark that $\partial_f \circ F(1) = f(1)F(1) = (fF)(1)$. Let $X \in \mathfrak{g}$ and $W \in S(\mathfrak{g})$. We suppose by induction that $\partial_f \circ G(W) = (fG)(W)$ for any derivation $G : S(\mathfrak{g}) \to S(\mathfrak{g})$. Using this hypothesis, the definitions and the notation 6.2.1 we have

$$\begin{aligned} & \partial_f \circ F(X \cdot W) = \partial_f (F(X) \cdot W + (-1)^{p(X)p(F)} X \cdot F(W)) = \\ = \; & (fF(X)^L)(W) + (-1)^{p(X)p(F)} \left(\partial_{f \circ X^L} + (-1)^{p(f)p(X)} X^L \circ \partial_f \right) \circ F(W) \\ = \; & (fF(X)^L)(W) + (-1)^{p(X)p(F)} \left((f \circ X^L) F + (-1)^{p(f)p(X)} X^L \circ (fF) \right) (W) \\ = \; & (fF(X)^L)(W) + (-1)^{p(X)p(F)} \left((f \circ X^L) F + f(X^L \circ F) \right) (W) \\ = \; & \left(f(F \circ X^L) + (-1)^{p(X)p(F)} (f \circ X^L) F \right) (W) \\ = \; & (fF)(X \cdot W). \; \blacksquare \end{aligned}$$

We recall that $j : \mathfrak{g} \to U(\mathfrak{g})$ is the inclusion of $\mathfrak{g}$ in the enveloping algebra $U(\mathfrak{g})$.

Corollary A.3.2. *For all $a \in \mathfrak{g}$, $\beta^{-1} \circ \frac{1}{2}(j(a)^L + j(a)^R) \circ \beta = a^L + \sum_i (-1)^{p(e_i)} \gamma(t(a), e^i) \circ \Phi_0^{e_i}$.*

Proof. By theorem 2.5.2 we know that $\beta^{-1} \circ (j(a)^L + j(a)^R) \circ \beta = \Phi_1^a - \Phi_{-1}^a$. The coderivation $\frac{1}{2}(\Phi_1^a - \Phi_{-1}^a) - a^L$ is associated to the formal series $\frac{1}{2}\left(\frac{z}{e^z - 1} - \frac{z}{e^{-z} - 1}\right) - 1$ which is equal to $-z\rho(z)$. By definitions we get

$$\begin{aligned} & \text{ad}x \circ \rho(\text{ad}x)(a) \equiv \text{ad}x \circ t(a) = \text{ad}x(e_i)\gamma(e^i, t(a)) = -\sum_i \varphi_0^{e_i} \gamma(e^i, t(a)) = \\ = \; & -\sum_i (-1)^{p(e_i)} \gamma(t(a), e^i) \varphi_0^{e_i}. \end{aligned}$$

The part i of theorem A.3.2 gives that the coderivation associated to $-\text{ad}x \circ \rho(\text{ad}x)(a)$ is $\sum_i (-1)^{p(e_i)} \gamma(t(a), e^i) \Phi_0^{e_i}$. From the part ii of theorem A.3.2 we get that $\gamma(t(a), e^i) \Phi_0^{e_i} = \gamma(t(a), e^i) \circ \Phi_0^{e_i}$. $\blacksquare$

This corollary, theorems A.1.3 and 2.5.2, the formula (A.2) for d', give

$$\begin{aligned} & Ad(\partial_{J^{\frac{1}{2}}} \circ e^{-W})(d') = \\ = \; & -\frac{1}{6} \sum_{i,k,l} \left(Cycl(L_{ikl}) + \frac{1}{4} \gamma(e^l, [(-1)^{p(e_k)} e^k, e^i]) \right) \circ i'(e_i) \circ i'(e_k) \circ i'(e_l) + \\ & + \beta^{-1} \circ \sigma_\gamma^{-1} \circ d^{\mathcal{W}} \circ \sigma_\gamma \circ \beta. \end{aligned}$$

In particular the identity (A.4) is equivalent to

$$\sum_{i,k,l} \left(Cycl(L_{ikl}) + \frac{1}{4}\gamma(e^l, [(-1)^{p(e_k)} e^k, e^i]) \right) \circ i^{'}(e_i) \circ i^{'}(e_k) \circ i^{'}(e_l) = 0.$$

Using the notations of chapter 5 we have

$$Cycl\big(L_{ikl}\big) = i(e^l) i((-1)^{p(e_k)} e^k) i(e^i) \left(-d\omega + (\rho(t)\rho(v) + \rho(u)\rho(v) + \rho(t)\rho(u) : \alpha)_x\right).$$

By theorem 5.2.2 we get $Cycl\big(L_{ikl}\big) + \frac{1}{4}\gamma(e^l, [(-1)^{p(e_k)} e^k, e^i]) = 0$ for any i, k, l, so the proof of identity (A.4) is finished.

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Errata Corrige

Title of paragraph 2.5.1: "Universal representations in the enveloping algebra"

PAGE 10, *third line of example 1.1.2 part a*: "Moreover"

PAGE 12, *first line after remark 1.2.2*: "$S(M)^*$-superalgebra"

PAGE 21, *third line of theorem 2.2.6*: replace "if and only if" by "if"

PAGE 23, *remark 2.3.5*: the right formula is "$(\varphi(t) + at, \psi(t) + bt, \rho(t) + abt + a\psi(t) + b\varphi(t))$"
after remark 2.3.5: "By this remark and by remark 2.3.2"

PAGE 24, *line 2*: "$\theta_c(t) = \sqrt{c}t \coth(\sqrt{c}t)$"
lemma 2.3.1: "there exists" ... "$(\sqrt{c}t \coth(\sqrt{c}t), ct)$."
proof of lemma 2.3.1: "as its derivative"
theorem 2.3.3: set $f = 1$

PAGE 26, *first line of remark 2.5.1*: "$gr(U(\mathfrak{g}))$ might not be commutative"
first line of definition 2.5.2: "$\mathfrak{g}$ verifies"

PAGE 28, *second line after corollary 2.5.1*: "if $\mathbb{K}$ does not contain $\mathbb{Q}$"

PAGE 30, *lemma 2.5.3*: "The symmetrization map verifies"
theorem 2.5.3: "and $\mathfrak{g} = \mathfrak{g}_0$ a Lie $\mathbb{K}$-algebra."

PAGE 33, *ninth line after definition 3.3.2*: the right signe of $i(a_1) \cdots i(a_n)\alpha$ is $(-1)^{p(\alpha)p(\pi a_1 + \cdots + \pi a_n)}$

PAGE 35, *corollary 3.4.1*: "Let $\mathbb{K} \supseteq \mathbb{Q}$. Any $\mathbb{K}$-superalgebra has the Maurer-Cartan forms"
second line of remark 3.4.2: "Moreover"

PAGE 37, *theorem 4.0.2*: set $N = 1$
end of corollary 4.0.2: replace "$\mathrm{ad}a$" by "$\mathrm{ad}j(a)$", "$\mathrm{ad}b$" by "$\mathrm{ad}j(b)$", "a^L" by "$j(a)^L$", "b^R" by "$j(b)^R$"

PAGE 39, *proof of theorem 4.1.3*: replace "a^L" by "$j(a)^L$", "$\theta(a)^R$" by "$j(\theta(a))^R$"

end of the proof of theorem 4.1.3: the right text is

$$"\Psi_a = \Phi_1^a + \Phi_{-1}^{\theta(a)} = \begin{cases} \Phi_0^a, \ a \in \mathfrak{h} \\ \Phi_1^a + \Phi_{-1}^{-a}, \ a \in \mathfrak{q} \end{cases} .$$

This means that $\beta^{-1} \circ \mathrm{ad}'(a) \circ \beta|_{S(\mathfrak{q})} = \Pi_2^a$."
corollary 4.1.1: replace "invariant" by stable

PAGE 41, *fourth line after remark 4.3.1* : "formal vector field"
line 18: $f \in \mathfrak{q}^*$

PAGE 42, *third line after theorem 4.3.1*: "$\in Hom(\mathfrak{g}, \mathfrak{g}_x)$"

PAGE 43, *last line*: replace "lg" by "ln"

PAGE 44: the right formula of $v_0^{(4)}$ is

$$\begin{aligned} v_0^{(4)} &= v_1 \cdot v_2 \cdot v_3 \cdot v_4 + \frac{1}{6} \sum_{\substack{s \in \Sigma_4 \\ s(1)<s(2) \\ s(3)<s(4)}} (-1)^{|s|} str_{\mathfrak{q}}(\mathrm{ad}v_{s(1)} \circ \mathrm{ad}v_{s(2)}) v_{s(3)} \cdot v_{s(4)} + \\ &+ \frac{1}{288} \sum_{\substack{s \in \Sigma_4 \\ s(1)<s(2) \\ s(3)<s(4)}} (-1)^{|s|} str_{\mathfrak{q}}(\mathrm{ad}v_{s(1)} \circ \mathrm{ad}v_{s(2)}) str_{\mathfrak{q}}(\mathrm{ad}v_{s(3)} \circ \mathrm{ad}v_{s(4)}) + \\ &- \frac{1}{2880} \sum_{s \in \Sigma_4} (-1)^{|s|} str_{\mathfrak{q}}(\mathrm{ad}v_{s(1)} \circ \mathrm{ad}v_{s(2)} \circ \mathrm{ad}v_{s(3)} \circ \mathrm{ad}v_{s(4)}) \end{aligned}$$

PAGE 47, *third and fourth lines* of section 5.1: replace "$\mathfrak{g}^*$" by "$Hom(M, N)$", "$\mathfrak{g}$" by "M"

PAGE 48, *second line of remark 5.2.1*: replace "$f(-\mathrm{ad}x)(Y)$" by "$f(\mathrm{ad}x)(Y)$"
line 8: "$i(X)i(Y)i(Z)(\alpha) = (-1)^{p(Y)}\gamma(X, Y, Z)$"

PAGE 49, *after theorem 5.2.1*: "if $t + u + v = 0$, equation (5.2) becomes"
lemma 5.2.2: this lemma assume that $\mathbb{K} \supseteq \mathbb{Q}$
proof of lemma 5.2.2: "We apply the limit $u \to 0$"

PAGE 50, *first line of remark 5.2.3*: "$\mathfrak{g}$ be a finite-dimensional Lie $\mathbb{K}$-algebra"
eighteenth line of remark 5.2.3: "Using the properties of γ"
nineteenth line of remark 5.2.3: replace "r_β" by "r_γ"
eighteenth line of remark 5.2.4: replace "$\coth(\frac{z}{2})$" by "$\coth(\frac{t}{2})$"