Scuola Dottorale in Scienze Astronomiche,
Chimiche, Fisiche, Matematiche e della Terra
"Vito Volterra"

Dottorato di Ricerca in Matematica

Dipartimento di Matematica "Guido Castelnuovo"

MEAN CURVATURE FLOW OF PINCHED SUBMANIFOLDS IN POSITIVELY CURVED SYMMETRIC SPACES

Candidate: *GIUSEPPE PIPOLI, XXVI ciclo.*
Thesis advisor: *Prof. CARLO SINESTRARI.*

June 2014

Giuseppe Pipoli
Mean curvature flow of pinched submanifolds
in positively curved symmetric spaces

Ph.D. thesis Sapienza – Univeristà di Roma

email: pipoli@mat.uniroma1.it
pipolig@libero.it

Contents

Chapter 1

Introduction

The main topic of this thesis is the *mean curvature flow*: it is a well-known, and probably the most important, geometric evolution equation of submanifolds in Riemannian manifolds. An important aim of this problem is the original purpose of Hamilton for the Ricci flow: study PDEs on manifolds in order to obtain geometric properties. One of the most relevant applications of mean curvature flow is, in fact, the classification of the submanifolds.

Roughly speaking in any point p the speed of the evolution of a submanifold is given by H_p the mean curvature vector in p. The mean curvature is characterized as the unique direction along which the volume of the submanifold would be decreased most effectively.

Formally, given $F_0 : \mathcal{M} \to (\overline{\mathcal{M}}, \bar{g})$ a smooth immersion of a differentiable manifold into a Riemannian manifold, the evolution by mean curvature of F_0 is the one-parameter family of smooth immersions $F : \mathcal{M} \times [0, T_{max}[\to (\overline{\mathcal{M}}, \bar{g})$ satisfying

$$\begin{cases} \dfrac{\partial}{\partial t} F(p,t) = H(p,t), \qquad p \in \mathcal{M}, \quad 0 \leq t < T_{max}, \\ F(\cdot, 0) = F_0, \end{cases} \tag{1.0.1}$$

where $H(p,t)$ is the mean curvature vector of the immersion $F(\cdot, t)$ at point p. Usually the Riemannian manifold $\overline{\mathcal{M}}$ is called the *ambient manifold* and the parameter t is thought as *time*. Very often we identify the immersion with the immersed submanifold, so we talk indifferently about the evolution of an immersion F_0 or the evolution of a submanifold $\mathcal{M}_0 = F_0(\mathcal{M})$. Consequently a solution of (1.0.1) can be expressed as a one-parameter family of submanifolds $\mathcal{M}_t = F(\mathcal{M}, t)$, for $t \in [0, T_{max}[$.

It can be checked that $H(p,t) = \Delta_{\mathcal{M}_t} F(p,t)$, where $\Delta_{\mathcal{M}_t}$ is the Laplace-Beltrami operator on $\mathcal{M}_t$. Thus, the mean curvature flow may be regarded as a kind of heat equation for the immersion. However the mean curvature flow is not really equivalent to a heat equation, since the operator $\Delta_{\mathcal{M}_t}$ is not the Laplacian with respect to a fixed metric, but it depends on $\mathcal{M}_t$ which is the unknown of the problem. Precisely (1.0.1) is a weakly parabolic quasilinear system of second order. The short-time existence of the solution of (1.0.1) was proved and if $\mathcal{M}_0$ is compact, the solution of the mean curvature flow exists and is unique up to a maximal time T_{max}.

In general it is very hard to find an exact solution of (1.0.1), in fact there are very few explicit examples. Minimal submanifolds, i.e. submanifolds with zero mean curvature everywhere, are trivial constant solutions. A simple non-trivial case is a sphere in the Euclidean space: it stays a sphere at any time and shrinks to a point in finite time. Other explicit examples are traslating solutions like the so colled "grim reaper". Then a very interesting problem is to understand the behavior of the solution, especially when the time t is approaching the maximal time. If T_{max} is finite the flow develops a *singularity*, that is $max_{x \in \mathcal{M}} |A|^2 (x,t) \to \infty$ as t goes to T_{max}, where A is the second fundamental form of the immersion and the norm is respect to the metric induced on the submanifold by the metric of the ambient manifold. If $T_{max} = \infty$ the flow can converge to a stationary limit, that is a minimal submanifold. The simplest singularity is called *round point*: the evolving submanifold shrinks to a single point with asymptotically round shape. Methods to go beyond the singularity were developed, like weak solutions or a mean curvature flow with surgeries introduced by Huisken and Sinestrari in [HS2], but we do not treat them in this thesis.

There are many ways to approach mean curvature flow. We use the perspective of partial differential equations and Riemannian geometry introduced by Huisken's seminal paper [H1] about the flow of convex hypersurfaces in the Euclidean space. This paper proves that the behavior of the sphere is typical of any convex hypersurface: they converge to a round point in finite time. Next Huisken again showed in [H2] that his approach works for a general ambient manifold too, and how the curvature of the ambient manifold can influence the flow. In particular he showed how the negative sectional curvature and the non-symmetry of the ambient manifold are obstructions for the convergence to a round point.

After the early works mentioned above, many other were produced. Many different ambient manifolds and different classes of submanifolds were considered, for example graphs of functions in the Euclidean space [EH] and in the hyperbolic space [Un]. One of the most studied classes of codimension greater than 1 is Lagrangian submanifolds. If the ambient manifold is Kähler-Einstein, the Lagrangian condition is preserved and we talk about *Lagrangian mean curvature flow*. It has been studied by several authors during the years, see e.g. [Ne, S2, S3, MW, Wa] and the references therein. The study of a general submanifold is more difficult because of the higher complexity of the normal bundle. First of all, for a hypersurface we can identify the second fundamental form with a symmetric tensor of type $(0,2)$ and the mean curvature with a scalar funciton. This is no longer true for a higher codimensional submanifold. Technically this translates to much more complicated evolution equations. Moreover the following comparison principle holds only for hypersurfaces: any two disjoint embedded hypersurfaces will avoid each other evolving by mean curvature flow. For these reasons, the study of singularities in higher codimension has started developing only in recent years.

The main source of inspiration for this thesis in another classical work of Huisken, [H3]. It treats the problem of evolution of hypersurfaces in the sphere. The main theorem proved is the following

Theorem 1.0.1 *(Huisken) Let $n \geq 2$ and $\mathbb{S}^{n+1}(K)$ be a spherical spaceform of sectional curvature $K > 0$. Let $\mathcal{M}_0$ be a compact connected hypersurface without boundary which* *is smoothly immersed in $\mathbb{S}^{n+1}(K)$, and suppose that we have on $\mathcal{M}_0$*

$$|A|^2 < \frac{1}{n-1}|H|^2 + 2K, \quad \text{if } n \geq 3, \qquad |A|^2 < \frac{3}{4}|H|^2 + \frac{4}{3}K, \quad \text{if } n = 2. \tag{1.0.2}$$

Then one of the following holds:

1) *equation (1.0.1) has a smooth solution $\mathcal{M}_t$ up to a finite maximal time T_{max} and the $\mathcal{M}_t$'s converge to a round point as $t \to T_{max}$.*

2) *equation (1.0.1) has a smooth solution $\mathcal{M}_t$ defined for all time, i.e. $T_{max} = \infty$, which converge in the $C^\infty-$topology to a smooth totally geodesic hypersurface $\mathcal{M}_\infty$.*

Inequalities of the form $|A|^2 < a|H|^2 + b$ for some constant a and b, are often called *pinching conditions* because, as we will see later, they can give informations about how the principal curvatures diverge each others. Then submanifolds that satisfies pinching conditions are called *pinched submanifolds*. Of course a pinching condition is interesting if it is preserved by the flow. We said that one of the main applications of the mean curvature flow is the classification of submanifolds. For example, from the result just discussed, we have that hypersurfaces of the sphere that satisfy (1.0.2) are diffeomorphic to a sphere.

The study of mean curvature flow of pinched submanifold is an active research field. Only in 2010 Baker, with his doctoral thesis [Ba], generalized theorem 1.0.1 for submanifolds of any codimension in the sphere: he proved in this case too the alternative between the round point and the totally geodesic limit. In [AB] is studied a class of pinched submanifolds of the Euclidean space which develops a singularity of type round point in finite time. After that some other works were produced: in [LXYZ] the extension of theorem 1.0.1 to higher codimension submanifold of the hyperbolic space is proved, but, because of the negative curvature of the ambient manifold, only the convergence to a round point is possible. The case of a general ambient manifold is the subject of [LXZ].

Theorem 1.0.2 *(K. Liu, H. Xu, E. Zhao) Let $\overline{\mathcal{M}}$ a Riemannian manifold of dimension $m+k$ satisfying the following bounds on sectional curvature $\bar{K}$, Riemannian curvature tensor $\bar{R}$ and injectivity radius $inj(\overline{\mathcal{M}})$:*

$$-K_1 \leq \bar{K} \leq K_2, \qquad \left|\bar{\nabla}\bar{R}\right| \leq L, \qquad ing(\overline{\mathcal{M}}) \geq i_{\overline{\mathcal{M}}},$$

for some nonnegative constant K_1, K_2, L and some positive constant $i_{\overline{\mathcal{M}}}$. Let $\mathcal{M}_0$ be a closed submanifold of $\overline{\mathcal{M}}$ with dimension m. There is an explicitly computable nonnegative constant b_0 depending on m, k, K_1, K_2 and L such that if $\mathcal{M}_0$ satisfies

$$|A|^2 < \begin{cases} \frac{4}{3m}|H|^2 - b_0 & \text{if} \quad m = 2, 3, \\ \frac{1}{m-1}|H|^2 - b_0 & \text{if} \quad m \geq 4, \end{cases}$$

then the mean curvature flow with $\mathcal{M}_0$ as initial value contracts to a round point in finite time.

Note that the sign of b_0 exclude the possibility of the convergence to a minimal limit.

The original results exposed in this thesis mainly concern the evolution of pinched submanifold in various ambient manifolds. In most cases, the ambient manifold is the complex projective space $\mathbb{CP}^n$ endowed with the Fubini-Study metric: it is a symmetric space with positive and bounded curvature, so it seems the right space for extending theorem 1.0.1, both for hypersurfaces and for higher codimension. Some of the results obtained for $\mathbb{CP}^n$ are easily proved also for the quaternionic projective space $\mathbb{HP}^n$, giving the generalization of theorem 1.0.1 for almost all CROSSes. In the last part of the thesis, after a general result that links the mean curvature flow with Riemannian submersions, we prove new examples of evolution of pinched submanifolds starting from results taken from the literature or proved for the first time in this thesis.

The content of this thesis is as follows. In chapter 2 we summarize the very essential preliminaries. First we recall some basic facts on the geometry of the submanifolds and Riemannian submersions fixing the notations. Then we present the CROSSes. It is an acronym meaning Compact Rank One Symmetric Spaces. This class of Riemannian manifolds includes the Euclidean sphere $\mathbb{S}^n$, the real projective space $\mathbb{RP}^n$, the complex projective space $\mathbb{CP}^n$, the quaternionic projective space $\mathbb{HP}^n$ and the Cayley plane $\mathbb{C}a\mathbb{P}^2$. They are the ambient spaces considered in most parts of this thesis. They can be characterized in many ways, the most useful for our purpose is that they are the symmetric spaces with strictly positive and bounded curvature. Finally some general results on the mean curvature flow are exposed. For example, we present the evolution equations for some important geometric quantities, like the norm of the second fundamental form and the norm of the mean curvature.

The first original result is the subject of chapter 3. We consider the evolution of pinched submanifold of $\mathbb{CP}^n$ with codimension k small enough respect to the dimension m.

Theorem 1.0.3 *Let $\mathcal{M}_0$ be a closed submanifold of $\mathbb{CP}^n$, with $n \geq 3$, of dimension m and codimension k . If k is sufficientely low, precisely $k = 1$ or $2 \leq k < \frac{2n-3}{5}$ (that is $k < \frac{m-3}{4}$) and $\mathcal{M}_0$ satisfies the pinching condition*

$$|A|^2 < \frac{1}{m-1}|H|^2 + b, \tag{1.0.3}$$

where

$$b = \begin{cases} 2 & \text{if } \quad k = 1, \\ \dfrac{m-3-4k}{m} & \text{if } \quad k \geq 2, \end{cases}$$

then (1.0.3) is preserved by the mean curvature flow. Moreover if k is odd the evolution of $\mathcal{M}_0$ shrinks to a point in finite time, while if k is even one of the following holds:

1) the evolution of $\mathcal{M}_0$ shrinks to a round point in finite time,

2) the evolution of $\mathcal{M}_0$ is defined for any time $0 \leq t < \infty$ and converges to a smooth totally geodesic submanifold, that is a $\mathbb{CP}^{n-\frac{k}{2}}$.

With respect to the general case of theorem 1.0.2, we are considering a specific ambient manifold, but we study a pinching condition that properly includes the one already known: in fact we changed the sign of last constant in the pinching condition. In this way, for even k we find the alternative between round point and totally geodesic limit discovered by Huisken and Baker for the sphere in [Ba, H3]. For small odd codimension, instead, there are no totally geodesic submanifolds of $\mathbb{CP}^n$, but we cannot exclude a priori the possibility of a stationary limit: the proof is the same for any k and only at the end we prove that if a stationary limit exists, then it is totally geodesic. The strategy for the proof is inspired by the analogous problem for submanifolds of the sphere. The curvature of the ambient manifold is no longer constant giving some technical complications. In order to efficiently estimate the reaction terms in the evolution equations, we build normal and tangent frames strongly linked with the geometry of $\mathbb{CP}^n$. An other help to overcome these difficulties is splitting the analisys in two cases: T_{max} finite and T_{max} infinite. The hypothesis T_{max} finite is essential to apply the integral estimates like in the previous papers, while for T_{max} infinite the analysis is quite direct. A further difference with the cited works is that, in our case, the submanifolds considered do not necessary have positive sectional curvature. However we prove that, if time is sufficiently close to T_{max}, the sectional curvatures of the evolving submanifolds become stricly positive everywhere. From theorem 1.0.3 the following classification results follows easily.

Corollary 1.0.4 *Under the hypothesis of theorem 3.0.1, let $\mathcal{M}_0$ satisfy (3.0.1). Then if k is odd, $\mathcal{M}_0$ is diffeomorphic to an $\mathbb{S}^{2n-k}$, if k is even, $\mathcal{M}_0$ is diffeomorphic to an $\mathbb{S}^{2n-k}$ or to a $\mathbb{CP}^{n-\frac{k}{2}}$. In every case $\mathcal{M}_0$ is simply connected.*

Moreover in theorem 3.0.3 we will find a class of pinched submanifolds preserved by the flow bigger than the one defined by (1.0.3). In this case we are able to understand the nature only of the possible stationary limit (if it exists, it is again a totally geodesic submanifold), but we are not able to classify the sigularities. In conclusion of this chapeter we show how theorem 1.0.3 can be easily extended for hypersurfaces of $\mathbb{HP}^n$ too. The pinching condition considered is again (1.0.3) and, since the condimension is 1, we have only the convergence to a round point in finite time. This is a generalization of Huisken's theorem 1.0.1 to almost all CROSSes: unfortunately we are not able to obtain an analogous statement for $\mathbb{C}a\mathbb{P}^2$, because its low dimension.

In chapter 4 we focus again on hypersurfaces of CROSSes. We consider a class which contains properly the class studied in the previous chapter and, with the further assumption that $H \neq 0$ everywhere, we are able to classify the singularities for this class. The main theorem proved is the following

Theorem 1.0.5 *Let $n \geq 4$ and $\mathcal{M}_0$ be a closed real hypersurface of $\mathbb{CP}^n$ or $\mathbb{HP}^n$, that satisfies*

$$|A|^2 < \frac{1}{m-2}|H|^2 + 4, \tag{1.0.4}$$

where m is the real dimension of $\mathcal{M}_0$. Then the mean curvature flow with initial data $\mathcal{M}_0$ develops a singularity in finite time. Moreover if $H \neq 0$ everywhere on $\mathcal{M}_0$, then for *every $\eta > 0$ there exists a constant C_η that depends only on η and $\mathcal{M}_0$ such that*

$$|\lambda_1| \leq \eta\,|H| \qquad \Rightarrow \qquad (\lambda_i - \lambda_j)^2 \leq \Lambda\eta H^2 + C_\eta, \quad \forall i, j \geq 2, \tag{1.0.5}$$

for a constant Λ that depends only on the ambient manifold.

This theorem is a generalization of Nguyen's result [Ng] on the sphere to almost all CROSSes. Once again the hypothesis $n \geq 4$ does not allows us to give the analogous statement for the Cayley plane. The further assumption $H \neq 0$ is preserved by the flow and used to apply the convexity estimates of Huisken and Sinestrari [HS1] and derive the second part of the theorem with integral estimates and Stampacchia iteration on a suitable function. Theorem 1.0.5 implies that, at a point where the curvature is large, either all principal curvatures are positive and comparable with each other, or the smallest is infinitesimal respect to the others and the others become closer and closer, giving a cylindrical profile. In [HS2] an analogous result was an important step towards the construction of a mean curvature flow with surgeries in Euclidean spaces. Then theorem 1.0.5 may be a first step for a future study about surgeries in CROSSes.

Last chapter starts with a general result that links mean curvature flow with Riemannian submersions. We consider submersions defined by the action of a group of isometries: let G be a Lie group acting by isometries on a Riemannian manifold $(\overline{\mathcal{M}}, g_{\overline{\mathcal{M}}})$. Suppose that the quotient space, obteined identifying the point of a orbit of the action of G on $\overline{\mathcal{M}}$, is a smooth manifold $\overline{\mathcal{B}} = \overline{\mathcal{M}}/G$ and consider the induced metric $g_{\overline{\mathcal{B}}}$ on it. The natural projection $\pi : \overline{\mathcal{M}} \to \overline{\mathcal{B}}$ is a Riemannian submersion with fibers the orbits of G. If the action of G is free we have the well-known principal bundles. In this case the fibers of π are isometric to the group G. Lifting a submanifold of $\overline{\mathcal{B}}$ we have a submanifold of $\overline{\mathcal{M}}$ G-invariant, vice versa projecting a G-invariant submanifold of $\overline{\mathcal{M}}$ we get a submanifold of $\overline{\mathcal{B}}$. We show a sufficient condition for the mean curvature flow commutes with the submersion.

Theorem 1.0.6 *Let $\pi : \overline{\mathcal{M}} \to \overline{\mathcal{B}} = \overline{\mathcal{M}}/G$ a Riemannian submersion. If π has closed and minimal fibers then the mean curvature flow of any closed submanifold commutes with the submersion. More precisely let $\mathcal{M}_0$ is a G-invariant submanifold of $\overline{\mathcal{M}}$ and $\mathcal{B}_0$ is a submanifold of $\overline{\mathcal{B}}$. If $\pi(\mathcal{M}_0) = \mathcal{B}_0$ then the mean curvature flow of $\mathcal{M}_0$ and $\mathcal{B}_0$ are defined up to the same maximal time T_{max} and $\pi(\mathcal{M}_t) = \mathcal{B}_t$ for any time $0 \leq t < T_{max}$.*

There are many examples of Riemannian submersions with these characteristics, the best known are, probabily, the Hopf fibrations. In [Pa] Pacini studied the evolution of a single orbit, that is the "lift" of single point of $\overline{\mathcal{B}}$. A very close problem was studied by Smoczyk in [S1]. He considered hypersurfaces invariant respect to a Lie group of isometries that acts freely and properly on the ambient manifold. He showed that the mean curvature flow of such hypersurfaces is ruled by its projection to the quotient manifold and that this projection is a new flow which depends on the nature of the fibers. If the fibers are minimal we find exactly the mean curvature flow. However theorem 1.0.6 holds for any codimensions and even if the action of the group is not free. Theorem 1.0.6 can be used to produce new examples of evolution by mean curvature flow lifting or projecting solutions already known. We can lift theorem 1.0.3 to the sphere $\mathbb{S}^{2n+1}$ via the Hopf fibration.

Proposition 1.0.7 *Consider $\mathcal{M}_0$ a closed $\mathbb{S}^1$-invariant submanifold of $\mathbb{S}^{2n+1}$ of dimension m and codimension $2 \leq k < \frac{2n-3}{5}$ satisfying the pinching condition*

$$|A|^2 < \frac{1}{m-2}\,|H|^2 + \frac{m-4-4k}{m-1}.$$

If k is odd, the evolution by mean curvature flow of $\mathcal{M}_0$ converges in finite time to a $\mathbb{S}^1$, while if k is even one of the following holds:

1) the evolution of $\mathcal{M}_0$ converges in finite time to a $\mathbb{S}^1$,

2) the evolution of $\mathcal{M}_0$ is defined for any time $0 \leq t < \infty$ and converges to a smooth totally geodesic submanifold, that is an $\mathbb{S}^{2n-k+1}$.

Note that the pinching condition in this theorem is weaker than the one of [Ba], but the $\mathbb{S}^1$-invariance is required. Another result of this kind is proposition 5.1.4: we consider $\mathbb{S}^3$-invariant pinched hypersurfaces of $\mathbb{S}^{4n+3}$, with $n \geq 3$ and prove that they are diffeomorphic to a $\mathbb{S}^3 \times \mathbb{S}^{4n-1}$. Finally we exhibit new examples even in case of ambient manifolds neither positively curved nor symmetric: propositions 5.1.7 and 5.1.8 concern submanifolds of the Heisenberg group, while proposition 5.1.10 is about pinched submanifolds of the tangent sphere bundle endowed with the Sasaki metric.

Chapter 2

Preliminaries

2.1 Geometry of immersed submanifolds

In this section we recall some basic notions and fix some notations used through all this thesis. Let $\mathcal{M}$ be a differential manifold of dimension m and $(\overline{\mathcal{M}}, \bar{g})$ a Riemannian manifold of dimension $\bar{m} = m + k > m$. $\overline{\mathcal{M}}$ is called *ambient manifold.* Consider $F : \mathcal{M} \to \overline{\mathcal{M}}$ a smooth immersion. The image $F(\mathcal{M})$ is a submanifold of $\overline{\mathcal{M}}$. Often we identify the immersion with the submanifold associated.

Notation 2.1.1 *Unless told otherwise, geometric quantities of the submanifolds are indicated in the usual way, while for the ambient manifold we use a line over the common symbol. For example g is the Riemannian metric induced by the immersion on $\mathcal{M}$ and $\bar{g}$ is the metric of the ambient manifold. Given a point $\bar{x} \in \overline{\mathcal{M}}$, we denote by $\langle ., . \rangle_{\bar{x}}$ the scalar product on the tangent space $T_{\bar{x}}\overline{\mathcal{M}}$. The subscript will be dropped if there is no risk o confusion. Moreover we will identify via the immersion F $T_x\mathcal{M}$, the tangent space to $\mathcal{M}$ in x, and $N_x\mathcal{M}$, the normal space to $\mathcal{M}$ in x as subspaces of $T_{F(x)}\overline{\mathcal{M}}$. In this way $T_{F(x)}\overline{\mathcal{M}} = T_x\mathcal{M} \oplus N_x\mathcal{M}$.*

Fix $(x_1, \cdots, x_n)$ a local coordinate system around a point $x \in \mathcal{M}$. The local expression of g is

$$g_{ij}(x) = \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \right\rangle_{F(x)} . \tag{2.1.1}$$

Let $\bar{\nabla}$ be the Levi-Civita connection of $(\overline{\mathcal{M}}, \bar{g})$. The *second fundamental form* A of the immersion F is defined for every X, Y tangent vectors of $\mathcal{M}$ by

$$A(X, Y) = \left(\overline{\nabla}_X Y\right)^{\perp},$$

where $\perp$ denote the normal component to $\mathcal{M}$.

Notation 2.1.2 *Unless specified otherwise, latin letters $i, j, k, ...$ run from 1 to m, greek letters $\alpha, \beta, \gamma, ...$ run from $m + 1$ to $m + k$.*

Let $(e_1, \cdots, e_{m+k})$ be an orthonormal frame tangent to $\overline{\mathcal{M}}$ in a point of $F(\mathcal{M})$ with the first m vectors tangent to $F(\mathcal{M})$ and the others normal. Respect to this frame, the second fundamental form can be written

$$A = \sum_{\alpha} h^{\alpha} \otimes e_{\alpha},$$

where the $h^{\alpha} = \left(h^{\alpha}_{ij}\right)$ are symmetric 2-tensors. The metric induces a natural isomorphism between tangent and cotangent space. In coordinates, this is expressed in terms of raising/lowering indexes by means of the matrices g_{ij} and g^{ij}, where g^{ij} is the inverse of g_{ij}. The scalar product on the tangent space extends to any tensor bundle, by contracting any pair of lower and upper indices with g_{ij} and g^{ij} respectively. Thus, for instance, the scalar product of two $(1,2)$-tensors T^i_{jk} and S^i_{jk} is defined by

$$\langle T^i_{jk}, S^i_{jk}\rangle := T^{jk}_i S^i_{jk} = T^l_{pq} S^i_{jk} g_{li} g^{pj} g^{qk}.$$

Notation 2.1.3 *Here and in the following, if there are no explicit signs of sum, we use Eistein notation, that is we sum over repeted indices.*

This also allows to define the norm of any tensor T as $|T|^2 := \langle T, T\rangle$. A function that we use very often is the norm of the second fundamental form

$$|A|^2 = \sum_{\alpha} |h^{\alpha}|^2$$

The trace respect to the metric g of the second fundamental form is the *mean curvature vector* H:

$$H = \sum_{\alpha} tr h^{\alpha} e_{\alpha} = \sum_{\alpha}\sum_{ij} g^{ij} h^{\alpha}_{ij} e_{\alpha}.$$

It is independent of the orientation and it is well defined globally even if $\mathcal{M}$ is non-orientable. Note that some outhors defines the mean curvature as the trace of A over m, of course this makes no substantial difference in the analysis. The *traceless second fundamental* form is $\mathring{A} = A - \dfrac{1}{m} H \otimes g = (\mathring{h}_{ij})$. Of course $|\mathring{A}|^2 \geq 0$, then

$$|A|^2 \geq \frac{1}{m} |H|^2 \tag{2.1.2}$$

holds for any immersion. If the codimension $k = 1$, we say that $F(\mathcal{M})$ is an *hypersurface*. If it is orientable, a normal unit vector field ν is defined everywhere on $F(\mathcal{M})$. Then the second fundamental form can be identify with a symmetric $(0,2)$ tensor that we denote always with A:

$$A(X,Y) = \bar{g}(A(X,Y), \nu).$$

Similarly the mean curvature vector is a multiple of ν:

$$\vec{H} = -H\nu.$$

The eigenvalues of A respect to the metric g are called *principal curvatures* of the hypersurface. We denote them by $\lambda_1 \leq \cdots \leq \lambda_n$. We have

$$H = \lambda_1 + \cdots + \lambda_n, \qquad |A|^2 = \lambda_1^2 + \cdots + \lambda_n^2.$$

Moreover, another equality that we will use is

$$|A|^2 - \frac{1}{m}|H|^2 = \frac{1}{m}\sum_{i<j}(\lambda_i - \lambda_j)^2\,. \tag{2.1.3}$$

We recall that the Laplace-Beltrami operator on functions $f : \mathcal{M} \to \mathbb{R}$ is

$$\Delta_{\mathcal{M}} f = g^{ij}\nabla_i\nabla_j f,$$

where ∇ is the Levi-Civita connection of $(\mathcal{M}, g)$. If there is no risk of confusion of ambiguity, we write simply Δ instead $\Delta_{\mathcal{M}}$.

2.2 Riemannian submersions

We introduce Riemanniana submersions and the most important related notions for our purpose. What follows is taken from the classical O'Neill's paper [O], many other interesting results can be found in chapter 9 of [Be2], while [FIP] is a extensive monography about Riemannian submersions.

Let $(\mathcal{M}, g_{\mathcal{M}})$ and $(\mathcal{B}, g_{\mathcal{B}})$ two Riemannian manifolds of dimension m and b respectively. A *Riemannian submersions* is a C^∞ map $\pi : \mathcal{M} \to \mathcal{B}$ satisfying the following axioms $S1$ and $S2$.

S1) π has maximal rank;

For every $p \in \mathcal{M}$, $\pi^{-1}(p)$ is a submanifold of $\mathcal{M}$ called *fiber* over p. A vector field on $\mathcal{M}$ is called *vertical* if it is always tangent to fibers, *horizontal* if always orthogonal to fibers. The second axiom is

S2) for every X, Y horizontal tangent vectors of $\mathcal{M}$ we have

$$g_{\mathcal{M}}(X, Y) = g_{\mathcal{B}}(\pi_* X, \pi_* Y) \circ \pi,$$

where π_* is the differential of π. $\mathcal{M}$ is called *total space* of the submersion and $\mathcal{B}$ is called *base*. Axiom $S1$ implies that $m \geq b$ and so the dimension of the fibers is $\hat{m} = m - b$. Axiom $S2$ says that π preserves lenghts of horizontal vectors.

Notation 2.2.1 *If not specified otherwise, we use the same simbols for geometric quantities of $\mathcal{M}$ and $\mathcal{B}$. It will be clear from the context in which manifold we are. While the same quantities of the fibers are distinguished by the superscript* $\hat{}$ *.*

Following the notations of O'Neill [O], the *vertical distribution* $\mathscr{V}$ is the distribution of vertical vector fields, that is $\mathscr{V} = \ker \pi_*$. Its orthogonal complement respect to $g_{\mathcal{M}}$ is the *horizontal distribution* $\mathscr{H}$. We denote with the same simbols $\mathscr{H}$ and $\mathscr{V}$ the projections of the tangent space of $\mathcal{M}$ to the subspaces of horizontal and vertical vectors, respectively. Then every X tangent to $\mathcal{M}$ can be decomposed in an unique way in the sum of a horizontal and a vertical vectors:

$$X = \mathscr{H}X + \mathscr{V}X.$$

An horizontal vector field X' is called *basic* if there exists a vector fields X on $\mathcal{B}$ such that $\pi_* X' = X$, in this case X and X' are said to be π-related. There is an one-to-one correspondence between basic vector fields on $\mathcal{M}$ and arbitrary vector fields on $\mathcal{B}$: every basic vector field gives a vector field on $\mathcal{B}$ by definition, while every X tangent to $\mathcal{B}$ has an unique horizontal lift $X^{\mathscr{H}}$ to $\mathcal{M}$ characterized by $\pi_* X^{\mathscr{H}} = X$. As showed by O'Neill's paper, submersions are ruled by two tensors. For every X and Y tangent to $\mathcal{M}$ we define

$$\begin{aligned} \mathcal{T}_X Y &= \mathscr{H}\nabla_{\mathscr{V}X}(\mathscr{V}Y) + \mathscr{V}\nabla_{\mathscr{V}X}(\mathscr{H}Y); \\ \mathcal{A}_X Y &= \mathscr{V}\nabla_{\mathscr{H}X}(\mathscr{H}Y) + \mathscr{H}\nabla_{\mathscr{H}X}(\mathscr{V}Y). \end{aligned}$$

Note that if X and Y are tangent to fibers, i.e. vertical, then $\mathcal{T}_X Y = \hat{A}(X,Y)$ the second fundamental form of the fibers as submanifolds of $\mathcal{M}$. We have that $\mathcal{T} \equiv 0$ if and only if each fiber is totally geodesic, while $\mathcal{A} \equiv 0$ if and only if $\mathscr{H}$ is integrable.

Now we consider the lift of a submanifold and we want to understand how it is related with the initial submanifold. Let $\pi : (\overline{\mathcal{M}}, g_{\overline{\mathcal{M}}}) \to (\overline{\mathcal{B}}, g_{\overline{\mathcal{B}}})$ a Riemannian submersion, and $F : \mathcal{B} \to \overline{\mathcal{B}}$ an immersion. $\pi^{-1}(F(\mathcal{B}))$ is a submanifold of $\overline{\mathcal{M}}$ of the same codimension of $F(\overline{\mathcal{B}})$. Formally there is a manifold $\mathcal{M}$, an immersion $F' : \mathcal{M} \to \overline{\mathcal{M}}$ and a sommersion that we indicate again with π, such that the following diagrams commutes.

$$\begin{array}{ccc} \overline{\mathcal{M}} & \xrightarrow{\pi} & \overline{\mathcal{B}} \\ F' \uparrow & & \uparrow F \\ \mathcal{M} & \xrightarrow{\pi} & \mathcal{B} \end{array}$$

We want to understand the link between A, the second fundamental form of F, and A', the second fundamental form of F'. The main tool is the following O'Neill's formulas.

Lemma 2.2.2 *[O] For every tangent vector fields on $\overline{\mathcal{B}}$ X and Y we have*

1) $[X,Y]^{\mathscr{H}} = \mathscr{H}\left[X^{\mathscr{H}}, Y^{\mathscr{H}}\right]$;

2) $\left(\bar{\nabla}_X Y\right)^{\mathscr{H}} = \mathscr{H}\left(\bar{\nabla}_{X^{\mathscr{H}}} Y^{\mathscr{H}}\right)$.

Lemma 2.2.3 *[O] Let X and Y be horizontal vector fields and V and W vertical vector fields. Then*

1) $\bar{\nabla}_V W = \mathcal{T}_V W + \hat{\nabla}_V W$;

2) $\bar\nabla_V X = \mathscr{H}\bar\nabla_V X + \mathcal{T}_V X$;

3) $\bar\nabla_X V = \mathcal{A}_X V + \mathscr{V}\bar\nabla_X V$;

4) $\bar\nabla_X Y = \mathscr{H}\bar\nabla_X Y + \mathcal{A}_X Y$.

Note that, by construction, $\mathcal{M} \equiv F'(\mathcal{M})$ is tangent to the fibers, then any vector normal to $\mathcal{M}$ is necessarily horizontal. From Lemma 2.2.2 and Gauss equation we have that for any X and Y tangent to $\mathcal{M}$

$$\begin{aligned}\bar\nabla_{X^{\mathscr{H}}} Y^{\mathscr{H}} &= \mathscr{H}\left(\bar\nabla_{X^{\mathscr{H}}} Y^{\mathscr{H}}\right) + \mathscr{V}\left(\bar\nabla_{X^{\mathscr{H}}} Y^{\mathscr{H}}\right)\\ &= \left(\bar\nabla_X Y\right)^{\mathscr{H}} + \mathscr{V}\left(\bar\nabla_{X^{\mathscr{H}}} Y^{\mathscr{H}}\right)\\ &= \left(\nabla_X Y\right)^{\mathscr{H}} + \left(A(X,Y)\right)^{\mathscr{H}} + \mathscr{V}\left(\bar\nabla_{X^{\mathscr{H}}} Y^{\mathscr{H}}\right).\end{aligned} \tag{2.2.1}$$

By definition $A'(X^{\mathscr{H}}, Y^{\mathscr{H}}) = \left(\bar\nabla_{X^{\mathscr{H}}} Y^{\mathscr{H}}\right)^{\perp}$ (where $(.)^{\perp}$ means the component normal to $\mathcal{M}$), then it is an horizontal vector field. By (2.2.1) we have

$$A'(X^{\mathscr{H}}, Y^{\mathscr{H}}) = \left((\nabla_X Y)^{\mathscr{H}}\right)^{\perp} + \left((A(X,Y))^{\mathscr{H}}\right)^{\perp}.$$

The vector field $(\nabla_X Y)^{\mathscr{H}}$ is the lift of a vector field tangent to $\mathcal{B}$, then it is tangent to $\mathcal{M}$. In the same way $(A(X,Y))^{\mathscr{H}}$ is normal to $\mathcal{M}$. Hence we have

$$A'(X^{\mathscr{H}}, Y^{\mathscr{H}}) = (A(X,Y))^{\mathscr{H}}. \tag{2.2.2}$$

Now consider two vertical vector fields V and W. They are tangent to $\mathcal{M}$ by construction. $A'(V,W)$ is normal to $\mathcal{M}$ then it is a horizontal vector field. By lemma 2.2.3 we have

$$A'(V,W) = \left(\bar\nabla_V W\right)^{\perp} = \left(\mathscr{H}\bar\nabla_V W\right)^{\perp} = (\mathcal{T}_V W)^{\perp} = (\hat{A}(V,W))^{\perp}. \tag{2.2.3}$$

Lemma 2.2.3 does not say anything about the mixed terms $A'(X,V)$ with X horizontal and V vertical: they strongly depend on the specific submersion considered as we will see in the examples of chapter 5.

Let $(X_1, \dots, X_m)$ a local orthonormal frame tangent to $\mathcal{B}$ around a point p. Consider $(V_1, \dots, V_{\hat m})$ a local orthonormal set of vertical vector fields. $(X_1^{\mathscr{H}}, \dots, X_m^{\mathscr{H}}, V_1, \dots V_{\hat m})$ is a local orthonormal basis tangent to $\mathcal{M}$ around any point of the fiber $\pi^{-1}(p)$. Summarizing what we found, with respect to this basis we have

$$A' = \left(\begin{array}{c|c} h_{ij}^{\mathscr{H}} & \text{mixed terms} \\ \hline \text{mixed terms} & \hat{h}_{ij}^{\perp} \end{array}\right) \tag{2.2.4}$$

where $h_{ij} = A(X_i, X_j)$ and $\hat{h}_{ij} = \hat{A}(V_i, V_j)$.

Starting from a fixed Riemannian submersion $\pi : (\mathcal{M}, g_{\mathcal{M}}) \to (\mathcal{B}, g_{\mathcal{B}})$ there is a standard way to deform the metric $g_{\mathcal{M}}$ to obtain again a Riemannian submersion. The *canonical variation* of $g_{\mathcal{M}}$ is the family of metrics $\{g_\lambda\}_{\lambda>0}$ on $\mathcal{M}$ such that

$$\begin{array}{ll} g_\lambda(U,V) = \lambda g_{\mathcal{M}}(U,V) & \text{if} \quad U, V \in \mathscr{V}, \\ g_\lambda(X,Y) = g_{\mathcal{M}}(U,V) & \text{if} \quad X, Y \in \mathscr{H}, \\ g_\lambda(U,X) = 0 & \text{if} \quad U \in \mathscr{V},\ X \in \mathscr{H}. \end{array}$$

Obviously $g_1 = g_{\mathcal{M}}$. For any $\lambda > 0$, g_λ makes π a Riemannian submersion with the same horizontal and vertical distributions and the same fibers. Let ∇^λ be the Levi-Civita connection of the metric g_λ. A straightforward computation gives:

$$\begin{aligned} &\mathscr{V}\left(\nabla^\lambda_U V\right) = \mathscr{V}\left(\nabla^1_U V\right), \quad \mathscr{H}\left(\nabla^\lambda_U V\right) = \lambda \mathscr{H}\left(\nabla^1_U V\right), \\ &\nabla^\lambda_X U = \nabla^1_X U, \quad \nabla^\lambda_U X = \nabla^1_U X, \quad \nabla^\lambda_X Y = \nabla^1_X Y, \end{aligned} \tag{2.2.5}$$

for every $U, V \in \mathscr{V}$ and $X, Y \in \mathscr{H}$. It follows that $\pi : (\mathcal{M}, g_{\mathcal{M}} = g_1) \to (\mathcal{B}, g_{\mathcal{B}})$ has minimal (resp. totally geodesic) fibers if and only if $\pi : (\mathcal{M}, g_\lambda) \to (\mathcal{B}, g_{\mathcal{B}})$ has minimal (resp. totally geodesic) fibers for every $\lambda > 0$. Moreover let $(X_1^{\mathscr{H}}, \dots, X_m^{\mathscr{H}}, V_1, \dots V_{\hat{m}})$ is a local $g_{\mathcal{M}}$-orthonormal basis tangent to $\mathcal{M}$ around any point of the fiber $\pi^{-1}(p)$, then for any $\lambda > 0$ $(X_1^{\mathscr{H}}, \dots, X_m^{\mathscr{H}}, \lambda^{-\frac{1}{2}} V_1, \dots \lambda^{-\frac{1}{2}} V_{\hat{m}})$ is a g_λ-orthonormal frame. Using (2.2.5) is easy to see that, respect to this basis, the equation 2.2.4 becomes

$$A'_\lambda = \left(\begin{array}{c|c} h^{\mathscr{H}}_{ij} & \lambda^{-\frac{1}{2}} \text{ mixed terms} \\ \hline \lambda^{-\frac{1}{2}} \text{ mixed terms} & \hat{h}^{\perp}_{ij} \end{array}\right). \tag{2.2.6}$$

2.3 CROSSes

The ambient manifold for the flow that we are going to use most often in the following are $\mathbb{CP}^n$, the complex projective space, and $\mathbb{HP}^n$, the quaternionic projective space. They are examples of CROSSes: an acronym meaning Compact Rank One Symmetric Spaces. In this class of Riemannian manifolds there are also the Euclidean sphere $\mathbb{S}^n$, the real projective space $\mathbb{RP}^n$ and the Cayley plane $\mathbb{C}a\mathbb{P}^2$. Their geometric structure is very rich and they can be characterized in many ways. For example they are the symmetric spaces with strictly positive curvature: these properties will be very important for the proofs of the later chapters. Here we are going to describe the most needed proprieties for our purpose. An excellent introduction to CROSSes can be found in chapter 3 of [Be1].

Let $\mathbb{K}$ be one of the field $\mathbb{C}$ or the associative algebra $\mathbb{H}$ and a be the real dimension of $\mathbb{K}$, that is

$$a = \begin{cases} 2, & \text{if } \mathbb{K} = \mathbb{C}; \\ 4, & \text{if } \mathbb{K} = \mathbb{H}. \end{cases}$$

We denote with $\mathbb{S}^n(c)$ the n-dimensional sphere with the canonical metric of constant curvature $c > 0$. The action $T : \mathbb{S}^{a-1}(1) \times \mathbb{S}^{na+a-1}(c) \to \mathbb{S}^{na+a-1}(c)$, $(\lambda, z) \mapsto \lambda z$ is by isometries which acts transitively on the fiber. $\mathbb{KP}^n$ can be identify with $\mathbb{S}^{na+a-1}/\mathbb{S}^{a-1}$. The *Hopf fibration* is $\pi : \mathbb{S}^{na+a-1}(c) \to \mathbb{KP}^n$, $z \mapsto [z]$, where $[z]$ is the class of z under the action T. The Riemannian metric that we consider on $\mathbb{KP}^n$ is the one induced from the metric of $\mathbb{S}^{na+a-1}(c)$ such that π becomes a Riemannian submersion. For $\mathbb{K} = \mathbb{C}$ it is the well-known Fubini-Study metric.

Notation 2.3.1 *We denote by $\mathbb{KP}^n(4c)$ the $\mathbb{K}-$ projective space endowed with this metric. We use simply $\mathbb{KP}^n$ instead of $\mathbb{KP}^n(4)$.*

Fix two tangent vectors X and Y, we denote by $pr_{Y\mathbb{K}}X$ the projection of X on the tangent subspace $Y\mathbb{K}$ of dimension a. The metrics defined in this way has positive bounded sectional curvatures. In fact, if X and Y are two orthogonal unit vector tangent to $\mathbb{KP}^n$, theorem 3.30 of [Be1] shows that $\overline{K}(X,Y)$, the sectiona curvature of the tangent plane spanned by X and Y is

$$\overline{K}(X,Y) = c(1 + 3\,|pr_{Y\mathbb{K}}X|^2), \tag{2.3.1}$$

where $|.|^2$ is the norm induced by the metric. It follows that $c \le \overline{K} \le 4c$ and $\overline{K} = c$ (respectively $\overline{K} = 4c$) if and only if X is orthogonal (respectively belongs) to $Y\mathbb{K}$. Another property that will be often used is that the CROSSes are Einstein manifolds. In particular, the Einstein constant of $\mathbb{KP}^n$ is

$$\bar{r} = \begin{cases} 2(n+1)c & \text{if } \mathbb{K} = \mathbb{C}; \\ 4(n+2)c & \text{if } \mathbb{K} = \mathbb{H}. \end{cases} \tag{2.3.2}$$

Notation 2.3.2 *For simplicity of notation we will use $c = 1$ through this thesis, giving only the statements of main theorems for the general case. The proofs are the same: just multiply by c each terms where the curvature of the ambient manifold occurs.*

Since we are going to study mean curvature flow, we are interested in submanifolds of the ambient manifold, with particular attention to minimal submanifolds. Theorem 3.25 of [Be1] characterizes the totally geodesic submanifolds of CROSSes.

Theorem 2.3.3 *Let $\mathbb{K} \in \{\mathbb{R},\ \mathbb{C},\ \mathbb{H}\}$ and $n \in \mathbb{N}$. $\mathcal{M}$ is a closed totally geodesic submanifold of $\mathbb{KP}^n$ if and only if there exist $\mathbb{K}' \in \{\mathbb{R},\ \mathbb{C},\ \mathbb{H}\}$ with $\mathbb{K}' \le \mathbb{K}$ and $n' \in \mathbb{N}$ with $n' \le n$ such that $\mathcal{M}$ is isometric to $\mathbb{K}'\mathbb{P}^{n'}$.*

In the follow we are mostly interested in submanifolds of $\mathbb{CP}^n$ and $\mathbb{HP}^n$. In particular we have that there are no totally geodesic hypersurfaces for these ambiente manifold or totally goedesic submanifolds of odd and low codimension.

We conclude this section giving some explicit formulae for the case $\mathbb{K} = \mathbb{C}$. $\mathbb{CP}^n$, with the Fubini-Study metric g_{FS} can be seen as a complex manifold, with complex dimension n and complex structure J. Moreovere it is a Kähler manifold. Let $\bar{R}$ be its Riemann curvature tensor. $\bar{R}$ has this explicit form

$$\begin{aligned} \bar{R}(X,Y,Z,W) \;=\; & g_{FS}(X,Z)g_{FS}(Y,W) - g_{FS}(X,W)g_{FS}(Y,Z) \\ & g_{FS}(X,JZ)g_{FS}(Y,JW) - g_{FS}(X,JW)g_{FS}(Y,JZ) \\ & +2g_{FS}(X,JY)g_{FS}(Z,JW), \end{aligned} \tag{2.3.3}$$

for all tangent vector fields $X, Y, Z, W,$. In particular, the sectional curvature of a tangent plane spanned by two orthonormal vector fields X and Y is

$$\bar{K}(X,Y) = 1 + 3g_{FS}(X,JY)^2, \tag{2.3.4}$$

therefore $1 \le \bar{K} \le 4$ and $\bar{K} = 1$ (resp. $\bar{K} = 4$) if and only if JY is orthogonal (resp. tangent) to X. This propriety makes $\mathbb{CP}^n$ a *complex space form* of constant holomorphic curvature 4.

2.4 Mean curvature flow

Let $F_0 : \mathcal{M} \to \overline{\mathcal{M}}$ be a smooth immersion of a real m-dimensional manifold in a Riemann manifold of metric $\bar{g}$ and dimension $\bar{m} = m + k$. The evolution of $\mathcal{M}_0 = F_0(\mathcal{M})$ by mean curvature flow is the one-parameter family of immersions $F : \mathcal{M} \times [0, T_{max}[\to \overline{\mathcal{M}}$ satisfying

$$\begin{cases} \dfrac{\partial}{\partial t} F(p,t) = H(p,t), \qquad p \in \mathcal{M},\, t \geq 0, \\ F(\cdot, 0) = F_0. \end{cases} \tag{2.4.1}$$

where $H(p,t)$ is the mean curvature vector of the immersion $F(\cdot, t)$ at point p.

Notation 2.4.1 *The Riemann manifold $(\overline{\mathcal{M}}, \bar{g})$ is called ambient manifold of the flow. We denote with $\mathcal{M}_t = F(\mathcal{M}, t)$ the immersed submanifold at time t. We talk indiferrently about the evolution of the immersion $F(., t)$ or the evolution of the associated immersed submanifold $\mathcal{M}_t$. Moreover we indicate with $A = A(t)$ the second fundamental form of $\mathcal{M}_t$.*

The existence of the solution is a well known property.

Theorem 2.4.2 *If the initial submanifold $\mathcal{M}_0$ is smooth and compact, then the solution of (2.4.1) exists, is unique and smooth up to a maximal time $0 < T_{max} \leq \infty$. If T_{max} is finite, the flow develops a singularity at the maximal time, that is $\max_{\mathcal{M}_t} |A|^2$ becomes unbounded as t approaches T_{max}.*

From the equation (2.4.1) that defines the mean curvature flow, one can derive the evolution equations for the other geometric functions.

Computation in [Ba] shows the evolution equations for important geometric quantities, for any ambient manifold and any codimension.

Lemma 2.4.3 *Along the mean curvature flow we have*

$$1)\quad \frac{\partial}{\partial t}|H|^2 = \Delta |H|^2 - 2|\nabla H|^2 + 2\sum_{i,j}\left(\sum_\alpha H^\alpha h^\alpha_{ij}\right)^2 + 2\sum_{k,\alpha,\beta}\bar{R}_{k\alpha k\beta}H^\alpha H^\beta,$$

$$\begin{aligned} 2)\quad \frac{\partial}{\partial t}|A|^2 = {} & \Delta |A|^2 - 2|\nabla A|^2 + 2\sum_{\alpha,\beta}\left(\sum_{i,j} h^\alpha_{ij}h^\beta_{ij}\right) + 2\sum_{i,j,\alpha,\beta}\left[\sum_p h^\alpha_{ip}h^\beta_{jp} - h^\beta_{ip}h^\alpha_{jp}\right]^2 \\ & +4\sum_{i,j,p,q}\bar{R}_{ipjq}\left(\sum_\alpha h^\alpha_{pq}h^\alpha_{ij}\right) - 4\sum_{j,k,p}\bar{R}_{kjkp}\left(\sum_{i,\alpha} h^\alpha_{pi}h^\alpha_{ij}\right) \\ & +2\sum_{k,\alpha,\beta}\bar{R}_{k\alpha k\beta}\left(\sum_{ij} h^\alpha_{ij}h^\beta_{ij}\right) - 8\sum_{j,p,\alpha,\beta}\bar{R}_{jp\alpha\beta}\left(\sum_i h^\alpha_{ip}h^\beta_{ij}\right) \\ & +2\sum_{i,j,p,\beta}\bar{\nabla}_p\bar{R}_{pij\beta}h^\beta_{ij} - 2\sum_{i,j,p,\beta}\bar{\nabla}_i\bar{R}_{jpp\beta}h^\beta_{ij}, \end{aligned}$$

$$3)\quad \frac{\partial}{\partial t}d\mu_t = -|H|^2 d\mu_t,$$

where $d\mu_t$ is the volume form of the metric induced by the immersion on the submanifold. It follows that the volume of the evolving submanifold is non increasing during the flow.

Since the ambient manifold that we are considering are often symmetric, we have that $\bar{\nabla}\bar{R} = 0$, and the last line in the evolution of $|A|^2$ vanishes in these cases. Of course, when the codimension is 1 these equations have a much simpler form.

Lemma 2.4.4 *The mean curvature flow of hypersurfaces in a symmetric ambient manifold satisfies:*

$$\begin{aligned} 1)\qquad & \frac{\partial}{\partial t}|H|^2 = \Delta |H|^2 - 2|\nabla H|^2 + 2|H|^2\left(|A|^2 + \bar{R}ic(\nu,\nu)\right), \\ 2)\qquad & \frac{\partial}{\partial t}|A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^2\left(|A|^2 + \bar{R}ic(\nu,\nu)\right) \\ & \qquad - 4\left(h_{ij}h_j^{\ p}\bar{R}_{pli}^{\ \ \ l} - h^{ij}h^{lp}\bar{R}_{pilj}\right), \end{aligned}$$

where $\bar{R}ic$ is the Ricci tensor of the ambient manifold and ν is the normal unit vector of the hypersurface.

The following result is indipendent of the flow, but we will use it often in the next chapters. It can be found in Lemma 2.2 of [H2] for hypersurfaces, or Lemma 3.2 of [LXZ] for any codimension. For our need, here we expose a simplified version that holds only on Einstein manifold.

Lemma 2.4.5 *Let $\overline{\mathcal{M}}$ an Einstein manifold and $\mathcal{M}$ a submanifold of $\overline{\mathcal{M}}$ of dimension n and arbitrary codimension. Then at every point of $\mathcal{M}$*

$$|\nabla A|^2 \geq \frac{3}{n+2} |\nabla H|^2 \tag{2.4.2}$$

holds.

Proof. It follows from Lemma 3.2 of [LXZ] with $w = \eta = 0$. For completeness we describe the proof. Set $E_{ijk} = \frac{1}{n+2} (\nabla_i H g_{jk} + \nabla_j H g_{ik} + \nabla_k H g_{ij})$ and $F_{ijk} = \nabla_i h_{jk} - E_{ijk}$. By the Codazzi equation $\langle E_{ijk}, F_{ijk} \rangle = 0$. Hence $|\nabla A|^2 \geq |E|^2 = \frac{3}{n+2} |\nabla H|^2$. □

An extremely important tool is the maximum principle. Here we expose only the version for scalar function that we use in the following.

Theorem 2.4.6 *(Maximum principle) Let $\mathcal{M}$ be a closed manifold and $g(t)$ a time-depending family of Riemannian metric on $\mathcal{M}$. We denote by $\Delta_{g(t)}$ the Laplace-Beltrami operator of the metric $g(t)$. Given a C^∞ function $f : \mathcal{M} \times [0, T[\to \mathbb{R}$, if*

$$\begin{cases} \dfrac{\partial}{\partial t} f - \Delta_{g(t)} = Q(f, t) \\ f_{|t=0} = f_0 \leq 0 \end{cases}$$

for some function Q such that $Q(x,t) \leq 0$ in the points where $f(x,t) = 0$, then $f(x,t) \leq 0$ for every (x,t).

Chapter 3

Low codimension submanifolds of $\mathbb{CP}^n$

The fist problem discussed is the evolution by mean curvature of a class of pinched submanifolds of the complex projective space with codimension small enough respect to the dimension. We will show that the same result holds in the quaternionic projective space, at least for hypersurfaces giving the generalization of theorem 1.0.1 to (almost) all CROSSes.

Theorem 3.0.1 *Let $\mathcal{M}_0$ be a closed submanifold of $\mathbb{CP}^n$, with $n \geq 3$, of dimension m and codimension k . If k is sufficientely low, precisely $k = 1$ or $2 \leq k < \frac{2n-3}{5}$ (that is $k < \frac{m-3}{4}$) and $\mathcal{M}_0$ satisfies the pinching condition*

$$|A|^2 < \frac{1}{m-1} |H|^2 + b, \tag{3.0.1}$$

where

$$b = \begin{cases} 2 & \text{if } \; k = 1, \\ \dfrac{m-3-4k}{m} & \text{if } \; k \geq 2, \end{cases}$$

then (3.0.1) is preserved by the mean curvature flow. Moreover if k is odd the evolution of $\mathcal{M}_0$ shrinks to a point in finite time, while if k is even one of the following holds:

1) *the evolution of $\mathcal{M}_0$ shrinks to a round point in finite time,*

2) *the evolution of $\mathcal{M}_0$ is defined for any time $0 \leq t < \infty$ and converges to a smooth totally geodesic submanifold, that is a $\mathbb{CP}^{n-\frac{k}{2}}$.*

The following result is a direct conseguence of theorem 3.0.1.

Corollary 3.0.2 *Under the hypothesis of theorem 3.0.1, let $\mathcal{M}_0$ satisfying (3.0.1). Then if k is odd, $\mathcal{M}_0$ is diffeomorphic to an $\mathbb{S}^{2n-k}$, if k is even, $\mathcal{M}_0$ is diffeomorphic to an $\mathbb{S}^{2n-k}$ or to a $\mathbb{CP}^{n-\frac{k}{2}}$. In every case $\mathcal{M}_0$ is simply connected.*

For a bigger class of submanifold we cannot classify the singularities, but we can say what is the shape of a stationary limit, if it exists.

Theorem 3.0.3 *Let $\mathcal{M}_0$ be a closed submanifold of $\mathbb{CP}^n$, with $n \geq 3$ of dimension m and codimension k . If k is sufficientely low, precisely $k = 1$ or $2 \leq k < \frac{2n-3}{5}$ and $\mathcal{M}_0$ satisfies the pinching condition*

$$|A|^2 < a\,|H|^2 + b, \tag{3.0.2}$$

where a and b are two positive constants satisfying

$$\frac{1}{m} < \quad a \quad \leq \frac{4}{3m},$$

and

$$0 < b < \begin{cases} (m-3-4k)\left(1 - \dfrac{1}{ma}\right) & \text{if } k \geq 2, \\ min\left(2,\ \dfrac{2}{a}(ma-1)\right) & \text{if } k = 1,\ n = 3, \\ \dfrac{2}{a}(ma-1) & \text{if } k = 1,\ n \geq 4. \end{cases}$$

then (3.0.2) is preserved by the mean curvature flow. Moreover if k is odd the evolution of $\mathcal{M}_0$ develops a singularity in finite time, while if k is even one of the following holds:

1) *the evolution of $\mathcal{M}_0$ develops a singularity in finite time,*

2) *the evolution of $\mathcal{M}_0$ is defined for any time $0 \leq t < \infty$ and converges to a smooth totally geodesic submanifold, that is a $\mathbb{CP}^{n-\frac{k}{2}}$.*

The strategy for the proof is inspired by the analogous problem for submanifolds of the sphere [H3, Ba]. The curvature of the ambient manifold is no longer constant giving some technical complications. In order to efficiently estimate the reaction terms in the evolution equations, we build normal and tangent frames strongly linked with the geometry of $\mathbb{CP}^n$. An other help to overcome these difficulties is splitting the analisys in two cases: T_{max} finite and T_{max} infinite. The hypothesis T_{max} finite is essential to apply the integral estimates like in the previous papers, while for T_{max} infinite the analysis is very much direct. Note that for small odd k we already known that there are no totally geodesic submanifolds, but we cannot exclude a priori the convergence to a stationary limit: the proof is the same for any k and we prove that if it exists, then it is totally geodesic.

Remark 3.0.4 *As said in notation 2.3.2, we can generalize considering as ambient manifold $\mathbb{CP}^n(c)$ for any $c > 0$. In this case the pinching condition of theorem 3.0.1 becomes*

$$|A|^2 < \frac{1}{m-1}\,|H|^2 + b,$$

where

$$b = \begin{cases} 2c & \text{if} \quad k = 1, \\ \dfrac{m-3-4k}{m}c & \text{if} \quad k \geq 2. \end{cases}$$

In the same way the pinching condition of theorem 3.0.3 becomes

$$|A|^2 < a\,|H|^2 + b,$$

where a is not changed and

$$0 < b < \begin{cases} (m-3-4k)\left(1-\dfrac{1}{ma}\right)c & \textit{if } k \geq 2, \\ min\left(2c,\ \dfrac{2}{a}(ma-1)c\right) & \textit{if } k=1,\ n=3, \\ \dfrac{2}{a}(ma-1)c & \textit{if } k=1,\ n\geq 4. \end{cases}$$

the result (and the proof as well) is the same in both cases: the only difference is that the totally geodesic limit is a $\mathbb{CP}^{n-\frac{k}{2}}(c)$.

3.1 Invarance of pinching

We want to prove that the pinching condition (3.0.1) is preserved by the mean curvature flow for any time the solution exists. We use the maximum principle. Indeed we can prove this kind of result directly for the bigger condition (3.0.2), that we will restrict later for technical reasons when we want to study the singularities of the flow.

Once fixed an orthonormal basis $(e_m+1,\ldots,e_{m+k})$ of the normal space, we can write the second fundamental form as $A = \sum_{\alpha=m+1}^{2n} h^\alpha e_\alpha$, where the h^α are symmetric 2-tensors. The traceless part of the second fundamental form is $\mathring{A} = A - \frac{1}{m} H \otimes g$ and its squared length is $|\mathring{A}|^2 = |A|^2 - \frac{1}{m}|H|^2$. In order to simplify computations we introduce two kind of frames that we consider for every time t and point $p \in \mathcal{M}_t$. In general the two basis are not the same and we will use the one that will be more convenient depending on the circumstances.

B1) In any point where $H \neq 0$ we can follow the notations [AB, LXZ] and choose a privileged normal direction defining

$$e_{m+1} = \frac{H}{|H|}. \tag{3.1.1}$$

Then we can built $(e_{m+1}, e_{m+2}, \ldots, e_{m+k})$ an orthonormal basis of $N_p\mathcal{M}_t$ and choose any orthonormal basis of $T_p\mathcal{M}_t$ $(e_1,\ldots,e_m)$. With this kind of frames, the second fundamental form and its traceless part satisfy

$$\begin{cases} \operatorname{tr} h^{m+1} = |H|, \\ \operatorname{tr} h^\alpha = 0, & \alpha \geq m+2. \end{cases}$$

and

$$\begin{cases} \mathring{h}^{m+1} = h^{m+1} - \dfrac{|H|}{m} g, \\ \mathring{h}^\alpha = h^\alpha, & \alpha \geq m+2 \end{cases}$$

We will adopt the following notation only when we use basis of kind B1:

$$|h_1|^2 := \left|h^{m+1}\right|^2, \qquad |\mathring{h}_1|^2 := \left|\mathring{h}^{m+1}\right|^2, \qquad |h_-|^2 = |\mathring{h}_-|^2 := \sum_{\alpha=m+2}^{2n} |\mathring{h}^\alpha|^2. \quad (3.1.2)$$

B2) A second kind of frames, more linked with the geometry of $\mathbb{CP}^n$, is useful when we have to compute terms involving the Riemann curvature tensor of the ambient manifold.

Lemma 3.1.1 *If $k \leq m$, for every time t and every point $p \in \mathcal{M}_t$ there exist $(e_1, \ldots, e_m)$ an orthonormal basis of $T_p\mathcal{M}_t$ and $(e_{m+1}, \ldots, e_{m+k})$ an orthonormal basis of $N_p\mathcal{M}_t$ such that:*

1. *for every $r \leq \frac{k}{2}$ we have*
$$\begin{cases} Je_{m+2r-1} &= \tau_r e_{2r-1} + \nu_r e_{m+2r}, \\ Je_{m+2r} &= \tau_r e_{2r} - \nu_r e_{m+2r-1}, \end{cases} \quad (3.1.3)$$
with $\tau_r, \nu_r \in \mathbb{R}$ and $\tau_r^2 + \nu_r^2 = 1$.
2. *If k is odd $Je_{m+k} = e_k$.*
3. *Finally the remaining vectors are*
$$e_{k+1}, e_{k+2} = Je_{k+1}, \ldots, e_{m-1}, e_m = Je_{m-1}. \quad (3.1.4)$$

Proof. For every time t and point $p \in \mathcal{M}_t$ the function

$$\begin{array}{rccl} \varphi: & N_p\mathcal{M}_t \times N_p\mathcal{M}_t & \to & \mathbb{R} \\ & (X,Y) & \mapsto & \varphi(X,Y) := g(JX,Y) \end{array}$$

is a skew-symmetric bilinear form. It is a well-known fact that there is an orthonormal basis of $N_p\mathcal{M}_t$ $(e_{m+1}, \ldots, e_{m+k})$ such that with respect to this basis φ is represented by the matrix

$$M_\varphi = \begin{pmatrix} \begin{matrix} 0 & \nu_1 \\ -\nu_1 & 0 \end{matrix} & 0 & \cdots & 0 \\ 0 & \begin{matrix} 0 & \nu_2 \\ -\nu_2 & 0 \end{matrix} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \begin{matrix} 0 & \nu_p \\ -\nu_p & 0 \end{matrix} \end{pmatrix} \qquad \text{if } k = 2p,$$

$$M_\varphi = \begin{pmatrix} \begin{matrix} 0 & \nu_1 \\ -\nu_1 & 0 \end{matrix} & 0 & \cdots & 0 & 0 \\ 0 & \begin{matrix} 0 & \nu_2 \\ -\nu_2 & 0 \end{matrix} & & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \begin{matrix} 0 & \nu_p \\ -\nu_p & 0 \end{matrix} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \qquad \text{if } k = 2p+1.$$

This means that for every $r \leq \frac{k}{2}$

$$\begin{cases} Je_{m+2r-1} & = & \mu_r T_{2r-1} + \nu_r e_{m+2r}, \\ Je_{m+2r} & = & \bar{\mu}_r T_{2r} - \nu_r e_{m+2r-1}, \end{cases} \tag{3.1.5}$$

where the T_i are unit vectors of $T_p\mathcal{M}_t$ and $\mu_r, \bar{\mu}_r \in \mathbb{R}$. Moreover if k is odd statement 2 follows easily. Since e_{m+i} is a unit vector for every i, we have that for every r

$$\mu_r^2 = 1 - \nu_r^2 = \bar{\mu}_r^2,$$

so, up to change the sign of T_{2r}, we get $\mu_r = \bar{\mu}_r =: \tau_r$. When $\tau_r = 0$ we can choose indipendently T_{2r-1} and define $T_{2r} = -JT_{2r-1}$, hence, in particular T_{2r-1} and T_{2r} are orthogonal. In general, since $(e_{m+1}, \ldots, e_{m+k})$ is an orthonormal basis, from equations (3.1.5), we have for any $i \neq j$

$$g(T_i, T_j) = 0.$$

Therefore we define for every $i = 1, \ldots, k$ $e_i = T_i$. Finally we can complete the basis of $T_p\mathcal{M}_t$ in an orthonormal way choosing

$$e_{k+1}, e_{k+2} = Je_{k+1}, \ldots, e_{m-1}, e_m = Je_{m-1}.$$

$\square$

In equations (3.1.3) it is meant that if for some r $\nu_r = 0$ we can choose indipendentely e_{m+2r-1} and e_{m+2r}, while if $\tau_r = 0$ we can choose indipendentely a tangent vector e_{2r-1} and define $e_{2r} = -Je_{2r-1}$.

Since $J^2 = -id$, from (3.1.3) it follows easily that when we use frames of kind B2

$$\begin{cases} Je_{2r-1} & = & -\nu_r e_{2r} - \tau_r e_{m+2r-1}, \\ Je_{2r} & = & \nu_r e_{2r-1} - \tau_r e_{m+2r}. \end{cases} \tag{3.1.6}$$

When we use frames of type B2, in general, there is no reason for the condition (3.1.1) is verified, then we set

$$H = \sum_\alpha H^\alpha e_\alpha.$$

Obviously when $k = 1$ these constructions are trivial: there is an unique (up to sign) normal unit vector e_{2n}, H is a multiple of such vector and $e_1 = Je_{2n}$ is a tangent vector. Then in the special case of hypersurfaces we can choose a basis that is at the same time of type B1 and B2.

When $k \geq 2$, we introduce the following notation taken from [AB]

$$R_1 := \sum_{\alpha,\beta} \left(\sum_{i,j} h_{ij}^\alpha h_{ij}^\beta \right)^2 + \sum_{i,j,\alpha,\beta} \left[\sum_p h_{ip}^\alpha h_{jp}^\beta - h_{ip}^\beta h_{jp}^\alpha \right]^2,$$

$$R_2 := \sum_{i,j} \left(\sum_{\alpha} H^{\alpha} h^{\alpha}_{ij} \right)^2 .$$

If we use a normal frame of kind B1, it is easily checked that

$$R_2 = \begin{cases} |\overset{\circ}{h}_1|^2 |H|^2 + \dfrac{1}{m} |H|^4 & \text{if } H \neq 0 \\ 0 & \text{if } H = 0. \end{cases} \tag{3.1.7}$$

The following result, proved in [AB, §3] and in [Ba, §5.2], is useful in the estimation of the reaction terms occurring in the evolution equations of lemma 2.4.3. It only uses the algebraic properties of R_1 and R_2 and is independent on the flow.

Lemma 3.1.2 *At a point where $H \neq 0$ we have, for any $a \in \mathbb{R}$*

$$\begin{aligned} 2R_1 - 2aR_2 \;\; \leq \;\; & 2|\overset{\circ}{h}_1|^4 - 2\left(a - \frac{2}{m}\right)|\overset{\circ}{h}_1|^2 |H|^2 - \frac{2}{m}\left(a - \frac{1}{m}\right)|H|^4 \\ & + 8|\overset{\circ}{h}_1|^2 |\overset{\circ}{h}_-|^2 + 3|\overset{\circ}{h}_-|^4. \end{aligned}$$

In addition, if $a > 1/m$ and if $b \in \mathbb{R}$ is such that $|A|^2 = a|H|^2 + b$, we have

$$\begin{aligned} 2R_1 - 2aR_2 \;\; \leq \;\; & \left(6 - \frac{2}{ma - 1}\right)|\overset{\circ}{A}|^2 |\overset{\circ}{h}_-|^2 - 3|\overset{\circ}{h}_-|^4 \\ & + \frac{2mab}{ma - 1}|\overset{\circ}{h}_1|^2 + \frac{4b}{ma - 1}|\overset{\circ}{h}_-|^2 - \frac{2b^2}{ma - 1}. \end{aligned}$$

We want to prove that the pinching condition of theorem 3.0.3 is preserved by the flow. The structure of the proof is the same for any codimension: we compute the evolution equation of the function $Q = |A|^2 - a\,|H|^2 - b$ showing that, if $Q(x,t) = 0$ at some point $(x,t) \in \mathcal{M} \times [0, T_{max}[$, then $\left(\dfrac{\partial}{\partial t} - \Delta\right) Q \leq 0$ at this point. By the maximum principle, the result will follow. Since the evolution equation is much simpler for hypersurfaces we exhibit two different proofs, one for hypersurfaces and one for higher codimension. In the latter case the two kind of basis introduced above are essential.

Proposition 3.1.3 *Let $\mathcal{M}_0$ be a closed hypersurface of $\mathbb{CP}^n$, with $n \geq 3$, then the pinching condition*

$$|A|^2 < a\,|H|^2 + b$$

is preserved by the mean curvature flow for every

$$\frac{1}{m} < a \leq \frac{4}{3m}, \qquad 0 < b \leq \begin{cases} \min\left(2,\ \dfrac{2}{a}(ma - 1)\right) & \text{if } n = 3, \\ \dfrac{2}{a}(ma - 1) & \text{if } n \geq 4. \end{cases}$$

where $m = 2n - 1$ is the dimension of $\mathcal{M}_0$.

Proof. Lemma 2.4.4 gives

$$
\begin{aligned}
\frac{\partial}{\partial t}Q &= \Delta Q - 2\left(|\nabla A|^2 - a|\nabla H|^2\right) + 2\left(|A|^2 - a|H|^2\right)\left(|A|^2 + \bar{r}\right)\\
&\quad -4\left(h_{ij}h_j{}^p\bar{R}_{pli}{}^l - h^{ij}h^{lp}\bar{R}_{pilj}\right)\\
&= \Delta Q - 2\left(|\nabla A|^2 - a|\nabla H|^2\right) + 2Q\left(|A|^2 + \bar{r}\right) + 2b\left(|A|^2 + \bar{r}\right)\\
&\quad -4\left(h_{ij}h_j{}^p\bar{R}_{pli}{}^l - h^{ij}h^{lp}\bar{R}_{pilj}\right),
\end{aligned}
\tag{3.1.8}
$$

where $\bar{r} = \bar{R}ic(\nu,\nu) = 2(n+1)$. By lemma 2.4.5 and our hypothesis on a,

$$
|\nabla A|^2 - a|\nabla H|^2 \geq \left(\frac{3}{m+2} - \frac{4}{3m}\right)|\nabla H|^2 \geq 0.
$$

Then the gradient terms in equation (3.1.8) are non-positive and it suffices to consider the contribution of the reaction terms. Fix an orthonormal basis tangent to $\mathcal{M}_t$ that diagonalizes the second fundamental form and call $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_m$ its eigenvalues. Recalling that $\bar{K} \geq 1$, we get

$$
\begin{aligned}
-4\left(h_{ij}h_j{}^p\bar{R}_{pli}{}^l - h^{ij}h^{lp}\bar{R}_{pilj}\right) &= -4\left(\lambda_j^2\delta_{ij}\delta_{jp}\bar{R}_{plil} - \lambda_j\lambda_l\delta_{ij}\delta_{lp}\bar{R}_{pilj}\right)\\
&= -4\sum_{j,l}\left(\lambda_j^2 - \lambda_j\lambda_l\right)\bar{R}_{jljl}\\
&= -4\sum_{j,l}\left(\lambda_j^2 - \lambda_j\lambda_l\right)\bar{K}_{jl}\\
&= -2\sum_{j,l}\left(\lambda_j - \lambda_l\right)^2\bar{K}_{jl}\\
&\leq -2\sum_{j,l}\left(\lambda_j - \lambda_l\right)^2 = -4m|\mathring{A}|^2.
\end{aligned}
\tag{3.1.9}
$$

With our hypothesis on b

$$
2b\left(|A|^2 + \bar{r}\right) - 4m\left(|A|^2 - \frac{1}{m}|H|^2\right) \leq -2\lambda Q \tag{3.1.10}
$$

holds with $\lambda = max\left(\frac{2}{a}, \bar{r}\right)$. Putting together all these informations, we have

$$
\frac{\partial}{\partial t}Q \leq \Delta Q + 2Q\left(|A|^2 + \bar{r} - \lambda\right).
$$

Then $\frac{\partial}{\partial t}Q \leq \Delta Q$ in the points where $Q = 0$ and so, by the maximum principle, the thesis follows. $\square$

Proposition 3.1.4 *Let $\mathcal{M}_0$ be a closed submanifold of $\mathbb{CP}^n$ of dimension m and codimension $2 \le k < \frac{2n-3}{5}$, then the pinching condition*

$$|A|^2 < a\,|H|^2 + b$$

is preserved by the flow for any

$$\frac{1}{m} < a \le \frac{4}{3m}, \quad 0 < b \le (m-3-4k)\left(1 - \frac{1}{ma}\right).$$

Proof. By lemma 2.4.3 we have

$$\frac{\partial}{\partial t} Q = \Delta Q - 2(|\nabla A|^2 - a\,|\nabla H|^2) + 2R_1 - 2aR_2 + P_a, \tag{3.1.11}$$

where $P_a = I + II + III$, with

$$\begin{aligned} I &= 4\sum_{i,j,p,q} \bar{R}_{ipjq}\left(\sum_{\alpha} h^{\alpha}_{pq} h^{\alpha}_{ij}\right) - 4\sum_{j,s,p} \bar{R}_{sjsp}\left(\sum_{i,\alpha} h^{\alpha}_{pi} h^{\alpha}_{ij}\right),\\ II &= 2\sum_{s,\alpha,\beta} \bar{R}_{s\alpha s\beta}\left(\sum_{ij} h^{\alpha}_{ij} h^{\beta}_{ij}\right) - 2a\sum_{s,\alpha,\beta} \bar{R}_{s\alpha s\beta} H^{\alpha} H^{\beta},\\ III &= -8\sum_{j,p,\alpha,\beta} \bar{R}_{jp\alpha\beta}\left(\sum_{i} h^{\alpha}_{ip} h^{\beta}_{ij}\right). \end{aligned}$$

By lemma 2.4.5

$$|\nabla A|^2 - a\,|\nabla H|^2 \ge \left(\frac{3}{m+2} - \frac{4}{3m}\right)|\nabla H|^2 \ge 0.$$

Then the gradient terms in equation (3.1.11) are non-positive and it suffices to consider the contribution of the reaction terms. Let us divide into two case: $H = 0$ and $H \neq 0$. Consider first a point where $Q = 0$ and $H \neq 0$. To estimate I, we fix α and choose a tangent basis $(\widetilde{e}_1, ..., \widetilde{e}_m)$, not necessarily of kind B1 or B2, that diagonalizes h^{α}, i.e. $h^{\alpha}_{ij} = \lambda^{\alpha}_i \delta_{ij}$. Likewise in estimate (3.1.9) we have

$$\begin{aligned} &4\sum_{i,j,p,q} \bar{R}_{ipjq} h^{\alpha}_{pq} h^{\alpha}_{ij} - 4\sum_{j,k,p} \bar{R}_{kjkp}\left(\sum_{i} h^{\alpha}_{pi} h^{\alpha}_{ij}\right)\\ &= 4\sum_{i,p} \bar{R}_{ipip}(\lambda^{\alpha}_i \lambda^{\alpha}_p - (\lambda^{\alpha}_i)^2)\\ &= -2\sum_{i,p} \bar{K}_{ip}(\lambda^{\alpha}_i - \lambda^{\alpha}_p)^2 \le -4m|\mathring{h}^{\alpha}|^2. \end{aligned}$$

Hence we get

$$I \le -4m|\mathring{A}|^2. \tag{3.1.12}$$

A basis satisfying of type B2 is useful for estimate the terms II and III. We recall that the curvature tensor of the Fubini-Study metric, for every X, Y, Z and W tangent vector field of $\mathbb{CP}^n$, is

$$\begin{aligned}\bar{R}(X,Y,Z,W) &= g_{FS}(X,Z)g_{FS}(Y,W) - g_{FS}(X,W)g_{FS}(Y,Z) \\ &\quad g_{FS}(X,JZ)g_{FS}(Y,JW) - g_{FS}(X,JW)g_{FS}(Y,JZ) \\ &\quad +2g_{FS}(X,JY)g_{FS}(Z,JW)\end{aligned} \quad (3.1.13)$$

In order to study the term II, note that, with our choice of the basis, we have that $\bar{R}_{s\alpha s\beta} = 0$ for any s if $\alpha \neq \beta$. Otherwise we have

$$\bar{R}_{s\alpha s\alpha} = 1 + 3g_{FS}(e_s, Je_\alpha)^2 = 1 + 3\tau_r^2 \delta_{s,m-\alpha},$$

where $\alpha = m + 2r - 1$ or $\alpha = m + 2r$. Since $\tau_r^2 \leq 1$ and $a \geq \frac{1}{m}$, we have

$$\begin{aligned}II &= 2\sum_{r\leq\frac{k}{2}}(m+3\tau_r^2)\left(\left|h^{m+2r-1}\right|^2 - a\left|H^{m+2r-1}\right|^2\right) \\ &\quad +2\sum_{r\leq\frac{k}{2}}(m+3\tau_r^2)\left(\left|h^{m+2r}\right|^2 - a\left|H^{m+2r}\right|^2\right) \\ &= 2\sum_{r\leq\frac{k}{2}}(m+3\tau_r^2)\left(\left|\mathring{h}^{m+2r-1}\right|^2 - \left(a-\frac{1}{m}\right)\left|H^{m+2r-1}\right|^2\right) \\ &\quad +2\sum_{r\leq\frac{k}{2}}(m+3\tau_r^2)\left(\left|\mathring{h}^{m+2r}\right|^2 - \left(a-\frac{1}{m}\right)\left|H^{m+2r}\right|^2\right) \\ &\leq 2\sum_{r\leq\frac{k}{2}}(m+3\tau_r^2)\left(\left|\mathring{h}^{m+2r-1}\right|^2 + \left|\mathring{h}^{m+2r}\right|^2\right) \\ &\leq 2(m+3)|\mathring{A}|^2.\end{aligned} \quad (3.1.14)$$

The most complicated term is III. Since $\bar{R}_{jp\alpha\beta}$ is anti-symmetric for j,p, while h_{jp}^α is symmetric, we have

$$\begin{aligned}III &= -8\sum_{j,p,\alpha,\beta}\bar{R}_{jp\alpha\beta}\left(\sum_i h_{ip}^\alpha h_{ij}^\beta\right) \\ &= -8\sum_{j,p,\alpha,\beta}\bar{R}_{jp\alpha\beta}\left(\sum_i \mathring{h}_{ip}^\alpha \mathring{h}_{ij}^\beta\right)\end{aligned}$$

By the simmetries of any curvature tensor, $\bar{R}_{jp\alpha\beta} = 0$ if $j = p$ or $\alpha = \beta$. First fix α and β coupled by (3.1.3). We can assume $\alpha = m + 2r - 1$ and $\beta = m + 2r$ for some r (if $\alpha = m+2r$ and $\beta = m+2r-1$, we have $\bar{R}_{jp\alpha\beta} = -\bar{R}_{jp\beta\alpha}$, so we fall in the previous case). We get

$$\begin{aligned}\bar{R}_{jp\alpha\beta} &= \tau_r^2\left(\delta_{j,2r-1}\delta_{p,2r} - \delta_{j,2r}\delta_{p,2r-1}\right) \\ &\quad -2\nu_r g_{FS}(e_j, Je_p),\end{aligned}$$

and

$$g_{FS}(e_j, Je_p) = \begin{cases} -\nu_s & \text{if} \quad j = 2s, \qquad p = 2s-1, \qquad s \leq \frac{k}{2}; \\ \nu_s & \text{if} \quad j = 2s-1, \qquad p = 2s, \qquad s \leq \frac{k}{2}; \\ 1 & \text{if} \quad j = k+2s, \qquad p = k+2s-1, \quad s \leq \frac{m-k}{2}; \\ -1 & \text{if} \quad j = k+2s-1, \quad p = k+2s, \qquad s \leq \frac{m-k}{2}; \\ 0 & \quad \text{otherwise} \end{cases}$$

If α and β are not coupled by (3.1.3), there are two index $r \neq s$ such that α is (or is coupled with) e_{m+2r} and β is (or is coupled with) e_{m+2s}. In this case we have

$$\bar{R}_{jp\alpha\beta} = \tau_r \tau_s \left(\delta_{j,\alpha-m} \delta_{p,\beta-m} - \delta_{j,\beta-m} \delta_{p,\alpha-m} \right).$$

Using what we have just found and summing all similar terms we have

$$\begin{aligned} III \quad = \quad & 16 \sum_r \left(2\nu_r^2 - \tau_r^2\right) \sum_i \left(\mathring{h}_{i\,2r}^{m+2r-1} \mathring{h}_{i\,2r-1}^{m+2r} - \mathring{h}_{i\,2r-1}^{m+2r-1} \mathring{h}_{i\,2r}^{m+2r} \right) \\ & -8 \sum_{r \neq s \leq \frac{k}{2}} \tau_r \tau_s \sum_i \left(\mathring{h}_{i\,2s}^{m+2r} \mathring{h}_{i\,2r}^{m+2s} - \mathring{h}_{i\,2r}^{m+2r} \mathring{h}_{i\,2s}^{m+2s} \right) \\ & -16 \sum_{r \neq s \leq \frac{k}{2}} \tau_r \tau_s \sum_i \left(\mathring{h}_{i\,2s-1}^{m+2r} \mathring{h}_{i\,2r}^{m+2s-1} - \mathring{h}_{i\,2r}^{m+2r} \mathring{h}_{i\,2s-1}^{m+2s-1} \right) \\ & -8 \sum_{r \neq s \leq \frac{k}{2}} \tau_r \tau_s \sum_i \left(\mathring{h}_{i\,2s-1}^{m+2r-1} \mathring{h}_{i\,2r-1}^{m+2s-1} - \mathring{h}_{i\,2r-1}^{m+2r-1} \mathring{h}_{i\,2s-1}^{m+2s-1} \right) \\ & +32 \sum_{r \neq s \leq \frac{k}{2}} \nu_r \nu_s \sum_i \left(\mathring{h}_{i\,2s}^{m+2r-1} \mathring{h}_{i\,2s-1}^{m+2r} - \mathring{h}_{i\,2s-1}^{m+2r-1} \mathring{h}_{i\,2s}^{m+2r} \right) \\ & +32 \sum_r \nu_r \sum_{s \leq \frac{m-k}{2}} \sum_i \left(\mathring{h}_{i\,k+2s-1}^{m+2r-1} \mathring{h}_{i\,k+2s}^{m+2r} - \mathring{h}_{i\,k+2s}^{m+2r-1} \mathring{h}_{i\,k+2s-1}^{m+2r} \right). \end{aligned}$$

Obviously $III \leq |III|$. Using triangle inequality and Young's inequality on many terms and the fact that for any r and s

$$\begin{cases} |2\nu_r^2 - \tau_r^2| \leq 2, \\ |\tau_r \tau_s| \leq 1, \\ |\nu_r \nu_s| \leq 1, \\ |\nu_r| \leq 1. \end{cases}$$

we have:

$$\begin{aligned}
III \quad \leq \quad & 16\sum_{i,r}\left(\left|\mathring{h}^{m+2r-1}_{i\;2r}\right|^2+\left|\mathring{h}^{m+2r}_{i\;2r-1}\right|^2+\left|\mathring{h}^{m+2r-1}_{i\;2r-1}\right|^2+\left|\mathring{h}^{m+2r}_{i\;2r}\right|^2\right)\\
& +4\sum_{i,r\neq s\leq\frac{k}{2}}\left(\left|\mathring{h}^{m+2r}_{i\;2s}\right|^2+\left|\mathring{h}^{m+2s}_{i\;2r}\right|^2+\left|\mathring{h}^{m+2r}_{i\;2r}\right|^2+\left|\mathring{h}^{m+2s}_{i\;2s}\right|^2\right)\\
& +8\sum_{i,r\neq s\leq\frac{k}{2}}\left(\left|\mathring{h}^{m+2r}_{i\;2s-1}\right|^2+\left|\mathring{h}^{m+2s-1}_{i\;2r}\right|^2+\left|\mathring{h}^{m+2r}_{i\;2r}\right|^2+\left|\mathring{h}^{m+2s-1}_{i\;2s-1}\right|^2\right)\\
& +4\sum_{i,r\neq s\leq\frac{k}{2}}\left(\left|\mathring{h}^{m+2r-1}_{i\;2s-1}\right|^2+\left|\mathring{h}^{m+2s-1}_{i\;2r-1}\right|^2+\left|\mathring{h}^{m+2r-1}_{i\;2r-1}\right|^2+\left|\mathring{h}^{m+2s-1}_{i\;2s-1}\right|^2\right)\\
& +16\sum_{i,r\neq s\leq\frac{k}{2}}\left(\left|\mathring{h}^{m+2r-1}_{i\;2s}\right|^2+\left|\mathring{h}^{m+2r}_{i\;2s-1}\right|^2+\left|\mathring{h}^{m+2r-1}_{i\;2s-1}\right|^2+\left|\mathring{h}^{m+2r}_{i\;2s}\right|^2\right)\\
& +16\sum_{i,r,s\leq\frac{m-k}{2}}\left(\left|\mathring{h}^{m+2r-1}_{i\;k+2s-1}\right|^2+\left|\mathring{h}^{m+2r}_{i\;k+2s}\right|^2+\left|\mathring{h}^{m+2r-1}_{i\;k+2s}\right|^2+\left|\mathring{h}^{m+2r}_{i\;k+2s-1}\right|^2\right).
\end{aligned}$$

Note that if $k=2$, there are no index $r\neq s$ then, in the expressions above many sums are empty and we easily find that

$$III \leq 16|\mathring{A}|^2$$

If $k>2$, collecting similar terms we found

$$\begin{aligned}
III \quad \leq \quad & \sum_{i,r}\left(16\left|\mathring{h}^{m+2r-1}_{i\;2r}\right|^2+16\left|\mathring{h}^{m+2r}_{i\;2r-1}\right|^2+8k\left|\mathring{h}^{m+2r}_{i\;2r}\right|^2+8k\left|\mathring{h}^{m+2r-1}_{i\;2r-1}\right|^2\right)\\
& +24\sum_{i,r\neq s\leq\frac{k}{2}}\left(\left|\mathring{h}^{m+2r}_{i\;2s}\right|^2+\left|\mathring{h}^{m+2r-1}_{i\;2s}\right|^2+\left|\mathring{h}^{m+2r}_{i\;2s-1}\right|^2+\left|\mathring{h}^{m+2r-1}_{i\;2s-1}\right|^2\right)\\
& +16\sum_{i,r,s\leq\frac{m-k}{2}}\left(\left|\mathring{h}^{m+2r-1}_{i\;k+2s-1}\right|^2+\left|\mathring{h}^{m+2r}_{i\;k+2s}\right|^2+\left|\mathring{h}^{m+2r-1}_{i\;k+2s}\right|^2+\left|\mathring{h}^{m+2r}_{i\;k+2s-1}\right|^2\right)\\
\leq \quad & 8k|\mathring{A}|^2.
\end{aligned}$$

So we can say that in any case

$$III \leq 8k|\mathring{A}|^2. \tag{3.1.15}$$

By (3.1.12), (3.1.14) and (3.1.15), $P_a = I + II + III \leq -2(m-3-4k)|\mathring{A}|^2$ holds. Let $R = 2R_1 - 2aR_2 + P_a$. From now on consider a frame of type B1: lemma 3.1.2 says that

$$\begin{aligned}
R \quad \leq \quad & \left(6-\frac{2}{ma-1}\right)|\mathring{A}|^2|\mathring{h}_-|^2+\left(\frac{2mab}{ma-1}-2(m-3-4k)\right)|\mathring{h}_1|^2\\
& -3|\mathring{h}_-|^4+\left(\frac{4b}{ma-1}-2(m-3-4k)\right)|\mathring{h}_-|^2-\frac{2b^2}{ma-1}.
\end{aligned}$$

Since $Q = 0$, we have $|\mathring{A}|^2 \geq b$. Using this estimate in the previous inequality we get:

$$\begin{aligned} R \;\leq\; & \left(\frac{2mab}{ma-1} - 2(m-3-4k)\right)|\mathring{h}_1|^2 \\ & -3|\mathring{h}_-|^4 + \left(\frac{2b}{ma-1} + 6b - 2(m-3-4k)\right)|\mathring{h}_-|^2 - \frac{2b^2}{ma-1}. \end{aligned}$$

Using again $|\mathring{A}|^2 \geq b$ we have that $R \leq R + \lambda|\mathring{A}|^2 - \lambda b$, for any $\lambda > 0$. With our assumpion about b, we can choose $\lambda = -\frac{2mab}{ma-1} + 2(m-3-4k) > 0$. Hence we get

$$R \leq -3|\mathring{h}_-|^4 + 4b|\mathring{h}_-|^2 + 2b(b-m+3+4k)$$

Then, with our choice of b, $R \leq 0$ for any value of $|\mathring{h}_-|^2$. Finally, in the point where $Q = |H|^2 = 0$ we have $|A|^2 = |\mathring{A}|^2 = b$, $R_2 = 0$, by [LL] $2R_1 \leq 3\,|A|^4 = b^2$ and, as before, $P_a \leq -2(m-3+4k)|\mathring{A}|^2 = -2(m-3+4k)b$. Therefore in this case too

$$R \leq 3b^2 - 2(m-3+4k)b < 0$$

for our choice of a and b. By the maximum principle, the thesis follows. □

3.2 Technical lemmata

In this section we collect some technical results essential for the following. We consider again the general pinching condition (3.0.2) preserved by the flow. We are interested to study the asymptotic behaviour of $|\mathring{A}|^2$. For hypersurfaces

$$|\mathring{A}|^2 = |A|^2 - \frac{1}{m}\,|H|^2 = \frac{1}{m}\sum_{i<j}(\lambda_i - \lambda_j)^2,$$

so it measure how far the eigenvalues of the second fundamental form diverge from each other.

Since $\mathcal{M}_0$ is compact, there is an $\varepsilon > 0$ small enough such that $\mathcal{M}_0$ satisfies

$$|A|^2 \leq \left(a\,|H|^2 + b\right)(1-\varepsilon). \tag{3.2.1}$$

For short, let $a_\varepsilon = a(1-\varepsilon)$ and $b_\varepsilon = b(1-\varepsilon)$.
For technical reasons it is more convenient to work initially with the auxiliary function $f_\sigma := \dfrac{|\mathring{A}|^2}{W^{1-\sigma}}$ where σ is a positive constant small enough and $W = \alpha\,|H|^2 + \beta$ for some constants

$$\begin{cases} max\left(\frac{a_\varepsilon b_\varepsilon}{b_\varepsilon + \bar{r}}, \frac{a_\varepsilon}{m}, a_\varepsilon - \frac{1}{m}, \frac{5(m-3-4k)}{3m(7m-12-16k)}, \frac{2(m-3-4k)}{3m(m+6+8k)}\right) < \alpha < \frac{3}{m+2} - \frac{1}{m}, \\ \beta = b_\varepsilon. \end{cases} \tag{3.2.2}$$

Note that the interval of definition for α is not empty, as we can see with trivial computations. First we derive the evolution equation for f_σ.

Proposition 3.2.1 *There is a* σ_1 *depending only on* $\mathcal{M}_0$ *that for all* $0 \leq \sigma \leq \sigma_1$

$$\frac{\partial}{\partial t} f_\sigma \leq \Delta f_\sigma + \frac{2\alpha(1-\sigma)}{W} \left\langle \nabla f_\sigma, \nabla |H|^2 \right\rangle - 2C_1 W^{\sigma-1} |\nabla H|^2 + 2\sigma |A|^2 f_\sigma - 2C_2 f_\sigma, \quad (3.2.3)$$

for some constants $C_1 > 0$ *and* $C_2 > 0$.

Proof. We start this proof computing Δf_σ: $f_\sigma = f_0 W^\sigma$ and so $\Delta f_\sigma = f_0 \Delta W^\sigma + W^\sigma \Delta f_0 + 2\langle \nabla f_0, \nabla W^\sigma \rangle$. Results

$$\begin{aligned}
2\langle \nabla f_0, \nabla W^\sigma \rangle &= 2\alpha\sigma W^{\sigma-1} \left\langle \nabla f_0, \nabla |H|^2 \right\rangle \\
&= \frac{2\alpha\sigma}{W} \left(\left\langle \nabla f_\sigma, \nabla |H|^2 \right\rangle - f_0 \sigma W^{\sigma-1} \alpha \left| \nabla |H|^2 \right|^2 \right) \\
&= \frac{2\alpha\sigma}{W} \left\langle \nabla f_\sigma, \nabla |H|^2 \right\rangle - 2\alpha^2\sigma^2 \frac{f_\sigma}{W^2} \left| \nabla |H|^2 \right|^2 .
\end{aligned}$$

$$\begin{aligned}
\Delta W^\sigma &= \nabla_i \nabla_i W^\sigma = \nabla_i \left(\alpha\sigma W^{\sigma-1} \nabla_i |H|^2 \right) \\
&= \alpha\sigma \left((\sigma-1)\alpha W^{\sigma-2} \left| \nabla |H|^2 \right|^2 + W^{\sigma-1} \Delta |H|^2 \right) \\
&= \alpha\sigma W^{\sigma-1} \Delta |H|^2 - \alpha^2 \sigma (1-\sigma) W^{\sigma-2} \left| \nabla |H|^2 \right|^2 .
\end{aligned}$$

$$\begin{aligned}
\Delta f_0 &= \nabla_i \nabla_i \left(\frac{|\mathring{A}|^2}{W} \right) \\
&= \nabla_i \left(\frac{\nabla_i |\mathring{A}|^2}{W} - \frac{|\mathring{A}|^2}{W^2} \nabla_i W \right) \\
&= \frac{\Delta |\mathring{A}|^2}{W} - \frac{2}{W^2} \left\langle \nabla_i |\mathring{A}|^2, \nabla_i W \right\rangle - \frac{1}{W^2} |\mathring{A}|^2 \Delta W + \frac{2|\mathring{A}|^2}{W^3} |\nabla W|^2 \\
&= \frac{\Delta |\mathring{A}|^2}{W} - \frac{2\alpha}{W} \left\langle \nabla_i f_0, \nabla_i |H|^2 \right\rangle - \frac{\alpha}{W^2} |\mathring{A}|^2 \Delta |H|^2 .
\end{aligned}$$

$$\nabla_i f_\sigma = W^\sigma \nabla_i f_0 + f_0 \nabla_i W^\sigma = W^\sigma \nabla_i f_0 + \alpha\sigma f_0 W^\sigma .$$

$$-\frac{2\alpha}{W} \left\langle W^\sigma \nabla_i f_0, \nabla_i |H|^2 \right\rangle = -\frac{2\alpha}{W} \left\langle \nabla_i f_\sigma, \nabla_i |H|^2 \right\rangle + 2\alpha^2 \sigma \frac{f_\sigma}{W^2} \left| \nabla |H|^2 \right| .$$

Then

$$
\begin{aligned}
\Delta f_\sigma &= f_0 \Delta W^\sigma + W^\sigma \Delta f_0 + 2\left\langle \nabla f_0, \nabla W^\sigma \right\rangle \\
&= f_0 \left(\alpha\sigma W^{\sigma-1} \Delta |H|^2 - \alpha^2 \sigma(1-\sigma) W^{\sigma-2} \left|\nabla |H|^2\right|^2 \right) \\
&\quad + W^\sigma \left(\frac{\Delta |\mathring{A}|^2}{W} - \frac{2\alpha}{W} \left\langle \nabla_i f_0, \nabla_i |H|^2 \right\rangle - \frac{\alpha}{W^2} |\mathring{A}|^2 \Delta |H|^2 \right) \\
&\quad + \frac{2\alpha\sigma}{W} \left\langle \nabla f_\sigma, \nabla |H|^2 \right\rangle - 2\alpha^2\sigma^2 \frac{f_\sigma}{W^2} \left|\nabla |H|^2\right|^2 \\
&= f_0 \left(\alpha\sigma W^{\sigma-1} \Delta |H|^2 - \alpha^2 \sigma(1-\sigma) W^{\sigma-2} \left|\nabla |H|^2\right|^2 \right) \\
&\quad + W^\sigma \left(\frac{\Delta |\mathring{A}|^2}{W} - \frac{\alpha}{W^2} |\mathring{A}|^2 \Delta |H|^2 \right) - \frac{2\alpha}{W} \left\langle \nabla_i f_\sigma, \nabla_i |H|^2 \right\rangle + 2\alpha^2 \sigma \frac{f_\sigma}{W^2} \left|\nabla |H|^2\right|^2 \\
&\quad + \frac{2\alpha\sigma}{W} \left\langle \nabla f_\sigma, \nabla |H|^2 \right\rangle - 2\alpha^2\sigma^2 \frac{f_\sigma}{W^2} \left|\nabla |H|^2\right|^2 \\
&= W^{\sigma-1} \Delta |\mathring{A}|^2 - \alpha(1-\sigma) \frac{f_\sigma}{W} \Delta |H|^2 - \frac{2\alpha(1-\sigma)}{W} \left\langle \nabla_i f_\sigma, \nabla_i |H|^2 \right\rangle \\
&\quad + \alpha^2 \sigma(1-\sigma) \frac{f_\sigma}{W^2} \left|\nabla |H|^2\right|^2 .
\end{aligned} \tag{3.2.4}
$$

Moreover

$$
\begin{aligned}
\frac{\partial}{\partial t} f_\sigma &= W^\sigma \frac{\partial}{\partial t} f_0 + f_0 \frac{\partial}{\partial t} W^\sigma \\
&= W^\sigma \left(\frac{1}{W} \frac{\partial}{\partial t} |\mathring{A}|^2 - \frac{\alpha |\mathring{A}|^2}{W^2} \frac{\partial}{\partial t} |H|^2 \right) + \alpha\sigma W^{\sigma-1} f_0 \frac{\partial}{\partial t} |H|^2 \\
&= W^{\sigma-1} \frac{\partial}{\partial t} |\mathring{A}|^2 - \alpha(1-\sigma) \frac{f_\sigma}{W} \frac{\partial}{\partial t} |H|^2 .
\end{aligned}
$$

Consider first the case of $k \geq 2$. From lemma 2.4.3 and the definition of the curvature tensor $\bar{R}$, using frames of kind B2 is easy to find that

$$
\begin{aligned}
\frac{\partial}{\partial t} |H|^2 &= \Delta |H|^2 - 2|\nabla H|^2 + 2R_2 + 2\sum_r (m + 3\tau_r^2) \left(\left|H^{m+2r-1}\right|^2 + \left|H^{m+2r}\right|^2 \right) \\
&\geq \Delta |H|^2 - 2|\nabla H|^2 + 2R_2 + 2m|H|^2 .
\end{aligned} \tag{3.2.5}
$$

Moreover, by lemma 2.4.3,

$$
\frac{\partial}{\partial t} |\mathring{A}|^2 = \Delta |\mathring{A}|^2 - 2\left(|\nabla A|^2 - \frac{1}{m} |\nabla H|^2 \right) + 2\left(R_1 - \frac{1}{m} R_2 \right) + P_{\frac{1}{m}}
$$

holds, where, like in the proof of proposition 3.1.4,

$$
P_{\frac{1}{m}} \leq -2(m - 3 - 4k)|\mathring{A}|^2 .
$$

Then

$$\begin{aligned}\frac{\partial}{\partial t} f_\sigma &\leq W^{\sigma-1}\left(\Delta|\mathring{A}|^2 - 2\left(|\nabla A|^2 - \frac{1}{m}|\nabla H|^2\right)\right)\\ &\quad + W^{\sigma-1}\left(2\left(R_1 - \frac{1}{m}R_2\right) - 2(m-3-4k)|\mathring{A}|^2\right)\\ &\quad - \alpha(1-\sigma)\frac{f_\sigma}{W}\left(\Delta|H|^2 - 2|\nabla H|^2 + 2R_2 + 2m|H|^2\right).\end{aligned}$$

Using the expression found previously for Δf_σ, for higher codimensions we get

$$\begin{aligned}\frac{\partial}{\partial t} f_\sigma &\leq \Delta f_\sigma + \frac{2\alpha(1-\sigma)}{W}\left\langle \nabla f_\sigma, \nabla |H|^2\right\rangle - 2W^{\sigma-1}|\nabla A|^2\\ &\quad + 2W^{\sigma-1}\left[\frac{1}{m} + f_0(1-\sigma)\alpha\right]|\nabla H|^2\\ &\quad + 2W^{\sigma-1}\left(R_1 - \frac{1}{n}R_2\right) - 2\alpha(1-\sigma)\frac{f_\sigma}{W}R_2\\ &\quad - 2m\alpha(1-\sigma)\frac{f_\sigma}{W}|H|^2 - 2(m-3-4k)W^{\sigma-1}.\end{aligned} \tag{3.2.6}$$

While, by lemma 2.4.4 for hypersurfaces, we have

$$\begin{aligned}\frac{\partial}{\partial t} f_\sigma &= \Delta f_\sigma + \frac{2\alpha(1-\sigma)}{W}\left\langle \nabla f_\sigma, \nabla |H|^2\right\rangle - 2W^{\sigma-1}|\nabla A|^2\\ &\quad + 2W^{\sigma-1}\left[\frac{1}{m} + f_0(1-\sigma)\alpha\right]|\nabla H|^2\\ &\quad + 2\beta\frac{(1-\sigma)}{W}f_\sigma\left(|A|^2 + \bar{r}\right) + 2\sigma f_\sigma\left(|A|^2 + \bar{r}\right)\\ &\quad - 4W^{\sigma-1}\left(h_{ij}h_j^{\ m}\bar{R}_{mli}^{\ \ \ \ l} - h^{ij}h^{lm}\bar{R}_{milj}\right).\end{aligned} \tag{3.2.7}$$

For any k, the choice of α and β gives $0 \leq f_0 < 1$ holds. Hence by Lemma 2.4.5

$$\begin{aligned}-|\nabla A|^2 &+ \left[\frac{1}{m} + f_0(1-\sigma)\alpha\right]|\nabla H|^2\\ &\leq \left(\frac{1}{m} + \alpha\right)|\nabla H|^2 - |\nabla A|^2\\ &\leq \left(\alpha + \frac{1}{m} - \frac{3}{m+2}\right)|\nabla H|^2 = -C_1|\nabla H|^2,\end{aligned} \tag{3.2.8}$$

with $C_1 = \frac{3}{m+2} - \frac{1}{m} - \alpha > 0$.

In order to complete the proof, we need to estimate the reaction terms. For hypersurfaces we have

$$R := 2\beta\frac{(1-\sigma)}{W}f_\sigma\left(|A|^2 + \bar{r}\right) + 2\sigma\bar{r}f_\sigma - 4W^{\sigma-1}\left(h_{ij}h_j^{\ m}\bar{R}_{mli}^{\ \ \ \ l} - h^{ij}h^{lm}\bar{R}_{milj}\right).$$

Using inequality (3.1.9) we have

$$R \leq 2f_\sigma\left[\beta(1-\sigma)\frac{|A|^2 + \bar{r}}{W} + \sigma\bar{r} - 2m\right]$$

From (3.0.2) and conditions (3.2.2) on α and β, we get

$$|A|^2 + \bar{r} \leq a_\varepsilon |H|^2 + b_\varepsilon + \bar{r} \leq \frac{b_\varepsilon + \bar{r}}{b_\varepsilon} W.$$

Since σ is small enough, we have :

$$R \leq 2\left[\beta(1-\sigma)\frac{b_\varepsilon + \bar{r}}{b_\varepsilon} + \bar{r}\sigma - 2m\right] f_\sigma \leq -2C_2 f_\sigma,$$

for some positive constant C_2. For $k \geq 2$ we have

$$\begin{aligned} 2\alpha\sigma\frac{f_\sigma}{W}R_2 &\leq 2\alpha\sigma\frac{f_\sigma}{W}|A|^2|H|^2 \\ &= 2\sigma f_\sigma |A|^2 - 2\sigma\beta\frac{f_\sigma}{W}|H|^2 \\ &\leq 2\sigma f_\sigma |A|^2 . \end{aligned} \tag{3.2.9}$$

We still have to estimate

$$R := 2W^{\sigma-2}\left[\left(R_1 - \frac{1}{m}R_2\right)W - \alpha|\mathring{A}|^2 R_2 - \alpha m(1-\sigma)|\mathring{A}|^2|H|^2 - (m-3-4k)|\mathring{A}|^2 W\right].$$

Let us call R' what we have in the square brackets. By lemma 3.1.2

$$R_1 - \frac{1}{m}R_2 \leq |\mathring{h}_1|^4 + \frac{1}{m}|\mathring{h}_1|^2|H|^2 + 4|\mathring{h}_1|^2|\mathring{h}_-|^2 + \frac{3}{2}|\mathring{h}_-|^4.$$

Moreover $|\mathring{A}|^2 = |\mathring{h}_1|^2 + |\mathring{h}_-|^2$ and $R_2 = |\mathring{h}_1|^2|H|^2 + \frac{1}{m}|H|^4$, so

$$\begin{aligned} R' \leq\ & 3\alpha|\mathring{h}_1|^2|\mathring{h}_-|^2|H|^2 + \frac{3}{2}\alpha|\mathring{h}_-|^4|H|^2 - \frac{\alpha}{m}|\mathring{h}_-|^2|H|^4 \\ & +\beta|\mathring{h}_1|^4 + 4\beta|\mathring{h}_1|^2|\mathring{h}_-|^2 + \frac{3}{2}\beta|\mathring{h}_-|^4 \\ & + \left(\frac{\beta}{m} - m\alpha(1-\sigma) - \alpha(m-3-4k)\right)|\mathring{h}_1|^2|H|^2 \\ & -\alpha\left(m(1-\sigma) - m + 3 + 4k\right)|\mathring{h}_-|^2|H|^2 \\ & -\beta(m-3-4k)\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2\right). \end{aligned} \tag{3.2.10}$$

Since the pinching condition (3.0.2) holds, we have that

$$|H|^2 \geq \left(a_\varepsilon - \frac{1}{m}\right)^{-1}\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2 - b_\varepsilon\right).$$

Then we have

$$\begin{aligned} R' &= R' + 3\alpha\left(a_\varepsilon - \frac{1}{m}\right)|\mathring{h}_-|^2|H|^4 + 2\beta\left(a_\varepsilon - \frac{1}{m}\right)\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2\right)|H|^2 \\ &\quad -3\alpha\left(a_\varepsilon - \frac{1}{m}\right)|\mathring{h}_-|^2|H|^4 - 2\beta\left(a_\varepsilon - \frac{1}{m}\right)\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2\right)|H|^2 \\ &\leq R' + 3\alpha\left(a_\varepsilon - \frac{1}{m}\right)|\mathring{h}_-|^2|H|^4 + 2\beta\left(a_\varepsilon - \frac{1}{m}\right)\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2\right)|H|^2 \\ &\quad -3\alpha(|\mathring{h}_1|^2 + |\mathring{h}_-|^2 - b_\varepsilon)|\mathring{h}_-|^2|H|^2 - 2\beta\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2\right)\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2 - b_\varepsilon\right) \end{aligned}$$

The hypothesis on a and b gives: $a_\varepsilon \leq \frac{4}{3m}$, then $b_\varepsilon \leq (m-3-4k)\left(1-\frac{1}{ma}\right) \leq \frac{1}{4}(m-3-4k)$.

Using these inequalities, together with (3.2.10), the hypotesis on α and β and the fact that σ is small, say $\sigma < \frac{1}{4}$, we have

$$\begin{aligned} R' &\leq \left(\beta\left(2a_\varepsilon - \frac{1}{m}\right) - \frac{\alpha}{4}(7m-12-16k)\right)|\mathring{h}_1|^2\,|H|^2 \\ &\quad + \left(3\alpha b_\varepsilon + 2\beta\left(a_\varepsilon - \frac{1}{m}\right) - \frac{3}{4}\alpha m - \alpha(m-3-4k)\right)|\mathring{h}_-|^2\,|H|^2 \\ &\quad + \beta\,(2b_\varepsilon - m + 3 + 4k)\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2\right) \\ &\leq -C_2|\mathring{A}|^2 W, \end{aligned}$$

for some positive constant C_2 small enough. Then

$$R = 2W^{\sigma-2}R' \leq -2C_2 f_\sigma.$$

$\square$

Lemma 3.2.2 *We have the estimates:*

1) $$\frac{\partial}{\partial t}|\mathring{A}|^2 \leq \Delta|\mathring{A}|^2 - \frac{4(m-1)}{3m}|\nabla A|^2 + 2|A|^2|\mathring{A}|^2,$$

2) $$\frac{\partial}{\partial t}|H|^4 \geq \Delta|H|^4 - 12|H|^2|\nabla H|^2 + \frac{4}{m}|H|^6.$$

Proof.

1) This inequality follows easily for hypersurfaces from lemma 2.4.4, lemma 2.4.5 and estimate (3.1.9). For higher codimension we use lemma 2.4.3:

$$\begin{aligned} \frac{\partial}{\partial t}|\mathring{A}|^2 &\leq \Delta|\mathring{A}|^2 - 2\left(|\nabla A|^2 - \frac{1}{m}|\nabla H|^2\right) \\ &\quad + 2\left(R_1 - \frac{1}{m}R_2\right) + P_{\frac{1}{m}}. \end{aligned}$$

By lemma 2.4.5 we have

$$-2\left(|\nabla A|^2 - \frac{1}{m}|\nabla H|^2\right) \leq -2\left(1 - \frac{m+2}{3m}\right)|\nabla A|^2 = -\frac{4(m-1)}{3m}|\nabla A|^2.$$

Moreover, using a computation in [Ba], we also have

$$\begin{aligned} R_1 - \frac{1}{m}R_2 &\leq |\mathring{h}_1|^4 + 4|\mathring{h}_1|^2|\mathring{h}_-|^2 + |\mathring{h}_-|^4 + \frac{1}{m}|\mathring{h}_1|^2|H|^2 \\ &\leq 2\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2\right)^2 + \frac{2}{m}|H|^2\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2\right) = 2|\mathring{A}|^2|A|^2. \end{aligned}$$

Finally, like in the proof of proposition 3.1.4

$$P_{\frac{1}{m}} \leq -2(m-3-4k)|\mathring{A}|^2 \leq 0.$$

Then we have the inequality desired.

2) Obviously we have $\frac{\partial}{\partial t}|H|^4 = 2|H|^2 \frac{\partial}{\partial t}|H|^2$. We need once again lemmata 2.4.4 and 2.4.3, together the fact that $|A|^2 \geq \frac{1}{m}|H|^2$ and

$$2R_2 = 2|H|^2 \left(|\mathring{h}_1|^2 + \frac{1}{m}|H|^2 \right) \geq \frac{2}{m}|H|^4 .$$

For hypersurfaces we get

$$\begin{aligned} \frac{\partial}{\partial t}|H|^4 &= \Delta|H|^4 - 2\left|\nabla|H|^2\right|^2 - 4|H|^2|\nabla H|^2 + 4|H|^4(|A|^2 + \bar{r}) \\ &\geq \Delta|H|^4 - 12|H|^2|\nabla H|^2 + \frac{4}{m}|H|^6 , \end{aligned}$$

while for higher codimensions

$$\begin{aligned} \frac{\partial}{\partial t}|H|^4 &= \Delta|H|^4 - 2\left|\nabla|H|^2\right|^2 - 4|H|^2|\nabla H|^2 \\ &\quad +2|H|^2\left(2R_2 + 2\sum_r (m+3\tau_r^2)\left(\left|H^{m+2r-1}\right|^2 + \left|H^{m+2r}\right|^2\right)\right) \\ &\geq \Delta|H|^4 - 12|H|^2|\nabla H|^2 + \frac{4}{m}|H|^6 . \end{aligned}$$

□

Now we need the evolution equation for $|\nabla H|^2$. With the same proof of Corollary 5.10 in [Ba], we have the following result.

Proposition 3.2.3 *There exists a constant C_3 depending only on $\mathcal{M}_0$ such that*

$$\frac{\partial}{\partial t}|\nabla H|^2 \leq \Delta|\nabla H|^2 + C_3(|H|^2+1)|\nabla A|^2 .$$

3.3 Finite maximal time

Our ambient manifold, $\mathbb{CP}^n$, has bounded, but non-costant sectional curvature. This complicates the equations of evolution of many geometric quantities, as we can see in lemmata 2.4.3 and 2.4.4. In order to overcome these difficulties, we have to divide into two cases depending on whether T_{max} is finite or infinite. We star with the case $T_{max} < \infty$. We follow the scheme of the proof used by Huisken [H3] and Baker [Ba] for a similar problem in the sphere, but some results holds only with the initial hypothesis that T_{max} is finite. Throughout this section we consider only the pinching condition (3.0.1). This restriction is essential for the study of the term Z in lemma 3.3.3.

One of the main result of this section is the following

Proposition 3.3.1 *If T_{max} is finite, there are constants $C_0 < \infty$ and $\sigma_0 > 0$ depending only on the initial submanifold $\mathcal{M}_0$ such that for all $0 \le t < T_{max}$ we have*

$$|\mathring{A}|^2 \le C_0(|H|^2 + 1)^{1-\sigma_0}.$$

When $\sigma \neq 0$, the small positive term $2\sigma f_\sigma |A|^2$ in 3.2.1 prevents us from using the maximum principle. To prove proposition 3.3.1 we procced by deriving integral estimates and an iteration procedure exploiting the good negative $|\nabla H|^2$ term by the divergence theorem. First we need a suitable lower bound for Δf_σ.

As in [LXZ], we have

$$\Delta|\mathring{A}|^2 \ge 2|\nabla \mathring{A}|^2 + 2\left\langle \mathring{h}_{ij}, \nabla_i\nabla_j H\right\rangle + 2Z - \gamma W, \tag{3.3.1}$$

where $\gamma = \gamma(n)$ is a positive constant and

$$Z = \sum_{i,j,p,\alpha,\beta} H^\alpha h^\alpha_{ip} h^\beta_{pj} h^\beta_{ij} - \sum_{\alpha,\beta}\left(\sum_{i,j} h^\alpha_{ij} h^\beta_{ij}\right)^2 - \sum_{i,j,\alpha,\beta}\left(\sum_p \left(h^\alpha_{ip}h^\beta_{pj} - h^\alpha_{jp}h^\beta_{ip}\right)\right)^2.$$

To understand the behaviour of Z, note that only for hypersurfaces and for any T_{max}, the pinching condition (3.0.1) implies that the submanifold has positive intrinsic sectional curvature. Like in [H3], we can use the following algebric property: for any square matrix M of order m and eigenvalues $\lambda_1 \le \cdots \le \lambda_m$ we have that for any $i \neq j$

$$\begin{aligned} |M|^2 - \frac{1}{m-1}(trM)^2 &= -2\lambda_i\lambda_j + \left(\lambda_1 + \lambda_j - \frac{trM}{m-1}\right)^2 + \sum_{l\neq i,j}\left(\lambda_l - \frac{trM}{m-1}\right)^2 \\ &\ge -2\lambda_i\lambda_j. \end{aligned} \tag{3.3.2}$$

Proposition 3.3.2 *If $k = 1$ there is a $\mu > 0$ that for any time $0 \le t < T_{max} \le \infty$, the intrinsic sectional curvature of $\mathcal{M}_t$ satisfies*

$$K > \mu W > 0.$$

Proof. Fix $(e_1, ..., e_m)$ a orthonormal tangent basis that diagonalizes the second fundamental form. For any $i \neq j$ the Gauss equation gives

$$K_{ij} = \bar{K}_{ij} + \lambda_i\lambda_j.$$

Moreover, by (3.3.2)

$$|A|^2 - \frac{1}{m-1}|H|^2 \ \ge\ -2\lambda_i\lambda_j \tag{3.3.3}$$

holds. Then we have

$$\begin{aligned} 2K_{ij} &\ge 2 - |A|^2 + \frac{1}{m-1}|H|^2 \\ &\ge \left(\frac{1}{m-1} - a_\varepsilon\right)|H|^2 + 2 - b_\varepsilon \\ &\ge 2\mu\left(\alpha|H|^2 + \beta\right) > 0 \end{aligned}$$

for every $\mu > 0$ small enough. $\square$

We cannot use directly this kind of arguments for higher codimensions because we can not diagonalize all together all the tensors h^α, for $\alpha = m+1, \dots, 2n$, but we will prove at the end of this section that, for big enough time, the sectional curvature of the evolving submanifold becomes positive.

Lemma 3.3.3 *There exist a positive constant ρ depending only on $\mathcal{M}_0$ such that the estimate*

$$Z + 2mb|\mathring{A}|^2 \geq \rho|\mathring{A}|^2 W$$

holds.

Proof. Once again the proof for hypersurfaces is much simpler. Choosing a basis that diagonalize the second fundamental form, using roposition 3.3.2, Gauss equation and $\bar{K} \leq 4$, we have

$$\begin{aligned} Z &= \left(\sum_i \lambda_i\right)\left(\sum_i \lambda_i^3\right) - \left(\sum_i \lambda_i^2\right)^2 \\ &= \sum_{i<j} \lambda_i \lambda_j \left(\lambda_i - \lambda_j\right)^2 \\ &= \sum_{i<j} K_{ij} \left(\lambda_i - \lambda_j\right)^2 - \sum_{i<j} \bar{K}_{ij} \left(\lambda_i - \lambda_j\right)^2 \\ &\geq \mu W |\mathring{A}|^2 - 4m|\mathring{A}|^2 = \mu W - 2bm|\mathring{A}|^2. \end{aligned}$$

For $k \geq 2$ we need to distinguish the cases $H = 0$ and $H \neq 0$. Let us examine first the case $H \neq 0$. Doing the same computation of [Ba], we have

$$Z \geq -\frac{m}{2}|\mathring{h}_1|^4 - \frac{3}{2}|\mathring{h}_-|^4 - \frac{m+2}{2}|\mathring{h}_1|^2|\mathring{h}_-|^2 + \frac{1}{2(m-1)}\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2\right)|H|^2.$$

Since (3.2.1) holds, we have $|H|^2 \geq \dfrac{m(m-1)}{1-m\varepsilon}\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2 - b_\varepsilon\right)$ and then

$$\begin{aligned} Z &\geq -\frac{m}{2}|\mathring{h}_1|^4 - \frac{3}{2}|\mathring{h}_-|^4 - \frac{m+2}{2}|\mathring{h}_1|^2|\mathring{h}_-|^2 \\ &\quad + \frac{m}{2(1-m\varepsilon)}\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2\right)\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2 - b_\varepsilon\right) \\ &= \frac{\varepsilon m^2}{2(1-m\varepsilon)}|\mathring{h}_1|^4 + \frac{m-3-3m\varepsilon}{2(1-m\varepsilon)}|\mathring{h}_-|^4 \\ &\quad + \frac{m-2+m\varepsilon(m+2)}{2(1-m\varepsilon)}|\mathring{h}_1|^2|\mathring{h}_-|^2 - \frac{m}{2(1-m\varepsilon)}b_\varepsilon|\mathring{A}|^2. \end{aligned}$$

Since $2m > \frac{m}{2(1-m\varepsilon)}$, there exist $\rho_1 > 0$ such that

$$Z + 2mb|\mathring{A}|^2 \geq \rho_1|\mathring{A}|^4.$$

By using Young's inequality on various terms of Z we can estimate

$$Z \geq \rho_2 |\mathring{A}|^2 \, |H|^2 - \rho_3 |\mathring{A}|^4,$$

for ρ_2 and ρ_3 positive constants. Combining these two inequalities gives that for any $0 \leq c \leq 1$

$$Z + 2mb|\mathring{A}|^2 \geq c \left(\rho_2 |\mathring{A}|^2 \, |H|^2 - \rho_3 |\mathring{A}|^4 + 2mb|\mathring{A}|^2 \right) + (1-c) \left(\rho_1 |\mathring{A}|^4 \right).$$

Choosing $\bar{c} = \frac{\rho_1}{\rho_1 + \rho_3}$ we have

$$Z + 2mb|\mathring{A}|^2 \geq \bar{c} \left(\rho_2 \, |H|^2 + 2mb \right) |\mathring{A}|^2.$$

Therefore the thesis follows for ρ small enough. When $H = 0$ we have $|A|^2 = |\mathring{A}|^2 \leq b_\varepsilon < b$ and $W = \beta$. By a computation in [LL]

$$Z \geq -\frac{3}{2} \, |A|^4 \geq -\frac{3}{2} b |\mathring{A}|^2.$$

Hence we have

$$Z + 2mb|\mathring{A}|^2 \geq \left(2m - \frac{3}{2} \right) b |\mathring{A}|^2 > \rho |\mathring{A}|^2 W,$$

for some $\rho > 0$ small enough. □

Next we derive the integral estimates.

Proposition 3.3.4 *For any $p \geq 2$ and $\eta > 0$ exists a constant C_4 indipendent from p such that*

$$\begin{aligned} \frac{\rho}{2} \int f_\sigma^p W d\mu \;\; &\leq \;\; (\eta(p+1) + 5) \int W^{\sigma-1} f_\sigma^{p-1} \, |\nabla H|^2 \, d\mu + \frac{p+1}{\eta} \int f_\sigma^{p-2} \, |\nabla f_\sigma|^2 \, d\mu \\ &\quad + 4mb \int f_\sigma^p d\mu + \frac{1}{p} C_4^p. \end{aligned}$$

Proof. Putting equation (3.3.1) into (3.2.4), we get

$$\begin{aligned} \Delta f_\sigma \;\; \geq \;\; & 2W^{\sigma-1} |\mathring{A}|^2 + 2W^{\sigma-1} \left\langle \mathring{h}_{ij}, \nabla_i \nabla_j H \right\rangle + 2W^{\sigma-1} Z - \gamma W^\sigma - \alpha(1-\delta) \frac{f_\sigma}{W} \Delta \, |H|^2 \\ & - \frac{2\alpha(1-\sigma)}{W} \left\langle \nabla_i f_\sigma, \nabla_i \, |H|^2 \right\rangle + \alpha^2 \sigma (1-\sigma) \frac{f_\sigma}{W^2} \left| \nabla \, |H|^2 \right|^2. \end{aligned}$$

The terms $2W^{\sigma-1}|\mathring{A}|^2$ and $\alpha^2\sigma(1-\sigma)\frac{f_\sigma}{W^2} \left| \nabla \, |H|^2 \right|^2$ are positive, so we can omit them. Thanks to Lemma 3.3.3 we have

$$\begin{aligned} \Delta f_\sigma \;\; \geq \;\; & 2W^{\sigma-1} \left\langle \mathring{h}_{ij}, \nabla_i \nabla_j H \right\rangle - \alpha(1-\delta) \frac{f_\sigma}{W} \Delta \, |H|^2 - \frac{2\alpha(1-\sigma)}{W} \left\langle \nabla_i f_\sigma, \nabla_i \, |H|^2 \right\rangle \\ & + 2\rho W^\sigma |\mathring{A}|^2 - 4mbf_\sigma - \gamma W^\sigma. \end{aligned}$$

Note that this is exactly like the case of the sphere, treated in [H3] for hypersurfaces and in [Ba] for higher codimension, except for the last two terms. So we can proceed like the old works for the first terms and then study separately last two terms. We multiply this inequality by f_σ^{p-1} and integrate. By the same computation in [H3] and in [Ba] we have for any $\eta > 0$

$$\begin{aligned}\rho \int f_\sigma^p W\, d\mu \;&\leq\; (\eta(p+1)+5)\int W^{\sigma-1} f_\sigma^{p-1}\left|\nabla H\right|^2 d\mu + \frac{p+1}{\eta}\int f_\sigma^{p-2}\left|\nabla f_\sigma\right|^2 d\mu \\ &\quad +4mb\int f_\sigma^p d\mu + \gamma\int W^\sigma f_\sigma^{p-1} d\mu.\end{aligned}$$

In order to estimate last term we use Young's inequality with conjugate exponents p and $\frac{p}{p-1}$:

$$\gamma W^\sigma f_\sigma^{p-1} \leq \gamma W\left(\frac{r^p}{p}W^{(\sigma-1)p} + \frac{p-1}{p}r^{-\frac{p}{p-1}} f_\sigma^p\right), \quad \forall r > 0.$$

Choose r such that $\frac{p-1}{p}\gamma r^{-\frac{p}{p-1}} = \frac{\rho}{2}$, then r is bounded by a constant indipendent from p. Moreover $\sigma > 0$ but small enough, so we have $(\sigma-1)p+1 < 0$. $W \geq \beta$, β is a positive constant. Then $W^{(\sigma-1)p+1} \leq \beta^{(\sigma-1)p+1}$. Hence we have

$$\begin{aligned}\frac{1}{p}\gamma r^p \int W^{(\sigma-1)p+1} d\mu \;&\leq\; \frac{1}{p}\gamma r^p \beta^{(\sigma-1)p+1} vol(\mathcal{M}_t) \\ &\leq\; \frac{1}{p}\gamma r^p \beta^{(\sigma-1)p+1} vol(\mathcal{M}_0) \leq \frac{1}{p}C_4^p.\end{aligned}$$

where C_4 is a finite constant depending on $\mathcal{M}_0$ and indipendent from p. □

In the following result the hypothesis $T_{max} < \infty$ is essential: with this assumption, we can bound high L^p-norms of f_σ, provided σ is of order $p^{-\frac{1}{2}}$.

Proposition 3.3.5 *There is a constant $C_5 < \infty$ depending only on $\mathcal{M}_0$ such that for all*

$$p \geq \frac{8}{C_1} + 1 \qquad \sigma \leq \frac{\sqrt{C_1}\rho}{2^6 m\sqrt{p}}$$

we have the inequality

$$\left(\int f_\sigma^p d\mu\right)^{\frac{1}{p}} \leq C_5.$$

Proof. We multiply inequality (3.2.3) by $p f_\sigma^{p-1}$, integrate and obtain

$$\begin{aligned}\frac{d}{dt}\int f_\sigma^p d\mu \;&+\; p(p-1)\int f_\sigma^{p-2}\left|\nabla f_\sigma\right|^2 d\mu + 2C_1 p\int \left|\nabla H\right|^2 W^{\sigma-1} f_\sigma^{p-1} d\mu \\ &\leq\; 4p\alpha\int |H|\, W^{-1}\left|\nabla H\right|\left|\nabla f_\sigma\right| f_\sigma^{p-1} d\mu + 2\sigma p\int |A|^2 f_\sigma^p d\mu \\ &\quad -2C_2 p\int f_\sigma^p d\mu.\end{aligned}$$

We have, for any $\eta > 0$

$$4p\alpha \int |H| W^{-1} |\nabla H| |\nabla f_\sigma| f_\sigma^{p-1} d\mu \leq 4p\alpha \int |H| W^{-1} f_\sigma^{p-1} \left(\frac{\eta}{2} |\nabla f_\sigma|^2 + \frac{1}{2\eta} |\nabla H|^2 \right) d\mu.$$

Taking $\eta = \frac{p-1}{4} W^{\frac{1}{2}-\sigma}$ and the facts that $\alpha |H| \leq W^{\frac{1}{2}}$ and $f_\sigma \leq W^\sigma$, we have

$$\begin{aligned}
4p\alpha & \int |H| W^{-1} |\nabla H| |\nabla f_\sigma| f_\sigma^{p-1} d\mu \\
& \leq \frac{p(p-1)}{2} \int \left(\frac{\alpha |H|}{W} f_\sigma \right) f_\sigma^{p-2} |\nabla f_\sigma|^2 W^{\frac{1}{2}-\sigma} d\mu \\
& \quad + \frac{8p}{p-1} \int \frac{\alpha |H|}{W} W^{\sigma - \frac{1}{2}} f_\sigma^{p-1} |\nabla H|^2 d\mu \\
& \leq \frac{p(p-1)}{2} \int f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu \\
& \quad + \frac{8p}{p-1} \int W^{\sigma-1} f_\sigma^{p-1} |\nabla H|^2 d\mu.
\end{aligned}$$

With our choice of p, we have $C_1 p \leq 2C_1 p - \frac{8p}{p-1}$. For the choice of α and β (3.2.2), we have also that $|A|^2 \leq mW$, and so

$$\begin{aligned}
\frac{d}{dt} \int f_\sigma^p d\mu & + \frac{p(p-1)}{2} \int f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu + C_1 p \int |\nabla H|^2 W^{\sigma-1} f_\sigma^{p-1} d\mu \\
& \leq 2\sigma p \int |A|^2 f_\sigma^p d\mu - 2C_2 p \int f_\sigma^p d\mu \\
& \leq 2\sigma p m \int W f_\sigma^p d\mu - 2C_2 p \int f_\sigma^p d\mu.
\end{aligned}$$

Thanks lemma 3.3.4 we get for any $\eta > 0$

$$\begin{aligned}
\frac{d}{dt} \int f_\sigma^p d\mu & + \frac{p(p-1)}{2} \int f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu + C_1 p \int |\nabla H|^2 W^{\sigma-1} f_\sigma^{p-1} d\mu \\
& \leq \frac{4\sigma p m}{\rho} \left[(\eta(p+1) + 5) \int W^{\sigma-1} f_\sigma^{p-1} |\nabla H|^2 d\mu \right. \\
& \qquad \left. + \frac{p+1}{\eta} \int f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu + 4mb \int f_\sigma^p d\mu + \frac{1}{p} C_4^p \right] \\
& \quad -2C_2 p \int f_\sigma^p d\mu.
\end{aligned}$$

Using $\eta = \frac{\sqrt{C_1}}{4\sqrt{p}}$ and assumptions on p and σ, we get

$$\frac{4\sigma p m}{\rho} (\eta(p+1) + 5) \leq C_1 p, \qquad \frac{4\sigma p(p+1)}{\rho\eta} \leq \frac{p(p-1)}{2}.$$

Then

$$\frac{d}{dt} \int f_\sigma^p d\mu \leq \bar{C}_2 \int f_\sigma^p d\mu + \bar{C}_4,$$

where $\bar{C}_2 = \dfrac{16m^2pb\sigma}{\rho} - 2C_2p$ and $\bar{C}_4 = \dfrac{4\sigma m}{\rho}C_4^p$ are costants. Since by hypothesis T_{max} is finite, we have the thesis for a constant C_5 indipendent from p. $\square$

Because σ has only to decay like $p^{-\frac{1}{2}}$ and not like p^{-1}, we also get

Corollary 3.3.6 *If $T_{max} < \infty$, for all q , $p \geq \frac{q^2 2^{14}}{C_1\rho^2}$, and $\sigma \leq \frac{\sqrt{C_1}\rho}{2^7\sqrt{p}}$ we have*

$$\left(\int |A|^{2q} f_\sigma^p d\mu\right)^{\frac{1}{p}} \leq C_5.$$

Proof. We have $|A|^2 \leq mW$ and so

$$\left(\int |A|^{2q} f_\sigma^p d\mu\right)^{\frac{1}{p}} \leq m^{\frac{q}{p}} \left(\int W^q f_\sigma^p d\mu\right)^{\frac{1}{p}} = m^{\frac{q}{p}} \left(\int f_{\sigma'}^p d\mu\right)^{\frac{1}{p}},$$

where

$$\sigma' = \sigma + \frac{q}{p} \leq \frac{\sqrt{C_1}\rho}{2^7\sqrt{p}} + qp^{-\frac{1}{2}} \left(\frac{\rho\sqrt{C_1}}{q2^7}\right) = \frac{\sqrt{C_1}\rho}{2^6\sqrt{p}}.$$

The assertion follows from Lemma 3.3.5. $\square$

To obtain the proof of the Theorem 3.3.1, we can now procced as in [H3] via a Stampacchia iteration procedure to uniformly bound the function f_σ when $T_{max} < \infty$.

Here we establish a gradient estimate for the mean curvature flow. This estimate is required to compare the mean curvature at different points of the submanifold. First we need some technical inequalities.

Lemma 3.3.7

$$\begin{aligned} \frac{\partial}{\partial t} |H|^2 |\mathring{A}|^2 &\leq \Delta(|H|^2 |\mathring{A}|^2) - \frac{2(m-1)}{3m} |H|^2 |\nabla A|^2 + C_6 |\nabla A|^2 \\ &\quad + 2 |H|^2 |\mathring{A}|^2 (2 |A|^2 + m) \end{aligned}$$

for some constant $C_6 > 0$.

Proof. By lemmata 2.4.3 and 3.2.2,

$$\begin{aligned} \frac{\partial}{\partial t} |H|^2 |\mathring{A}|^2 &\leq \Delta(|H|^2 |\mathring{A}|^2) - 2\left\langle \nabla |H|^2, \nabla |\mathring{A}|^2 \right\rangle - \frac{4(m-1)}{3m} |H|^2 |\nabla A|^2 \\ &\quad - 4|\mathring{A}|^2 |\nabla H|^2 + 2|\mathring{A}|^2 |H|^2 (2|A|^2 + m). \end{aligned}$$

holds. Furthermore we have

$$\begin{aligned} -2\left\langle \nabla |H|^2, \nabla |\mathring{A}|^2 \right\rangle &\leq 4|H| \left\langle |\nabla H|, \nabla |\mathring{A}|^2 \right\rangle \\ &\leq 8|H| |\nabla H| |\mathring{A}| |\nabla A| \\ &\leq 8|H| \sqrt{\frac{m+2}{3}} |\nabla A|^2 |\mathring{A}|^2. \end{aligned}$$

In order to estimate this last term we can use Theorem 3.3.1, so there exists a constant $C_6 > 0$ such that

$$\begin{aligned} 8\,|H|\sqrt{\frac{m+2}{3}}\,|\nabla A|^2\,|\mathring{A}|^2 &\leq 8\,|H|\sqrt{\frac{n+2}{3}}\,|\nabla A|^2\sqrt{C_0}\left(|H|^2+1\right)^{\frac{1-\sigma}{2}} \\ &\leq \frac{2(m-1)}{3m}\,|H|^2\,|\nabla A|^2 + C_6\,|\nabla A|^2\,. \end{aligned}$$

□

Now we consider the function

$$g = |H|^2\,|\mathring{A}|^2 + 2(C_6+1)|\mathring{A}|^2. \tag{3.3.4}$$

Using lemmata 3.2.2 and 3.3.7, the estimates $\frac{2(m-1)}{3m} > \frac{1}{4}$ and $|H|^2 \leq m\,|A|^2$ we get

$$\begin{aligned} \frac{\partial}{\partial t}g &\leq \Delta g - \frac{2(m-1)}{3m}\,|H|^2\,|\nabla A|^2 + C_6\,|\nabla A|^2 + 2|\mathring{A}|^2\,|H|^2\,(2\,|A|^2+m) \\ &\quad +2(C_6+1)\left(-\frac{4(m-1)}{3m}\,|\nabla A|^2 + 4\,|A|^2\,|\mathring{A}|^2\right) \\ &\leq \Delta g - \frac{2(m-1)}{3m}\,|H|^2\,|\nabla A|^2 - \frac{8(m-1)}{3m}\,|\nabla A|^2 + 2|\mathring{A}|^2\,|H|^2\,(2\,|A|^2+m) \\ &\quad +8(C_6+1)|\mathring{A}|^2\,|A|^2 \\ &\leq \Delta g - \frac{1}{4}(|H|^2+1)\,|\nabla A|^2 + 2|\mathring{A}|^2\,|A|^2\,(2m\,|A|^2+C_7), \end{aligned} \tag{3.3.5}$$

where $C_7 = m^2 + 4C_6 + 4$ is a constant.

Proposition 3.3.8 *If $T_{max} < \infty$, for every $\eta > 0$ small enough there exists a constant $C_\eta > 0$ depending only on η such that for all time*

$$|\nabla H|^2 \leq \eta\,|H|^4 + C_\eta$$

holds.

Proof. Let $f = |\nabla H|^2 + 4(C_3+1)g - \eta\,|H|^4$ wiht $\eta > 0$. By proposition 3.2.3, lemma 3.3.7 and inequality (3.3.5) we have

$$\begin{aligned} \frac{\partial}{\partial t}f &\leq \Delta f + C_3(|H|^2+1)\,|\nabla A|^2 - (C_3+1)(|H|^2+1)\,|\nabla A|^2 \\ &\quad +8(C_3+1)|\mathring{A}|^2\,|A|^2\,(2m\,|A|^2 + m^2 + 2C_6 + 2) \\ &\quad -\eta\left(\frac{4}{m}\,|H|^6 - 12\,|H|^2\,|\nabla H|^2\right). \end{aligned}$$

By Lemma 2.4.5, the gradient terms are

$$-\left(|H|^2+1\right)|\nabla A|^2 + 12\eta\,|H|^2\,|\nabla H|^2 \leq \left(-\,|H|^2 - 1 + 4(m+2)\eta\right)|\nabla A|^2\,,$$

that are non-positive for η sufficiently small. The remaining terms are

$$R = 8(C_3+1)|\mathring{A}|^2 |A|^2 \left(2m|A|^2 + m^2 + 2C_6 + 2\right) - \frac{4\eta}{m}|H|^6 .$$

Using the pinching condition (3.2.1) we have

$$R \leq 8(C_3+1)|\mathring{A}|^2 \left(a|H|^2 + b\right)\left(2ma|H|^2 + C_8\right) - \frac{4\eta}{m}|H|^6 ,$$

where $C_8 = 2mb + m^2 + C_6 + 2$ is a constant. Hence, thanks to Theorem 3.3.1, we get

$$\begin{aligned} R &\leq 8(C_3+1)C_0\left(|H|^2+1\right)^{1-\sigma}\left(a|H|^2+b\right)\left(2ma|H|^2+C_8\right) - \frac{4\eta}{m}|H|^6 \\ &\leq 8(C_3+1)C_0\left(\mu(1-\sigma)\left(|H|^2+1\right)+\sigma\mu^{\frac{\sigma-1}{\sigma}}\right)\left(a|H|^2+b\right)\left(2ma|H|^2+C_8\right) \\ &\quad -\frac{4\eta}{m}|H|^6 \\ &\leq C_9, \end{aligned}$$

for some constant C_9 if μ is small enough. Putting these informations together, we have $\frac{\partial}{\partial t}f \leq \Delta f + C_9$. Since $T_{max} < \infty$, we can conclude that there exists a constant C_η depending only on η such that $f \leq C_\eta$. Then, from the definition of f, we have

$$|\nabla H|^2 \leq |\nabla H|^2 + 4(C_3+1)g \leq \eta|H|^4 + C_\eta .$$

□

As seen at the beginning of this section, when the codimension is greater than 1 we cannot repeat the proof of proposition 3.3.2, but using propositions 3.3.1 and 3.3.8 we can prove that, after waitin enough time, the sectional curvature of the evolving submanifold becomes positive.

Proposition 3.3.9 *There is a $\mu > 0$ and a time $\vartheta > 0$ such that for any time $\vartheta < t < T_{max} < \infty$, the intrinsic sectional curvature of $\mathcal{M}_t$ satisfies*

$$K > \mu W > 0.$$

Proof. From Gauss equation we have that

$$2K_{ij} = 2\bar{K}_{ij} + 2\sum_{\alpha=m+1}^{2n}\left(h^\alpha_{ii}h^\alpha_{jj} - \left(h^\alpha_{ij}\right)^2\right), \tag{3.3.6}$$

where K_{ij} is the sectional curvature of $\mathcal{M}_t$ of the plane spanned by two orthonormal vectors (e_i, e_j), and $\bar{K}_{ij}$ is the sectional curvature of the same plane, but in $\mathbb{CP}^n$. The idea is to use (3.3.2) with only one normal direction: fix a orthonormal basis $(e_1, \ldots, e_m)$

tangent to $\mathcal{M}_t$ that diagonalize h^{m+1} and let $\lambda_1^{m+1} \leq \cdots \leq \lambda_m^{m+1}$ its eigenvalues. Recalling that $\bar{K} \geq 1$, (3.3.6) becomes

$$\begin{aligned} 2K_{ij} &\geq 2 + 2\lambda_i^{m+1}\lambda_j^{m+1} + 2\sum_{\alpha=m+2}^{2n}\left(\mathring{h}_{ii}^{\alpha}\mathring{h}_{jj}^{\alpha} - \left(\mathring{h}_{ij}^{\alpha}\right)^2\right) \\ &\geq 2 + \frac{1}{m-1}\left|H\right|^2 - \left|h_1\right|^2 - 2|\mathring{h}_-|^2 \\ &= 2 + \frac{1}{m(m-1)}\left|H\right|^2 - |\mathring{h}_1|^2 - 2|\mathring{h}_-|^2 \\ &\geq 2 + \frac{1}{m(m-1)}\left|H\right|^2 - 2|\mathring{A}|^2. \end{aligned} \tag{3.3.7}$$

By proposition 3.3.1 we get

$$2K_{ij} \geq 2 + \frac{1}{m(m-1)}\left|H\right|^2 - 2C_0\left(\left|H\right|^2 + 1\right)^{1-\sigma}. \tag{3.3.8}$$

Fix some $0 < \mu < \frac{1}{2\alpha m(m-1)}, \frac{1}{\beta}$, then

$$2 + \frac{1}{m(m-1)}\left|H\right|^2 - 2C_0\left(\left|H\right|^2 + 1\right)^{1-\sigma} \geq 2\mu W = 2\mu(\alpha\left|H\right|^2 + \beta),$$

in the points (x,t) where $\left|H\right|^2(x,t)$ is big enough. Let $\bar{H} = \bar{H}(t) = \max_{\mathcal{M}_t}\left|H\right|$. Since $T_{max} < \infty$, the flow develops for sure a singularity, then $\bar{H}$ becomes unbounded as t tends to T_{max}. Hence there exists a ϑ such that for all $\vartheta \leq t < T_{max}$

$$2 + \frac{1}{m(m-1)}x^2 - 2C_0\left(x^2+1\right)^{1-\sigma} \geq 2\mu(\alpha x^2 + \beta), \qquad \forall \frac{\bar{H}}{2} \leq x \leq \bar{H}. \tag{3.3.9}$$

Fix some $0 < \eta < \frac{1}{2}$. From Theorem 3.3.8, there is a constant C_η with $|\nabla H| \leq \frac{1}{2}\eta^2\left|H\right|^2 + C_\eta$ for all t. Up to increasing ϑ, and hence $\bar{H}$ too, we can assume that $C_\eta \leq \frac{1}{2}\eta^2\bar{H}^2$ and so $|\nabla H| \leq \eta^2\bar{H}^2$. Now fix some $\vartheta \leq t < T_{max}$ and let x be a point on $\mathcal{M}_t$ where $|H|$ assume its maximum. Along any geodesic starting at x of length at most $r = \eta^{-1}\bar{H}^{-1}$, we have $|H| \geq (1-\eta)\bar{H} > \frac{1}{2}\bar{H}$. By inequalities (3.3.8) and (3.3.9) we find that in all $B_r(x)$

$$K > \mu W > \mu\alpha\left|H\right|^2 \geq \mu\alpha\frac{\bar{H}^2}{4} > 0$$

holds, with μ indipendent of the choice of η. Then in $B_r(x)$ we have $Ric_{ij} \geq (n-1)\frac{\mu\alpha}{4}\bar{H}^2 g_{ij}$. Using Myers' theorem A.0.1 to geodesics in $B_r(x)$ we have that if such a geodesic is long at least $\frac{2\pi}{\bar{H}\sqrt{\mu\alpha}}$, then it has a conjugate point. So if η is small, precisely

$$\frac{2\pi}{\bar{H}\sqrt{\mu\alpha}} < r = \frac{1}{\eta\bar{H}} \quad \Leftrightarrow \quad \eta < \frac{\sqrt{\mu\alpha}}{2\pi},$$

$B_r(x)$ covers all $\mathcal{M}_t$. $\square$

To conclude with the convergence to a round point we use theorem 1.0.2 the main result of [LXZ] there is a constant $b_0 > 0$ such that if a submanifold of dimension m satisties

$$|A|^2 < \frac{1}{m-1}|H|^2 - b_0, \tag{3.3.10}$$

then the mean curvature flow of this submanifold contracts to a round point (in finite time). Our pinching condition (3.0.1) says that in general $\mathcal{M}_0$ does not satisfies (3.3.10), but we have that it holds on $\mathcal{M}_t$ for t sufficiently close to T_{max}.

Proposition 3.3.10 *There exist a time $0 < \vartheta < T_{max}$ such that for all $\vartheta < t < T_{max}$ (3.3.10) holds on $\mathcal{M}_t$.*

Proof. By proposition (3.3.1) we have

$$\begin{aligned} |A|^2 - \frac{1}{m-1}|H|^2 + b_0 &= |\mathring{A}|^2 - \frac{1}{m(m-1)}|H|^2 + b_0 \\ &\leq C_0\left(|H|^2+1\right)^{1-\sigma} - \frac{1}{m(m-1)}|H|^2 + b_0 \end{aligned}$$

which is negative only in the points (x,t) where $|H|^2(x,t)$ is big enough. Using Myers' theorem A.0.1 exactly in the same way of the proof of proposition 3.3.9 we have the thesis. $\square$

3.4 Infinite maximal time

Throughout this section we consider $T_{max} = \infty$ and the bigger pinching condition 3.0.2. In this case, the way to proceed is similar to that of the case T_{max} finite, but it is rather simpler because we do not need integral estimates, as shown in the next result.

Proposition 3.4.1 *There are positive constants C_0 and δ_0 depending only on the initial manifold $\mathcal{M}_0$ such that*

$$|\mathring{A}|^2 \leq C_0\left(|H|^2+1\right)e^{-\delta_0 t}$$

holds for any time $0 \leq t < T_{max} = \infty$.

Proof. Using proposition 3.2.1 with $\sigma = 0$ and the maximum principle, we have that

$$f_0 \leq C_0' e^{-\delta_0 t},$$

for some positive constants C_0' and δ_0 that depends only on the initial submanifold. Recalling that

$$f_0 = \frac{|\mathring{A}|^2}{\alpha|H|^2+\beta},$$

we have the thesis for an appropriate costant C_0. $\square$

Note that this result is meaningful when we can considere time t arbitrarily large, while it does not say much more than the original pinching condition for small t. Like in the previous section, we can prove that, after waiting enough time, the sectional curvature of the evolving submanifold becomes positive as consequence of theorem 3.4.1. Since with the hypothesis $T_{max} = \infty$ we have an exponential decay, the proof is much more direct than the proof of proposition 3.3.9 and does not involve Myers' theorem.

Proposition 3.4.2 *There is a $\mu > 0$ and a time $\vartheta > 0$ such that for any time $\vartheta < t < T_{max} = \infty$, the intrinsic sectional curvature of $\mathcal{M}_t$ satisfies*

$$K > \mu W > 0.$$

Proof. Like in the proof of proposition 3.3.9 we have $2K_{ij} \geq 2 + \frac{1}{m(m-1)} |H|^2 - 2|\mathring{A}|^2$. By the exponential decay of $|\mathring{A}|^2$ proved in proposition 3.4.1, finally we have

$$\begin{aligned} 2K_{ij} &\geq 2 + \frac{1}{m(m-1)} |H|^2 - 2C_0 \left(|H|^2 + 1\right) e^{-\delta_0 t} \\ &\geq 2\mu W > 0, \end{aligned}$$

for $\mu > 0$ small enough and t sufficiently big. □

Now we need to compare the mean curvature in different points of the submanifold. Like in the previous section we have the same estimate for $\frac{\partial}{\partial t} |H|^2 |\mathring{A}|^2$.

Lemma 3.4.3

$$\begin{aligned} \frac{\partial}{\partial t} |H|^2 |\mathring{A}|^2 &\leq \Delta(|H|^2 |\mathring{A}|^2) - \frac{2(m-1)}{3m} |H|^2 |\nabla A|^2 + C_6 |\nabla A|^2 \\ &\quad +2 |H|^2 |\mathring{A}|^2 (2 |A|^2 + m) \end{aligned}$$

for some constant $C_6 > 0$.

Proof. We proceed like in the proof of lemma 3.3.7, but this time we use proposition 3.4.1: there existis a constant $C_6 > 0$ such that

$$\begin{aligned} 8 |H| \sqrt{\frac{m+2}{3}} |\nabla A|^2 |\mathring{A}|^2 &\leq 8 |H| \sqrt{\frac{n+2}{3}} |\nabla A|^2 \sqrt{C_0(|H|^2+1)} e^{-\delta_0 t/2} \\ &\leq \frac{2(m-1)}{3m} |H|^2 |\nabla A|^2 + C_6 |\nabla A|^2 . \end{aligned}$$

Note that this inequality is certainly true if t in big enough, because the exponential decay. If t is small, the flow can be extended over t because we assumed $T_{max} = \infty$, then, at that time, $|H|^2$ is bounded. □

Now we consider the function g defined in (3.3.4). Using lemmata 3.2.2, 3.4.3, the estimates $\frac{2(m-1)}{3m} > \frac{1}{4}$ and $|H|^2 \leq m |A|^2$ we get that also for $T_{max} = \infty$ (3.3.5) holds too.

Theorem 3.4.4 *For every $\eta > 0$ small enough there exists a constant $C_\eta > 0$ depending only on η such that for all time*

$$|\nabla H|^2 \leq \left(\eta\,|H|^4 + C_\eta\right) e^{-\delta_0 t/4}$$

holds.

Proof. We proceed in the same way of the proof of proposition 3.3.8, but in this case we need an exponential decay, hence we define

$$f = e^{\delta_0 t/2}\left(|\nabla H|^2 + 4(C_3 + \delta_0 m)g\right) - \eta\,|H|^4 .$$

By proposition 3.2.3, lemma 3.2.2 and inequality (3.3.5) we have

$$\begin{aligned}
\frac{\partial}{\partial t} f \quad &\leq \quad \Delta f + \left[\frac{\delta_0}{2}\,|\nabla H|^2 + 2\delta_0\,(C_3 + \delta_0 m)\left(|H|^2\,|\mathring{A}|^2 + 2(C_6+1)|\mathring{A}|^2\right)\right] e^{\delta_0 t/2} \\
&\quad + \left[-\delta_0 m(|H|^2+1)\,|\nabla A|^2 + 8(C_3+\delta_0 m)|\mathring{A}|^2\,|A|^2\,(2m\,|A|^2 + C_7)\right] e^{\delta_0 t/2} \\
&\quad -\eta\left(\frac{2}{m}\,|H|^6 - 12\,|H|^2\,|\nabla H|^2\right).
\end{aligned}$$

By lemma 2.4.5, the gradient terms are

$$\begin{aligned}
&\left[\frac{\delta_0}{2}\,|\nabla H|^2 - \delta_0 m\left(|H|^2+1\right)|\nabla A|^2\right] e^{\delta_0 t/2} + 12\eta\,|H|^2\,|\nabla H|^2 \\
&\qquad \leq \left[\frac{\delta_0}{2} - \frac{3\delta_0 m}{m+2}(|H|^2+1) + 12\eta\,|H|^2\right] |\nabla H|^2\, e^{\delta_0 t/2}
\end{aligned}$$

that are non-positive for η sufficiently small. We call R the remaining terms. Using condition (3.0.2) and the Theorem 3.4.1, we can find a constant Λ such that

$$\begin{aligned}
R \quad &\leq \quad C_0\Lambda\left(|H|^2+1\right)\left(|H|^4+1\right) e^{-\delta_0 t/2} - \frac{2\eta}{m}\,|H|^6 \\
&\leq \quad \left[C_0\Lambda\left(|H|^2+1\right)\left(|H|^4+1\right) e^{-\delta_0 t/4} - \frac{2\eta}{m}\,|H|^6\right] e^{-\delta_0 t/4} \\
&\leq \quad C_9 e^{-\delta_0 t/4},
\end{aligned}$$

for some constant C_9. Note that this is true, because $e^{-\delta_0 t/4}$ is small, for t big enough, but because $|H|^2$ is bounded, for t small. Then we have that there exist a constant C_η such that $f \leq C_\eta$. Recalling the definition of f we conclude the proof. □

We show that, if $T_{max} = \infty$, there are no formation of singularities.

Lemma 3.4.5 *If $T_{max} = \infty$, $|H|^2$ is bounded and then there are no formation of singularities approaching the maximal time.*

Proof. Let b_0 the constant used in the main result of [LXZ] and suppose that $|H|^2$ is unbouded. From theorem 3.4.1 we have

$$\begin{aligned} |A|^2 - \frac{1}{m-1}|H|^2 + b_0 &= |\mathring{A}|^2 - \frac{1}{m(m-1)}|H|^2 + b_0 \\ &\leq C_0\left(|H|^2 + 1\right)e^{-\delta_0 t} - \frac{1}{m(m-1)}|H|^2 + b_0 \end{aligned}$$

which is negative for t sufficiently big in the part of $\mathcal{M}_t$ where $|H|^2$ is big enough. Since in proposition 3.4.2 we proved that the sectional curvature of $\mathcal{M}_t$ is positive for t big enough, we can apply Myers' theorem A.0.1 like in the proof of proposition 3.3.10. We have that, for t big enough,

$$|A|^2 - \frac{1}{m-1}|H|^2 + b_0 < 0$$

everywhere on $\mathcal{M}_t$. The main result of [LXZ] says that the mean curvature flow of initial value $\mathcal{M}_t$ shrinks to a point in finite time, giving a contraddiction. □

Now we have all the ingredients to understand the convergence in the case $T_{max} = \infty$. Since $|H|^2$ stay bounded, theorems 3.4.1 and 3.4.4 give that there is a constant C such that

$$|\mathring{A}|^2 \leq Ce^{-\delta_0 t}, \qquad |\nabla H|^2 \leq Ce^{-\delta_0 t/2}.$$

Applying once again Myers' theorem A.0.1, the diameter of $\mathcal{M}_t$ is uniformly bounded and so $|H|^2_{max} - |H|^2_{min} \leq Ce^{-\delta_0 t/2}$. Moreover $|H|^2_{min} = 0$ otherwise the flow could only have a solution on a finite time interval. Then $|H|^2$ decays exponentially fast and

$$|A|^2 = |\mathring{A}|^2 + \frac{1}{m}|H|^2 \leq Ce^{-\delta_0 t/2},$$

for some positive constant C. Then

$$\int_0^\infty \left|\frac{\partial}{\partial t} g_{ij}\right| dt = \int_0^\infty |H|\,|A|\,dt \leq \sqrt{m}\int_0^\infty |A|^2\,dt \leq \sqrt{m}C\int_0^\infty e^{-\delta_0 t/2} \leq \bar{C},$$

for some finite constant $\bar{C}$. So we can apply Hamilton's lemma A.0.2 to obtain that there is a continuos limit metric $g_{ij}(\infty)$. It remains to show that the limit hypersurface $\mathcal{M}_\infty$ is smooth. With a well-known method which dates back to the famous work of Hamilton [Ha], used in [H1, §10] too, the exponential decay for $|A|^2$ gives the exponential decay for all derivatives $\nabla^k A$. This finally gives $C^\infty-$convergence to a smooth totally geodesic submanifold $\mathcal{M}_\infty$. But, since the codimension k is low, the only possibility is that $\mathcal{M}_\infty = \mathbb{CP}^{n'}$ for some $n' < n$ as seen in theorem 2.3.3. This means that, if k is odd this possibility cannot happen and then we have only a singularity in finite time. This conclude the proof of theorems 3.0.1 and 3.0.3.

3.5 Extensions to CROSSes

We conclude this chapter extending the main theorem 3.0.1 to the hypersurfaces of (almost) all CROSSes and giving some examples. Let $\mathbb{K}$ be the field $\mathbb{C}$ of complex number or the associative algebra $\mathbb{H}$ of quaternions and c a positive constant. Let $\mathcal{M}_0$ be a real hypersurface of $\mathbb{KP}^n(4c)$, the projective space over $\mathbb{K}$ with sectional curvature $c \leq \bar{K} \leq 4c$.

Theorem 3.5.1 *Let $n \geq 3$, $c > 0$ and $\mathcal{M}_0$ be a closed real hypersurface of $\mathbb{KP}^n(4c)$. Let m the real dimension of $\mathcal{M}_0$ and suppose that $\mathcal{M}_0$ satisfies*

$$|A|^2 < \frac{1}{m-1}|H|^2 + 2c, \tag{3.5.1}$$

then the mean curvature flow with initial condition $\mathcal{M}_0$ has a smooth solution $\mathcal{M}_t$ on a finite time interval $0 \leq t < T_{max} < \infty$ and the flow converges to a round point as t goes to T_{max}.

The proof is the same exposed in the previous sections for the case of hypersurfaces of $\mathbb{CP}^n = \mathbb{CP}^n(4)$. The constants used are

$$m = \begin{cases} 2n-1 & \text{if } \mathbb{K} = \mathbb{C}, \\ 4n-1 & \text{if } \mathbb{K} = \mathbb{H}, \end{cases} \qquad \text{and} \qquad \bar{r} = \begin{cases} 2(n+1)c & \text{if } \mathbb{K} = \mathbb{C}, \\ 4(n+2)c & \text{if } \mathbb{K} = \mathbb{H}. \end{cases}$$

Note that, even if we already know that for $\mathbb{KP}^n(4c)$ there are no totally geodesic real hypersurfaces, we cannot exclude the possibility that the flow is defined for all time until we prove that if T_{max} were infinite the flow would converge to a totally geodesic submanifold. From theorem 3.5.1 follows this classification result.

Corollary 3.5.2 *Let $n \geq 3$ and $c > 0$.*

1. *If $\mathcal{M}_0$ is a closed real hypersurface of $\mathbb{KP}^n(4c)$ satisfying the pinching condition (3.5.1), then $\mathcal{M}_0$ is diffeomorphic to a sphere.*

2. *For any minimal closed real hypersurface of $\mathbb{KP}^n(4c)$, $|A|^2 \geq 2c$ holds.*

Theorem 3.5.1 is the generalization of the main theorem of [H3] about pinched hypersurfaces of the sphere to (almost) all CROSSes. Unfortunately, at least with these techniques, we are not able to find an analogous result for the Cayley plane $\mathbb{C}a\mathbb{P}^2$ due to its fixed low dimension. We are not able to find an interesting pinching condition preserved by the flow. In fact the real dimension of $\mathbb{C}a\mathbb{P}^2$ is 16, then $m = 15$. Moreover the Einstein constant is $\bar{r} = 36$. To get an inequality like (3.1.10) in this case, we want to find a constant λ such that

$$2b\left(|A|^2 + 36\right) - 60\left(|A|^2 - \frac{1}{15}|H|^2\right) \leq 2\lambda\left(|A|^2 - a|H|^2 - b\right),$$

which gives

$$\begin{cases} b - 30 \leq \lambda, \\ 2 \leq -a\lambda, \\ 36b \leq -\lambda b. \end{cases}$$

The condition $b > 0$ is incompatible with this system. Futhermore the class of hypersurfaces defined with $b < 0$ is noteless in the sense that such hypersurfaces, in particular, satisfies

$$|A|^2 - \frac{1}{m-1}|H|^2 < 0.$$

Recalling (3.3.3), this means that for every i and j we have $\lambda_i\lambda_j \geq 0$, then the principal curvatures have all the same sign, that is the hypersurfaces considered are convex. Since the ambient manifold is symmetric with positive scalar curvature, we can use theorem A.0.3 concluding immediately that the flow converges to a round point in finite time. We can see that theorem 3.5.1 is not trivial, that is there are non-convex hypersurfaces in the class considered.

Example 3.5.3 *Consider for semplicity $c = 1$ and let $\mathcal{M}_0$ a geodesic sphere in $\mathbb{CP}^n$. In [NR] it is proved that $\mathcal{M}_0$ has two distinct principal curvatures: $\lambda_1 = 2cot(2u)$ with multiplicity 1 and $\lambda_2 = cot(u)$ with multiplicity $2(n-1)$, for some $0 < u < \frac{\pi}{2}$. For any $u > \frac{\pi}{4}$, we have $\lambda_1 < 0$ and $\lambda_2 > 0$, so $\mathcal{M}_0$ is not convex. Moreover, it is easy to compute that in this case condition (3.5.1) is equivalent to*

$$\frac{2(2n-3)}{n-1}cot^2(2u) - 2cot^2(u) < 0.$$

Then there are non-convex examples in our class for every n. In the same way a geodesic sphere in $\mathbb{HP}^n$ has principal curvature $\lambda_1 = 2cot(2u)$ with multiplicity 3 and $\lambda_2 = cot(u)$ with multiplicity $4(n-1)$, for some $0 < u < \frac{\pi}{2}$ (see for example [MP]). Condition (3.5.1) in this case becomes

$$3(4n-5)cot^2(2u) - 4(n-1)cot^2(u) + 4n - 5 < 0,$$

so we have non-convex examples in our class for $\mathbb{K} = \mathbb{H}$ too.

However, even if the initial hypersurface is not convex, it becomes convex approaching the maximal time.

Proposition 3.5.4 *Under the hypothesis of theorem 3.5.1 there is a $0 < \vartheta < T_{max}$ such that for any time $\vartheta < t < T_{max}$ $\mathcal{M}_t$ is convex.*

Proof. The proof uses Myers' theorem A.0.1 and theorem 3.3.1 (that holds for any $\mathbb{K}$ and for any $c > 0$) like in the proof of propisition 3.3.9, showing that for time t close enough to T_{max} we have

$$|A|^2 - \frac{1}{m-1}|H|^2 \leq C_0(|H|^2 + 1)^{1-\sigma} - \frac{1}{m(m-1)}|H|^2 < 0$$

everywhere on $\mathcal{M}_t$. □

Chapter 4

Cylindrical estimates in CROSSes

In this chapter we focus on hypersurfaces of CROSSes. We consider a class which contains properly the class studied in theorem 3.5.1: with the further assumption that $H \neq 0$ everywhere we are able to classify the singularities for this class. Like in the previous chapter $\mathbb{K}$ is one of $\mathbb{C}$ or $\mathbb{H}$. Since the codimension is 1, the mean curvature can be seen as a function: fix an unit vector field ν normal to the hypersurface (up to sign ν there is an unique choice for ν), then the mean curvature vector is a scalar multiple of ν.

Notation 4.0.1 *Only in this chapter we change the notation for the mean curvature. We denote with $\vec{H}$ the mean curvature vector and with H the mean curvature function, that is*

$$\vec{H} = -H\nu.$$

In [H2] is proved the evolution equation of the function H.

Lemma 4.0.2

$$\frac{\partial}{\partial t} H = \Delta H + H \left(|A|^2 + \bar{R}ic(\nu, \nu) \right).$$

The main theorem proved is the following

Theorem 4.0.3 *Let $n \geq 4$ and $\mathcal{M}_0$ be a closed real hypersurface of $\mathbb{KP}^n$, that satisfies*

$$|A|^2 < \frac{1}{m-2} |H|^2 + 4, \tag{4.0.1}$$

then the mean curvature flow with initial data $\mathcal{M}_0$ develops a singularity in finite time. Moreover if $H \neq 0$ everywhere on $\mathcal{M}_0$, then for every $\eta > 0$ there exists a constant C_η that depends only on η and $\mathcal{M}_0$ such that

$$|\lambda_1| \leq \eta |H| \qquad \Rightarrow \qquad (\lambda_i - \lambda_j)^2 \leq \Lambda \eta H^2 + C_\eta, \quad \forall i, j \geq 2, \tag{4.0.2}$$

for a constant Λ that depends only on the ambient manifold.

This theorem is a generalization of Nguyen's result [Ng] on the sphere to (almost) all CROSSes. Once again the hypothesis $n \geq 4$ does not allows us to give the analogous statement for the Cayley plane.

The first step of the proof is to show that (4.0.1) is preserved by the flow. Since we assume inequality (4.0.1) on $\mathcal{M}_0$, by compactness

$$|A|^2 \leq a_\varepsilon |H|^2 + b_\varepsilon \tag{4.0.3}$$

holds everywhere on $\mathcal{M}_0$, where this time

$$a_\varepsilon = \frac{1}{m-2}(1-\varepsilon), \qquad b_\varepsilon = 4(1-\varepsilon),$$

for some $\varepsilon > 0$ small enough depending only on $\mathcal{M}_0$. Proposition 3.1.3 shows that (4.0.3) is preserved by the flow for hypersurfaces of $\mathbb{CP}^n$, but the same proof holds for hypersurfaces of $\mathbb{HP}^n$ too. Moreover theorem 3.0.3 gives that, at least for $\mathbb{K} = \mathbb{C}$, the mean curvature flow of any hypersurface in the class considered develops a singularity in finite time. The same proof holds for $\mathbb{K} = \mathbb{H}$ too. With the further assumption $H \neq 0$ we can use the convexity estimates of Huisken and Sinestrari [HS1] and derive the second part of the main theorem with integral estimates and Stampacchia iteration on a suitable function. Theorem 4.0.3 allows us to classify the singularity for this class of hypersurfaces. In fact condition (4.0.2) means that the only singularities that can occurs are the round point and the cylindrical singularity.

Note that the hypothesis $H \neq 0$ implies that $H > 0$ everywhere for a suitable choice of the sign of ν. From lemma 4.0.2 is easy to see tha the mean convexity is preserved by the flow.

4.1 A technical lemma

For technical reason let us introduce the following auxiliary function

$$f_{\sigma,\eta} := \frac{|A|^2 - (\frac{1}{m-1} + \eta)|H|^2}{W^{1-\sigma}},$$

where σ and η are two positive constant small enough and, likewise the previous chapter, $W = \alpha |H|^2 + \beta$ for some positive constants

$$\begin{cases} \dfrac{1}{m-2} - \dfrac{1}{m-1} - \eta < \alpha < \dfrac{3}{m+2} - \dfrac{1}{m-1} - \eta, \\ 4(1-\varepsilon) < \beta < 4. \end{cases} \tag{4.1.1}$$

For simplicity we denote by $f_0 = f_{0,\eta}$. First we derive the evolution equation for $f_{\sigma,\eta}$.

Proposition 4.1.1 *There is a $0 < \sigma_1 < 1$ depending only on $\mathcal{M}_0$ such in the points where $f_{\sigma,\eta} > 0$*

$$\begin{aligned} \frac{\partial}{\partial t} f_{\sigma,\eta} \leq\ & \Delta f_{\sigma,\eta} + \frac{2\alpha(1-\sigma)}{W}\left\langle \nabla f_{\sigma,\eta}, \nabla |H|^2 \right\rangle - 2C_1 W^{\sigma-1} |\nabla H|^2 \\ & + 2\sigma(|A|^2 + \bar{r}) f_{\sigma,\eta} - 2C_2 f_{\sigma,\eta}, \end{aligned} \tag{4.1.2}$$

holds for all $0 \leq \sigma \leq \sigma_1$ with $C_1 > 0$ and $C_2 > 0$ constants.

Proof. Making similar calculations of the proof of proposition 3.2.6 we have

$$
\begin{aligned}
\frac{\partial}{\partial t} f_{\sigma,\eta} \ \leq\ & \Delta f_{\sigma,\eta} + \frac{2\alpha(1-\sigma)}{W} \left\langle \nabla f_{\sigma,\eta}, \nabla |H|^2 \right\rangle \\
& -2W^{\sigma-1} |\nabla A|^2 + 2W^{\sigma-1} \left[\frac{1}{m-1} + \eta + f_0(1-\sigma)\alpha \left(1 - 2\alpha\sigma \frac{|H|^2}{W} \right) \right] \\
& 2\frac{f_{\sigma,\eta}}{W} \left(|A|^2 + \bar{r} \right) - \alpha(1-\sigma) \frac{f_{\sigma,\eta} |H|^2}{W} (|A|^2 + \bar{r}) - 4mW^{\sigma-1} |\mathring{A}|^2 \\
& +\beta(1-\sigma) \frac{|A|^2 + \bar{r}}{W} - \beta(1-\sigma) \frac{|A|^2 + \bar{r}}{W} \\
=\ & \Delta f_{\sigma,\eta} + \frac{2\alpha(1-\sigma)}{W} \left\langle \nabla f_{\sigma,\eta}, \nabla |H|^2 \right\rangle \\
& -2W^{\sigma-1} |\nabla A|^2 + 2W^{\sigma-1} \left[\frac{1}{m-1} + \eta + f_0(1-\sigma)\alpha \left(1 - 2\alpha\sigma \frac{|H|^2}{W} \right) \right] \\
& +2\beta(1-\sigma) \frac{f_{\sigma,\eta}}{W} \left(|A|^2 + \bar{r} \right) + 2\sigma f_{\sigma,\eta} (|A|^2 + \bar{r}) - 4mW^{\sigma-1} |\mathring{A}|^2.
\end{aligned}
$$

With the choice (4.1.1) of α and β we have $f_0 < 1$. Hence by Lemma 2.4.5

$$
\begin{aligned}
-|\nabla A|^2 \ +\ & \left[\frac{1}{m-1} + \eta + f_0(1-\sigma)\alpha \left(1 - 2\alpha\sigma \frac{|H|^2}{W} \right) \right] |\nabla H|^2 \\
\leq\ & \left(\frac{1}{m-1} + \eta + \alpha \right) |\nabla H|^2 - |\nabla A|^2 \\
\leq\ & \left(\alpha + \frac{1}{m-1} + \eta - \frac{3}{n+2} \right) |\nabla H|^2 \\
=\ & -C_1 |\nabla H|^2,
\end{aligned}
\tag{4.1.3}
$$

with $C_1 = \dfrac{3}{m+2} - \dfrac{1}{m-1} - \eta - \alpha > 0$. In order to complete the proof, we need to estimate the reaction terms. Let us call

$$
\begin{aligned}
R \ =\ & 2\beta(1-\sigma) \frac{f_{\sigma,\eta}}{W} \left(|A|^2 + \bar{r} \right) - 4mW^{\sigma-1} |\mathring{A}|^2 \\
=\ & 2\beta(1-\sigma) \frac{f_{\sigma,\eta}}{W} \left(|A|^2 + \bar{r} \right) - 4m f_{\sigma,\eta} - 4 \left(\frac{1}{m-1} + m\eta \right) W^{\sigma-1}.
\end{aligned}
$$

Using conditions (4.0.3), (4.1.1) and the fact that $f_{\sigma,\eta} \leq W^\sigma$, we get

$$\begin{aligned} R &\leq 2\frac{f_{\sigma,\eta}}{W}\left[\beta(1-\sigma)\left(\frac{1-\varepsilon}{m-2}|H|^2 + 4(1-\varepsilon) + \bar{r}\right)\right] \\ &\quad -4m\frac{f_{\sigma,\eta}}{W}\left[\left(\frac{1}{m-2} - \frac{1}{m-1} - \eta\right)|H|^2 + \beta\right] \\ &\quad -4\left(\frac{1}{m-1} + m\eta\right)|H|^2 W^{\sigma-1} \\ &\leq 2\frac{f_{\sigma,\eta}}{W}\left[\beta(1-\sigma)\left(\frac{1-\varepsilon}{m-2}|H|^2 + 4(1-\varepsilon) + \bar{r}\right)\right] \\ &\quad -4m\frac{f_{\sigma,\eta}}{W}\left[\left(\frac{1}{m-2} - \frac{1}{m-1} - \eta\right)|H|^2 + \beta\right] \\ &\quad -4\left(\frac{1}{m-1} + m\eta\right)|H|^2 \frac{f_{\sigma,\eta}}{W} \\ &\leq -2C_2 f_{\sigma,\eta}, \end{aligned}$$

for some positive constant C_2 for both choice of $\mathbb{K}$ if η is small enough. □

4.2 Cylindrical estimates

From now on consider the further hypothesis $H > 0$. Once proved that there is a singularity in finite time, in this section we want to show that if the hypersurface does not become spherical then it becomes cylindrical. The strategy of the proof is inspired by analogous problems in [HS2, Ng] and consists in showing that the function $f_{\sigma,\eta}$ introduced in the previous section is bounded. This will imply the following result.

Theorem 4.2.1 *Let $n \geq 4$ and $\mathcal{M}_0$ a closed hypersurface of $\mathbb{KP}^n$. If $\mathcal{M}_0$ satisfies conditions (4.0.1) and $H > 0$, then for any $\eta > 0$ there exists a constant C_η depending on η and the initial data such that*

$$|A|^2 - \frac{1}{m-1}|H|^2 \leq \eta|H|^2 + C_\eta,$$

for every time $t \in [0, T_{max}[$.

The above theorem has this simple and meaningful consequence.

Corollary 4.2.2 *Under the hypotesis of theorem 4.2.1, for every $\eta > 0$ there exists a constant C_η that depends only on η and $\mathcal{M}_0$ such that*

$$|\lambda_1| \leq \eta H \qquad \Rightarrow \qquad (\lambda_i - \lambda_j)^2 \leq \Lambda\eta H^2 + C_\eta, \quad \forall i, j \geq 2, \tag{4.2.1}$$

for a constant Λ that depends only on the ambient manifold.

Proof. The following identity holds in general

$$|A|^2 - \frac{1}{m-1}|H|^2 = \frac{1}{m-1}\left(\sum_{1<i<j}(\lambda_i-\lambda_j)^2 + \lambda_1(m\lambda_1 - 2H)\right), \qquad (4.2.2)$$

where $\lambda_1 \leq \cdots \leq \lambda_m$ are the principal curvature of $\mathcal{M}_t$. Then for every $i > j > 1$ we have

$$(\lambda_i-\lambda_j)^2 \leq (m-1)\left(|A|^2 - \frac{1}{m-1}|H|^2\right) - \lambda_1(m\lambda_1-2H).$$

Using theorem 4.2.1 and $|\lambda_1| < \eta H$ we have

$$(\lambda_i-\lambda_j)^2 \leq (m-1)\left(\eta|H|^2 + C'_\eta\right) + \eta(m\eta+2)|H|^2,$$

that is the thesis for suitable constants Λ and C_η. □

The meaning of this corollary is that, if near a singularity, the first principal curvature is small compared to the others, then after rescaling the λ_i's, with $i \neq 1$, become close each other, that is the profile of the singularity is a cylinder.

Hypotesis $H > 0$ allows us to use the Huisken-Sinestrari convexity estimates proved in [HS1] for the Euclidean space, but, as the authors said, it works in a general ambient space with small changes in the proof. Let us call S_k the k-th elementary symmetric polynomial evaluated at the principal curvature of an hypersurface. Explicitly

$$S_k = \sum_{1\leq i_1<i_2<\cdots<i_k\leq m} \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_m}.$$

Theorem 4.2.3 *(Huisken, Sinestrari) Let $F_0 : \mathcal{M} \to \mathbb{KP}^n$ a smooth closed hypersurface immersion with nonnegative mean curvature. For each k, $2 \leq k \leq m$, and any $\eta > 0$ there is a positive constant $C_{\eta,k}$ depending only on k, η, the initial data and the ambient space, such that everyhere on $\mathcal{M} \times [0, T_{max}[$ we have*

$$S_k \geq -\eta H^k - C_{\eta,k}.$$

In particular it follows this bound on the first principal curvature.

Corollary 4.2.4 *Under the hypotesis of the previous theorem, for any $\eta > 0$ there is a C_η such that*

$$\lambda_1 \geq -\eta H - C_\eta,$$

everyhere on $\mathcal{M} \times [0, T_{max}[$.

To prove that $f_{\sigma,\eta}$ is bounded we want to use integral estimates and Stampacchia iteration like for the proof of theorem 3.3.1. First we need a lower bound for $\Delta|A|^2$. We recall the computations done in the previous chapter:

$$\begin{aligned}
\Delta|A|^2 &= 2\langle h_{ij}, \nabla_i\nabla_j H\rangle + 2|\nabla A|^2 + 2Z + 2\left(Hh^{ij}\bar{R}_{0i0j} - |A|^2\bar{R}_{0l0}{}^l\right)\\
&\quad + 4\left(h_{ij}h_j{}^p\bar{R}_{pli}{}^l - h^{ij}h^{lp}\bar{R}_{pilj}\right),\\
4&\left(h_{ij}h_j{}^p\bar{R}_{pli}{}^l - h^{ij}h^{lp}\bar{R}_{pilj}\right) \geq 4m|\mathring{A}|^2 \geq 0,\\
2&\left(Hh^{ij}\bar{R}_{0i0j} - |A|^2\bar{R}_{0l0}{}^l\right) \geq -C_3W,
\end{aligned}$$

Where C_3 is some positive constant. Since we consider only hypersurfaces, Z has the simpler form: $Z = (trA)(trA^3) - |A|^4$. Then

$$\Delta |A|^2 \geq 2 \langle h_{ij}, \nabla_i \nabla_j H \rangle + 2 |\nabla A|^2 + 2Z - 2C_3 W. \tag{4.2.3}$$

We cannot prove thate Z is nonnegative, but combining the pinching condition (4.0.1) with the convexity estimate 4.2.4 we have the following inequality for Z which shows that the negative part is of lower order.

Lemma 4.2.5 *Assuming the pinching condition* (4.0.1) *and* $H > 0$, *there is a constant* γ *depending only on* n *and* $\mathcal{M}_0$ *such that for any* $\eta > 0$ *there exists a* C_η *such that*

$$Z \geq \gamma \alpha |H|^2 \left(|A|^2 - \frac{1}{m-1} |H|^2 - \eta |H|^2 \right) - C_\eta \left(H^3 + 1 \right).$$

In particular there is a constant K_η *such that*

$$Z \geq \gamma W^2 f_0 - K_\eta \left(H^3 + 1 \right). \tag{4.2.4}$$

Proof. Suppose $\lambda_1 < 0$ otherwise the hypersurface is convex and this estimate has been proven in [H2]. Chosing a basis that diagonalize the second fundamental form we have

$$\begin{aligned} Z &= \sum_{i<j} \lambda_i \lambda_j (\lambda_i - \lambda_j)^2 \\ &= \sum_{j} \lambda_1 \lambda_j (\lambda_1 - \lambda_j)^2 + \sum_{1<i<j} \lambda_i \lambda_j (\lambda_i - \lambda_j)^2 . \end{aligned}$$

The way of treating the second term is suggested by the following: for every distinct index i, j, k we have

$$\begin{aligned} |A|^2 - \frac{1}{m-2} |H|^2 &= -2(\lambda_i\lambda_j + \lambda_i\lambda_k + \lambda_j\lambda_k) + \left(\lambda_i + \lambda_j + \lambda_k - \frac{H}{m-2} \right)^2 \\ &\quad + \sum_{l \neq i,j,k} \left(\lambda_l - \frac{H}{m-2} \right)^2 \\ &\geq -2(\lambda_i\lambda_j + \lambda_i\lambda_k + \lambda_j\lambda_k) \end{aligned} \tag{4.2.5}$$

We decompose

$$\begin{aligned} \sum_{1<i<j} \lambda_i \lambda_j (\lambda_i - \lambda_j)^2 &= \sum_{1<i<j} (\lambda_i\lambda_j + \lambda_i\lambda_1 + \lambda_j\lambda_1 + 2) (\lambda_i - \lambda_j)^2 \\ &\quad - \sum_{1<i<j} (\lambda_1 (\lambda_i + \lambda_j) + 2) (\lambda_i - \lambda_j)^2 . \end{aligned}$$

From (4.0.3), (4.2.2) and (4.2.5) we have

$$\begin{aligned} &\sum_{1<i<j} (\lambda_i\lambda_j + \lambda_i\lambda_1 + \lambda_j\lambda_1 + 2) (\lambda_i - \lambda_j)^2 \\ &\quad \geq \frac{1}{2} \left(4 + \frac{1}{m-2} |H|^2 - |A|^2 \right) \left[(m-1) \left(|A|^2 - \frac{1}{m-1} |H|^2 \right) - \lambda_1 (m\lambda_1 - 2H) \right] \\ &\quad \geq \frac{\varepsilon(m-1)}{2(m-2)(m-2+\varepsilon)} |H|^2 \left(|A|^2 - \frac{1}{m-1} |H|^2 \right) \\ &\qquad + \frac{\lambda_1}{2} \left(4 + \frac{1}{m-2} |H|^2 - |A|^2 \right) (2H - m\lambda_1) . \end{aligned}$$

Let $\gamma = \dfrac{\varepsilon(m-1)}{2(m-2)(m-2+\varepsilon)\alpha}$. For any fixed $\eta' > 0$, by 4.2.4, there exists a $C_{\eta'}$ such that $\lambda_1 > -\eta' H - C_{\eta'}$. Since $\lambda_i \geq \lambda_1$, then $\lambda_i + \lambda_j \geq 2\lambda_1$. Moreover from pinching condition (4.0.1) we have

$$\sum_{1<i<j} (\lambda_i - \lambda_j)^2 \leq \sum_{1\leq i<j} (\lambda_i - \lambda_j)^2 = m|\mathring{A}|^2 \leq \frac{2}{m-2}|H|^2 + 4m.$$

Using these inequalities we get

$$\begin{aligned}
&-\sum_{1<i<j} \left(\lambda_1\left(\lambda_i + \lambda_j\right) + 2\right)\left(\lambda_i - \lambda_j\right)^2 \\
&\qquad \geq \left(2\lambda_1(\eta' H + C_{\eta'}) - 2\right)\sum_{1<i<j}(\lambda_i - \lambda_j)^2 \\
&\qquad \geq 4\lambda_1(\eta' H + C_{\eta'})\left(\frac{1}{m-2}|H|^2 + 2m\right) - 4\left(\frac{1}{m-2}|H|^2 + 2m\right).
\end{aligned}$$

Collect all the terms multiplied by λ_1:

$$\begin{aligned}
\lambda_1 \Bigg[& 4(\eta' H + C_{\eta'})\left(\frac{1}{m-2}|H|^2 + 2m\right) + \sum_j \lambda_j(\lambda_1 - \lambda_j)^2 \\
& + \frac{1}{2}\left(4 + \frac{1}{m-2}|H|^2 - |A|^2\right)(2H - m\lambda_1)\Bigg].
\end{aligned}$$

Once again we estimate $\lambda_1 > -\eta' H - C_{\eta'}$. Since $H > 0$ we have that for every i

$$\lambda_i^2 \leq |A|^2 \leq \frac{1}{m-2}|H|^2 + 4 \leq \left(\frac{1}{\sqrt{m-2}}H + 2\right)^2,$$

then

$$-\lambda_1, \lambda_n \leq \frac{1}{\sqrt{m-2}}H + 2.$$

Futhermore $\sum_j (\lambda_1 - \lambda_j)^2 \leq m\,|A|^2$. Hence for the terms in square brackets we get

$$\begin{aligned}
\sum_j \lambda_j(\lambda_1 - \lambda_j)^2 &\leq \lambda_n \sum_j (\lambda_1 - \lambda_j)^2 \\
&\leq m\lambda_n\,|A|^2 \\
&\leq m\left(\frac{1}{\sqrt{m-2}}H + 2\right)\left(\frac{1}{m-2}|H|^2 + 4\right),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\left(4 + \frac{1}{m-2}|H|^2 - |A|^2\right)(2H - m\lambda_1) \\
&\qquad \leq \frac{1}{2}\left(4 + \frac{1}{m-2}|H|^2\right)\left(2H + \frac{m}{\sqrt{m-2}}|H|^2 + 4m\right).
\end{aligned}$$

Putting together all these informations we have that, for every $\eta' > 0$

$$\begin{aligned} Z \quad \geq \quad & \gamma\alpha \left|H\right|^2 \left(\left|A\right|^2 - \frac{1}{m-1}\left|H\right|^2\right) - 4\left(\frac{1}{m-2}\left|H\right|^2 + 2m\right) \\ & - (\eta' H - C_{\eta'})\left[4(\eta' H + C_{\eta'})\left(\frac{1}{m-2}\left|H\right|^2 + 2m\right)\right. \\ & + m\left(\frac{1}{\sqrt{m-2}}H + 2\right)\left(\frac{1}{m-2}\left|H\right|^2 + 4\right) \\ & \left. + \frac{1}{2}\left(4 + \frac{1}{m-2}\left|H\right|^2\right)\left(2H + \frac{m}{\sqrt{m-2}}\left|H\right|^2 + 4m\right)\right]. \end{aligned}$$

Finally for any $\eta > 0$ we can find an η' small enough such that the terms of degree 4 in H in the formula above are bigger than $-\gamma\left(\frac{1}{m-1} + \eta\right)$. Moreover there is a constant C_η such that the lower-order terms in H are bigger than $-C_\eta\left(H^3 + 1\right)$. In particular,

$$\gamma\alpha \left|H\right|^2 \left(\left|A\right|^2 - \frac{1}{m-1}\left|H\right|^2 - \eta\left|H\right|^2\right) = \gamma W^2 f_0 - \gamma\beta W f_0.$$

By the pinching condition (4.0.1) we have

$$\gamma W^2 f_0 - \gamma\beta W f_0 \leq \gamma W^2 f_0 - \bar{\gamma}(\left|H\right|^2 + 1),$$

for some positive constant $\bar{\gamma}$. The thesis follows for a suitable constant K_η. □

Since we want to prove that the function $f_{\sigma,\eta}$ is bounded from above, we focus on f_+ the positive part $f_{\sigma,\eta}$.

Lemma 4.2.6 *There are positive constants C_4 and C_5 such that for every $\delta > 0$ the following holds:*

$$C_4 \int W f_+^p d\mu \quad \leq \quad (1+\delta)\int f_+^{p-1} W^{\sigma-1}\left|\nabla H\right|^2 d\mu + \frac{1}{\delta}\int f_+^{p-2}\left|\nabla f_{\sigma,\eta}\right|^2 d\mu + C_5.$$

Proof. For any $\eta > 0$ we call $h_{ij}^\eta = h_{ij} - \left(\frac{1}{m-1} + \eta\right) H g_{ij}$. From the definition of $f_{\sigma,\eta}$ and estimate (4.2.3) on $\Delta\left|A\right|^2$ we have

$$\begin{aligned} \Delta f_{\sigma,\eta} \quad \geq \quad & 2W^{\sigma-1}\left\langle h_{ij}^\eta, \nabla_i\nabla_j H\right\rangle + 2ZW^{\sigma-1} - 2C_3 W^\sigma \\ & -2(1-\sigma)\frac{H f_{\sigma,\eta}}{W}\Delta H - 4\alpha(1-\sigma)\frac{H}{W}\left\langle\nabla_i H, \nabla_i f_{\sigma,\eta}\right\rangle. \end{aligned} \tag{4.2.6}$$

From now on, we consider only the postive part of $f_{\sigma,\eta}$. We multiply by f_+^{p-1} and integrate. Since

$$\int f_+^{p-1}\Delta f_+ d\mu = -(p-1)\int f_+^{p-2}\left|\nabla f_+\right|^2 d\mu$$

from (4.2.6) we have

$$\begin{aligned}\int 2Zf_+^{p-1}W^{\sigma-1}d\mu &\le \int 2C_3 f_+^{p-1}W^{\sigma}d\mu - \int 2W^{\sigma-1}f_+^{p-1}\left\langle h_{ij}^{\eta}, \nabla_i\nabla_j H\right\rangle d\mu \\ &+2(1-\sigma)\int \frac{f_+^p H}{W}\Delta H d\mu + 4\alpha(1-\sigma)\int \frac{Hf_+^{p-1}}{W}\left\langle \nabla_i H, \nabla_i f_+\right\rangle d\mu \\ &-(p-1)\int f_+^{p-2}\left|\nabla f_+\right|^2 d\mu.\end{aligned}$$

Integrating by parts we have

$$\begin{aligned}-\int 2f_+^{p-1}W^{\sigma-1}\left\langle h_{ij}^{\eta}, \nabla_i\nabla_j H\right\rangle d\mu &= 2(p-1)\int f_+^{p-2}W^{\sigma-1}\left\langle h_{ij}^{\eta}, \nabla_i H\nabla_j f_+\right\rangle d\mu \\ &-4\alpha(1-\sigma)\int Hf_+^{p-1}W^{\sigma-2}\left\langle h_{ij}^{\eta}, \nabla_i H\nabla_j H\right\rangle d\mu \\ &+2\int f_+^{p-1}W^{\sigma-1}\left|\nabla H\right|^2 d\mu.\end{aligned}$$

Using the fact that $\alpha\left|H\right|^2 \le W$ and $f_{\sigma,\eta} \le W^{\sigma}$ and itegrating by parts again we have

$$\begin{aligned}\int \frac{f_+^p H}{W}\Delta H d\mu &= -p\int \frac{Hf_+^{p-1}}{W}\left\langle \nabla H, \nabla f_+\right\rangle d\mu - \int \frac{f_+^p}{W}\left|\nabla H\right|^2 d\mu \\ &+2\alpha\int \frac{\left|H\right|^2 f_+^p}{W^2}\left|\nabla H\right|^2 d\mu \\ &\le -p\int \frac{Hf_+^{p-1}}{W}\left\langle \nabla H, \nabla f_+\right\rangle d\mu - \int \frac{f_+^p}{W}\left|\nabla H\right|^2 d\mu \\ &+2\int \frac{f_+^p}{W}\left|\nabla H\right|^2 d\mu \\ &= \int \frac{f_+^p}{W}\left|\nabla H\right|^2 d\mu - p\int \frac{Hf_+^{p-1}}{W}\left\langle \nabla H, \nabla f_+\right\rangle d\mu.\end{aligned}$$

Then

$$\begin{aligned}
\int 2Zf_+^{p-1}W^{\sigma-1}d\mu \ \le\ & \int 2C_3 f_+^{p-1}W^{\sigma} + 2\int f_+^{p-1}W^{\sigma-1}\left|\nabla H\right|^2 d\mu \\
& +2(p-1)\int f_+^{p-2}W^{\sigma-1}\left\langle h_{ij}^{\eta}, \nabla_i H \nabla_j f_+\right\rangle d\mu \\
& -4\alpha(1-\sigma)\int H f_+^{p-1}W^{\sigma-2}\left\langle h_{ij}^{\eta}, \nabla_i H \nabla_j H\right\rangle d\mu \\
& +2(1-\sigma)\int \frac{f_+^p}{W}\left|\nabla H\right|^2 d\mu \\
& -2p(1-\sigma)\int \frac{H f_+^{p-1}}{W}\left\langle \nabla H, \nabla f_+\right\rangle d\mu \\
& +4\alpha(1-\sigma)\int \frac{H f_+^{p-1}}{W}\left\langle \nabla_i H, \nabla_i f_+\right\rangle d\mu \\
& -(p-1)\int f_+^{p-2}\left|\nabla f_{\sigma,\eta}\right|^2 d\mu.
\end{aligned}$$

Now we split the gradient terms into two types: $|\nabla H|^2$ and $|\nabla f_+|\,|\nabla H|$. We have $H \le C\sqrt{W}$ for some constant C, $f_0 < 1$ and $f_{\sigma,\eta} \le W^{\sigma}$, then

$$\begin{aligned}
\left|f_+^{p-2}W\left\langle h_{ij}^{\eta}, \nabla_i H\nabla_j f_+\right\rangle\right| &\le f_+^{p-2}\sqrt{f_0}W^{\sigma-\frac{1}{2}}\left|\nabla f_+\right|\left|\nabla H\right| \\
&\le f_+^{p-2}f_0 W^{\sigma-\frac{1}{2}}\left|\nabla f_+\right|\left|\nabla H\right| \\
&= f_+^{p-1}W^{-\frac{1}{2}}\left|\nabla f_+\right|\left|\nabla H\right|, \\
\left|H f_+^{p-1}W^{\sigma-2}\left\langle h_{ij}^{\eta}, \nabla_i H\nabla_j H\right\rangle\right| &\le H f_+^{p-1}\sqrt{f_0}W^{\sigma-1-\frac{1}{2}}\left|\nabla H\right|^2 \\
&\le C f_+^{p-1}W^{\sigma-1}\left|\nabla H\right|^2, \\
\frac{f_+^p}{W} &= f_+^{p-1}\frac{f_+}{W} \le f_+^{p-1}W^{\sigma-1}, \\
\left|\frac{H f_+^{p-1}}{W}\left\langle \nabla_i H, \nabla_i f_+\right\rangle\right| &\le \frac{H f_+^{p-1}}{W}\left|\nabla f_+\right|\left|\nabla H\right| \\
&\le C f_+^{p-1}W^{-\frac{1}{2}}\left|\nabla f_+\right|\left|\nabla H\right|,
\end{aligned}$$

for some constant C. Hence the first type terms are

$$\begin{aligned}
2\int f_+^{p-1}W^{\sigma-1}\left|\nabla H\right|^2 d\mu + 2(1-\sigma)\int \frac{f_+^p}{W}\left|\nabla H\right|^2 d\mu \\
-4\alpha(1-\sigma)\int H f_+^{p-1}W^{\sigma-2}\left\langle h_{ij}^{\eta}, \nabla_i H\nabla_j H\right\rangle d\mu,
\end{aligned}$$

which are smaller then $C\int f_+^{p-1}W^{\sigma-1}\left|\nabla H\right|^2 d\mu$ for some constant C. The second type terms are

$$\begin{aligned}
+2(p-1)\int f_+^{p-2}W^{\sigma-1}\left\langle h_{ij}^{\eta}, \nabla_i H\nabla_j f_+\right\rangle d\mu - 2p(1-\sigma)\int \frac{H f_+^{p-1}}{W}\left\langle \nabla H, \nabla f_+\right\rangle d\mu \\
+4\alpha(1-\sigma)\int \frac{H f_+^{p-1}}{W}\left\langle \nabla_i H, \nabla_i f_+\right\rangle d\mu
\end{aligned}$$

which are smaller then $2C\int f_+^{p-1}W^{-\frac{1}{2}}|\nabla f_+||\nabla H|\,d\mu$. We use Young's inequality on this integral, then for every $\delta > 0$ we have

$$2f_+^{p-1}W^{-\frac{1}{2}}|\nabla f_+||\nabla H| \leq \delta f_+^{p-1}W^{\sigma-1}|\nabla H|^2 + \frac{1}{\delta}f_+^{p-2}|\nabla f_{\sigma,\eta}|^2.$$

Now recall lemma 4.2.5, then for H big enough we have that $Z \geq \gamma W^2 f_0 - K_\eta H^3$. Multiply by $2W^{\sigma-1}f_+^{p-1}$ and integrate, we get

$$\begin{aligned}\frac{2\gamma}{C}\int Wf_+^p d\mu &\leq (1+\delta)\int f_+^{p-1}W^{\sigma-1}|\nabla H|^2 d\mu + \frac{1}{\delta}\int f_+^{p-2}|\nabla f_+|^2 d\mu \\ &\quad + \frac{2C_3}{C}\int f_+^{p-1}W^\sigma d\mu + \frac{2K_\eta}{C}\int f_+^{p-1}W^{\sigma-1}H^3 d\mu.\end{aligned}$$

We estimate $\int f_+^{p-1}W^\sigma d\mu$ and $\int f_+^{p-1}W^{\sigma-1}H^3 d\mu$ like similar terms in the proof of lemma 3.3.4.

By Young's inequality, for any $r > 0$, we have

$$f_+^{p-1}W^\sigma = W\left(f_+^{p-1}W^{\sigma-1}\right) \leq W\left(\frac{r^{\frac{p}{p-1}}}{2}f_+^p + \frac{r^{-p}}{2}W^{(\sigma-1)p}\right).$$

Moreover $H^3 = \left(\frac{\alpha H^2}{\alpha}\right)^{\frac{3}{2}} \leq \alpha^{-\frac{3}{2}}W^{\frac{3}{2}}$, then for any $s > 0$ we get

$$\begin{aligned}f_+^{p-1}W^{\sigma-1}H^3 &\leq \alpha^{-\frac{3}{2}}f_+^{p-1}W^{\sigma+\frac{1}{2}} \\ &\leq \alpha^{-\frac{3}{2}}W\left(\frac{s^{\frac{p}{p-1}}}{2}f_+^p + \frac{s^{-p}}{2}W^{(\sigma-\frac{1}{2})p}\right)\end{aligned}$$

Fix $r = s$ such that $\left(\frac{C_3}{C} + \frac{K_\eta\alpha^{-\frac{3}{2}}}{C}\right)\frac{r^{\frac{p}{p-1}}}{2} = \frac{\gamma}{C}$. Furthermore we have that σ is small, then $(\sigma-1)p + 1 \leq (\sigma - \frac{1}{2})p + 1 < 0$, $W > \beta$, β is a positive constant, then by lemma 2.4.3 we have

$$\begin{aligned}&\frac{2C_3}{C}\int \frac{r^{-p}}{2}W^{(\sigma-1)p+1}d\mu + \frac{2K_\eta}{C\alpha^{\frac{3}{2}}}\int \frac{r^{-p}}{2}W^{(\sigma-\frac{1}{2})p+1}d\mu \\ &\leq \frac{r^{-p}}{C}\left(C_3\beta^{(\sigma-1)p+1} + \frac{K_\eta}{\alpha^{\frac{3}{2}}}\beta^{(\sigma-\frac{1}{2})p+1}\right)vol(\mathcal{M}_t) \\ &\leq \frac{r^{-p}}{C}\left(C_3\beta^{(\sigma-1)p+1} + \frac{K_\eta}{\alpha^{\frac{3}{2}}}\beta^{(\sigma-\frac{1}{2})p+1}\right)vol(\mathcal{M}_0) =: C_5.\end{aligned}$$

Choosing $C_4 = \frac{\gamma}{C}$ we have the thesis. □

Since we have already proved that T_{max} is finite, we can bound high L^p-norms of f_+ provided σ is of order $p^{-\frac{1}{2}}$.

Theorem 4.2.7 *There are positive constants C_6, C_7 and C_8 depending only on $\mathcal{M}_0$ such that for all $p \geq C_7$ and $\sigma \leq \frac{C_8}{\sqrt{p}}$*

$$\left(\int f_+^p d\mu\right)^{\frac{1}{p}} \leq C_6.$$

holds.

Multiply inequality found in theorem 4.1.1 by pf_+^{p-1} and integrate, we have

$$\begin{aligned}\frac{\partial}{\partial t}\int f_+^p d\mu + p(p-1)\int f_+^{p-2}\left|\nabla f_+\right|^2 d\mu + 2C_1 p\int f_+^{p-1}W^{\sigma-1}\left|\nabla H\right|^2 d\mu \\ \leq 4p\int \frac{H}{W} f_+^{p-1}\left|\nabla f_+\right|\left|\nabla H\right| d\mu + 2p\sigma\int(|A|^2+\bar{r})f_+^p d\mu - 2pC_2\int f_+^p d\mu.\end{aligned}$$

Now we can proceed in a similar manner of the proof of theorem 3.3.5 to estimate $\int \frac{H}{W} f_+^{p-1}\left|\nabla f_+\right|\left|\nabla H\right| d\mu$. We get

$$\begin{aligned}\frac{\partial}{\partial t}\int f_+^p d\mu + \frac{p(p-1)}{2}\int f_+^{p-2}\left|\nabla f_+\right|^2 d\mu + C_1 p\int f_+^{p-1}W^{\sigma-1}\left|\nabla H\right|^2 d\mu \\ \leq 2p\sigma\int |A|^2 f_+^p d\mu + 2p(\sigma\bar{r}-C_2)\int f_+^p d\mu.\end{aligned}$$

From the pinching inequality (4.0.1), $|A|^2 \leq mW$. Then, by lemma 4.2.6,

$$\begin{aligned}2p\sigma\int |A|^2 f_+^p d\mu \leq 2pm\sigma\int W f_+^p d\mu \\ \leq \frac{2pm\sigma}{C_4}\left[(1+\delta)\int f_+^{p-1}W^{\sigma-1}\left|\nabla H\right|^2 d\mu + \frac{1}{\delta}\int f_+^{p-2}\left|\nabla f_{\sigma,\eta}\right|^2 d\mu + C_5\right]\end{aligned}$$

If p is big enough and σ is small enough, we can find a δ such that

$$\begin{cases}\dfrac{p(p-1)}{2} \geq \dfrac{4\sigma pm}{C_4\delta}, \\ C_1 p \geq \dfrac{4\sigma pm}{C_4}(1+\delta).\end{cases}$$

Define $\bar{C}_2 = 2p(\sigma\bar{r}-C_2)$ and $\bar{C}_5 = \dfrac{2\sigma pm}{C_4}C_5$, then we have

$$\frac{\partial}{\partial t}\int f_+^p d\mu \leq \bar{C}_2\int f_+^p d\mu + \bar{C}_5.$$

Since $T_{max} < \infty$ we have the thesis. □

This result, together a Stampacchia iteration procedure, gives an uniform bound of the function $f_{\sigma,\eta}$. This prove theorem 4.2.1 and so we have the validity of the corollary 4.2.2 too. Then the proof of theorem 4.0.3 is conclused.

4.3 Gradient estimate

To conclude the study of the singularities we need an estimate for the gradient of the second fundamental form. It allows us to comapre the norm of the second fundamental form in different points of the submanifold giving that the cylindrical estimate 4.2.2 holds not only in the singular point but in a neighborhood too. First we need the evolution equation for the gradient of the second fundamental form as in [Ha, H1].

Lemma 4.3.1 *There is a positive constant C_9 such that*

$$\frac{\partial}{\partial t}|\nabla A|^2 \le \Delta |\nabla A|^2 - \left|\nabla^2 A\right|^2 + C_9\left(|A|^2+1\right)|\nabla A|^2$$

holds everywhere on $\mathcal{M}_t$, for any $0 \le t < T_{max}$.

The principal result of this section is the following gradient estimate.

Proposition 4.3.2 *There is a positive constant C_{10} such that*

$$|\nabla A|^2 \le C_{10}(|A|^4+1)$$

holds everywhere on $\mathcal{M}_t$, for any $0 \le t < T_{max}$.

Proof. The proof is inspired by theorem 6.1 of [HS2]. Define $k_m = \frac{1}{2}\left(\frac{3}{m+2} - \frac{1}{m-1}\right)$. It is a positive constant. Cylindrical estimate 4.2.1 with $\eta = k_m$ says that there is a positive constant c_m such that

$$\left(\frac{1}{m-1}+k_m\right)|H|^2 - |A|^2 + c_m \ge 0.$$

Define the following functions:

$$\begin{aligned} g_1 &= \left(\frac{1}{m-1}+k_m\right)|H|^2 - |A|^2 + 2c_m; \\ g_2 &= \frac{3}{m+2}|H|^2 - |A|^2 + 2c_m, \end{aligned}$$

then $g_2 \ge g_1 \ge c_m > 0$ and for every i $g_i - 2c_m = 2(g_i - c_m) - g_i \ge -g_i$. For short set

$$a_i = \begin{cases} \dfrac{1}{m-1}+k_m & \text{if} \quad i=1, \\ \dfrac{3}{m+2} & \text{if} \quad i=2. \end{cases}$$

By lemma 2.4.4 we find that for every i we have the following evolution equation:

$$\begin{aligned} \frac{\partial}{\partial t}g_i &\ge \Delta g_i - 2\left(a_i|\nabla H|^2 - |\nabla A|^2\right) \\ &\quad +2a_i|H|^2\left(|A|^2+\bar{r}\right) - 2|A|^2\left(|A|^2+\bar{r}\right) + 4m|\mathring{A}|^2 \\ &= \Delta g_i - 2\left(a_i|\nabla H|^2 - |\nabla A|^2\right) \\ &\quad +2g_i(|A|^2+\bar{r}) - 4c_n(|A|^2+\bar{r}) + 4m|\mathring{A}|^2. \end{aligned}$$

From lemma 2.4.5 in particular we have

$$\frac{\partial}{\partial t} g_1 \geq \Delta g_1 + \frac{2}{3}(m+2)k_m |\nabla A|^2 - 2|A|^2 g_1, \tag{4.3.1}$$

$$\frac{\partial}{\partial t} g_2 \geq \Delta g_2 - 2|A|^2 g_2. \tag{4.3.2}$$

For every functions f and g we have the following general formula

$$\frac{\partial}{\partial t}\left(\frac{f}{g}\right) - \Delta\left(\frac{f}{g}\right) = \frac{2}{g}\left\langle \nabla g, \nabla\left(\frac{f}{g}\right)\right\rangle + \frac{1}{g}\left(\frac{\partial}{\partial t} f - \Delta f\right) - \frac{f}{g^2}\left(\frac{\partial}{\partial t} g - \Delta g\right). \tag{4.3.3}$$

We want to find the evolution equation for $\frac{|\nabla A|^2}{g_1 g_2}$. We use twice the formula (4.3.3): for the first time consider $f = |\nabla A|^2$ and $g = g_1$. From lemma 4.3.1 and inequality (4.3.1) we have

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{|\nabla A|^2}{g_1}\right) - \Delta\left(\frac{|\nabla A|^2}{g_1}\right) \leq{}& \frac{2}{g_1}\left\langle \nabla g_1, \nabla\left(\frac{|\nabla A|^2}{g_1}\right)\right\rangle \\ &+ \frac{1}{g_1}\left(-2\left|\nabla^2 A\right|^2 + C_9(|A|^2+1)|\nabla A|^2\right) \\ &- \frac{|\nabla A|^2}{g_1^2}\left(\frac{2}{3}(m+2)k_m|\nabla A|^2 - 2|A|^2 g_1\right).\end{aligned}$$

From Schwarz inequality and Young inequality we have

$$\begin{aligned}-\left|\nabla^2 A\right|^2 + \left\langle \nabla g_1, \nabla\left(\frac{|\nabla A|^2}{g_1}\right)\right\rangle ={}& -\left|\nabla^2 A\right|^2 - \frac{1}{g_1^2}|\nabla H|^2|\nabla g_1|^2 + \frac{1}{g_1}\left\langle \nabla g_1, \nabla|\nabla A|^2\right\rangle \\ \leq{}& -\left|\nabla^2 A\right|^2 - \frac{1}{g_1^2}|\nabla H|^2|\nabla g_1|^2 + \frac{1}{g_1}|\nabla g_1||\nabla H|\left|\nabla^2 H\right| \\ \leq{}& -\left|\nabla^2 A\right|^2 - \frac{1}{g_1^2}|\nabla H|^2|\nabla g_1|^2 + \frac{1}{g_1}\left(\frac{1}{g_1}|\nabla g_1|^2|\nabla H|^2 + g_1\left|\nabla^2 H\right|^2\right) \\ \leq{}& 0.\end{aligned}$$

Then

$$\frac{\partial}{\partial t}\left(\frac{|\nabla A|^2}{g_1}\right) - \Delta\left(\frac{|\nabla A|^2}{g_1}\right) \leq (C_9+2)\frac{|\nabla A|^2}{g_1}\left(|A|^2+1\right)|\nabla A|^2 - \frac{2}{3}(m+2)k_m\frac{|\nabla A|^4}{g_1^2}. \tag{4.3.4}$$

Now we apply again the general formula (4.3.3) with, this time, $f = \frac{|\nabla A|^2}{g_1}$ and $g = g_2$: from (4.3.2) and (4.3.4) we have:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{|\nabla A|^2}{g_1 g_2}\right) - \Delta\left(\frac{|\nabla A|^2}{g_1 g_2}\right) - \frac{2}{g_2}\left\langle \nabla g_2, \nabla\left(\frac{|\nabla A|^2}{g_1 g_2}\right)\right\rangle \\
&\quad \le \frac{1}{g_2}\left((C_9+2)\frac{|\nabla A|^2}{g_1}\left(|A|^2+1\right)|\nabla A|^2 - \frac{2}{3}(m+2)k_m\frac{|\nabla A|^4}{g_1^2}\right) + 2\frac{|\nabla A|^2}{g_1 g_2}|A|^2 \\
&\quad \le (C_9+4)(|A|^2+1)\frac{|\nabla A|^2}{g_1 g_2} - \frac{2}{3}(m+2)k_m\frac{|\nabla A|^2}{g_1^2 g_2}.
\end{aligned}$$

Using the pinching condition (4.0.1) and choosing a c_m sufficiently big, there is a positive constant C_9' such that

$$\begin{aligned}
g_2 &\ge \left(\frac{3(m-2)}{m+2} - 1\right)|A|^2 + 2c_m - 4(m-2) \\
&\ge C_9'(|A|^2+1).
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{|\nabla A|^2}{g_1 g_2}\right) - \Delta\left(\frac{|\nabla A|^2}{g_1 g_2}\right) - \frac{2}{g_2}\left\langle \nabla g_2, \nabla\left(\frac{|\nabla A|^2}{g_1 g_2}\right)\right\rangle \\
&\quad \le (|A|^2+1)\frac{|\nabla A|^2}{g_1 g_2}\left(C_9 + 4 - C_9'\frac{|\nabla A|^2}{g_1 g_2}\right).
\end{aligned}$$

From the maximum principle it follows that

$$\frac{|\nabla A|^2}{g_1 g_2} \le max\left(m_0, \frac{C_9+4}{C_9'}\right),$$

where $m_0 = max_{\mathcal{M}_0}\frac{|\nabla A|^2}{g_1 g_2}$. In any case the function $\frac{|\nabla A|^2}{g_1 g_2}$ is bounded. Together with the inequality $|H|^2 \le m|A|^2$ we can conclude that there is a positive constant C_{10} such that $\frac{|\nabla A|^2}{g_1 g_2} \le C_{10}\left(|A|^4+1\right)$ holds. □

Chapter 5

Mean curvature flow and Riemannian submersions

We consider submersion defined by the action of a group of isometries : let G be a Lie group acting as isometries of a Riemannian manifold $(\overline{\mathcal{M}}, g_{\overline{\mathcal{M}}})$. Suppose that the quotient space, obteined identifying the points of a orbit of the action of G on $\overline{\mathcal{M}}$, is a smooth manifold $\overline{\mathcal{B}} = \overline{\mathcal{M}}/G$ and consider the induced metric $g_{\overline{\mathcal{B}}}$ on it. The natural projection $\pi : \overline{\mathcal{M}} \to \overline{\mathcal{B}}$ is a submersion with fibers the orbits of G. If the action of G is free we have the well-known principal bundles. In this case the fiber of π are isometric to the group G.

Lifting a submanifold of $\overline{\mathcal{B}}$ we have a submanifold of $\overline{\mathcal{M}}$ G-invariant, vice versa projecting a G-invariant submanifold of $\overline{\mathcal{M}}$ we get a submanifold of $\overline{\mathcal{B}}$. We want to study how the mean curvature flow is related to a submersion, in particular we show a sufficient condition for the mean curvature flow commutes with the submersion.

Theorem 5.0.1 *Let $\pi : \overline{\mathcal{M}} \to \overline{\mathcal{B}} = \overline{\mathcal{M}}/G$ a submersion. If π has closed and minimal fibers then the mean curvature flow of any closed submanifold commutes with the submersion. More precisely let $\mathcal{M}_0$ is a G-invariant submanifold of $\overline{\mathcal{M}}$ and $\mathcal{B}_0$ is a submanifold of $\overline{\mathcal{B}}$. If $\pi(\mathcal{M}_0) = \mathcal{B}_0$ then the mean curvature flow of $\mathcal{M}_0$ and $\mathcal{B}_0$ are defined up to the same maximal time T_{max} and $\pi(\mathcal{M}_t) = \mathcal{B}_t$ for any time $0 \leq t < T_{max}$.*

Note that consider closed fibers and closed initial immersions guarantee the uniqueness of the solution of mean curvature flow of the submanifold $\mathcal{B}_0$ and its lift.

Lemma 5.0.2 *Let $F_0 : \mathcal{M} \to \overline{\mathcal{M}}$ be a closed immersion and φ an isometry of $\overline{\mathcal{M}}$, then φ commutes with the mean curvature flow. Formally if $G_0 = \varphi \circ F_0$ and F_t and G_t are the evolutions of F_0 and G_0 respectively, we have that $G_t = \varphi \circ F_t$ for any time t that the flow is defined.*

Proof. Since φ is an isometry we have

$$\frac{\partial}{\partial t}(\varphi \circ F_t)(p,t) = \varphi_* H^F(p,t) = H^{\varphi \circ F}(p,t),$$

where H^ψ is the mean curvature vector of ψ for any immersion ψ. Then $\varphi \circ F_t$ is a solution of the mean curvature flow of initial data $\varphi \circ F_0 = G_0$. For the uniqueness of the solution we have the thesis. $\square$

It follows immediately that

Corollary 5.0.3 *Let F_0 and φ like in the statement of Lemma 5.0.2 and G a group of isometries of $\overline{\mathcal{M}}$. We have*

1) if F_0 is φ-invariant, then F_t is φ-invariant for any t,

2) if F_0 is G-invariant then F_t is G-invariant for every time t.

Proof of Theorem 5.0.1. Let $F_0 : \mathcal{B} \to \overline{\mathcal{B}}$ and $F_0' : \mathcal{M} \to \overline{\mathcal{M}}$ two immersions for $\mathcal{B}_0$ and $\mathcal{M}_0$ respectively. By hypothesis we have that F_0' is G-invariant and $\pi \circ F_0' = F_0 \circ \pi$. The crucial point is that, since the fibers are minimal, we have that H' is basic and is π-related with H, where H is the mean curvature vector of any submanifold of $\overline{\mathcal{B}}$ and H' is the mean curvature vector of its lift to $\overline{\mathcal{M}}$. In fact H' is horizontal because it is normal to $\mathcal{M}$. Moreover let $(X_1, \ldots, X_m)$ a local orthonormal frame tangent to $\mathcal{B}$ around a point p and consider $(V_1, \ldots, V_{\hat{m}})$ a local orthonormal set of vertical vector fields. Then around any point q of the fiber $\pi^{-1}(p)$ we use the orthonormal basis $(X_1^{\mathscr{H}}, \ldots, X_m^{\mathscr{H}}, V_1, \ldots V_{\hat{m}})$ tangent to $\mathcal{M}$. By (2.2.2) and (2.2.3) we have

$$
\begin{aligned}
H' &= \sum_i A'(X_i^{\mathscr{H}}, X_i^{\mathscr{H}}) + \sum_i A'(V_i, V_i) \\
&= \sum_i A(X_i, X_i)^{\mathscr{H}} + \sum_i \hat{A}(V_i, V_i)^{\perp} \\
&= \left(\sum_i A(X_i, X_i) \right)^{\mathscr{H}} + \left(\sum_i \hat{A}(V_i, V_i) \right)^{\perp} \\
&= H^{\mathscr{H}} + \hat{H}^{\perp}.
\end{aligned}
$$

If the fibers are minimal we get $H' = H^{\mathscr{H}}$, that is H' and H are π-related. In particular $\pi_* H' = H$ holds. Now let F_t the evolution of F_0, F_t' the lift of F_t, $\widetilde{F}_t'$ the evolution of F_0' and $\widetilde{F}_t$ the projection of $\widetilde{F}_t'$. We want to prove that $F_t' = \widetilde{F}_t'$ and $F_t = \widetilde{F}_t$. By construction we have that for any t

$$\pi \circ F_t' = F_t \circ \pi, \tag{5.0.1}$$

and F_t' is G-invariant. Then in particular H' is horizontal. Deriving (5.0.1) we have

$$\pi_* \frac{\partial}{\partial t} F_t' = \frac{\partial}{\partial t}(F_t \circ \pi) = H = \pi_* H'.$$

Then $\frac{\partial}{\partial t} F_t' = H' + V'$ for some vertical vector field V'. Since F_t' is G-invariant, V' is tangent to $F_t'(\mathcal{M}')$. Therefore

$$\left(\frac{\partial}{\partial t} F_t' \right)^{\perp} = H'.$$

This means that, up to a tangential diffeomorphism, F'_t is the solution of the mean curvature flow of initial data F'_0. Then $F'_t = \widetilde{F}'_t$. Vice versa

$$\frac{\partial}{\partial t}\left(\widetilde{F}_t \circ \pi\right) = \frac{\partial}{\partial t}\left(\pi \circ \widetilde{F}'_t\right) = \pi_* \frac{\partial}{\partial t}\widetilde{F}'_t = \pi_* \widetilde{H}'.$$

Corollary 5.0.3 says that $\widetilde{F}'_t$ is G-invariant as its initial data F'_0, then $\pi_* \widetilde{H}' = \widetilde{H}$, the mean curvature vector of $\widetilde{F}_t$. Then $\widetilde{F}_t$ is the evolution of initial data F_0, that is $\widetilde{F}_t = F_t$ for any time t. □

Remark 5.0.4 *If fibers are not closed could happen that there are no unique solution of the mean curvature flow of the lift. But if they are minimal, the same proof given for theorem 5.0.1 shows that the lift of the mean curvature flow is, in any case, a G-invariant solution of the mean curvature flow. In the same way the projection of a G-invariant solution is again an evolution by mean curvature. Then if the projection of the initial data $\mathcal{M}_0$ is a closed submanifold $\mathcal{B}_0$ then there exists only one G-invariant solution of initial data $\mathcal{M}_0$.*

5.1 Examples and applications

A trivial example is given by the product manifold: consider a Riemannian manifold $(\overline{\mathcal{M}}, g_{\overline{\mathcal{M}}})$, a Lie group G endowed with a left-invariant metric g_G and the product manifold $(\overline{\mathcal{M}} \times G, g_{\overline{\mathcal{M}}} + g_G)$. The projection to the first factor $\pi : \overline{\mathcal{M}} \times G \to \overline{\mathcal{M}}$ is a Riemannian submersion with fiber isometric to G and totally geodesic. In this case the lift of a submanifold $\mathcal{M}_0$ of $\overline{\mathcal{M}}$ is $\mathcal{M}_0 \times G$ and theorem 5.0.1 says that the mean curvature flow of $\mathcal{M}_0 \times G$ is given by $\mathcal{M}_t \times G$, where $\mathcal{M}_t$ is the evolution of $\mathcal{M}_0$ in $\overline{\mathcal{M}}$. Moreover in this trivial case the mixed terms $A'(X, V) = 0$ for every X horizontal and V vertical.

Notation 5.1.1 *For any submersion π considered below $(X_1, \dots, X_m)$ denote a local orthonormal frame tangent to a submanifold of the base space around a point p and $(V_1, \dots, V_{\hat{m}})$ is a local orthonormal set of vertical vector fields. Then around any point q of the fiber $\pi^{-1}(p)$ we use the orthonormal basis $(X_1^{\mathscr{H}}, \dots, X_m^{\mathscr{H}}, V_1, \dots V_{\hat{m}})$ tangent to the lift of the submanifold. Moreover $(\xi_1, \dots, \xi_k)$ is a local orthonormal frame normal to a submanifold of the base, then $(\xi_1^{\mathscr{H}}, \dots, \xi_k^{\mathscr{H}})$ is a local orthonormal frame normal to the lift of the submanifold considered.*

One of the best known examples of submersions is the family of the Hopf fibrations introduced in section 2.3. Let us consider the Hopf fibration $\pi : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$. In this case $V = J\nu$ is the vertical unit vector field, where J is the complex structure of $\mathbb{C}^{n+1}$ and ν is the outward normal unict vector field of the sphere as submanifold of $\mathbb{R}^{2n+2} \equiv \mathbb{C}^{n+1}$. Let $\mathcal{B}_0$ a submanifold of $\mathbb{CP}^n$ of dimension m and codimension k and $\mathcal{M}_0$ its lift to $\mathbb{S}^{2n+1}$. The fibers $\mathbb{S}^1$ are geodesics, hence of course minimal. For every i define $J\xi_i = -U_i + N_i$ where U_i is tangent to $\mathcal{B}_0$, while N_i is normal. As shown in [O] for every horizontal lift we have

$$\mathcal{A}_{X^{\mathscr{H}}} V = \mathscr{H} \bar{\nabla}_{X^{\mathscr{H}}}(J\nu) = \mathscr{H} J \bar{\nabla}_{X^{\mathscr{H}}} \nu = \mathscr{H} J X^{\mathscr{H}} = (JX)^{\mathscr{H}},$$

where with J we denote both the complex structure of $\mathbb{C}^{n+1}$ and the one induced on $\mathbb{CP}^n$. If X is tangent to $\mathcal{M}_0$ then $A'(X,V)$ is an horizontal vector field then, by lemma 2.2.3

$$\begin{aligned} A'(X,V) &= (\mathcal{A}_X V)^\perp = (JX)^\perp \\ &= \sum_i \bar{g}(JX, \xi_i^{\mathscr{H}})\xi_i^{\mathscr{H}} \\ &= -\sum_i \bar{g}(X, (J\xi_i)^{\mathscr{H}})\xi_i^{\mathscr{H}} \\ &= \sum_i \bar{g}(X, U_i^{\mathscr{H}})\xi_i^{\mathscr{H}}. \end{aligned}$$

Togeter to (2.2.2) that holds in general we have

$$|A'|^2 = |A|^2 + 2\sum_i \left|U_i^{\mathscr{H}}\right|^2 = |A|^2 + 2\sum_i |U_i|^2 .$$

For any $\lambda > 0$ we can consider $\bar{g}_\lambda$, the metric on $\mathbb{S}^{2n+1}$ obtained deforming the standard metric of constant sectional curvature via the canonical variation of the Hopf fibration. Respect to this metric, a unit vertical vector field is $V_\lambda = \lambda^{-\frac{1}{2}} J\nu$, then with the same computation seen above we have that

$$|A'|^2 = |A|^2 + 2\lambda^{-\frac{1}{2}} \sum_i |U_i|^2 .$$

Then, for any $\lambda > 0$

$$|A|^2 \leq |A'|^2 \leq |A|^2 + 2\lambda^{-\frac{1}{2}} cod\mathcal{M}_0 = |A|^2 + 2\lambda^{-\frac{1}{2}} cod\mathcal{B}_0$$

holds, with $|A'|^2 = |A|^2$ if and only if $U_i = 0$ for every i, that is $\mathcal{B}_0$ is a complex submanifold and hence minimal, and $|A'|^2 = |A|^2 + 2\lambda^{-\frac{1}{2}} cod\mathcal{B}_0$ if and only if $U_i = -J\xi_i^{\mathscr{H}}$ for every i, that is $\mathcal{B}_0$ is a CR-submanifold of CR-dimension $m-k$. Obviously since $\pi_* H' = H$ and H' is horizontal we have that $|H'|^2 = |H|^2$ in any case.

In the same way we can study Hopf fibration $\pi : \mathbb{S}^{4n+3} \to \mathbb{HP}^n$. In this case the fibers are $\mathbb{S}^3$ which are totally geodesic. Let J_1, J_2 and J_3 the complex stuctures of $\mathbb{H}^{n+1}$ given by the multiplication of the quaternionic imaginary units. Then $(V_1 = J_1\nu, V_2 = J_2\nu, V_3 = J_3\nu)$ is an orthonormal basis of $\mathscr{V}$. Following the same notations and the same computations of the previous case we define for every i and α define $J_\alpha \xi_i^{\mathscr{H}} = -U_{i\alpha} + N_{i\alpha}$ where $U_{i\alpha}$ is tangent to $\mathcal{M}_0$, while $N_{i\alpha}$ is normal. Moreovere

$$\mathcal{A}_{X^{\mathscr{H}}} V_\alpha = (J_\alpha X)^{\mathscr{H}} ,$$

and

$$A'(X, V_\alpha) = \sum_i \bar{g}(X, U_{i\alpha}^{\mathscr{H}})\xi_i^{\mathscr{H}}.$$

Then we get

$$|A'|^2 = |A|^2 + 2\sum_{i,\alpha} \left|U_{i\alpha}^{\mathscr{H}}\right|^2 = |A|^2 + 2\sum_{i,\alpha} |U_{i\alpha}|^2 .$$

The canonical variation of this Hopf fibration gives a second family $\{\tilde{g}_\lambda\}_{\lambda>0}$ of metric on $\mathbb{S}^{4n+3}$. Likewise to the previous case we have

$$|A'|^2 = |A|^2 + 2\lambda^{-\frac{1}{2}} \sum_{i,\alpha} |U_{i\alpha}|^2 .$$

Then for every $\lambda > 0$

$$|A|^2 \leq |A'|^2 \leq |A|^2 + 6\lambda^{-\frac{1}{2}} cod\mathcal{M}_0 = |A|^2 + 6\lambda^{-\frac{1}{2}} cod\mathcal{B}_0.$$

As application of theorem 5.0.1 we have the following results.

Proposition 5.1.2 *Let $\mathcal{M}_0$ be a closed $\mathbb{S}^1$-invariant hypersurface of $(\mathbb{S}^{2n+1}, \bar{g}_\lambda)$, with $n \geq 3$. If $\mathcal{M}_0$ satisfies*

$$|A'|^2 < \frac{1}{2n-2} |H'|^2 + 2 + 2\lambda^{-\frac{1}{2}}, \tag{5.1.1}$$

then the mean curvature flow of $\mathcal{M}_0$ develops a singularity in finite time and converges to a $\mathbb{S}^1$, then such an $\mathcal{M}_0$ is diffeomorphic to a $\mathbb{S}^1 \times \mathbb{S}^{2n-1}$.

Proof. Considere the Hopf fibration $\pi : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$. Since $\mathcal{M}_0$ is $\mathbb{S}^1$ invariant we can project it to an hypersurface $\mathcal{B}_0$ of $\mathbb{CP}^n$. For hypersurfaces we have necessarily $|A'|^2 = |A|^2 + 2\lambda^{-\frac{1}{2}}$. Then $\mathcal{B}_0$ satisfies

$$|A|^2 < \frac{1}{2n-2} |H|^2 + 2$$

By theorem 3.5.1 the evolution of $\mathcal{B}_0$ converges in finite time to a round point p. By theorem 5.0.1 we have that the mean curvature flow commutes with π then the lift of the evolution of $\mathcal{B}_0$ is the evolution of $\mathcal{M}_0$. We have the convergence in finite time to $\pi^{-1}(p) = \mathbb{S}^1$. □

Note that, for $\lambda = 1$, the pinching condition 5.1.1 is the same studied by Nguyen in [Ng]. We reached a similar result: a cylindrical singularity. Note that Nguyen used the further hypothesis $H \neq 0$ everywhere, while we have the $\mathbb{S}^1$-invariance. In our case we have a more complete result, in fact we found the global behavior of the evolution and not only around a singularity.

Proposition 5.1.3 *Consider $\mathcal{M}_0$ a closed $\mathbb{S}^1$-invariant submanifold of $(\mathbb{S}^{2n+1}, \bar{g}_\lambda)$ of dimension m and codimension $2 \leq k < \frac{2n-3}{5}$ satisfying the pinching condition*

$$|A|^2 < \frac{1}{m-2} |H|^2 + \frac{m-4-4k}{m-1}.$$

If k is odd, the evolution by mean curvature flow of $\mathcal{M}_0$ converges in finite time to a $\mathbb{S}^1$, while if k is even one of the following holds:

1) *the evolution of $\mathcal{M}_0$ converges in finite time to a $\mathbb{S}^1$,*

2) the evolution of $\mathcal{M}_0$ is defined for any time $0 \leq t < \infty$ and converges to a smooth totally geodesic submanifold, that is an $\mathbb{S}^{2n-k+1}$.

Proof. Projecting $\mathcal{M}_0$ via the Hopf fibration π to $\mathbb{CP}^n$, we have a closed submanifold $\mathcal{B}_0 = \pi(\mathcal{M}_0)$ of dimension $m' = m - 1$ and codimension $k' = k$. Hence $\mathcal{B}_0$ satisfies

$$|A|^2 < \frac{1}{m'-1}|H|^2 + \frac{m'-3-4k'}{m'}.$$

Then the thesis is a consequence of theorem 3.0.1 and theorem 5.0.1, since $\pi^{-1}\left(\mathbb{CP}^{n-\frac{k}{2}}\right) = \mathbb{S}^{2n-k+1}$. □

Proposition 5.1.4 *Let $\mathcal{M}_0$ be a closed $\mathbb{S}^3$-invariant hypersurface of $(\mathbb{S}^{4n+3}, \tilde{g}_\lambda)$, with $n \geq 3$. If $\mathcal{M}_0$ satisfies*

$$|A'|^2 < \frac{1}{4n-2}|H'|^2 + 2 + 6\lambda^{-\frac{1}{2}},$$

then the mean curvature flow of $\mathcal{M}_0$ develops a singularity in finite time and converges to a $\mathbb{S}^3$, then such an $\mathcal{M}_0$ is diffeomorphic to a $\mathbb{S}^3 \times \mathbb{S}^{4n-1}$.

Proof. This time consider the Hopf fibration $\pi : \mathbb{S}^{4n+3} \to \mathbb{HP}^n$. We can project $\mathcal{M}_0$ to an hypersurface $\mathcal{B}_0$ of $\mathbb{HP}^n$. For hypersurfaces we have necessarily $|A'|^2 = |A|^2 + 6\lambda^{-\frac{1}{2}}$. Then $\mathcal{B}_0$ satisfies

$$|A|^2 < \frac{1}{4n-2}|H|^2 + 2.$$

Applyng again theorem 3.5.1 and theorem 5.0.1 we have the thesis. □

Note that for $\mathbb{S}^{4n+3}$ we have both Hopf fibrations: $\pi_1 : \mathbb{S}^{4n+3} \to \mathbb{CP}^{2n+1}$ and $\pi_2 : \mathbb{S}^{4n+3} \to \mathbb{HP}^n$. Considering only the case $\lambda = 1$, we have that $\bar{g}_1 = \tilde{g}_1$ is the standard metric on the sphere. Moreover $\mathbb{S}^1$ is a subgroup of $\mathbb{S}^3$, then if a submanifold of $\mathbb{S}^{4n+3}$ is $\mathbb{S}^3$-invariant, we can project it both to $\mathbb{CP}^{2n+1}$ and $\mathbb{HP}^n$. Putting together propositions 5.1.2 and 5.1.4 we have a negative result.

Corollary 5.1.5 *There are no closed $\mathbb{S}^3$-invariant hypersurfaces of $\mathbb{S}^{4n+3}$ such that*

$$|A'|^2 < \frac{1}{4n}|H'|^2 + 4.$$

Proof. If such a hypersurface exists, propositions 5.1.2 and proposition 5.1.4 can be applied together giving a contradiction. □

A further example is the submersiond $\rho : \mathbb{CP}^{2n+1} \to \mathbb{HP}^n$ described in [E2]: it is the submersion that makes commutative the following diagrams

$$\begin{array}{ccc} \mathbb{S}^{4n+3} & \xrightarrow{\pi_1} & \mathbb{CP}^{2n+1} \\ \pi_2 \searrow & & \swarrow \rho \\ & \mathbb{HP}^n & \end{array} \tag{5.1.2}$$

where π_1 and π_2 are the usual Hopf fibrations. The fibers of ρ are $\mathbb{CP}^1 \equiv \mathbb{S}^2(4)$ and hence they are totally geodesic by theorem 2.3.3. The commutativity of (5.1.2) and the results obteined before for Hopf fibration gives that when we lift an hypersurface of $\mathbb{HP}^n$ to an hypersurface of $\mathbb{CP}^{2n+1}$ via ρ we have that $|A'|^2 = |A|^2 + 4$. In the same way of the previous propositions we can prove the following result.

Proposition 5.1.6 *Let $\mathcal{M}_0$ be a closed $\mathbb{CP}^1$-invariant hypersurface of $\mathbb{CP}^{2n+1}$. If $\mathcal{M}_0$ satisfies*

$$|A'|^2 < \frac{1}{4n-2} |H'|^2 + 6,$$

then the mean curvature flow of $\mathcal{M}_0$ develops a singularity in finite time and converves to a fiber $\mathbb{CP}^1$, then such an $\mathcal{M}_0$ is diffeomorphic to a $\mathbb{S}^2 \times \mathbb{S}^{4n-1}$.

Note that we obtain the same result if we lift an hypersurface of $\mathbb{HP}^n$ satisfying $|A|^2 < \frac{1}{4n-2} |H|^2 + 2$ via π_2 to an hypersurface of the sphere, or if first we lift it to $\mathbb{CP}^{2n+1}$ via ρ and then via π_1 to the sphere.

The examples seen before are all principal bundles with compact fibers. An interesting case with non-compact fibers comes from the Heisenberg group $\mathbb{H}^n$ (not to be confused with the algebra of quaternions!). The Heisenberg group is the Lie group of dimension $2n+1$

$$\mathbb{H}^n = \left\{ \left(\begin{array}{ccc} 1 & \vec{a} & c \\ 0 & I_n & \vec{b}^T \\ 0 & 0 & 1 \end{array} \right) \middle| \; \vec{a}, \vec{b} \in \mathbb{R}^n, c \in \mathbb{R} \right\}$$

endowed with the matrix product. The exponential coordiantes give an other model more useful for computations: $\mathbb{H}^n$ is the Lie group $\mathbb{R}^{2n} \times \mathbb{R}$ endowed with the following product:

$$(x, y, z)(x', y', z') = \left(x + x', y + y', z + z' + \frac{1}{2} \left(\langle x, y' \rangle - \langle y, x' \rangle \right) \right),$$

where $x, x', y, y' \in \mathbb{R}^n$, $z, z' \in \mathbb{R}$ and $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product of $\mathbb{R}^n$. Respect to these cohordinates $(x, y, z) = (x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ we define the following left invariant vector fields on $\mathbb{H}^n$:

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial z}, \\ Y_j &= \frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial z}, \\ V &= \frac{\partial}{\partial z}. \end{aligned}$$

Declaring orthonormal the basis $(X_j, Y_j, V)_j$ we have a left-invariant metric $\bar{g}$ on $\mathbb{H}^n$. On $\mathbb{C}^n$ consider the Euclidean metric, then

$$\pi : (x, y, z) \in \mathbb{H}^n \mapsto (x + iy) \in \mathbb{C}^n$$

is a Riemannian submersion. The fibers are the vertical line:

$$\pi^{-1}(x_0+iy_0)=\{(x_0,y_0,t)|\, t\in\mathbb{R}\}\,.$$

Moreover $\mathscr{V}=span\,\langle V\rangle$ and $\mathscr{H}=span\,\langle X_j,Y_j\rangle_{j=1,\dots,n}$. The structural group is the group of vertical traslations, that is the multiplication by a point of the tipe $(0,0,t)$. It is a group of isometries and it is isomorphic to $(\mathbb{R},+)$. The Levi-Civita connection associated to $\bar{g}'$ is determined by

$$\begin{aligned}
\bar{\nabla}_{X_j}Y_j &= -\bar{\nabla}_{Y_j}X_j &= \frac{1}{2}V,\\
\bar{\nabla}_{X_j}V &= \bar{\nabla}_V X_j &= -\frac{1}{2}Y_j\\
\bar{\nabla}_{Y_j}V &= \bar{\nabla}_V Y_j &= \frac{1}{2}X_j
\end{aligned}$$

and zero for all others pairs of vector of the basis $(X_j,Y_j,V)_{j=1,\dots,n}$. A proof can be found in [Ma]. In particular $\bar{\nabla}_V V$ vanishes, hence the fiber of π are geodesics. On the horizontal distribution $\mathscr{H}$ we have a complex structure J defined on the vector of the basis by $JX_j=Y_j$ and $JY_j=-X_j$ for all j. Then more succinctly, for any horizontal vector field Z on $\mathbb{H}^n$ we have

$$\bar{\nabla}_Z V=\bar{\nabla}_V Z=-JZ. \tag{5.1.3}$$

Now consider $\mathcal{B}_0$ a submanifold of the Euclidean space $\mathbb{C}^n$ of dimension m and codimension k. Its lift via π is a submanifold $\mathcal{M}_0$ invariant respect to vertical translations. Using notation 5.1, by (5.1.3) we have:

$$\begin{aligned}
A'(X_j^{\mathscr{H}},V) &= \sum_{\alpha=1}^{k}\bar{g}\left(\bar{\nabla}_{X_j^{\mathscr{H}}}V,\xi_\alpha^{\mathscr{H}}\right)\xi_\alpha^{\mathscr{H}}\\
&= \frac{1}{2}\sum_{\alpha=1}^{k}\bar{g}\left(-JX_j^{\mathscr{H}},\xi_\alpha^{\mathscr{H}}\right)\xi_\alpha^{\mathscr{H}}\\
&= \frac{1}{2}\sum_{\alpha=1}^{k}\bar{g}\left(X_j^{\mathscr{H}},J\xi_\alpha^{\mathscr{H}}\right)\xi_\alpha^{\mathscr{H}}\\
&= \frac{1}{2}\sum_{\alpha=1}^{k}\bar{g}\left(X_j,J\xi_\alpha\right)\xi_\alpha^{\mathscr{H}},
\end{aligned}$$

where J in the last line is the usual complex structure of $\mathbb{C}^n$. This result is very similar to what we have for Hopf fibration $\pi:\mathbb{S}^{2n+1}\to\mathbb{CP}^n$. It follows that

$$|A'|^2=|A|^2+\frac{1}{2}\sum_{\alpha=1}^{k}\left|J\xi_\alpha^\top\right|^2,$$

then

$$|A|^2\le|A'|^2\le|A|^2+\frac{k}{2}, \tag{5.1.4}$$

with $|A'|^2 = |A|^2$ if and only if for every α $J\xi_\alpha$ is normal to $\mathcal{B}_0$, that is $\mathcal{B}_0$ is a complex submanifold of $\mathbb{C}^n$, while $|A'|^2 = |A|^2 + \frac{k}{2}$ if and only if for every α $J\xi_{n+\alpha}$ is tangent to $\mathcal{B}_0$, that is $\mathcal{B}_0$ is CR-submanifold of $\mathbb{C}^n$ of CR-dimension $m-k$. In the first case, in particular, $\mathcal{B}_0$ is a minimal submanifold. The classical Huisken's result [H1] about evolution of convex hypersurfaces of the Euclidean space gives the followig result for hypersurfaces of the Heisenberg group.

Proposition 5.1.7 *Let $\mathcal{M}_0$ an hypersurface of $\mathbb{H}^n$. If $\mathcal{M}_0$ is a cylinder with vertical axis, without boundary and its projection via π is a convex hypersurface of $\mathbb{R}^{2n}$, then there is an unique solution of the mean curvature flow of $\mathcal{M}_0$ invariant respect to vertical translations. Moreover this solution develops a singularity in finite time and converges to a vertical line. Then such an $\mathcal{M}_0$ is diffeomorphic to a cylinder $\mathbb{S}^{2n-1} \times \mathbb{R}$.*

Proof. Such an $\mathcal{M}_0$ is invariant respect to vertical traslation, the fiber of π are not closed so we can apply theorem 5.0.1 in the sense of remark 5.0.4. Let $\mathcal{B}_0 = \pi(\mathcal{M}_0)$. By the main result of [H1], $\mathcal{B}_0$ shrinks to a point in finite time. The thesis follows lifting this result to $\mathcal{M}_0$. □

Using the main theorem of [AB], we have the following result for submanifolds of arbitrary codimension in the Heisenberg group.

Proposition 5.1.8 *Let $\mathcal{M}_0$ a cylinder with vertical axis of $\mathbb{H}^n$ of dimension $m \geq 3$, without boundary and whose horizontal section is a closed submanifold. If $\mathcal{M}_0$ has $H' \neq 0$ everywhere and satisfies $|A'|^2 \leq c\,|H'|^2$ with*

$$c \leq \begin{cases} \frac{4}{3(m-1)} & \text{if} \quad 3 \leq m \leq 5, \\ \frac{1}{m-2} & \text{if} \quad m > 5, \end{cases}$$

then the mean curvature flow of initial data $\mathcal{M}_0$ has an unique $\mathbb{R}$-invariant solution and this solution converges in finite time to a vertical line. Hence such an $\mathcal{M}_0$ is diffeomorphic to a cylinder $\mathbb{S}^{m-1} \times \mathbb{R}$.

Proof. We have that $\mathcal{B}_0 = \pi(\mathcal{M}_0)$ is a closed submanifold of $\mathbb{R}^{2n}$ of dimension $m-1$. By (5.1.4), $\mathcal{B}_0$ satisfies

$$|A|^2 \leq |A'|^2 \leq c\,|H'|^2 = c\,|H|^2 .$$

The main result of [AB] says that the evolution by mean curvature of $\mathcal{B}_0$ shrinks to a point in finite time. $\mathcal{M}_0$ is $\mathbb{R}$-invariant and then it is not closed, so its evolution could have more then a solution. Since its projection via π is closed, as seen in remark 5.0.4, we can apply theorem 5.0.1 to the unique $\mathbb{R}$-invariant solution obtaining the convergence of $\mathcal{M}_0$ to a fiber of π, that is to a vertical line of $\mathbb{H}^n$. □

Another interesting submersion is the one that arise with the tangent sphere bundle of a Riemannian manifold equipped with the Sasaki metric. For any Riemannian manifold $(\overline{\mathcal{B}}, \bar{g})$ let $T\overline{\mathcal{B}}$ its tangent bundle. The natural projection

$$\pi : (p, u) \in T^r\overline{\mathcal{B}} \mapsto p \in \overline{\mathcal{B}}$$

is a submersion. In this special case, for any X vector field on $\overline{\mathcal{B}}$ we can define also a vertical lift $X^{\mathscr{V}} \in \mathscr{V}$: see [KS] for an exhaustive description. For any $r > 0$ let $T^r\overline{\mathcal{B}} = \left\{(p,u) \in T\overline{\mathcal{B}} \,\middle|\, |u|_{\bar{g}} = r\right\}$ be the tangent sphere bundle of radius r. For any vector field X on $\overline{\mathcal{B}}$ define the tangential lift $X^{\mathscr{T}}$ of X as the component of $X^{\mathscr{V}}$ tangent to $T^r\overline{\mathcal{B}}$. The Sasaki metric is a natural metric $\bar{g}'$ on $T\overline{\mathcal{B}}$, restricted to $T^r\overline{\mathcal{B}}$ has the following form:

$$
\begin{aligned}
\bar{g}'_{(p,u)}(X^{\mathscr{H}}, Y^{\mathscr{H}}) &= \bar{g}_p(X,Y), \\
\bar{g}'_{(p,u)}(X^{\mathscr{T}}, Y^{\mathscr{T}}) &= \bar{g}_p(X,Y) - \frac{1}{r^2}\bar{g}_p(X,u)\bar{g}_p(Y,u), \\
\bar{g}'_{(p,u)}(X^{\mathscr{H}}, Y^{\mathscr{T}}) &= 0,
\end{aligned}
\tag{5.1.5}
$$

for any X and Y tangent to $\overline{\mathcal{B}}$. With this metric the projection $\pi : T^r\overline{\mathcal{B}} \to \overline{\mathcal{B}}$ is a Riemannian submersion with fibers $\pi^{-1}(p) = T^r_p\overline{\mathcal{B}}$, the sphere of radius r tangent to $\overline{\mathcal{B}}$ in p. The horizontal distribution of π is generated by the horizontal lifts and the vertical distribution is generated by the tangential lifts introduced before. The group of isometries that we are considering acts only on the vectorial part as an isometry of $T^r_p\overline{\mathcal{B}}$ and is isomorphic to $O(n)$, where n is the dimension of $\overline{\mathcal{B}}$. Note that in this case the action of the group is not free in fact the orbits are not isometric to the group, but the quotient manifold $T^r\overline{\mathcal{B}}/O(n) \equiv \overline{\mathcal{B}}$ is a well defined manifold. The Levi-Civita connection of the Sasaki on metric on $T^r\overline{\mathcal{B}}$ is

Lemma 5.1.9 *[KS] For any X and Y vector fields tangent to $\overline{\mathcal{B}}$ we have:*

1) $\left(\bar{\nabla}_{X^{\mathscr{H}}} Y^{\mathscr{H}}\right)_{(p,u)} = \left(\bar{\nabla}_X Y\right)^{\mathscr{H}}_{(p,u)} - \frac{1}{2}\left(\bar{R}_p(X,Y)u\right)^{\mathscr{T}}$,

2) $\left(\bar{\nabla}_{X^{\mathscr{H}}} Y^{\mathscr{T}}\right)_{(p,u)} = \left(\bar{\nabla}_X Y\right)^{\mathscr{T}}_{(p,u)} + \frac{1}{2}\left(\bar{R}_p(u,Y)X\right)^{\mathscr{H}}$,

3) $\left(\bar{\nabla}_{X^{\mathscr{T}}} Y^{\mathscr{H}}\right)_{(p,u)} = \frac{1}{2}\left(\bar{R}_p(u,X)Y\right)^{\mathscr{H}}$,

4) $\left(\bar{\nabla}_{X^{\mathscr{T}}} Y^{\mathscr{T}}\right)_{(p,u)} = -\frac{1}{r^2}\bar{g}_p(u,Y)X^{\mathscr{T}}$,

where $\bar{R}$ is the Riemann curvature tensor of $\overline{\mathcal{B}}$.

The fibers are closed and last equation shows that they are also totally geodesic: $\hat{A}(X^{\mathscr{T}}, Y^{\mathscr{T}})$ is the horizontal part of $\bar{\nabla}_{X^{\mathscr{T}}} Y^{\mathscr{T}}$. From now on consider a submanidold $\mathcal{B}_0$ of dimension n and codimension k and $\mathcal{M}_0$ its $O(n+k)$-invariant lift to $T^r\overline{\mathcal{B}}$. Since in this case we have a way to lift vector fields on $\mathcal{B}$ to vector fields tangent to the fibers, we modify notation 5.1. For any $p \in \mathcal{B}_0$ and any $(p,u) \in \pi^{-1}\{p\}$, let $(X_1,\ldots,X_n)$ an orthonormal basis tangent to $\mathcal{B}_0$ in p, $(\xi_1,\ldots,\xi_k)$ an orthonormal basis normal to $\mathcal{B}_0$ in p such that

$$u = r\cos(\vartheta)X_1 + r\sin(\vartheta)\xi_1,$$

for some ϑ. Let $Z = \sin(\vartheta)X_1 - \cos(\vartheta)\xi_1$, then $(u, Z, X_2, \ldots, X_n, \xi_2, \ldots, \xi_k)$ is an orthogonal basis of $T_p\mathcal{B}$. By (5.1.5) we have that $(X_1^{\mathscr{H}}, \ldots, X_n^{\mathscr{H}}, Z^{\mathscr{T}}, X_2^{\mathscr{T}}, \ldots, X_n^{\mathscr{T}}, \xi_2^{\mathscr{T}}, \ldots, \xi_k^{\mathscr{T}})$

is an orthonormal basis tangent to $\mathcal{M}_0$ in (p,u), while $(\xi_1^{\mathscr{H}},\dots,\xi_k^{\mathscr{H}})$ is an orthonormal basis normal to $\mathcal{M}_0$ in (p,u). As concrete example, consider $\overline{\mathcal{B}}=\mathbb{S}^{n+k}(c)$ the sphere of costant curvature $c>0$. By lemma 5.1.9 we have

$$\begin{aligned} A'(X_i^{\mathscr{H}},Z^{\mathscr{T}})(p,u) &= \frac{1}{2}\sum_{\alpha=1}^{k}\bar{R}_p(u,Z,X_i,\xi_\alpha)\xi_\alpha^{\mathscr{H}} \\ &= \frac{c}{2}\sum_{\alpha=1}^{k}\left(\langle u,X_i\rangle\langle Z,\xi_\alpha\rangle-\langle u,\xi_\alpha\rangle\langle Z,X_i\rangle\right)\xi_\alpha^{\mathscr{H}} \\ &= -\frac{cr}{2}\delta_{i1}\xi_1^{\mathscr{H}}. \end{aligned}$$

Similarly

$$\begin{aligned} A'(X_i^{\mathscr{H}},X_j^{\mathscr{T}})(p,u) &= -\frac{cr}{2}\sin(\vartheta)\delta_{ij}\xi_1^{\mathscr{H}}, \\ A'(X_i^{\mathscr{H}},\xi_j^{\mathscr{T}})(p,u) &= \frac{cr}{2}\cos(\vartheta)\delta_{i1}\xi_j^{\mathscr{H}}. \end{aligned}$$

Then

$$\begin{aligned} \left|A'\right|^2(p,u) &= \left|A\right|^2(p)+\frac{c^2r^2}{2}\left(1+(n-1)\sin^2(\vartheta)+(k-1)\cos^2(\vartheta)\right) \\ &= \left|A\right|^2(p)+\frac{c^2}{2}\left(r^2+(n-1)\left|u^{\perp}\right|^2+(k-1)\left|u^{\top}\right|^2\right), \end{aligned}$$

where $\perp$ (respectively $\top$) indicates the normal (respectively the tangent) component respect to $\mathcal{B}_0$. In particular we have

$$\left|A\right|^2(p)+\frac{c^2r^2}{2}min(k,n)\le\left|A'\right|^2(p,u)\le\left|A\right|^2(p)+\frac{c^2r^2}{2}max(k,n).$$

Lifting the submanifolds of the sphere considerede by Huisken [H3] and Baker [Ba] we have the following result as consequence of theorem 5.0.1.

Proposition 5.1.10 *For any $r>0$, $n\ge 3$ and $k\ge 1$, let $\mathcal{M}_0$ be a $2n+k-1$-dimensional $O(n+k)$-invariant submanifold of $T^r\mathbb{S}^{n+k}(c)$. Suppose that for any $(p,u)\in\mathcal{M}_0$ $\mathcal{M}_0$ satisfies the pinching condition*

$$\left|A'\right|^2(p,u)<\frac{1}{n-1}\left|H'\right|^2(p,u)+2c+\frac{c^2}{2}\left(r^2+(n-1)\left|u^{\perp}\right|^2+(k-1)\left|u^{\top}\right|^2\right),$$

where $\perp$ (respectively $\top$) indicates the normal (respectively the tangent) component respect to $\mathcal{B}_0=\pi(\mathcal{M}_0)$. Then the mean curvature flow with initial data $\mathcal{M}_0$ converges in finite time to a fiber $\pi^{-1}(p)=T_p^r\mathbb{S}^{n+k}(c)$ or the flow is defined for any time and converges to $\pi^{-1}(\mathbb{S}^n(c))$ that is a minimal, but not totally geodesic, submanifold of $T^r\mathbb{S}^{n+k}(c)$.

Appendix A

Appendix

Here we collect some theorems alredy known in literature used in the proof of the previous chapter.

Theorem A.0.1 *(Myers) Let $\mathcal{M}$ be a Riemannian manifold of dimension m, if its Ricci curvature satisfies*

$$Ric_{ij} \geq (n-1)Bg_{ij}$$

for some positive constant B along a geodesic of lenght at least $\pi B^{-\frac{1}{2}}$, then the geodesic has a conjugate point.

Theorem A.0.2 *(Hamilton [Ha]) Let $g_{ij} = g_{ij}(t)$ be a time-dependent metric on a manifold $\mathcal{M}$, with $0 \leq t < T_{max} \leq \infty$. Suppose that*

$$\int_0^{T_{max}} \left| g'_{ij} \right| dt \leq C < \infty.$$

Then the metrics $g_{ij}(t)$ for all different times are equivalent and they converge as $t \to T_{max}$ uniformly to a positive-definite metric tensor $g_{ij}(T_{max})$ which is continuous and also equivalent.

Theorem A.0.3 *(Huisken [H2]) Let $\overline{\mathcal{M}}$ a Riemannian manifold of dimension $m+1$ satisfying the following bounds on sectional curvature $\bar{K}$, Riemannian curvature tensor $\bar{R}$ and injectivity radius $inj(\overline{\mathcal{M}})$:*

$$-K_1 \leq \bar{K} \leq K_2, \qquad \left| \bar{\nabla} \bar{R} \right| \leq L, \qquad ing(\overline{\mathcal{M}}) \geq i_{\overline{\mathcal{M}}},$$

for some nonnegative constant K_1, K_2, L and some positive constant $i_{\overline{\mathcal{M}}}$. Let $\mathcal{M}_0$ be a closed hypersurface of $\overline{\mathcal{M}}$. Suppose that on $\mathcal{M}_0$ we have

$$Hh_{ij} > mK_1 g_{ij} + \frac{m^2}{H} L g_{ij}.$$

Then the mean curvature flow of initial value $\mathcal{M}_0$ converges to a round point in finite time.

If the ambient manifold is, like CROSSes, symmetric and with positive sectional curvature, then we can apply this theorem with $K_1 = L = 0$ and then any closed convex hypersurfaces shrinks to a round point in finite time.

Bibliography

[AB] B. ANREWS, C. BAKER, *Mean curvature flow of pinched submanifolds to spheres*, J. Differential Geometry **85** (2010), 357-395.

[Ba] C. BAKER, *The mean curvature flow of submanifolds of high codimension*, Ph.D. thesis. Australian National University. arXiv:1104.4409v1 [math.DG] (2011).

[Be1] A. L. BESSE, *Manifolds all of whose geodesics are closed* Springer-Verlag, Berlin, Hidelberg, New York, 1978.

[Be2] A. L. BESSE, *Einstein manifolds* Springer-Verlag, Berlin, Hidelberg, New York, 1987.

[CH] X. CHANGYU, *Minimal submanifolds with bounded second fundamental form*, Math. Z. **208** (1991), 537-543.

[DO] M. DJORIC, M. OKUMURA, *CR submanifolds of complex projective space* Springer, New York (2010).

[E1] R. H. ESCOBALES JR., *Riemannian submersions with totally geodesic fibers* J. Diff. Geom. **10** (1975), 253 - 276.

[E2] R. H. ESCOBALES JR., *Riemannian submersions from complex projective space* J. Diff. Geom. **13** (1978), 93 - 107.

[EH] K. ECKER, G. HUISKEN, *Mean curvature evolution of entire graphs*, Ann. of Math. (2) **130** (1989), no. 3, 453 - 471.

[FIP] M. FALCITELLI, S. IANUS, A.M. PASTORE, *Riemannian submersions and related topics*, World Scientific (2004).

[GK] S. GUDMUNDSSON E. KAPPOS, *On the geometry of tangent bundles* Expo. Math. **20** (2002), 1 - 41.

[Ha] R. S. HAMILTON *Tree-manifolds with positive Ricci curvature*, J. Diff. Geom. **17** (1982), 255-306.

[H1] G. HUISKEN, *Flow by mean curvature of convex surfaces into spheres*, J. Diff. Geom. **20** (1984), 237-266.

[H2] G. HUISKEN, *Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature*, Invent. Math. **84** (1986), 463-480.

[H3] G. HUISKEN, *Deforming hypersurfaces of the sphere by their mean curvature*, Math. Z. **195** (1987), 205-219.

[HS1] G. HUISKEN, C. SINESTRARI *Convexity estimates for mean curvature flow and singularities of mean convex surfaces*, Acta Math. **183 (1)** (1999), 45–70.

[HS2] G. HUISKEN, C. SINESTRARI *Mean curvature flow with surgeries of two-convex hypersurfaces*, Invent. Math. **(175 (1)** (2009), 137–221.

[KS] O. KOWALSKI, M. SEKIZAWA, *On tangent sphere bundles with small or large constant radius* Annals of Global Analysis and Geometry **18** (2000), 207 - 219.

[LL] A. M. LI, J. LI, *An intrinsic rigidity theorem for minimal submanifold in a sphere*, Arch. Math. (Basel) **58** (1992), no. 6 582-594.

[LXYZ] K. LIU, H. XU, F. YE, E. ZHAO, *Mean Curvature Flow of Higher Codimension in Hyperbolic Spaces*, Comm. Anal. Geom. **21** (2013), n. 3, 651 - 669.

[LXZ] K. LIU, H. XU, E. ZHAO, *Mean curvature flow of higher codimension in Riemannian manifolds*, arXiv:1204.0107v1 [math.DG] (2012).

[Ma] V. MARENICH *Geodesics in Heisenberg groups*, Geom. Dedicata **66** (1997), no. 2, 175–185.

[MP] A. MARTINEZ, J. D. PEREZ *Real hypersurfaces in quaternionic projective space*, Annali di matematica pura ed applicata **145 (1)** (1986), 355–384

[MW] I. MEDOŠ, M.-T. WANG, *Deforming symplectomorphisms of complex projective spaces by the mean curvature flow*, J. Differential Geometry **87** (2011), 309–342.

[Ne] A. NEVES, *Recent progress on singularities of Lagrangian mean curvature flow*, in "Surveys in Geometric Analysis and Relativity" (H.L Bray, W.P. Minicozzi II, Eds.), pp. 413–437, International Press (2011).

[Ng] H. T. NGUYEN *Convexity and cylindrical estimates for mean curvature flow in the sphere*, to appear in Trans. Amer. Math. Soc.

[NR] R. NIEBERGALL, P. J. RYAN, *Real Hypersurfaces in Complex Space Forms*, Tight and Taut Submanifolds MSRI Publications **32** (1997), 233–305.

[O] B. O'NEILL, *The fundamental equations of a submersion* Michigan Math. J. **13** (1966) 459 - 469.

[Pa] T. PACINI, *Mean curvature flow, orbits, moment maps* Trans. Amer. Math. Soc. **355** (2003), no. 8, 3343-3357

[S1] K. SMOCZYK, *Symmetric hypersurfaces in Riemannian manifolds contracting to Lie-groups by their mean curvature* Calc. Var. Partial Differential Equations **4** (1996), no. 2, 155 - 170.

[S2] K. SMOCZYK, *The Lagrangian mean curvature flow*, Habilitation Thesis, University of Leipzig (2000),
available at http://service.ifam.uni-hannover.de/ smoczyk/publications.html.

[S3] K. SMOCZYK, *Mean curvature flow in higher codimension - Introduction and survey*, in "Global Differential Geometry" (C. Bär, J. Lohkamp and M. Schwarz, Eds.), pp. 231-274, Springer-Verlag (2012).

[Un] P. UNTERBERGER, *Evolution of radial graphs in hyperbolic space by their mean curvature* Comm. Anal. Geom. **11** (2003), no. 4, 675 - 695.

[Wa] M.-T. WANG, *Some recent developments in Lagrangian mean curvature flows* in "Surveys in Differential Geometry, Vol. XII. Geometric flows" (H.-D. Cao, S.-T. Yau, Eds.), pp. 333–347, International Press, (2008).