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WELL-POSEDNESS OF DEGENERATE ELLIPTIC AND PARABOLIC PROBLEMS

Ph.D. Thesis in Mathematics

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Contents

Introduction

I.1. The problems

In this thesis we investigate uniqueness and nonuniqueness of solutions to degenerate elliptic and parabolic equations with possibly unbounded coefficients. Specifically, we address equations of the following types:

$$\mathcal{L}\,u - cu = \phi \quad \text{in } \Omega\,, \tag{I.1.1}$$

$$\mathcal{L}\,u - cu - \partial_t u = f(x,t,u) \quad \text{in } \Omega \times (0,T] =: Q_T \qquad (T>0), \tag{I.1.2}$$

$$\rho\,\partial_t u = \Delta\big[G(u)\big] \quad \text{in } \; Q_T\,. \tag{I.1.3}$$

In the last chapter, also fully nonlinear elliptic equations will be considered.

Here $\Omega \subseteq I\!R^n$ is an open connected, possibly unbounded set with boundary $\partial\Omega$ and c, ϕ, f, ρ, G are given functions. The function c is supposed to be nonnegative when dealing with equation (I.1.1), $f \in C(\bar{Q}_T \times I\!R)$ is Lipschitz continuous with respect to $u \in I\!R$, uniformly for $(x,t) \in \bar{Q}_T$. The coefficient ρ is positive and depends only on the space variables; a typical choice for the function G is $G(u) = |u|^{m-1}u$, $m \geq 1$. Finally, the operator $\mathcal{L}$ is formally defined as follows:

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i}\,;$$

we always assume

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq 0 \qquad \text{for any } x \in \Omega\,,\ (\xi_1,..,\xi_n) \in I\!R^n\,. \tag{I.1.4}$$

A great deal of work has been devoted to investigate uniqueness of solutions, both of the Dirichlet boundary value problem associated to equation (I.1.1) and to Cauchy or Cauchy-Dirichlet problems for equations (I.1.2)-(I.1.3). Also probabilistic methods have been extensively employed to study *linear* degenerate equations. In the following section we give a short overview of such known results. In the last section of this Introduction the results of the thesis are outlined.

I.2. A survey of the literature

I.2.1. Degenerate elliptic equations. As is well-known, to formulate a well-posed Dirichlet problem associated to equation (I.1.1) it is natural, in view of the loss of ellipticity, not to prescribe boundary conditions on some portion of the boundary. In fact, early in [**46**] (see also [**57**]), equations degenerating only at the boundary were treated and uniqueness of solutions to this problem was proved, without giving boundary conditions on a certain region of the boundary, where the coefficients of the operator $\mathcal{L}$ converge to zero exponentially. Later on, a general formulation of the Dirichlet problem, when the operator $\mathcal{L}$ possibly degenerates also in the interior of Ω, was given in the pioneering work [**26**]. In this context, when the domain is bounded and regular and the coefficients of the operator sufficiently smooth, the points of the boundary $\partial\Omega$ were classified in *characteristic* and *non-characteristic* points for the operator $\mathcal{L}$. In addition, at characteristic points the *drift vector-field* can point either outward or inward, or it can vanish. Then the Dirichlet problem for equation (I.1.1) was investigated, prescribing boundary data only at non-characteristic points and at characteristic points where drift points outward. By the so-called Gauss-Green formula for the operator $\mathcal{L}$ and integration by parts, a convenient L^p- estimate was derived for any classical solution to this Dirichlet problem, ensuring uniqueness of smooth and bounded solutions.

The above formulation of the Dirichlet problem attracted much attention and many related existence, uniqueness and regularity results were proven (see [**56**]). In particular, in [**56**] uniqueness results were obtained, by an approach different from that of [**26**], which makes use of suitable supersolutions to equation (I.1.1) with $\phi \equiv 0$. Assuming that the operator coefficients are sufficiently smooth and bounded, it was proved that if a nonnegative supersolution to equation (I.1.1) with $\phi \equiv 0$, diverging at some region Σ of the boundary, exists, then the Dirichlet problem for equation (I.1.1) admits at most one classical bounded solution, without specifying boundary conditions at Σ. In [**56**] also the relationship between these two different methods is pointed out. In fact, consider a smooth portion Σ of the boundary made of characteristic points where the drift vector-field does not point outward. Then by the results in [**26**] the Dirichlet problem has at most one smooth bounded solution, without giving data at Σ. The same conclusion can be reached by the uniqueness criterion of [**56**]. In fact, in the proof of Theorem 2.7.1 of [**56**] a nonnegative supersolution to equation (I.1.1) diverging at Σ is explicitly constructed, mainly using the distance function from Σ, hence uniqueness follows.

Furthermore, in Lemma 2.7.1 and Theorem 2.7.1 of [**56**] the existence of classical solutions to the Dirichlet problem for equation (I.1.1) is addressed, under suitable regularity hypotheses on the domain and on the operator. Here the construction of a *barrier* at any point where boundary conditions are specified plays an important role. It turns out that, under appropriate assumptions, such barriers can be constructed both at non-characteristic points and at characteristic points where the drift points outward, in agreement with the above uniqueness results.

It is informative to observe that in [**61**] similar uniqueness results are given. Here uniqueness classes larger than $L^\infty(\Omega)$ are considered and the function c may change sign, yet the operator is supposed to be *uniformly elliptic*. Such results have been obtained in consequence of the Phragmèn-Lindelöf principle, a well-known generalization of the maximum principle. Loosely speaking, the Phragmèn-Lindelöf principle states that if there exists a positive supersolution to equation (I.1.1) with $\phi \equiv 0$, diverging at a portion Σ of the boundary, then any subsolution to equation (I.1.1), nonpositive in $\partial\Omega \setminus \Sigma$, is necessarily nonpositive in Ω, provided that it possibly diverges at Σ with an order lower than that of the mentioned supersolution. In [**61**] similar results are established for unbounded domains under suitable growth conditions at infinity on the subsolution.

In recent years also elliptic equations with *unbounded* coefficients have been widely investigated - mostly in the case $\Omega = \mathbb{R}^n$ - both by analytical methods and stochastic calculus (see [**17**], [**52**] and references therein). Also the corresponding parabolic equations have attracted much interest, particularly concerning the uniqueness of solutions to the relative Cauchy problem (*e.g.*, see [**20**], [**34**], [**52**], [**71**], [**44**], [**60**]). We outline, for further purposes, the results of [**60**], which are in the same spirit of those described before for elliptic equations, although unbounded coefficients are considered.

In [**60**] operators of the form

$$\tilde{\mathcal{L}}v \equiv \frac{1}{\rho}\big\{div[A(x)\nabla v] \;+\; \langle b(x), \nabla v\rangle\big\}$$

are considered. The boundary $\partial\Omega$ is expressed as the disjoint union of the *regular boundary* $\mathcal{R}$, where the coefficients of $\tilde{\mathcal{L}}$ are well-behaved (in particular, bounded), and the *singular boundary* $\mathcal{S}$, where the coefficients can diverge, vanish or need not to have a limit; the condition $\overline{\mathcal{R}} \cap \overline{\mathcal{S}} = \emptyset$ is always assumed.

A key role to establish uniqueness or nonuniqueness of (very weak) bounded solutions to the parabolic problem

$$\text{(I.2.1)} \qquad \begin{cases} \tilde{\mathcal{L}}[G(u)] - \partial_t u = f(x,u) & \text{in } Q_T \\ u = 0 & \text{on } \mathcal{R} \times (0,T] \\ u = u_0 & \text{in } \Omega \times \{0\} \end{cases}$$

with $u_0 \in L^\infty(\Omega)$ is played by the so-called *first exit time problem*:

$$\text{(I.2.2)} \qquad \begin{cases} \tilde{\mathcal{L}}U = -1 & \text{in } \Omega \\ U = 0 & \text{on } \mathcal{R}\,. \end{cases}$$

It was proved that if there exists a supersolution V to problem (I.2.2), bounded from below such that

$$\inf_{\Omega\cup\mathcal{R}} V < \inf_{\mathcal{R}} V,$$

then problem (I.2.1) admits infinitely many bounded solutions. On the contrary, if there exists a subsolution Z to problem (I.2.2) such that

$$\lim_{dist(x,\mathcal{S})\to 0} Z(x) = -\infty\,,$$

then problem (I.2.1) with $G(u) = u$ admits at most one bounded solution.

Another interesting contribution to study uniqueness for the Dirichlet problem associated to equation (I.1.1) is given in [**7**], using methods different from those introduced above. Here uniformly elliptic operators in bounded domains are considered with coefficients which might not be regular approaching the boundary. Relying on the uniform ellipticity of $\mathcal{L}$ and the boundedness of the coefficients, the minimal positive solution U_0 to the *first exit time equation* (*e.g.*, see [**35**]) is constructed. Then the so-called *refined maximum principle* is proven -namely, any subsolution to equation (I.1.1) with $\phi \equiv 0$ bounded from above, nonpositive only at those points of the boundary where the function U_0 can be prolonged to zero in a suitable sense, is necessarily nonpositive in Ω. Notice that, in particular, U_0 can be prolonged to zero at $x_0 \in \partial\Omega$ if x_0 has a barrier. As a consequence of the refined maximum principle, we can infer uniqueness of bounded solutions to the Dirichlet problem in which boundary conditions are imposed only on the portion of the boundary where the function U_0 vanishes in a proper sense. For, the uniqueness result given in [**7**] is in its very nature of the same kind of those described above.

Not surprisingly, the uniqueness results introduced above are connected with the strong maximum principle for linear degenerate elliptic equations (see [**10**], [**19**], [**67**], [**70**]). In particular, in [**19**] (see also [**67**]) the *propagation set* of any point $x_0 \in \Omega$ is defined; it is the collection of all points of Ω which can be connected to x_0, running *diffusion* and/or *drift trajectories*. Then it is proved that if a subsolution u to equation (I.1.1) with $\phi \equiv 0$ attains at some $x_0 \in \Omega$ the maximum of u restricted to the propagation set of x_0, then u is identically equal to $u(x_0)$ in the whole propagation set of x_0, provided that the coefficients of the operator are smooth enough. Using this characterization of the propagation set of local maxima, it is also shown that, if every point lying in Ω and every characteristic point where drift is zero, or is directed inward can be joined, by a finite number of diffusion or drift trajectories, either to some non-characteristic point, or to some characteristic point where drift is directed outward, then uniqueness for the Dirichlet problem formulated in [**26**] holds true.

Analogously, in [**70**], Theorem 7.2.1, it is established that global maxima propagate along both *subunit* trajectories and *drift* trajectories. Then it is proved that this propagation set indeed is the same as that considered in [**19**].

Let us finally mention that similar strong maximum principles have been proven also for fully nonlinear equations in [**3**]-[**4**].

I.2.2. Connections with probabilistic methods. In this Subsection, we discuss briefly the relationship between stochastic calculus and the results recalled in Subsection I.2.1, whose proof was obtained solely by analytical methods.

In [**48**] (see also [**29**]), the general question is addressed, whether boundary data are needed on some region of the boundary to get uniqueness of solutions to linear elliptic degenerate problems. In the light of the probabilistic representation of the solution of elliptic equations, uniqueness holds without prescribing boundary data at some region $\Sigma \subseteq \partial\Omega$, if Σ is *unattainable* by the Markov process generated by the operator. On the contrary, if the Markov process can reach Σ, then at Σ boundary data must be imposed.

Interestingly to decide whether the Markov process does attain or not some region Σ of the boundary, sub– and supersolutions of the same type considered in Subsection I.2.1 are expedient. More precisely, if a supersolution to equation (I.1.1) with $\phi \equiv 0$, diverging at Σ, exists, then Σ is unattainable. On the contrary, if near some point $x_0 \in \partial\Omega$ a *barrier* exists, then the Markov process reaches x_0. In this case the point x_0 is said to be a *regular* point for the stochastic process.

In [**48**] the notion of *attracting* regions of the boundary is also introduced. In general terms, the set Σ is *attracting* if there exists a *barrier* for the whole Σ (see Definition 1.5.1). It turns out that *constant* data can always be specified at attracting regions $\Sigma \subseteq \partial\Omega$. In addition, if the coefficients of the operator are bounded, it is possible to construct a barrier at any point $x_0 \in \Sigma$; hence *continuous* data can be prescribed at Σ, too (see [**48**], Theorem 3.1).

Many of the results collected in [**31**] can be regarded form the same viewpoint. In particular, in [**31**], the actual construction of the mentioned sub– supersolutions is investigated, always supposing that the operator $\mathcal{L}$ has bounded coefficients.

In Chapter 11, Theorem 4.1 of [**31**] it is proved that, if Σ is a smooth submanifold of $I\!R^n$ of codimension 1, containing solely characteristic points where drift does not point outward, then it is unattainable by the Markov process. To prove this property, a supersolution W to the problem

$$\text{(I.2.1)} \qquad \begin{cases} \mathcal{L}\,U = \mu U & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \setminus \Sigma \end{cases}$$

(for some $\mu \geq 0$) diverging at Σ is explicitly constructed.

Moreover, it is proved that the manifold Σ is always unattainable, if $dim\,\Sigma$ is small enough (roughly speaking, in this case Σ is "too thin" to be reached by the process). More precisely, if Σ is a k-dimensional smooth submanifold of $I\!R^n$ with $k \leq n-3$, and the diffusion matrix does not degenerate too much along orthogonal directions to Σ at any $x_0 \in \Sigma$-namely, the *orthogonal rank* of A at x_0 is greater or equal to 3 (see Definition 3.2.11)- then Σ is unattainable (see [**31**], Chapter 11, Theorem 3.1). The argument goes by contradiction. In fact, first it is supposed that the Markov process associated to the operator, starting in Ω, touches a point $x_0 \in \Sigma$, after a certain finite time. Then it is proved that this is impossible, using a suitable nonnegative supersolution W of the equation

$$\mathcal{L} U = \mu U \quad \text{in } W_\delta \setminus K \quad (\mu \geq 0) \tag{I.2.2}$$

such that

$$\lim_{dist(x,K)\to 0} W(x) = +\infty$$

(here W_δ is a neighborhood of $K := \Sigma \cap B_\delta(x_0)$; $\delta > 0$).

Also the already mentioned strong maximum principle for degenerate equations (see [**19**], [**70**]) has a deep probabilistic interpretation. In fact, in the first part of [**69**] it is proved that the set of propagation of maxima of subsolution to elliptic degenerate equations coincides with the closure of the *support* of the Markov process generated by the operator, provided that the coefficients are sufficiently regular and, in particular, bounded. This support -by definition, the closure of trajectories of a Markovian particle starting in Ω- actually coincides with the propagation set defined in [**70**]. Relying on this characterization, in the second part of [**69**] solutions to the Dirichlet problem introduced in [**26**] are proven to be unique.

I.2.3. Nonlinear equations. Equation (I.1.3) (which arises in several physical problems; *e.g.*, see [**45**]) has raised much interest. In particular, several papers have been devoted to the Cauchy problem associated with equation (I.1.3), namely

$$\begin{cases} \rho\, \partial_t u = \Delta\big[G(u)\big] & \text{in } I\!R^n \times (0,T] =: S_T \\ u = u_0 & \text{in } I\!R^n \times \{0\}\,, \end{cases} \tag{I.2.1}$$

assuming $\rho \in C(I\!R^n)$, $u_0 \in L^\infty(I\!R^n)$. It is well-known that problem (I.2.1) is well-posed in the class of bounded solutions when $n \leq 2$ and ρ is sufficiently smooth, or when $n \geq 3$ and ρ is constant (see [**6**], [**37**], [**42**]; see also [**11**]). On the contrary, when $n \geq 3$ and $\rho \to 0$ sufficiently fast as $|x| \to \infty$, some conditions at infinity are needed to restore well-posedness (see [**21**]-[**23**], [**43**], [**45**], [**68**]). In the above references conditions at infinity are of Dirichlet type and *homogeneous*. Moreover, it can be proven that when $G(u) = u$ and $\Gamma \star \rho \in L^\infty(I\!R^n)$ (Γ being the fundamental solution to the Laplace equation in $I\!R^n$) problem (I.2.1) is well-posed in the class of bounded solutions, even if *non-homogeneous* conditions of Dirichlet type or homogeneous condition of Neumann type are prescribed at infinity (see [**44**]).

For $n = 1$, in [**43**], the following generalization of problem (I.2.1) was investigated:

$$\begin{cases} \rho\, \partial_t u = \{a[G(u)]_x\}_x & \text{in } S_T \\ u = u_0 & \text{in } I\!R \times \{0\}\,; \end{cases} \tag{I.2.2}$$

here $a \in C^1(I\!R), a > 0$. Conditions for the well-posedness were given, which depend on the behaviour as $|x| \to \infty$ of solutions to the associated *first exit time equation* to problem (I.2.2). In this one-dimensional case, this becomes the ordinary differential equation

$$(ay')' = -\rho \quad \text{in } I\!R\,. \tag{I.2.3}$$

It turns out that, if any solution of equation (I.2.3) is bounded in $I\!R$, then there exists a unique nonnegative bounded solution to problem (I.2.2) satisfying an extra constraint at infinity, hence the solution to problem (I.2.2) in the class $L^\infty(S_T)$ is not unique. Instead, if there exists a solution y to equation (I.2.3) such that $|y| \to \infty$ as $|x| \to \infty$, then there exist at most one bounded nonnegative solution to problem (I.2.2) .

Since the solution to equation (I.2.3) can be written explicitly, the above conditions for uniqueness or nonuniqueness can be also formulated in terms of the behaviour of ρ and a at infinity.

Moreover, from such uniqueness criteria, conditions for uniqueness of bounded *radial* nonnegative solutions to problem (I.2.1) are deduced, in case ρ and u_0 are radial. According to them, problem (I.1.3) admits at most one bounded nonnegative *radial* solution, when ρ at infinity diverges not too fast; in the opposite case, nonuniqueness holds true.

In [**47**] results similar to the previous ones are obtained for the initial-boundary value problem:

$$\text{(I.2.4)} \qquad \begin{cases} \rho\, \partial_t u = \{a[G(u)]_x\}_x & \text{in } (0,R)\times(0,T] = Q_T \\ u = 0 & \text{in } \{R\}\times(0,T] \\ u = u_0 & \text{in } (0,R)\times\{0\}\,; \end{cases}$$

here $\rho \in C((0,R]), a \in C^1((0,R]),\ a > 0, \rho > 0$ in $(0,R]$. Observe that the coefficients ρ, a can either vanish or diverge, or else they need not to have a limit as $|x| \to 0$. In this case, the point 0 can be regarded as the counterpart of the *point at infinity* in the Cauchy problem (I.2.2). Then it is natural to expect that uniqueness or nonuniqueness criteria for problem (I.2.4) depend on the behaviour of ρ and a near 0. In fact, in [**47**] it is proved that if any solution of problem

$$\text{(I.2.5)} \qquad \begin{cases} (ay')' = -\rho & \text{in } (0,R) \\ y(R) = 0\,, \end{cases}$$

is bounded, then there exists a unique nonnegative bounded solution to problem (I.2.4) satisfying an extra condition at 0, hence the solution to problem (I.2.4) is not unique in the class $L^\infty(Q_T)$. Instead, if there exists a solution to problem (I.2.5) which diverges as $x \to 0$, then problem (I.2.5) is well-posed in $L^\infty(Q_T)$.

Furthermore, since $n = 1$, such conditions for uniqueness or nonuniqueness of solutions are reformulated in dependence of the behaviour of ρ and a at 0. Hence, in the linear case $G(u) = u$, they reduce to classical results proven in [**25**] by probabilistic methods.

Finally, results in the same spirit as those above have been established for fully nonlinear elliptic equations of the type:

$$\text{(I.2.6)} \qquad F(x,u,Du,D^2u) = 0 \quad \text{in } \Omega\,;$$

here F is a real-valued continuous function defined in $\Omega \times I\!R \times I\!R^n \times \Sigma^n$, Σ^n being the linear space of $n \times n$ symmetric matrices with real entries.

For F fulfilling natural structural conditions, Phragmèn-Lindelöf principles for the Dirichlet problems associated to equation (I.2.6) have been proved in [**15**]-[**16**], when Ω is unbounded and satisfies specific geometric conditions; roughly speaking, it is necessary that there is "enough boundary" near any point of Ω. In [**15**]-[**16**] at first a boundary weak Harnack inequality is shown, then Alexandrov-Bakelman-Pucci estimates and Phragmèn-Lindelöf principles are obtained.

On the other hand, in case of bounded domains and $F \in C(\bar\Omega \times I\!R \times I\!R^n \times \Sigma^n)$, in [**2**] comparison principles for the Dirichlet problem for equation (I.2.6) are given, without imposing boundary conditions on some regions of the boundary where F satisfies suitable conditions, which extend to the nonlinear case those introduced in [**26**].

I.3. Outline of results

In Chapter 1 (where [**65**] is reproduced) we address the degenerate elliptic equation (I.1.1), allowing the coefficients a_{ij}, b_i, c and the function ϕ to be *unbounded*. As in [**60**] (see Section I.2.1), we always express the boundary $\partial\Omega$ as the disjoint union of the *regular boundary* $\mathcal{R}$ and the *singular boundary* $\mathcal{S}$. The portion $\mathcal{R}$ is called "regular", for the coefficients a_{ij}, b_i, c are well-behaved and the operator is elliptic in the set $\Omega \cup \mathcal{R}$; on the contrary, the coefficients can vanish or diverge or need not to have a limit, or ellipticity can be lost when $dist(x, \mathcal{S}) \to 0$ and/or $|x| \to \infty$, if Ω is unbounded (see assumptions $(H_1)-(H_2)$ in Chapter 1). Consequently, we study the problem

$$\text{(I.3.1)} \qquad \begin{cases} \mathcal{L}u - cu = \phi & \text{in } \Omega \\ u = g & \text{on } \mathcal{R}, \end{cases}$$

where Dirichlet boundary conditions are specified only at $\mathcal{R}$. Let us notice that the case $\partial\Omega \neq \partial\bar{\Omega}$ is allowed, thus $\mathcal{S}$ can be a submanifold of $I\!R^n$ of dimension less then $n-1$.

We state sufficient criteria for uniqueness or nonuniqueness of solutions to problem (I.3.1); whereas our nonuniqueness results require that $\mathcal{L}$ does not have points of degeneracy in Ω, the uniqueness results are also valid when ellipticity is lost in the interior of Ω.

Assuming the existence of suitable supersolutions to the *first exit time problem*

$$\text{(I.3.2)} \qquad \begin{cases} \mathcal{L}U = -1 & \text{in } \Omega \\ U = 0 & \text{on } \mathcal{R}, \end{cases}$$

we prove nonuniqueness of solution to problem (I.3.1) (in particular, see Theorem 1.2.5). Such supersolutions can be regarded as a sort of *barriers* for the whole $\mathcal{S}$; in fact, nonuniqueness prevails if it is possible to prescribe, in a proper sense, the value of the solution of problem (I.3.1) at some point of the singular boundary $\mathcal{S}$, or at infinity if Ω is unbounded. This suggests that, if uniqueness does not hold, it might be restored by assigning boundary data on some subset $\mathcal{S}_1 \subseteq \mathcal{S}$ and/or a *condition at infinity*, if Ω is unbounded. Hence we study the problems:

$$\text{(I.3.3)} \qquad \begin{cases} \mathcal{L}u - cu = \phi & \text{in } \Omega \\ u = g & \text{on } \mathcal{R} \cup \mathcal{S}_1, \end{cases}$$

respectively

$$\text{(I.3.4)} \qquad \begin{cases} \mathcal{L}u - cu = \phi & \text{in } \Omega \\ u = g & \text{on } \mathcal{R} \cup \mathcal{S}_1 \\ \lim_{|x|\to\infty} u(x) = L & (L \in I\!R). \end{cases}$$

At first we extend the classical Phragmèn-Lindelöf principle (see [**61**]; see also Section I.2.1) to the present case of degenerate operators with unbounded coefficients (see Propositions 1.2.10, 1.2.11). We prove that the sign of a generic subsolution u to problem (I.3.3) is preserved in the whole Ω, if it is prescribed both in $\mathcal{R}$ and in $\mathcal{S}_1$, provided that u satisfies a suitable growth condition at $\mathcal{S}_2 := \mathcal{S} \setminus \mathcal{S}_1$. More precisely, at $\mathcal{S}_2$, u could diverge, yet with an order lower than that of a certain subsolution to the *homogeneous problem*:

$$\text{(I.3.5)} \qquad \begin{cases} \mathcal{L}U = cU & \text{in } \Omega \\ U = 0 & \text{on } \mathcal{R}, \end{cases}$$

which is assumed to exist (*e.g.*, see Theorem 1.2.13). Observe that in the corresponding results mentioned in Subsection I.2.1 we had $\mathcal{S}_1 = \emptyset$.

Further interesting considerations arise when studying existence of solutions to problem (I.3.3). In fact, when $\mathcal{L}$ has bounded coefficients in a neighbourhood of $\mathcal{S}_1$ and $\mathcal{S}_1$ is *attracting* in the sense of [**48**] (see Section I.2.2), it is shown, using standard tools, that the solution of problem (I.3.3) can take any continuous function g at $\mathcal{S}_1$. In general, this is not the case when the coefficients of $\mathcal{L}$ can become unbounded at $\mathcal{S}_1$. Indeed, we show by an example that general Dirichlet boundary data cannot be prescribed on an attracting portion $\mathcal{S}_1$ of the boundary (see Example (c) in Section 1.5). However, it is always possible to prescribe constant data on an attracting portion $\mathcal{S}_1$ of the boundary.

Similar uniqueness and nonuniqueness results are proved in Chapter 2 (where [**59**] is reproduced) for semilinear degenerate parabolic equations

$$\text{(I.3.6)} \qquad \begin{cases} \mathcal{L}u - cu - \partial_t u = f(x,t,u) & \text{in } Q_T \\ u = g & \text{in } (\mathcal{R} \cup \mathcal{S}_1) \times (0,T] \\ u = u_0 & \text{in } (\Omega \cup \mathcal{R} \cup \mathcal{S}_1) \times \{0\}\,; \end{cases}$$

here $f \in C(\bar{Q}_T \times I\!R)$ is Lipschitz continuous with respect to $u \in I\!R$, uniformly for $(x,t) \in \bar{Q}_T$. Such results extend in several respects those given in [**60**], recalled in Section I.2.1 (in particular, the assumptions $\mathcal{S}_1 = \emptyset, \overline{\mathcal{R}} \cap \overline{\mathcal{S}} = \emptyset$ considered in [**60**] are not made).

Concerning existence of solutions to problem (I.3.6), we show as in the elliptic case that general Dirichlet boundary data cannot be prescribed on an attracting portion $\mathcal{S}_1$ of the boundary (see Example (c) in Section 2.5). However, on any attracting region $\mathcal{S}_1$ of the boundary it is possible to prescribe a function which depends on the time variable.

So far, the question of the existence of sub- and supersolutions of auxiliary problems used to prove uniqueness and nonuniqueness has not been addressed. To address this question, the concrete construction of such sub– supersolutions is made in Chapter 3 (where [**58**] is reproduced), under the following main hypotheses: Ω is bounded and $\mathcal{S}$ is a sufficiently smooth submanifold of $I\!R^n$ satisfying $\overline{\mathcal{S}} \cap \overline{\mathcal{R}} = \emptyset$. Once this construction is made, explicit uniqueness and nonuniqueness results both for elliptic and parabolic problems can be derived from the results proved in [**59**], [**65**].

In the construction of such sub– supersolutions, a key role is played by the dimension of the manifold $\mathcal{S}$; in particular, the value $dim\,\mathcal{S} = n-2$ is critical to recognize whether boundary data has to be given or not at $\mathcal{S}$.

In fact, if $n \geq 2$, $dim\,\mathcal{S} \leq n-2$ and the orthogonal rank of the diffusion matrix A is at least 2 on $\mathcal{S}$, there exists at most one bounded solution of problem (I.3.1) (actually, the uniqueness class is larger; see Definition 3.2.11 and Theorem 3.2.12), without giving data on $\mathcal{S}$. This result extends Theorem 4.1, Ch.11 in [**31**], which was proved under more restrictive assumptions by stochastic methods (see Subsection I.2.2). It also extends the results in [**40**], where A was uniformly elliptic.

If $dim\,\mathcal{S} = n-1$, the portions of the boundary where we cannot impose data depend on the behaviour of the coefficients of $\mathcal{L}$ near $\mathcal{S}$. In general terms, if "diffusion and drift near $\mathcal{S}$ are low" (see Theorem 3.2.16, in particular conditions (3.2.16) - (3.2.17)), no extra conditions at $\mathcal{S}$ are needed to ensure uniqueness of problem (I.3.1). Instead, when "diffusion and drift near $\mathcal{S}$ are high", boundary conditions on $\mathcal{S}$ are necessary to make the problem well-posed (see Theorem 3.2.18 and conditions (3.2.20) - (3.2.21)). Connections with analogous results stated in [**26**], [**31**], [**56**] (see also Section I.2.1) are discussed (see Chapter 3, Subsection 3.2.1.3 and Section 3.6).

In Chapter 4 (whose main results are contained in [**66**]), the refined maximum principle of [**7**] we recalled in Section I.2.1 is extended to degenerate linear elliptic and semilinear parabolic equations, with possibly unbounded coefficients. In this framework we also discuss the existence of solutions to the Dirichlet problem. In particular, we show that we do not necessarily have existence of solutions, without a particular choice of the boundary data (see Theorem 4.3.1 and condition (4.3.1); here use of some results of Chapters 1-2 is made). This implies that the refined maximum principle is in general inaccurate, for it suggests to impose boundary conditions on a larger subset of the boundary than it is necessary to make the problem well-posed.

Equation (I.1.3) is studied in Chapters 5-6, where the uniqueness and nonuniqueness results given in [**43**] and [**47**] and already described in Section I.2.3, are generalized to the case of several space dimensions.

In Chapter 5 (where [**62**] is reproduced) we prove uniqueness of bounded solutions to problem (I.2.1), not satisfying any additional condition at infinity, when $\rho(x) \to 0$ slowly, or ρ does not go to zero, as $|x| \to \infty$ (see Theorem 5.2.3). Moreover, we prove existence of bounded solutions to problem (I.2.1), satisfying at infinity possibly *inhomogeneous* conditions of Dirichlet type, when $\rho(x) \to 0$ sufficiently fast as $|x| \to \infty$ (see Theorems 5.2.8, 5.2.11 and 5.2.15). Observe that these existence results, in particular, imply nonuniqueness of bounded solutions to problem (I.2.1).

Equation (I.1.3) is studied in bounded domains in Chapter 6 (whose main results are contained in [**63**]). Inspired by [**47**], we allow the density ρ either to vanish or to diverge, or not to have a limit as the distance $d(x, \mathcal{S})$ goes to zero, $\mathcal{S}$ being a subset of the boundary $\partial\Omega$ referred to as the *singular boundary.* On the other hand, ρ is supposed to be well-behaved both in Ω and on the *regular boundary* $\mathcal{R} := \partial\Omega \setminus \mathcal{S}$ (see assumptions $(H_0) - (H_1)$ in Chapter 6). Throughout Chapter 6 we always assume that $\mathcal{R}$ and $\mathcal{S}$ are compact smooth disjoint submanifolds of $\mathbb{R}^n$ of dimension $n-1$.

As in the linear case, we consider the following initial-boundary value problem associated to equation (I.1.3):

$$\begin{cases} \rho\, \partial_t u = \Delta\big[G(u)\big] & \text{in } Q_T \\ u = 0 & \text{in } \mathcal{R} \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\}\,. \end{cases} \tag{I.3.7}$$

We prove uniqueness of bounded solutions to problem (I.3.7), not satisfying any additional condition at $\mathcal{S}$, when $\rho(x) \to \infty$ sufficiently fast as $d(x, \mathcal{S}) \to 0$ (see Theorem 6.2.2). Moreover, we prove existence of bounded solutions to problem (I.3.7), satisfying at $\mathcal{S}$ possibly *inhomogeneous* conditions of Dirichlet type, when $\rho(x) \to \infty$ sufficiently slow as $d(x, \mathcal{S}) \to 0$, or ρ does not diverge as $d(x, \mathcal{S}) \to 0$ (see Theorems 6.2.5 and 6.2.9).

In Chapter 7, regarding again the boundary $\partial\Omega$ as the disjoint union of the *regular boundary* $\mathcal{R}$ and the *singular boundary* $\mathcal{S}$ (see assumptions $(F_1) - (F_2)$ in Chapter 7), we study the following problem:

$$\begin{cases} F(x, u, Du, D^2u) = 0 & \text{in } \Omega \\ u = g & \text{in } \mathcal{R}\,. \end{cases} \tag{I.3.8}$$

We establish Phragmèn-Lindelöf type results, when F need not to be bounded or to have a limit as $dist(x, \mathcal{S}) \to 0$. Beside problem (I.3.8), as in the linear case (see (I.3.3)-(I.3.4)) and for the same reason, we also consider the problems:

$$\text{(I.3.9)} \qquad \begin{cases} F(x,u,Du,D^2u) = 0 & \text{in } \Omega \\ u = g & \text{in } \mathcal{R} \cup \mathcal{S}_1 \end{cases}$$

and

$$\text{(I.3.10)} \qquad \begin{cases} F(x,u,Du,D^2u) = 0 & \text{in } \Omega \\ u = g & \text{in } \mathcal{R} \cup \mathcal{S}_1 \\ \lim_{|x|\to\infty} u(x) = L & (L \in I\!\!R). \end{cases}$$

As in Chapters 1 and 3, under suitable hypotheses on F, (in particular requiring the existence of suitable supersolutions to a companion problem to problem (I.3.8)), we prove that the sign of u at $\mathcal{R} \cup \mathcal{S}_1$ propagates in the whole Ω, even if we do not require a sign condition on u on the portion $\mathcal{S}_2$ of the singular boundary $\mathcal{S}$ (see Theorem 7.3.1).

The actual construction of such supersolutions is made for special classes of equations, such as semilinear degenerate equations (see Theorem 7.3.4) or fully nonlinear equations related to extremal Pucci operators (see Theorem 7.3.6). We also discuss some generalizations of the previous results to singular fully nonlinear operators, which need not to be defined where the gradient vanishes .

Remark. As already mentioned, in every chapter (except Chapter 7) a paper submitted for publication, or in press, is reproduced. However, some minor variations have been made in this thesis, to avoid repetitions.

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CHAPTER 1

Uniqueness of solutions to degenerate elliptic problems with unbounded coefficients

1.1. Introduction

In this paper we address linear *degenerate* elliptic equations of the form

$$\mathcal{L}\,u - cu = f \quad \text{in } \Omega\,. \tag{1.1.1}$$

Here $\Omega \subseteq I\!R^n$ is an open connected, possibly unbounded set with boundary $\partial\Omega$ and c, f are given functions, $c \geq 0$ in Ω; the operator $\mathcal{L}$ is formally defined as follows:

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i}\,.$$

We assume

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq 0 \qquad \text{for any } x \in \Omega\,,\ (\xi_1,..,\xi_n) \in I\!R^n\,;$$

in particular, for equations *degenerating at the boundary* we have

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j > 0 \qquad \text{for any } x \in \Omega\,,\ (\xi_1,..,\xi_n) \neq 0\,.$$

The coefficients a_{ij}, b_i, c and the function f may be *unbounded* (see assumptions (H_2)-(H_3) below).

We study existence and uniqueness of solutions to the Dirichlet boundary value problem for equation (1.1.1). Special attention will be paid to the case of bounded solutions.

(i) In the case of *bounded coefficients* much work has been devoted to this classical problem, using both analytical methods and stochastic calculus. For equations degenerating at the boundary, it was early recognized that the Dirichlet problem may be well posed prescribing boundary data only on a portion of the boundary, which depends on the behaviour of the coefficients of the operator $\mathcal{L}$ ([**46**]; see also [**57**]). Introducing a classification of the boundary points based on such behaviour, a general formulation of the Dirichlet problem for equation (1.1.1) was given in the pioneering paper [**26**]; existence, uniqueness and a priori estimates of solutions to the problem were also proved under suitable assumptions. A comprehensive account of such results can be found in [**56**] (see also [**27**], [**55**]).

Clearly, uniqueness of solutions to the Dirichlet problem for equation (1.1.1) is related to the validity of the maximum principle for degenerate elliptic operators. Assume $a_{ij} \in C^2(\overline{\Omega})$, $b_i \in C^1(\overline{\Omega})$, $D^\alpha a_{ij} \in L^\infty(\Omega)$ for $|\alpha| \leq 2$, $D^\alpha b_i \in L^\infty(\Omega)$ for $|\alpha| \leq 1$; let $u \in C^2(\overline{\Omega})$ satisfy $\mathcal{L}u \geq cu$ in Ω. For any $x_0 \in \Omega$ such that $u(x_0) = \sup_\Omega u > 0$ consider the *propagation set* $\mathcal{P}(x_0) := \{x \in \Omega \,|\, u(x) = u(x_0)\}$. As proved in [**70**], $\mathcal{P}(x_0)$ contains the closure (in the relative topology) of the set $\mathcal{P}'(x_0)$ consisting of points, which can be joined to x_0 by a finite number of *subunitary* and/or *drift trajectories* (see [**10**], [**19**], [**56**], [**67**] for the proof in particular cases; see also [**1**]). By a local version of the same result a sufficient condition for the uniqueness of solutions to the Dirichlet problem, as formulated in [**26**], can be derived (see [**19**]).

Remarkably, the above mechanism for propagation of maxima of subsolutions is closely related to the Markov process corresponding to the operator $\mathcal{L}$. In fact, the set $\mathcal{P}'(x_0)$ coincides with the *support* of this process, namely with the closure of the collection of all trajectories of a Markovian particle, starting at x_0, with generator $\mathcal{L}$ (see [**69**], [**70**]). Hence, roughly speaking, the above uniqueness criterion for the Dirichlet problem can be rephrased by saying that the boundary data have to be specified only at *attainable* boundary points (see [**29**], [**31**]).

The same idea underlies the so-called *refined maximum principle* in [**7**]. Consider the minimal positive solution U_0 of the *first exit time equation*

$$\mathcal{L}U = -1 \quad \text{in} \quad \Omega \tag{1.1.2}$$

(*e.g.*, see [**35**]); consider those point of $\partial\Omega$ where U_0 can be prolonged to zero. It was proved in [**7**] that a sub- and a supersolution of equation (1.1.1) degenerating at the boundary are ordered in Ω, if they are ordered at these points; as a consequence, prescribing the boundary data at such points is sufficient for the uniqueness of the Dirichlet problem. Observe that prolonging U_0 to zero is possible at any point of $\partial\Omega$ where a *local barrier* for equation (1.1.2) exists, or, equivalently, at any *attracting* point of $\partial\Omega$ (see [**48**]; see also Definition 1.5.1 and Proposition 1.5.3 below).

Before discussing the results of the present paper, it is worth recalling the main assumptions made in the above literature:

- boundedness of the coefficients a_{ij}, b_i, c is always assumed;
- in [**26**], [**27**], [**55**], [**56**] Ω is bounded; $\partial\Omega = \partial\overline{\Omega}$ is a finite union of smooth manifolds; $a_{ij} \in C^2(\overline{\Omega})$, $b_i \in C^1(\overline{\Omega})$, $c \in C(\overline{\Omega})$, $\min_{\overline{\Omega}} c > 0$;
- in [**70**] $a_{ij} \in C^2(\overline{\Omega})$, $b_i \in C^1(\overline{\Omega})$, $D^\alpha a_{ij} \in L^\infty(\Omega)$ for $|\alpha| \leq 2$, $D^\alpha b_i \in L^\infty(\Omega)$ for $|\alpha| \leq 1$. Moreover, subsolutions are meant in the classical sense;
- in [**7**] uniform ellipticity of the operator $\mathcal{L}$ is assumed.

(*ii*) In the present study the above assumptions are relaxed in several respects. In particular, as already remarked, we allow the coefficients of equation (1.1.1) to be *unbounded* (motivations for this hypothesis come from many problems; *e.g.*, think of the Ornstein-Uhlenbeck process). Elliptic equations with unbounded coefficients have been widely investigated in recent years - mostly in the case $\Omega = I\!R^n$ - both by analytical and by probabilistic methods (see [**17**], [**52**] and references therein). Also the corresponding parabolic equations have attracted much attention, particularly studying uniqueness of solutions to the Cauchy problem (*e.g.*, see [**20**], [**34**], [**71**] and references therein; see also [**44**], [**60**] for different initial-boundary value problems).

We always think of the boundary $\partial\Omega$ as the disjoint union of the *regular boundary* $\mathcal{R}$ and the *singular boundary* $\mathcal{S}$ (see assumption (H_1)). In view of assumptions (H_2)-(H_3) below, it is natural to prescribe the Dirichlet boundary condition on $\mathcal{R}$. This leads to the problem

$$\begin{cases} \mathcal{L}u - cu = f & \text{in } \Omega \\ u = g & \text{on } \mathcal{R}, \end{cases} \tag{1.1.3}$$

where the coefficients of $\mathcal{L}$ and the function c can either vanish or diverge, or need not have a limit, when $dist(x, \mathcal{S}) \to 0$ and/or $|x| \to \infty$, if Ω is unbounded. In addition, ellipticity is possibly lost in Ω and/or when $dist(x, \mathcal{S}) \to 0$, and/or when $|x| \to \infty$, if Ω is unbounded.

The assumptions concerning the regular boundary $\mathcal{R}$ and the singular boundary $\mathcal{S}$ are summarized as follows:

$$(H_1) \quad \begin{cases} (i) & \partial\Omega = \mathcal{R} \cup \mathcal{S},\ \mathcal{R} \cap \mathcal{S} = \emptyset,\ \mathcal{S} \neq \emptyset\,; \\ (ii) & \mathcal{R} \subseteq \partial\overline{\Omega} \text{ is open, } \Omega \text{ satisfies the outer sphere condition at } \mathcal{R}\,. \end{cases}$$

It is natural to choose $\mathcal{R}$ as *the largest subset* of $\partial\Omega$ where ellipticity of the operator $\mathcal{L}$ holds (see assumptions $(H_2)-(ii)$, $(H_3)-(iii)$ below), as we do in the following. Observe that no regularity assumption concerning $\mathcal{S}$ is made (see $(H_1)-(ii)$).

Our nonuniqueness results only address the case of degeneracy at the boundary (see Subsection 1.2.1). To prove these results, we always assume the following about the coefficients a_{ij}, b_i and the functions c, f, g:

$$(H_2)\quad \begin{cases} (i)\quad a_{ij}=a_{ji}\in C^{1,1}(\Omega\cup\mathcal{R}),\ b_i\in C^{0,1}(\Omega\cup\mathcal{R})\ \ (i,j=1,\dots,n)\,; \\ (ii)\ \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j>0 \text{ for any } x\in\Omega\cup\mathcal{R} \text{ and } (\xi_1,..,\xi_n)\neq 0\,; \\ (iii)\ c\in C(\Omega\cup\mathcal{R}),\ c\geq 0\,; \\ (iv)\ f\in C(\Omega)\,; \\ (v)\ \ g\in C(\mathcal{R})\,. \end{cases}$$

On the other hand, the uniqueness results in Subsection 1.2.2 hold for the general degenerate equation (1.1.1). In this case we replace assumption (H_2) by the following:

$$(H_3)\quad \begin{cases} (i)\quad a_{ij}=a_{ji}\in C^{1,1}(\Omega\cup\mathcal{R}),\ \sigma_{i,j}\in C^1(\Omega), \\ \qquad b_i\in C^{0,1}(\Omega\cup\mathcal{R})\ \ (i,j=1,\dots,n)\,; \\ (ii)\ \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j\geq 0 \text{ for any } x\in\Omega \text{ and } (\xi_1,..,\xi_n)\in I\!R^n\,; \\ (iii)\ \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j>0 \text{ for any } x\in\mathcal{R} \text{ and } (\xi_1,..,\xi_n)\neq 0\,; \\ (iv) \text{ either } c>0 \text{ in } \Omega\cup\mathcal{R}, \text{ or } c\geq 0,\ c+\sum_{i=1}^n\sigma_{ji}^2>0 \text{ in } \Omega\cup\mathcal{R} \\ \qquad \text{for some } j=1,\dots,n \text{ and } c\in C(\Omega\cup\mathcal{R})\,; \\ (v)\ f\in C(\Omega)\,; \\ (vi)\ \ g\in C(\mathcal{R})\,; \end{cases}$$

here $\sigma\equiv(\sigma_{ij})$ denotes the square root of the matrix $A\equiv(a_{ij})$ (namely, $A(x)=\sigma(x)\sigma(x)^T$; $x\in\Omega\cup\mathcal{R}$). Assumption (H_3) (in particular, $(H_3)-(iv)$) enables us to use comparison results for *viscosity* sub- and supersolutions to second order degenerate elliptic equations, via an equivalence result proved in [**38**] (see Propositions 1.2.3-1.2.4).

(iii) The results of the paper can be described as follows. First we prove sufficient conditions for nonuniqueness of solutions to problem (1.1.3), which require the existence of suitable supersolutions to the first exit time problem:

$$(1.1.4)\qquad \begin{cases} \mathcal{L}U=-1 & \text{in } \Omega \\ U=0 & \text{on } \mathcal{R} \end{cases}$$

(in particular, see Theorem 1.2.5 below). Nonuniqueness depends on the need of prescribing the value of the solution of problem (1.1.3) at some point of the singular boundary $\mathcal{S}$, or at infinity if Ω is unbounded. Therefore, if uniqueness fails, it is natural to try and recover it by assigning boundary data on some subset $\mathcal{S}_1\subseteq\mathcal{S}$ and/or a *condition at infinity*, if Ω is unbounded. Hence we study the problems:

$$(1.1.5)\qquad \begin{cases} \mathcal{L}u-cu=f & \text{in } \Omega \\ u=g & \text{on } \mathcal{R}\cup\mathcal{S}_1\,, \end{cases}$$

respectively

$$(1.1.6)\qquad \begin{cases} \mathcal{L}u-cu=f & \text{in } \Omega \\ u=g & \text{on } \mathcal{R}\cup\mathcal{S}_1 \\ \lim_{|x|\to\infty}u(x)=L & (L\in I\!R) \end{cases}$$

(where possibly $\mathcal{S}_1 = \emptyset$; see (1.2.9)). The following assumption will be made:

$$(H_4) \quad \begin{cases} (i) \quad \mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2, \quad \mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset\,; \\ (ii) \;\; \mathcal{S}_j = \cup_{k=1}^{k_j} \mathcal{S}_j^k \text{ , where every } \mathcal{S}_j^k \text{ is connected and, if } k_j \geq 2, \\ \qquad \overline{\mathcal{S}_j^k} \cap \overline{\mathcal{S}_j^l} = \emptyset \quad \text{for any } k, l = 1, .., k_j,\ k \neq l \ \ (k_j \in I\!N; j = 1, 2)\,. \end{cases}$$

We prove sufficient conditions for uniqueness of solutions to problems (1.1.5) and (1.1.6), extending the classical Phragmèn-Lindelöf principle to the present degenerate case (see Propositions 1.2.10, 1.2.11). Such conditions depend on the existence of subsolutions to the *homogeneous problem*:

$$\begin{cases} \mathcal{L}U = cU & \text{in } \Omega \\ U = 0 & \text{on } \mathcal{R} \end{cases} \tag{1.1.7}$$

and on their behaviour as $dist(x, \mathcal{S}_2) \to 0$ (*e.g.*, see Theorem 1.2.13).

Let us mention that the main step in the nonuniqueness proof concerning problem (1.1.3) is to prove existence of nontrivial solutions to the homogeneous problem (1.1.7) (see Subsection 1.2.1). Also observe that existence for problem (1.1.5) implies nonuniqueness for problem (1.1.3), if $\mathcal{S}_1 \neq \emptyset$; similarly for problems (1.1.6) and (1.2.9) below.

In Section 5 we apply our general results to some examples. The applicability of these results relies on the actual construction of suitable super- and subsolutions to problems (1.1.4), respectively (1.1.7) (or (1.2.22) below; see Subsection 1.2.2); in turn, this depends both on the behaviour of the coefficients of the operator $\mathcal{L}$ at the boundary and on properties of the boundary itself (*e.g.*, the Hausdorff dimension of the subset $\mathcal{S}_2$). Concerning this point, we refer the reader to the paper [**58**].

1.2. Mathematical framework and results

Let us first make precise the definition of solution to the problems introduced above. Denote by $\mathcal{L}^*$ the formal adjoint of the operator $\mathcal{L}$, namely:

$$\mathcal{L}^* v \equiv \sum_{i,j=1}^{n} \frac{\partial^2 (a_{ij} v)}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \frac{\partial (b_i v)}{\partial x_i}\,.$$

DEFINITION 1.2.1. *By a* subsolution *to equation (1.1.1) we mean any function* $u \in C(\Omega)$ *such that*

$$\int_\Omega u\,\{\mathcal{L}^*\psi - c\psi\}\,dx \;\geq\; \int_\Omega f\psi\,dx \tag{1.2.1}$$

for any $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$. Supersolutions *of (1.1.1) are defined replacing* "$\geq$" *by* '$\leq$" *in (1.2.1). A function u is a* solution *of (1.1.1) if it is both a sub- and a supersolution.*

DEFINITION 1.2.2. *Let* $\mathcal{R} \subseteq \mathcal{E} \subseteq \partial\Omega$, $g \in C(\mathcal{E})$. *By a* subsolution *to the problem*

$$\begin{cases} \mathcal{L}u - cu = f & \textit{in } \Omega \\ u = g & \textit{on } \mathcal{E} \end{cases} \tag{1.2.2}$$

we mean any function $u \in C(\Omega \cup \mathcal{E})$ *such that:*
(i) u is a subsolution of equation (1.1.1);
(ii) $u \leq g$ *on* $\mathcal{E}$.
Supersolutions *and* solutions *of (1.2.2) are defined similarly.*

Let us mention the following result (for the definition of viscosity subsolution of equation (1.1.1), see *e.g.* [**18**], [**38**]).

PROPOSITION 1.2.3. *Let either assumption* (H_2) *or* (H_3) *hold; let* $u \in C(\Omega)$. *Then the following statements are equivalent:*
(i) *u is a subsolution of equation (1.1.1);*
(ii) *u is a viscosity subsolution of equation (1.1.1).*

Proof. $(i) \Rightarrow (ii)$: Under the present regularity assumptions the square root σ of the matrix A is in $C^1(\Omega)$ (actually, assumptions $(H_2)-(i)$ and $(H_2)-(ii)$ imply $\sigma_{ij} \in C^{1,1}(\Omega)$; see [**31**], Ch. 6, Lemma 1.1). Hence the claim follows by Theorem 2 in [**38**].
$(ii) \Rightarrow (i)$: Follows by Theorem 1 in [**38**], due to the present regularity assumptions. □

In view of the above proposition, we obtain the following comparison result (see [**50**],[**51**] for a related maximum principle).

PROPOSITION 1.2.4. *Let either assumption* (H_2) *or* (H_3) *hold; let* Ω_1 *be any open bounded subset of* Ω *such that* $\overline{\Omega}_1 \subseteq \Omega \cup \mathcal{R}$. *Let* $\underline{u} \in C(\overline{\Omega}_1)$ *be a subsolution,* $\overline{u} \in C(\overline{\Omega}_1)$ *a supersolution of the equation*

$$\mathcal{L}u - cu = f \quad in\ \Omega_1\,. \tag{1.2.3}$$

If $\underline{u} \leq \overline{u}$ *on* $\partial\Omega_1$, *then* $\underline{u} \leq \overline{u}$ *on* $\overline{\Omega}_1$.

Proof. By Proposition 1.2.3 $\underline{u}$ is a viscosity subsolution, $\overline{u}$ a viscosity supersolution of equation (1.2.3). Then the claim follows:
(a) by the comparison results in Subsection V.1 of [**39**], if (H_2) holds;
(b) by Theorem II.2 in [**39**], if (H_3) holds and $c > 0$ in $\Omega \cup \mathcal{R}$;
(c) by a slight refinement of Theorem 3.3 in [**5**], if (H_3) holds and $c \geq 0$, $c + \sum_{i=1}^n \sigma_{ji}^2 > 0$ in $\Omega \cup \mathcal{R}$ for some $j = 1, \ldots, n$. □

1.2.1. Existence and nonuniqueness results. Concerning problem (1.1.3), we shall prove the following

THEOREM 1.2.5. *Let assumptions* $(H_1)-(H_2)$ *be satisfied; suppose* $c \in L^\infty(\Omega)$. *Let there exist a supersolution* V *of problem (1.1.4) such that*

$$\inf_{\Omega\cup\mathcal{R}} V = 0 < \inf_{\mathcal{R}} V\,. \tag{1.2.4}$$

Then either no solutions, or infinitely many solutions of problem (1.1.3) exist.

The assumption $c \in L^\infty(\Omega)$ is necessary for the above theorem to hold (see Example (c) in Subsection 1.5.2). Let us also observe the following:
(a) if $c = 0$, in Theorem 1.2.5 we can assume V to be a supersolution of problem (1.1.7);
(b) if V is a supersolution of problem (1.1.4) bounded from below, then $\overline{V} := V - \inf_{\Omega\cup\mathcal{R}} V$ is a supersolution of the same problem with $\inf_{\Omega\cup\mathcal{R}} \overline{V} = 0$.

It is informative to outline the proof of Theorem 1.2.5. Suppose first Ω bounded. The existence of a supersolution V of problem (1.1.4) satisfying (1.2.4) implies

$$\liminf_{dist(x,\mathcal{S})\to 0} V(x) = \inf_{\Omega\cup\mathcal{R}} V = 0 \tag{1.2.5}$$

(see Lemma 1.3.1). Then there exists a sequence $\{x_m\} \subseteq \Omega$ such that

$$\lim_{m\to\infty} dist(x_m, \mathcal{S}) = 0\,, \tag{1.2.6}$$

with the following property: for any $\beta \in I\!R$ there exists a bounded solution U_β of the homogeneous problem (1.1.7) such that

$$\lim_{m\to\infty} U_\beta(x_m) \;=\; \beta\,. \tag{1.2.7}$$

This gives the existence of infinitely many bounded solutions of the homogeneous problem (1.1.7) (see Proposition 1.3.4). Plainly, the existence of infinitely many nontrivial bounded solutions of (1.1.7) implies a corresponding nonuniqueness result for problem (1.1.3), if at least one solution of the latter exists (in this respect, see Proposition 1.2.7 below).

If Ω is unbounded, condition (1.2.4) implies either equality (1.2.5), or

$$\liminf_{|x|\to\infty} V(x) = \inf_{\Omega\cup\mathcal{R}} V = 0 \tag{1.2.8}$$

(see Lemma 1.3.2). In the latter case the limit (1.2.7) is attained along a *diverging* sequence $\{x_m\} \subseteq \Omega$, thus nonuniqueness depends on the absence of a "condition at infinity".

To rule out this possibility, it is natural to consider the problem:

$$\begin{cases} \mathcal{L}u - cu = f & \text{in } \Omega \\ u = g & \text{on } \mathcal{R} \\ \lim_{|x|\to\infty} u(x) = L & (L \in \mathbb{R}). \end{cases} \tag{1.2.9}$$

The following nonuniqueness result can be proved.

THEOREM 1.2.6. *Let Ω be unbounded and assumptions $(H_1)-(H_2)$ be satisfied; suppose $c \in L^\infty(\Omega)$. Let there exist a supersolution V of problem (1.1.4) such that*

$$\inf_{\Omega\cup\mathcal{R}} V = 0 < \min\Big\{ \inf_{\mathcal{R}} V \,, \liminf_{|x|\to\infty} V(x) \Big\}. \tag{1.2.10}$$

Moreover, let there exist a positive supersolution F of the equation

$$\mathcal{L}u - cu = 0 \quad in \ \Omega \tag{1.2.11}$$

such that $\lim_{|x|\to\infty} F(x) = 0$. Then either no solutions, or infinitely many solutions of problem (1.2.9) exist.

The proof of Theorem 1.2.6 is analogous to that of Theorem 1.2.5. In this case the stricter inequality (1.2.10) implies the existence of a *bounded* sequence $\{x_m\} \subseteq \Omega$ satisfying (1.2.6), such that for any $\beta \in \mathbb{R}$ equality (1.2.7) holds. Now the bounded solution U_β of the homogeneous problem (1.1.7) satisfies the additional condition

$$\lim_{|x|\to\infty} U_\beta(x) = 0\,; \tag{1.2.12}$$

this follows from the properties of the function F, which plays the rôle of a *barrier at infinity.* This entails the existence of infinitely many bounded solutions to problem (1.2.9) with $f = g = L = 0$ (see Proposition 1.3.5), whence Theorem 1.2.6 follows.

Theorems 1.2.5 and 1.2.6 show that infinitely many solutions to problems (1.1.3), respectively (1.2.9) exist, if one does. Therefore the following existence results, combined with the above theorems, imply nonuniqueness for such problems.

PROPOSITION 1.2.7. *Let assumptions $(H_1)-(H_2)$ be satisfied; suppose $f \in L^\infty(\Omega)$, $g \in L^\infty(\mathcal{R})$. Let there exist a positive supersolution $F \in C(\Omega\cup\mathcal{R})$ of the equation*

$$\mathcal{L}u - cu = -1 \quad in \ \Omega\,. \tag{1.2.13}$$

Then there exists a solution of problem (1.1.3).

REMARK 1.2.8. In connection with Proposition 1.2.7 observe that, if $c(x) \geq c_0 > 0$ for any $x \in \Omega\cup\mathcal{R}$, $F := 1/c_0$ is a bounded supersolution of equation (1.2.13).

Concerning problem (1.2.9), we have the following

PROPOSITION 1.2.9. *Let Ω be unbounded and assumptions $(H_1) - (H_2)$ be satisfied. Let $f \in L^\infty(\Omega)$, $g \in L^\infty(\mathcal{R})$, $c \in L^\infty(\Omega \setminus \overline{B_M})$ for some $M > 0$; if $\mathcal{R}$ is unbounded, suppose*

$$\lim_{|x|\to\infty} g(x) = L\,.$$

Let there exist a positive supersolution $F \in C(\Omega \cup \mathcal{R})$ of equation (1.2.13) such that $\lim_{|x|\to\infty} F(x) = 0$. Then there exists a solution of problem (1.2.9).

The proofs of Propositions 1.2.7 , 1.2.9 make use of a local barrier at the points of $\mathcal{R}$ (which exists by assumptions $(H_1) - (ii)$, $(H_2) - (ii)$; *e.g.*, see [**32**]).

It is immediately seen that, if the supersolution F in the above statements is bounded, the solution u is bounded, too. Then by Propositions 1.3.4, 1.3.5 we obtain nonuniqueness in $L^\infty(\Omega)$ for problems (1.1.3), respectively (1.2.9).

Existence results analogous to Propositions 1.2.7, 1.2.9 hold for problems (1.1.5) and (1.1.6), respectively (however, see Proposition 1.5.2 and Example (c) in Subsection 1.5.1).

1.2.2. Comparison and uniqueness results. In this subsection we address uniqueness of solutions to problem (1.1.5). In the particular case $\mathcal{S}_1 = \emptyset$, $\mathcal{S}_2 = \mathcal{S}$ we recover uniqueness criteria for problem (1.1.3).

Set $B_r(\bar{x}) := \{|x - \bar{x}| < r\}$ $(\bar{x} \in \mathbb{R}^n)$, $B_r(0) \equiv B_r$. We shall prove the following Phragmèn-Lindelöf principle (*e.g.*, see [**61**] for the classical case, where ellipticity of the operator, smoothness of the coefficients and a classical notion of supersolution are assumed).

PROPOSITION 1.2.10. *Let assumptions (H_1) and (H_4) hold, and either (H_2) or (H_3) be satisfied; suppose $\mathcal{S}_2 \neq \emptyset$. Let there exist a subsolution $Z \leq H < 0$ of problem (1.1.7). Let u be a subsolution of problem (1.1.5) with $f = g = 0$, such that*

$$\liminf_{dist(x,\mathcal{S}_2)\to 0} \frac{u(x)}{Z(x)} \geq 0\,. \tag{1.2.14}$$

If Ω is unbounded, assume also

$$\liminf_{|x|\to\infty} \frac{u(x)}{Z(x)} \geq 0\,. \tag{1.2.15}$$

Then $u \leq 0$ in Ω.

If Ω is unbounded and condition (1.2.14) is satisfied, the same conclusion of Proposition 1.2.10 holds true if we "prescribe the sign at infinity". In fact, the following result can be proved.

PROPOSITION 1.2.11. *Let Ω be unbounded, assumptions (H_1) and (H_4) hold, and either (H_2) or (H_3) be satisfied; suppose $\mathcal{S}_2 \neq \emptyset$. Let there exist a subsolution $Z \leq H < 0$ of problem (1.1.7) Let u be a subsolution of problem (1.1.5) with $f = g = 0$ such that*

$$\liminf_{dist(x,\mathcal{S}_2)\to 0} \frac{u(x)}{Z(x)} \geq 0\,, \qquad \limsup_{|x|\to\infty} u(x) \leq 0\,. \tag{1.2.16}$$

Then $u \leq 0$ in Ω.

REMARK 1.2.12. In the above propositions we can replace condition (1.2.14) by the weaker assumption

$$\limsup_{\varepsilon\to 0} \left\{ \inf_{\mathcal{A}_2^\varepsilon \setminus \mathcal{S}} \frac{u}{Z} \right\} \geq 0\,, \tag{1.2.17}$$

where

$$\mathcal{A}_2^\varepsilon := \{x \in \overline{\Omega} \mid dist(x, \mathcal{S}_2) = \varepsilon\}$$

($\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 > 0$ suitably small). Similarly, condition (1.2.15) can be replaced by the weaker assumption

$$\limsup_{\varepsilon \to 0} \Big\{ \inf_{[\Omega \cup \mathcal{R}] \cap \partial B_{\frac{1}{\varepsilon}}} \frac{u}{Z} \Big\} \geq 0\,, \tag{1.2.18}$$

and the second inequality in (1.2.16) by

$$\liminf_{\varepsilon \to 0} \Big\{ \sup_{[\Omega \cup \mathcal{R}] \cap \partial B_{\frac{1}{\varepsilon}}} u \Big\} \leq 0\,. \tag{1.2.19}$$

In fact, the proof of Propositions 1.2.10-1.2.11 will be given using assumptions (1.2.17)-(1.2.19) instead of (1.2.14)-(1.2.16).

Observe that, if Z is a subsolution of problem (1.1.7) bounded from above with $H_0 := \sup_{\Omega \cup \mathcal{R}} Z \geq 0$ and $M > H_0$, then $\overline{Z} := Z - M \leq H_0 - M < 0$ is a subsolution of the same problem. The same remark holds for problems (1.1.4) and (1.2.22) below.

The following uniqueness result is an immediate consequence of Proposition 1.2.10.

Theorem 1.2.13. *Let assumptions* (H_1) *and* (H_4) *hold, and either* (H_2) *or* (H_3) *be satisfied. Suppose* $\mathcal{S}_2 \neq \emptyset$, $g \in C(\mathcal{R} \cup \mathcal{S}_1)$. *Let there exist a subsolution* $Z \leq H < 0$ *of problem (1.1.7). Then:*
(i) *if* Ω *is bounded, there exists at most one solution* u *of problem (1.1.5) such that*

$$\lim_{dist(x, \mathcal{S}_2) \to 0} \frac{u(x)}{Z(x)} = 0\,; \tag{1.2.20}$$

(ii) *if* Ω *is unbounded, there exists at most one solution* u *of problem (1.1.5) such that*

$$\lim_{dist(x, \mathcal{S}_2) \to 0} \frac{u(x)}{Z(x)} = \lim_{|x| \to +\infty} \frac{u(x)}{Z(x)} = 0\,. \tag{1.2.21}$$

Remark 1.2.14. (i) It is easily seen that in the proof of Proposition 1.2.10 (thus in Theorem 1.2.13) the homogeneous problem (1.1.7) can be replaced by the *eigenvalue problem*:

$$\begin{cases} \mathcal{L}U = \mu U & \text{in } \Omega \\ U = 0 & \text{on } \mathcal{R} \end{cases} \tag{1.2.22}$$

with $\mu \in [0, \inf_{\Omega \cup \mathcal{R}} c]$.
(ii) If $c(x) \geq c_0 > 0$ for any $x \in \Omega \cup \mathcal{R}$, we can replace problem (1.2.22) by (1.1.4), obtaining uniqueness results analogous to Theorem 1.2.13. In fact, let Z be a subsolution of problem (1.1.4); it is not restrictive to assume

$$Z \leq -\frac{1}{c_0} \qquad \text{in } \Omega \cup \mathcal{R}\,.$$

Then by Definition 1.2.2 we have

$$\int_\Omega Z\, \mathcal{L}^* \psi\, dx \geq -\int_\Omega \psi\, dx \geq c_0 \int_\Omega Z\, \psi\, dx \geq \int_\Omega cZ\, \psi\, dx$$

for any $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$; moreover, $Z \leq 0$ on $\mathcal{R}$. Hence Z is a subsolution of problem (1.1.7); thus by Theorem 1.2.13 and the above remark (i) the claim follows.

Concerning problem (1.1.6), from Proposition 1.2.11 we obtain the following uniqueness result. The elementary proof is omitted.

Theorem 1.2.15. *Let* Ω *be unbounded, assumptions* (H_1) *and* (H_4) *hold, and either* (H_2) *or* (H_3) *be satisfied; suppose* $\mathcal{S}_2 \neq \emptyset$, $g \in C(\mathcal{R} \cup \mathcal{S}_1)$. *Let there exist a subsolution* $Z \leq H < 0$ *of problem (1.1.7). Then there exists at most one solution* u *of problem (1.1.6) such that condition (1.2.20) is satisfied.*

Uniqueness results in $L^\infty(\Omega)$ for problems (1.1.5), (1.1.6) follow immediately from those above, if the subsolution Z diverges as $dist(x, \mathcal{S}_2) \to 0$. We state below such consequences of Theorems 1.2.13 and 1.2.15.

PROPOSITION 1.2.16. *Let assumptions (H_1) and (H_4) hold, and either (H_2) or (H_3) be satisfied. Suppose $\mathcal{S}_2 \neq \emptyset$, $g \in C(\mathcal{R} \cup \mathcal{S}_1)$. Let there exist a subsolution $Z \le H < 0$ of problem (1.1.7) such that*

$$\lim_{dist(x,\mathcal{S}_2)\to 0} Z(x) = -\infty \tag{1.2.23}$$

if Ω is bounded, or

$$\lim_{dist(x,\mathcal{S}_2)\to 0} Z(x) = \lim_{|x|\to\infty} Z(x) = -\infty \tag{1.2.24}$$

if Ω is unbounded. Then there exists at most one solution $u \in L^\infty(\Omega)$ of problem (1.1.5).

PROPOSITION 1.2.17. *Let Ω be unbounded, assumptions (H_1) and (H_4) hold, and either (H_2) or (H_3) be satisfied; suppose $\mathcal{S}_2 \neq \emptyset$, $g \in C(\mathcal{R} \cup \mathcal{S}_1)$. Let there exist a subsolution $Z \le H < 0$ of problem (1.1.7), satisfying condition (1.2.23). Then for any $L \in I\!R$ there exists at most one solution $u \in L^\infty(\Omega)$ of problem (1.1.6).*

1.3. Existence and nonuniqueness results: Proofs

To prove Theorem 1.2.5 we need a few preliminary results; the proofs are adapted from [**60**].

LEMMA 1.3.1. *Let Ω be bounded and assumptions $(H_1) - (H_2)$ be satisfied. Let V be a supersolution of problem (1.1.7) with $c = 0$ satisfying condition (1.2.4). Then equality (1.2.5) holds.*

Proof. By absurd, suppose

$$\liminf_{dist(x,\mathcal{S})\to 0} V(x) =: \gamma > 0\,;$$

then $V(x) \ge \gamma/2$ for any $x \in \mathcal{S}^\varepsilon := \{x \in \Omega \,|\, dist(x, \mathcal{S}) < \varepsilon\}$ ($\varepsilon \in (0, \varepsilon_0)$ sufficiently small). It follows that

$$\inf_{\Omega\setminus\overline{\mathcal{S}^\varepsilon}} V = \inf_{\Omega\cup\mathcal{R}} V = 0\,.$$

On the other hand, V is a supersolution of the problem

$$\begin{cases} \mathcal{L}U = 0 & \text{in } \Omega\setminus\overline{\mathcal{S}^\varepsilon} \\ U = \alpha & \text{on } \partial[\Omega\setminus\overline{\mathcal{S}^\varepsilon}] \end{cases}$$

where $\alpha := \min\{\frac{\gamma}{2}, \inf_{\mathcal{R}\setminus\overline{\mathcal{S}^\varepsilon}} V\}$, while $V_1 := \alpha$ is a solution of the same problem. By Proposition 1.2.4 we obtain $V \ge \alpha > 0$ in $\Omega \setminus \overline{\mathcal{S}^\varepsilon}$, a contradiction. Hence the conclusion follows. □

If Ω is unbounded, a slight modification of the previous proof gives the following[1]

LEMMA 1.3.2. *Let Ω be unbounded and assumptions $(H_1) - (H_2)$ be satisfied. Let V be a supersolution of problem (1.1.7) with $c = 0$ satisfying condition (1.2.4). Then either equality (1.2.5), or equality (1.2.8) holds.*

[1] Lemmas 1.3.1-1.3.2 can be proved using the strong maximum principle in [**50**], if more regularity of the coefficients is assumed.

COROLLARY 1.3.3. *Let Ω be unbounded and assumptions $(H_1)-(H_2)$ be satisfied. Let V be a supersolution of problem (1.1.7) with $c=0$ satisfying condition (1.2.10). Then equality (1.2.5) holds.*

Now we can prove the following result.

PROPOSITION 1.3.4. *Let the assumptions of Theorem 1.2.5 be satisfied. Then there exist infinitely many bounded solutions of the homogeneous problem (1.1.7).*

Proof. (i) If $c=0$ and $\mathcal{R}=\emptyset$ any constant is a solution of problem (1.1.7), thus the conclusion follows in this case. Otherwise define

$$l := \begin{cases} \inf_{\mathcal{R}} V & \text{if } c=0\,, \\ \min\{\inf_{\mathcal{R}} V, \frac{1}{||c||_\infty}\} & \text{if } c\neq 0\,; \end{cases}$$

then condition (1.2.4) implies $l\in(0,\infty)$.

Set

$$\mathcal{R}_j := \{x\in\mathcal{R}\,|\, dist(x,\mathcal{S})>1/j\} \qquad (j\in I\!N)\,.$$

Consider a sequence of bounded domains $\{H_j\}_{j\in I\!N}$ satisfying an exterior sphere condition at each point of the boundary ∂H_j, such that

$$\begin{cases} \overline{H}_j\subseteq\Omega\cup\overline{\mathcal{R}}_j, \quad H_j\subseteq H_{j+1}, \quad \bigcup_{j=1}^\infty H_j=\Omega\cup\mathcal{R}\,, \\ \partial H_j=\mathcal{R}_j\cup\mathcal{T}_j, \quad \mathcal{R}_j\cap\mathcal{T}_j=\emptyset\,; \end{cases} \tag{1.3.1}$$

observe that by assumption $(H_2)-(ii)$ the operator $\mathcal{L}$ is strictly elliptic in H_j $(j\in I\!N)$.

It is easily seen that $\underline{W}:=\max\{l-V,0\}$ is a subsolution of the problem

$$\begin{cases} \mathcal{L}u=cu & \text{in } H_j \\ u=\underline{W} & \text{on } \partial H_j \end{cases} \tag{1.3.2}$$

for any $j\in I\!N$. In fact, for any $\psi\in C_0^\infty(H_j)$, $\psi\geq 0$ there holds:

$$\int_{H_j}(l-V)\{\mathcal{L}^*\psi-c\psi\}\,dx =$$

$$= -l\int_{H_j}c\,\psi\,dx-\int_{H_j}V\{\mathcal{L}^*\psi-c\psi\}\,dx\;\geq\;\int_{H_j}(1-lc)\,\psi\,dx\;\geq\;0\,.$$

Since $u=0$ is also a (classical) subsolution, the claim follows. It is also immediately seen that $\overline{W}:=\sup_{\Omega\cup\mathcal{R}}\underline{W}=l$ is a classical supersolution of problem (1.3.2).

(ii) By usual arguments (*e.g.*, see [**32**]) for any $j\in I\!N$ there exists a solution $W_j\in C(\overline{H}_j)$ $(\alpha\in(0,1))$ of problem (1.3.2), such that

$$0\leq\underline{W}\leq W_j\leq\overline{W}=l \qquad \text{in } H_j\;; \tag{1.3.3}$$

observe that $\underline{W}=0$, thus $W_j=0$ on $\mathcal{R}_j$.

By compactness arguments there exists a subsequence $\{W_{j_k}\}\subseteq\{W_j\}$, which converges uniformly in any compact subset of Ω. Set

$$W:=\lim_{k\to\infty}W_{j_k}\,. \tag{1.3.4}$$

We shall prove the following

Claim: The function W is a bounded solution of the homogeneous problem (1.1.7). Moreover, W is nontrivial, for there exists a sequence $\{x_m\}\subseteq\Omega$ such that

$$\lim_{m\to\infty}W(x_m)\;=\;l\;>\;0\,. \tag{1.3.5}$$

The above Claim leads easily to the conclusion. In fact, define

$$U_\beta := \frac{\beta}{l}\, W\,. \tag{1.3.6}$$

Then U_β solves (1.1.7) and by (1.3.3)-(1.3.4)

$$|U_\beta| \leq |\beta| \qquad \text{in } \Omega \cup \mathcal{R}\,;$$

moreover, along the sequence $\{x_m\}$ there holds

$$\lim_{m\to\infty} U_\beta(x_m) \;=\; \beta\,.$$

Since β is arbitrary the result follows.

(iii) Let us now prove the Claim. Clearly, by its very definition W is a solution of equation (1.2.11). We use a local barrier argument to prove that $W \in C(\Omega \cup \mathcal{R})$ and $W = 0$ on $\mathcal{R}$.

Let $x_0 \in \mathcal{R}$; take $j_0 \in I\!N$ so large that $x_0 \in \mathcal{R}_j$ for any $j \geq j_0$. Choose $\delta_0 > 0$ so small that

$$\overline{N_\delta(x_0)} \subseteq \overline{H}_j \subseteq \Omega \cup \mathcal{R}$$

for any $j \geq j_0$, where $N_\delta(x_0) := B_\delta(x_0) \cap \Omega$; observe that

$$\partial N_\delta(x_0) = \Big[\partial B_\delta(x_0) \cap [\Omega \cup \mathcal{R}]\Big] \cup \Big[B_\delta(x_0) \cap \mathcal{R}\Big] \qquad (\delta \in (0, \delta_0))\,.$$

Since the operator $\mathcal{L}$ is strictly elliptic in $N_\delta(x_0)$ and an exterior sphere condition is satisfied at $x_0 \in \mathcal{R}$ by assumption $(H_1) - (ii)$, there exists a local barrier at x_0 - namely, a function $h \in C^2(N_\delta(x_0)) \cap C(\overline{N_\delta(x_0)})$ such that

$$\mathcal{L}h - ch \leq -1 \text{ in } N_\delta(x_0)\,, \tag{1.3.7}$$

$$h > 0 \text{ in } \overline{N_\delta(x_0)} \setminus \{x_0\}, \quad h(x_0) = 0 \tag{1.3.8}$$

(*e.g.*, see [**32**]). Set

$$m := \min_{\partial B_\delta(x_0) \cap [\Omega \cup \mathcal{R}]} h \;>\; 0\,.$$

Plainly, from inequality (1.3.3) we obtain

$$0 \leq W_j \leq \frac{l}{m}\, h \text{ on } \partial N_\delta(x_0) \tag{1.3.9}$$

for any $j \geq j_0$ (recall that $W_j = 0$ on $\mathcal{R}_j$ for any $j \in I\!N$, thus $W_j = 0$ on $B_\delta(x_0) \cap \mathcal{R}$ if $j \geq j_0$). In view of inequality (1.3.9), it is easily seen that

$$F_j :=\; -W_j + \frac{l}{m}\, h \qquad (j \geq j_0)$$

is a supersolution of the problem

$$\begin{cases} \mathcal{L}\, u = cu & \text{in } \; N_\delta(x_0) \\ \;u \;= 0 & \text{on } \; \partial N_\delta(x_0)\,; \end{cases} \tag{1.3.10}$$

then by Proposition 1.2.4 we obtain

$$0 \leq W_j \leq \frac{l}{m}\, h \qquad \text{in } \; N_\delta(x_0)$$

for any $j \geq j_0$. Rewriting the above inequality with $j = j_k$ and letting $k \to \infty$, we obtain

$$0 \leq W(x) \leq \frac{l}{m}\, h(x) \;\; \text{ for any } x \in N_\delta(x_0)\,,$$

whence $\lim_{x\to x_0} W(x) = 0$; then the Claim follows.

It remains to prove equality (1.3.5). Let $\{x_m\} \subseteq \Omega$ be a sequence such that

$$\lim_{m\to\infty} V(x_m) = \inf_{\Omega \cup \mathcal{R}} V = 0\,; \tag{1.3.11}$$

such a sequence exists by Lemmas 1.3.1-1.3.2. By inequality (1.3.3) we have:

$$l - V \leq \underline{W} \leq W \leq \overline{W} = l \quad \text{in } \Omega \cup \mathcal{R}\,, \tag{1.3.12}$$

thus equality (1.3.11) implies (1.3.5). This completes the proof. □

Proof of Theorem 1.2.5. Let $U_\beta \in L^\infty(\Omega)$ be the solution of problem (1.1.7) satisfying (1.2.7) constructed in the above proof ($\beta \in I\!R$). Since U_1 is nontrivial and $U_\beta = \beta\, U_1$ (see (1.3.6)), there exist $\overline{x} \in \Omega$ such that $U_1(\overline{x}) \neq 0$, thus $U_{\beta_1}(\overline{x}) \neq U_{\beta_2}(\overline{x})$ for any $\beta_1, \beta_2 \in I\!R$, $\beta_1 \neq \beta_2$.

Let there exist a solution $\overline{u}$ of problem (1.1.3). Then $u_\beta := \overline{u} + U_\beta$ is a solution of problem (1.1.3) for any $\beta \in I\!R$; moreover, $u_{\beta_1}(\overline{x}) \neq u_{\beta_2}(\overline{x})$ for any $\beta_1, \beta_2 \in I\!R$, $\beta_1 \neq \beta_2$. Hence the conclusion follows. □

The proof of Theorem 1.2.6 is the same of Theorem 1.2.5, using the following proposition instead of Proposition 1.3.4.

PROPOSITION 1.3.5. *Let the assumptions of Theorem 1.2.6 be satisfied. Then there exist infinitely many bounded solutions of problem (1.2.9) with $f = g = L = 0$.*

Proof. Define

$$l_\infty := \min\{\inf_{\mathcal{R}} V, \liminf_{|x|\to\infty} V(x)\} \quad \text{if} c = 0\,,$$

$$l_\infty := \min\{\inf_{\mathcal{R}} V, \liminf_{|x|\to\infty} V(x), \frac{1}{||c||_\infty}\} \quad \text{otherwise};$$

then condition (1.2.10) implies $l_\infty > 0$.

Fix $l \in (0, l_\infty)$; consider the family of problems (1.3.2) with H_j, $\underline{W}$ defined as above. Arguing as in the proof of Proposition 1.3.4 (using Corollary 1.3.3 instead of Lemmas 1.3.1-1.3.2), we prove the following: there exists a sequence $\{x_m\} \subseteq \Omega$ satisfying (1.2.6) and a bounded solution $W \geq 0$ of problem (1.1.7), defined by (1.3.4), such that equality (1.3.5) holds. We prove below the additional property:

$$\lim_{|x|\to\infty} W(x) = 0\,; \tag{1.3.13}$$

then defining the family U_β ($\beta \in I\!R$) as in (1.3.6) the conclusion follows.

To prove equality (1.3.13), observe preliminarly that

$$\limsup_{|x|\to\infty} \underline{W}(x) = \lim_{|x|\to\infty} \underline{W}(x) = 0 \tag{1.3.14}$$

(this follows from the above definition of l, since $\underline{W} := \max\{l - V, 0\}$). Then for any $\sigma > 0$ there exists $M > 0$ such that

$$0 \leq \underline{W}(x) < \sigma \quad \text{in} [\Omega \cup \mathcal{R}] \setminus \overline{B_M}\,. \tag{1.3.15}$$

Consider the subsequence $\{j_k\} \subseteq I\!N$ such that (1.3.4) holds. Fix k so large that

$$N_k := H_{j_k} \cap \Big[[\Omega \cup \mathcal{R}] \setminus \overline{B_M}\Big] \neq \emptyset\,;$$

observe that

$$\partial N_k = \Big[\partial H_{j_k} \cap \Big[[\Omega \cup \mathcal{R}] \setminus \overline{B_M}\Big]\Big] \cup \Big[\overline{H_{j_k}} \cap \partial B_M\Big] \qquad (k \in I\!N)\,.$$

By (1.3.15) there holds

$$0 \leq W_{j_k} = \underline{W} < \sigma\,,$$

on $\partial H_{j_k} \cap \Big[[\Omega \cup \mathcal{R}] \setminus \overline{B_M}\Big]$ (see (1.3.2)). Besides, for any k sufficiently large in $\overline{H_{j_k}} \cap \partial B_M$ there holds (see inequality (1.3.3)):

$$0 \leq W_{j_k} \leq l \leq \frac{l}{m} F\,,$$

where

$$m := \min_{\partial B_M} F > 0 .$$

From the above inequalities we get

$$0 \leq W_{j_k} < \sigma + \frac{l}{m} F \qquad \text{on} \partial N_k \tag{1.3.16}$$

for any k sufficiently large.

It is easily seen that for such values of k the function

$$Z_k := W_{j_k} - \sigma - \frac{l}{m} F$$

is a subsolution of the problem

$$\begin{cases} \mathcal{L} u = cu & \text{in } N_k \\ u = 0 & \text{on } \partial N_k . \end{cases}$$

In fact, for any $\psi \in C_0^\infty(N_k)$, $\psi \geq 0$ we have:

$$\int_{N_k} Z_k \{\mathcal{L}^* \psi - c\psi\} \, dx = \sigma \int_{N_k} c\, \psi \, dx - \frac{l}{m} \int_{N_k} F \{\mathcal{L}^* \psi - c\psi\} \, dx \geq 0 ;$$

moreover, $Z_k \leq 0$ on ∂N_k by (1.3.16), thus the claim follows.

In view of Proposition 1.2.4, this implies

$$0 \leq W_{j_k} < \sigma + \frac{l}{m} F \qquad \text{in } N_k \tag{1.3.17}$$

for any $k \in I\!N$ sufficiently large. As $k \to \infty$ we obtain

$$0 \leq W(x) < \sigma + \frac{l}{m} F(x)$$

for any $x \in [\Omega \cup \mathcal{R}] \setminus \overline{B_M}$. This obtains

$$0 \leq \limsup_{|x| \to \infty} W(x) \leq \sigma ;$$

since $\sigma > 0$ is arbitrary, equality (1.3.13) follows. This completes the proof. □

Let us now prove Proposition 1.2.7.

Proof of Proposition 1.2.7. (*i*) If $\overline{\mathcal{R}} \cap \overline{\mathcal{S}} \neq \emptyset$, let $\zeta_j \in C_0^\infty(\mathcal{R}_j)$, $0 \leq \zeta_j \leq 1$, $\zeta_j = 1$ in $\mathcal{R}_{j-1}$ ($j \in I\!N$; $\mathcal{R}_0 := \emptyset$). If $\overline{\mathcal{R}} \cap \overline{\mathcal{S}} = \emptyset$, we have that $\mathcal{R}_j = \mathcal{R}$, for any $j \geq j_0$, for some $j_0 \in I\!N$; in this case we set $\zeta_j \equiv 1$ on $\mathcal{R}_j = \mathcal{R}$, for any $j \geq j_0$.

For any $j \geq j_0$ consider the problem

$$\begin{cases} \mathcal{L} u - cu = f & \text{in } H_j \\ u = \phi_j & \text{on } \partial H_j ; \end{cases} \tag{1.3.18}$$

here $\{H_j\}$ is the sequence of domains used in the proof of Proposition 1.3.4 and the boundary data

$$\phi_j := \begin{cases} \zeta_j g + (1 - \zeta_j) F & \text{on } \mathcal{R}_j \\ F & \text{in } \mathcal{T}_j \end{cases} \tag{1.3.19}$$

are continuous on ∂H_j ($j \geq j_0$).

It is easily seen that the function

$$\tilde{F} := \max\{||f||_\infty, 1\} \, (F + ||g||_\infty) \tag{1.3.20}$$

is a supersolution of problem (1.3.18)-(1.3.19) for any $j \geq j_0$. In fact, for any $\psi \in C_0^\infty(H_j)$, $\psi \geq 0$ we have:

$$\int_{H_j} \tilde{F}\,\{\mathcal{L}^*\psi - c\psi\}\,dx \,=\, \max\{||f||_\infty, 1\}\Big\{\int_{H_j} F\,\{\mathcal{L}^*\psi - c\psi\}\,dx -$$
$$-||g||_\infty \int_{H_j} c\,\psi\,dx\Big\} \,\leq\, -\max\{||f||_\infty, 1\}\int_{H_j}\psi\,dx \,\leq\, \int_{H_j} f\,\psi\,dx\,;$$

moreover, $\tilde{F} \geq F + ||g||_\infty \geq \phi_j$ on ∂H_j. Hence the claim follows. It is similarly checked that $-\tilde{F}$ is a subsolution of the same problem.

(ii) In view of (i) above, there exists a solution $u_j \in C(\overline{H}_j)$ $(\alpha \in (0,1))$ of problem (1.3.18)-(1.3.19), such that

$$|u_j| \leq \tilde{F} \quad \text{in } H_j \tag{1.3.21}$$

for any $j \geq j_0$. By standard compactness arguments there exists a subsequence $\{u_{j_k}\} \subseteq \{u_j\}$, which converges uniformly in any compact subset of Ω. Clearly, $u := \lim_{k\to\infty} u_{j_k}$ is a solution of equation (1.1.1); moreover, $|u| \leq \tilde{F}$ in Ω.

(iii) It remains to prove that $u \in C(\Omega \cup \mathcal{R})$ and $u = g$ on $\mathcal{R}$. To this purpose, we use a local barrier argument as in the proof of Proposition 1.3.4.

Let $x_0 \in \mathcal{R}$ be arbitrarily fixed; take $j_0 \in I\!N$ so large that $x_0 \in \mathcal{R}_{j_0-1}$. Since each $\mathcal{R}_j$ is open and $\mathcal{R}_{j_0-1} \subseteq \mathcal{R}_j$ for $j \geq j_0$, there exists $\delta_0 > 0$ such that:

$$u_j = g \quad \text{in } B_{\delta_0}(x_0) \cap \mathcal{R} \tag{1.3.22}$$

for any $j \geq j_0$ (see (1.3.19)). Moreover, we can choose $\delta_0 > 0$ so small that

$$\overline{N_\delta(x_0)} \subseteq \overline{H}_j \subseteq \Omega \cup \mathcal{R} \tag{1.3.23}$$

for any $j \geq j_0$, where $N_\delta(x_0) := B_\delta(x_0) \cap \Omega$. Observe that

$$\partial N_\delta(x_0) = \Big[\partial B_\delta(x_0) \cap [\Omega \cup \mathcal{R}]\Big] \cup \Big[B_\delta(x_0) \cap \mathcal{R}\Big] \qquad (\delta \in (0, \delta_0))\,.$$

Since $g \in C(\mathcal{R})$ (see $(H_2) - (v)$), in view of (1.3.22) for any $\sigma > 0$ there exists $\delta \in (0, \delta_0)$ such that

$$|u_j(x) - g(x_0)| < \sigma \quad \text{for any } x \in B_{\delta_0}(x_0) \cap \mathcal{R}\,,\ j \geq j_0\,. \tag{1.3.24}$$

Let $h \in C^2(N_\delta(x_0)) \cap C(\overline{N_\delta(x_0)})$ satisfy (1.3.7)-(1.3.8). For any $x \in \partial B_\delta(x_0) \cap [\Omega \cup \mathcal{R}]$ $(\delta \in (0, \delta_0))$ and $j \geq j_0$ there holds

$$|u_j(x) - g(x_0)| \leq \max_{N_\delta(x_0)} \tilde{F} + |g(x_0)| \,\leq\, mM \,\leq\, Mh(x)\,, \tag{1.3.25}$$

where

$$m \,:=\, \min_{\partial B_\delta(x_0) \cap [\Omega \cup \mathcal{R}]} h \,>\, 0\,,$$
$$M \,:=\, \frac{2}{m}\max\Big\{\max_{N_\delta(x_0)} \tilde{F},\, ||g||_\infty,\, m||f||_\infty,\, m\,||g||_\infty \max_{N_\delta(x_0)} c\Big\}.$$

(see (1.3.21), (1.3.23)).

In view of inequalities (1.3.24)-(1.3.25), we conclude that for any $\sigma > 0$ there exists $\delta \in (0, \delta_0)$ such that

$$|u_j(x) - g(x_0)| < \sigma + Mh(x) \quad \text{for any } x \in \partial N_\delta(x_0)\,,\ j \geq j_0\,.$$

Then it is easily seen that for such values of j

$$E_j \,:=\, -u_j + g(x_0) - \sigma - Mh$$

is a subsolution,

$$F_j \,:=\, -u_j + g(x_0) + \sigma + Mh$$

a supersolution of problem (1.3.10). By Proposition 1.2.4 this implies $E_j \leq 0 \leq F_j$ in $N_\delta(x_0)$, namely

$$|u_j(x) - g(x_0)| < \sigma + Mh(x) \quad \text{for any } x \in N_\delta(x_0)\,,\ j \geq j_0\,. \tag{1.3.26}$$

Set $j = j_k$ in inequality (1.3.26), then let $k \to \infty$. This obtains the following: for any $\sigma > 0$ there exists $\delta \in (0, \delta_0)$ such that

$$|u(x) - g(x_0)| < \sigma + Mh(x) \quad \text{for any } x \in N_\delta(x_0)\,,$$

whence

$$\limsup_{x \to x_0} |u(x) - g(x_0)| \leq \sigma$$

for any $\sigma > 0$. Then the conclusion follows. □

To prove Proposition 1.2.9, first a solution u of problem (1.1.3) is constructed as for Proposition 1.2.7. Then, arguing as in the proof of Proposition 1.3.5, it is proved that $\lim_{|x|\to\infty} u(x) = L$. We leave the details to the reader.

1.4. Comparison and uniqueness results: Proofs

Let us first introduce some notations. Set for any $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \frac{\varepsilon}{2})$ ($\varepsilon_0 > 0$ suitably small):

$$\mathcal{S}_{1,\varepsilon} := \{x \in \mathcal{S}_1 \,|\, dist(x, \mathcal{S}_2) \geq \varepsilon\},$$
$$\mathcal{S}^\varepsilon := \{x \in \Omega \,|\, dist(x, \mathcal{S}) < \varepsilon\},$$
$$\mathcal{S}_2^\varepsilon := \{x \in \Omega \,|\, dist(x, \mathcal{S}_2) < \varepsilon\},$$
$$\mathcal{A}_2^\varepsilon := \{x \in \overline{\Omega} \,|\, dist(x, \mathcal{S}_2) = \varepsilon\}\,.$$
$$\mathcal{R}^{\varepsilon,\delta} := \{x \in \mathcal{R} \,|\, dist(x, \mathcal{S}_1) > \delta,\ dist(x, \mathcal{S}_2) > \varepsilon\}\,.$$

If $\mathcal{S}_{1,\varepsilon} \neq \emptyset$, we also define:

$$\mathcal{I}_1^{\varepsilon,\delta} := \{x \in \Omega \,|\, dist(x, \mathcal{S}_{1,\varepsilon}) < \delta\},$$
$$\mathcal{F}_1^{\varepsilon,\delta} := \{x \in \overline{\Omega} \,|\, dist(x, \mathcal{S}_{1,\varepsilon}) = \delta, dist(x, \mathcal{S}_2) \geq \varepsilon\},$$
$$\mathcal{I}_2^{\varepsilon,\delta} := \{x \in \mathcal{S}_2^\varepsilon \,|\, dist(x, \mathcal{S}_{1,\varepsilon}) \geq \delta\},$$
$$\mathcal{F}_2^{\varepsilon,\delta} := \{x \in \mathcal{A}_2^\varepsilon \,|\, dist(x, \mathcal{S}_{1,\varepsilon}) > \delta\}\,;$$

otherwise we set $\mathcal{I}_1^{\varepsilon,\delta} = \mathcal{F}_1^{\varepsilon,\delta} := \emptyset$, $\mathcal{I}_2^{\varepsilon,\delta} := \mathcal{S}_2^\varepsilon$, $\mathcal{F}_2^{\varepsilon,\delta} := \mathcal{A}_2^\varepsilon$. Finally, define:

$$\mathcal{I}^{\varepsilon,\delta} := \mathcal{I}_1^{\varepsilon,\delta} \cup \mathcal{I}_2^{\varepsilon,\delta}\,, \quad \mathcal{F}^{\varepsilon,\delta} := \mathcal{F}_1^{\varepsilon,\delta} \cup \mathcal{F}_2^{\varepsilon,\delta}$$

The above sets are depicted in Figure 1 for the case of bounded Ω. Observe that $\mathcal{S}_1 \subseteq \overline{\mathcal{I}^{\varepsilon,\delta}}$.

Lemma 1.4.1. *For any $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \frac{\varepsilon}{2})$:*
(i) there holds

$$\overline{\Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}} \subseteq \Omega \cup \mathcal{R}\,, \tag{1.4.1}$$

$$\partial[\Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}] = \mathcal{R}^{\varepsilon,\delta} \cup \mathcal{F}^{\varepsilon,\delta}\,; \tag{1.4.2}$$

(ii) for any open subset $\Omega_1 \subseteq \Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}$ there holds

$$\partial\Omega_1 \setminus [\mathcal{R} \cup \mathcal{S}_1] = \partial\Omega_1 \setminus \mathcal{R} = \partial\Omega_1 \setminus \mathcal{R}^{\varepsilon,\delta}\,. \tag{1.4.3}$$

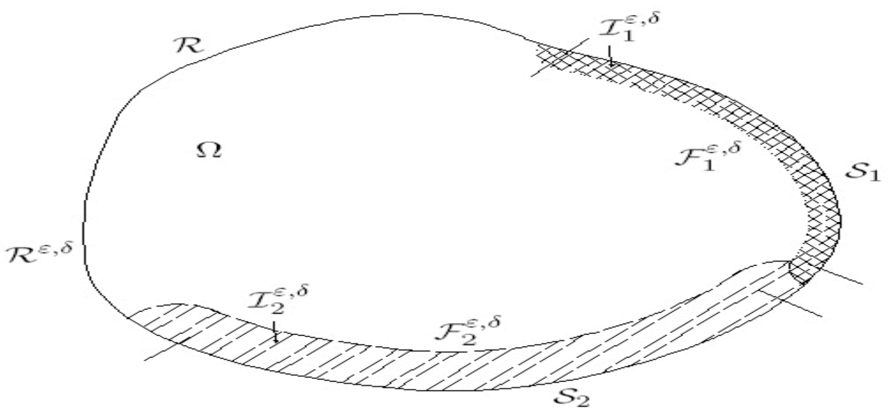

FIGURE 1. Bounded Ω.

Proof. We only check (1.4.1), since equalities (1.4.2)-(1.4.3) are clear. In fact, there holds:

$$\overline{\Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}} \subseteq [\overline{\Omega} \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}] \cup \mathcal{F}^{\varepsilon,\delta} \subseteq \overline{\Omega} \setminus \mathcal{S} = \Omega \cup \mathcal{R}\,, \tag{1.4.4}$$

since $\mathcal{S} \subseteq \overline{\mathcal{I}^{\varepsilon,\delta}}$, $\mathcal{F}^{\varepsilon,\delta} \subseteq \overline{\Omega} \setminus \mathcal{S}$. □

When Ω is unbounded, we also use the following family of subsets of Ω (see Figure 2):

$$\Omega^{\varepsilon,\delta,\beta} := (\Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}) \cap B_{\frac{1}{\beta}}$$

($\varepsilon \in (0,\varepsilon_0)$, $\delta \in (0,\frac{\varepsilon}{2})$, $\beta > 0$); observe that by (1.4.2)

$$\partial\Omega^{\varepsilon,\delta,\beta} = \left[\partial[\Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}] \cap \overline{B_{\frac{1}{\beta}}}\right] \cup \left[\overline{\Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}} \cap \partial B_{\frac{1}{\beta}}\right] =$$

$$= \left[[\mathcal{R}^{\varepsilon,\delta} \cup \mathcal{F}^{\varepsilon,\delta}] \cap \overline{B_{\frac{1}{\beta}}}\right] \cup \left[\overline{\Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}} \cap \partial B_{\frac{1}{\beta}}\right].$$

Now we can prove Proposition 1.2.10.

Proof of Proposition 1.2.10. Let us distinguish two cases: (a) Ω bounded, and (b) Ω unbounded.

(a) *Ω bounded*: (i) In view of inequality (1.2.17), there exists a sequence $\{\varepsilon_k\} \subseteq (0,\varepsilon_0)$, $\varepsilon_k \to 0$ as $k \to \infty$, such that

$$\lim_{k\to+\infty} \left\{ \inf_{\mathcal{A}_2^{\varepsilon_k}\setminus\mathcal{S}} \frac{u}{Z} \right\} \geq 0\,. \tag{1.4.5}$$

Then for any $\alpha > 0$ there exists $\bar{k} = \bar{k}(\alpha) \in I\!N$ such that for any $k > \bar{k}$ there holds

$$\frac{u}{Z} > -\alpha \quad \text{in} \quad \mathcal{A}_2^{\varepsilon_k} \setminus \mathcal{S}. \tag{1.4.6}$$

(ii) Define for any $\alpha > 0$

$$V_\alpha(x) := -\alpha Z(x) = \alpha|Z(x)| \qquad (x \in \Omega \cup \mathcal{R})\,. \tag{1.4.7}$$

Observe that

$$\alpha|H| \leq V_\alpha \quad \text{in } \Omega \cup \mathcal{R}\,. \tag{1.4.8}$$

In view of (1.4.1)-(1.4.2) and (1.4.8), the following claim is easily seen to hold.

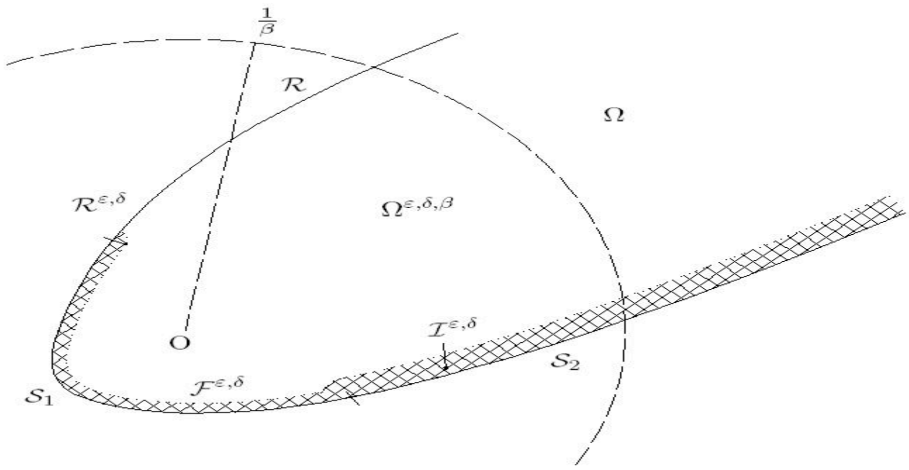

FIGURE 2. Unbounded Ω.

Claim 1: For any $\alpha > 0$, $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \frac{\varepsilon}{2})$ the function V_α defined in (1.4.7) is a supersolution of the problem

$$\begin{cases} \mathcal{L}\, u - cu = 0 & \text{in } \Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}} \\ u = 0 & \text{on } \mathcal{R}^{\varepsilon,\delta} \\ u = V_\alpha & \text{on } \mathcal{F}^{\varepsilon,\delta}\,. \end{cases} \tag{1.4.9}$$

(*iii*) We shall prove the following
Claim 2: For any $\alpha > 0$ there exists $\bar{k} = \bar{k}(\alpha) \in \mathbb{N}$ with the following property: for any $k > \bar{k}$ there exists $\delta_k \in (0, \frac{\varepsilon_k}{2})$ such that the function u is a subsolution of problem (1.4.9) with $\varepsilon = \varepsilon_k, \delta = \delta_k$, where $\{\varepsilon_k\}$ is the infinitesimal sequence of inequality (1.4.6).

From Claims 1 and 2 the conclusion follows immediately. In fact, by Proposition 1.2.4 we obtain for any $\alpha > 0$, $k > \bar{k}$

$$u \le V_\alpha \qquad \text{in } \Omega \setminus \overline{\mathcal{I}^{\varepsilon_k,\delta_k}}\,.$$

Letting $\alpha \to 0$ in the latter inequality we obtain $u \le 0$ in any compact subset of Ω (observe that $\bar{k} \to \infty$, thus $\varepsilon_k \to 0$ as $\alpha \to 0$); hence the result follows.

To prove Claim 2 we use the following facts:

- for any $\alpha > 0$, $\varepsilon \in (0, \varepsilon_0)$ there exists $\bar{\delta} \in (0, \frac{\varepsilon}{2})$ such that for any $\delta \in (0, \bar{\delta})$ there holds

$$u < \alpha |H| \qquad \text{in } \mathcal{F}_1^{\varepsilon,\delta}\,; \tag{1.4.10}$$

- for any $\alpha > 0$ there exists $\bar{k} = \bar{k}(\alpha) \in \mathbb{N}$ such that for any $k > \bar{k}$ and for any $\delta \in (0, \frac{\varepsilon_k}{2})$ the function V_α satisfies

$$u < V_\alpha \qquad \text{in } \mathcal{F}_2^{\varepsilon_k,\delta}\,. \tag{1.4.11}$$

Let us put off the proof of (1.4.10)-(1.4.11) and complete the proof of Claim 2. Plainly, from (1.4.8) and (1.4.10)-(1.4.11) we obtain

$$u < V_\alpha \qquad \text{in } \mathcal{F}^{\varepsilon_k,\delta_k} \tag{1.4.12}$$

for any $\alpha > 0$, $k > \bar{k}$ and some $\delta_k \in (0, \frac{\varepsilon_k}{2})$. On the other hand, the function u is by assumption a subsolution of the problem

$$\text{(1.4.13)} \qquad \begin{cases} \mathcal{L}\, u = cu & \text{in } \Omega \\ \\ u = 0 & \text{on } \mathcal{R} \cup \mathcal{S}_1 \,, \end{cases}$$

thus in particular $u \leq 0$ on $\mathcal{R}^{\varepsilon_k, \delta_k} \subseteq \mathcal{R}$. Hence Claim 2 follows.

It remains to prove inequalities (1.4.10)-(1.4.11). Concerning (1.4.10), observe that $u \leq 0$ on $\mathcal{S}_1$ and $u \in C(\mathcal{S}_1)$, thus in particular $u \leq 0$ on $\mathcal{S}_{1,\varepsilon}$ and $u \in C(\mathcal{S}_{1,\varepsilon})$ (recall that u is a subsolution of (1.4.13), hence $u \in C(\Omega \cup \mathcal{R} \cup \mathcal{S}_1)$ by Definition 1.2.2). As a consequence, for any $\bar{x} \in \mathcal{S}_{1,\varepsilon}$ and any $\sigma > 0$ there exists $\delta = \delta(\bar{x}, \sigma) > 0$ such that

$$u(x) < \sigma \quad \text{for any } x \in [\Omega \cup \mathcal{R}] \cap B_\delta(\bar{x}) \,.$$

It is immediately seen that $\mathcal{S}_{1,\varepsilon}$ is closed, thus compact. Hence from the covering $\{B_\delta(\bar{x})\}_{\bar{x} \in \mathcal{S}_{1,\varepsilon}}$ we can extract a finite covering $\{B_{\delta_n}(\bar{x}_n)\}_{n=1,\dots,\bar{n}}$ ($\bar{n} \in I\!N$), namely

$$\mathcal{S}_{1,\varepsilon} \subseteq \cup_{n=1}^{\bar{n}} B_{\delta_n}(\bar{x}_n) =: \mathcal{B}_{\varepsilon,\sigma} \,.$$

Set

$$\bar{\delta} := \min\{\delta_1, \dots, \delta_{\bar{n}}, \frac{\varepsilon}{3}\} \,;$$

then

$$\{x \in \Omega \cup \mathcal{R} \,|\, dist(x, \mathcal{S}_{1,\varepsilon}) \leq \bar{\delta}\} \subseteq [\Omega \cup \mathcal{R}] \cap \mathcal{B}_{\varepsilon,\sigma} \,,$$

thus in particular

$$\mathcal{F}_1^{\varepsilon,\delta} \subseteq [\Omega \cup \mathcal{R}] \cap \mathcal{B}_{\varepsilon,\sigma} \quad \text{for any } \delta \in (0, \bar{\delta}) \,.$$

This shows that for any $\sigma > 0$, $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \bar{\delta})$ there holds

$$u < \sigma \qquad \text{in } \mathcal{F}_1^{\varepsilon,\delta} \,;$$

choosing $\sigma = \alpha |H|$ we obtain (1.4.10).

Inequality (1.4.11) follows immediately from (1.4.6), since $\mathcal{F}_2^{\varepsilon_k,\delta} \subseteq \mathcal{A}_2^{\varepsilon_k} \setminus \mathcal{S}$ for any $\delta \in (0, \frac{\varepsilon_k}{2})$. This completes the proof when Ω is bounded.

(b) <u>Ω unbounded</u>: (i) In view of inequalities (1.2.17) and (1.2.18), there exist two sequences $\{\varepsilon_k\} \subseteq (0, \varepsilon_0)$, $\varepsilon_k \to 0$ as $k \to \infty$ and $\{\beta_k\} \subseteq (0, \infty)$, $\beta_k \to 0$ as $k \to \infty$, such that

$$\text{(1.4.14)} \qquad \lim_{k \to +\infty} \left\{ \inf_{\mathcal{A}_2^{\varepsilon_k} \setminus \mathcal{S}} \frac{u}{Z} \right\} \geq 0 \,, \qquad \lim_{k \to +\infty} \left\{ \inf_{[\Omega \cup \mathcal{R}] \cap \partial B_{\frac{1}{\beta_k}}} \frac{u}{Z} \right\} \geq 0 \,.$$

Then for any $\alpha > 0$ there exists $\bar{k} = \bar{k}(\alpha) \in I\!N$ such that for any $k > \bar{k}$

$$\text{(1.4.15)} \qquad \frac{u}{Z} > -\alpha \ \text{ in } \ \mathcal{A}_2^{\varepsilon_k} \setminus \mathcal{S} \,, \qquad \frac{u}{Z} \geq -\alpha \ \text{ on } \ [\Omega \cup \mathcal{R}] \cap \partial B_{\frac{1}{\beta_k}} \,.$$

(ii) As in the above case of bounded Ω, it is easily seen that the function $V_\alpha := -\alpha Z$ is a supersolution of the problem

$$\text{(1.4.16)} \qquad \begin{cases} \mathcal{L}\, u - cu = 0 & \text{in } \Omega^{\varepsilon,\delta,\beta} \\ \\ u = 0 & \text{on } \mathcal{R}^{\varepsilon,\delta} \cap \overline{B_{\frac{1}{\beta}}} \\ \\ u = V_\alpha & \text{on } \left[\mathcal{F}^{\varepsilon,\delta} \cap \overline{B_{\frac{1}{\beta}}}\right] \cup \left[\overline{\Omega \setminus \mathcal{I}^{\varepsilon,\delta}} \cap \partial B_{\frac{1}{\beta}}\right] \end{cases}$$

for any $\alpha > 0$, $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \frac{\varepsilon}{2})$, $\beta > 0$.

Arguing as in (a) above the conclusion follows from the

Claim 3: For any $\alpha > 0$ there exists $\bar{k} = \bar{k}(\alpha) \in I\!N$ with the following property: for any $k > \bar{k}$ there exists $\delta_k \in (0, \frac{\varepsilon_k}{2})$ such that the function u is a subsolution of problem (1.4.16) with

$\varepsilon = \varepsilon_k, \delta = \delta_k, \beta = \beta_k$, where $\{\varepsilon_k\}$ and $\{\beta_k\}$ are the infinitesimal sequences of inequalities (1.4.15).

To prove Claim 3, it suffices to prove that

$$u < V_\alpha \quad \text{on } \left[\mathcal{F}^{\varepsilon_k,\delta_k} \cap \overline{B_{\frac{1}{\beta_k}}}\right] \cup \left[\overline{\Omega \setminus \overline{\mathcal{I}^{\varepsilon_k,\delta_k}}} \cap \partial B_{\frac{1}{\beta_k}}\right] \tag{1.4.17}$$

with $\alpha, k, \varepsilon_k, \delta_k, \beta_k$ as above. Notice that (1.4.8) and (1.4.11) are still valid. Moreover, in view of the compactness of $\mathcal{S}_{1,\varepsilon} \cap \overline{B_{\frac{1}{\beta}}}$ ($\varepsilon \in (0, \varepsilon_0)$, $\beta > 0$), arguing as in the proof of (1.4.10), we get that

- for any $\alpha > 0$, $\varepsilon \in (0, \varepsilon_0)$, $\beta > 0$ there exists $\bar{\delta} \in (0, \frac{\varepsilon}{2})$ such that for any $\delta \in (0, \bar{\delta})$ there holds

$$u < \alpha|H| \qquad \text{in } \mathcal{F}_1^{\varepsilon,\delta} \cap \overline{B_{\frac{1}{\beta}}}\,. \tag{1.4.18}$$

Then by (1.4.8), (1.4.11), (1.4.18), the inequality

$$u < V_\alpha \qquad \text{in } \mathcal{F}^{\varepsilon_k,\delta_k} \cap \overline{B_{\frac{1}{\beta_k}}} \tag{1.4.19}$$

follows. Concerning the inequality

$$u < V_\alpha \qquad \text{in } \overline{\Omega \setminus \overline{\mathcal{I}^{\varepsilon_k,\delta_k}}} \cap \partial B_{\frac{1}{\beta_k}}\,, \tag{1.4.20}$$

it follows immediately from (1.4.15) since $\overline{\Omega \setminus \overline{\mathcal{I}^{\varepsilon_k,\delta}}} \subseteq \Omega \cup \mathcal{R}$ for any $\delta \in (0, \frac{\varepsilon_k}{2})$ (see (1.4.1)). Then inequality (1.4.17) and the conclusion for unbounded Ω follow. This completes the proof. □

Proof of Proposition 1.2.11. The proof is the same of Proposition 1.2.10 in the case of unbounded Ω, the only difference being that to prove (1.4.20) we use (1.4.8) and the following inequality:

$$u < \alpha|H| \qquad \text{in } \overline{\Omega \setminus \overline{\mathcal{I}^{\varepsilon_k,\delta_k}}} \cap \partial B_{\frac{1}{\beta_k}}\,. \tag{1.4.21}$$

As for the latter, by the second inequality in (1.2.16) there exists a sequence $\{\beta_k\} \subseteq (0, \infty)$, $\beta_k \to 0$ as $k \to \infty$, such that

$$\lim_{k\to+\infty} \left\{ \sup_{[\Omega \cup \mathcal{R}] \cap \partial B_{\frac{1}{\beta_k}}} u \right\} \leq 0\,.$$

Then for any $\alpha > 0$ there exists $\bar{k} = \bar{k}(\alpha) \in \mathbb{N}$ such that for any $k > \bar{k}$ there holds

$$u < \alpha|H| \quad \text{in} \quad [\Omega \cup \mathcal{R}] \cap \partial B_{\frac{1}{\beta_k}}\,,$$

which implies (1.4.21). Then the conclusion follows. □

Proof of Theorem 1.2.13. Let u_1, u_2 solve problem (1.1.5); then both $u_1 - u_2$ and $u_2 - u_1$ are solutions of the same problem with $f = g = 0$. In view of Proposition 1.2.10 and Remark 1.2.12, conditions (1.2.14), (1.2.15) with $u = u_1 - u_2$, $u = u_2 - u_1$ yields $u_1 \leq u_2$, respectively $u_2 \leq u_1$. Then the conclusion follows. □

1.5. Examples and remarks

In this section we discuss some applications of the above general results.

1.5.1. Nonuniqueness and existence. According to the assumptions made in Subsection 1.2.1, only degeneracy at the boundary is allowed in the examples of this subsection.
(a) Consider the problem

$$\begin{cases} x^2 u_{xx} + y^2 u_{yy} - u_y - u = f & \text{in } (0,\infty)\times(0,1) = \Omega \\ u = g & \text{on } (0,\infty)\times\{1\} = \mathcal{R} \end{cases} \tag{1.5.1}$$

with $f \in C(\Omega)\cap L^\infty(\Omega)$, $g \in C(\mathcal{R})\cap L^\infty(\mathcal{R})$. The function $V(x,y) = y$ satisfies

$$\mathcal{L}V = -1 \quad \text{in } \Omega\,, \qquad \inf_{\Omega\cup\mathcal{R}} V = 0 < \inf_{\mathcal{R}} V = 1\,;$$

moreover,

$$\mathcal{L}V - V = -1 - y \le -1 \quad \text{in } \Omega\,.$$

By Theorem 1.2.5 and Proposition 1.2.7 (applied with $F = V$) problem (1.5.1) has infinitely many solutions in $L^\infty(\Omega)$.

(b) Consider the problem

$$\begin{cases} \frac{1}{2}x^2 u_{xx} + (x-1)^2 y^2 u_{yy} + 2x^2 u_x - (2x^2+1)u_y - u = f & \text{in } (1,\infty)\times(0,1) = \Omega \\ u = g & \text{on } (1,\infty)\times\{1\} = \mathcal{R} \\ \lim_{x\to\infty} u(x,y) = L & (y \in (0,1)) \end{cases} \tag{1.5.2}$$

with $f \in C(\Omega)\cap L^\infty(\Omega)$, $g \in C(\mathcal{R})\cap L^\infty(\mathcal{R})$, $L \in I\!R$ and

$$\lim_{x\to\infty} g(x) = L\,.$$

The function $V(x,y) = x + y - 1$ satisfies

$$\mathcal{L}V = -1 \quad \text{in } \Omega\,, \quad \inf_{\Omega\cup\mathcal{R}} V = 0 < \min\Big\{\inf_{\mathcal{R}} V\,, \lim_{x\to\infty} V(x,y)\Big\} = 1 \quad (y \in (0,1))\,.$$

Moreover, the function $F \in L^\infty(\Omega)$, $F(x,y) = \dfrac{1}{x}$ satisfies

$$\mathcal{L}F - F \le -1 \quad \text{in } \Omega\,, \qquad \lim_{x\to\infty} F(x) = 0\,.$$

In view of Theorem 1.2.6 and Proposition 1.2.9, problem (1.5.2) has infinitely many solutions in $L^\infty(\Omega)$.

Let us show by an example that general Dirichlet boundary data cannot be prescribed on a portion of the boundary, which is *attracting* in the sense of the following definition (see [**48**]).

Let $\Sigma \subseteq \partial\Omega$; for any $\varepsilon \in (0,\varepsilon_0)$ ($\varepsilon_0 > 0$ suitably small) set

$$\Sigma^\varepsilon := \{x \in \Omega \,|\, dist(x,\Sigma) < \varepsilon\}\,.$$

Definition 1.5.1. *A subset $\Sigma \subseteq \partial\Omega$ is* attracting *if there exist $\varepsilon \in (0,\varepsilon_0)$ and a supersolution $V \in C(\overline{\Sigma^\varepsilon})$ of the equation:*

$$\mathcal{L}V - cV = -1 \quad in\ \Sigma^\varepsilon\,, \tag{1.5.3}$$

such that

$$V > 0 \quad in\ \overline{\Sigma^\varepsilon}\setminus\Sigma\,, \qquad V = 0 \quad on\ \Sigma\,.$$

Sufficient conditions for the attractivity of Σ can be given adapting results in [**31**], [**56**]. The proof of the following result is very similar to that of Proposition 1.2.7, thus we omit it.

PROPOSITION 1.5.2. *Let $\mathcal{S}_1 \subseteq \partial\Omega$. Let assumptions (H_1), (H_2) and (H_4) be satisfied; suppose $f \in L^\infty(\Omega)$, $g \in C(\overline{\mathcal{R} \cup \mathcal{S}_1}) \cap L^\infty(\mathcal{R} \cup \mathcal{S}_1)$, $c \in L^\infty(\mathcal{S}_1^\varepsilon)$ for some $\varepsilon \in (0, \varepsilon_0)$. Let there exist a positive supersolution $F \in C(\Omega \cup \mathcal{R}) \cap L^\infty(\mathcal{S}_1^\varepsilon)$ of equation (1.2.13). If $\mathcal{S}_1$ is attracting and bounded, there exists a solution $u \in C(\Omega \cup \mathcal{R} \cup \mathcal{S}_1)$ of problem (1.1.5), provided that*

$$g = \text{constant} \qquad \text{on } \mathcal{S}_1 . \tag{1.5.4}$$

Condition (1.5.4) and the boundedness of $\mathcal{S}_1$ are unnecessary, if a local barrier exists at any point $x_0 \in \mathcal{S}_1$.

In view of the above proposition, the function V can be regarded as a barrier *for the whole of* $\mathcal{S}_1$, if the latter is bounded (clearly, V is a local barrier at some point $x_0 \in \Sigma$ if and only if x_0 is isolated in the relative topology of $\partial\Omega$). In such case *constant* Dirichlet data can be prescribed on $\mathcal{S}_1$. However, this need not be the case for general Dirichlet data, as the following example shows.

(c) Consider the problem

$$\begin{cases} \frac{1}{y \sin x}(u_{xx} + y^2 u_{yy}) = f & \text{in } (\frac{\pi}{4}, \frac{3\pi}{4}) \times (0,1) = \Omega \\ \\ u = g & \text{on } \partial\Omega \setminus \Big([\frac{\pi}{4}, \frac{3\pi}{4}] \times \{0\} \Big) = \mathcal{R} \end{cases} \tag{1.5.5}$$

with $f \in C(\Omega) \cap L^\infty(\Omega)$, $g \in C(\mathcal{R}) \cap L^\infty(\mathcal{R})$. Here we take $\mathcal{S}_1 = \mathcal{S} = [\frac{\pi}{4}, \frac{3\pi}{4}] \times \{0\}$, $\mathcal{S}_2 = \emptyset$.

It is easily checked that the function $Z(x, y) := x^2 + \log y - \pi^2$ satisfies

$$Z < 0 \text{ in } \Omega\,, \qquad \mathcal{L}Z = \frac{1}{y \sin x} > 0 \text{ in } \Omega\,, \qquad \lim_{y \to 0} Z(x,y) = -\infty\,.$$

Then by Proposition 1.2.16 there exists at most one solution $u \in L^\infty(\Omega)$ of problem (1.5.5).

On the other hand, the function $V \in C(\overline{\Omega})$, $V(x, y) := y \sin x$ satisfies

$$V > 0 \quad \text{in } \Omega \cup \mathcal{R}\,, \qquad V = 0 \quad \text{on } \mathcal{S}_1\,, \qquad \mathcal{L}V = -1 \quad \text{in } \Omega\,,$$

thus $\mathcal{S}_1$ is attracting (see Definition 1.5.1). By Proposition 1.5.2 there exists a solution $u_0 \in L^\infty(\Omega)$ of the problem

$$\begin{cases} \frac{1}{y \sin x}(u_{xx} + y^2 u_{yy}) = f & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

In view of the above uniqueness result, this implies that there exists no solution $u_g \in L^\infty(\Omega)$ of the problem

$$\begin{cases} \frac{1}{y \sin x}(u_{xx} + y^2 u_{yy}) = f & \text{in } \Omega \\ \\ u = g & \text{on } \partial\Omega \end{cases}$$

with $g \in C(\partial\Omega)$, $g = 0$ on $\mathcal{R}$, $g(\bar{x}) \neq 0$ at some point $\bar{x} \in \mathcal{S}_1$.

Let us add some remarks concerning Proposition 1.5.2. If some subset $\Sigma \subseteq \partial\overline{\Omega}$ is attracting and the coefficients $a_{i,j}$, b_i are bounded in Σ^ε for some $\varepsilon > 0$, for any $x_0 \in \Sigma$ a local barrier does exist, thus general Dirichlet data g can be assigned on Σ. This is the content of the following proposition (see [**48**]).

PROPOSITION 1.5.3. *Let assumptions (H_1), (H_2) and (H_4) be satisfied. Let $\Sigma \subseteq \partial\overline{\Omega}$ be attracting; suppose $a_{i,j}$, $b_i \in L^\infty(\Sigma^\varepsilon)$ for some $\varepsilon \in (0, \varepsilon_0)$ $(i, j = 1, ...n)$. Then for any $x_0 \in \Sigma$ there exists a local barrier.*

Finally, let us mention a nonuniqueness result for problem (1.1.3), which immediately follows from Proposition 1.5.2.

Corollary 1.5.4. *Let the assumptions of Proposition 1.5.2 be satisfied; suppose $\overline{\mathcal{R}} \cap \overline{\mathcal{S}_1} = \emptyset$. Then there exist infinitely many solutions of problem (1.1.3).*

1.5.2. Uniqueness. (a) Consider the problem

$$\begin{cases} u_{xx} + y^2 u_{yy} + y u_y = f & \text{in } (0,1)\times(0,1) = \Omega \\ \\ u = g & \text{on } \partial\Omega \setminus \Big([0,1]\times\{0\}\Big) = \mathcal{R} \end{cases} \tag{1.5.6}$$

with $f \in C(\Omega) \cap L^\infty(\Omega)$, $g \in C(\mathcal{R}) \cap L^\infty(\mathcal{R})$. Here we take $\mathcal{S}_1 = \emptyset$, $\mathcal{S}_2 = \mathcal{S} = [0,1]\times\{0\}$.

It is easily checked that the function $Z(x,y) := x^2 + \log y - 2$ satisfies

$$Z \le -1 \ \text{ in } \Omega\,, \qquad \mathcal{L}Z = 0 \ \text{ in } \Omega\,, \qquad \lim_{y\to 0} Z(x,y) \;=\; -\infty\,.$$

Then by Proposition 1.2.16 there exists at most one solution $u \in L^\infty(\Omega)$ of problem (1.5.6). Moreover, observe that the function $F(x,y) = -x^2 + 1$ satisfies

$$F > 0 \ \text{ in } \Omega\,, \qquad \mathcal{L}F < -1 \quad \text{in } \Omega\,.$$

Then by Proposition 1.2.7 and the above uniqueness result problem (1.5.6) is well posed in $L^\infty(\Omega)$.

(b) Consider the problem

$$\begin{cases} (x-\frac{1}{2})^4\,[u_{xx} + y^2(1-y)u_{yy}] - u = f & \text{in } (0,1)\times(0,1) = \Omega \\ \\ u = g & \text{on } \mathcal{R}\cup\mathcal{S}_1 \end{cases} \tag{1.5.7}$$

with $f \in C(\Omega)$, $g \in C(\mathcal{R}\cup\mathcal{S}_1)$; here $\mathcal{R} = [\{0\}\times(0,1)] \cup [\{1\}\times(0,1)]$, $\mathcal{S}_1 = [0,1]\times\{1\}$, $\mathcal{S}_2 = [0,1]\times\{0\}$. Observe that the operator $\mathcal{L}$ degenerates on $\{\frac{1}{2}\}\times(0,1) \subseteq \Omega$.

The function $Z(x,y) := x^2 + \log y - 2$ satisfies

$$Z \le -1 \ \text{ in } \Omega\,, \qquad \mathcal{L}Z \ge Z \ \text{ in } \Omega\,, \qquad \lim_{y\to 0} Z(x,y) \;=\; -\infty\,.$$

Then by Proposition 1.2.16 there exists at most one solution $u \in L^\infty(\Omega)$ of problem (1.5.7).

(c) Consider the problem

$$\begin{cases} u_{xx} + y^2 u_{yy} - u_y - \frac{y+1}{y|\log y|}u = f & \text{in } (0,1)\times(0,1) = \Omega \\ \\ u = g & \text{on } \partial\Omega \setminus \Big([0,1]\times\{0\}\Big) = \mathcal{R} \end{cases} \tag{1.5.8}$$

with $f \in C(\Omega) \cap L^\infty(\Omega)$, $g \in C(\mathcal{R}) \cap L^\infty(\mathcal{R})$. Here we take $\mathcal{S}_1 = \emptyset$, $\mathcal{S}_2 = \mathcal{S} = [0,1]\times\{0\}$. Observe that the coefficient c does not belong to $L^\infty(\Omega)$.

The function $Z(x,y) = \log y - 1$ satisfies

$$Z \le -1 \text{ in } \Omega\cup\mathcal{R}, \qquad \mathcal{L}Z - cZ = c \ge 0 \text{ in } \Omega, \qquad \lim_{y\to 0} Z(x,y) = -\infty\,;$$

hence by Proposition 1.2.16 there exists at most one bounded solution u of problem (1.5.8). Further observe that the function $V(x,y) := (x-\frac{1}{2})^2 + 3y$ is a positive supersolution of equation (1.2.13). Then, in view of Proposition 1.2.7 and the above uniqueness result, problem (1.5.8) is well posed in $L^\infty(\Omega)$.

It is worth observing that

$$\inf_{\Omega\cup\mathcal{R}} V = 0 < \inf_{\mathcal{R}} V = \frac{1}{4}\,, \qquad \mathcal{L}V = -1 \ \text{ in } \Omega\,;$$

however, Theorem 1.2.5 does not apply since the coefficient c is unbounded.

(d) Consider the problem

$$\text{(1.5.9)} \qquad \begin{cases} u_{xx} + y^2(1-y)^2 u_{yy} - \frac{(y+1)^2(y^2+2)}{x} u_x = f & \text{in } (0,\infty) \times (0,1) = \Omega \\ u = g & \text{on } \{0\} \times (0,1) = \mathcal{S}_1 \end{cases}$$

with $f \in C(\Omega)$, $g \in C(\mathcal{S}_1)$; here we take $\mathcal{R} = \emptyset$, $\mathcal{S} = \partial\Omega$, $\mathcal{S}_1 = \{0\} \times (0,1)$, $\mathcal{S}_2 = [0,\infty) \times \{0,1\}$.

The function

$$Z(x,y) = -(x^2 + y^2) + \log[y(1-y)] - 1$$

satisfies

$$Z \leq -1 \text{ in } \Omega, \qquad \mathcal{L}Z \geq 0 \text{ in } \Omega,$$
$$\lim_{y\to 0} Z(x,y) = \lim_{y\to 1} Z(x,y) = \lim_{x\to\infty} Z(x,y) = -\infty.$$

In view of Proposition 1.2.16, there exists at most one solution in $L^\infty(\Omega)$ of problem (1.5.9).

(e) Consider the problem[2]

$$\text{(1.5.10)} \qquad \begin{cases} x^2 u_{xx} + u_{yy} + 3x u_x = f & \text{in } (0,\infty) \times I\!R = \Omega \\ \lim\limits_{|x|+|y|\to\infty} u(x,y) = L \end{cases}$$

with $f \in C(\Omega)$, $L \in I\!R$. In this case $\mathcal{R} = \mathcal{S}_1 = \emptyset$, $\mathcal{S} = \mathcal{S}_2 = \{0\} \times I\!R$. Consider the function

$$Z(x,y) = -\frac{1}{dist((x,y),\mathcal{S})} - 1 = -\frac{1}{x} - 1 \qquad ((x,y) \in \Omega).$$

It is easily seen that

$$Z \leq -1 \text{ in } \Omega, \qquad \mathcal{L}Z > 0 \text{ in } \Omega, \qquad \lim_{x\to 0} Z(x,y) = -\infty.$$

In view of Proposition 1.2.17, for any $L \in I\!R$ problem (1.5.10) admits at most one bounded solution. Observe that the "condition at infinity" is necessary to ensure uniqueness: in fact, any constant is a bounded solution of the differential equation in (1.5.10) with $f = 0$.

The sub- and supersolutions constructed in the above examples are smooth in Ω; however, less regularity is needed for the general results to hold (see Definition 1.2.2). This is expedient in several respects; for instance, the subsolution used to prove uniqueness is often a function of the distance from the boundary, thus its smoothness depends on that of the latter. A simple example is given below.

(f) Consider the problem

$$\text{(1.5.11)} \qquad \begin{cases} a_{11} u_{xx} + a_{22} u_{yy} + 2x u_x + 2y u_y - \left|\log|1-x^2|\right| u = f & \text{in } \Omega \\ u = g & \text{on } \mathcal{R} \cup \mathcal{S}_1, \end{cases}$$

where

$$\Omega = ((-1,1) \times [0,1)) \cup ((-1,0) \times (-1,0)),$$
$$\mathcal{R} = ([-1,0] \times \{-1\}) \cup ([-1,1] \times \{1\}),$$
$$\mathcal{S}_1 = (\{-1\} \times (-1,1)) \cup (\{1\} \times (0,1)),$$
$$\mathcal{S}_2 = (\{0\} \times (-1,0)) \cup ([0,1] \times \{0\});$$
$$a_{11}(x,y) = \begin{cases} x^2 + y^2 & \text{if } x \in (-1,1), y \in [0,1) \\ x^2 & \text{if } x \in (-1,0), y \in (-1,0), \end{cases}$$

[2]This example was suggested by X. Cabré.

$$a_{22}(x,y) = \begin{cases} x^2+y^2 & \text{if} \quad x \in (-1,0), y \in (-1,1) \\ \\ y^2 & \text{if} \quad x \in [0,1), y \in (0,1) \end{cases}$$

and $f \in C(\Omega)$, $g \in C(\mathcal{R} \cup \mathcal{S}_1)$. Since

$$dist((x,y),\mathcal{S}_2) = \begin{cases} y & \text{if} \quad x \in [0,1), y \in (0,1) \\ \\ \sqrt{x^2+y^2} & \text{if} \quad x \in (-1,0), y \in (0,1) \\ \\ -x & \text{if} \quad x \in (-1,0), y \in (-1,0] \end{cases}$$

for any $(x,y) \in \Omega)$, it is easily seen that the function

$$Z(x,y) = 2\log[dist((x,y),\mathcal{S}_2)] - \log 3 \qquad ((x,y) \in \Omega))$$

belongs to $C^1(\Omega) \cap C(\Omega \cup \mathcal{R})$, but not to $C^2(\Omega)$. However, $Z \in C^2(\Omega \setminus [\{0\} \times (0,1) \cup (-1,0) \times \{0\}])$ and there holds

$$Z \leq \log \frac{2}{3} \ \text{ in } \ \Omega\,, \qquad \mathcal{L}Z \geq 0 \ \text{ a.e. in } \ \Omega\,;$$

hence we have

$$\int_\Omega Z\,\{\mathcal{L}^*\psi - c\psi\}\,dx \;=\; \int_\Omega \{\mathcal{L}Z - cZ\}\,\psi\,dx \;\geq\; 0$$

for any $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$. Clearly, condition (1.2.23) is satisfied; hence by Proposition 1.2.16 problem (1.5.11) admits at most one bounded solution.

CHAPTER 2

Uniqueness and nonuniqueness of solutions to parabolic problems with singular coefficients

2.1. Introduction

We investigate uniqueness and nonuniqueness of solutions to semilinear degenerate parabolic problems, with possibly unbounded coefficients, of the following form:

$$(2.1.1)\qquad \begin{cases} \mathcal{L}u - cu - \partial_t u = f(x,t,u) & \text{in } \Omega\times(0,T) =: Q_T \\ u = g & \text{in } \mathcal{R}\times(0,T] \\ u = u_0 & \text{in } (\Omega\cup\mathcal{R})\times\{0\}\,. \end{cases}$$

Here $\Omega \subseteq I\!R^n$ is an open connected set with boundary $\partial\Omega$, $\mathcal{R}$ is a subset of the boundary which we call the *regular boundary*, and the operator $\mathcal{L}$ is formally defined as follows:

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i}\,.$$

Special attention will be paid to *bounded* solutions of problem (2.1.1).

The portion $\mathcal{R}$ of the boundary is called "regular", since we address situations in which the coefficients a_{ij}, b_i, c are well-behaved and the operator $\mathcal{L}$ is elliptic only in the set $\Omega\cup\mathcal{R}$ (see assumptions (H_0), (H_2)). In such case it is natural to prescribe the Dirichlet boundary condition $u = g$ only on $\mathcal{R}\times(0,T]$, as assumed in problem (2.1.1). The complementary set $\mathcal{S} := \partial\Omega\setminus\mathcal{R}$ is called the *singular boundary*, for the coefficients can vanish (or diverge, or possibly do not have a limit), or ellipticity is possibly lost when $dist(x,\mathcal{S})\to 0$ (and/or as $|x|\to\infty$, if Ω is unbounded). Let us notice that the case $\partial\Omega \neq \partial\overline{\Omega}$ is allowed, thus $\mathcal{S}$ can be a manifold of dimension less than $n-1$, while $\mathcal{R}\subseteq\partial\overline{\Omega}$ (see (H_0)).

If uniqueness of problem (2.1.1) fails, in typical cases additional conditions are needed on some part of $\mathcal{S}\times(0,T]$ to obtain a well–posed problem. Hence it is natural to address the following generalization of problem (2.1.1):

$$(2.1.2)\qquad \begin{cases} \mathcal{L}u - cu - \partial_t u = f(x,t,u) & \text{in } Q_T \\ u = g & \text{in } (\mathcal{R}\cup\mathcal{S}_1)\times(0,T] \\ u = u_0 & \text{in } (\Omega\cup\mathcal{R}\cup\mathcal{S}_1)\times\{0\}\,, \end{cases}$$

$\mathcal{S}_1$ being a suitable subset of the singular boundary $\mathcal{S}$.

We give sufficient conditions for uniqueness of solutions to problem (2.1.2) (see Subsection 2.2.1 and Section 2.3). Such conditions depend on the existence for some $\mu\geq 0$ of subsolutions to the elliptic problem

$$(2.1.3)\qquad \begin{cases} \mathcal{L}U - cU = \mu U & \text{in } \Omega \\ U = 0 & \text{in } \mathcal{R}\,, \end{cases}$$

and on their behaviour as $dist(x, \mathcal{S} \setminus \mathcal{S}_1) \to 0$ (see Theorem 2.2.5). In the particular case $\mathcal{S}_1 = \emptyset$ we recover uniqueness criteria for problem (2.1.1).

We also provide sufficient conditions for nonuniqueness of *bounded* solutions to problem (2.1.1) (see Subsection 2.2.2 and Section 2.4). These conditions involve suitable supersolutions to the *first exit time problem*

$$\begin{cases} \mathcal{L}U = -1 & \text{in } \Omega \\ U = 0 & \text{in } \mathcal{R} \end{cases} \tag{2.1.4}$$

(see Theorem 2.2.16; for the probabilistic interpretation of problem (2.1.4), see *e.g.* [**35**]). Both uniqueness and nonuniqueness results are used in Section 2.5 to discuss some selected examples, which exhibit singularities of different kinds.

Analogous results hold for the Cauchy problem

$$\begin{cases} \mathcal{L}u - cu - \partial_t u = f(x,t,u) & \text{in } I\!R^n \times (0,T] \\ u = u_0 & \text{in } I\!R^n \times \{0\}, \end{cases} \tag{2.1.5}$$

although the latter does not fit formally within the present framework. We leave their formulation to the reader (however, see Proposition 2.2.13; here the "point at infinity" plays the role of the singular boundary).

The present work extends in several respects the results proved in [**60**] for parabolic, and in [**65**] for elliptic problems. The applicability of the present results, and of those in [**65**], relies on the actual construction of sub- and supersolutions to problems (2.1.3), (2.1.4) with suitable properties. This point is addressed in the paper [**58**]; the construction depends both on the dimension of the singular boundary $\mathcal{S}$ and on the behaviour of the coefficients of $\mathcal{L}$ as $dist(x, \mathcal{S}) \to 0$.

Let us add some comments on the wide literature concerning degenerate elliptic and parabolic equations, and its relationship with the present approach.

The case of *bounded coefficients* was first investigated. Since the Dirichlet problem for degenerate second-order equations may be well-posed prescribing boundary data only on a subset of the boundary, much work was devoted to the general characterization of such subset, using both analytical methods and stochastic calculus (in particular, see [**7**], [**26**], [**31**], [**56**], [**69**], [**70**]). A typical result is the elliptic *refined maximum principle* (see [**7**]), which implies uniqueness with boundary data prescribed at those points of $\partial\Omega$, where the minimal positive solution of the equation $\mathcal{L}U = -1$ in Ω can be extended to zero. On the other hand, the relationship between the *propagation set* of subsolutions of the equation $\mathcal{L}u = cu$ in Ω, and the *support* of the Markov process associated with $\mathcal{L}$, implies that uniqueness holds for the Dirichlet problem, if the boundary data are specified at the *attainable* boundary points (see [**31**], [**69**], [**70**]). Similar ideas underly our distinction between the regular and the singular boundary, although the present approach is purely analytical in nature.

More recently, parabolic equations with *unbounded coefficients* - mostly nondegenerate, and considered either in $\Omega = I\!R^n$ or in unbounded domains - have been widely investigated by an approach using Markov semigroups and related stochastic methods (see [**17**], [**52**] and references therein). In this approach sub- and supersolutions of the resolvent equation (often called Lyapunov functions) are used as barriers, as we do in the following with sub- and supersolutions of problems (2.1.3), (2.1.4). Since the Cauchy problem can be treated like problem (2.1.1), our uniqueness and nonuniqueness criteria generalize those mentioned in [**43**] –[**53**].

2.2. Mathematical framework and results

Our assumptions concerning the regular boundary $\mathcal{R}$ and the singular boundary $\mathcal{S}$ are stated as follows:

$$(H_0) \qquad \begin{cases} (i) \quad \partial\Omega = \mathcal{R} \cup \mathcal{S},\ \mathcal{R} \cap \mathcal{S} = \emptyset,\ \mathcal{S} \neq \emptyset\,; \\ (ii) \quad \mathcal{R} \subseteq \partial\overline{\Omega} \text{ is open, } \Omega \text{ satisfies the outer sphere condition at } \mathcal{R}\,; \end{cases}$$

$$(H_1) \qquad \begin{cases} (i) \ \ \mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2,\ \ \mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset\,; \\ (ii) \ \ \mathcal{S}_1 \text{ and } \mathcal{S}_2 \text{ have a finite number of connected components.} \end{cases}$$

Denote by $C^{0,1}(\Omega \cup \mathcal{R})$ the space of functions which are locally Lipschitz continuous in $\Omega \cup \mathcal{R}$, and by $C^{1,1}(\Omega \cup \mathcal{R})$ the subspace of those with locally Lipschitz continuous first derivatives. Concerning the coefficients of the operator $\mathcal{L}$ and the function c, we assume the following:

$$(H_2) \qquad \begin{cases} (i) \ \ a_{ij} = a_{ji} \in C^{1,1}(\Omega \cup \mathcal{R}),\ b_i \in C^{0,1}(\Omega \cup \mathcal{R}) \ \ (i,j = 1, \dots, n)\,; \\ (ii) \ \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j > 0 \text{ for any } x \in \Omega \cup \mathcal{R} \text{ and } (\xi_1, .., \xi_n) \neq 0\,; \\ (iii) \ c \in C(\Omega \cup \mathcal{R})\,. \end{cases}$$

It is natural to choose $\mathcal{R}$ as *the largest subset* of $\partial\Omega$ where the operator $\mathcal{L}$ is elliptic (see $(H_2) - (ii)$). We shall always do so in the following.

Concerning the functions f, g and u_0, we make the following assumption:

$$(H_3) \qquad \begin{cases} (i) \ f \ \in C(\bar{Q}_T \times I\!R) \ \text{ is Lipschitz continuous} \\ \qquad \text{with respect to } u \in I\!R, \text{ uniformly for } (x,t) \in \bar{Q}_T\,; \\ (ii) \ \ g \in C(\mathcal{R} \times [0,T]),\ u_0 \in C(\Omega \cup \mathcal{R})\,; \\ (iii) \ \ g(x,0) = u_0(x) \text{ for any } x \in \mathcal{R}\,. \end{cases}$$

Let us observe that the assumption of global Lipschitz continuity for f can be replaced by local Lipschitz continuity (uniformly in $\bar{Q}_T$), if bounded solutions are considered.

The uniqueness results in Subsection 2.2.1 hold true, if (H_2) is replaced by the following weaker assumption:

$$(H_2') \qquad \begin{cases} (i) \ \ a_{ij} = a_{ji} \in C^1(\Omega \cup \mathcal{R}),\ \sigma_{ij} \in C^1(\Omega),\ b_i \in C^{0,1}(\Omega \cup \mathcal{R}) \ \ (i,j = 1, \dots, n)\,; \\ (ii) \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq 0 \text{ for any } x \in \Omega \text{ and } (\xi_1, .., \xi_n) \in I\!R^n\,; \\ (iii) \ \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j > 0 \text{ for any } x \in \mathcal{R} \text{ and } (\xi_1, .., \xi_n) \neq 0\,. \end{cases}$$

Here $\Sigma \equiv (\sigma_{ij})$ denotes the square root of the matrix $A \equiv (a_{ij})$, namely $A(x) = \Sigma(x)\Sigma(x)^T$ $(x \in \Omega \cup \mathcal{R})$. Observe that assumption $(H_2') - (ii)$ is more general than $(H_2) - (ii)$, for it allows the operator $\mathcal{L}$ to degenerate in Ω.

Before going on, let us make precise the definition of solution, sub- and supersolution to problems (2.1.1)-(2.1.4). Denote by $\mathcal{L}^*$ the formal adjoint of the operator $\mathcal{L}$, namely:

$$\mathcal{L}^* v \equiv \sum_{i,j=1}^n \frac{\partial^2(a_{ij}v)}{\partial x_i \partial x_j} - \sum_{i=1}^n \frac{\partial(b_i v)}{\partial x_i}\,.$$

Sub- supersolutions and solutions to elliptic equations or problems are meant in the sense of Definition 1.2.1, respectively Definition 1.2.2; moreover, we have the following definitions.

Definition 2.2.1. *By a* subsolution *to the equation*

$$\mathcal{L}u - cu - \partial_t u = f(x,t,\,u) \quad \text{in } Q_T \tag{2.2.1}$$

we mean any function $u \in C(Q_T)$ *such that*

$$\int_{Q_T} u\,\{\mathcal{L}^*\psi - c\psi + \partial_t\psi\}\,dxdt \ \geq \int_{Q_T} f(x,t,\,u)\psi\,dxdt \tag{2.2.2}$$

for any $\psi \in C_0^\infty(Q_T)$, $\psi \geq 0$. Supersolutions *of (2.2.1) are defined reversng the inequality sign in (2.2.2). A function u is a* solution *of (2.2.1) if it is both a sub- and a supersolution.*

DEFINITION 2.2.2. *Let* $\mathcal{R} \subseteq \mathcal{E} \subseteq \partial\Omega$, $g \in C(\mathcal{E} \times [0,T]))$, $u_0 \in C(\Omega \cup \mathcal{E})$, $g(x,0) = u_0(x)$ *for any* $x \in \mathcal{E}$. *By a* subsolution *to the problem*

$$\begin{cases} \mathcal{L}u - cu - \partial_t u = f(x,t,u) & \text{in } Q_T \\ u = g & \text{in } \mathcal{E} \times (0,T] \\ u = u_0 & \text{in } (\Omega \cup \mathcal{E}) \times \{0\} \end{cases} \tag{2.2.3}$$

we mean any function $u \in C((\Omega \cup \mathcal{E}) \times [0,T])$ *such that:*
(i) u is a subsolution of equation (2.2.1);
(ii) $u \leq g$ *in* $\mathcal{E} \times (0,T]$;
(iii) $u \leq u_0$ *in* $(\Omega \cup \mathcal{E}) \times \{0\}$.
Supersolutions *and* solutions *of (2.2.3) are defined similarly.*

2.2.1. Comparison and uniqueness results. The following comparison result generalizes the classical Phragmèn-Lindelöf principle (*e.g.*, see [**61**]).

PROPOSITION 2.2.3. *Assume* $\mathcal{S}_2 \neq \emptyset$, (H_0), (H_1), (H_3) *and either* (H_2) *or* (H_2'). *Let there exist a subsolution* $Z \leq H < 0$ *of problem (2.1.3) for some* $\mu \geq 0$. *If* Ω *is bounded, then any subsolution u of problem (2.1.2) with* $f = g = u_0 = 0$ *such that*

$$\limsup_{dist(x,\mathcal{S}_2)\to 0} \frac{\sup_{t\in(0,T]} u(x,t)}{|Z(x)|} \leq 0 \tag{2.2.4}$$

satisfies $u \leq 0$ *in* Q_T.
If Ω *is unbounded, the same conclusion holds true under the additional condition*

$$\limsup_{|x|\to\infty} \frac{\sup_{t\in(0,T]} u(x,t)}{|Z(x)|} \leq 0\,. \tag{2.2.5}$$

In the above proposition the sign condition on the portion $\mathcal{S}_2$ of the singular boundary, where the subsolution u need not be defined, is replaced by a growth rate condition with respect to a suitable subsolution of the elliptic problem (2.1.3). Then the result follows by standard arguments. Observe that condition (2.2.4) reduces to a sign condition for u at $\mathcal{S}_2$ whenever Z is bounded in a neighbourhood of $\mathcal{S}_2$.

REMARK 2.2.4. Proposition 2.2.3 holds true also in the following cases:
(i) if condition (2.2.5) is replaced by the sign condition

$$\limsup_{|x|\to\infty} \Big(\sup_{t\in(0,T]} u(x,t) \Big) \leq 0\,; \tag{2.2.6}$$

(ii) if condition (2.2.4), respectively (2.2.5), is replaced by the weaker assumption

$$\liminf_{\varepsilon\to 0} \left\{ \sup_{x\in\mathcal{A}_2^\varepsilon\setminus\mathcal{S}} \frac{\sup_{t\in(0,T]} u(x,t)}{|Z(x)|} \right\} \leq 0\,, \tag{2.2.7}$$

respectively

$$\liminf_{\varepsilon\to 0} \left\{ \sup_{(x\in[\Omega\cup\mathcal{R}]\cap\partial B_{\frac{1}{\varepsilon}})} \frac{\sup_{t\in(0,T]} u(x,t)}{|Z(x)|} \right\} \leq 0\,. \tag{2.2.8}$$

In the above inequalities and hereafter we set $B_r(\bar{x}) := \{|x-\bar{x}| < r\} \subseteq I\!R^n$, $B_r(0) \equiv B_r$ and

$$\mathcal{A}_2^{\varepsilon} := \{x \in \overline{\Omega} \mid dist(x, \mathcal{S}_2) = \varepsilon\}$$

for any $\varepsilon \in (0, \varepsilon_0)$ ($\varepsilon_0 > 0$ suitably small).

In view of Proposition 2.2.3, we have the following uniqueness result.

THEOREM 2.2.5. *Let $\mathcal{S}_2 \neq \emptyset$, $g \in C((\mathcal{R} \cup \mathcal{S}_1) \times [0, T])$, $u_0 \in C(\Omega \cup \mathcal{R} \cup \mathcal{S}_1)$, $g(x, 0) = u_0(x)$ for any $x \in \mathcal{R} \cup \mathcal{S}_1$. Assume (H_0), (H_1), (H_3) and either (H_2) or (H_2'). Let there exist a subsolution $Z \leq H < 0$ of problem (2.1.3) for some $\mu \geq 0$. If Ω is bounded, then any two solutions u_1, u_2 of problem (2.1.2) such that*

$$\lim_{dist(x,\mathcal{S}_2)\to 0} \frac{\sup_{t\in(0,T]} |u_2(x,t) - u_1(x,t)|}{Z(x)} = 0 \tag{2.2.9}$$

coincide in Q_T.
If Ω is unbounded, the same conclusion holds true under the additional condition

$$\lim_{|x|\to+\infty} \frac{\sup_{t\in(0,T]} |u_2(x,t) - u_1(x,t)|}{Z(x)} = 0\,. \tag{2.2.10}$$

REMARK 2.2.6. Theorem 2.2.5 holds true, if (2.2.10) is replaced by the stronger condition

$$\lim_{|x|\to\infty} \sup_{t\in(0,T]} |u_2(x,t) - u_1(x,t)| \,=\, 0\,. \tag{2.2.11}$$

REMARK 2.2.7. If $c \geq c_1 > -\infty$ in $\Omega \cup \mathcal{R}$, in Proposition 2.2.3 and in Theorem 2.2.5 problem (2.1.3) can be replaced either by the *eigenvalue problem*

$$\begin{cases} \mathcal{L}U \,=\, \mu U & \text{in } \Omega \\ \\ U \,=0 & \text{in } \mathcal{R} \end{cases} \tag{2.2.12}$$

with $\mu \in [0, +\infty)$, or by problem (2.1.4).

REMARK 2.2.8. In connection with the above remark observe that, if $c \geq c_1 > -\infty$ in $\Omega \cup \mathcal{R}$, in Proposition 2.2.3 and Theorem 2.2.5 the function Z may be any subsolution bounded from above of problem (2.2.12). In fact, if Z is a subsolution of problem (2.2.12) bounded from above and $M > H_0 := \sup_{\Omega\cup\mathcal{R}} Z \geq 0$, then $\overline{Z} := Z - M \leq H_0 - M = H < 0$ is a subsolution of the same problem satisfying the required sign condition (see Definition 1.2.2). The same holds for problem (2.1.4).

REMARK 2.2.9. By compactness arguments, it is easily seen that in Theorem 2.2.5 condition (2.2.9) can be replaced by

$$\lim_{dist(x,\mathcal{S}_2)\to 0} \frac{|u_2(x,t) - u_1(x,t)|}{Z(x)} = 0 \qquad (t \in (0, T])\,. \tag{2.2.13}$$

Uniqueness results concerning *bounded* solutions of problem (2.1.2) follow immediately from the above, if the subsolution Z diverges as $dist(x, \mathcal{S}_2) \to 0$. We state them below for convenience of the reader.

THEOREM 2.2.10. *Let $\mathcal{S}_2 \neq \emptyset$, $g \in C((\mathcal{R} \cup \mathcal{S}_1) \times [0, T])$, $u_0 \in C(\Omega \cup \mathcal{R} \cup \mathcal{S}_1)$, $g(x, 0) = u_0(x)$ for any $x \in \mathcal{R} \cup \mathcal{S}_1$. Assume (H_0), (H_1), (H_3) and either (H_2) or (H_2'). Let there exist a subsolution $Z \leq H < 0$ of problem (2.1.3) for some $\mu \geq 0$, such that*

$$\lim_{dist(x,\mathcal{S}_2)\to 0} Z(x) \,=\, -\infty\,. \tag{2.2.14}$$

If Ω is unbounded, also assume

$$\lim_{|x|\to\infty} Z(x) = -\infty \,. \tag{2.2.15}$$

Then there exists at most one solution $u \in L^\infty(Q_T)$ of problem (2.1.2).

REMARK 2.2.11. If Ω is unbounded and a *condition at infinity* is imposed, condition (2.2.15) is not needed for uniqueness.

REMARK 2.2.12. As for Remark 2.2.7, let us mention that in Theorem 2.2.10 problem (2.1.3) (with $\mu \geq 0$) can be replaced by (2.2.12) or (2.1.4), if $c \geq c_1 > -\infty$ in $\Omega \cup \mathcal{R}$.

Let us mention the following counterpart of Theorem 2.2.10 for the Cauchy problem (2.1.5) in the case of "degeneracy at infinity" (see [**20**], [**43**] - [**53**]).

PROPOSITION 2.2.13. *Let $\Omega = I\!R^n$, $\mathcal{R} = \emptyset$ and the assumptions (H_2'), (H_3) be satisfied. Let there exist a subsolution $Z \leq H < 0$ of problem (2.1.3) for some $\mu \geq 0$, such that condition* (2.2.15) *is satisfied. Then there exists at most one solution $u \in L^\infty(S_T)$ of problem (2.1.5).*

2.2.2. Nonuniqueness results. Let us first state some standard existence results, which require the existence of suitable supersolutions of auxiliary problems.

PROPOSITION 2.2.14. *Let assumptions (H_0), (H_2), (H_3) be satisfied. Suppose $g \in L^\infty(\mathcal{R}\times(0,T))$, $u_0 \in L^\infty(\Omega)$, $f(\cdot,\cdot,0) \in L^\infty(Q_T)$. Let there exist a supersolution $F \in C((\Omega\cup\mathcal{R})\times[0,T])$, $F \geq c_2 > 0$ of the equation*

$$\mathcal{L}\,u - cu - \partial_t u = -1 \quad \textit{in } \; Q_T \,. \tag{2.2.16}$$

Then there exists a solution u of problem (2.1.1). If $F \in L^\infty(Q_T)$, then $u \in L^\infty(Q_T)$.

If $c \geq c_1$ in $\Omega \cup \mathcal{R}$ for some $c_1 \in I\!R$, it is easy to check that the function $F := \exp\{(|c_1| + 1)t\} \geq 1 \, (t \in [0,T])$ is a *bounded* supersolution of equation (2.2.16). However, it should be noticed that problem (2.1.1) can admit a solution in $L^\infty(Q_T)$, even if c is not bounded from below.

If Ω is unbounded and a *"barrier at infinity"* exists, we can prescribe a *conditon at infinity* for the *time mean* of the solution, as stated in the following proposition.

PROPOSITION 2.2.15. *Let Ω be unbounded and the assumptions of Proposition 2.2.14 be satisfied. Moreover, suppose $c \in L^\infty(\Omega \setminus \bar{B}_R)$, $F \in L^\infty((\Omega\setminus B_R)\times(0,T))$, $(R>0)$, $l \in I\!R$ and, if $\mathcal{R}$ is unbounded,*

$$\lim_{|x|\to\infty} \frac{1}{T}\int_0^T g(x,t)\,dt = l \,. \tag{2.2.17}$$

Let there exist a positive supersolution H of the equation

$$\mathcal{L}H = -1 \quad \textit{in } \Omega \setminus \bar{B}_R \,, \tag{2.2.18}$$

such that

$$\lim_{|x|\to+\infty} H(x) = 0 \,. \tag{2.2.19}$$

Then there exists a solution u of problem (2.1.1) such that

$$\lim_{|x|\to+\infty} \frac{1}{T}\int_0^T u(x,t)\,dt = l \,. \tag{2.2.20}$$

If $F \in L^\infty(Q_T)$, then $u \in L^\infty(Q_T)$.

Observe that Proposition 2.2.15 gives a nonuniqueness result for problem (2.1.1), if Ω is unbounded and $\mathcal{R}$ bounded. In fact, in this case condition (2.2.17) is not needed and any $l \in I\!R$ can be prescribed at infinity (see (2.2.20)).

Now we can state our nonuniqueness results. In the following theorem nonuniqueness follows from existence of a supersolution V to the first exit time problem (2.1.4), such that

$$\inf_{\Omega\cup\mathcal{R}} V = 0 < \inf_{\mathcal{R}} V\,. \tag{2.2.21}$$

THEOREM 2.2.16. *Let assumptions* (H_0), (H_2), (H_3) *be satisfied. Suppose* $g \in L^\infty(\mathcal{R}\times(0,T))$, u_0, $c \in L^\infty(\Omega), f(\cdot,\cdot,0) \in L^\infty(Q_T)$. *Let there exist a supersolution V of problem (2.1.4) such that* (2.2.21) *is satisfied. Then there exist infinitely many bounded solutions of problem (2.1.1).*

More precisely, there exists a sequence $\{x_m\} \subseteq \Omega$ with the following property: for any $\beta \in I\!R$ there exists a solution $u_\beta \in L^\infty(Q_T)$ of problem (2.1.1) such that

$$\lim_{m\to\infty} \frac{1}{T}\int_0^T u_\beta(x_m,t)\,dt \;=\; \beta\,. \tag{2.2.22}$$

REMARK 2.2.17. The assumption $V \geq 0$ in (2.2.21) is equivalent to the (apparently weaker) assumption that V be a supersolution of problem (2.1.4) bounded from below. Indeed, in the latter case $\overline{V} := V - \inf_{\Omega\cup\mathcal{R}} V$ is a supersolution of the same problem such that $\inf_{\Omega\cup\mathcal{R}} \overline{V} = 0$ (see Definition 1.2.2).

If Ω is unbounded and there exists a "barrier at infinity", nonuniqueness of solutions to problem (2.1.1) satisfying (2.2.20) follows as in Theorem 2.2.16, provided that the supersolution V of problem (2.1.4) satisfies the stronger assumption:

$$\inf_{\Omega\cup\mathcal{R}} V = 0 < \min\Big\{\inf_{\mathcal{R}} V\,, \liminf_{|x|\to\infty} V(x)\Big\}. \tag{2.2.23}$$

In fact, inequality (2.2.23) ensures nonuniqueness, also when a condition at infinity is prescribed.

THEOREM 2.2.18. *Let Ω be unbounded, $l \in I\!R$ and the assumptions of Theorem 2.2.16 be satisfied, with condition* (2.2.21) *replaced by* (2.2.23). *Moreover, if $\mathcal{R}$ is unbounded, assume (2.2.17). Let there exist a positive supersolution H of equation* (2.2.18) *satisfying (2.2.19) for some $R > 0$. Then there exist infinitely many bounded solutions of problem (2.1.1) satisfying (2.2.20) .*

More precisely, there exists a bounded sequence $\{x_m\} \subseteq \Omega$ with the following property: for any $\beta \in I\!R$ there exists a solution $u_\beta \in L^\infty(Q_T)$ of problem (2.1.1) such that (2.2.20) and (2.2.22) are satisfied.

Let us observe that, as shown by Example (a) in Subsection 2.5.1, the hypothesis $c \in L^\infty(\Omega)$ in Theorems 2.2.16 and 2.2.18 is necessary.

In the following we deal with cases in which boundary data can be prescribed also at $\mathcal{S}_1 \times (0,T]$. Let us notice that existence of solutions of problem (2.1.2) with $\mathcal{S}_1 \neq \emptyset$ implies nonuniqueness of solutions of problem (2.1.1).

We will use the following definition, where for any $\Sigma \subseteq \partial\Omega$ and $\varepsilon \in (0,\varepsilon_0)$ ($\varepsilon_0 > 0$ suitably small) we set

$$\Sigma^\varepsilon := \{x \in \Omega \mid dist(x,\Sigma) < \varepsilon\}\,.$$

DEFINITION 2.2.19. *We say that $\Sigma \subseteq \partial\Omega$ is* attracting *for the operator $\mathcal{L}$ if there exist $\varepsilon \in (0,\varepsilon_0)$ and a supersolution $V \in C(\overline{\Sigma^\varepsilon})$ of the equation:*

$$\mathcal{L}V = -1 \quad in\ \Sigma^\varepsilon \tag{2.2.24}$$

such that

$$V > 0 \quad in\ \overline{\Sigma^\varepsilon}\setminus\Sigma\,, \qquad V = 0 \quad on\ \Sigma\,.$$

For the sake of simplicity, in the following two statements we suppose Ω to be bounded. Similar results can be proved when Ω is unbounded.

THEOREM 2.2.20. *Let Ω be bounded and assumptions $(H_0)-(H_3)$ be satisfied; suppose $c \in L^\infty(\mathcal{S}_1^\varepsilon)$ for some $\varepsilon > 0$, $f(\cdot,\cdot,0) \in L^\infty(Q_T)$. Let the following assumptions be satisfied:*

- *$\mathcal{S}_1 \subseteq \partial\Omega$ is attracting;*
- *there exists $\phi \in C((\Omega\cup\mathcal{R}\cup\overline{\mathcal{S}_1})\times[0,T])\cap L^\infty(Q_T)$, such that $g=\phi$ in $(\mathcal{R}\cup\mathcal{S}_1)\times[0,T]$ and $u_0(x)=\phi(x,0)$ for any $x \in \Omega\cup\mathcal{R}\cup\mathcal{S}_1$;*
- *ϕ does not depend on x in $\mathcal{S}_1\times[0,T]$, namely*

$$\phi(x,t)=\phi_1(t) \quad \textit{for any } (x,t)\in \mathcal{S}_1\times[0,T]\,, \tag{2.2.25}$$

  for some $\phi_1 \in C([0,T])$;
- *there exists a supersolution $F \in C((\Omega\cup\mathcal{R})\times[0,T])\cap L^\infty(\mathcal{S}_1^\varepsilon\times(0,T))$, $F \geq c_2 > 0$ of equation* (2.2.16).

Then there exists a solution u of problem (2.1.2). In addition, if $F \in L^\infty(Q_T)$, then $u \in L^\infty(Q_T)$.

Observe that, in the light of the above theorem, *arbitrary Dirichlet boundary data cannot be prescribed on an attracting portion $\mathcal{S}_1$ of the singular boundary $\mathcal{S}$*. In fact, in this case a solution of problem (2.1.2) need not exist unless (2.2.25) is satisfied, *i.e.*, unless the initial data are *constant* in $\mathcal{S}_1$ and the boundary data *only depend on time* in $\mathcal{S}_1\times[0,T]$ (see Example (c) in Subsection 2.5.1).

REMARK 2.2.21. It is easily seen that in Theorem 2.2.20 the restriction (2.2.25) can be removed, if the attractivity of $\mathcal{S}_1$ is replaced by the stronger assumption that a *local barrier* exists at any point of $\mathcal{S}_1\times[0,T]$ - namely, if for any $(x_0,t_0)\in\mathcal{S}_1\times[0,T]$ there exist $\delta>0$ and a supersolution $h \in C(\overline{K_\delta(x_0,t_0)})$ of the equation

$$\mathcal{L}h - ch - \partial_t h = -1 \quad \text{in } K_\delta(x_0,t_0)$$

such that

$$h>0 \quad \text{in } \overline{K_\delta(x_0,t_0)}\setminus\{(x_0,t_0)\} \quad \text{ and } \quad h(x_0,t_0)=0\,;$$

here $K_\delta(x_0,t_0) := \big(B_\delta(x_0)\times(t_0-\delta,t_0+\delta)\big)\cap Q_T$ (*e.g.*, see [**30**]). A similar situation for elliptic problems was pointed out in [**65**].

The following proposition gives sufficient conditions for the existence of a local barrier at any point of $\Sigma\times[0,T]$, Σ denoting an *attracting* subset of the boundary $\partial\Omega$.

PROPOSITION 2.2.22. *Let assumptions $(H_0)-(H_3)$ be verified. Let $\Sigma\subseteq\partial\Omega$ be attracting for the operator $\mathcal{L}$; suppose $a_{i,j}$, $b_i \in L^\infty(\Sigma^\varepsilon)$, $c\geq c_1$ in Σ^ε for some $c_1 \in I\!R$ ($\varepsilon\in(0,\varepsilon_0)$; $i,j=1,\dots,n$). Then for any $(x_0,t_0)\in\Sigma\times[0,T]$ there exists a local barrier.*

Let us mention that the boundedness of the coefficients $a_{i,j}$, b_i, assumed in the above proposition, is not necessary to construct a local barrier at any point of $\mathcal{S}_1\times[0,T]$ (see Examples (d), (e) in Subsection 2.5.1).

2.3. Proof of comparison and uniqueness results

Let us first discuss some auxiliary results to be used in the sequel. We refer the reader to [**18**]) for the definition of viscosity subsolution of equation (2.2.1).

PROPOSITION 2.3.1. *Assume (H_3) and either (H_2) or (H_2'). Let $u\in C(Q_T)$. Then the following statements are equivalent:*
(i) u is a subsolution of equation (2.2.1);
(ii) u is a viscosity subsolution of equation (2.2.1).

Proof. $(i) \Rightarrow (ii)$: Let (H_2) and (H_3) hold. Since the matrix $A \equiv (a_{ij})$ is positive definite in Ω and $a_{ij} \in C^{1,1}(\Omega)$, its square root $\Sigma \equiv (\sigma_{i,j})$ is in $C^{1,1}(\Omega)$, too (see [**31**], Ch. 6, Lemma 1.1). Set $(x_1, ..., x_n, x_{n+1}) \equiv (x_1, ..., x_n, t)$; consider the $(n+1) \times (n+1)$ matrices $\tilde{A} \equiv (\tilde{a}_{ij})$, $\tilde{\Sigma} \equiv (\tilde{\sigma}_{ij})$ defined as follows:

$$\tilde{a}_{ij} := \begin{cases} a_{ij} & \text{if } i, j \in \{1, ..., n\} \\ 0 & \text{otherwise;} \end{cases} \tag{2.3.1}$$

$$\tilde{\sigma}_{ij} := \begin{cases} \sigma_{ij} & \text{if } i, j \in \{1, ..., n\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3.2}$$

Then $\tilde{\Sigma}$ is the square root of $\tilde{A}$ and $\tilde{\Sigma} \in C^{1,1}(\Omega)$. Moreover, let

$$\tilde{b}_i := \begin{cases} b_i & \text{if } i \in \{1, ..., n\} \\ -1 & \text{if } i = n+1, \end{cases} \tag{2.3.3}$$

$$\tilde{\mathcal{L}}u \equiv \sum_{i,j=1}^{n+1} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n+1} b_i \frac{\partial u}{\partial x_i},$$

$$(\tilde{\mathcal{L}})^* v \equiv \sum_{i,j=1}^{n+1} \frac{\partial^2 (a_{ij} v)}{\partial x_i \partial x_j} - \sum_{i=1}^{n+1} \frac{\partial (b_i v)}{\partial x_i}.$$

It is easily seen that $u \in C(Q_T)$ is subsolution of equation (2.2.1) if and only if

$$\int_{Q_T} u \{(\tilde{\mathcal{L}})^* \psi - c\psi\} \, dx \geq \int_{Q_T} f(x, t, u) \psi \, dx$$

for any $\psi \in C_0^\infty(Q_T)$, $\psi \geq 0$. Hence (ii) follows from Theorem 2 in [**38**] (which holds also for $f = f(x, t, u(x,t)))$ and Remark 8.1 in [**18**].
Assuming (H_2') instead of (H_2) the proof is analogous.
$(ii) \Rightarrow (i)$: Follows from Theorem 1 in [**38**] and Remark 8.1 in [**18**], in view of the present regularity assumptions. □

From the above proposition we obtain the following comparison result.

Proposition 2.3.2. *Assume (H_3) and either (H_2) or (H_2'). Let Ω_1 be any open bounded subset of Ω such that $\overline{\Omega}_1 \subseteq \Omega \cup \mathcal{R}$. Let $\underline{u}$ be a subsolution, $\overline{u}$ a supersolution of equation*

$$\mathcal{L}u - cu - \partial_t u = f(x, t, u) \quad \text{in } \Omega_1 \times (0, T),$$

with

$$\underline{u} \leq \overline{u} \quad \text{in } \big(\partial\Omega_1 \times (0, T]\big) \cup \big(\bar{\Omega}_1 \times \{0\}\big).$$

Then $\underline{u} \leq \overline{u}$ in $\overline{\Omega}_1 \times [0, T]$.

Proof. It is easily seen that $u^* := \underline{u} - \overline{u}$ is a subsolution of the problem

$$\begin{cases} \mathcal{L}\, u - cu - \partial_t u = -L|u| & \text{in } \partial\Omega_1 \times (0, T) \\ u = 0 & \text{in } \partial\Omega_1 \times (0, T] \\ u = 0 & \text{in } \overline{\Omega}_1 \times \{0\}, \end{cases} \tag{2.3.4}$$

where $L > 0$ is the Lipschitz constant of $f = f(x, t, \cdot)$, uniform for $(x, t) \in \bar{Q}_T$ (see $(H_3) - (i)$). In fact, for any $\psi \in C_0^\infty(\Omega_1 \times (0, T))$, $\psi \geq 0$, there holds

$$\int_{\Omega_1 \times (0,T)} u^* \{\mathcal{L}^* \psi - c\psi + \partial_t \psi\} \, dxdt \geq \int_{\Omega_1 \times (0,T)} f(x, t, \underline{u}) - f(x, t, \overline{u}) \psi \, dxdt \geq$$

$$\geq - \int_{\Omega_1 \times (0,T)} L|u^*|\, \psi \, dxdt.$$

Moreover $u^* \leq 0$ in $\partial\Omega_1 \times (0,T)$, $u^* \leq 0$ in $\overline{\Omega}_1 \times \{0\}$. In view of Proposition **??**, u^* is a viscosity subsolution of problem (2.3.4); since the null function is a viscosity solution of the same problem, the function $u^*_+ := \sup\{u^*, 0\}$ is a subsolution of the problem (2.3.4), too. Since u^*_+ is nonnegative, it is a subsolution of problem

$$(2.3.5) \qquad \begin{cases} \mathcal{L}\, u - cu - \partial_t u = -Lu & \text{in } \partial\Omega_1 \times (0,T) \\ u = 0 & \text{in } \partial\Omega_1 \times (0,T] \\ u = 0 & \text{in } \overline{\Omega}_1 \times \{0\}\,. \end{cases}$$

Then the claim follows by comparison results concerning viscosity sub- and supersolutions (*e.g.*, see [**39**]). □

We can now prove Proposition 2.2.3. To this purpose we use the notations introduced in Section 1.4.

Proof of Proposition 2.2.3. (a) Let Ω *be bounded.* We give the proof when condition (2.2.4) is replaced by the weaker assumption (2.2.7) (see Remark 2.2.4-(*ii*)). Define for any $\alpha > 0$

$$(2.3.6) \qquad V_\alpha(x,t) := -\alpha Z(x) \exp\{\mu t\} = \alpha |Z(x)| \exp\{\mu t\}$$

$\Big((x,t) \in (\Omega \cup \mathcal{R}) \times [0,T]\Big)$. Clearly,

$$(2.3.7) \qquad V_\alpha \geq \alpha |H| > 0 \quad \text{in}\, (\Omega \cup \mathcal{R}) \times [0,T]$$

for any $\alpha > 0$. By inequality (2.2.7) there exists a sequence $\{\varepsilon_k\} \subseteq (0, \varepsilon_0)$, $\varepsilon_k \to 0$ as $k \to \infty$, such that

$$(2.3.8) \qquad \lim_{k \to +\infty} \left\{ \sup_{x \in \mathcal{A}_2^{\varepsilon_k} \setminus \mathcal{S}} \frac{\sup_{t \in (0,T]} u(x,t)}{|Z(x)|} \right\} \leq 0\,.$$

Let us prove the following:

(*i*) For any $\alpha > 0$, $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \frac{\varepsilon}{2})$ the function V_α defined in (2.3.6) is a supersolution of the problem

$$(2.3.9) \qquad \begin{cases} \mathcal{L}\, u - cu - \partial_t u = 0 & \text{in } (\Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}) \times (0,T) \\ u = V_\alpha & \text{in } \partial(\Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}) \times (0,T] \\ u = V_\alpha & \text{in } \overline{\Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}} \times \{0\}\,. \end{cases}$$

(*ii*) For any $\alpha > 0$ there exists $\bar{k} = \bar{k}(\alpha) \in I\!N$ with the following property: for any $k > \bar{k}$ there exists $\delta_k \in (0, \frac{\varepsilon_k}{2})$ such that the function u is a subsolution of problem (2.3.9) with $\varepsilon = \varepsilon_k, \delta = \delta_k$.

From (*i*) and (*ii*) the conclusion follows immediately. In fact, in view of Proposition 2.3.2, for any $\alpha > 0$, $k > \bar{k}$ we obtain

$$u \leq V_\alpha \qquad \text{in } (\Omega \setminus \overline{\mathcal{I}^{\varepsilon_k,\delta_k}}) \times (0,T)\,.$$

Letting $\alpha \to 0$ in the latter inequality obtains $u \leq 0$ in any compact subset of Q_T (observe that $\bar{k} \to \infty$, thus $\varepsilon_k \to 0$ as $\alpha \to 0$); hence the result follows.

To prove (i), take ψ as in Definition 2.2.1. For any fixed $t \in [0,T]$ the function $\psi(\cdot,t)$ is nonnegative and belongs to $C_0^\infty(\Omega)$; therefore from Definition 1.2.1 we get

$$\int_\Omega Z\,\{\mathcal{L}^*\psi - c\psi\}\,dx \,\geq\, \mu \int_\Omega Z\psi\,dx\,.$$

Hence multiplying by $-\alpha\exp\{\mu t\}$ and integrating in $(0,T)$ we obtain easily

$$\int_{Q_T} V_\alpha\,\{\mathcal{L}^*\psi - c\psi\}\,dxdt \,\leq \mu \int_{Q_T} V_\alpha\psi\,dxdt\,. \tag{2.3.10}$$

On the other hand, it is easily checked that

$$\int_{Q_T} V_\alpha\,\partial_t\,\psi\,dxdt = -\mu\int_{Q_T} V_\alpha\,\psi\,dxdt, \tag{2.3.11}$$

for any ψ as above. From (1.4.1), (2.3.10), (2.3.11), we can infer that V_α is a supersolution to equation

$$\mathcal{L}u\,-cu\,-\partial_t u = 0 \quad \text{ in } (\Omega\setminus\overline{\mathcal{I}^{\varepsilon,\delta}})\times(0,T),$$

thus (i) follows.

To prove (ii) we shall make use of the following

Claim: The following statements hold true:

- for any $\alpha>0$, $\varepsilon\in(0,\varepsilon_0)$ and $\delta\in(0,\frac{\varepsilon}{2})$ there holds

$$u\leq V_\alpha \qquad \text{in } \overline{\Omega\setminus\overline{\mathcal{I}^{\varepsilon,\delta}}}\times\{0\}\,; \tag{2.3.12}$$

- for any $\alpha>0$, $\varepsilon\in(0,\varepsilon_0)$ and $\delta\in(0,\frac{\varepsilon}{2})$ there holds

$$u\leq V_\alpha \qquad \text{in } \mathcal{R}^{\varepsilon,\delta}\times(0,T]\,; \tag{2.3.13}$$

- for any $\alpha>0$, $\varepsilon\in(0,\varepsilon_0)$ and $\delta\in(0,\frac{\varepsilon}{2})$ there holds

$$\alpha|H|\,\leq\, V_\alpha \qquad \text{in } \mathcal{F}_1^{\varepsilon,\delta}\times(0,T]\,; \tag{2.3.14}$$

- for any $\alpha>0$, $\varepsilon\in(0,\varepsilon_0)$ there exists $\bar\delta\in(0,\frac{\varepsilon}{2})$ such that for any $\delta\in(0,\bar\delta)$ there holds

$$u\,\leq\,\alpha|H| \qquad \text{in } \mathcal{F}_1^{\varepsilon,\delta}\times(0,T]\,; \tag{2.3.15}$$

- for any $\alpha>0$ there exists $\bar k=\bar k(\alpha)\in I\!N$ such that for any $k>\bar k$ and for any $\delta\in(0,\frac{\varepsilon_k}{2})$ the function V_α satisfies

$$u\,\leq\,V_\alpha \qquad \text{in } \mathcal{F}_2^{\varepsilon_k,\delta}\times(0,T]\,. \tag{2.3.16}$$

Let us prove the Claim. Observe that u is subsolution to problem (2.1.2) with $f=g=u_0=0$, moreover $\mathcal{R}^{\varepsilon,\delta}\subseteq\mathcal{R}$ and (1.4.1), (2.3.7) hold, thus (2.3.12) and (2.3.13) are valid.

Inequality (2.3.14) follows from the very definition of V_α and H (actually, the same inequality holds in $(\Omega\cup\mathcal{R})\times(0,T]$).

To prove (2.3.15) observe that $u\leq 0$ on $\mathcal{S}_1\times[0,T]$ and $u\in C((\Omega\cup\mathcal{R}\cup\mathcal{S}_1)\times[0,T])$, thus in particular $u\leq 0$ on $\mathcal{S}_{1,\varepsilon}\times[0,T]$ and $u\in C((\Omega\cup\mathcal{R}\cup\mathcal{S}_{1,\varepsilon})\times[0,T])$. Moreover $\mathcal{S}_{1,\varepsilon}\times[0,T]$ is compact since $\mathcal{S}_{1,\varepsilon}$ is closed and bounded. Then it is not difficult to see that for any $\sigma>0$, $\varepsilon\in(0,\varepsilon_0)$ there exists $\bar\delta=\bar\delta(\sigma,\varepsilon)\in(0,\frac{\varepsilon}{2})$ such that for any $\delta\in(0,\bar\delta)$ there holds

$$u\leq\sigma \qquad \text{in } \mathcal{F}_1^{\varepsilon,\delta}\times(0,T]\,.$$

Choosing $\sigma=\alpha|H|$ we obtain (2.3.15).

To prove (2.3.16) observe that, in view of (2.3.8), for any $\alpha>0$ there exists $\bar k=\bar k(\alpha)\in I\!N$ such that for any $k>\bar k$

$$\frac{\sup_{t\in(0,T]}u(x,t)}{|Z(x)|}\leq\alpha \quad \text{for any} \quad x\in\mathcal{A}_2^{\varepsilon_k}\setminus\mathcal{S}\,. \tag{2.3.17}$$

Hence (2.3.16) follows immediately from (2.3.17), since $\mathcal{F}_2^{\varepsilon_k,\delta} \subseteq \mathcal{A}_2^{\varepsilon_k} \setminus \mathcal{S}$ for any $\delta \in (0, \frac{\varepsilon_k}{2})$. This completes the proof of the Claim.

Let us go back to the proof of (ii). There holds

$$u \leq V_\alpha \qquad \text{in } \partial(\Omega \setminus \overline{\mathcal{I}^{\varepsilon_k,\delta_k}}) \times (0,T] \tag{2.3.18}$$

for any $\alpha > 0$, $k > \bar{k}$ and some $\delta_k \in (0, \frac{\varepsilon_k}{2})$; this follows from (1.4.2), (2.3.13)-(2.3.16), for $\varepsilon = \varepsilon_k$ and $\delta_k = \frac{\bar{\delta}}{2}$. On the other hand, the function u is a subsolution of the problem (2.1.2) with $f = g = u_0 = 0$ thus, in particular, u is subsolution to equation

$$\mathcal{L}u - cu - \partial_t u = 0 \quad \text{in } (\Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}) \times (0,T).$$

Combining this fact with the initial and boundary conditions (2.3.12) and (2.3.18) we obtain (ii). This completes the proof when Ω is bounded.

(b) Let Ω be unbounded. We give the proof when conditions (2.2.4), (2.2.5) are replaced by the weaker assumptions (2.2.7), (2.2.8) (see Remark 2.2.4-(ii)). We shall use the family $\Omega^{\varepsilon,\delta,\beta}$ of subsets of Ω, introduced in 1.4.

(i) In view of inequalities (2.2.7) and (2.2.8), there exist two sequences $\{\varepsilon_k\} \subseteq (0, \varepsilon_0)$, $\varepsilon_k \to 0$ as $k \to \infty$ and $\{\beta_k\} \subseteq (0, \infty)$, $\beta_k \to 0$ as $k \to \infty$, such that (2.3.8) and

$$\lim_{k\to+\infty} \left\{ \sup_{x \in ([\Omega\cup\mathcal{R}]\cap\partial B_{\frac{1}{\beta_k}})} \frac{\sup_{t\in(0,T]} u(x,t)}{|Z(x)|} \right\} \leq 0 \tag{2.3.19}$$

hold true. Then for any $\alpha > 0$ there exists $\bar{k} = \bar{k}(\alpha) \in I\!N$ such that for any $k > \bar{k}$

$$\frac{\sup_{t\in(0,T]} u(x,t)}{|Z|} \leq \alpha \quad \text{in } \mathcal{A}_2^{\varepsilon_k} \setminus \mathcal{S}\,, \tag{2.3.20}$$

and

$$\frac{\sup_{t\in(0,T]} u(x,t)}{|Z|} \leq \alpha \;\text{ in }\; [\Omega \cup \mathcal{R}] \cap \partial B_{\frac{1}{\beta_k}}. \tag{2.3.21}$$

(ii) As in case (a), it is easily seen that the function $V_\alpha := -\alpha \exp\{\mu t\} Z$ is a supersolution of the problem

$$\begin{cases} \mathcal{L}u + cu - \partial_t u = 0 & \text{in } \Omega^{\varepsilon,\delta,\beta} \times (0,T) \\ u = V_\alpha & \text{in } \partial\Omega^{\varepsilon,\delta,\beta} \times (0,T] \\ u = V_\alpha & \text{in } \overline{\Omega^{\varepsilon,\delta,\beta}} \times \{0\} \end{cases} \tag{2.3.22}$$

for any $\alpha > 0$, $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \frac{\varepsilon}{2})$, $\beta > 0$.

Arguing as in case (a) the conclusion follows from the following
Claim: For any $\alpha > 0$ there exists $\bar{k} = \bar{k}(\alpha) \in I\!N$ with the following property: for any $k > \bar{k}$ there exists $\delta_k \in (0, \frac{\varepsilon_k}{2})$ such that the function u is a subsolution of problem (2.3.22) with $\varepsilon = \varepsilon_k, \delta = \delta_k, \beta = \beta_k$, where $\{\varepsilon_k\}$ and $\{\beta_k\}$ are the infinitesimal sequences of inequalities (2.3.20)-(2.3.21).

To prove the Claim, it suffices to prove that

$$u \leq V_\alpha \quad \text{on } \left[\left(\mathcal{F}^{\varepsilon_k,\delta_k} \cap \overline{B_{\frac{1}{\beta_k}}}\right) \cup \left(\overline{\Omega \setminus \overline{\mathcal{I}^{\varepsilon_k,\delta_k}}} \cap \partial B_{\frac{1}{\beta_k}}\right)\right] \times (0,T] \tag{2.3.23}$$

with $\alpha, k, \varepsilon_k, \delta_k, \beta_k$ as above. Notice that (2.3.12)-(2.3.14) and (2.3.16) are still valid. Moreover, in view of the compactness of $\mathcal{S}_{1,\varepsilon} \cap \overline{B_{\frac{1}{\beta}}}$ ($\varepsilon \in (0, \varepsilon_0)$, $\beta > 0$), arguing as in the proof of (2.3.15) we conclude that

- for any $\alpha > 0$, $\varepsilon \in (0, \varepsilon_0)$, $\beta > 0$ there exists $\bar{\delta} \in (0, \frac{\varepsilon}{2})$ such that for any $\delta \in (0, \bar{\delta})$ there holds

$$u < \alpha|H| \qquad \text{in } \big(\mathcal{F}_1^{\varepsilon,\delta} \cap \overline{B_{\frac{1}{\beta}}}\big) \times (0, T]\,. \tag{2.3.24}$$

Then, in view of (2.3.12)-(2.3.14), (2.3.16) and (2.3.24), for any $\alpha > 0$ there exists $\overline{k} = \overline{k}(\alpha)$ with the following property: for any $k > \overline{k}$ there exists $\delta_k = \frac{\bar{\delta}}{2} \in (0, \frac{\varepsilon_k}{2})$ such that

$$u \le V_\alpha \qquad \text{in } \big(\mathcal{F}^{\varepsilon_k,\delta_k} \cap \overline{B_{\frac{1}{\beta_k}}}\big) \times (0, T]\,. \tag{2.3.25}$$

Concerning the inequality

$$u \le V_\alpha \qquad \text{in } \big(\overline{\Omega \setminus \overline{\mathcal{I}^{\varepsilon_k,\delta_k}}} \cap \partial B_{\frac{1}{\beta_k}}\big) \times (0, T]\,, \tag{2.3.26}$$

it follows immediately from (2.3.21) since $\overline{\Omega \setminus \overline{\mathcal{I}^{\varepsilon_k,\delta}}} \subseteq \Omega \cup \mathcal{R}$ for any $\delta \in (0, \frac{\varepsilon_k}{2})$ (see (1.4.1)). Then inequality (2.3.23) and the conclusion follow. This completes the proof when when Ω is unbounded. Then the conclusion follows. □

Proof of Remark 2.2.4. We get (i) proving that (2.2.6) implies (2.2.5). Since $|Z(x)| \ge H > 0$, we have

$$\liminf_{\varepsilon \to 0} \Bigg\{ \sup_{(x \in [\Omega \cup \mathcal{R}] \cap \partial B_{\frac{1}{\varepsilon}})} \frac{\sup_{t \in (0,T]} u(x,t)}{|Z(x)|} \Bigg\} \le \limsup_{|x| \to \infty} \Bigg\{ \frac{\sup_{t \in (0,T]} u(x,t)}{|Z(x)|} \Bigg\} \le$$

$$\le \limsup_{|x| \to \infty} \Bigg\{ \frac{\big(\sup_{t \in (0,T]} u(x,t)\big)_+}{|Z(x)|} \Bigg\} \le \limsup_{|x| \to \infty} \Bigg\{ \frac{\big(\sup_{t \in (0,T]} u(x,t)\big)_+}{|H|} \Bigg\} \le 0\,,$$

where the last inequality is due to (2.2.6).

Case (ii) has been dealt with already in the proof of Proposition 2.2.3. □

Proof of Theorem 2.2.5. Let $u^* := u_2 - u_1$. Then u^* is a subsolution of the problem

$$\begin{cases} \mathcal{L} u - cu + L|u| - \partial_t u = 0 & \text{in } Q_T \\ u = 0 & \text{in } (\mathcal{R} \cup \mathcal{S}_1) \times (0, T] \\ u = 0 & \text{in } (\Omega \cup \mathcal{R} \cup \mathcal{S}_1) \times \{0\}\,, \end{cases} \tag{2.3.27}$$

where $L > 0$ denotes the Lipschitz constant of $f = f(x, t, \cdot)$, uniform for $(x, t) \in \bar{Q}_T$. In fact, for any $\psi \in C_0^\infty(Q_T)$, $\psi \ge 0$, there holds

$$\int_{Q_T} u^* \{\mathcal{L}^* \psi - c\psi + \partial_t \psi\}\, dxdt \ge \int_{Q_T} f(x, t, u_2) - f(x, t, u_1)\psi\, dxdt \ge$$

$$\ge - \int_{Q_T} L|u^*|\, \psi\, dxdt\,.$$

Clearly, the function identically equal to 0 is a subsolution to problem (2.3.27), too. Then the function $u^*_+ := \sup\{u^*, 0\} \ge 0$ is a subsolution of problem (2.3.27), thus also of the problem

$$
\begin{cases} \mathcal{L}\,u + (L-c)u - \partial_t u = 0 & \text{in } \ Q_T \\ u = 0 & \text{in } \ (\mathcal{R}\cup\mathcal{S}_1)\times(0,T] \\ u = 0 & \text{in } (\Omega\cup\mathcal{R}\cup\mathcal{S}_1)\times\{0\}\,. \end{cases} \tag{2.3.28}
$$

Moreover, in view of (2.2.9),

$$
\limsup_{dist(x,\mathcal{S}_2)\to 0} \frac{\sup_{t\in(0,T]} u_+^*(x,t)}{|Z(x)|} \le \lim_{dist(x,\mathcal{S}_2)\to 0} \frac{\sup_{t\in(0,T]} |u^*(x,t))|}{Z(x)} = 0.
$$

Then by Proposition 2.2.3 we obtain that $u_2 \le u_1$. Similarly we get that $u_1 \le u_2$; this completes the proof. □

Proof of Remark 2.2.7. Suppose that $Z \le H < 0$ is a subsolution to problem (2.2.12) for some $\mu \equiv \mu_1 \ge 0$. Let $c_1 \ge 0$. Then obviously Z is a subsolution to problem (2.1.3) with $\mu = \mu_1$ too. If $c_1 < 0$, it is easily seen that Z is a subsolution to problem (2.1.3) with $\mu = \mu_1 - c_1$. Then the conclusion follows by Proposition 2.2.3.

Now suppose that $Z \le H < 0$ is a subsolution to problem (2.1.4). In this case, it is direct to see that Z is a subsolution to problem (2.1.3) with $\mu = \frac{1}{|H|}$. Then the conclusion follows. □

2.4. Proof of nonuniqueness results

Proof of Proposition 2.2.14. (i) Set

$$
\mathcal{R}_j := \{x\in\mathcal{R}\,|\,dist(x,\mathcal{S}) > 1/j\} \qquad (j\in I\!N)\,.
$$

Consider a sequence of bounded domains $\{H_j\}_{j\in I\!N}$ satisfying the exterior sphere condition at each point of the boundary ∂H_j, such that

$$
\begin{cases} \overline{H}_j \subseteq \Omega\cup\overline{\mathcal{R}}_j, \quad H_j\subseteq H_{j+1}, \quad \bigcup_{j=1}^\infty H_j = \Omega\cup\mathcal{R}\,, \\ \partial H_j = \mathcal{R}_j\cup\mathcal{T}_j, \quad \mathcal{R}_j\cap\mathcal{T}_j = \emptyset\,. \end{cases} \tag{2.4.1}
$$

Moreover, let $\{H'_j\}$ be a sequence of domains such that

$$
H'_j \subseteq H'_{j+1}, \quad \partial H'_j \text{ smooth}, \quad \bar{H}'_j\subseteq\Omega\,, \quad H'_j\subseteq H_j\,, \quad \bigcup_{j=1}^\infty H'_j = \Omega\,.
$$

Let $\eta_j\in C_0^\infty(H_j)$, $\eta_j\equiv 1$ in H'_j. We also need to define suitable functions ζ_j on $\mathcal{R}_j$. Observe that if $\overline{\mathcal{R}}\cap\mathcal{S}=\emptyset$, then we have $\mathcal{R}_j=\mathcal{R}$ for any $j\ge j_0$, for some $j_0\in I\!N$. In this case we take $\zeta_j\equiv 1$ on $\mathcal{R}_j=\mathcal{R}$ for any $j\ge j_0$. Otherwise let $\zeta_j\in C_0^\infty(\mathcal{R}_j)$, $0\le\zeta_j\le 1$, $\zeta_j = 1$ in $\mathcal{R}_{j-1}$ ($j\in I\!N$; $\mathcal{R}_0:=\emptyset$). Define for any $j\ge j_0$

$$
\phi_j := \begin{cases} \zeta_j g + (1-\zeta_j)F & \text{in } \ \mathcal{R}_j\times[0,T] \\ F & \text{in } \ \mathcal{T}_j\times[0,T]\,, \end{cases} \tag{2.4.2}
$$

and

$$
\chi_j := \eta_j u_0 + (1-\eta_j)\phi_j(\cdot,0) \quad \text{in } \bar{H}_j\,. \tag{2.4.3}
$$

Then $\phi_j\in C(\partial H_j\times[0,T]), \chi_j\in C(\bar{H}_j)$ and $\phi_j(x,0)=\chi_j(x)$ for any $x\in\partial H_j$ and any $j\ge j_0$.

For any $j \geq j_0$ consider the problem

$$(2.4.4)\qquad \begin{cases} \mathcal{L}\,u - cu - \partial_t u = f(x,t,\,u) & \text{in } \; H_j \times (0,T) \\ u \; = \phi_j & \text{in } \; \partial H_j \times (0,T] \\ u \; = \chi_j & \text{in } \; \bar{H}_j \times \{0\}\,. \end{cases}$$

It is easily seen that the function

$$(2.4.5)\qquad \tilde{F} := \frac{2}{c_2} F \exp\{Lt\} \max\{||u_0||_\infty,\, ||g||_\infty,\, c_2(1 + ||f(\cdot,\cdot,0)||_\infty)\}$$

is a supersolution of problem (2.4.4) for any $j \geq j_0$, while $-\tilde{F}$ is a subsolution of the same problem. In fact, take ψ as in Definition 2.2.1 and

$$\bar{c} := \frac{2}{c_2} \max\{||u_0||_\infty,\, ||g||_\infty,\, c_2(1 + ||f(\cdot,\cdot,0)||_\infty)\}\,.$$

Since $\exp\{Lt\}\psi$ belongs to $C_0^\infty(Q_T)$ and is nonnegative, by Definition 2.2.1 we get:

$$\int_{Q_T} \tilde{F}\,\{\mathcal{L}^*\psi - c\psi + \partial_t\psi\}\,dxdt \; =$$

$$= \int_{Q_T} \bar{c}F\,\{\mathcal{L}^*(\exp\{Lt\}\psi) - c\exp\{Lt\}\psi + \partial_t(\exp\{Lt\}\psi) - L\exp\{Lt\}\psi\}\,dxdt \; \leq$$

$$\leq -\int_{Q_T} \bar{c}\exp\{Lt\}(1 + LF)\psi\,dxdt \leq -\int_{Q_T} (||f(\cdot,\cdot,0)||_\infty + L\tilde{F})\psi\,dxdt \leq$$

$$\leq \int_{Q_T} f(x,t,\,\tilde{F})\psi\,dxdt\,.$$

Moreover, there holds

$$\tilde{F} \geq ||g||_\infty + F \geq \phi_j \quad \text{ in } \; \partial H_j \times (0,T]$$

and

$$\tilde{F} \geq (||u_0||_\infty + ||g||_\infty) \geq \chi_j \quad \text{ in } \; \bar{H}_j \times \{0\}\,.$$

It is similarly seen that $-\tilde{F}$ is a subsolution of (2.4.4).

(ii) Problem (2.4.4) is regular, hence by classical results it has a solution u_j for any $j \geq j_0$. In view of (i) above and of Proposition 2.3.2,

$$(2.4.6)\qquad |u_j| \leq \tilde{F} \quad \text{in } H_j \times (0,T]$$

for any $j \geq j_0$. By standard compactness arguments there exists a subsequence $\{u_{j_k}\} \subseteq \{u_j\}$, which converges uniformly in any compact subset of $\Omega \times (0,T]$. Clearly, $u := \lim_{k\to\infty} u_{j_k}$ is a solution of equation (2.2.1); moreover $|u| \leq \tilde{F}$ in $\Omega \times (0,T]$.

(iii) It remains to prove that $u \in C((\Omega \cup \mathcal{R}) \times [0,T])$ and takes the boundary and the initial data - namely, $u = g$ in $\mathcal{R} \times (0,T]$ and $u = u_0$ in $(\Omega \cup \mathcal{R}) \times \{0\}$. This will be made by a local barrier argument. To this purpose, we use the following notations:

$$N_\delta(x_0) := B_\delta(x_0) \cap \Omega\,,$$

$$C_\delta(x_0,t_0) := \Big(B_\delta(x_0) \times (t_0 - \delta, t_0 + \delta)\Big) \cap Q_T$$

for any $(x_0,t_0) \in ((\Omega \cup \mathcal{R}) \times \{0\}) \cup (\mathcal{R} \times (0,T]), \delta > 0$.

(a) *(Boundary data)* Let $(x_0,t_0) \in \mathcal{R} \times [0,T]$ be arbitrarily fixed. By $(H_1) - (ii)$ Ω satisfies the outer sphere condition at any point of $\mathcal{R}$ and the operator is regular in $\Omega \cup \mathcal{R}$ by (H_2). Then (see [**30**]) we can choose $\delta > 0$ small enough, so that we can exhibit a function $h \in C^2(C_\delta(x_0,t_0) \cap C(\overline{C_\delta(x_0,t_0)})$ satisfying

$$(2.4.7)\qquad \mathcal{L}h - ch - \partial_t h \leq -1 \quad \text{in } C_\delta(x_0,t_0),$$

$$h > 0 \quad \text{in } \overline{C_\delta(x_0,t_0)} \setminus \{(x_0,t_0)\}, \qquad h(x_0,t_0) = 0\,. \tag{2.4.8}$$

Take $j_0 \in I\!N$ so large that $x_0 \in \mathcal{R}_{j_0-1}$. Since each $\mathcal{R}_j$ is open and $\mathcal{R}_{j_0-1} \subseteq \mathcal{R}_j$ for $j \geq j_0$, there exists $\delta_0 > 0$ such that

$$u_j = g \quad \text{in } \; (B_{\delta_0}(x_0) \cap \mathcal{R}) \times [0,T]) \tag{2.4.9}$$

for any $j \geq j_0$. Moreover, we can choose $\delta_0 > 0$ so small that

$$\overline{B_{\delta_0}(x_0) \cap \Omega} \subseteq \overline{H}_j \subseteq \Omega \cup \mathcal{R} \tag{2.4.10}$$

for any $j \geq j_0$.

Since by assumption $g \in C(\mathcal{R} \times [0,T])$, in view of (2.4.9) for any $\sigma > 0$ there exists $\delta \in (0, \delta_0)$ such that, for any $(x,t) \in \Big(B_\delta(x_0) \times (t_0-\delta, t_0+\delta)\Big) \cap \Big(\mathcal{R} \times [0,T]\Big)$ and any $j \geq j_0$,

$$|u_j(x,t) - g(x_0,t_0)| = |g(x,t) - g(x_0,t_0)| < \sigma \;. \tag{2.4.11}$$

Observe that for $\delta \in (0, \delta_0)$

$$\partial N_\delta(x_0) = \Big[\partial B_\delta(x_0) \cap [\Omega \cup \mathcal{R}]\Big] \cup \Big[B_\delta(x_0) \cap \mathcal{R}\Big]\,. \tag{2.4.12}$$

and that the parabolic boundary of the cylinder $C_\delta(x_0,t_0)$ is

$$\partial_p C_\delta(x_0,t_0) = \Big((\partial N_\delta(x_0) \times (\underline{t}_\delta, \overline{t}_\delta)\Big) \cup \Big(\overline{N_\delta(x_0)} \times \{\underline{t}_\delta\}\Big)\,, \tag{2.4.13}$$

where

$$\underline{t}_\delta := \max\{t_0 - \delta, 0\}, \; \overline{t}_\delta := \min\{t_0 + \delta, T\}.$$

We will use a comparison argument in the cylinder $C_\delta(x_0,t_0)$. Observe that on $\partial_p C_\delta(x_0,t_0) \cap (\mathcal{R} \times [0,T])$ inequality (2.4.11) holds.

For any $(x,t) \in \partial_p C_\delta(x_0,t_0) \setminus (\mathcal{R} \times [0,T])$ and $j \geq j_0$ there holds

$$|u_j(x,t) - g(x_0,t_0)| \leq \max_{C_\delta(x_0,t_0)} \tilde{F} + |g(x_0,t_0)| \;\leq\; mM \;\leq\; Mh(x,t)\,, \tag{2.4.14}$$

where

$$m \;:=\; \min_{\partial_p C_\delta(x_0,t_0) \setminus (\mathcal{R} \times [0,T])} h \;>\; 0\,;$$

$$M \;:=\; \frac{4}{m} \max\Big\{(1+mL) \max_{C_\delta(x_0,t_0)} \tilde{F},\; ||g||_\infty,\; m||f(\cdot,\cdot,0)||_\infty,\; m\,(||g||_\infty + 1) \max_{C_\delta(x_0,t_0)} |c|\Big\}.$$

(see (2.4.6), (2.4.10)).

In view of (2.4.11)-(2.4.14) we conclude that for any $\sigma > 0$ there exist $\delta \in (0, \delta_0)$ and $M > 0$ such that

$$|u_j(x,t) - g(x_0,t_0)| < \sigma + Mh(x,t) \quad \text{for any } (x,t) \in \partial_p C_\delta(x_0,t_0) \tag{2.4.15}$$

and any $j \geq j_0$.

Using inequality (2.4.15) it is easily seen that for such values of j, for any $0 < \sigma < 1$

$$E_j := -u_j + g(x_0,t_0) - \sigma - Mh$$

is a subsolution and

$$F_j := -u_j + g(x_0,t_0) + \sigma + Mh$$

is a supersolution of problem

$$\begin{cases} \mathcal{L}\,u - cu - \partial_t u = 0 & \text{in } C_\delta(x_0,t_0) \\ u = 0 & \text{in } \; \partial N_\delta(x_0) \times (\underline{t}_\delta, \overline{t}_\delta] \\ u = 0 & \text{in } \; \overline{N_\delta(x_0)} \times \{\underline{t}_\delta\}\,, \end{cases} \tag{2.4.16}$$

(see (2.4.13)). By Proposition 2.3.2, this implies $E_j \leq 0 \leq F_j$ in $C_\delta(x_0, t_0)$, namely

$$|u_j(x,t) - g(x_0,t_0)| < \sigma + Mh(x,t) \tag{2.4.17}$$

for any $(x,t) \in C_\delta(x_0,t_0)$ and $j \geq j_0$.

Taking inequality (2.4.17) with $j = j_k$ and letting $k \to \infty$, we see that for any $\sigma \in (0,1)$ there exist $\delta \in (0, \delta_0)$ and $M > 0$ such that

$$|u(x,t) - g(x_0,t_0)| < \sigma + Mh(x,t) \quad \text{for any } (x,t) \in C_\delta(x_0,t_0)\,.$$

This implies

$$\limsup_{(x,t)\to(x_0,t_0)} |u(x,t) - g(x_0,t_0)| \leq \sigma$$

for any $\sigma \in (0,1)$; then

$$\lim_{(x,t)\to(x_0,t_0)} u(x,t) = g(x_0,t_0), \tag{2.4.18}$$

for any $(x_0,t_0) \in \mathcal{R} \times (0,T]$, and

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = g(x_0,0) = u_0(x_0), \tag{2.4.19}$$

for any $x_0 \in \mathcal{R}$.

(*b*) *(Initial data)* Let $x_0 \in \Omega$. Take j_0 so large that $x_0 \in H'_{j_0}$. Since each H'_j is open and $H'_{j_0} \subseteq H'_j$ for $j \geq j_0$, there exists $\delta_0 > 0$ such that:

$$u_j = \chi_j = u_0 \quad \text{in } B_{\delta_0}(x_0) \subseteq \overline{H}'_j \subseteq \Omega \tag{2.4.20}$$

for any $j \geq j_0$. Moreover (see [**30**]), we can choose $\delta_0 > 0$ so small that it is possible to exhibit a function $\tilde{h} \in C^2(C_\delta(x_0,0)) \cap C(\overline{C_\delta(x_0,0)})$ satisfying conditions (2.4.7)-(2.4.8). Notice that in this case $C_\delta(x_0,0) = B_\delta(x_0) \times (0,\delta)$. Since by assumption $u_0 \in C(\Omega \cup \mathcal{R})$, in view of (2.4.20) for any $\sigma > 0$ there exists $\delta \in (0,\delta_0)$ such that

$$|u_j(x,0) - u_0(x_0)| = |u_0(x) - u_0(x_0)| < \sigma \quad \text{for any } x \in B_\delta(x_0) \subseteq \Omega \tag{2.4.21}$$

and any $j \geq j_0$. Moreover, for any $j \geq j_0$, on the lateral boundary of $C_\delta(x_0,0)$ (*i.e.*, for any $(x,t) \in \partial B_\delta(x_0) \times [0,\delta]$) there holds

$$|u_j(x,t) - u_0(x_0)| \leq \max_{\overline{C_\delta(x_0,0)}} \tilde{F} + |u_0(x_0)| \;\leq\; \tilde{m}\tilde{M} \;\leq\; \tilde{M}\tilde{h}(x,t)\,, \tag{2.4.22}$$

where

$$\tilde{m} := \min_{\partial B_\delta(x_0)\times[0,\delta]} \tilde{h} > 0\,,$$

$$\tilde{M} := \frac{4}{m} \max\Big\{(1+mL)) \max_{\overline{C_\delta(x_0,0)}} \tilde{F},\, ||u_0||_\infty,\, m||f(\cdot,\cdot,0)||_\infty,\, m\,(||u_0||_\infty + 1) \max_{\overline{C_\delta(x_0,0)}} |c|\Big\}$$

(see (2.4.6), (2.4.20)).

In view of inequalities (2.4.21)-(2.4.22), we conclude that for any $\sigma \in (0,1)$ there exist $\delta \in (0,\delta_0)$ and $\tilde{M} > 0$ such that

$$|u_j(x,t) - u_0(x_0)| < \sigma + \tilde{M}\tilde{h}(x,t) \quad \text{for any } (x,t) \in \partial_p C_\delta(x_0,0) \tag{2.4.23}$$

and any $j \geq j_0$.

Using inequality (2.4.23) it is easily seen that for such values of j, for any $0 < \sigma < 1$

$$\tilde{E}_j := -u_j + u_0(x_0) - \sigma - \tilde{M}\tilde{h}$$

is a subsolution and

$$\tilde{F}_j := -u_j + u_0(x_0) + \sigma + \tilde{M}\tilde{h}$$

is a supersolution of problem (2.4.16).

By Proposition 2.3.2, this implies $\tilde{E}_j \leq 0 \leq \tilde{F}_j$ in $C_\delta(x_0, 0)$, namely

$$|u_j(x,t) - u_0(x_0)| < \sigma + \tilde{M}\tilde{h}(x,t) \tag{2.4.24}$$

for any $(x,t) \in C_\delta(x_0, 0)$ and $j \geq j_0$. The conclusion follows as in case (a). This completes the proof. □

Proof of Proposition 2.2.15. We can argue as in the proof of Proposition 2.2.14, up to some changes needed to impose the condition at infinity (2.2.20). First, a different constant must be chosen in the definition of the supersolution $\tilde{F}$, namely

$$\tilde{F} := \frac{3}{c_2} F \exp\{Lt\} \max\{||u_0||_\infty,\, ||g||_\infty,\, c_2(1 + ||f(\cdot,\cdot,0)||_\infty), |l| + 1\}\,.$$

Moreover, the boundary data for the approximating functions u_j should take the value $l \in I\!R$ into account. To this aim, choose $\bar{M} > 2M$ such that

$$H(x) \leq 1 \quad \text{in } (\Omega \cup \mathcal{R}) \setminus B_{\bar{M}}\,. \tag{2.4.25}$$

Let

$$\xi \in C^\infty(I\!R^n),\ \xi \equiv 1 \quad \text{in } B_{\bar{M}},\ \xi \equiv 0 \quad \text{in } I\!R^n \setminus B_{2\bar{M}},\ 0 \leq \xi \leq 1\,, \tag{2.4.26}$$

$$\tilde{H} := H \quad \text{in } (\Omega \cup \mathcal{R}) \setminus \bar{B}_R,\ \tilde{H} \equiv 0 \quad \text{in } (\Omega \cup \mathcal{R}) \cap \bar{B}_R\,, \tag{2.4.27}$$

$$\hat{F} := \xi F + (1 - \xi)(\tilde{H} + l) \ \text{ in } (\Omega \cup \mathcal{R}) \times [0, T]\,. \tag{2.4.28}$$

Then we have

$$\hat{F}(x,t) = H(x) \quad ((x,t) \in ((\Omega \cup \mathcal{R}) \setminus B_{2\bar{M}}) \times [0,T])\,. \tag{2.4.29}$$

Taking H_j, H'_j, $\mathcal{R}_j$, $\mathcal{T}_j$, η_j, ζ_j, j_0, $(j \geq j_0)$ as in part (i) of the proof of Proposition 2.2.14, define

$$\phi_j := \begin{cases} \zeta_j g + (1 - \zeta_j)\hat{F} & \text{in } \mathcal{R}_j \\ \hat{F} & \text{in } \mathcal{T}_j \end{cases} \tag{2.4.30}$$

and χ_j as before in (2.4.3). As in the proof of Proposition 2.2.14, from the sequence $\{u_j\}$ of solutions to the auxiliary problem (2.4.4) $(j \geq j_0)$ we can extract a subsequence $\{u_{j_k}\}$, which converges uniformly in compact subsets of $\Omega \times (0, T]$ and satisfies inequality (2.4.6) for each j_k. Clearly, the function $u := \lim_{k\to\infty} u_{j_k}$ solves equation (2.2.1) and satisfies $|u| \leq \tilde{F}$. As before, u takes continuously the boundary data g at $\mathcal{R} \times (0, T]$ and the initial data u_0 at $(\Omega \cup \mathcal{R}) \times \{0\}$.

It remains to show that (2.2.20) is satisfied. To this purpose, set

$$v_j := \int_0^T u_j(x,t)\,dt\,, \qquad v := \int_0^T u(x,t)\,dt$$

for any $j \geq j_0$. Clearly, there holds

$$v(x) = \lim_{k\to+\infty} v_{j_k}(x) \quad (x \in \Omega)\,.$$

We limit ourselves to the case of unbounded $\mathcal{R}$, the proof being similar when $\mathcal{R}$ is bounded. In view of (2.2.19), (2.2.17) and (2.4.29), for any $\sigma \in (0,1)$ there exists $\gamma > 2\bar{M}$ such that

$$\left|\frac{1}{T}\int_0^T g(x,t)\,dt - l\right| < \sigma \qquad (x \in \mathcal{R} \setminus B_\gamma), \tag{2.4.31}$$

$$|\hat{F}(x,t)| = |H(x)| < \sigma \qquad ((x,t) \in ((\Omega \cup \mathcal{R}) \setminus B_\gamma) \times [0,T])\,. \tag{2.4.32}$$

Define $N_{j,\gamma} := H_j \setminus \bar{B}_\gamma$; then

$$\partial N_{j,\gamma} = [\mathcal{R}_j \setminus B_\gamma] \cup [\mathcal{T}_j \setminus B_\gamma] \cup [\bar{H}_j \cap \partial B_\gamma].$$

By (2.4.31), (2.4.32) for any $x \in \mathcal{R}_j \setminus B_\gamma$, we have

$$\frac{1}{T}v_j(x) - l = \frac{1}{T}\int_0^T \phi_j(x,t)\,dt - l = \frac{1}{T}\int_0^T \zeta_j(x)g(x,t)\,dt +$$
$$+\frac{1}{T}\int_0^T (1-\zeta_j(x))(H(x)+l)\,dt - l =$$
$$= \frac{1}{T}\zeta_j(x)\int_0^T g(x,t)\,dt + (1-\zeta_j(x))(H(x)+l) - l \leq$$
$$\leq \zeta_j(x)(\sigma+l) + (1-\zeta_j(x))(\sigma+l) - l = \sigma\,.$$

On the other hand, there holds

$$\frac{1}{T}v_j(x) - l = \frac{1}{T}\zeta_j(x)\int_0^T g(x,t)\,dt + (1-\zeta_j(x))(H(x)+l) - l \geq$$
$$\geq \zeta_j(x)(-\sigma+l) + (1-\zeta_j(x))(-\sigma+l) - l = -\sigma,$$

thus

$$|\frac{1}{T}v_j(x) - l| \leq \sigma \quad (x \in \mathcal{R}_j \setminus B_\gamma). \tag{2.4.33}$$

In view of (2.4.32), for any $x \in \mathcal{T}_j \setminus B_\gamma$ we have

$$\Big|\frac{1}{T}v_j(x) - l\Big| = \Big|\frac{1}{T}\int_0^T \phi_j(x,t)\,dt - l\Big| = \Big|\frac{1}{T}\int_0^T (H(x)+l)\,dt - l\Big| = |H(x)| \leq \sigma\,. \tag{2.4.34}$$

For any $x \in \bar{H}_j \cap \partial B_\gamma$, $j \geq j_0$, we have

$$\Big|\frac{1}{T}v_j(x) - l\Big| \leq \tilde{M}H(x), \tag{2.4.35}$$

where

$$m := \min_{x\in(\Omega\cup\mathcal{R})\cap\partial B_\gamma} H(x)\,,$$
$$\tilde{M} := \frac{2}{m}\max\Big\{(1 + m(L + \frac{2}{T} + \sup_{(\Omega\cup\mathcal{R})\setminus\bar{B}_R}|c|)) \sup_{[(\Omega\cup\mathcal{R})\setminus B_R]\times[0,T]} \tilde{F},\ |l|,\ m||f(\cdot,\cdot,0)||_\infty\Big\}\,.$$

From (2.4.33), (2.4.34), (2.4.35) we obtain, for any $\sigma > 0$, $j \geq j_0$, $x \in \partial N_{j,\gamma}$,

$$|\frac{1}{T}v_j(x) - l| \leq \sigma + \tilde{M}H(x)\,. \tag{2.4.36}$$

We claim that for any $\sigma > 0$, $j \geq j_0$ the function

$$E_j := -\frac{1}{T}v_j + l - \sigma - \tilde{M}H$$

is a subsolution of the problem

$$\begin{cases} \mathcal{L}\,u = 0 & \text{in } N_{j,\gamma} \\ \ u\ = 0 & \text{in } \partial N_{j,\gamma}\,, \end{cases} \tag{2.4.37}$$

whereas

$$F_j := -\frac{1}{T}v_j + l + \sigma + \tilde{M}H$$

is a supersolution. To prove the claim observe that, in view of Definition 2.2.2, any solution u_j of problem (2.4.4) satisfies inequality (2.2.2) with ψ in $C_0^\infty(N_{j,\gamma}\times(0,T))$. However, it can be proved that any u_j also satisfies the inequality

$$\int_{N_{j,\gamma}\times(0,T)} u_j\,\{\mathcal{L}^*\psi - c\psi + \partial_t\psi\}\,dxdt \ \geq \int_{N_{j,\gamma}\times(0,T)} f(x,t,\,u_j)\psi\,dxdt +$$

$$+\int_{N_{j,\gamma}} u_j(x,T)\psi(x,T)\,dx - \int_{N_{j,\gamma}} u_j(x,0)\psi(x,0)\,dx$$

for any ψ in a wider class, namely $\psi \in C^\infty(N_{j,\gamma}\times(0,T))$ such that $\psi(\cdot,t)\in C_0^\infty(\Omega)$ for any $t\in[0,T]$. In particular, choosing $\psi=\psi(x)\in C_0^\infty(N_{j,\gamma}), \psi\geq 0,\ j\geq j_0$, we have

$$\int_{N_{j,\gamma}} E_j\,\mathcal{L}^*\psi\,dx \geq -\frac{1}{T}\int_{N_{j,\gamma}\times(0,T)} u_j\,\mathcal{L}^*\psi\,dxdt \,+\, \tilde{M}\int_{N_{j,\gamma}}\psi\,dx \geq$$

$$\geq -\frac{1}{T}\int_{N_{j,\gamma}\times(0,T)} f(x,t,\,u_j)\,\psi\,dxdt - \frac{1}{T}\int_{N_{j,\gamma}\times(0,T)} u_j\,c\,\psi\,dxdt +$$

$$+\frac{1}{T}\int_{N_{j,\gamma}}[u_j(x,0)-u_j(x,T)]\psi(x)\,dx \,+\tilde{M}\int_{N_{j,\gamma}}\psi\,dx \geq$$

$$\geq \int_{N_{j,\gamma}}\Big[-\|f(\cdot,\cdot,0)\|_\infty - \Big(L+\sup_{(\Omega\cup\mathcal{R})\setminus\bar{B}_R}|c|+\frac{2}{T}\Big)\sup_{[(\Omega\cup\mathcal{R})\setminus B_R]\times[0,T]}\tilde{F}\,+\,\tilde{M}\Big]\psi(x)\,dx \geq 0\,.$$

The above inequality, combined with (2.4.36), implies that E_j is a subsolution to problem (2.4.37). It is similarly seen that F_j is a supersolution to problem (2.4.37). Then Proposition 1.2.4 implies $E_j\leq 0\leq F_j$ in $N_{j,\gamma}$, namely

$$\Big|\frac{1}{T}v_j(x)-l)\Big| < \sigma+\tilde{M}H(x) \tag{2.4.38}$$

for any $x\in N_{j,\gamma}$.

Set $j=j_k$ in inequality (2.4.38) and let $k\to\infty$. Then we obtain that for any $\sigma>0$ there exist $\gamma>0$ and $\tilde{M}>0$ such that

$$\Big|\frac{1}{T}v(x)-l\Big| < \sigma+\tilde{M}H(x) \quad \text{for any } x\in N_{j,\gamma}\,,$$

whence

$$\limsup_{|x|\to+\infty}\Big|\frac{1}{T}v(x)-l\Big|\leq\sigma$$

for any $\sigma>0$. Then the conclusion follows. □

The proofs of Theorems 2.2.16 and 2.2.18 make use of the following two propositions.

Proposition 2.4.1. *Let the assumptions of Theorem 2.2.16 be satisfied and* $\gamma\in C(\mathcal{R})\cap L^\infty(\mathcal{R})$. *Then there exist infinitely many bounded solutions of the problem*

$$\begin{cases} \mathcal{L}U\,=\,0 & \text{in } \Omega \\ \\ U\,=\gamma & \text{in } \mathcal{R}\,. \end{cases} \tag{2.4.39}$$

More precisely, there exists a sequence $\{x_m\}\subseteq\Omega$ *with the following properties:*
(a) $\lim_{m\to+\infty}V(x_m)=0$*;*
(b) for any $\beta\in I\!R$ *there exists a solution* W_β *of problem (2.4.39), such that* $|W_\beta|\leq|\beta|+\|\gamma\|_\infty$ *in* $\bar{\Omega}\setminus\mathcal{S}$ *and*

$$\lim_{m\to\infty}W_\beta(x_m)\,=\,\beta\,.$$

Proposition 2.4.2. *Let the assumptions of Theorem 2.2.18 be satisfied and* $\gamma\in C(\mathcal{R})\cap L^\infty(\mathcal{R})$. *Then there exist infinitely many bounded solutions of problem (2.4.39) satisfying*

$$\lim_{|x|\to\infty}W(x)=l\,. \tag{2.4.40}$$

More precisely, there exists a bounded sequence $\{x_m\}\subseteq\Omega$ *with the following properties:*
(a) $\lim_{m\to+\infty}V(x_m)=0$*;*

(*b*) *for any* $\beta \in I\!R$ *there exists a solution* W_β *of problem (2.4.39), such that (2.4.40) is satisfied,* $|W_\beta| \leq |\beta| + ||\gamma||_\infty + |l|$ *in* $\bar{\Omega} \setminus \mathcal{S}$ *and*

$$\lim_{m\to\infty} W_\beta(x_m) \,=\, \beta\,.$$

Up to minor changes, the proof of Proposition 2.4.1 (respectively, of Proposition 2.4.2) is the same of Theorems 2.5 and 2.9 (respectively, of Theorems 2.7 and 2.11) in [**65**], hence it is omitted.

Proof of Theorem 2.2.16. Let $\beta \in I\!R$. By Proposition 2.4.1, applied with $\gamma(x) := \frac{1}{T}\int_0^T g(x,t)\,dt$ $(x \in \mathcal{R})$, there exist $\{x_m\} \subseteq \Omega$ and a solution $W_\beta \in L^\infty(\Omega)$ of the problem

$$\begin{cases} \mathcal{L}W = 0 & \text{in } \Omega \\ W = \gamma & \text{on } \mathcal{R}\,, \end{cases} \tag{2.4.41}$$

such that

$$\lim_{m\to+\infty} V(x_m) = 0\,,$$

$$\lim_{m\to\infty} W_\beta(x_m) = \beta\,,$$

$$|W_\beta| \leq |\beta| + ||g||_\infty \quad \text{in } \bar{\Omega}\setminus\mathcal{S}\,.$$

Choose $H_j, H_j', \mathcal{R}_j, \mathcal{T}_j, \zeta_j, \eta_j, j_0\,(j \geq j_0)$ as in the proof of Proposition 2.2.14. Set

$$\phi_j := \begin{cases} \zeta_j g + (1-\zeta_j)W_\beta & \text{in } \mathcal{R}_j \times [0,T] \\ W_\beta & \text{in } \mathcal{T}_j \times [0,T]\,. \end{cases} \tag{2.4.42}$$

For any $j \geq j_0$, let $u_{j,\beta}$ be the solution of problem

$$\begin{cases} \mathcal{L}u - cu - \partial_t u = f(x,t,\,u) & \text{in } H_j \times (0,T) \\ u = \phi_j & \text{in } \partial H_j \times (0,T] \\ u = \chi_j & \text{in } \bar{H}_j \times \{0\}\,, \end{cases} \tag{2.4.43}$$

where $\phi_j \in C(\partial H_j \times [0,T])$ and χ_j is the function (2.4.3), thus $\chi_j \in C(\bar{H}_j)$. As before, by standard arguments we deduce that there exists a subsequence $\{u_{j_k,\beta}\}$ of $\{u_{j,\beta}\}$ which converges uniformly in any compact subset of $\Omega \times (0,T]$. Let

$$u_\beta := \lim_{k\to\infty} u_{j_k,\beta} \quad \text{in } \Omega \times (0,T].$$

As in Proposition 2.2.14, we infer that $u_\beta \in C((\Omega \cup \mathcal{R}) \times [0,T])$ and u_β solves (2.1.1). Since $c \in L^\infty(\Omega)$, by Proposition 2.2.14 there exists $K > 0$ such that

$$|u_{j,\beta}| \leq K \quad \text{in } \overline{H}_j \times [0,T] \quad \text{for any } j \geq j_0\,,$$

thus

$$|u_\beta| \leq K \quad \text{in } (\Omega \cup \mathcal{R}) \times [0,T]\,.$$

For any $j \geq j_0$ define

$$v_{j,\beta} := \int_0^T u_{j,\beta}(x,t)\,dt \quad (x \in \bar{H}_j),\;\; v_\beta(x) := \int_0^T u_\beta(x,t)\,dt \qquad (x \in \Omega \cup \mathcal{R})\,.$$

Clearly, there holds

$$v_\beta(x) = \lim_{k\to\infty}\int_0^T u_{j_k,\beta}(x,t)\,dt \quad (x \in \Omega)\,. \tag{2.4.44}$$

Let $M := T\,[||f(\cdot,\cdot,0)||_\infty + K(L+||c||_\infty)] + 2K$. We claim that

$$-MV + TW_\beta \le v_\beta \le MV + TW_\beta \qquad \text{in } \bar\Omega \setminus \mathcal{S}\,. \tag{2.4.45}$$

From (2.4.45) we obtain

$$\lim_{m\to+\infty} v_\beta(x_m) = T\beta\,,$$

namely (2.2.22).

It remains to prove inequalities (2.4.45). To this aim, observe that $F_{1,\beta} := \frac{v_{j,\beta}}{M}$ is a subsolution, while $F_{2,\beta} := V + \frac{T}{M}W_\beta$ is a supersolution of the problem

$$\begin{cases} \mathcal{L}U = -1 & \text{in } H_j \\ U = \frac{T}{M}W_\beta & \text{in } \partial H_j. \end{cases} \tag{2.4.46}$$

In fact, arguing as in the proof of Proposition 2.2.15 we obtain

$$\int_{H_j} v_{j,\beta}\,\mathcal{L}^*\psi\,dx = \int_{H_j\times(0,T)} f(x,t,\,u_{j,\beta})\,\psi\,dxdt + \int_{H_j\times(0,T)} u_{j,\beta}\,c\,\psi\,dxdt +$$
$$+\int_{H_j}[u_{j,\beta}(x,0)-u_{j,\beta}(x,T)]\psi\,dx \ge -M\int_{H_j}\psi\,dx$$

for any ψ as in Definition 1.2.1. In addition, since W_β satisfies the boundary condition in (2.4.39) with $\gamma(x) = \frac{1}{T}\int_0^T g(x,t)dt$ $(x\in\mathcal{R})$, there holds

$$v_{j,\beta}(x) = \int_0^T u_{j,\beta}(x,t)\,dt = \int_0^T [\zeta_j(x)g(x,t) + (1-\zeta_j(x)W_\beta(x))]\,dt =$$
$$= \zeta_j(x)\int_0^T g(x,t)\,dt + (1-\zeta_j(x))TW_\beta(x) = TW_\beta(x) \qquad (x\in\mathcal{R}_j)\,.$$

Plainly, there holds

$$v_{j,\beta}(x) = \int_0^T u_{j,\beta}(x,t)\,dt = \int_0^T W_\beta(x)dt = TW_\beta(x) \qquad (x\in\mathcal{T}_j)\,.$$

Therefore $F_{1,\beta}$ is a subsolution to problem (2.4.46). On the other hand, for any ψ as in Definition 1.2.1 we have

$$\int_{H_j}(V+\frac{T}{M}W_\beta)\mathcal{L}^*\psi\,dx \le -\int_{H_j}\psi\,dx.$$

In addition, since $V\ge 0$ there holds

$$V(x) + \frac{T}{M}W_\beta(x) \ge \frac{T}{M}W_\beta(x) \qquad (x\in\partial H_j)\,,$$

thus $F_{2,\beta}$ is a supersolution of (2.4.46). In view of Proposition 1.2.4, we conclude that

$$v_{j,\beta} \le MV + TW_\beta \quad \text{in } H_j\,. \tag{2.4.47}$$

Analogously, it is not difficult to see that $F_{1,\beta}$ is a supersolution, while $F_{3,\beta} := -V + \frac{T}{M}W_\beta$ is a subsolution of the problem

$$\begin{cases} \mathcal{L}U = 1 & \text{in } H_j \\ U = \frac{T}{M}W_\beta & \text{in } \partial H_j\,. \end{cases} \tag{2.4.48}$$

By Proposition 1.2.4 we obtain

$$v_{j,\beta} \ge -MV + TW_\beta \quad \text{in } H_j\,. \tag{2.4.49}$$

By (2.4.47) and (2.4.49) with $j = j_k$ and by (2.4.44), letting $k \to +\infty$ we get (2.4.45), since $v_\beta \in C(\Omega \cup \mathcal{R})$. The proof is complete. $\square$

Proof of Theorem 2.2.18 Let $\beta \in I\!R$. By Proposition 2.4.2, applied with $\gamma(x) := \int_0^T g(x,t)\, dt$ $(x \in \mathcal{R})$, there exist a bounded sequence $\{x_m\} \subseteq \Omega$ and a solution W_β of problem (2.4.41) such that

$$\lim_{m\to+\infty} V(x_m) = 0\,,$$

$$\lim_{m\to\infty} W_\beta(x_m) = \beta\,,$$

$$|W_\beta| \leq |\beta| + ||\gamma||_\infty + |l| \quad \text{in } \bar{\Omega} \setminus \mathcal{S}\,, \qquad \lim_{|x|\to+\infty} W_\beta(x) = l\,.$$

Arguing as in the proof of Theorem 2.2.16, we can construct a solution u_β of problem (2.1.1) which fulfills (2.2.22). Moreover, using the function H as a barrier at infinity, as in the proof of Proposition 2.2.15 we prove the equality $\lim_{|x|\to+\infty} \frac{1}{T}\int_0^T u_\beta(x,t)\, dt = l\,.$ Hence the conclusion follows. $\square$

Proof of Theorem 2.2.20 Consider a sequence of bounded domains $\{H_j\}_{j\in I\!N}$ with smooth boundary ∂H_j, such that

$$\overline{H}_j \subseteq \Omega, \quad H_j \subseteq H_{j+1}, \quad \bigcup_{j=1}^{\infty} H_j = \Omega\,. \tag{2.4.50}$$

Then we argue as in the proof of Proposition 2.2.14, taking $\phi_j \equiv \phi$, $\chi_j = u_0$ $(j \in I\!N)$. In the definition (2.4.5) of $\tilde{F}$ we replace $||u_0||_\infty$ and $||g||_\infty$ by $||\phi||_\infty$. Up to minor changes of the above proof, the approximating problems (2.4.4) can be uniquely solved. The sequence of solutions $\{u_j\}$ admits a subsequence $\{u_{j_k}\}$ which converges to a solution u of the parabolic problem in (2.1.2), uniformly in compact subsets of $\Omega \times (0,T]$. Moreover,

$$|u_j| \leq \tilde{F} \quad \text{in } H_j \times (0,T], \qquad |u| \leq \tilde{F} \quad \text{in } \Omega \times (0,T]. \tag{2.4.51}$$

As in the proof of Proposition 2.2.14 there holds $u \in C(\Omega \times [0,T])$, and $u = u_0$ in $\Omega \times \{0\}$. In order to prove that the boundary condition is attained on $(\mathcal{R} \cup \mathcal{S}_1) \times [0,T]$, we have to prove that $u \in C((\Omega \cup \mathcal{R} \cup \mathcal{S}_1) \times [0,T])$ and $u(x,t) = g(x,t)$ for $(x,t) \in (\mathcal{R} \cup \mathcal{S}_1) \times [0,T]$. We prove in detail this result on $\mathcal{S}_1 \times [0,T]$, since on $\mathcal{R} \times [0,T]$ the proof is analogous, but should be done at each point of the set, since g is not constant in space on it as it is on $\mathcal{S}_1 \times [0,T]$.

To this aim, recall that by hypothesis $\mathcal{S}_1$ is attracting, hence there exists $\varepsilon > 0$ and a function V satisfying the properties in Definition 2.2.19 for $\Sigma = \mathcal{S}_1$. Without loss of generality, we can take ε so small that the assumptions of the theorem.are satisfied. Finally, recall that g is independent of the spatial variable on $\mathcal{S}_1 \times [0,T]$ and $g_1(t) = g(x,t) = \phi(x,t)$ $((x,t) \in \mathcal{S}_1 \times [0,T])$. Take $t_0 \in [0,T]$ and $\sigma > 0$. Since ϕ is continuous in $(\Omega \cup \mathcal{R} \cup \overline{\mathcal{S}_1}) \times [0,T]$, and $\overline{\mathcal{S}_1} \times \{t_0\}$ is compact, we can find $\delta \in (0,\varepsilon)$ such that

$$|\phi(x,t) - g_1(t_0)| < \sigma\,, \quad \text{for any } (x,t) \in \overline{\mathcal{S}_1^\delta} \times [\underline{t}_\delta, \overline{t}_\delta]. \tag{2.4.52}$$

We claim that

$$v(x,t) := \lambda_1[\exp\{c_3 t\} V(x) + \lambda_2 (t-t_0)^2], \quad ((x,t) \in \overline{\mathcal{S}_1^\delta} \times [0,T]) \tag{2.4.53}$$

is a supersolution of equation

$$\mathcal{L}v - cv - \partial_t v = -1 \quad \text{in } \mathcal{S}_1^\delta \times (0,T)\,, \tag{2.4.54}$$

such that

$$v \in C(\overline{\mathcal{S}_1^\delta} \times [0,T])\,, \quad v > 0 \quad \text{in } \left(\overline{\mathcal{S}_1^\delta} \times [0,T]\right) \setminus \left(\mathcal{S}_1 \times \{t_0\}\right), \qquad v = 0 \quad \text{in } \mathcal{S}_1 \times \{t_0\}\,; \tag{2.4.55}$$

here $c_3 := \sup_{\mathcal{S}_1^\varepsilon} |c|\,,\ \lambda_1 > 1,\ 0 < \lambda_2 \leq \dfrac{\lambda_1 - 1}{\lambda_1 T(2 + c_3 T)}$. In fact, take ψ as in Definition 2.2.1. For any fixed $t \in [0,T]$ the function $\psi(\cdot,t)$ is nonnegative and belongs to $C_0^\infty(\Omega)$, thus by Definition 1.2.1 we have

$$\int_{\mathcal{S}_1^\delta} V\,\{\mathcal{L}^*\psi\}\,dx \,\leq\, -\int_{\mathcal{S}_1^\delta} \psi\,dx\,,$$

whence

$$\int_{\mathcal{S}_1^\delta \times (0,T)} \exp\{c_3 t\} V\,\{\mathcal{L}^*\psi\}\,dxdt \,\leq -\int_{\mathcal{S}_1^\delta \times (0,T)} \exp\{c_3 t\}\psi\,dxdt\,. \tag{2.4.56}$$

On the other hand, it is easily checked that

$$\int_{\mathcal{S}_1^\delta \times (0,T)} \exp\{c_3 t\} V\,\partial_t\,\psi\,dxdt = -c_3 \int_{\mathcal{S}_1^\delta \times (0,T)} \exp\{c_3 t\} V\,\psi\,dxdt\,. \tag{2.4.57}$$

From (2.4.56), (2.4.57) we get

$$\int_{\mathcal{S}_1^\delta \times (0,T)} \exp\{c_3 t\} V\,\{\mathcal{L}^*\psi - c\psi + \partial_t\psi\}\,dxdt \,\leq -\int_{\mathcal{S}_1^\delta \times (0,T)} [1 + c_3 V + cV]\exp\{c_3 t\}\psi\,dxdt\,.$$

Then

$$\int_{\mathcal{S}_1^\delta \times (0,T)} v\,\{\mathcal{L}^*\psi - c\psi + \partial_t\psi\}\,dxdt \,\leq\, \int_{\mathcal{S}_1^\delta \times (0,T)} (-\lambda_1 + 2\lambda_1\lambda_2 T + c_3\lambda_1\lambda_2 T^2)\psi\,dxdt \,\leq$$

$$\leq -\int_{\mathcal{S}_1^\delta \times (0,T)} \psi\,dxdt,$$

for any ψ as above, hence the claim follows.

Define for any $j \in I\!N$

$$N_{\delta,j}(t_0) := \mathcal{S}_1^\delta \cap H_j\,, C_{\delta,j}(t_0) := N_{\delta,j}(t_0) \times (\underline{t}_\delta, \bar{t}_\delta)\,.$$

Take $j_0 \in I\!N$ such that $N_{\delta,j_0}(t_0) \neq \emptyset$. Observe that for any $j \geq j_0$

$$\partial N_{\delta,j}(t_0) = \left(\partial\mathcal{S}_1^\delta \cap \bar{H}_j\right) \cup \left(\mathcal{S}_1^\delta \cap \partial H_j\right), \tag{2.4.58}$$

and that the parabolic boundary of the cylinder $C_{\delta,j}(t_0)$ is given by

$$\partial_p C_{\delta,j}(t_0) = \Big((\partial N_{\delta,j}(t_0) \times (\underline{t}_\delta, \bar{t}_\delta)\Big) \cup \Big(\overline{N_{\delta,j}(t_0)} \times \{\underline{t}_\delta\}\Big)\,. \tag{2.4.59}$$

Set

$$m := \begin{cases} \inf_{\left((\partial\mathcal{S}_1^\delta \cap \Omega) \times [0,\bar{t}_\delta]\right)} v & \text{if } t_0 = 0 \\[2ex] \inf_{\left(\mathcal{S}_1^\delta \times \{\underline{t}_\delta\}\right) \cup \left((\partial\mathcal{S}_1^\delta \cap \Omega) \times [\underline{t}_\delta, \bar{t}_\delta]\right)} v & \text{if } t_0 > 0\,, \end{cases} \tag{2.4.60}$$

$$M := \frac{4}{m} \max\Big\{(1 + mL) \max_{C_{\delta,j}(t_0)} \tilde{F},\, ||\phi||_\infty,\, m||f(\cdot,\cdot,0)||_\infty,\, m\,(||\phi||_\infty + 1) \max_{C_{\delta,j}(t_0)} |c|\Big\}.$$

First suppose $t_0 = 0$. Then for any $(x,t) \in (\partial\mathcal{S}_1^\delta \cap \overline{H}_j) \times [0, \bar{t}_\delta]$ and $j \geq j_0$ we have

$$|u_j(x,t) - g_1(t_0)| \leq \max_{C_{\delta,j}(t_0)} \tilde{F} + \|\phi\|_\infty \leq mM \leq Mv(x,t)\,. \tag{2.4.61}$$

On the other hand, in view of (2.4.52), for any $(x,t) \in \partial_p C_{\delta,j}(t_0) \setminus \big((\partial \mathcal{S}_1^\delta \cap \overline{H}_j) \times [0, \bar{t}_\delta]\big)$ and $j \geq j_0$ there holds

$$|u_j(x,t) - g_1(t_0)| = |\phi(x,t) - g_1(t_0)| < \sigma \,. \tag{2.4.62}$$

Now suppose $t_0 > 0$. Then inequality (2.4.61) holds for any $(x,t) \in \partial_p C_{\delta,j}(t_0) \setminus \big(\partial H_j \times (\underline{t}_\delta, \bar{t}_\delta)\big)$ and $j \geq j_0$, while inequality (2.4.62) is satisfied for any $(x,t) \in (\mathcal{S}_1^\delta \cap \partial H_j) \times (\underline{t}_\delta, \bar{t}_\delta)$ and $j \geq j_0$.

By (2.4.58), (2.4.59), (2.4.61) and (2.4.62) we conclude that for any $\sigma > 0$ there exist $\delta \in (0, \varepsilon)$ and $M > 0$ such that

$$|u_j(x,t) - g_1(t_0)| < \sigma + Mv(x,t) \tag{2.4.63}$$

for any $(x,t) \in \partial_p C_{\delta,j}(t_0)$ and $j \geq j_0$.

Using inequality (2.4.63) it is easily seen that for such values of j, for any $0 < \sigma < 1$

$$E_j := -u_j + g_1(t_0) - \sigma - Mv$$

is a subsolution, and

$$F_j := -u_j + g_1(t_0) + \sigma + Mv$$

a supersolution of problem (2.4.16) with $C_\delta(x_0, t_0)$ and $N_\delta(x_0, t_0)$ replaced by $C_{\delta,j}(t_0)$ and $N_{\delta,j}(t_0)$, respectively. Then the conclusion follows as in Proposition 2.2.14. □

Proof of Proposition 2.2.22 Let $(x_0, t_0) \in \Sigma \times [0, T]$ and V as in Definition 2.2.19. It is easily seen that the function

$$h(x,t) := \lambda_1[\exp\{|c_1|t\}V(x) + \lambda_2(|x - x_0|^2 + (t - t_0)^2)] \quad ((x,t) \in C_\delta((x_0, t_0))$$

is a local barrier at (x_0, t_0), provided that $\lambda_1 > 0$ is big enough, $\lambda_2 > 0$ is small enough and $\delta \in (0, \varepsilon)$. □

2.5. Examples

In the sequel we always suppose that assumption (H_3) is satisfied.

2.5.1. Uniqueness. (a) Consider the problem

$$\begin{cases} u_{xx} + y^3 u_{yy} - u_y - \frac{3}{y}u - \partial_t u = f & \text{in } Q_T \\ u = g & \text{in } \mathcal{R} \times (0, T] \\ u = u_0 & \text{in } (\Omega \cup \mathcal{R}) \times \{0\} \end{cases} \tag{2.5.1}$$

with $\Omega = (0,1) \times (0,1)$, $\mathcal{R} = \partial\Omega \setminus ([0,1] \times \{0\})$, $\mathcal{S}_1 = \emptyset$, $\mathcal{S}_2 = \mathcal{S} = [0,1] \times \{0\}$, $f(\cdot,\cdot,0) \in L^\infty(Q_T)$. Observe that the coefficient c does not belong to $L^\infty(\Omega)$.

The function

$$Z(x,y) := -\frac{1}{y} - 1$$

satisfies

$$Z \leq -1 \text{ in } \Omega \cup \mathcal{R}, \qquad \mathcal{L}Z - cZ \geq 0 \text{ in } \Omega, \qquad \lim_{y \to 0} Z(x,y) = -\infty\,;$$

hence by Theorem 2.2.5 and Remark 2.2.9 *uniqueness* holds in the class of solutions which diverge at a rate lower than $|Z|$ as $y \to 0^+$ $(t \in (0, T])$. Moreover, observe that the function

$$F(x,y) := (x - \frac{1}{2})^2 + 3y + 1 \geq 1$$

is a supersolution of equation (2.2.16). Then, in view of Proposition 2.2.14 and the above uniqueness result, problem (2.5.1) is *well-posed* in $L^\infty(Q_T)$.

It is worth observing that the function

$$V(x,y) := F(x,y) - 1 = (x - \frac{1}{2})^2 + 3y$$

satisfies

$$\inf_{\Omega \cup \mathcal{R}} V = 0 < \inf_{\mathcal{R}} V = \frac{1}{4}, \qquad \mathcal{L}V = -1 \text{ in } \Omega;$$

however, Theorem 2.2.16 does not apply to prove non–uniqueness since the coefficient c is unbounded.

(b) Consider the equation

$$x^2 u_{xx} + u_{yy} + 3x u_x - \partial_t u = f \quad \text{in } Q_T\,. \tag{2.5.2}$$

In this case $\Omega = (0,\infty) \times I\!R$, $\mathcal{R} = \mathcal{S}_1 = \emptyset$, $\mathcal{S} = \mathcal{S}_2 = \{0\} \times I\!R$. Consider the function

$$Z(x,y) := -\frac{1}{dist((x,y),\mathcal{S})} - 1 = -\frac{1}{x} - 1 \qquad ((x,y) \in \Omega)\,.$$

It is easily seen that

$$Z \le -1 \text{ in } \Omega\,, \qquad \mathcal{L}Z > 0 \text{ in } \Omega\,, \qquad \lim_{x \to 0} Z(x,y) = -\infty\,.$$

In view of Theorem 2.2.5 and Remarks 2.2.6, 2.2.9 *uniqueness* holds in the class of solutions that satisfy

$$\lim_{|x|+|y| \to \infty} \sup_{t \in (0,T]} u(x,y,t) = 0\,,$$

and diverge at a rate lower than $|Z|$ as $x \to 0^+$ $(t \in (0,T])$.

(c) Consider the problem

$$\begin{cases} \frac{1}{y \sin x}(u_{xx} + y^2 u_{yy}) - \partial_t u = f & \text{in } Q_T \\ u = g & \text{in } \mathcal{R} \times (0,T] \\ u = u_0 & \text{in } (\Omega \cup \mathcal{R}) \times \{0\}\,. \end{cases} \tag{2.5.3}$$

Here we take $\Omega = (\frac{\pi}{4}, \frac{3\pi}{4}) \times (0,1)$, $\mathcal{R} = \partial\Omega \setminus \Big([\frac{\pi}{4}, \frac{3\pi}{4}] \times \{0\}\Big)$, $\mathcal{S} = [\frac{\pi}{4}, \frac{3\pi}{4}] \times \{0\}$, $f(\cdot,\cdot,0) \in L^\infty(Q_T)$.

It is easily checked that the function

$$Z(x,y) := x^2 + \log y - \pi^2$$

satisfies

$$Z < 0 \text{ in } \Omega\,, \qquad \mathcal{L}Z = \frac{1}{y \sin x} > 0 \text{ in } \Omega\,, \qquad \lim_{y \to 0} Z(x,y) = -\infty\,.$$

Then by Theorem 2.2.5 and Remark 2.2.9 *uniqueness* holds in the class of solutions which diverge at a rate lower than $|Z|$ as $y \to 0^+$ $(t \in (0,T])$.

On the other hand, the function

$$F(x,y) := y \sin x + 1 \ge 1$$

is a supersolution of equation (2.2.16). Moreover $V(x,y) := F(x,y) - 1 = y \sin x$ satisfies $V \in C(\overline{\Omega})$,

$$V > 0 \quad \text{in } \Omega \cup \mathcal{R}\,, \qquad V = 0 \quad \text{on } \mathcal{S}\,, \qquad \mathcal{L}V = -1 \quad \text{in } \Omega\,,$$

thus $\mathcal{S}$ is attracting (for $\mathcal{L}$) (see Definition 2.2.19). By Theorem 2.2.20 there exists a solution $\overline{u} \in L^\infty(Q_T)$ of the problem

$$\begin{cases} \frac{1}{y\sin x}(u_{xx}+y^2u_{yy})-\partial_t u=f & \text{in } Q_T \\ u=\varphi & \text{in } \partial\Omega\times(0,T] \\ u=u_0 & \text{in } \bar\Omega\times\{0\}, \end{cases}$$

where $\varphi\in C([0,T])$, $u_0=\varphi(0)$ on $\partial\Omega$.

In view of the above uniqueness result, this implies that there exists *no solution* $u_g\in L^\infty(Q_T)$ of the problem

$$\begin{cases} \frac{1}{y\sin x}(u_{xx}+y^2u_{yy})-\partial_t u=f & \text{in } Q_T \\ u=g & \text{in } \partial\Omega\times(0,T] \\ u=u_0 & \text{in } \bar\Omega\times\{0\}, \end{cases}$$

with $g\in C(\partial\Omega\times[0,T])$, $g=\varphi$ in $\mathcal{R}\times(0,T]$, $u_0=\varphi(0)$ on $\mathcal{R}$, if $g(\bar x,\bar t)\neq\varphi(\bar t)$ at some point $(\bar x,\bar t)\in\mathcal{S}\times(0,T)$.

(d) Consider the problem

$$\text{(2.5.4)}\qquad \begin{cases} 5u_{xx}+y^2u_{yy}+(\log x)u_x+yu_y-\partial_t u=f & \text{in } \Omega \\ u=g & \text{in } (\mathcal{R}\cup\mathcal{S}_1)\times(0,T] \\ u=u_0 & \text{in } (\Omega\cup\mathcal{R}\cup\mathcal{S}_1)\times\{0\}. \end{cases}$$

Here we take $\Omega=(0,e)\times(0,1)$, $\mathcal{R}=(\{e\}\times(0,1])\cup((0,e)\times\{1\})$, $\mathcal{S}_1=\{0\}\times(0,1]$, $\mathcal{S}_2=[0,e]\times\{0\}$, $f(\cdot,\cdot,0)\in L^\infty(Q_T)$, $g\in C((\mathcal{R}\cup\mathcal{S}_1)\times[0,T])$, $u_0\in C(\mathcal{R}\cup\mathcal{S}_1)$ and $u_0(x)=g(x,0)$ for any $x\in\mathcal{R}\cup\mathcal{S}_1$. Observe that in this problem the coefficient b_1 is unbounded at $\mathcal{S}_1$. However, as we see in the following, besides a supersolution F of (2.2.16), we can exhibit barrier functions h for all points of $\mathcal{S}_1\times[0,T]$. Indeed the function

$$F(x,y,t):=\exp\{t\}\ \ ((x,y,t)\in(\Omega\cup\mathcal{R})\times[0,T])$$

is a bounded supersolution of equation

$$\mathcal{L}F-\partial_t F=-1 \quad \text{in } Q_T$$

and $F\geq 1$. Moreover, barriers can be constructed. In fact, take $(x_0,y_0,t_0)\in\mathcal{S}_1\times[0,T]$. Then $x_0=0$. Define

$$h(x,y,t):=\sigma[x+\tau(x^2+(y-y_0)^2+(t-t_0)^2)]$$

$((x,y,t)\in C_\delta((0,y_0,t_0))$, where σ,τ,δ are positive constants to be chosen and

$$\text{(2.5.5)}\qquad C_\delta((0,y_0,t_0)):=\big(B_\delta((0,y_0))\times(t_0-\delta,t_0+\delta)\big)\cap\big(Q_T\big).$$

Simple computations show that

$$\mathcal{L}h-\partial_t h=\sigma[\log x+2\tau(y^2+5+x\log x+y(y-y_0)-(t-t_0))]\leq -1 \text{ in } C_\delta((0,y_0,t_0)),$$

if $\tau>0$ and $\delta>0$ are small enough and $\sigma>0$ is big enough. Clearly

$$\text{(2.5.6)}\qquad h\in C^2(\overline{C_\delta((x_0,y_0,t_0))}),\ h>0 \text{ in } \overline{C_\delta((x_0,y_0,t_0))}\setminus\{(x_0,y_0.t_0)\},\ h(x_0,y_0,t_0)=0$$

for $x_0=0$. Then by Remark 2.2.21 there exists a solution $u\in L^\infty(Q_T)$ of problem (2.5.4). Moreover, the function

$$Z(x,y):=\log y-1$$

satisfies

$$Z\leq -1 \text{ in } \Omega\,,\qquad \mathcal{L}Z=0 \text{ in } \Omega\,,\qquad \lim_{y\to 0}Z(x,y)\ =\ -\infty\,.$$

Then by Theorem 2.2.5 and the above existence result, problem (2.5.4) is *well-posed* in $L^\infty(Q_T)$.

(e) Consider the problem

$$
\text{(2.5.7)} \qquad \begin{cases} -(\log x)u_{xx} + y^2 u_{yy} + u_x + y u_y - \partial_t u = f & \text{in } \Omega \\ u = g & \text{in } (\mathcal{R} \cup \mathcal{S}_1) \times (0,T] \\ u = u_0 & \text{in } (\Omega \cup \mathcal{R} \cup \mathcal{S}_1) \times \{0\}\,. \end{cases}
$$

Here we take $\Omega = (0,1) \times (0,1)$, $\mathcal{R} = (0,1) \times \{1\}$, $\mathcal{S}_1 = \{0,1\} \times (0,1]$, $\mathcal{S}_2 = [0,1] \times \{0\}$, $f(\cdot,\cdot,0) \in L^\infty(Q_T)$, $g \in C((\mathcal{R} \cup \mathcal{S}_1) \times [0,T])$, $u_0 \in C(\mathcal{R} \cup \mathcal{S}_1)$ and $u_0(x) = g(x,0)$ for any $x \in \mathcal{R} \cup \mathcal{S}_1$. Observe that in this problem the coefficient a_{11} is unbounded at $\mathcal{S}_1$. However, as we see in the following, besides a supersolution F of (2.2.16), we can exhibit barrier functions h for all points of $\mathcal{S}_1 \times [0,T]$. In fact, the function

$$F(x,y,t) := \exp\{t\} \ ((x,y,t) \in (\Omega \cup \mathcal{R}) \times [0,T])$$

is a bounded supersolution of equation

$$\mathcal{L}F - \partial_t F = -1 \quad \text{in } Q_T$$

and $F \geq 1$. Moreover, barriers can be constructed. Take $y_0 \in (0,1]$, thus $(0,y_0) \in \mathcal{S}_1$. For $\delta' > 0$ small enough, let $\psi \in C^2([y_0 - 2\delta', y_0 + 2\delta']; I\!R)$ with

$$\psi(y_0) = 1, \psi(y) \equiv 0 \ (y \notin (y_0 - \delta, y_0 + \delta))\,,$$
$$0 < \psi < 1 \ \text{ otherwise}\,.$$

For $\tau > 0$ set

$$Q_{\delta',\tau}((0,y_0)) := \{(x,y) \in \Omega | 0 < x < \tau\psi(y)\}\,.$$

Define

$$V(x,y) := \exp\{\alpha\tau\} - \exp\{\alpha(\tau\psi(y) - x)\} \quad \big((x,y) \in Q_{\delta',\tau}((0,y_0))\big),$$

where $\alpha > 0, \tau > 0$ are constants to be chosen. We easily obtain:

$$\mathcal{L}V = \exp\{\alpha(\tau\psi(y) - x)\}[\alpha^2 \log x - \alpha^2\tau^2(\psi'(y))^2 y^2 - \alpha\tau\psi''(y)y^2 +$$
$$+ \alpha - y\alpha\tau\psi'(y)] \leq -1 \text{ in } Q_{\delta',\tau}((0,y_0)),$$

provided that we take, as we do in the following, $\tau > 0$ small enough and $\alpha > 0$ big enough.

Let $t_0 \in [0,T]$. Define

$$h(x,y,t) := \lambda_1[V(x,y) + \lambda_2(t-t_0)^2]$$

$((x,y,t) \in C_\delta((0,y_0,t_0))$, where $\lambda_1 > 0$, $\lambda_2 > 0$ are positive constants to be chosen, $C_\delta((0,y_0,t_0))$ is defined as in (2.5.5) and $\delta > 0$ is so small that $B_\delta((0,y_0)) \cap \Omega \subseteq Q_{\delta',\tau}(0,y_0)$. It is not difficult to see that

$$\mathcal{L}h - \partial_t h \leq -1 \quad \text{in } C_\delta((0,y_0,t_0)),$$

provided that $\lambda_1 > 0$ is big enough and $\lambda_2 > 0$ is small enough. Clearly the function h satisfies (2.5.6) for $x_0 = 0$. Now take $(1,y_0,t_0) \in \mathcal{S}_1 \times [0,T]$. Define

$$h(x,y,t) := \sigma[1 - x + \tau((x-1)^2 + (y-y_0)^2 + (t-t_0)^2)]$$

$((x,y,t) \in C_\delta((x_0,y_0,t_0))$, where σ, τ, δ are positive constants to be chosen. It is easily checked that

$$\mathcal{L}h - \partial_t h = \sigma[-1 + 2\tau(-\log x + y^2 + (x-1) + y(y-y_0) - (t-t_0))] \leq -1 \text{ in } C_\delta((1,y_0,t_0)),$$

if $\tau > 0$ and $\delta > 0$ are small enough and $\sigma > 0$ is big enough. Clearly h satisfies (2.5.6) for $x_0 = 1$.

Then by Remark 2.2.21 there exists a solution $u \in L^\infty(Q_T)$ of problem (2.5.7). Uniqueness follows by Theorem 2.2.5 since the function $Z(x,y) := \log y - 1$ satisfies

$$Z \leq -1 \text{ in } \Omega\,, \qquad \mathcal{L}Z = 0 \text{ in } \Omega\,, \qquad \lim_{y\to 0} Z(x,y) \;=\; -\infty\,.$$

Then problem (2.5.7) is *well-posed* in $L^\infty(Q_T)$.

2.5.2. Nonuniqueness. According to the assumptions made in Subsection 2.2.2, only degeneracy at the boundary is allowed in the examples of this subsection.

(a) Consider the problem

$$\begin{cases} u_{xx} + y^2 u_{yy} - u_y + \frac{1}{x} u_x - \partial_t u = f & \text{in } Q_T \\ u = g & \text{in } \mathcal{R} \times (0,T] \\ u = u_0 & \text{in } (\Omega \cup \mathcal{R}) \times \{0\}\,, \end{cases} \tag{2.5.8}$$

where $\Omega = (0,\infty)\times(0,1)$, $\mathcal{R} = (0,\infty)\times\{1\}$, $\mathcal{S} = ((0,\infty)\times\{0\}) \cup (\{0\}\times[0,1])$, $f(\cdot,\cdot,0) \in L^\infty(Q_T)$. The function $V(x,y) := y$ satisfies

$$\mathcal{L}V = -1 \quad \text{in } \Omega\,, \qquad \inf_{\Omega\cup\mathcal{R}} V = 0 < \inf_{\mathcal{R}} V = 1\,.$$

By Theorem 2.2.16 problem (2.5.8) has *infinitely many solutions* in $L^\infty(Q_T)$.

(b) Take Ω, $\mathcal{R}$, $\mathcal{S}$, f as in case (a) and consider the problem

$$\begin{cases} \frac{1}{2}x^2 u_{xx} + y^2 u_{yy} + 2x^2 u_x - (2x^2+1)u_y - u - \partial_t u = f & \text{in } Q_T \\ u = g & \text{in } \mathcal{R}\times(0,T] \\ u = u_0 & \text{in } (\Omega\cup\mathcal{R})\times\{0\} \\ \lim_{x\to\infty} \int_0^T u(x,y,t)\,dt = l & (y \in (0,1))\,, \end{cases} \tag{2.5.9}$$

where $l \in \mathbb{R}$, and g satisfies the condition

$$\lim_{x\to+\infty} \frac{1}{T}\int_0^T g(x,y,t)\,dt = l\,.$$

The function $V(x,y) := x + y - 1$ satisfies

$$\mathcal{L}V = -1 \quad \text{in } \Omega\,, \quad \inf_{\Omega\cup\mathcal{R}} V = 0 < \min\Big\{\inf_{\mathcal{R}} V\,, \lim_{x\to\infty} V(x,y)\Big\} = 1 \quad (y \in (0,1))\,.$$

Moreover, the function $H(x,y) := \dfrac{1}{x}$ $(x > 1)$ satisfies

$$\mathcal{L}H \leq -1 \quad \text{in } \Omega \setminus \bar{B}_2\,, \qquad \lim_{x\to\infty} H(x) = 0\,.$$

In view of Theorem 2.2.18, problem (2.5.9) has *infinitely many solutions* in $L^\infty(Q_T)$.

CHAPTER 3

Criteria for well-posedness of degenerate elliptic and parabolic problems

3.1. Introduction

We study existence and uniqueness of solutions to linear *degenerate* elliptic equations of the following form:

$$\mathcal{L}u - cu = \phi \quad \text{in} \quad \Omega\,. \tag{3.1.1}$$

Here $\Omega \subseteq I\!\!R^n$ is an open connected bounded set with boundary $\partial\Omega$ and c, ϕ are given functions, $c \geq 0$ in Ω; the operator $\mathcal{L}$ is formally defined as follows:

$$\mathcal{L}\,u \equiv \frac{1}{\rho}\,\mathcal{M}\,u \equiv \frac{1}{\rho(x)}\Big[\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i}\Big]\,.$$

We assume $\rho > 0$ in Ω,

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq 0 \qquad \text{for any } x \in \Omega\,,\ (\xi_1, .., \xi_n) \in I\!\!R^n\,.$$

Precise assumptions on the coefficients of equation (3.1.1) are made below (see assumptions (A_2), (E_1) and (E_2)).

Our methods also apply to companion parabolic equations of the form:

$$\mathcal{L}u - cu - \partial_t u = f \quad \text{in} \quad \Omega \times (0,T) =: Q_T \tag{3.1.2}$$

with $T > 0$, $f = f(x,t)$ given; no sign condition on $c = c(x)$ is needed in this case (see Section 3.3). Quasilinear parabolic equations can be dealt with similarly (see [**60**], [**59**].

We always regard the boundary $\partial\Omega$ as the disjoint union of the *regular boundary* $\mathcal{R}$ and the *singular boundary* $\mathcal{S}$. The case $\partial\Omega \neq \partial\bar{\Omega}$ is possible, thus $\mathcal{S}$ can be a manifold of dimension less than $n-1$ (while $\mathcal{R} \subseteq \partial\bar{\Omega}$; see assumption (A_1)). In general, the coefficients of $\mathcal{L}$ and the function c can either vanish or diverge, or need not have a limit, when $dist(x,\mathcal{S}) \to 0$; moreover, ellipticity is possibly lost in Ω and/or when $dist(x,\mathcal{S}) \to 0$ (see (A_2), (E_1) and (E_2)). Then it is natural to prescribe the Dirichlet boundary condition on the regular boundary $\mathcal{R}$; this leads to the following problem for equation (3.1.1):

$$\begin{cases} \mathcal{L}\,u - cu = \phi & \text{in } \ \Omega \\ u = \gamma & \text{in } \ \mathcal{R}\,. \end{cases} \tag{3.1.3}$$

Similarly, for equation (3.1.2) we address the problem:

$$\begin{cases} \mathcal{L}\,u - cu - \partial_t u = f & \text{in } Q_T \\ u = g & \text{in } \ \mathcal{R} \times (0,T] \\ u = u_0 & \text{in } \ \big[\Omega \cup \mathcal{R}\big] \times \{0\}\,. \end{cases} \tag{3.1.4}$$

Sufficient conditions for uniqueness or nonuniqueness of solutions to problems (3.1.3)-(3.1.4) have been given in [**60**], [**?**], [**59**]. These conditions are implicit in character, for they depend on the existence of suitable sub- and supersolutions to related elliptic problems, like the *first exit time problem*:

$$(3.1.5) \qquad \begin{cases} \mathcal{L}U = -1 & \text{in } \Omega \\ U = 0 & \text{on } \mathcal{R}. \end{cases}$$

In this paper we address the actual construction of such sub- and supersolutions, aiming to give explicit criteria for well-posedness of problems (3.1.3)-(3.1.4). Not surprisingly, the feasibility of this program depends on geometrical properties of the singular boundary $\mathcal{S}$ (in particular, on its dimension), as well as on the behaviour of the coefficients of the operator $\mathcal{L}$ as the distance $d(x, \mathcal{S})$ goes to zero.

3.1.1. Assumptions. Our assumptions concerning the set Ω, the regular boundary $\mathcal{R}$ and the singular boundary $\mathcal{S}$ are summarized as follows:

$$(A_1) \qquad \begin{cases} (i) \quad \Omega \subseteq I\!R^n \text{ is open, bounded and connected;} \\ (ii) \ \partial\Omega = \mathcal{R} \cup \mathcal{S},\ \overline{\mathcal{R}} \cap \overline{\mathcal{S}} = \emptyset,\ \mathcal{S} \neq \emptyset; \\ (iii)\ \mathcal{R} \subseteq \partial\overline{\Omega},\ \ \Omega \text{ satisfies the outer sphere condition at } \mathcal{R}; \\ (iv) \ \ \mathcal{S} \text{ is a compact } k-\text{dimensional submanifold of } I\!R^n \text{ of} \\ \qquad \text{class } C^3 \ (k = 0, 1, \dots, n-1) \end{cases}$$

(we say that $dim\,\mathcal{S} = 0$, if $\mathcal{S}$ is a finite union of points). Further assumptions will be needed below (see (A_3)).

Cases where different connected components of $\mathcal{S}$ are submanifolds of different dimension can also be considered; we omit the details.

It is natural to choose $\mathcal{R}$ as *the largest subset* of $\partial\Omega$ where ellipticity of the operator $\mathcal{L}$ holds (see assumptions (E_1), $(E_2) - (ii)$ below); we do so in the following.

Denote by $C^{k,1}(B)$ the space of functions defined in a subset $B \subseteq \overline{\Omega}$, whose derivatives of order $\leq k$ $(k = 0, 1)$ are locally Lipschitz continuous in B. Concerning coefficients and data of the elliptic problem (3.1.3), we make the following assumptions:

$$(A_2) \qquad \begin{cases} (i) \quad \rho \in C^{1,1}(\Omega \cup \mathcal{R}),\ \rho > 0 \ \text{ in } \Omega \cup \mathcal{R}; \\ (ii) \quad a_{ij} = a_{ji} \in C^{1,1}(\Omega \cup \mathcal{R}) \cap C^{0,1}(\overline{\Omega}),\ b_i \in C^{0,1}(\Omega \cup \mathcal{R}) \cap \\ \qquad L^\infty(\Omega)\ \ (i, j = 1, \dots, n); \\ (iii)\ c \in C(\Omega \cup \mathcal{R}),\ c \geq 0; \\ (iv)\ \phi \in C(\Omega); \\ (v) \ \ \gamma \in C(\mathcal{R}) \end{cases}$$

(the assumption $b_i \in L^\infty(\Omega)$ will be omitted in Theorem 3.2.12).

Our nonuniqueness results for problem (3.1.3) require ellipticity of the operator $\mathcal{L}$ in Ω; therefore we assume:

$$(E_1) \qquad \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j > 0 \text{ for any } x \in \Omega \cup \mathcal{R} \text{ and } (\xi_1, .., \xi_n) \neq 0\,.$$

On the other hand, the uniqueness results hold true even if ellipticity of $\mathcal{L}$ in Ω is lost. In this case we replace assumption (E_1) by the following:

$$(E_2) \qquad \begin{cases} (i) \ \ \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq 0 \text{ for any } x \in \Omega \text{ and } (\xi_1, .., \xi_n) \in I\!R^n, \\ \qquad \sigma_{ij} \in C^1(\Omega)\ \ (i, j = 1, \dots, n); \\ (ii)\ \ \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j > 0 \text{ for any } x \in \mathcal{R} \text{ and } (\xi_1, .., \xi_n) \neq 0; \\ (iii) \text{ either } c > 0 \text{ in } \Omega \cup \mathcal{R},\ \text{ or } c \geq 0 \text{ and } c + \sum_{i=1}^{n} \sigma_{ji}^2 > 0 \\ \qquad \text{in } \Omega \cup \mathcal{R} \text{ for some } j = 1, \dots, n; \end{cases}$$

here $\sigma \equiv (\sigma_{ij})$ denotes the square root of the matrix $A \equiv (a_{ij})$ (namely, $A(x) = \sigma(x)\sigma(x)^T$; $x \in \overline{\Omega}$). Assumption (E_2) (in particular, $(E_2) - (iii)$) enables us to use comparison results for *viscosity* sub- and supersolutions to second order degenerate elliptic equations, via an equivalence result proved in [?] (see [?]; for the parabolic case [**59**]).

3.1.2. Well-posedness conditions. The following result was proved in [**60**] (see also [**65**]).

THEOREM 3.1.1. *Let assumptions* $(A_1) - (A_2)$ *and* (E_1) *be satisfied; suppose* $c \in L^\infty(\Omega)$. *Let there exist a supersolution* V *of problem (3.1.5) such that*

$$\inf_{\Omega\cup\mathcal{R}} V = 0 < \inf_{\mathcal{R}} V\,. \tag{3.1.6}$$

Then either no solutions, or infinitely many solutions of problem (3.1.3) exist.

The proof of the above theorem shows that nonuniqueness depends on the possibility of prescribing the value of the solution of problem (3.1.3) at some point of the singular boundary $\mathcal{S}$. Typically, to have a well-posed problem boundary conditions must be prescribed on some subset $\mathcal{S}_1 \subset \mathcal{S}$, while on the complementary subset $\mathcal{S}_2$ the singular character of the operator does not allow to impose boundary data. Hence we make the following assumption:

$$(A_3)\qquad \begin{cases} (i)\;\; \mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2,\;\; \overline{\mathcal{S}_1} \cap \overline{\mathcal{S}_2} = \emptyset\,; \\ (ii)\;\; \mathcal{S}_j = \cup_{k=1}^{k_j} \mathcal{S}_j^k\,, \text{where every}\, \mathcal{S}_j^k \text{ is connected and}\, \overline{\mathcal{S}_j^k} \cap \overline{\mathcal{S}_j^l} = \emptyset \\ \qquad \text{for any}\, k,l = 1,..,k_j,\; k \neq l\,,\; \text{if}\; k_j \geq 2\;\; (j=1,2)\,. \end{cases}$$

Then we consider the problem:

$$\begin{cases} \mathcal{L}u - cu = \phi & \text{in } \Omega \\ u = \gamma & \text{in } \mathcal{R} \cup \mathcal{S}_1\,. \end{cases} \tag{3.1.7}$$

For the same reason we associate to the parabolic problem (3.1.4) the following:

$$\begin{cases} \mathcal{L}u - cu - \partial_t u = f & \text{in } Q_T \\ u = g & \text{in } \left[\mathcal{R} \cup \mathcal{S}_1\right] \times (0,T] \\ u = u_0 & \text{in } \left[\Omega \cup \mathcal{R} \cup \mathcal{S}_1\right] \times \{0\}\,. \end{cases} \tag{3.1.8}$$

In the elliptic case, sufficient conditions for uniqueness of solutions to problem (3.1.7) have been proved in [**65**] (analogous results hold for the parabolic case, see [**59**]). Such conditions depend on the existence of subsolutions to the *homogeneous problem*:

$$\begin{cases} \mathcal{L}U = cU & \text{in } \Omega \\ U = 0 & \text{on } \mathcal{R} \end{cases} \tag{3.1.9}$$

and on their behaviour as the distance $d(x, \mathcal{S}_2)$ goes to zero. Let us mention the following result.

THEOREM 3.1.2. *Let assumptions* $(A_1) - (A_3)$, *and either* (E_1) *or* (E_2) *be satisfied. Suppose* $\mathcal{S}_2 \neq \emptyset$, $\gamma \in C(\mathcal{R} \cup \mathcal{S}_1)$. *Let there exist a subsolution* $Z \leq H < 0$ *of problem (3.1.9). Then there exists at most one solution* u *of problem (3.1.7) such that*

$$\lim_{d(x,\mathcal{S}_2)\to 0} \frac{u(x)}{Z(x)} = 0\,. \tag{3.1.10}$$

Clearly, if $\mathcal{S}_1 = \emptyset$ we recover uniqueness conditions for problem (3.1.3). If $\mathcal{S}_1 \neq \emptyset$, existence of solutions to problem (3.1.7) implies nonuniqueness for problem (3.1.3), since $\overline{\mathcal{R}} \cap \overline{\mathcal{S}} = \emptyset$ by assumption and the boundary data on $\mathcal{S}_1$ can be arbitrarily chosen. This remark will be used below, *e.g.* in the proof of Theorems 3.2.20-3.2.21.

Let us mention for further purposes that problem (3.1.9) can be replaced by problem (3.1.5) in the above statement, if $c(x) \geq c_0 > 0$ in Ω.

3.1.3. Outline of results. As already pointed out, applying Theorems 3.1.1-3.1.2 to concrete cases calls for the actual construction of the sub- and supersolutions V, Z; this is the point we address below. Results analogous to Theorems 3.1.1-3.1.2 have been proved in [**59**] for the parabolic problems (3.1.4), (3.1.8), relying on the existence of the same functions V, Z as above. Therefore our criteria for well-posedness are conceptually the same both for the elliptic and the parabolic case (see Sections 3.2-3.3).

Our main results for the elliptic case can be described as follows (for the parabolic case, see Section 3.3).

(a) If $n \geq 2$, $dim\,\mathcal{S} \leq n-2$ and the *orthogonal rank* of the diffusion matrix A is at least 2 on $\mathcal{S}$, there exists at most one bounded solution of problem (3.1.3) (actually, the uniqueness class is larger; see Definition 3.2.11 and Theorem 3.2.12). This result extends Theorem 4.1, Ch.11 in [**31**], which was proved under more restrictive assumptions by stochastic methods; it also extends the results in [**40**], where A was uniformly elliptic.

We stress that the above uniqueness result holds *without imposing any additional condition at* $\mathcal{S}$. In the parlance of [**31**], conditions at $\mathcal{S}$ are unnecessary for uniqueness since $\mathcal{S}$ is "too thin", hence *non-attainable* by trajectories of the Markov process generated by the operator $\mathcal{L}$.

(b) If $dim\,\mathcal{S} = n-1$, well-posedness crucially depends on the behaviour of the coefficients of $\mathcal{L}$ near $\mathcal{S}$. Roughly speaking, if "diffusion near $\mathcal{S}$ is low" (see Theorem 3.2.16, in particular condition (3.2.16) - (3.2.17)), no additional conditions at $\mathcal{S}$ are needed to ensure uniqueness of problem (3.1.3), much as in the case $n \geq 2$, $dim\,\mathcal{S} \leq n-2$. The opposite holds when "diffusion near $\mathcal{S}$ is high": in this case boundary conditions on some part of $\mathcal{S}$ are necessary to make the problem well posed (see Theorem 3.2.18 and condition (3.2.20) - (3.2.21)). An interesting model case is when $\mathcal{L} = \frac{1}{\rho}\Delta$: if $\rho(x) \sim [d(x,\mathcal{S})]^{-\alpha}$ for some $\alpha \geq 2$, Theorem 3.2.16 applies and no additional conditions at $\mathcal{S}$ are needed; the opposite holds, if $\rho(x) \sim [d(x,\mathcal{S})]^{-\alpha}$ with $\alpha < 2$, so that Theorem 3.2.18 applies (see Example (c) in Section 6).

In the light of the previous results, dim $\mathcal{S} = n-2$ is critical for well-posedness of problem (3.1.3) in the class of bounded solutions. This is not surprising, since $n-2$ is the critical dimension for studying sets of zero capacity with respect to uniformly elliptic second order operators (see [**73**]). In such case the role of capacity to study uniqueness of the bounded Cauchy problem is well understood (*e.g.*, see [**33**]). We are not aware of similar results in the present more general case (however, see Remark 3.2.15 below for the particular case $\mathcal{L} = \frac{1}{\rho}\Delta$).

Also the role of the behaviour of ρ near the singular boundary when $dim\,\mathcal{S} = n-1$ is not unexpected. In fact, the above condition $\rho(x) \sim [d(x,\mathcal{S})]^{-\alpha}$ ($\alpha < 2$) was considered in [**72**], where the generation of semigroups in $L^\infty(\Omega)$ by second order operators, with coefficients possibly vanishing at $\partial\Omega$, was investigated.

Let us mention that results analogous to Theorems 3.1.1-3.1.2 also hold for unbounded domains (see [**65**]). Accordingly, several results we state below can be extended to domains of this kind; we leave their formulation to the reader.

The paper is organized as follows. In Section 2 first we introduce some definitions and related results existing in the literature, then we state our main results concerning elliptic problems. The same is done for parabolic problems in Section 3. Proofs are given in Sections 4 and 5, depending on the assumption made on the dimension of the singular manifold. Finally, a few examples are discussed in Section 6.

3.2. Elliptic problems

3.2.1. Mathematical framework and auxiliary results.

3.2.1.1. *Sub- and supersolutions.* Let us make precise the definition of solution to the above mentioned elliptic problems. Denote by $\mathcal{M}^*$ the formal adjoint of the operator $\mathcal{M}$, namely:

$$\mathcal{M}^* u \equiv \sum_{i,j=1}^{n} \frac{\partial^2 (a_{ij} u)}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \frac{\partial (b_i u)}{\partial x_i} .$$

DEFINITION 3.2.1. *By a* subsolution *to equation (3.1.1) we mean any function $u \in C(\Omega)$ such that*

$$\int_\Omega u \{\mathcal{M}^* \psi - \rho c \psi\} \, dx \ \geq \int_\Omega \rho \phi \psi \, dx \tag{3.2.1}$$

for any $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$. Supersolutions *of (3.1.1) are defined replacing "$\geq$" by '$\leq$" in (3.2.1). A function u is a* solution *of (3.1.1) if it is both a sub- and a supersolution.*

DEFINITION 3.2.2. *Let $\mathcal{R} \subseteq \mathcal{E} \subseteq \partial\Omega$, $\gamma \in C(\mathcal{E})$. By a* subsolution *to the problem*

$$\begin{cases} \mathcal{L} u - cu = \phi & \text{in } \Omega \\ \\ u = \gamma & \text{on } \ \mathcal{E} \end{cases} \tag{3.2.2}$$

we mean any function $u \in C(\Omega \cup \mathcal{E})$ such that:
(i) u is a subsolution of equation (3.1.1);
(ii) $u \leq \gamma$ on $\mathcal{E}$.
Supersolutions *and* solutions *of (3.2.2) are similarly defined.*

3.2.1.2. *Attracting boundaries and barriers.* Let $\Sigma \subseteq \partial\Omega$; define

$$\Sigma^\varepsilon := \{x \in \Omega \,|\, dist(x, \Sigma) < \varepsilon\} \qquad (\varepsilon > 0) .$$

Also set

$$B_r(x^0) := \{x \in I\!R^n \,|\, |x - x^0| < r\} \qquad (x^0 \in I\!R^n) ,$$
$$\mathcal{B}_r(y^0) := B_r(y^0) \cap \mathcal{S} \qquad (y^0 \in \mathcal{S}) .$$

Let us introduce, for use in the sequel, the following definitions (see [**?**]).

DEFINITION 3.2.3. *We say that $\Sigma \subseteq \partial\overline{\Omega}$ is* attracting, *if there exist $\varepsilon > 0$ and a supersolution $V \in C(\overline{\Sigma^\varepsilon})$ of the equation*

$$\mathcal{L}u - cu = -1 \quad \text{in } \Sigma^\varepsilon \tag{3.2.3}$$

such that

$$V > 0 \quad \text{in } \overline{\Sigma^\varepsilon} \setminus \Sigma , \qquad V = 0 \quad \text{on } \Sigma .$$

DEFINITION 3.2.4. *Let $x^0 \in \partial\overline{\Omega}$. A function $h \in C(\overline{\Omega \cap B_r(x^0)})$ is called a* barrier *at x^0 if:*
(i) h is a supersolution of

$$\mathcal{L}u - cu = -1 \quad \text{in } \Omega \cap B_r(x^0) ;$$

(ii) there holds

$$h > 0 \text{ in } \overline{\Omega \cap B_r(x^0)} \setminus \{x^0\} , \quad h(x^0) = 0 .$$

If Σ is attracting, the function V can be viewed as a barrier for the whole of Σ.

Let us state the following result, concerning existence of solutions to problem (3.1.7) (or (3.1.3), if $\mathcal{S}_1 = \emptyset$; see [**65**])).

THEOREM 3.2.5. *Let $\mathcal{S}_1 \subseteq \partial\overline{\Omega}$ be attracting and assumptions $(A_1)-(A_3)$, (E_1) be satisfied. In addition, suppose:*
(a) $c \in L^\infty(\mathcal{S}_1^\varepsilon)$ for some $\varepsilon > 0$;
(b) $\phi \in L^\infty(\Omega)$;
(c) $\gamma \in C(\mathcal{R} \cup \mathcal{S}_1)$.
Let there exist a positive supersolution $F \in C(\Omega \cup \mathcal{R}) \cap L^\infty(\mathcal{S}_1^\varepsilon)$ of the equation

$$\mathcal{L}u - cu = -1 \quad \text{in } \Omega\,. \tag{3.2.4}$$

Then there exists a solution of problem (3.1.7), provided that

$$\gamma = \textit{constant} \qquad \text{on } \mathcal{S}_1\,. \tag{3.2.5}$$

Condition (3.2.5) is unnecessary, if a barrier exists at any point of $\mathcal{S}_1$.

REMARK 3.2.6. If $c(x) \geq c_0 > 0$ in Ω, we can take $F \equiv \frac{1}{c_0}$ in $\bar{\Omega}$ as a supersolution of (3.2.4).

Under the assumptions of Theorem 3.2.5, since $\mathcal{S}_1$ is attracting, *constant* Dirichlet data can be prescribed on it. Moreover, if a barrier exists at any point of $\mathcal{S}_1$ also general Dirichlet data can be prescribed, but this need not be the case without this additional requirement (e, g,, see [**65**] for an example). If the coefficients a_{ij}, b_i are bounded and ρ is bounded away from zero in $\mathcal{S}_1^\varepsilon$ for some $\varepsilon > 0$, a barrier exists at any point of $\mathcal{S}_1$ (see [**?**], [**65**]; see also the proof of Proposition 3.2.7 below).

3.2.1.3. *Revisiting classical results.* The above remarks are deeply connected with the approach developed in [**26**] to investigate uniqueness for problem (3.2.2). As in [**26**], let the following assumptions be satisfied:

$$(F_1) \qquad \begin{cases} (i) \quad \Omega \text{ open, bounded and connected, } \partial\Omega = \partial\overline{\Omega}\,; \\ (ii) \;\; \partial\Omega = \bigcup_{h=1}^m \Gamma_h; \text{ each } \Gamma_h \text{ is a regular } \; (n-1)-\text{dimensional} \\ \qquad \text{submanifold with boundary } \partial\Gamma_h \; (h = 1, 2, ..., m)\,; \\ (iii) \;\; \Gamma_h \cap \Gamma_k = \partial\Gamma_h \cap \partial\Gamma_k \text{ for any } h, k = 1, \ldots, m, \; h \neq k\,; \end{cases}$$

$$(F_2) \qquad \begin{cases} (i) \quad \rho \in C^2(\overline{\Omega}), \;\; \rho > 0 \text{ in } \overline{\Omega}\,; \\ (ii) \; a_{ij} = a_{ji} \in C^2(\overline{\Omega}), \, b_i \in C^1(\overline{\Omega})\,; \\ (iii) \; c \in C(\overline{\Omega}), \;\; c \geq 0\,; \end{cases}$$

$$(F_3) \qquad \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq 0 \quad \text{for any } x \in \Omega \text{ and } (\xi_1, .., \xi_n) \in I\!\!R^n\,.$$

Define for any $x \in \Gamma_h \setminus \partial\Gamma_h$ ($h = 1, \ldots, m$):

$$\alpha_F(x) := \sum_{i,j=1}^n a_{ij}(x)\nu_i(x)\nu_j(x)\,,$$

$$\beta_F(x) := \sum_{i=1}^n \Big[b_i(x) - \sum_{j=1}^n \frac{\partial a_{ij}(x)}{\partial x_j}\Big]\nu_i(x)\,,$$

where $\nu(x) \equiv (\nu_1(x), \ldots, \nu_n(x))$ denotes the outer normal to Ω at $x \in \Gamma_h \setminus \partial\Gamma_h$; then extend the definition of α_F, β_F to $\partial\Gamma_h$ by continuity. Observe that the extensions of α_F, β_F to $\partial\Gamma_h$ for different values of $h = 1, \ldots, m$ need not agree on the intersection of the boundaries.

Set

$$\Sigma_1 := \{x \in \partial\Omega \,|\, \alpha_F(x) = 0, \beta_F(x) \leq 0\}\,, \tag{3.2.6}$$

$$\Sigma_2 := \{x \in \partial\Omega \,|\, \alpha_F(x) = 0, \beta_F(x) > 0\}\,, \tag{3.2.7}$$

$$\Sigma_3 := \{x \in \partial\Omega \,|\, \alpha_F(x) > 0\} = \partial\Omega \setminus [\Sigma_1 \cup \Sigma_2]\,. \tag{3.2.8}$$

Observe that Σ_3 contains the regular boundary $\mathcal{R}$, if (F_1) holds. Moreover, the *drift trajectories* (*e.g.*, see [**19**]) do not point outwards at the points of Σ_1, but they do at those of Σ_2.

The following result will be proved (see Section 3.5).

PROPOSITION 3.2.7. *Let assumptions $(F_1) - (F_3)$ be satisfied; let $\sigma_{ij} \in C^1\big(\overline{\Sigma_2{}^\varepsilon}\big)$ for some $\varepsilon > 0$ $(i, j = 1, \dots, n)$. Let Σ be a smooth connected component of $\partial\Omega$, such that $\Sigma \subseteq \Sigma_2$. Then:*
(i) Σ is attracting;
(ii) for any $x^0 \in \Sigma$ there exists a barrier.

The proof of claim (i) relies on the fact that some multiple of the distance $d(\cdot, \Sigma)$ is a supersolution of equation (3.2.3) (see Definition 3.2.3); claim (ii) follows by a standard argument from the attractivity of Σ and the boundedness of the coefficients ρ, a_{ij}, b_i.

The proof of the following result is similar to that of Proposition 3.2.7, thus it will be omitted. A related result can be found in Lemma 2.7.1 of [**56**].

PROPOSITION 3.2.8. *Let assumptions $(F_1) - (F_3)$ be satisfied. Let Σ be a smooth connected component of $\partial\Omega$, such that $\Sigma \subseteq \Sigma_3$. Then for any $x^0 \in \Sigma$ there exists a barrier.*

If $\Sigma \subseteq \Sigma_1$, the distance $d(\cdot, \Sigma)$ can be used to construct a subsolution of problem (3.1.9). This is the content of the following proposition, where $\chi \in C^2(\overline{\Omega})$, $0 \le \chi \le 1$ is any function such that

$$\chi(x) = \begin{cases} 1 & \text{if } \; x \in \Sigma^{\varepsilon/2} \\ 0 & \text{if } \; x \in \Omega \setminus \Sigma^{\varepsilon} \qquad (\varepsilon > 0)\,. \end{cases} \tag{3.2.9}$$

Similar results can be found in Theorem 2.7.1 of [**56**] and in Chapter 9, Vol. I of [**31**].

PROPOSITION 3.2.9. *Let assumptions $(F_1) - (F_3)$ be satisfied; for some $\varepsilon_0 > 0$ let $\sigma_{ij} \in C^1\big(\overline{\Sigma_1{}^{\varepsilon_0}}\big)$ $(i, j = 1, \dots, n)$. Let Σ be a smooth connected component of $\partial\Omega$, such that $\Sigma \subseteq \Sigma_1$; suppose $c(x) \ge c_0 > 0$ for any $x \in \Omega$. Then for any $\alpha > 0$ sufficiently small and $H > 0$ large enough there exists $\varepsilon \in (0, \varepsilon_0)$ such that the function*

$$Z(x) := -[d(x, \Sigma)]^{-\alpha}\chi(x) - H \qquad (x \in \overline{\Omega} \setminus \Sigma) \tag{3.2.10}$$

(where $\chi = \chi_\varepsilon$ satisfies (3.2.9)) is a subsolution of problem (3.1.9).

In view of Theorem 3.1.2, if $\overline{\mathcal{S}_2} \subseteq \Sigma_1$ and the assumptions of Proposition 3.2.9 are satisfied, we expect uniqueness of solutions to problem (3.1.7) such that

$$\lim_{d(x, \mathcal{S}_2) \to 0} \frac{u(x)}{[d(x, \mathcal{S}_2)]^{-\alpha}} = 0 \tag{3.2.11}$$

(with $\alpha > 0$ sufficiently small), thus in particular uniqueness of bounded solutions. In fact, this is the content of Theorem 3.2.23 below. Observe that Proposition 3.2.9 is in agreement with the following uniqueness result, which was proved in [**26**].

THEOREM 3.2.10. *Let assumptions $(F_1) - (F_3)$ be satisfied and $c > 0$ in $\overline{\Omega}$. Suppose that the Gauss-Green identity applies in $\overline{\Omega}$. Then problem (3.2.2) with $\mathcal{E} = \Sigma_2 \cup \Sigma_3$ admits at most one solution in the space $C_{\mathcal{L}} := \{u \in C^1(\overline{\Omega}) \cap C^2(\Omega) \,|\, \mathcal{L}u \in L^\infty(\Omega)\}$.*

3.2.2. Main results: Singular manifolds of low dimension. To state our results we need some preliminary remarks. Set $k \equiv dim\,\mathcal{S}$; denote by $\mathcal{M}_m$ the linear space of $m \times m$ matrices with real entries ($m \in \mathbb{N}$). For any fixed $y \in \mathcal{S}$ there exist orthonormal vectors $\eta^{(1)}(y), \dots, \eta^{(n-k)}(y) \in \mathbb{R}^n$, which are orthogonal to $\mathcal{S}$ at y. Consider the matrix $A_\perp(y) \equiv (\alpha_{lm}(y)) \in \mathcal{M}_{n-k}$, where

$$\alpha_{lm}(y) := \sum_{i,j=1}^{n} a_{ij}(y)\eta_i^{(l)}(y)\eta_j^{(m)}(y) \qquad (l, m = 1, \dots, n-k\,; y \in \mathcal{S})\,.$$

Let us make the following definition (see [**31**]).

DEFINITION 3.2.11. *Let $y \in \mathcal{S}$. The rank $r(y)$ of the matrix $A_\perp(y)$ is called the* orthogonal rank of the diffusion matrix A at y.

The above definition is well posed, for $r(y)$ is independent of the choice of the set $\{\eta^{(l)}(y)\,|\,l = 1, \dots, n-k\}$; observe that $r(y) \le n-k$. In view of assumption $(A_1)-(iv)$, there exist $y^1, \dots, y^N \in \mathcal{S}$ such that:

$$(3.2.12) \qquad \begin{cases} \mathcal{S} \text{ is the union of the graphs } \mathcal{U}_i \text{ of } C^3 \text{ functions, say} \\ \phi^{(i)} : B_{R_i}(y_1^i, \dots, y_k^i) \subset \mathbb{R}^k \to \mathbb{R}^{n-k}, \phi^{(i)} \equiv (\phi_{k+1}^{(i)}, \dots, \phi_n^{(i)}) \\ (i = 1, \dots, N), \text{ up to reorderings of the coordinates.} \end{cases}$$

We shall use the following assumption:

$$(A_4) \qquad \begin{cases} (i) \quad n \ge 2,\ dim\,\mathcal{S} \le n-2\,; \\ (ii) \ \ r(y) \ge 2 \text{ for any } y \in \mathcal{S}\,; \\ (iii) \text{ for any } y \in \mathcal{U}_i \ (i = 1, \dots, N) \text{ there exist orthonormal vectors} \\ \qquad \eta^{(1)}(y), \dots, \eta^{(n-k)}(y) \in \mathbb{R}^n, \text{ which are orthogonal to } \mathcal{S} \text{ at } y, \\ \qquad \eta^{(l)}(\cdot) \in C^2(\mathcal{U}_i\,; \mathbb{R}^n) \ (l = 1, \dots, n-k), \text{ such that the matrix} \\ \qquad A_\perp(\cdot) \text{ has unit eigenvectors of class } C^2(\mathcal{U}_i; \mathbb{R}^{n-k})\,. \end{cases}$$

(here the notation in (3.2.12) has been used).

Now we can state the following

THEOREM 3.2.12. *Let assumptions $(A_1)-(A_4)$ be satisfied, with $(A_2)-(ii)$ replaced by the following:*

$$(A_2)-(ii)' \qquad \begin{cases} a_{ij} = a_{ji} \in C^2(\overline{\Omega}),\ b_i \in C^{0,1}(\Omega \cup \mathcal{R})\,; \\ \text{there exist } B_0 > 0 \text{ and } \beta \in [0,1) \text{ such that} \\ |b_i(x)| \le \dfrac{B_0}{[d(x,\mathcal{S})]^\beta} \quad \text{for any } x \in \Omega \ (i,j = 1, \dots, n). \end{cases}$$

Moreover, let either (E_1) or (E_2) hold; if (E_2) holds, let $c(x) > 0$ for any $x \in \Omega$. Then:
(i) *there exists at most one solution u of problem (3.1.3) such that*

$$(3.2.13) \qquad \lim_{d(x,\mathcal{S}) \to 0} \frac{u(x)}{\log[d(x,\mathcal{S})]} = 0\,;$$

(ii) *if $\alpha := \inf_{y \in \mathcal{S}} r(y) - 2 \ge 1$, there exists at most one solution u of problem (3.1.3) such that*

$$(3.2.14) \qquad \lim_{d(x,\mathcal{S}) \to 0} \frac{u(x)}{[d(x,\mathcal{S})]^{-\alpha}} = 0\,.$$

In particular, problem (3.1.3) has at most one bounded solution.

A simple application of Theorem 3.2.12 is given in Section 3.6, Example (a). Example (b) in the same Section shows that the limit value $\beta = 1$ in assumption $(A_2)-(ii)'$ above is not allowed.

Theorem 3.2.12 follows from Theorem 3.1.2 (with $\mathcal{S}_2 = \mathcal{S}$), if we exhibit a subsolution Z of problem (3.1.9) diverging like $\log[d(x,\mathcal{S})]$, or like $[d(x,\mathcal{S})]^{-\alpha}$ if $\alpha = \inf_{y\in\mathcal{S}} r(y) - 2 \geq 1$, as $d(x,\mathcal{S}) \to 0$; this is done in Section 3.4. Remarkably, the construction of Z does not require any assumption on the behaviour of ρ near $\mathcal{S}$ (which instead plays a role when $\mathit{dim}\,\mathcal{S} = n-1$; see Subsection 3.2.3).

Let us also mention the following well-posedness result, which follows immediately from Theorem 3.2.12, Theorem 3.2.5 (with $\mathcal{S}_1 = \emptyset$) and Remark 3.2.6.

THEOREM 3.2.13. *Let assumptions $(A_1)-(A_4)$, with $(A_2)-(ii)$ replaced by $(A_2)-(ii)'$, and (E_1) be satisfied; suppose $\phi \in L^\infty(\Omega)$ and $c(x) \geq c_0 > 0$ for any $x \in \Omega$. Then there exists a unique bounded solution of problem (3.1.3).*

As an example, it is informative to discuss the above situation when $A = (\delta_{ij})$. In this case problem (3.1.3) reads:

$$\begin{cases} \frac{1}{\rho}\,\Delta\, u \,-\, cu = \,\phi & \text{in } \;\Omega \\ \\ \,u \,=\gamma & \text{in } \;\mathcal{R}\,. \end{cases} \tag{3.2.15}$$

Since $r(y) = n-k$ $(y \in \mathcal{S})$, assumption (A_4) reduces to $(A_4)-(i)$; then we have the following refinement of Theorem 3.2.12.

COROLLARY 3.2.14. *Let assumptions $(A_1)-(A_3)$ be satisfied, with $(A_2)-(ii)$ replaced by $(A_2)-(ii)'$; suppose $\mathit{dim}\,\mathcal{S} \leq n-2$. Then:*
(i) there exists at most one solution u of problem (3.2.15) satisfying (3.2.13);
(ii) if $\mathit{dim}\,\mathcal{S} \equiv k \leq n-3$, there exists at most one solution u of problem (3.2.15) such that

$$\lim_{d(x,\mathcal{S})\to 0} \frac{u(x)}{[d(x,\mathcal{S})]^{2-(n-k)}} = 0\,.$$

In particular, there exists at most one bounded solution of problem (3.2.15).

REMARK 3.2.15. For $n \geq 2$, if dim $\mathcal{S} < n-2$, or if dim $\mathcal{S} = n-2$ and the Hausdorff measure $\mathcal{H}^{N-2}(\mathcal{S})$ is finite, the capacity $\mathrm{cap}_\Delta\,\mathcal{S}$ is zero and it is well known that

$$\mathrm{cap}_\Delta\,\mathcal{S} = 0 \quad \Leftrightarrow \quad \begin{cases} \text{there exists } u \in C^2(\Omega\cup\mathcal{R}) \text{ such that } u > 0 \text{ in } \Omega\cup\mathcal{R}, \\ \Delta u \leq 0 \text{ in } \Omega\cup\mathcal{R}, u(x) \to \;+\infty \;\text{ as } d(x,\mathcal{S}) \to 0\,. \end{cases}$$

Hence $Z := -u-1$ is a subsolution of problem (3.1.9) which diverges as $d(x,\mathcal{S}) \to 0$, and by Theorem 3.1.2 there exists at most one bounded solution of problem (3.2.15), in agreement with the above corollary. Similar remarks hold for uniformly elliptic operators with sufficiently smooth coefficients (*e.g.*, see [**73**] and references therein).

3.2.3. Main results: Singular manifolds of high dimension.

Let us now address the case $\mathit{dim}\,\mathcal{S} = n-1$. We shall prove the following uniqueness result.

THEOREM 3.2.16. *Let $\mathit{dim}\,\mathcal{S} = n-1$; let assumptions $(A_1)-(A_3)$, and either (E_1) or (E_2) be satisfied. Assume $\mathcal{S}_2 \neq \emptyset$. In addition, suppose the following:*
(a) there exist $\bar\varepsilon > 0$ and a positive, continuous function $\underline{\rho}$ satisfying

$$\int_0^{\bar\varepsilon} \eta\,\underline{\rho}(\eta)\,d\eta = +\infty\,, \tag{3.2.16}$$

such that

$$\rho(x) \geq \underline{\rho}\big(d(x,\mathcal{S}_2)\big) \quad \textit{for any } x \in \mathcal{S}_2^{\bar\varepsilon}\,; \tag{3.2.17}$$

(b) $c(x) \geq c_0 > 0$ for any $x \in \Omega$;
(c) $\gamma \in C(\mathcal{R}\cup\mathcal{S}_1)$.

Then:

(i) *the function*

$$P(\zeta) := \int_{\zeta}^{\bar{\varepsilon}} (\eta - \zeta)\, \underline{\rho}(\eta)\, d\eta \qquad (\zeta \in (0, \bar{\varepsilon})) \tag{3.2.18}$$

diverges as $\zeta \to 0^+$*;*

(ii) *there exists at most one solution* u *of problem (3.1.7) such that*

$$\lim_{d(x,\mathcal{S}_2)\to 0} \frac{u(x)}{P\big(d(x,\mathcal{S}_2)\big)} = 0\,. \tag{3.2.19}$$

In particular, problem (3.1.7) has at most one bounded solution.

In view of Theorem 3.1.2, Theorem 3.2.16 follows by constructing a suitable subsolution Z of problem (3.1.9) such that $|Z(x)|$ diverges with the same order of $P\big(d(x,\mathcal{S}_2)\big)$ as $d(x,\mathcal{S}_2) \to 0$ (see Section 3.5).

REMARK 3.2.17. A natural choice in Theorem 3.2.16 is $\underline{\rho}(\eta) = \eta^{-\sigma}$, $\sigma \geq 2$. Then:
(i) if $\sigma = 2$, there exists at most one solution u of problem (3.1.7) such that

$$\lim_{d(x,\mathcal{S}_2)\to 0} \frac{u(x)}{\log[d(x,\mathcal{S}_2)]} = 0\,;$$

(ii) if $\sigma > 2$, there exists at most one solution u of problem (3.1.7) such that

$$\lim_{d(x,\mathcal{S}_2)\to 0} \frac{u(x)}{[d(x,\mathcal{S}_2)]^{2-\sigma}} = 0\,.$$

In such cases the order of divergence of $P(\zeta)$ as $\zeta \to 0^+$ is the same as $Q(\zeta) := \displaystyle\int_{\zeta}^{\bar{\varepsilon}} \eta\, \underline{\rho}(\eta)\, d\eta$, although in general it can be lower (*e.g.*, take $\underline{\rho}(\eta) := e^{\frac{1}{\eta}}/\eta^3$).

In view of Theorem 3.2.16, no additional conditions at $\mathcal{S}_2$ are needed to ensure uniqueness of bounded solutions to problem (3.1.7), if (3.2.16)-(3.2.17) hold. If $\mathcal{S}_1 = \emptyset$, the situation is qualitatively the same as for $dim\,\mathcal{S} \leq n-2$ (see Theorem 3.2.12).

It is natural to investigate the complementary situation - namely, when conditions (3.2.20)-(3.2.21) below are satisfied. As it can be expected, boundary conditions at $\mathcal{S}_1$ are necessary in this case to have a well posed problem. In other words, nonuniqueness holds for problem (3.1.3), which lacks such conditions.

We address this situation strengthening assumption (E_1), *i.e.* requiring it in $\overline{\Omega}$ and not only in $\Omega \cup \mathcal{R}$. This is equivalent to assume that there exists $\alpha > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \quad \text{for any } x \in \overline{\Omega} \text{ and } (\xi_1, .., \xi_n) \neq 0\,. \tag{E_3}$$

Then we have the following

THEOREM 3.2.18. *Let* $dim\,\mathcal{S} = n-1$*. Let assumptions* $(A_1)-(A_3)$ *and* (E_3) *be satisfied; assume* $\mathcal{S}_2 = \emptyset$*. In addition, suppose the following:*
(a) *there exist* $\bar{\varepsilon} > 0$ *and a positive continuous function* $\overline{\rho}$ *satisfying*

$$\int_0^{\bar{\varepsilon}} \eta\, \overline{\rho}(\eta)\, d\eta < +\infty\,, \tag{3.2.20}$$

such that

$$\rho(x) \leq \overline{\rho}\big(d(x,\mathcal{S}_1)\big) \quad \textit{for any } x \in \mathcal{S}_1^{\bar{\varepsilon}}\,; \tag{3.2.21}$$

(b) $c \in L^\infty(\Omega)$.

Then either no solutions, or infinitely many solutions of problem (3.1.3) exist.

REMARK 3.2.19. A natural choice in Theorem 3.2.18 is $\overline{\rho}(\eta) = \eta^{-\sigma}$, $\sigma < 2$.

Theorem 3.2.18 is proved by constructing a positive supersolution V of problem (3.1.5) with the properties assumed in Theorem 3.1.1. Clearly, V is also a supersolution of equation (3.2.4) - namely, we can take $F = V$ in Theorem 3.2.5, which gives (with $\mathcal{S}_1 = \emptyset$) sufficient conditions for the existence of solutions to problem (3.1.3). Then from Theorem 3.2.18 we immediately obtain the following nonuniqueness result.

THEOREM 3.2.20. *Let the assumptions of Theorem 3.2.18 be satisfied. In addition, suppose* $\phi \in L^\infty(\Omega)$. *Then infinitely many solutions of problem (3.1.3) exist.*

Indeed in the proof of Theorem 3.2.18 we construct a supersolution V which shows that $\mathcal{S}_1 \subseteq \partial\overline{\Omega}$ is attracting. This suggests a different approach, *i.e.*, assuming $\mathcal{S}_1 \subseteq \Sigma_2$ and using Proposition 3.2.7 (with Σ a connected component of $\mathcal{S}_1$) to prove its attractivity, instead of assuming (E_3) and "high diffusion near $\mathcal{S}_1$" as in (3.2.20)-(3.2.21). In this way we obtain the following nonuniqueness result.

THEOREM 3.2.21. *Let* $\dim \mathcal{S} = n-1$, $\mathcal{S}_1 \neq \emptyset$ *and* $\mathcal{S}_1 \subseteq \Sigma_2$. *Let assumptions* (A_1), (E_1), $(F_1) - (F_2)$ *be satisfied and* $\sigma_{ij} \in C^1\big(\overline{\mathcal{S}_1^\varepsilon}\big)$ *for some* $\varepsilon > 0$ $(i,j = 1,\dots,n)$. *In addition, suppose:*
(a) $c(x) \geq c_0 > 0$ *for any* $x \in \Omega$;
(b) $\phi \in L^\infty(\Omega)$;
(c) $\gamma \in C(\mathcal{R})$.
Then infinitely many solutions of problem (3.1.3) exist.

REMARK 3.2.22. The assumptions of Proposition 3.2.7 and hence of Theorem 3.2.21 can be weakened assuming $(A_2) - (A_3)$ instead of $(F_1) - (F_2)$, and supposing in addition:
(a) $a_{ij} \in C^{1,1}(\Omega \cup \mathcal{R} \cup \mathcal{S}_1)$, $b_i \in C^{0,1}(\Omega \cup \mathcal{R} \cup \mathcal{S}_1)$;
(b) $0 < \rho_0 \leq \rho(x) \leq \rho_1 < +\infty$ for any $x \in \mathcal{S}_1^\varepsilon$;
(c) $\rho \in C^{1,1}\big(\overline{\mathcal{S}_1^\varepsilon}\big)$, $c \in L^\infty(\mathcal{S}_1^\varepsilon)$ for some $\varepsilon > 0$.

Theorem 3.2.21 establishes nonuniqueness for problem (3.1.3), if boundary data are not prescribed on points of $\mathcal{S}$ where drift trajectories point outwards. On the other hand, there is uniqueness of *bounded* solutions to (3.1.3), if drift trajectories do not point outwards at any point of $\mathcal{S}$; this is a particular consequence of the following theorem, which relies on Proposition 3.2.9 and Remark 3.2.6.

THEOREM 3.2.23. *Let* $\dim \mathcal{S} = n-1$, $\mathcal{S}_2 \neq \emptyset$ *and* $\mathcal{S}_2 \subseteq \Sigma_1$. *Let assumptions* (A_1), (E_2), $(F_1) - (F_2)$ *be satisfied and* $\sigma_{ij} \in C^1\big(\overline{\mathcal{S}_2^\varepsilon}\big)$ *for some* $\varepsilon > 0$ $(i,j = 1,\dots,n)$; *moreover, suppose* $c(x) \geq c_0 > 0$ *for any* $x \in \Omega$, $\gamma \in C(\mathcal{R} \cup \mathcal{S}_1)$. *Then there exists at most one solution of problem* (3.1.7) *satisfying condition (3.2.11) ($\alpha > 0$ sufficiently small). In particular, there exists at most one bounded solution of problem* (3.1.7).

A comparison between the results of Theorems 3.2.16, 3.2.20 and those of Theorems 3.2.21, 3.2.23 is given in Section 3.6, Example (c).

3.3. Parabolic problems

Results analogous to those above hold for parabolic problems (3.1.4), (3.1.8); the present section is devoted to the statement of the main of them.

We always assume the coefficients of the operator $\mathcal{L}$ to be independent of time. Concerning coefficients and data of the problems, let us state the counterpart of assumption (A_2) for the parabolic problem, namely

$$(A_5)\qquad \begin{cases} (i)\ \ \rho\in C^{1,1}(\Omega\cup\mathcal{R}),\ \rho>0\ \text{ in }\Omega\cup\mathcal{R}\,;\\ (ii)\ \ a_{ij}=a_{ji}\in C^{1,1}(\Omega\cup\mathcal{R})\cap C^{0,1}(\overline{\Omega}),\\ \qquad b_i\in C^{0,1}(\Omega\cup\mathcal{R})\cap L^\infty(\Omega)\ (i,j=1,\dots,n)\,;\\ (iii)\ \ c\in C(\Omega\cup\mathcal{R}),\ c\geq c_1>-\infty;\\ (iv)\ \ f\in C(\bar\Omega\times[0,T])\,;\\ (v)\ \ g\in C(\mathcal{R}\times[0,T]),\ u_0\in C(\Omega\cup\mathcal{R})\,;\\ (vi)\ \ g(x,0)=u_0(x)\ \text{for any}\ x\in\mathcal{R}\ , \end{cases}$$

where (i), (ii) and (iii) coincide with those in (A_2), apart from the sign condition on c which is not needed anymore. Concerning ellipticity, we require the following weaker assumption (which coincides with $(E_2)-(i)$, (ii)):

$$(E_4)\qquad \begin{cases} (i)\ \ \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j\geq 0 \text{ for any } x\in\Omega \text{ and } (\xi_1,..,\xi_n)\in I\!R^n\,,\\ \quad\ \sigma_{ij}\in C^1(\Omega)\ \ (i,j=1,\dots,n)\,;\\ (ii)\ \ \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j>0 \text{ for any } x\in\mathcal{R} \text{ and } (\xi_1,..,\xi_n)\neq 0\,. \end{cases}$$

Let us make the following definitions.

Definition 3.3.1. *By a* subsolution *to equation (3.1.2) we mean any function* $u\in C(\Omega\times(0,T])$ *such that*

$$\int_{\Omega\times(0,T)} u\,\{\mathcal{M}^*\psi-\rho c\psi+\rho\partial_t\psi\}\,dx\,dt\ \geq\int_{\Omega\times(0,T)}\rho f\psi\,dx\,dt \tag{3.3.1}$$

for any $\psi\in C_0^\infty(\Omega\times(0,T))$, $\psi\geq 0$. Supersolutions *of (3.1.2) are defined replacing "$\geq$" by '$\leq$" in (3.3.1). A function u is a* solution *of (3.1.2) if it is both a sub- and a supersolution.*

Definition 3.3.2. *Let* $\mathcal{R}\subseteq\mathcal{E}\subseteq\partial\Omega$, $g\in C(\mathcal{E}\times[0,T])$, $u_0\in C(\Omega\cup\mathcal{E})$, $g(x,0)=u_0(x)$ $(x\in\mathcal{E})$. *By a* subsolution *to the problem*

$$\begin{cases} \mathcal{L}u-cu-\partial_t u=f & \text{in }\Omega\times(0,T)\\ u=g & \text{in }\ \mathcal{E}\times(0,T]\\ u=u_0 & \text{in }\ \big(\Omega\cup\mathcal{E}\big)\times\{0\} \end{cases} \tag{3.3.2}$$

we mean any function $u\in C((\Omega\cup\mathcal{E})\times[0,T])$ *such that:*
(i) *u is a subsolution of equation (3.1.2);*
(ii) $u\leq g$ *in* $\mathcal{E}\times(0,T]$, $u\leq u_0$ *in* $(\Omega\cup\mathcal{E})\times\{0\}$.
Supersolutions *and* solutions *of (3.3.2) are defined accordingly.*

Our results rely on the following theorems, which are the parabolic counterpart of Theorem 3.1.1 and 3.1.2, respectively (see [**59**] for the proof).

Theorem 3.3.3. *Let assumptions* (A_1), (A_3), (A_5) *and* (E_1) *be satisfied. Suppose* $g\in L^\infty(\mathcal{R}\times(0,T))$, u_0, $c\in L^\infty(\Omega)$. *Let there exist a supersolution V of problem* (3.1.5) *such that* (3.1.6) *is satisfied. Then there exist infinitely many bounded solutions of problem (3.1.4).*

Theorem 3.3.4. *Let assumptions* (A_1), (A_3), (A_5) *and* (E_4) *be satisfied. Suppose* $\mathcal{S}_2\neq\emptyset$, $g\in C([\mathcal{R}\cup\mathcal{S}_1]\times[0,T])$, $u_0\in C(\Omega\cup\mathcal{R}\cup\mathcal{S}_1)$, $g(x,0)=u_0(x)$ *for any* $x\in\mathcal{R}\cup\mathcal{S}_1$. *Let there exist a subsolution* $Z\leq H<0$ *of problem*

$$\begin{cases} \mathcal{L}u=\mu u & \text{in }\ \Omega\\ u=0 & \text{on }\ \mathcal{R}\,, \end{cases} \tag{3.3.3}$$

for some $\mu \geq 0$*, or of problem* (3.1.5)*. Then there exists at most one solution* u *of problem (3.1.8) such that*

$$\lim_{dist(x,\mathcal{S}_2)\to 0} \frac{\sup_{t\in(0,T]}|u(x,t)|}{Z(x)} = 0\,. \tag{3.3.4}$$

If $dim\,\mathcal{S} \leq n-2$, from Theorem 3.3.4 we obtain the following analogous of Theorem 3.2.12.

THEOREM 3.3.5. *Let assumptions* (A_1)*,* $(A_3)-(A_5)$ *and* (E_4) *be satisfied, with* $(A_5)-(ii)$ *replaced by* $(A_2)-(ii)'$*. Then:*
(i) *there exists at most one solution* u *of problem (3.1.4) such that*

$$\lim_{d(x,\mathcal{S})\to 0} \frac{\sup_{t\in(0,T]}|u(x,t)|}{\log[d(x,\mathcal{S})]} = 0\,; \tag{3.3.5}$$

(ii) *if* $\alpha := \inf_{y\in\mathcal{S}} r(y) - 2 \geq 1$*, there exists at most one solution* u *of problem (3.1.4) such that*

$$\lim_{d(x,\mathcal{S})\to 0} \frac{\sup_{t\in(0,T]}|u(x,t)|}{[d(x,\mathcal{S})]^{-\alpha}} = 0\,. \tag{3.3.6}$$

In particular, problem (3.1.4) has at most one bounded solution.

If $dim\,\mathcal{S} = n-1$, the following results (to be compared with Theorems 3.2.16 and 3.2.20) can be proved.

THEOREM 3.3.6. *Let* $dim\,\mathcal{S} = n-1$*; let assumptions* (A_1)*,* (A_3)*,* (A_5) *and* (E_4) *be satisfied. Assume* $\mathcal{S}_2 \neq \emptyset$*,* $g \in C([\mathcal{R}\cup\mathcal{S}_1]\times[0,T])$*,* $u_0 \in C(\Omega\cup\mathcal{R}\cup\mathcal{S}_1)$*,* $g(x,0) = u_0(x)$ *for any* $x \in \mathcal{R}\cup\mathcal{S}_1$*. In addition, let there exist* $\overline{\varepsilon} > 0$ *and a positive, continuous function* $\underline{\rho}$ *satisfying* (3.2.16)-(3.2.17)*. Then there exists at most one solution* u *of problem (3.1.8) such that*

$$\lim_{d(x,\mathcal{S}_2)\to 0} \frac{\sup_{t\in(0,T]}|u(x,t)|}{P\big(d(x,\mathcal{S}_2)\big)} = 0\,, \tag{3.3.7}$$

with P *defined in* (3.2.18)*.*
In particular, problem (3.1.8) has at most one bounded solution.

THEOREM 3.3.7. *Let* $dim\,\mathcal{S} = n-1$*. Let assumptions* (A_1)*,* (A_3)*,* (A_5) *and* (E_3) *be satisfied. Assume* $\mathcal{S}_2 = \emptyset$*,* $g \in L^\infty(\mathcal{R}\times(0,T))$*,* $u_0, c \in L^\infty(\Omega)$*. In addition, let there exist* $\overline{\varepsilon} > 0$ *and a positive continuous function* $\overline{\rho}$ *satisfying* (3.2.20)-(3.2.21)*. Then infinitely many bounded solutions of problem (3.1.4) exist.*

Let us mention also the parabolic counterpart of Theorem 3.2.21 and 3.2.23, respectively.

THEOREM 3.3.8. *Let* $dim\,\mathcal{S} = n-1$*,* $\mathcal{S}_1 \neq \emptyset$ *and* $\mathcal{S}_1 \subseteq \Sigma_2$*. Let assumptions* (A_1)*,* $(A_5)(iv)-(vi)$*,* (E_1)*,* $(F_1)-(F_2)$ *be satisfied and* $\sigma_{ij} \in C^1\big(\overline{\mathcal{S}_1^{\varepsilon}}\big)$ *for some* $\varepsilon > 0$ $(i,j = 1,\ldots,n)$*. In addition, suppose* $c, u_0 \in L^\infty(\Omega)$*. Then infinitely many solutions of problem (3.1.4) exist.*

THEOREM 3.3.9. *Let* $dim\,\mathcal{S} = n-1$*,* $\mathcal{S}_2 \neq \emptyset$ *and* $\mathcal{S}_2 \subseteq \Sigma_1$*. Let assumptions* (A_1)*,* $(A_5)(iv)-(vi)$*,* (E_4)*,* $(F_1)-(F_2)$ *be satisfied and* $\sigma_{ij} \in C^1\big(\overline{\mathcal{S}_2^{\varepsilon}}\big)$ *for some* $\varepsilon > 0$ $(i,j = 1,\ldots,n)$*; moreover, suppose* $g \in C([\mathcal{R}\cup\mathcal{S}_1]\times[0,T])$*,* $u_0 \in C(\Omega\cup\mathcal{R}\cup\mathcal{S}_1)$*,* $u_0(x) = g(x,0)$ *for any* $x \in \mathcal{R}\cup\mathcal{S}_1$*. Then there exists at most one solution of problem* (3.1.8) *satisfying*

$$\lim_{d(x,\mathcal{S}_2)\to 0} \frac{\sup_{t\in(0,T]}|u(x,t)|}{[d(x,\mathcal{S}_2)]^{-\alpha}} = 0\,, \tag{3.3.8}$$

for some $\alpha > 0$*. In particular, there exists at most one bounded solution of problem* (3.1.8)*.*

REMARK 3.3.10. In Theorem 3.2.23 the parameter $\alpha > 0$ must be sufficiently small, while in Theorem it is arbitrary. This depends on the fact that to prove the former a subsolution of problem (3.1.9) must be constructed, while for the latter a subsolution of problem (3.3.3) for some $\mu \geq 0$ is needed.

The proofs of Theorems 3.3.5-3.3.9 are the same of those given below for the elliptic case, with obvious changes.

3.4. Singular manifolds of low dimension: Proofs

This section is devoted to the proof of Theorem 3.2.12. As already said, this follows from Theorem 3.1.2 if we exhibit a suitable subsolution Z of problem (3.1.9). The construction of Z is rather technical and lengthy; it requires the following steps:
(a) first we construct a local subsolution of the equation $\mathcal{L}u = 0$ - namely, for any $\hat{y} \in \mathcal{S}$ we construct a subsolution z_0 in $B_{\hat{R}}(\hat{y}) \cap \Omega$, where $B_{\hat{R}}(\hat{y})$ is a ball of radius $\hat{R} > 0$ sufficiently small;
(b) using the compactness of $\mathcal{S}$ and (a) above, we construct a subsolution z of the same equation in a neighbourhood $\mathcal{S}^\varepsilon$ ($\varepsilon > 0$ sufficiently small);
(c) finally, we extend the subsolution z in $\mathcal{S}^\varepsilon$ to a subsolution Z of problem (3.1.9) with the desired properties.

3.4.1. Technical preliminaries. In view of the compactness and regularity of $\mathcal{S}$ assumed in $(A_1) - (iv)$, the following holds (see [**28**] for the proof).

LEMMA 3.4.1. *Let assumption (A_1) be satisfied. Then there exists $\sigma > 0$ with the following properties:*
(i) for any $x \in \mathcal{S}^\sigma$ there exists a unique point $x^*(x) \in \mathcal{S}$ such that

$$d(x, \mathcal{S}) = |x - x^*(x)|\,;$$

(ii) $x^*(\cdot) \in C^2(\mathcal{S}^\sigma; \mathcal{S})$, $d(\cdot, \mathcal{S}) \in C^3(\mathcal{S}^\sigma)$ and

$$\nabla[d(x, \mathcal{S})]^2 = 2\big[x - x^*(x)\big] \qquad (x \in \mathcal{S}^\sigma)\,.$$

Let $y^0 \in \mathcal{S}$; let $T_{y^0}\,\mathcal{S}$ and $\perp_{y^0}\mathcal{S}$ denote the tangent, respectively the orthogonal space to $\mathcal{S}$ at y^0.

In view of the compactness of $\mathcal{S}$, we can choose possibly smaller R_i in the representation (3.2.12), such that

$$\left|\frac{\partial^{|\alpha|}\phi^{(i)}}{\partial y_1^{\alpha_1} \dots \partial y_k^{\alpha_k}}\right| \leq C_0 \quad \text{in } \overline{B}_{R_i}(y_1^i, \dots, y_k^i) \qquad (|\alpha| \leq 3\,;\, i = 1, \dots, N) \tag{3.4.1}$$

for some $C_0 > 0$ ($\alpha \equiv (\alpha_1, \dots, \alpha_k)$ denoting a multiindex).

It is convenient to point out for further reference a few technical observations; this is the content of the following remark.

REMARK 3.4.2. Let $\bar{y} \in \mathcal{S}$. Then there exists $i \in \{1, \dots, N\}$ such that

$$\bar{y} = \big(\bar{y}_1, \dots, \bar{y}_k, \phi^{(i)}(\bar{y}_1, \dots, \bar{y}_k)\big)\,.$$

Take the orthonormal vectors in $(A_4) - (iii)$ and construct a complete basis of orthonormal vectors $\eta^1(\cdot), \dots, \eta^n(\cdot) \in C^2(B_R(\bar{y}_1, \dots, \bar{y}_k))$, where we may assume that, for $\zeta \in B_R(\bar{y}_1, \dots, \bar{y}_k)$ (for some $R = R(i, \bar{y}) > 0$), $\eta^1(\zeta), \dots, \eta^k(\zeta)$ form a basis in the tangential subspace and $\eta^{k+1}(\zeta), \dots, \eta^n(\zeta)$ form a basis in the orthogonal subspace to $\mathcal{S}$ at $(\zeta, \phi^{(i)}(\zeta))$. In the following we use for simplicity the notation $\eta^l(y), y \in \mathcal{S}$ $(l = 1, \dots, n)$ as in $(A_4) - (iii)$.

The tangent space $T_{\bar{y}}\,\mathcal{S}$ to $\mathcal{S}$ at $\bar{y}$ can be taken into the linear subspace $\{Y \in I\!R^n \,|\, Y_{k+1} = ... = Y_n = 0\}$ by a transformation of coordinates $Y := M(\bar{y})\big(y - \bar{y}\big)$, where the rotation matrix is given by $M(\bar{y}) = (\eta^1(\bar{y}) \dots \eta^n(\bar{y}))^T \in \mathcal{M}_n$. This matrix valued function belongs to C^2 in some set $B_R(\bar{y}) \cap \mathcal{S}$, *i.e.*, its composition with the local representation of $\mathcal{S}$ is $C^2(\overline{B}_R(\bar{y}_1, \ldots, \bar{y}_k))$. Let the C^3 functions $p = p^{(i,\bar{y})} : \overline{U_{\bar{y}}^{(i)}} \to I\!\!R^{n-k}$, $p \equiv (p_{k+1}, \ldots, p_n)(Y_1, \ldots, Y_k)$ give a local representation of $\mathcal{S}$ in the closure of a bounded neighbourhood of 0, say in $\overline{U_{\bar{y}}^{(i)}} \subset I\!\!R^k$. Then

$$(Y_1, \ldots, Y_k, p_{k+1}, \ldots, p_n) = \bar{M}^{(i)}\big((y_1, \ldots, y_k, \phi^i_{k+1}, \ldots, \phi^i_n) - \bar{y}\big),$$

$p_{k+1}, \ldots, p_n$ being evaluated at $(Y_1, \ldots, Y_k)$ and $(\phi^i_{k+1}, \ldots, \phi^i_n)$ at $(y_1, \ldots, y_k)$. Moreover, since we compose regular functions in compact sets, there exists $C_1 > 0$ such that

$$\Big|\frac{\partial^{|\alpha|} p^{(i,\bar{y})}}{\partial Y_1^{\alpha_1} \ldots \partial Y_k^{\alpha_k}}\Big| \leq C_1 \quad \text{in } \overline{U_{\bar{y}}^{(i)}} \quad (|\alpha| \leq 3\,;\, i = 1, \ldots, N\,;\, \bar{y} \in \mathcal{S}). \tag{3.4.2}$$

In fact, by a change of $\bar{y}$ the tangent space $T_{\bar{y}}\mathcal{S}$, the $Y-$ coordinates and functions $p^{(i,\bar{y})}$ also change; however, inequality (3.4.2) holds true with a suitable choice of the constant C_1 independent from $\bar{y}$, i and α.

Let $\sigma > 0$ as in Lemma 3.4.1 and $x^0 \in \mathcal{S}^\sigma$ be fixed; then the projection $x^*(x^0) \in \mathcal{S}$ is well defined and there exists $i \in \{1, \ldots, N\}$ such that

$$x^*(x^0) = \big(\bar{y}_1, \ldots, \bar{y}_k, \phi^{(i)}(\bar{y}_1, \ldots, \bar{y}_k)\big).$$

With the notations of the above remark, let $p^{(i,x^*(x^0))}$ be the local representation of $\mathcal{S}$ in the neighbourhood $\overline{U}_0 \equiv \overline{U^{(i)}_{x^*(x^0)}}$ of 0 in $I\!\!R^k$. As we make below, the new coordinate system $X \equiv (X_1, \ldots, X_n)$ can be chosen in $I\!\!R^n$ so that, if $p : \overline{U}_0 \to I\!\!R^{n-k}$ denotes the local representation of $\mathcal{S}$ with respect to this system, the following holds:

$$(C) \qquad \begin{cases} (i) & X^*(X^0) = 0\,; \\ (ii) & \perp_0 \mathcal{S} = \{X \in I\!\!R^n \,|\, X_1 = \ldots = X_k = 0\}\,; \\ (iii) & X^0 \equiv (0, \ldots, 0, X^0_n),\ d(X^0, \mathcal{S}) = X^0_n\,; \\ (iv) & \dfrac{\partial^2 p_n}{\partial X_i \partial X_j}(0) = p^{ii}_n \delta_{ij} \qquad (i, j = 1, \ldots, k); \end{cases}$$

here X^0, $X^*(X^0)$ denote the new coordinates of the points x^0, $x^*(x^0)$. In fact, equalities (i) and (ii) also hold in the $Y-$ coordinates, whereas (iii) can be obtained up to a rotation in the orthogonal space $\{Y \in I\!\!R^n \,|\, Y_1 = \ldots = Y_k = 0\}$ and (iv) by a rotation in the tangent space $\{Y \in I\!\!R^n \,|\, Y_{k+1} = \ldots = Y_n = 0\}$. Hence we have the analogous of inequality (3.4.2) for some constant $C_2 > 0$ independent of $x^0 \in \mathcal{S}^\sigma$, namely

$$\Big|\frac{\partial^{|\alpha|} p_s}{\partial X_1^{\alpha_1} \ldots \partial X_k^{\alpha_k}}\Big| \leq C_2 \quad \text{in } \overline{U}_0 \qquad (|\alpha| \leq 3; s = k+1, \ldots, n). \tag{3.4.3}$$

In the following we set

$$p^l_s \equiv \frac{\partial p_s}{\partial X_l}(0), \qquad p^{lq}_s \equiv \frac{\partial^2 p_s}{\partial X_l \partial X_q}(0)\,, \tag{3.4.4}$$

and so on. Then the choice (C) implies

$$p_s(0) = 0\,, \quad p^l_s = 0 \qquad (s = k+1, \ldots, n;\ l = 1, \ldots, k)\,. \tag{3.4.5}$$

The following lemma deals with derivatives of the projection map $X^*(\cdot)$, respectively of the distance $d(\cdot, \mathcal{S})$. Part of its proof was given in [**54**]; we reproduce it here for convenience of the reader.

LEMMA 3.4.3. *Let assumption (A_1) be satisfied. There exists ε_0 such that, if $\varepsilon \in \big(0, \varepsilon_0\big)$, $x^0 \in \mathcal{S}^\varepsilon$ is fixed and the choice (C) is made, the following holds:*

(i) for any $i = 1, \dots, n$

$$\frac{\partial X_l^*}{\partial X_i}(X^0) = \begin{cases} \dfrac{\delta_{il}}{1 - X_n^0 p_n^{ll}} & \text{if } \; l = 1, \dots, k\,, \\ 0 & \text{if } \; l = k+1, \dots, n\,; \end{cases} \tag{3.4.6}$$

(ii) for any $i, j = 1, \dots, n$

$$\frac{\partial^2 X_l^*}{\partial X_i \partial X_j}(X^0) = \frac{1}{1 - X_n^0 p_n^{ll}} \Big\{ \sum_{s=k+1}^{n} \sum_{q=1}^{k} p_s^{lq} \, \frac{\delta_{is}\delta_{jq} + \delta_{iq}\delta_{js}}{1 - X_n^0 p_n^{qq}} +$$

$$+ p_n^{ijl} \, \frac{X_n^0 \sum_{m,q=1}^{k} \delta_{im}\delta_{jq}}{(1 - X_n^0 p_n^{ii})(1 - X_n^0 p_n^{jj})} \Big\} \qquad \text{if } \; l = 1, \dots, k\,, \tag{3.4.7}$$

respectively

$$\frac{\partial^2 X_l^*}{\partial X_i \partial X_j}(X^0) = \sum_{q,r=1}^{k} p_l^{qr} \, \frac{\delta_{iq}\delta_{jr}}{(1 - X_n^0 p_n^{ii})(1 - X_n^0 p_n^{jj})} \qquad \text{if } \; l = k+1, \dots, n\,. \tag{3.4.8}$$

(iii) for any $i = 1, \dots, n$

$$\frac{\partial d(X, \mathcal{S})}{\partial X_i}\Big|_{X = X^0} = \delta_{in}\,; \tag{3.4.9}$$

(iv) there holds

$$\frac{\partial^2 d(X, \mathcal{S})}{\partial X_i \partial X_j}\Big|_{X = X^0} = \begin{cases} -\frac{p_n^{ii}}{1 - d(X^0, \mathcal{S}) p_n^{ii}} \, \delta_{ij} & \text{if } i, j = 1, \dots, k\,, \\ \dfrac{\delta_{ij} - \delta_{in}\delta_{jn}}{d(X^0, \mathcal{S})} & \text{if } i, j = k+1, \dots, n\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{3.4.10}$$

In the following by $O(|X|)$, $O(|X|^2)$, ... we denote functions of X such that for some constant $D > 0$

$$|O(|X|)| \le D\,|X|, \; O(|X|^2) \le D\,|X|^2, \; \dots \; . \tag{3.4.11}$$

Remark 3.4.4. Omitting assumptions $(C) - (iii)$ and $(C) - (iv)$ in Lemma 3.4.3 amounts to deal, for any $x^0 \in \mathcal{S}^\varepsilon$, with the $Y-$ coordinates mentioned in Remark 3.4.2 with $\bar{y} = x^*(x^0)$. Hence, if Y^0, Y, $Y^*(Y)$ respectively correspond to x^0, x, $x^*(x)$ in the new coordinates, we have

$$\begin{gathered} Y^0 \equiv (0, \dots, 0, Y_{k+1}^0, \dots, Y_n^0)\,, \qquad Y^*(Y^0) = 0\,, \\ d(Y^0, \mathcal{S}) = |Y^0 - Y^*(Y^0)| = |Y^0|\,. \end{gathered} \tag{3.4.12}$$

Therefore equalities (3.4.5) are still valid. From Lemma 3.4.3, going back from $X-$ coordinates to $Y-$ coordinates, *i.e.* performing rotations in the tangential and orthogonal subspaces to $\mathcal{S}$ at X^0, gives:

$$\begin{aligned} &\frac{\partial (Y_l - Y_l^*(Y))}{\partial Y_i}\Big|_{Y = Y^0} = \\ &\qquad = \delta_{il} - \frac{\partial Y_l^*(Y^0)}{\partial Y_i} = \begin{cases} O(|Y^0|) & \text{if } \; i, \, l \le k \\ \delta_{il} & \text{if } \; l \ge k+1\,. \end{cases} \end{aligned} \tag{3.4.13}$$

In the above equalities the constant D related to $O(|Y^0|)$ can be chosen independent from $x^0 \in \mathcal{S}^\epsilon$.

Let us introduce for further reference the following notation. For any $i, j = 1, \dots, n$ set $[i, j] := \{i, i+1, \dots, j-1, j\} \subseteq I\!N$,

$$\chi_{[i,j]}(l) := \begin{cases} 1 & \text{if} \quad i \le l \le j \\ 0 & \text{otherwise} \end{cases} \qquad (l \in I\!N)$$

(namely, $\chi_{[i,j]}$ is the characteristic function of the string $[i, j]$). Then (3.4.13) reads

$$\frac{\partial (Y_l - Y_l^*(Y))}{\partial Y_i}\Big|_{Y^0} = \delta_{il} - \frac{\partial Y_l^*}{\partial Y_i}(Y^0) = O(|Y^0|)\chi_{[1,k]}(l)\chi_{[1,k]}(i) + \delta_{il}\chi_{[k+1,n]}(l)\,, \tag{3.4.14}$$

where we also used (3.4.12). Likewise,

$$\begin{aligned} \frac{\partial^2 Y_l^*(Y)}{\partial Y_i\, \partial Y_j}\Big|_{Y^0} = B_{lij}(Y^0)\Big\{ & \chi_{[1,k]}(l)\big(1 - \chi_{[k+1,n]}(i)\chi_{[k+1,n]}(j)\big) + \\ & + \chi_{[k+1,n]}(l)\chi_{[1,k]}(i)\chi_{[1,k]}(j)\Big\}\,, \end{aligned} \tag{3.4.15}$$

where the fuctions $B_{lij}(.)$ are uniformly bounded for Y^0 corresponding to any $x^0 \in \mathcal{S}^\varepsilon$.

Now we can prove Lemma 3.4.3.

Proof of Lemma 3.4.3. Let

$$\varepsilon_0 := \min\{\sigma, \frac{1}{2C_2}\}\,, \tag{3.4.16}$$

where $\sigma > 0$ is given in Lemma 3.4.1 and $C_2 > 0$ in (3.4.3). Suppose that $x^0 \in \mathcal{S}^\varepsilon$ ($\varepsilon \in (0, \varepsilon_0)$) is fixed and the choice (C) has been made.

(i) Let e_i be the unit vector of the i-th coordinate axis ($i = 1, \dots, n$). Since the map $X^*(\cdot)$ is continuous by Lemma 3.4.1-(ii), there exists $\eta > 0$ such that $X^*(X^0 + h_i e_i + h_j e_j) \in \{(X, p(X)) \,|\, X \in U_0\}$ for any $h_i, h_j \in (-\eta, \eta)$ $(i, j = 1, \dots, n)$. This implies:

$$\big|X^0 + h_i e_i + h_j e_j - X^*(X^0 + h_i e_i + h_j e_j)\big|^2 = \min_{X \in U_0} \big|X^0 + h_i e_i + h_j e_j - (X, p(X))\big|^2\,,$$

thus

$$\frac{\partial}{\partial X_l}\big|X^0 + h_i e_i + h_j e_j - (X, p(X))\big|^2\Big|_{(X,p(X)) = X^*(X^0 + h_i e_i + h_j e_j)} = 0 \tag{3.4.17}$$

for any $l = 1, \dots, k$ and i, j, h_i, h_j as above.

Let us prove the following

Claim: For any $i, j = 1, \dots, n$ there exist $\eta' \in (0, \min\{\eta, R_m\})$ and $\varphi \equiv (\varphi_1, \dots, \varphi_k) \in C^2((-\eta', \eta')^2; I\!R^k)$, $\varphi = \varphi(h_i, h_j)$ such that

$$X^*(X^0 + h_i e_i + h_j e_j) = (\varphi(h_i, h_j), p(\varphi(h_i, h_j))) \tag{3.4.18}$$

for any $(h_i, h_j) \in (-\eta', \eta')^2$.

In fact, for any fixed $i, j = 1, \dots, n$ define $F \equiv F^{i,j} : (-\eta, \eta)^2 \times U_0 \to I\!R^k$, $F \equiv (F_1, \dots, F_k)$ as follows:

$$\begin{aligned} F_l(h_i, h_j, X) &:= \frac{\partial}{\partial X_l}\big|(X^0 + h_i e_i + h_j e_j) - (X, p(X))\big|^2 = \\ &= -2\big(h_i \delta_{il} + h_j \delta_{jl} - X_l\big) - 2\sum_{s=k+1}^{n} \big(X_s^0 + h_i \delta_{is} + h_j \delta_{js} - p_s(X)\big)\frac{\partial p_s}{\partial X_l}(X)\,. \end{aligned}$$

An elementary calculation gives:

$$\frac{\partial F_l}{\partial X_h}(h_i, h_j, X) = 2\delta_{hl} + 2\sum_{s=k+1}^{n} \Big[\frac{\partial p_s}{\partial X_h}(X)\frac{\partial p_s}{\partial X_l}(X) - \big(X_s^0 + h_i \delta_{is} + h_j \delta_{js} - p_s(X)\big)\frac{\partial^2 p_s}{\partial X_h \partial X_l}(X)\Big]\,,$$

$$\frac{\partial F_l}{\partial h_i}(h_i,h_j,X) = -2\delta_{il} - 2\sum_{s=k+1}^{n}\delta_{is}\frac{\partial p_s}{\partial X_l}(X)$$

for any $h,l=1,\dots,k$, $i=1,\dots,n$. In particular,

$$\frac{\partial F_l}{\partial X_h}(0,0,0) = 2\big(1 - X_n^0 p_n^{hh}\big)\delta_{hl} \qquad (h,l=1,\dots,k)\,, \tag{3.4.19}$$

$$\frac{\partial F_l}{\partial h_i}(0,0,0) = -2\delta_{il} \qquad (l=1,\dots,k;\ i=1,\dots,n) \tag{3.4.20}$$

(here use of (C) and (3.4.5) has been made).

By (3.4.16) and (3.4.3)

$$0 \le X_n^0 \le \varepsilon \le \frac{1}{2C_2} \le \frac{1}{2|p_n^{hh}|}\,,$$

hence from (3.4.3) and (3.4.19) we obtain

$$\Big|\frac{\partial(F_1,\dots,F_k)}{\partial(X_1,\dots,X_k)}\Big|(0,0,0) \ge 1\,.$$

Then by the Implicit Function Theorem there exists $\eta' \in (0,\eta)$ and uniquely defined functions $\varphi_1,\dots,\varphi_k \in C^2((-\eta',\eta')^2)$ such that $\varphi_l(0,0)=0$, $(l=1,\dots,k)$ and

$$F(h_j,h_j,\varphi_1(h_i,h_j),\dots,\varphi_k(h_i,h_j)) = 0, \tag{3.4.21}$$

for any $(h_i,h_j)\in(-\eta',\eta')^2$. Thus

$$X_l^*(X^0 + h_i e_i + h_j e_j) = \varphi_l(h_i,h_j)$$

for any $s=1,\dots,k$, $(h_i,h_j)\in(-\eta',\eta')^2$. Hence equality (3.4.18) and the Claim follow.

Let $i=1,\dots,n$. From (3.4.18)-(3.4.20) we plainly have:

$$\frac{\partial X_l^*}{\partial X_i}(X^0) = \frac{\partial X_l^*}{\partial h_i}(X^0 + h_i e_i)\Big|_{h_i=0} = \frac{\partial \varphi_l}{\partial h_i}(0,0) \;=\; \frac{\delta_{il}}{1 - X_n^0 p_n^{ll}}\,, \tag{3.4.22}$$

if $l=1,\dots,k$, respectively:

$$\frac{\partial X_l^*}{\partial X_i}(X^0) = \frac{\partial}{\partial h_i} p_l(\varphi(h_i,0))\Big|_{h_i=0} = \sum_{m=1}^{k} p_l^m \frac{\partial \varphi_m}{\partial h_i}(0,0) = 0\,, \tag{3.4.23}$$

if $l=k+1,\dots,n$ (here use of (3.4.5) has been made). Hence equality (3.4.6) follows.

(ii) In view of (3.4.18), for any $i,j=1,\dots,n$, $l=1,\dots,k$, we have:

$$\frac{\partial^2}{\partial h_i \partial h_j} F_l(h_i,h_j,\varphi(h_i,h_j))\Big|_{h_i=h_j=0} = 0. \tag{3.4.24}$$

As a lengthy calculation shows, the above equality reads:

$$\frac{\partial^2 \varphi_l}{\partial h_i \partial h_j}(0,0) - \sum_{s=k+1}^{n}\sum_{q=1}^{k} p_s^{lq}\,\frac{\delta_{is}\delta_{jq} + \delta_{iq}\delta_{js}}{1 - X_n^0 p_n^{qq}} +$$

$$-p_n^{ijl}\,\frac{X_n^0\sum_{m,q=1}^{k}\delta_{im}\delta_{jq}}{(1 - X_n^0 p_n^{ii})(1 - X_n^0 p_n^{jj})} - X_n^0\sum_{q=1}^{k} p_n^{lq}\,\frac{\partial^2 \varphi_q}{\partial h_i \partial h_j}(0,0) = 0\,; \tag{3.4.25}$$

hence by (C)–(iv) equality (3.4.7) follows.

In view of (3.4.18), for any $i,j=1,\dots,n$, $l=k+1,\dots,n$ we have:

$$\frac{\partial^2 X_l^*}{\partial X_i \partial X_j}(X^0) = \frac{\partial^2 p_l(\varphi(h_i,h_j))}{\partial h_i \partial h_j}\Big|_{h_i=h_j=0} = \sum_{q,r=1}^{k} p_l^{qr}\,\frac{\delta_{iq}\delta_{jr}}{(1 - X_n^0 p_n^{qq})(1 - X_n^0 p_n^{rr})}$$

(here use of (3.4.5) has been made). Hence (3.4.8) follows by $(C)-(iv)$.

(iii) Observe preliminarly that for any $X \in \mathcal{S}^\varepsilon$

$$d^2(X,\mathcal{S}) = \sum_{l=1}^{n} [X_l - X_l^*(X)]^2 \,,$$

the projection $X^*(X) \in \mathcal{S}$ being well defined by Lemma 3.4.1-(i).

For any $i = 1, \dots, n$

$$\sum_{l=1}^{n} \frac{\partial\,[X_l - X_l^*(X)]^2}{\partial X_i} = 2\sum_{l=1}^{n} \big(X_l - X_l^*(X)\big)\Big(\delta_{il} - \frac{\partial X_l^*}{\partial X_i}(X)\Big) . \tag{3.4.26}$$

By (C) there holds $X_l^0 - X_l^*(X^0) = X_n^0 \delta_{ln}$ $(l = 1, \dots, n)$, thus we get

$$\frac{\partial d^2(X,\mathcal{S})}{\partial X_i}\Big|_{X=X^0} = 2X_n^0 \delta_{in} = 2d(X^0,\mathcal{S})\delta_{in}\,; \tag{3.4.27}$$

here use of (3.4.6) has been made. Hence equality (3.4.9) follows.

(iv) It is easily seen that

$$\frac{\partial^2 d}{\partial X_i \partial X_j} = \frac{1}{2d}\Big(\frac{\partial^2 d^2}{\partial X_i \partial X_j} - \frac{1}{2d^2}\frac{\partial d^2}{\partial X_i}\frac{\partial d^2}{\partial X_j}\Big) \tag{3.4.28}$$

for any $X \in \mathcal{S}^\varepsilon$, $i, j = 1, \dots, n$, (here $d \equiv d(X,\mathcal{S})$ for simplicity). From (3.4.26) we get

$$\sum_{l=1}^{n} \frac{\partial^2\,[X_l - X_l^*(X)]^2}{\partial X_i \partial X_j} = \tag{3.4.29}$$

$$= 2\sum_{l=1}^{n} \Big[\Big(\delta_{jl} - \frac{\partial X_l^*}{\partial X_j}(X)\Big)\Big(\delta_{il} - \frac{\partial X_l^*}{\partial X_i}(X)\Big) - \big(X_l - X_l^*(X)\big)\frac{\partial^2 X_l^*}{\partial X_i \partial X_j}(X)\Big] =: 2(S_1 - S_2)\,.$$

For $X = X^0$, using the choice (C), (3.4.6) and (3.4.8), we obtain easily:

$$S_1 = \begin{cases} \dfrac{(X_n^0 p_n^{ii})^2}{(1 - X_n^0 p_n^{ii})^2}\,\delta_{ij} & \text{if } i,j = 1,\dots,k\,, \\ \delta_{ij} & \text{if } \; i,j = k+1,\dots,n\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

$$S_2 = \begin{cases} \dfrac{X_n^0 p_n^{ii}}{(1 - X_n^0 p_n^{ii})^2}\,\delta_{ij} & \text{if } i,j = 1,\dots,k\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then from (3.4.27)-(3.4.29) equality (3.4.10) easily follows. This completes the proof. □

3.4.2. Proof of Theorem 3.2.12.

3.4.2.1. Let us prove the following proposition, which corresponds to the first step of the costruction of the subsolution Z.

Proposition 3.4.5. *Let the assumptions of Theorem 3.2.12 be satisfied; let $\hat{y} \in \mathcal{S}$ be fixed. Then there exist $\hat{R} > 0$ and $z_0 \in C^2(B_{\hat{R}}(\hat{y}) \cap \Omega)$ such that z_0 is a subsolution of the equation*

$$\mathcal{L}u = 0 \quad \text{in } B_{\hat{R}}(\hat{y}) \cap \Omega \,. \tag{3.4.30}$$

Moreover,

(i) $\sup\limits_{B_{\hat{R}}(\hat{y}) \cap \Omega} z_0 < +\infty$;

(ii) $z_0(x) \sim \log[d(x,\mathcal{S})]$ *as* $d(x,\mathcal{S}) \to 0 \qquad (x \in B_{\hat{R}}(\hat{y}) \cap \Omega)$.

If $\alpha := \inf\limits_{y\in\mathcal{S}} r(y) - 2 \geq 1$, *the same conclusion holds with (ii) replaced by*

(iii) $z_0(x) \sim -[d(x,\mathcal{S})]^{-\alpha}$ *as* $d(x,\mathcal{S}) \to 0 \qquad (x \in B_{\hat{R}}(\hat{y}) \cap \Omega)$.

To prove Proposition 3.4.5 we have to construct suitable matrix functions. Let $\hat{y} \in \mathcal{S}$ be fixed; take $\hat{R} \leq \varepsilon$, with ε as in Lemma 3.4.3. Taking possibly smaller $\hat{R}$ we have $\mathcal{B}_{\hat{R}}(\hat{y}) := B_{\hat{R}}(\hat{y}) \cap \mathcal{S} \subseteq \mathcal{U}_i$ ($i \in \{1,\dots,N\}$; see (3.2.12)). Moreover, by $(A_4)-(ii)$ there exists $\hat{r} \in \{2,\dots,n-k\}$ such that $\hat{r}$ is a lower bound for the orthogonal rank of A in $\mathcal{S}$.

For any $y \in \mathcal{B}_{\hat{R}}(\hat{y})$ set $A \equiv A(y)$ and, using the notation in Remark 3.4.2, $M \equiv M(y) \equiv M^{(i)}(y)$; then $M(\cdot) \in C^2(\mathcal{B}_{\hat{R}}(\hat{y}))$. In the new coordinate system $X := M\big(x - y\big)$ there holds $X^*(y) = 0$,

$$\perp_0 \mathcal{S} = \{X \in I\!R^n \,|\, X_1 = ... = X_k = 0\}.$$

To obtain a convenient representation of the diffusion matrix $A \equiv (a_{ij})$ in the new system, define

$$\tilde{A} := M\,A\,M^T\,; \tag{3.4.31}$$

let $\tilde{\alpha} \in \mathcal{M}_{n-k}$ denote the matrix with entries $(\tilde{A})_{k+i,k+j}$ ($i,j = 1,\dots,n-k$). It is immediately seen that the rank of the matrix $\tilde{\alpha}$ coincides with the orthogonal rank of the matrix A at y. In fact, there holds:

$$(\tilde{A})_{k+i,k+j} \;=\; \langle \tilde{A}\,e_{k+i}, e_{k+j}\rangle \;=\; \langle A\,\eta^{(k+i)}, \eta^{(k+j)}\rangle\,;$$

here $\langle\cdot,\cdot\rangle$ denotes the scalar product in $I\!R^n$, e_h the unit vector of the h-th coordinate axis and $\eta^{(k+i)} := M^T e_{k+i} \in \perp_y \mathcal{S}$ ($i = 1,\dots,n-k$) by the choice of M. Thus $\tilde{\alpha} = A_\perp$ and the claim clearly follows from Definition 3.2.11.

Let $\xi^{(k+i)} \in C^2(\mathcal{B}_{\hat{R}}(\hat{y}))$ denote the unit eigenvectors considered in $(A_4)-(iii)$, λ_{k+i} the corresponding eigenvalues of the matrix $\tilde{\alpha}$ ($i = 1,\dots,n-k$). Recall that $\tilde{\alpha}$ is symmetric and $\tilde{\alpha}(\cdot) \in C^2(\mathcal{B}_{\hat{R}}(\hat{y}); \mathcal{M}_{n-k})$. Then $\lambda_{k+i} = \langle \tilde{\alpha}\xi^{(k+i)}, \xi^{(k+i)}\rangle$ ($i = 1,\dots,n-k$) are $C^2(\mathcal{B}_{\hat{R}}(\hat{y}))$, too. Set

$$q_1 := ((\xi_j^{(k+i)}))_{i,j=1,\dots,n-k} \in \mathcal{M}_{n-k}$$

(where $\xi^{(k+i)}$ denotes the i-th row of the matrix q_1); clearly, the rank of the matrix $q_1\tilde{\alpha}\,q_1^T$ is not less than $\hat{r}$ and there holds:

$$\big(q_1\,\tilde{\alpha}\,q_1^T\big)_{ij} = \lambda_{k+i}\delta_{ij} \qquad (i,j = 1,\dots,n-k)\,.$$

Since at least $\hat{r}$ eigenvalues of $\tilde{\alpha}$ are strictly positive and continuous, choosing possibly smaller $\hat{R}$ we may assume that $\lambda_{n-\hat{r}+1},\dots,\lambda_n$, are strictly positive in $\overline{\mathcal{B}_{\hat{R}}(\hat{y})}$. Let $\beta_1 > 0$ be a lower bound for $\lambda_{n-\hat{r}+1},\dots,\lambda_n$ and $\beta_2 > 0$ an upper bound for all the eigenvalues of $\tilde{\alpha}$ in $\overline{\mathcal{B}_{\hat{R}}(\hat{y})}$, namely

$$\beta_1 \leq \lambda_{n-\hat{r}+1},\dots,\lambda_n, \qquad \text{and} \qquad \lambda_{k+1},\dots,\lambda_n \leq \beta_2, \quad \text{in } \overline{\mathcal{B}_{\hat{R}}(\hat{y})}\,. \tag{3.4.32}$$

Define the matrix $q_2 \in \mathcal{M}_{n-k}$ as follows:

$$(q_2)_{ij} = \begin{cases} \dfrac{1}{\sqrt{\beta_2}} & \text{if } i = j = 1, \dots, n-k-\hat{r}\,, \\ \dfrac{1}{\sqrt{\lambda_{k+i}}} & \text{if } i = j = n-k-\hat{r}+1, \dots, n-k\,, \\ 0 & \text{elsewhere}\,; \end{cases}$$

set also $q_0 := q_2 q_1$. Then we have:

$$\big(q_0\, \tilde{\alpha}\, q_0^T\big)_{ij} = \begin{cases} \dfrac{\lambda_{k+i}}{\beta_2} & \text{if } i = j = 1, \dots, n-k-\hat{r}\,, \\ 1 & \text{if } i = j = n-k-\hat{r}+1, \dots, n-k\,, \\ 0 & \text{elsewhere}\,. \end{cases} \tag{3.4.33}$$

Finally, define the matrix $Q_0 \in \mathcal{M}_n$ as follows:

$$\big(Q_0\big)_{ij} := \begin{cases} \delta_{ij} & \text{if } i, j = 1, \dots, k\,, \\ (q_0)_{i-k,\, j-k} & \text{if } i, j = k+1, \dots, n\,, \\ 0 & \text{elsewhere}\,. \end{cases} \tag{3.4.34}$$

Then by (3.4.33) we have:

$$\big(Q_0\, \tilde{A}\, Q_0^T\big)_{ij} = \begin{cases} \frac{\lambda_i}{\beta_2} & \text{if } i = j = k+1, \dots, n-\hat{r}\,, \\ \delta_{ij} & \text{elsewhere for } i, j = k+1, \dots, n\,; \end{cases} \tag{3.4.35}$$

moreover, the matrix function Q_0 belongs to $C^2(\mathcal{B}_{\hat{R}}(\hat{y}); \mathcal{M}_n)$.

Now we can prove Proposition 3.4.5.

Proof of Proposition 3.4.5 (i) Let us first prove the result assuming $\beta = 0$ in $(A_2) - (ii)'$, namely $b_i \in L^\infty(\Omega)$ $(i = 1, \dots, n)$. Fix $\hat{y} \in \mathcal{S}$; let $\hat{R} \in (0, \varepsilon)$ be as required in the above construction. For any $y \in \mathcal{B}_{\hat{R}}(\hat{y})$ set

$$E(y) := [M(y)]^T\, Q_0(y)\, M(y)\,,$$

$B_{\hat{R}} \equiv B_{\hat{R}}(\hat{y})$ and, for any $x \in B_{\hat{R}} \cap \Omega$,

$$z_0(x) := \log \big|E\big(x^*(x)\big)\big[x - x^*(x)\big]\big|^2 + K\, \big|E\big(x^*(x)\big)\big[x - x^*(x)\big]\big|\,, \tag{3.4.36}$$

with $K > 0$ to be chosen. Since $E \in C^2(\mathcal{B}_{\hat{R}}(\hat{y}); \mathcal{M}_n)$ by the above remarks and $x^* \in C^2(\mathcal{S}^\sigma; \mathcal{S})$ by Lemma 3.4.1-(ii), we have $z_0 \in C^2(B_{\hat{R}} \cap \Omega)$. Clearly, z_0 satisfies $(i) - (ii)$ (observe that $d(x, \mathcal{S}) = \big|x - x^*(x)\big|$ and (3.4.32) - (3.4.34) hold). The conclusion will follow, if we prove that

$$(\mathcal{M} z_0)(x^0) \geq 0 \quad \text{for any } x^0 \in B_{\hat{R}} \cap \Omega\,. \tag{3.4.37}$$

To this purpose, for any $x^0 \in B_{\hat{R}} \cap \Omega$ a change of variables as in Remark 3.4.2 is expedient. As above, denote by $X \equiv (X_1, \dots, X_n)$ the new coordinate system, where

$$X := M(\bar{y})\big(x - \bar{y}\big)\,, \quad \bar{y} := x^*(x^0)\,. \tag{3.4.38}$$

Set $\bar{M} \equiv M(\bar{y})$; then

$$X^0 := \bar{M}(x^0 - \bar{y}) \equiv (0, \dots, 0, X^0_{k+1}, \dots, X^0_n), \quad d(X^0, \mathcal{S}) = |X^0|\,. \tag{3.4.39}$$

Define also

$$Z_0(X) := z_0(x) \qquad (x \in B_{\hat{R}} \cap \Omega)\,;$$

it is easily seen that

$$Z_0(X) = \log \big|Q\big(X^*(X)\big)(X - X^*)\big|^2 + K\, \big|Q\big(X^*(X)\big)(X - X^*)\big|\,, \tag{3.4.40}$$

for any $x \in B_{\hat{R}} \cap \Omega$, where

$$X^*(X) := \bar{M}\big(x^*(x) - \bar{y}\big)\,,$$

$$Q\big(X^*(X)\big) := \bar{M}\, E\big(x^*(x)\big)\, \bar{M}^T = \bar{M}\big[M\big(x^*(x)\big]^T Q_0\big(x^*(x)\big)\, M\big(x^*(x)\big)\bar{M}^T\,. \tag{3.4.41}$$

For simplicity we always write $X^* \equiv X^*(X)$, omitting the dependence on X; we also set

$$Q(X^*) \equiv ((q_{ij})) \qquad (x \in B_{\hat{R}} \cap \Omega;\ i,j = 1, \ldots, n)\,, \tag{3.4.42}$$

In particular for $x = x^0$, by (3.4.41), we get

$$Q(0) = Q_0(\bar{y})\,. \tag{3.4.43}$$

Moreover, we set

$$W_i = W_i(X) := \sum_{j=1}^{n} q_{ij}(X_j - X_j^*) \qquad (i = 1, \ldots, n)\,,$$

thus

$$\big|Q(X^*)(X - X^*)\big| \, = \Big(\sum_{i=1}^{n} W_i^2\Big)^{1/2}\,. \tag{3.4.44}$$

It is also easily seen that

$$(\mathcal{M} z_0)(x^0) = (\tilde{\mathcal{M}} Z_0)(X^0) \qquad (x^0 \in B_{\hat{R}} \cap \Omega)\,, \tag{3.4.45}$$

where by $\tilde{\mathcal{M}}$ we denote the formal differential operator

$$\tilde{\mathcal{M}}\, u \equiv \sum_{i,j=1}^{n} \tilde{a}_{ij}(X) \frac{\partial^2 u}{\partial X_i \partial X_j} + \sum_{i=1}^{n} \tilde{b}_i(X) \frac{\partial u}{\partial X_i}\,,$$

$$\tilde{a}_{ij}(X) = \tilde{a}_{ji}(X) := \sum_{k.l=1}^{n} (\bar{M})_{ik} a_{ij}(x) (\bar{M})_{jl} \quad (i,j = 1, \ldots, n)\,, \tag{3.4.46}$$

$$\tilde{b}_i(X) := \sum_{j=1}^{n} (\bar{M})_{ij} b_j(x) \quad (i = 1, \ldots, n)\,.$$

Observe that, setting $\tilde{A} \equiv ((\tilde{a}_{ij}))_{i,j=1,\ldots,n}$, $\tilde{b} \equiv (\tilde{b}_i)_{i=1,\ldots,n}$, equality (3.4.46) reads:

$$\tilde{A}(X) := \bar{M} A(x) \bar{M}^T\,, \qquad \tilde{b}(X) := \bar{M} b(x)$$

which coincides with (3.4.31) if $x = \bar{y}, X = 0$. Now we prove that

$$(\tilde{\mathcal{M}} Z_0)(X^0) \geq 0 \quad \text{ for any } \ x^0 \in B_{\hat{R}} \cap \Omega\,, \tag{3.4.47}$$

whence the conclusion will follow.

For $h = 1, \cdots, n$, we have

$$\frac{\partial Z_0}{\partial X_h}(X) = \Big\{ \frac{2}{\big|Q(X^*)(X - X^*)\big|^2} + \frac{K}{\big|Q(X^*)(X - X^*)\big|} \Big\}\, I_h^{(1)}\,, \tag{3.4.48}$$

where

$$I_h^{(1)} := \sum_{i=1}^{n} W_i\, \frac{\partial W_i}{\partial X_h}\,,$$

$$\frac{\partial W_i}{\partial X_h}(X) = \sum_{j=1}^{n} \Big[\frac{\partial q_{ij}}{\partial X_h}\, \big(X_j - X_j^*\big) + q_{ij} \Big(\delta_{jh} - \frac{\partial X_j^*}{\partial X_h} \Big) \Big]\,. \tag{3.4.49}$$

For $h, l = 1, \cdots, n$ there holds

$$\frac{\partial^2 Z_0}{\partial X_h \partial X_l}(X) = \Big\{ \frac{2}{\big|Q(X^*)(X - X^*)\big|^2} + \frac{K}{\big|Q(X^*)(X - X^*)\big|} \Big\} I_{hl}^{(2)} + \tag{3.4.50}$$

$$-\Big\{\frac{4}{\big|Q(X^*)(X-X^*)\big|^4}+\frac{K}{\big|Q(X^*)(X-X^*)\big|^3}\Big\}I_h^{(1)}I_l^{(1)},$$

where

$$I_{hl}^{(2)}:=\frac{\partial I_h^{(1)}}{\partial X_l}=\sum_{i=1}^n\Big[\frac{\partial W_i}{\partial X_h}\frac{\partial W_i}{\partial X_l}+W_i\,\frac{\partial^2 W_i}{\partial X_h\partial X_l}\Big],$$

(3.4.51)
$$\frac{\partial^2 W_i}{\partial X_h\partial X_l}(X)=\sum_{j=1}^n\Bigg[\frac{\partial^2 q_{ij}}{\partial X_h\partial X_l}\,(X_j-X_j^*)+$$
$$+\frac{\partial q_{ij}}{\partial X_h}\Big(\delta_{jl}-\frac{\partial X_j^*}{\partial X_l}\Big)+\frac{\partial q_{ij}}{\partial X_l}\Big(\delta_{jh}-\frac{\partial X_j^*}{\partial X_h}\Big)-q_{ij}\,\frac{\partial^2 X_j^*}{\partial X_h\partial X_l}\Bigg].$$

To prove (3.4.47) the above quantities must be calculated at $X=X^0$, hence (3.4.43) and (3.4.42) must be used.

In the following we use the notation $O(|X^0|)$, $O(|X^0|^2)$, ... introduced above; the same symbol $O(\cdot)$ will denote any function satisfying (3.4.11). The constant D, although not explicitly given, will not depend on the specific choice of $x^0\in B_{\hat R}\cap\Omega$. In fact, it will only depend on the following quantities:

(α) the functions $q_{ij}(\cdot)$ (hence on the eigenvalues and eigenvectors of the matrix $\tilde a(\cdot)$) and their first and second derivatives in $\overline{B_{\hat R}\cap\Omega}$;

(β) the function $p(\cdot)$ which gives a local representation of $\mathcal{S}$ in $\overline{B_{\hat R}(\hat y)}$, and its first and second derivatives;

(γ) the lower and upper bounds β_1, β_2 of the eigenvalues of the orthogonal matrix.

Similar remarks hold for the positive constant H encountered below (see (3.4.60) and the following formulas).

(a) In view of (3.4.41), (3.4.43) and (3.4.34), we have

(3.4.52)
$$W_i(X^0)=\sum_{j=k+1}^n q_{ij}(0)X_j^0\qquad(i=1,\dots,n),$$

and

(3.4.53)
$$\big|Q(X^*)(X-X^*)\big|\Big|_{X=X^0}=\big|Q(0)X^0\big|.$$

(b) From (3.4.49) and (3.4.13) we obtain

(3.4.54)
$$\frac{\partial W_i}{\partial X_h}(X^0)=\sum_{j=k+1}^n\frac{\partial q_{in}}{\partial X_h}(0)X_j^0+\sum_{j=1}^k q_{ij}(0)O(|X^0|)+$$
$$+\sum_{j=k+1}^n q_{ij}(0)\delta_{jh}=q_{ih}(0)E\chi_{[k+1,n]}(h)+O(|X^0|).$$

Equalities (3.4.52) and (3.4.54), together with (3.4.42), (3.4.34) give:

(3.4.55)
$$I_h^{(1)}(X^0)=\sum_{i=1}^n\Big(\sum_{j=k+1}^n q_{ij}(0)X_j^0\Big)q_{ih}(0)\chi_{[k+1,n]}(h)+O(|X^0|^2)=$$
$$=\sum_{i,j=k+1}^n q_{ij}(0)X_j^0 q_{ih}(0)+O(|X^0|^2).$$

Concerning the above equality, observe that $q_{ih}(0)=0$ for $i=1,\dots,k$ and $h\ge k+1$.

(c) The formulas in Remark 3.4.4 and equality (3.4.51) imply that $\dfrac{\partial^2 W_i}{\partial X_h \partial X_l}(X)$ is bounded independently from i, h, l and the choice of $x^0 \in B_{\hat{R}} \cap \Omega$. Then (3.4.52) and (3.4.54) entail:

$$I_{hl}^{(2)}(X^0) = \sum_{i=k+1}^{n} q_{ih}(0) q_{il}(0) + O(|X^0|) \,. \tag{3.4.56}$$

From (3.4.55) we also obtain:

$$\begin{aligned} &\big(I_h^{(1)} I_l^{(1)}\big)(X^0) = \\ &= \Big(\sum_{i,j,r,s=k+1}^{n} q_{ij}(0) X_j^0 q_{ih}(0) q_{rs}(0) X_s^0 q_{rl}(0) \Big) + O(|X^0|^3) \,. \end{aligned} \tag{3.4.57}$$

To prove (3.4.37) we need an estimate from below of the quantity

$$\sum_{h,l=1}^{n} \tilde{a}_{hl}(0) \frac{\partial^2 Z_0}{\partial X_h \partial X_l}(X^0) \,, \tag{3.4.58}$$

(see (3.4.60)). To this aim, the following computations will be needed, where (3.4.33), (3.4.34) and (3.4.42) are used:

$$\begin{aligned} \sum_{h,l=1}^{n} \sum_{i=k+1}^{n} \tilde{a}_{hl}(0) q_{ih}(0) q_{il}(0) &= \sum_{i=k+1}^{n} \sum_{h,l=k+1}^{n} \tilde{a}_{hl}(0) q_{ih}(0) q_{il}(0) = \\ = \sum_{j=1}^{n-k} \sum_{s,t=1}^{n-k} (q_0)_{js} \tilde{a}_{s+k,t+k}(0) (q_0)_{jt} &= \sum_{j=1}^{n-k} \big(q_0 \, \tilde{\alpha} \, q_0^T\big)_{jj} \geq \hat{r} \,; \end{aligned}$$

here $\hat{r}$ is the lower bound for the orthogonal rank of A in $\mathcal{S}$ used in the construction of the matrix Q_0. It is similarly seen that (see (3.4.32)):

$$\begin{aligned} &\sum_{h,l=1}^{n} \tilde{a}_{hl}(0) \sum_{i,j,s,t=k+1}^{n} q_{ij}(0) X_j^0 q_{ih}(0) q_{st}(0) X_t^0 q_{sl}(0) = \\ &= \sum_{i,j,s,t=k+1}^{n} q_{ij}(0) X_j^0 q_{st}(0) X_t^0 \Big[\sum_{h,l=k+1}^{n} q_{ih}(0) \tilde{a}_{hl}(0) q_{sl}(0) \Big] = \\ &= \sum_{i,s=k+1}^{n} (Q(0)X^0)_i (Q(0)X^0)_s (q_0 \tilde{\alpha} q_0^T)_{i-k\, s-k} = \\ &= \sum_{i=k+1}^{n-\hat{r}} \frac{\lambda_i}{\beta_2} (Q(0)X^0)_i^2 + \sum_{i=n-\hat{r}+1}^{n} (Q(0)X^0)_i^2 \leq |Q(0)X^0|^2 \,. \end{aligned}$$

Further observe that

$$\frac{|X^0|}{\sqrt{\beta_2}} \leq |Q(0)X^0| \leq \frac{|X^0|}{\sqrt{\beta_1}} \,. \tag{3.4.59}$$

In view of equalities (3.4.50), (3.4.53), (3.4.56), (3.4.57) and the above computations, since $\hat{r}E \geq 2$ by assumption $(A_4) - (ii)$, we obtain:

$$\begin{aligned} &\sum_{h,l=1}^{n} \tilde{a}_{hl}(0) \frac{\partial^2 Z_0}{\partial X_h \partial X_l}(X^0) \geq \\ &\geq \frac{1}{|Q(0)X^0|^2} \Big\{ 2\hat{r} + O(|X^0|) + K|Q(0)X^0|(\hat{r}-1) - 4 + KO(|X^0|^2) \Big\} \geq \end{aligned} \tag{3.4.60}$$

$$\geq \frac{1}{|Q(0)X^0|}\Big\{ -H + K + KO(|X^0|)\Big\}.$$

Here H is a positive constant, independent on $x^0 \in B_{\hat{R}} \cap \Omega$, such that

$$|O(|X^0|)| \leq H|Q(0)X^0|$$

and $K > 0$ is the constant in the definition of z_0 to be chosen (see (3.4.36)). We also have

$$\Big|\frac{\partial^2 Z_0}{\partial X_h \partial X_l}(X^0)\Big| \leq \frac{1}{|Q(0)X^0|^2}\Big\{H + KO(|X^0|)\Big\}. \tag{3.4.61}$$

Since $\tilde{b}_i$ and q_{ij} are uniformly bounded for $x^0 \in B_{\hat{R}} \cap \Omega$, by (3.4.48), (3.4.55) we get

$$|\sum_{i=1}^n \tilde{b}_i(X^0)\frac{\partial Z_0}{\partial X_i}(X^0)| \leq \frac{1}{|Q(0)X^0|}\Big\{H + K|O(|X^0|)|\Big\}. \tag{3.4.62}$$

Inequalities (3.4.60)-(3.4.62) imply:

$$\begin{aligned}(\tilde{\mathcal{M}}Z_0)(X^0) &= \sum_{h,l=1}^n \big[\tilde{a}_{hl}(X^0) - \tilde{a}_{hl}(0)\big]\frac{\partial^2 Z_0}{\partial X_h \partial X_l}(X^0) + \\ &+ \sum_{h,l=1}^n \tilde{a}_{hl}(0)\frac{\partial^2 Z_0}{\partial X_h \partial X_l}(X^0) + \sum_{i=1}^n \tilde{b}_i(X^0)\frac{\partial Z_0}{\partial X_i}(X^0) \geq \\ &\geq \frac{1}{|Q(0)X^0|}\Big\{ -H + K - K|O(|X^0|)|\Big\},\end{aligned} \tag{3.4.63}$$

for the coefficients $\tilde{a}_{hl}$ are of class C^1, thus locally Lipschitz continuous in $B_{\hat{R}} \cap \Omega$. For a sufficiently small $\hat{R}$ we have $|O(|X^0|)| < \frac{1}{2}$, then (see (3.4.59))

$$(\tilde{\mathcal{M}}Z_0)(X^0) \geq \frac{\sqrt{\beta_1}}{2|X^0|}\big(K - 2H\big) > 0, \tag{3.4.64}$$

choosing $K > 2H$.

Finally, instead of definition (3.4.36) set

$$z_0(x) := -\frac{1}{\big|E\big(x^*(x)\big)\big(x - x^*(x)\big)\big|} + K \log \big|E\big(x^*(x)\big)\big(x - x^*(x)\big)\big|^2 \tag{3.4.65}$$

if $\inf_{y \in \mathcal{S}} r(y) \geq 3$, respectively

$$z_0(x) := -\frac{1}{\big|E\big(x^*(x)\big)\big(x - x^*(x)\big)\big|^\alpha} - \frac{K}{\big|E\big(x^*(x)\big)\big(x - x^*(x)\big)\big|^{\alpha-1}} \tag{3.4.66}$$

if $\inf_{y \in \mathcal{S}} r(y) \geq 4$. Arguing as in the case $\inf_{y \in \mathcal{S}} r(y) \geq 2$, inequality (3.4.47) is seen to hold in this cases, too; we omit the details. The proof for the case $\beta = 0$ is complete.

(ii) Now we assume $\beta \in (0, 1)$ in $(A_2) - (ii)'$. In this case the definition (3.4.36) of z_0 is replaced by

$$z_0(x) := \log \big|E\big(x^*(x)\big)\big(x - x^*(x)\big)\big|^2 + K\big|E\big(x^*(x)\big)\big(x - x^*(x)\big)\big|^{1-\beta}. \tag{3.4.67}$$

Then inequalities (3.4.60)-(3.4.63) are replaced by

$$\begin{aligned}&\sum_{h,l=1}^n \tilde{a}_{hl}(0)\frac{\partial^2 Z_0}{\partial X_h \partial X_l}(X^0) \geq \\ &\geq \frac{1}{|Q(0)X^0|}\Big\{ -H + \frac{K(1-\beta)(\hat{r} - 1 - \beta)}{|Q(0)X^0|^\beta} + KO(|X^0|^{1-\beta})\Big\}\end{aligned} \tag{3.4.68}$$

$$\Big|\frac{\partial^2 Z_0}{\partial X_h \partial X_l}(X^0)\Big| \leq \frac{1}{|Q(0)X^0|^2}\Big\{H + KO(|X^0|^{1-\beta})\Big\}. \tag{3.4.69}$$

Similarly, recalling also $(A_2)-(ii)'$ instead of (3.4.62) we get:

$$|\sum_{i=1}^n \tilde{b}_i(X^0)\frac{\partial Z_0}{\partial X_i}(X^0)| \leq \frac{1}{|Q(0)X^0|^{1+\beta}}\Big\{H + K|O(|X^0|^{1-\beta})|\Big\}. \tag{3.4.70}$$

Therefore we have in the present case

$$\begin{aligned}(\tilde{\mathcal{M}}Z_0)(X^0) &= \sum_{h,l=1}^n \big[\tilde{a}_{hl}(X^0) - \tilde{a}_{hl}(0)\big]\frac{\partial^2 Z_0}{\partial X_h \partial X_l}(X^0) + \\ &+ \sum_{h,l=1}^n \tilde{a}_{hl}(0)\frac{\partial^2 Z_0}{\partial X_h \partial X_l}(X^0) + \sum_{i=1}^n \tilde{b}_i(X^0)\frac{\partial Z_0}{\partial X_i}(X^0) \geq \\ &\geq \frac{1}{|Q(0)X^0|^{1+\beta}}\Big\{-H + K - K|O(|X^0|^{1-\beta})|\Big\},\end{aligned} \tag{3.4.71}$$

which corresponds to (3.4.63). Hence the conclusion follows as before, choosing $\hat{R}$ possibly smaller and K sufficiently large.

Finally, if $\inf_{y\in\mathcal{S}} r(y) \geq 3$, instead of definition (3.4.36) set:

$$z_0(x) := -\frac{1}{\big|E\big(x^*(x)\big)\big(x - x^*(x)\big)\big|} - \frac{K}{\big|E\big(x^*(x)\big)\big(x - x^*(x)\big)\big|^{\beta}}, \tag{3.4.72}$$

or respectively, if $\inf_{y\in\mathcal{S}} r(y) \geq 4$:

$$z_0(x) := -\frac{1}{\big|E\big(x^*(x)\big)\big(x - x^*(x)\big)\big|^{\alpha}} - \frac{K}{\big|E\big(x^*(x)\big)\big(x - x^*(x)\big)\big|^{\alpha-1+\beta}}. \tag{3.4.73}$$

Arguing as in the previous cases inequality (3.4.47) is seen to hold in this cases, too; we omit the details. The proof is complete. □

3.4.2.2. The following proposition corresponds to the second step in the costruction of the subsolution Z.

PROPOSITION 3.4.6. *Let the assumptions of Theorem 3.2.12 be satisfied. Then there exist* $\varepsilon > 0$ *and* $z \in C^2(\mathcal{S}^\varepsilon)$ *such that* z *is a subsolution of the equation*

$$\mathcal{L}u = 0 \quad \text{in } \mathcal{S}^\varepsilon. \tag{3.4.74}$$

Moreover,

(i) $\sup_{\mathcal{S}^\varepsilon} z < \infty$;

(ii) $z(x) \sim \log[d(x,\mathcal{S})]$ *as* $d(x,\mathcal{S}) \to 0 \quad (x \in \mathcal{S}^\varepsilon)$.

If $\alpha := \inf_{y\in\mathcal{S}} r(y) - 2 \geq 1$, *the same conclusion holds with* (ii) *replaced by*

(iii) $z(x) \sim -[d(x,\mathcal{S})]^{-\alpha}$ *as* $d(x,\mathcal{S}) \to 0 \qquad (x \in \mathcal{S}^\varepsilon)$.

Proof. (i) Assume first $\beta = 0$ in assumption $(A_2)-(ii)'$. In view of the compactness of $\mathcal{S}$ and of Proposition 3.4.5, there exist $\hat{y}^i \in \mathcal{S}$, H_i, h_i, $R_i > 0$ and $z_i \in C^2(B_{R_i}(\hat{y}^i)\cap\Omega)$ $(i = 1, \dots, \hat{N}$, for some $\hat{N} \in I\!N)$ of the form

$$z_i(x) := \log\big|E_i\big(x^*(x)\big)\big[x - x^*(x)\big]\big|^2 + K\,\big|E_i\big(x^*(x)\big)\big[x - x^*(x)\big]\big|,$$

with the following properties:

(a) $\mathcal{S} \subseteq \bigcup_{i=1}^{\hat{N}} B_{R_i}$;

(b) z_i $(i = 1, \ldots, \hat{N})$ is a subsolution of the equation

$$\mathcal{L}u = 0 \quad \text{in } B_{R_i} \cap \Omega\,,$$

if $K > H_i$ (see (3.4.64); we set $B_{R_i} \equiv B_{R_i}(\hat{y}^i)$ for simplicity). In fact, there holds:

$$\mathcal{M}z_i \geq \frac{h_i}{d(x, \mathcal{S})}\big(K - H_i\big) \quad \text{in } B_{R_i} \cap \Omega\,. \tag{3.4.75}$$

Observe that the matrix-valued functions E_i, E_j with $i \neq j$ may be different at the same point, since they depend on the local representation of $\mathcal{S}$ and on the choice and ordering of the nonzero eigenvalues.

Choose $\varepsilon \in (0, \min\{R_1, \ldots, R_{\hat{N}}\})$, so that $\overline{\mathcal{S}^\varepsilon} \subseteq \bigcup_{i=1}^{\hat{N}} B_{R_i}$. Let $\{\psi_i\}_{i=1}^N$ be a partition of unity subordinate to $\{B_{R_i}\}_{i=1}^N$, namely

$$label c3PUnita \psi_i \in C^\infty(\overline{\Omega})\,, \quad supp\, \psi_i \subseteq B_{R_i}\,, \quad 0 \leq \psi_i \leq 1\,, \quad \sum_{i=1}^{\hat{N}} \psi_i = 1 \text{ in } \overline{\mathcal{S}^\varepsilon}\,. \tag{3.4.76}$$

Define

$$z(x) := \sum_{i=1}^{\hat{N}} \psi_i(x) z_i(x) \qquad (x \in \mathcal{S}^\varepsilon)\,,$$

where we define $z_i := 0$ outside its domain $B_{R_i} \cap \Omega$ $(i = 1, \ldots, \hat{N})$. There holds:

$$\mathcal{M}z = \sum_{i=1}^{\hat{N}} \big(\mathcal{M}\psi_i\big) z_i + \sum_{i=1}^{\hat{N}} \sum_{h,l=1}^{n} a_{hl} \Big(\frac{\partial \psi_i}{\partial x_h} \frac{\partial z_i}{\partial x_l} + \frac{\partial \psi_i}{\partial x_l} \frac{\partial z_i}{\partial x_h}\Big) + \sum_{i=1}^{\hat{N}} \psi_i \big(\mathcal{M}z_i\big)\,. \tag{3.4.77}$$

The proof of Proposition 3.4.5, together with the boundedness in $\mathcal{S}^\varepsilon$ of the functions p_i, q_{ij}, a_{ij}, b_i and their first and second derivatives, easily give the following

Claim: There exist positive constants C_1, C_2 such that, taking $\varepsilon > 0$ possibly smaller and $C_3 := \min\{h_1, \ldots, h_{\hat{N}}\}$, there holds

$$\sum_{i=1}^{\hat{N}} \big(\mathcal{M}\psi_i\big) z_i \geq C_1 \log[d(\cdot, \mathcal{S})] \quad \text{in } \mathcal{S}^\varepsilon\,; \tag{3.4.78}$$

$$\sum_{i=1}^{\hat{N}} \sum_{h,l=1}^{n} a_{hl} \Big(\frac{\partial \psi_i}{\partial x_h} \frac{\partial z_i}{\partial x_l} + \frac{\partial \psi_i}{\partial x_l} \frac{\partial z_i}{\partial x_h}\Big) \geq -\frac{C_2}{d(\cdot, \mathcal{S})} \quad \text{in } \mathcal{S}^\varepsilon\,; \tag{3.4.79}$$

$$\sum_{i=1}^{\hat{N}} \psi_i \big(\mathcal{M}z_i\big) \geq \frac{C_3}{d(\cdot, \mathcal{S})}\big(K - K_0\big) \quad \text{in } \mathcal{S}^\varepsilon\,, \tag{3.4.80}$$

for any $K \geq K_0 := \max\{H_1, \ldots, H_{\hat{N}}\}$.

Inequality (3.4.78) follows from the very definition of z, while (3.4.79), (3.4.80) are a consequence of (3.4.48) and (3.4.55), respectively of (3.4.75) and (**??**).

From (3.4.77)-(3.4.80) we obtain $(x \in \mathcal{S}^\varepsilon)$

$$(\mathcal{M}z)(x) \geq \frac{1}{d(x, \mathcal{S})}\big(C_3 K - C_3 K_0 - C_2 - C_1 d(x, \mathcal{S}) |\log[d(x, \mathcal{S})]|\big) \geq 0\,,$$

for any K sufficiently large, thus the result follows in this case. The same argument applies when $\inf_{y \in \mathcal{S}} r(y) \geq 3$, whence the conclusion.

(ii) Now suppose $\beta \in (0,1)$ in assumption $(A_2)-(ii)'$. Following the proof of Proposition 3.4.5, we have z_i of the form (3.4.67); hence inequalities (3.4.78) and (3.4.80) become respectively

$$\sum_{i=1}^{\hat{N}} (\mathcal{M}\psi_i) z_i \geq C_1 \frac{\log[d(\cdot,\mathcal{S})]}{d(\cdot,\mathcal{S})^{\beta}} \quad \text{in } \mathcal{S}^{\varepsilon}, \tag{3.4.81}$$

$$\sum_{i=1}^{\hat{N}} \psi_i (\mathcal{M} z_i) \geq \frac{C_3}{d(\cdot,\mathcal{S})^{1+\beta}} (K - K_0) \quad \text{in } \mathcal{S}^{\varepsilon}. \tag{3.4.82}$$

Then the conclusion follows as in the previous case for sufficiently large K. The case $\inf_{y \in \mathcal{S}} r(y) \geq 3$ can be dealt with similarly. This completes the proof. □

3.4.2.3. The third step in the outlined costruction of Z allows us to complete the proof of Theorem 3.2.12.

Proof of Theorem 3.2.12. (i) Let us first address the case when (E_1) holds. Let z be the subsolution of equation (3.4.74) exhibited in Proposition 3.4.6. Consider the problem

$$\begin{cases} \mathcal{L} w = 0 & \text{in } \mathcal{S}^{\varepsilon/2} \\ w = -z & \text{in } \partial\mathcal{S}^{\varepsilon/2} \setminus \mathcal{S}; \end{cases} \tag{3.4.83}$$

a solution $w \in C^2(\mathcal{S}^{\varepsilon/2}) \cap C^1(\overline{\mathcal{S}^{\varepsilon/2}} \setminus \mathcal{S}) \cap L^{\infty}(\mathcal{S}^{\varepsilon/2})$ is easily constructed by standard compactness arguments. Indeed, observe that any constant $C \geq |z|_{L^{\infty}(\partial\mathcal{S}^{\varepsilon/2}\setminus\mathcal{S})}$ is a supersolution of (3.4.83), whereas $-C$ is a subsolution. Then $\tilde{Z} := z + w$ is a subsolution of the equation

$$\mathcal{L} u = 0 \quad \text{in } \mathcal{S}^{\varepsilon/2},$$

such that $\tilde{Z} = 0$ in $\partial\mathcal{S}^{\varepsilon/2} \setminus \mathcal{S}$ and

$$\tilde{Z}(x) \sim \log d(x,\mathcal{S}) \quad \text{as } d(x,\mathcal{S}) \to 0.$$

Then $\tilde{Z}(x) \leq 0$ by the maximum principle (applied in sets of the form $\mathcal{S}^{\varepsilon/2} \setminus \mathcal{S}^{\delta}$, $\delta \in (0, \varepsilon/2)$ and such that $\tilde{Z}(x) < 0$ on $\partial\mathcal{S}^{\delta} \cap \mathcal{S}^{\varepsilon/2}$).

Let $W \in C^2(\Omega \setminus \overline{\mathcal{S}^{\varepsilon/2}}) \cap C^1(\Omega \setminus \mathcal{S}^{\varepsilon/2})$ be the solution of the problem:

$$\begin{cases} \mathcal{L}W = 0 & \text{in } \Omega \setminus \overline{\mathcal{S}^{\varepsilon/2}} \\ W = 1 & \text{on } \mathcal{R} \\ W = 0 & \text{on } \partial\mathcal{S}^{\varepsilon/2} \setminus \mathcal{S}, \end{cases}$$

hence $0 \leq W \leq 1$ in $\Omega \setminus \mathcal{S}^{\varepsilon/2}$. Set

$$Z := \begin{cases} H\tilde{Z} - 2 & \text{in } \mathcal{S}^{\varepsilon/2} \\ W - 2 & \text{in } (\Omega \cup \mathcal{R}) \setminus \mathcal{S}^{\varepsilon/2} \end{cases}$$

with $H > 0$ to be chosen. Then:

(a) $Z \in C(\Omega \cup \mathcal{R})$, $Z \leq -1$ in $\Omega \cup \mathcal{R}$;

(b) it is easily seen that

$$\int_{\Omega} Z \mathcal{M}^* \psi \, dx \geq \int_{\Gamma} \psi \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial}{\partial x_i} (W - H\,\tilde{Z}) \nu_j(x) \, dx,$$

for any $\psi \in C_0^{\infty}(\Omega)$, $\psi \geq 0$; here $\Gamma := \partial\big(\mathcal{S}^{\varepsilon/2} \cap supp\ \psi\big)$ and $\nu(x) \equiv (\nu_1(x), \dots, \nu_n(x))$ denotes the outer normal to $\mathcal{S}^{\varepsilon/2}$ at $x \in \partial\mathcal{S}^{\varepsilon/2} \setminus \mathcal{S}$.

In view of the strong maximum principle, since $\mathcal{L}$ is uniformly elliptic in $\Omega \setminus \overline{\mathcal{S}^{\varepsilon/2}}$ and $\partial\mathcal{S}^{\varepsilon/2} \setminus \mathcal{S}$ is compact, there exists $\alpha > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial W}{\partial x_i} \nu_j \geq \alpha \qquad \text{in } \partial\mathcal{S}^{\varepsilon} \setminus \mathcal{S}\,.$$

Then choosing $H := \alpha / \left\{ \left(\sum_{i,j=1}^{n} |a_{ij}|_{L^\infty(\Omega)} \right) |\nabla \tilde{Z}|_{L^\infty(\partial\mathcal{S}^{\varepsilon/2}\setminus\mathcal{S})} \right\}$ we obtain

$$\int_\Omega Z\mathcal{M}^*\psi\, dx \,\geq\, 0\,.$$

Moreover, by $Z \leq -1$, ρ, ψ, $c \geq 0$ there holds

$$\int_\Omega Z\mathcal{M}^*\psi\, dx \,\geq\, 0 \geq \int_\Omega \rho\, c\, Z\, \psi\, dx\,.$$

Then Z is a subsolution of (3.1.9), hence the conclusion follows by Theorem 3.1.2.

(ii) Now suppose that (E_2) is satisfied and $c(x) > 0$ for any $x \in \Omega$. Let z be the subsolution of equation (3.4.74) exhibited in Proposition 3.4.6. Since z is defined in $\mathcal{S}^\varepsilon$, set $Z := 0$ in $\Omega \setminus \mathcal{S}^\varepsilon$. Consider as in (3.2.9) a function $\chi \in C^2(\overline{\Omega})$, $0 \leq \chi \leq 1$ such that

$$\chi(x) = \begin{cases} 1 & \text{if } \; x \in \mathcal{S}^{\varepsilon/2}, \\ \\ 0 & \text{if } \; x \in \Omega \setminus \mathcal{S}^\varepsilon\,; \end{cases} \tag{3.4.84}$$

then define $Z(x) := \chi(x)\, z(x)$, $x \in \Omega$. Clearly, there holds:

$$\mathcal{L}Z \geq 0 \quad \text{in } \mathcal{S}^{\varepsilon/2}\,, \tag{3.4.85}$$

$$\mathcal{L}Z = 0 \quad \text{in } \Omega \setminus \mathcal{S}^\varepsilon\,, \tag{3.4.86}$$

$$\mathcal{L}Z(x) \geq -K \quad \text{in } \mathcal{S}^\varepsilon \setminus \mathcal{S}^{\varepsilon/2}\,, \tag{3.4.87}$$

where $K := \max\limits_{\mathcal{S}^\varepsilon \setminus \mathcal{S}^{\varepsilon/2}} |\mathcal{L}Z|$. Since Z is bounded from above, it is not restrictive to assume

$$Z \leq -\frac{\max\{1, K\}}{c_0} \qquad \text{in } \Omega\,,$$

where $c_0 := \min\limits_{\mathcal{S}^\varepsilon \setminus \mathcal{S}^{\varepsilon/2}} c > 0$. Hence by inequalities (3.4.85)-(3.4.87) we conclude that Z is a subsolution of problem (3.1.9). Then by Theorem 3.1.2 the conclusion follows. □

3.5. Singular manifolds of high dimension: Proofs

The present section is devoted to prove the results stated in Subsections 3.2.1, paragraph 3.2.1.3, and 3.2.3.

Proof of Theorem 3.2.16. To prove claim (i) observe that $P(\zeta)$ is a decreasing positive function of $\zeta \in (0, \bar{\varepsilon})$, hence it has a limit as $\zeta \to 0^+$. By contradiction, let this limit be finite. Then for any $\sigma > 0$ there exists $\delta = \delta(\sigma)$ such that for any ζ_0, ζ with $0 < \zeta_0 \leq \zeta \leq \delta$

$$\begin{aligned} \sigma &\geq \int_{\zeta_0}^{\bar{\varepsilon}} (\eta - \zeta_0)\underline{\rho}(\eta)\, d\eta - \int_{\zeta}^{\bar{\varepsilon}} (\eta - \zeta)\underline{\rho}(\eta)\, d\eta = \\ &= \int_{\zeta_0}^{\zeta} (\eta - \zeta_0)\underline{\rho}(\eta)\, d\eta + \int_{\zeta}^{\bar{\varepsilon}} (\zeta - \zeta_0)\underline{\rho}(\eta)\, d\eta \geq 0\,. \end{aligned} \tag{3.5.1}$$

In particular, for any ζ_0, ζ as above there holds:

$$0 \leq \int_{\zeta}^{\bar{\varepsilon}} (\zeta - \zeta_0)\underline{\rho}(\eta)\, d\eta \leq \sigma \quad \big(\zeta \in (0,\delta)\big).$$

As $\zeta_0 \to 0^+$, the above inequality gives

$$0 \leq \zeta \int_{\zeta}^{\bar{\varepsilon}} \underline{\rho}(\eta)\, d\eta \leq \sigma \quad \text{for any} \quad \zeta \in (0,\delta). \tag{3.5.2}$$

Since σ is arbitrary and $\delta = \delta(\sigma)$, we get

$$\lim_{\zeta \to 0^+} \zeta \int_{\zeta}^{\bar{\varepsilon}} \underline{\rho}(\eta)\, d\eta = 0\,, \tag{3.5.3}$$

whence by (3.2.16)

$$\lim_{\zeta E \to 0^+} P(\zeta) = \lim_{\zeta E \to 0^+} \Big[\int_{\zeta}^{\bar{\varepsilon}} \eta \underline{\rho}(\eta)\, d\eta - \zeta \int_{\zeta}^{\bar{\varepsilon}} \underline{\rho}(\eta)\, d\eta \Big] = +E\infty\,, \tag{3.5.4}$$

a contradiction. This proves the claim.

To prove (ii), set

$$\mathcal{Z}(\zeta) := e^{\frac{C}{\beta}\zeta} - \frac{1}{C} \int_{\zeta}^{\bar{\varepsilon}} [1 - e^{\frac{C}{\beta}(\zeta - \eta)}] \underline{\rho}(\eta)\, d\eta \qquad (\zeta \in (0, \bar{\varepsilon}))$$

with positive constants C, β to be chosen. Plainly, $|Z(\zeta)|$ diverges with the same order as $P(\zeta)$ for $\zeta \to 0^+$. Hence we have:

$$\mathcal{Z}' > 0\,, \qquad \beta \mathcal{Z}'' - C\mathcal{Z}' = -\underline{\rho} \quad \text{in } (0, \bar{\varepsilon})\,;$$

$$\mathcal{Z}(\zeta) \to -\infty\,, \qquad \mathcal{Z}'(\zeta) \to \infty\,, \qquad \mathcal{Z}''(\zeta) \to -\infty \quad \text{as } \zeta \to 0\,.$$

Set $\tilde{Z}(x) := \mathcal{Z}\big(d(x, \mathcal{S}_2)\big)$ $(x \in \mathcal{S}_2^{\bar{\varepsilon}})$. Let us show that

$$\mathcal{L}\tilde{Z}(x) \geq -1 \quad \text{in } \mathcal{S}_2^{\varepsilon} \tag{3.5.5}$$

with a proper choice of C, β and for $\varepsilon \in (0, \bar{\varepsilon}]$ possibly smaller then $\bar{\varepsilon}$.

To check (3.5.5) fix any $x \in \mathcal{S}_2^{\varepsilon}$ $(\varepsilon \in (0, \bar{\varepsilon}])$. As in the proof of Theorem 3.2.12, take new coordinates, still denoted by x, satisfying condition (C) in Subsection 3.4.1. In particular, we have

$$x = (0, \dots, 0, x_n), \quad x^*(x) = 0, \quad d(x, \mathcal{S}_2) = x_n\,, \tag{3.5.6}$$

$$T_0\, \mathcal{S}_2 = \{(x_1, ..., x_n) \in I\!R^n \,|\, x_n = 0\}\,. \tag{3.5.7}$$

By abuse of notation, denote again by a_{ij}, b_i the coefficients of $\mathcal{M}$ in the new coordinates. We can choose C and β such that

$$|a_{nn}| \leq \beta, \qquad |\mathcal{M}d(x, \mathcal{S}_2)| \leq C \quad \text{in } \mathcal{S}_2^{\bar{\varepsilon}}\,;$$

here use of (3.4.9)-(3.4.10) with $k = n-1$ has been made. Using (3.2.17), (3.4.9) and taking $\varepsilon \in (0, \bar{\varepsilon}]$ so small that $\mathcal{Z}(\zeta), \mathcal{Z}''(\zeta) < 0$ and $\mathcal{Z}'(\zeta) > 0$ for any $\zeta \in (0, \varepsilon]$, we obtain:

$$\begin{aligned}\mathcal{M}\tilde{Z}(x) &= a_{nn}(x)\mathcal{Z}''\big(d(x, \mathcal{S}_2)\big) + \big[\mathcal{M}d(x, \mathcal{S}_2)\big]\, \mathcal{Z}'\big(d(x, \mathcal{S}_2)\big) \geq \\ &\geq \beta \mathcal{Z}''\big(d(x, \mathcal{S}_2)\big) - C\mathcal{Z}'\big(d(x, \mathcal{S}_2)\big) = -\underline{\rho}\big(d(x, \mathcal{S}_2)\big) \geq -\rho(x)\end{aligned}$$

for any $x \in \mathcal{S}_2^{\varepsilon}$; then inequality (3.5.5) follows.

In view of inequality (3.5.5), $\tilde{Z} - H$ with suitable H is a subsolution in $\mathcal{S}_2^{\varepsilon}$ of the differential equation in (3.1.9). To complete the proof, we must extend its definition to $(\Omega \cup \mathcal{R}) \setminus \mathcal{S}_2^{\varepsilon}$ so as to exhibit a subsolution of problem (3.1.9). This is easily done arguing as in part (ii) of the proof of Theorem 3.2.12; we leave the details to the reader. Hence the result follows. □

Proof of Theorem 3.2.18. (i) Fix $\varepsilon \in (0, \bar{\varepsilon}]$ so small that $\varepsilon < \sigma$, with σ as in Lemma 3.4.1, and $\overline{\mathcal{S}_1^{\varepsilon}} \cap \mathcal{R} = \emptyset$. Fix any $x \in \overline{\mathcal{S}_1^{\varepsilon}} \setminus \mathcal{S}_1$; as in the proof of Theorem 3.2.16, it is not restrictive to assume (3.5.6)-(3.5.7). In view of the assumptions about the coefficients a_{ij}, b_i, we can choose $C > 0$ such that

$$|\mathcal{M} d(x, \mathcal{S}_1)| \le C \quad \text{in } \overline{\mathcal{S}_1^{\varepsilon}} \setminus \mathcal{S}_1 \,. \tag{3.5.8}$$

Then choose $c_1 > 0$ such that

$$c_1 \le \frac{1}{C} \int_{\varepsilon/2}^{\varepsilon} e^{\frac{C}{\alpha}\eta}\, \overline{\rho}(\eta) d\eta \,. \tag{3.5.9}$$

Define

$$\mathcal{V}(\zeta) := c_1 e^{-\frac{C}{\alpha}\zeta} - \frac{1}{C} \int_{\zeta}^{\varepsilon} \left[e^{\frac{C}{\alpha}(\eta - \zeta)} - 1 \right] \overline{\rho}(\eta)\, d\eta \qquad (\zeta \in (0, \varepsilon))\,,$$

with C, c_1 as above, $\alpha > 0$ being the ellipticity constant in (E_3). It is easily seen that

$$\alpha\, \mathcal{V}'' + C\, \mathcal{V}' = -\overline{\rho} \qquad \text{in } (0, \varepsilon)\,,$$
$$\mathcal{V}' > 0\,, \quad \mathcal{V}'' < 0 \qquad \text{in } (0, \varepsilon/2)$$

(use of the choice (3.5.9) is made for the latter). Using assumption (3.2.20), it is also easily checked that $\mathcal{V}$ is bounded from below in $(0, \varepsilon/2)$; moreover $\mathcal{V}$ is increasing, there, hence there exists $\mathcal{V}(0) := \lim_{\zeta \to 0^+} \mathcal{V}(\zeta)$.

(ii) Set $\tilde{V}(x) := \mathcal{V}(d(x, \mathcal{S}_1)) - \mathcal{V}(0)$ $(x \in \overline{\mathcal{S}_1^{\varepsilon/2}} \setminus \mathcal{S}_1)$; let us prove that

$$\mathcal{L}\tilde{V}(x) \le -1 \quad \text{in } \overline{\mathcal{S}_1^{\varepsilon/2}} \setminus \mathcal{S}_1 \,. \tag{3.5.10}$$

In fact, by (3.5.8) we have:

$$\mathcal{M}\tilde{V}(x) = \mathcal{V}''\big(d(x, \mathcal{S}_1)\big) \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial d(x, \mathcal{S}_1)}{\partial x_i} \frac{\partial d(x, \mathcal{S}_1)}{\partial x_j} + \big[\mathcal{M} d(x, \mathcal{S}_1)\big]\, \mathcal{V}'\big(d(x, \mathcal{S}_1)\big) \le$$
$$\le \alpha \mathcal{V}''\big(d(x, \mathcal{S}_1)\big) + C\mathcal{V}'\big(d(x, \mathcal{S}_1)\big) = -\overline{\rho}\big(d(x, \mathcal{S}_1)\big) \le -\rho(x)$$

for any $x \in \overline{\mathcal{S}_1^{\varepsilon/2}} \setminus \mathcal{S}_1$; here use of (3.2.21), assumption (E_3) and the above properties of $\mathcal{V}$ has been made. Then inequality (3.5.10) follows.

(iii) Consider the solution $W \ge 0$ of the problem

$$\begin{cases} \mathcal{L}\, W = -1 & \text{in } \Omega \setminus \overline{\mathcal{S}_1^{\varepsilon/2}} \\ W = 1 & \text{on } \mathcal{R} \\ W = 0 & \text{on } \partial \mathcal{S}_1^{\varepsilon/2} \setminus \mathcal{S}_1 \,. \end{cases} \tag{3.5.11}$$

Define

$$V := \begin{cases} c_0\, \tilde{V} & \text{in } \mathcal{S}_1^{\varepsilon/2}, \\ W + k & \text{in } \Omega \setminus \overline{\mathcal{S}_1^{\varepsilon/2}}\,, \end{cases}$$

where

$$c_0 := \max \Big\{ 1\,,\ \frac{1}{\alpha \mathcal{V}'(\varepsilon/2)} \big(\sum_{i,j=1}^{n} |a_{ij}|_{L^\infty(\Omega)} \big) |\nabla W|_{L^\infty(\partial \mathcal{S}_1^{\varepsilon/2} \setminus \mathcal{S}_1)} \Big\}\,,$$
$$k := c_0 \big(\mathcal{V}(\varepsilon/2) - \mathcal{V}(0) \big) > 0\,.$$

We shall prove the following

Claim: The function V is a positive supersolution of problem (3.1.5) satisfying condition (3.1.6).

In view of Theorem 3.1.1, from the above Claim the conclusion follows.

To prove the Claim we must show the following:
(a) $V \in C(\Omega \cup \mathcal{R})$;
(b) $\displaystyle\int_\Omega V\mathcal{M}^*\psi\, dx \,\leq\, -\int_\Omega \rho\psi\, dx$ for any $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$;
(c) condition (3.1.6) is satisfied.

To check (a), observe that the condition

$$c_0\,\tilde{V} = W + k \quad \text{on} \quad \partial\mathcal{S}_1^{\varepsilon/2} \setminus \mathcal{S}_1$$

reads

$$c_0\big(\mathcal{V}(\varepsilon/2) - \mathcal{V}(0)\big) = k\,,$$

thus it follows from the choice of k. The continuity of V elsewhere is clear from its very definition; hence (a) follows.

Concerning (b), an easy calculation gives:

$$\int_\Omega V\mathcal{M}^*\psi\, dx \,\leq\, -\int_\Omega \rho\psi\, dx \,-\int_\Gamma \sum_{i,j=1}^n a_{ij}(x)\frac{\partial}{\partial x_i}\big(c_0\,\tilde{V} - W\big)\nu_j(x)\, dx$$

for any $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$; here $\Gamma := \partial\mathcal{S}_1^{\varepsilon/2} \cap \mathit{supp}\ \psi$ and $\nu(x) \equiv (\nu_1(x), \ldots, \nu_n(x))$ denotes the outer normal to $\mathcal{S}_1^{\varepsilon/2}$ at $x \in \partial\mathcal{S}_1^{\varepsilon/2} \setminus \mathcal{S}_1$. Since

$$\sum_{i,j=1}^n a_{ij}(x)\frac{\partial\tilde{V}}{\partial x_i}\nu_j(x) = \mathcal{V}'(\varepsilon/2)\sum_{i,j=1}^n a_{ij}(x)\nu_i(x)\nu_j(x) \geq \alpha\mathcal{V}'(\varepsilon/2) > 0\,,$$

inequality (b) follows from the above choice of c_0.

Finally, concerning (c) observe that

$$\inf_{\Omega\cup\mathcal{R}} V = c_0 \inf_{\mathcal{S}_1^{\varepsilon/2}} \tilde{V} = 0 < 1 + k = \inf_{\mathcal{R}}\,(W + k) = \inf_{\mathcal{R}} V\,;$$

hence (3.1.6) is satisfied. This completes the proof. □

Proof of Theorem 3.2.21. By Proposition 3.2.7-(i) $\mathcal{S}_1$ is attracting. Then, in view of Theorem 3.2.5, there exists a solution of problem (3.1.7) satisfying (3.2.5). Since the constant in (3.2.5) is arbitrary and $\overline{\mathcal{R}} \cap \overline{\mathcal{S}_1} = \emptyset$ (see assumption $(A_1) - (ii)$), the conclusion follows. □

Proof of Theorem 3.2.23. By Proposition 3.2.9 the function Z defined in (3.2.10) is a subsolution of problem (3.1.9). In view of Theorem 3.1.2, the conclusion follows. □

It remains to prove Propositions 3.2.7-3.2.9. To this purpose we need the following lemma (see [**31**] for the proof).

LEMMA 3.5.1. *Let assumptions $(F_1) - (F_3)$ be satisfied. Suppose that $\sigma_{ij} \in C^1(\overline{\Sigma^\varepsilon})$ for some $\varepsilon > 0$ (i,j,=1..., n), and let $\Sigma \subseteq \Sigma_1 \cup \Sigma_2$ be a smooth connected component of $\partial\Omega$. Then:*
(i) for any $x \in \Sigma$ there holds

$$\frac{\partial d(x,\Sigma)}{\partial x_i} = -\nu_i(x)\,, \tag{3.5.12}$$

$$\sum_{i,j=1}^n a_{ij}(x)\,\frac{\partial^2 d(x,\Sigma)}{\partial x_i \partial x_j} \;=\; \sum_{i,j=1}^n \frac{\partial a_{ij}(x)}{\partial x_j}\,\nu_i(x)\,; \tag{3.5.13}$$

(ii) there exist $\varepsilon > 0$, $C > 0$ such that, for any $x \in \Sigma^\varepsilon$,

$$\sum_{i,j=1}^n a_{ij}(x)\,\frac{\partial d(x,\Sigma)}{\partial x_i}\,\frac{\partial d(x,\Sigma)}{\partial x_j} \;\leq\; C\,[d(x,\Sigma)]^2\,; \tag{3.5.14}$$

moreover, if $\Sigma \subseteq \Sigma_1$, *there holds*

$$\mathcal{M}d(x,\Sigma) \geq -C\, d(x,\Sigma)\,. \tag{3.5.15}$$

Proof of Proposition 3.2.7. (i) Since $\Sigma \subseteq \Sigma_2$ is compact, there holds

$$C := \min_{x\in\Sigma} \frac{\beta_F(x)}{\rho(x)} > 0\,.$$

Hence from (3.5.12)-(3.5.13) we obtain

$$\mathcal{L}\, d(x,\Sigma) = -\frac{1}{\rho(x)} \sum_{i=1}^{n} \Big[b_i(x) - \sum_{j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j} \Big] \nu_i(x) = -\frac{\beta_F(x)}{\rho(x)} < -C$$

for any $x \in \Sigma$. By continuity we get

$$\mathcal{L}d(x,\Sigma) \leq -C \quad \text{in } \Sigma^{\varepsilon} \tag{3.5.16}$$

with $\varepsilon > 0$ suitably small. Choosing $V := \dfrac{d(\cdot,\Sigma)}{C}$ in Definition 3.2.3, the claim follows.
(ii) Let $x^0 \in \Sigma$; fix $r > 0$ so small that $\overline{\Omega \cap B_r(x^0)} \subseteq \Sigma^{\varepsilon}$. Define

$$h(x) := \lambda_1 \big[d(x,\Sigma) + \lambda_2\, |x - x^0|^2 \big] \qquad (x \in \overline{\Omega \cap B_r(x^0)}; \lambda_1, \lambda_2 > 0)\,.$$

It is easily seen that

$$\mathcal{L}h(x) = \lambda_1 \Big\{ \mathcal{L}d(x,\Sigma) + \frac{2\lambda_2}{\rho(x)} \Big[\sum_{i=1}^{n} \Big(a_{ii}(x) + b_i(x)(x_i - x_i^0) \Big) \Big] \Big\} \tag{3.5.17}$$

for any $x \in \Omega \cap B_r(x^0)$. In view of (3.5.16) and the boundedness of the coefficients, by a proper choice of λ_1, λ_2 the conclusion follows. □

Proof of Proposition 3.2.9. Take $\varepsilon \in (0, \varepsilon_0)$ as in Lemma 3.5.1 - (ii) and such that $\overline{\Sigma^{\varepsilon}} \subseteq \Omega \cup \Sigma$. Then it is easily seen that for any $x \in \Sigma^{\varepsilon/2}$

$$\begin{aligned}\mathcal{L}Z(x) = \frac{[d(x,\Sigma)]^{-\alpha}}{\rho(x)} \Big\{ \alpha [d(x,\Sigma)]^{-1} \mathcal{M}d(x,\Sigma) - \\ - \alpha(\alpha+1)[d(x,\Sigma)]^{-2} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial d(x,\Sigma)}{\partial x_i} \frac{\partial d(x,\Sigma)}{\partial x_j} \Big\} \geq \\ \geq \frac{[d(x,\Sigma)]^{-\alpha}}{\rho(x)} \Big\{ -\alpha\, C - \alpha(\alpha+1) C \Big\} \geq -C_1 [d(x,\Sigma)]^{-\alpha}\,,\end{aligned}$$

where $C_1 = C_1(\alpha) := [\alpha(\alpha+2)C] / \min_{\overline{\Omega}} \rho$; here use of (3.5.14)-(3.5.15) has been made. Then

$$\mathcal{L}Z \geq C_1(Z + H) \geq C_1 Z \quad \text{in } \Sigma^{\varepsilon/2} \tag{3.5.18}$$

with $\varepsilon > 0$ suitably small (see (3.2.9)-(3.2.10)). On the other hand, there holds

$$\mathcal{L}Z = 0 \quad \text{in } \Omega \setminus \Sigma^{\varepsilon}\,, \tag{3.5.19}$$

$$\mathcal{L}Z(x) \geq -C_2 \quad \text{in } \Sigma^{\varepsilon} \setminus \Sigma^{\varepsilon/2} \tag{3.5.20}$$

for some $C_2 > 0$. Choosing $H \geq C_2/c_0$ and $\alpha > 0$ so small that $C_1 \leq c_0$, from (3.5.18)-(3.5.20) we obtain

$$\mathcal{L}Z \geq c_0 Z \quad \text{in } \Omega\,;$$

then the conclusion follows. □

3.6. Examples

In this section we give a few applications of the above results, limiting ourselves to the elliptic case. We always assume $\phi \in C(\Omega) \cap L^\infty(\Omega)$, $\gamma \in C(\mathcal{R})$.

(a) *An application of Theorem 3.2.12.* Consider the problem

$$\text{(3.6.1)} \qquad \begin{cases} (x^2+d^2)u_{xx} + 2xyu_{xy} + (y^2+d^2)u_{yy} + u_{zz} - u = \phi & \text{in } \Omega \\ \\ u = \gamma & \text{on } \mathcal{R}\,, \end{cases}$$

where $\Omega := B_2(0) \setminus \{(x,y,0) \,|\, x^2+y^2=1\} \subseteq I\!R^3$, $\mathcal{R} = \partial B_2(0)$, $\mathcal{S} = \{(x,y,0) \,|\, x^2+y^2=1\}$, $d \equiv d\big((x,y,z), \mathcal{S}\big)$.

Here $dim\, \mathcal{S} = n-2$; moreover, the diffusion matrix of the operator considered in (3.6.1) is elliptic in $\Omega \cup \mathcal{R}$, but has points of degeneracy on $\mathcal{S}$.

It is easily seen that in this case

$$M(x,y,z) = \begin{pmatrix} -y & x & 0 \\ -x & -y & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_\perp(x,y,z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \big((x,y,z) \in \mathcal{S}\big).$$

By Theorems 3.2.5 (with $\mathcal{S}_1 = \emptyset$, $F \equiv 1$), 3.2.12 and Remark 3.4.2, problem (3.6.1) is well posed in $L^\infty(\Omega)$.

(b) *The limiting value $\beta = 1$ in condition $(A_2) - (ii)'$ of Theorem 3.2.12 is not allowed.* Consider the problem

$$\text{(3.6.2)} \qquad \begin{cases} \Delta\, u - \frac{4}{x^2+y^2}(xu_x + yu_y) - u = \phi & \text{in } \Omega \\ \\ u = \gamma & \text{on } \mathcal{R}\,, \end{cases}$$

where $\Omega = B_1(0) \setminus \{0\} \subseteq I\!R^2$, $\mathcal{R} = \partial B_1(0)$, $\mathcal{S} = \{0\}$.

The function $V(x,y) := x^2+y^2$ is a supersolution of equation

$$\mathcal{L}V = -1 \quad \text{in } \Omega\,;$$

moreover, it satisfies

$$0 = \inf_{\Omega \cup \mathcal{R}} V < \inf_{\mathcal{R}} V = 1.$$

By Theorem 3.1.1, problem (3.6.2) has infinitely many bounded solutions.

Observe that $dim\, \mathcal{S} = 0 = n-2$ and $r(0) = 2$, but Theorem 3.2.12 does not apply. In fact, condition $(A_2) - (ii)'$ is not satisfied, since

$$|b(x,y)| = \frac{4}{\sqrt{x^2+y^2}} \qquad \big((x,y) \in \Omega\big).$$

Similar remarks can be made when $r(y) \geq 3$ or $r(y) \geq 4$ for any $y \in \mathcal{S}$; we omit the details.

(c) *A comparison between the results of Theorems 3.2.16, 3.2.20 and those of Theorems 3.2.21, 3.2.23.* Consider the problem

$$\text{(3.6.3)} \qquad \begin{cases} d^\alpha \Delta\, u - u = \phi & \text{in } \Omega \\ \\ u = \gamma & \text{on } \mathcal{R}\,, \end{cases}$$

where $\alpha \in I\!R$, $\alpha \neq 0$, $\Omega = B_2(0) \setminus \overline{B}_1(0) \subseteq I\!R^3$, $\mathcal{R} = \partial B_2(0)$, $\mathcal{S} = \partial B_1(0)$, $d \equiv d(x) \equiv d(x, \mathcal{S})\, (x \in \Omega \cup \mathcal{R})$. Depending on the values of the parameter $\alpha \neq 0$ different situations arise, as discussed below.

Nonuniqueness case: $\alpha < 2$. Set $\rho(x) := [d(x)]^{-\alpha}$ $(x \in \Omega \cup \mathcal{R})$. By Theorem 3.2.20, problem (3.6.3) admits infinitely many bounded solutions in $L^\infty(\Omega)$.

It is worth observing that Theorems 3.2.21 and 3.2.23 do not apply in this case, due to the lack of regularity. In fact, problem (3.6.3) can be regarded as a particular case of problem (3.1.7) with

$$\mathcal{S} = \mathcal{S}_1, \quad \rho = c \equiv 1, \quad a_{ij} = d^{\alpha}\delta_{ij}, \quad b_i \equiv 0 \qquad (i, j = 1, 2, 3)\,.$$

When $\alpha < 1$ condition $(F_2) - (ii)$ is not satisfied; hence the functions α_F and β_F are not well defined and Theorem 3.2.21 cannot be applied. However, its conclusion holds true, as seen above.

When $\alpha = 1$ conditions $(F_1) - (F_3)$ are satisfied; moreover, it is easily seen that $\mathcal{S}_1 \subseteq \Sigma_2$. However, Theorem 3.2.21 cannot be applied, since $\sigma_{ij} = d^{1/2}\delta_{ij} \notin C^1\big(\overline{\mathcal{S}_1^{\varepsilon}}\big)$ (although its conclusion holds true, as already remarked).

When $\alpha \in (1, 2)$ condition $(F_2) - (ii)$ is not satisfied, yet in this case the functions α_F and β_F are well defined and $\mathcal{S}_1 \subseteq \Sigma_1$. Theorem 3.2.23 cannot be applied (observe that moreover $\sigma_{ij} = d^{\alpha/2}\delta_{ij} \notin C^1\big(\overline{\mathcal{S}_1^{\varepsilon}}\big)$); in fact, as already seen, its conclusion is false in this case.

Uniqueness case: $\alpha \geq 2$. Set $\rho(x) := [d(x)]^{-\alpha}$ $(x \in \Omega \cup \mathcal{R})$. In view of Theorem 3.2.16, there exists at most one bounded solution of problem (3.6.3).

Observe that problem (3.6.3) is a particular case of problem (3.1.7) with

$$\mathcal{S} = \mathcal{S}_2, \quad \rho = c \equiv 1, \quad a_{ij} = d^{\alpha}\delta_{ij}, \quad b_i \equiv 0 \qquad (i, j = 1, 2, 3)\,.$$

In this case conditions $(F_1) - (F_3)$ are satisfied; moreover, it is easily seen that $\mathcal{S}_2 \subseteq \Sigma_1$. By Theorem 3.2.23 there exists at most one bounded solution of problem (3.6.3), in agreement with the above conclusion obtained by Theorem 3.2.16.

CHAPTER 4

On the refined maximum principle for degenerate elliptic and parabolic problems

4.1. Introduction

Consider an elliptic operator of the form $\mathcal{L}-c$, where

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i}$$

and c is a given function, in an open subset $\Omega \subseteq I\!R^n$ with boundary $\partial\Omega$. In the following we assume Ω to be bounded, although the case of unbounded domains can also be treated.

Concerning the coefficients of the operator $\mathcal{L}$ and the function c, in [**7**] the following assumption was made:

$$(H_0) \qquad \begin{cases} (i) \quad a_{ij}=a_{ji}\in C(\Omega),\ b_i,\ c\in L^\infty(\Omega) \ \ (i,j=1,\dots,n)\,; \\ (ii) \text{ there exist } c_0,\ C_0>0 \text{ such that} \\ \qquad c_0|\xi|^2 \le \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \le C_0|\xi|^2 \text{ for any } x\in\Omega \text{ and } (\xi_1,..,\xi_n)\in I\!R^n\,. \end{cases}$$

Under assumption (H_0), the so-called *refined maximum principle* for the operator $\mathcal{L}-c$ was proved. By standard argument, this implies uniqueness in $L^\infty(\Omega)$ of solutions to the corresponding Dirichlet problem, where boundary data need not be prescribed at any point of the boundary $\partial\Omega$ (see Section 4.2 for details).

In fact, uniqueness in $L^\infty(\Omega)$ is ensured whenever Dirichlet boundary conditions are specified in the subset of $\partial\Omega$ where the *minimal* positive solution U_0 to the *first exit time equation*

$$\mathcal{L}U = -1 \quad \text{in } \Omega \tag{4.1.1}$$

can be prolonged to zero in a suitable sense. Therefore, in view of the well-known probabilistic interpretation of equation (4.1.1), boundary conditions are only needed at those points of $\partial\Omega$ which can be attained by trajectories of a Markovian particle, starting at some $x_0\in\Omega$, with generator $\mathcal{L}$ (see [**31**], [**35**]).

The purpose of the present paper is twofold:

- A) We prove the refined maximum principle, and its analogue for the *parabolic* operator $\mathcal{L}-c-\partial/\partial t$, under more general assumptions on the coefficients a_{ij}, b_i, c (see assumption (H_1) below). In fact, our assumptions (although slightly more demanding as for regularity with respect to $(H_0)-(i)$) allow the coefficients of $\mathcal{L}$ and the function c to vanish or diverge when $dist(x,\partial\Omega)\to 0$. In addition, ellipticity of $\mathcal{L}$ is possibly lost in Ω and/or when $dist(x,\partial\Omega)\to 0$.
- B) Relying on A) above, we obtain uniqueness of solutions in $L^\infty(\Omega)$ for the corresponding Dirichlet boundary value problem, and in $L^\infty(\Omega\times(0,T])$ for the Dirichlet initial-boundary value problem (see (4.2.11)). We also discuss the link between such results and those obtained in the papers [**60**] - [**65**] by a different approach.

Further interesting information comes from considering *existence* of the Dirichlet boundary value problem. For instance, existence of solutions to problem (4.2.11) cannot be ensured without a particular choice of the boundary data (see Theorem 4.3.1 and condition(4.3.1)). This reveals that the refined maximum principle is in general inaccurate, for it leads to prescribe boundary data on a larger subset than it is required to make the problem well-posed. Not surprisingly, prescribing boundary data on a set "too large" gives rise to nonexistence (see Section 4.5)..

Let us finally observe that both the refined maximum principle and the approach to uniqueness problems developed in [**60**]- [**65**] are deeply connected with classical results of the wide literature concerning degenerate elliptic and parabolic problems (in particular, see [**26**], [**56**] and references therein). We refer the reader to [**65**] for a discussion of this point.

4.2. The refined maximum principle

Concerning the coefficients of $\mathcal{L}$ and the function c, in this section we assume the following:

$$(H_1)\qquad \begin{cases} (i)\ \ a_{ij}=a_{ji}\in C^{1,1}(\Omega),\ b_i\in C^{0,1}(\Omega)\ \ (i,j=1,\dots,n)\,; \\ (ii)\ \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j>0 \text{ for any } x\in\Omega \text{ and any nonzero } (\xi_1,..,\xi_n)\in I\!R^n\,; \\ (iii)\ c\in C(\Omega)\,. \end{cases}$$

In the sequel we denote by $\mathcal{L}^*$ the formal adjoint of the operator $\mathcal{L}$:

$$\mathcal{L}^*v\equiv\sum_{i,j=1}^n\frac{\partial^2(a_{ij}v)}{\partial x_i\partial x_j}-\sum_{i=1}^n\frac{\partial(b_iv)}{\partial x_i}\,.$$

Moreover, notice that sub– supersolutions and solutions of the equation

$$\mathcal{L}u-cu=\phi\quad\text{in }\Omega\,,\tag{4.2.1}$$

respectively,

$$\mathcal{L}u-cu-\partial_tu=f\quad\text{in }Q_T\tag{4.2.2}$$

are meant in the sense of Definition 1.2.1, respectively Definition 2.2.1; here $\phi\in C(\Omega)$, $f\in C(Q_T)$.

Remark 4.2.1. (i) The above definitions of solution to equations (4.2.1) and (4.2.2) are equivalent to those of *viscosity solutions* of the same equations (see [**59**]-[**65**]).
(ii) If $c\geq 0$, comparison principles hold true for equation (4.2.1) in any open $\Omega_1\subseteq\Omega$ such that $\overline{\Omega}_1\subseteq\Omega\cup\mathcal{R}$. The same statement holds for equation (4.2.2) in any open cylinder $\Omega_1\times(0,T]\subseteq Q_T$, without assumptions about the sign of c (see [**59**]-[**65**]).

Inspired by [**7**], we introduce the following notation. Let $\{x_n\}\subseteq\Omega$ and $U\in C(\Omega)$; then

$$x_n\overset{U}{\to}\partial\Omega\qquad\overset{\text{def}}{\Longleftrightarrow}\qquad\begin{cases}\text{there exists } x_0\in\partial\Omega\\ \text{such that } x_n\to x_0,\ U(x_n)\to 0\,.\end{cases}\tag{4.2.3}$$

If $F\in C(\Omega)$, $G\in C(\partial\Omega)$,

$$F\overset{U}{=}G\text{ on }\partial\Omega\qquad\overset{\text{def}}{\Longleftrightarrow}\qquad\begin{cases}F(x_n)\to G(x_0)\ \text{ for any } x_0\in\partial\Omega\\ \text{such that } x_n\to x_0,\ U(x_n)\to 0\,.\end{cases}\tag{4.2.4}$$

Denote by $\mathcal{Z}\subseteq\partial\Omega$ the set of limiting points of sequences $\{x_n\}\subseteq\Omega$ such that $x_n\overset{U}{\to}\partial\Omega$, namely

$$\mathcal{Z}:=\{x_0\in\partial\Omega\,|\,x_n\to x_0,\ U(x_n)\to 0\text{ for some}\,\{x_n\}\subseteq\Omega\}\,.$$

In this parlance,

$$F\overset{U}{=}G\text{ on }\partial\Omega\qquad\Longleftrightarrow\qquad F(x_n)\to G(x_0)\,\text{for any }\ x_0\in\mathcal{Z}\,,\tag{4.2.5}$$

whenever $x_n\to x_0$ and $U(x_n)\to 0$. If $U>0$ in Ω, it is easily seen that

$$\mathcal{Z}=\{x_0\in\partial\Omega\,|\,\liminf_{x\to x_0}U(x)=0\}\,.\tag{4.2.6}$$

REMARK 4.2.2. In [**7**] the following definition was made:

$$x_n \stackrel{U}{\to} \partial\Omega \qquad \stackrel{\text{def}}{\Longleftrightarrow} \qquad dist(x_n, \partial\Omega) \to 0,\, U(x_n) \to 0 \text{ as } n \to \infty\,.$$

Since Ω is bounded, this is equivalent to (4.2.3).

A similar notation can be used in the parabolic case. For any $\{(x_n, t_n)\} \subseteq \Omega \times [0, T]$ and $U \in C(\Omega)$ we set

$$(x_n, t_n) \stackrel{U}{\to} \partial\Omega \times [0, T] \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \begin{cases} \text{there exists } (x_0, t_0) \in \partial\Omega \times [0, T] \\ \text{such that } (x_n, t_n) \to (x_0, t_0),\ U(x_n) \to 0\,, \end{cases} \tag{4.2.7}$$

and, if $f \in C(Q_T)$, $g \in C(\partial\Omega \times [0, T])$,

$$f \stackrel{U}{=} g \text{ in } \partial\Omega \times [0, T] \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \begin{cases} f(x_n, t_n) \to g(x_0, t_0) \text{ for any } (x_0, t_0) \in \partial\Omega \times [0, T] \\ \text{such that } (x_n, t_n) \to (x_0, t_0),\ U(x_n) \to 0\,. \end{cases} \tag{4.2.8}$$

The following lemma will be needed.

LEMMA 4.2.3. *Let assumption* (H_1) *be satisfied; moreover, let there exist a supersolution* $F > 0$ *of equation (4.1.1). Then there exists a minimal positive solution* U_0 *of (4.1.1) such that* $0 < U_0 \leq F$ *in* Ω.

The proof is the same as in [**7**]; we sketch it for further reference. Consider for any $j \in I\!N$ the elliptic problem:

$$\begin{cases} \mathcal{L}\, U_j \; = -1 & \text{in } \; H_j \\ \\ \; U_j \; = 0 & \text{on } \; \partial H_j\,, \end{cases} \tag{4.2.9}$$

where $\{H_j\}$ is a sequence of bounded domains with smooth boundary ∂H_j such that

$$\overline{H}_j \subseteq \Omega, \qquad H_j \subseteq H_{j+1}, \qquad \bigcup_{j=1}^{\infty} H_j = \Omega\,. \tag{4.2.10}$$

By standard existence and comparison results, there exists a sequence of functions $\{U_j\}$ with the following properties:

- U_j is a classical solution of problem (4.2.9);
- $0 < U_j \leq U_{j+1} \leq F$ in H_j $\; (j \in I\!N)$.

By compactness arguments there exists a subsequence of $\{U_j\}$ (also denoted by $\{U_j\}$) which converges uniformly in any compact subset of Ω. It is easily seen that its limit

$$U_0 := \lim_{j \to +\infty} U_j \; \text{ in } \; \Omega.$$

has the properties stated in Lemma 4.2.3, thus the result follows.

In [**7**] the existence of a *bounded* supersolution $F > 0$ of equation (4.1.1) was ensured by the boundedness of the coefficients and the uniform ellipticity of $\mathcal{L}$ (see assumption (H_0)). Therefore the function U_0 was also bounded, whereas in the present case both F and U_0 are possibly unbounded (see Definition 1.2.1).

In the following we always assume the existence of the solution U_0 considered in Lemma 4.2.3. Then we address the parabolic initial-boundary value problem

$$\begin{cases} \mathcal{L}\, u \; - cu - \partial_t u = f(x, t,\, u) & \text{in } \; Q_T \\ \\ \; u \stackrel{U_0}{=} g & \text{in } \; \partial\Omega \times (0, T] \\ \\ \; u \; = u_0 & \text{in } \; \Omega \times \{0\}\,. \end{cases} \tag{4.2.11}$$

Observe that in the above problem boundary conditions are only prescribed on the set $\mathcal{Z} \times [0,T]$ (see (4.2.5)).The following assumption will be made:

$$(H_2) \qquad \begin{cases} (i) \quad f \in C(\overline{Q}_T \times I\!R) \text{ Lipschitz continuous} \\ \qquad \text{with respect to } u \in I\!R, \text{ uniformly for } (x,t) \in \overline{Q}_T\ ; \\ (ii) \quad g \in C(\partial\Omega \times [0,T]),\ u_0 \in C(\Omega)\,; \\ (iii)\, u_0 \overset{U_0}{=} g(\cdot,0) \text{ on } \partial\Omega\,. \end{cases}$$

Solutions to problem (4.2.11) are meant in the following sense.

Definition 4.2.4. *Let assumptions* (H_1) *and* (H_2) *be satisfied. By a* solution *to problem* (4.2.11) *we mean any function* $u \in C(\Omega \times [0,T])$ *such that:*
(i) u *is a solution of equation* (4.2.2)*;*
(ii) $u \overset{U_0}{=} g$ *in* $\partial\Omega \times [0,T]$*;*
(iii) $u = u_0$ *in* $\Omega \times \{0\}$.
Subsolutions *and* supersolutions *of (4.2.11) are similarly defined.*

Now we can state the following *parabolic refined maximum principle.*

Theorem 4.2.5. *Let assumptions* (H_1) *and* (H_2) *be satisfied; suppose* $c \geq c_1$ *for some* $c_1 \in I\!R$. *Let* u *be a subsolution of problem* (4.2.11) *with* $f = g = u_0 = 0$ *bounded from above. Then* $u \leq 0$ *in* Q_T.

From Theorem 4.2.5 we deduce by usual arguments the following uniqueness result.

Theorem 4.2.6. *Let assumptions* (H_1) *and* (H_2) *be satisfied; suppose* $c \geq c_1$ *for some* $c_1 \in I\!R$. *Then there exists at most one bounded solution to problem* (4.2.11).

Similar results hold for the elliptic problem

$$(4.2.12) \qquad \begin{cases} \mathcal{L}u - cu = f(x) & \text{in } \Omega \\ \\ u \overset{U_0}{=} \gamma & \text{in } \partial\Omega\,, \end{cases}$$

assuming (H_1) and

$$(H_3) \qquad f \in C(\overline{\Omega})\,, \quad \gamma \in C(\partial\Omega)\,.$$

Definition 4.2.7. *Let assumptions* (H_1) *and* (H_3) *be satisfied. By a* solution *to problem* (4.2.12) *we mean any function* $u \in C(\Omega)$ *such that:*
(i) u *is a solution of equation* (4.2.1) *with* $\phi = f$*;*
(ii) $u \overset{U_0}{=} \gamma$ *in* $\partial\Omega$.
Subsolutions *and* supersolutions *of (4.2.12) are similarly defined.*

Theorem 4.2.8. *Let assumptions* (H_1) *and* (H_3) *be satisfied; suppose* $c \geq 0$. *Let* u *be a subsolution of problem* (4.2.11) *with* $f = \gamma = 0$ *bounded from above. Then* $u \leq 0$ *in* Ω.

Theorem 4.2.9. *Let assumptions* (H_1) *and* (H_3) *be satisfied; suppose* $c \geq 0$. *Then there exists at most one bounded solution to problem* (4.2.12).

4.3. Existence results

Concerning existence of solutions of problem (4.2.11) we shall prove the following result.

THEOREM 4.3.1. *Let assumptions* (H_1) *and* (H_2) *be satisfied with* g *independent of* x, *namely*

$$g(x,t) = g_1(t) \qquad ((x,t) \in \partial\Omega \times [0,T]) \tag{4.3.1}$$

for some $g_1 \in C([0,T])$. *Suppose* $f(\cdot,\cdot,0) \in L^\infty(Q_T)$ *and* c, $u_0 \in L^\infty(\Omega)$. *Then there exists a bounded solution to problem* (4.2.11).

In connection with the above theorem, let us mention that in [**7**] the existence of bounded solutions to problem (4.2.12) was proven in the particular case $\gamma = 0$. The proof extends to the case $\gamma =$ *constant*, which is the counterpart of condition (4.3.1) in the elliptic case. However, existence cannot be expected for problem (4.2.12) with any $\gamma \in C(\partial\Omega)$, nor for problem (4.2.11) if condition (4.3.1) is not satisfied (see Section 4.5).

A similar constraint on boundary conditions was already pointed out in [**60**], [**59**] in a more general framework, whenever a subset $\Sigma \subseteq \partial\Omega$ is *attracting* in the sense of the following definition. The reason for this is that the function V below can be regarded as a barrier *for the whole of* Σ.

DEFINITION 4.3.2. *A subset* $\Sigma \subseteq \partial\Omega$ *is* attracting *for the operator* $\mathcal{L}$ *if there exist* $\varepsilon \in (0,\varepsilon_0)$ *and a supersolution* $V \in C(\overline{\Sigma^\varepsilon})$ *of the equation:*

$$\mathcal{L}V = -1 \quad in\ \Sigma^\varepsilon \tag{4.3.2}$$

such that

$$V > 0 \quad in\ \overline{\Sigma^\varepsilon} \setminus \Sigma\,, \qquad V = 0 \quad on\ \Sigma\,.$$

Here

$$\Sigma^\varepsilon := \{x \in \Omega \,|\, dist(x,\Sigma) < \varepsilon\} \qquad (\varepsilon > 0)\,.$$

In the present case, U_0 plays the role of the function V, and the whole of $\partial\Omega$ can be regarded as "attracting in the sense of U_0". In fact, the proof of Theorem 4.3.1 is very similar to that of Theorem 2.32 in [**59**]; therefore it will be omitted.

In connection with the above remarks let us observe that, if Σ is attracting and the coefficients $a_{i,j}$, b_i are bounded in Σ^ε for some $\varepsilon > 0$, for any $(x_0,t_0) \in \Sigma \times [0,T]$ a *local barrier* can be constructed - namely, for any $(x_0,t_0) \in \Sigma \times [0,T]$ there exist $\delta > 0$ and a supersolution $h \in C(\overline{K_\delta(x_0,t_0)})$ of the equation

$$\mathcal{L}h - ch - \partial_t h = -1 \quad \text{in } K_\delta(x_0,t_0)\,,$$

such that

$$h > 0 \quad \text{in } \overline{K_\delta(x_0,t_0)} \setminus \{(x_0,t_0)\} \quad \text{and} \quad h(x_0,t_0) = 0$$

(here $K_\delta(x_0,t_0) := \big(B_\delta(x_0) \times (t_0 - \delta, t_0 + \delta)\big) \cap Q_T$; *e.g.*, see [**30**]). In fact, it is easily seen that the function

$$h(x,t) := \lambda_1[\exp\{|c_1|t\}V(x) + \lambda_2(|x - x_0|^2 + (t - t_0)^2)]$$

has the above properties for $\lambda_1 > 0$ big enough, $\lambda_2 > 0$ sufficiently small and $\delta \in (0,\varepsilon)$.

As a consequence, general Dirichlet data g can be assigned on $\Sigma \times [0,T]$. This gives the following theorem, whose standard proof is omitted.

THEOREM 4.3.3. *Let assumptions* (H_1) *and* (H_2) *be satisfied; suppose* $a_{ij}, b_i, c, u_0 \in L^\infty(\Omega)$ $(i,j = 1,\dots,n)$ *and* $f(\cdot,\cdot,0) \in L^\infty(Q_T)$. *Then there exists a bounded solution to problem* (4.2.11).

An analogous result holds for the elliptic problem (4.2.12). In this connection, observe that assumption (H_0) made in [**7**] implies $a_{ij},\ b_i \in L^\infty(\Omega)$.

4.4. Proofs

The following lemma is an easy extension of Remark 1.2 in [**7**].

LEMMA 4.4.1. *Let* $w : \Omega \times [0,T] \to I\!R$ *and* $U \in C(\Omega)$, $U > 0$ *in* Ω. *Let*

$$\limsup_{n\to\infty} w(x_n,t_n) \le 0 \quad \text{whenever} \quad (x_n,t_n) \overset{U}{\to} \partial\Omega \times [0,T]\,. \tag{4.4.1}$$

Then for any $\varepsilon > 0$ *there exists* $\delta > 0$ *such that:*
(i) $w(x,t) \le \varepsilon$ *at any* $(x,t) \in \Omega \times [0,T]$ *where* $U(x) < \delta$*;*
(ii) if moreover $w \le N$ *in* $\Omega \times [0,T]$ *for some* $N > 0$, *there holds*

$$w(x,t) \le \varepsilon + \frac{N}{\delta}\, U(x) \quad \text{for any } (x,t) \in \Omega \times [0,T]\,. \tag{4.4.2}$$

Proof. (i) By contradiction, let there exist $\varepsilon_0 > 0$ such that for any $n \in I\!N$ there exists $(x_n,t_n) \in \Omega \times [0,T]$ such that

$$0 < U(x_n) < \frac{1}{n}\,, \quad w(x_n,t_n) \ge \varepsilon_0\,. \tag{4.4.3}$$

Since Ω is bounded, there exists a converging subsequence $\{(x_{n_k},t_{n_k})\} \subseteq \{(x_n,t_n)\}$. We claim that its limit (x_0,t_0) belongs to $\partial\Omega \times [0,T]$. In fact, were $(x_0,t_0) \in \Omega \times [0,T]$, by the first inequality in (4.4.3) we would have

$$\lim_{k\to\infty} U(x_{n_k}) = 0 = U(x_0)\,,$$

since $U \in C(\Omega)$. However, $U(x_0) > 0$ since $x_0 \in \Omega$ and $U > 0$ in Ω by assumption. The contradiction proves the claim.
It follows that

$$(x_{n_k},t_{n_k}) \overset{U}{\to} \partial\Omega \times [0,T] \quad \text{and} \quad w(x_n,t_n) \ge \varepsilon_0 \quad \text{for any} \quad k \in I\!N\,,$$

which contradicts assumption (4.4.1). Then claim (i) follows.
(ii) Fix any $(x,t) \in \Omega \times [0,T]$. If $U(x) < \delta$, then by (i) and the positivity of U in Ω we get

$$w(x,t) \le \varepsilon < \varepsilon + \frac{N}{\delta}\, U(x)\,.$$

Otherwise, we have

$$w(x,t) \le N \le \frac{N}{\delta}\, U(x) < \varepsilon + \frac{N}{\delta}\, U(x)\,.$$

This completes the proof. □

Proof of Theorem 4.2.5 By assumption there exists $N > 0$ such that

$$u(x,t) \le N \quad \text{for any } (x,t) \in \Omega \times [0,T].$$

Let U_j $(j \in I\!N)$ and U_0 be defined as in the proof of Lemma 4.2.3. Since $U_0 > 0$ and $u \overset{U_0}{=} 0$ in $\partial\Omega \times [0,T]$ (see assumption (H_2)), we can apply Lemma 4.4.1 with $v = U_0, w = u$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$u(x,t) \le \varepsilon + \frac{N}{\delta}\, U_0(x) \text{ for any } (x,t) \in \Omega \times [0,T]\,. \tag{4.4.4}$$

We shall prove the following
Claim: For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $j \in I\!N$ the function

$$v(x,t) := \left\{\frac{N}{\delta}[U_0(x) - U_j(x)] + \varepsilon\right\} \exp(|c_1|t) - u(x,t) \qquad \big((x,t) \in \bar{H}_j \times [0,T]\big)$$

is a supersolution of problem

$$\left\{\begin{array}{lll}\mathcal{L}u - cu - \partial_t u = 0 & \text{in} & H_j \times (0,T) \\ u = 0 & \text{in} & \partial H_j \times (0,T] \\ u = 0 & \text{in} & H_j \times \{0\}\,. \end{array}\right. \tag{4.4.5}$$

Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $j \in I\!N$

$$u(x,t) \le \Big\{\frac{N}{\delta}[U_0(x) - U_j(x)] + \varepsilon\Big\} \exp(|c_1|t) \qquad \big((x,t) \in H_j \times [0,T]\big). \tag{4.4.6}$$

Fix any $(x,t) \in Q_T$, and $j_0 \in I\!N$ such that $(x,t) \in H_j \times (0,T)$ for any $j \ge j_0$. Letting first $j \to \infty$, then $\varepsilon \to 0$ in (4.4.6) we obtain $u(x,t) \le 0$; then the conclusion follows.

It remains to prove the Claim. By construction there holds $U_0 \ge U_j$, thus v is positive in $\Omega \times [0,T]$. To prove that v is a supersolution of the differential equation in (4.4.5), fix $\varepsilon > 0$ arbitrarily and choose $\delta > 0$ such that (4.4.4) holds; then fix any $j \in I\!N$. Since both U_j and U_0 solve the equation

$$\mathcal{L}U \;=\; -1 \quad \text{in } H_j\,,$$

there holds

$$\int_{H_j} \Big[\frac{N}{\delta}(U_0 - U_j) + \varepsilon\Big]\, \mathcal{L}^*\psi\, dx = 0$$

for any fixed $t \in [0,T]$ and $\psi = \psi(\cdot,t) \in C_0^\infty(H_j \times (0,T))$, $\psi \ge 0$ (see Definition 1.2.1). Hence multiplying by $\exp(|c_1|t)$ and integrating in $(0,T)$ gives

$$\int_{H_j \times (0,T)} v\, \mathcal{L}^*\psi\, dxdt = 0. \tag{4.4.7}$$

On the other hand,

$$\int_{H_j \times (0,T)} v\, \partial_t\psi\, dxdt = -\int_{H_j \times (0,T)} |c_1|\, v\, \psi\, dxdt. \tag{4.4.8}$$

From (4.4.7)-(4.4.8) we obtain

$$\int_{H_j \times (0,T)} (v - u)\, \{\mathcal{L}^*\psi - c\psi + \partial_t\psi\}\, dxdt \le 0\,, \tag{4.4.9}$$

since u is a subsolution of problem (4.2.11) with $f = g = u_0 = 0$. This completes the proof.□

Proof of Theorem 4.2.6. Let u_1 and u_2 be two bounded solutions of problem (4.2.11). Set $\hat{u} := u_2 - u_1$, $M := \max\{\|u_1\|_\infty, \|u_2\|_\infty\}$. Then for any $\psi \in C_0^\infty(Q_T)$, $\psi \ge 0$ there holds

$$\int_{Q_T} \hat{u}\, \{\mathcal{L}^*\psi - c\psi + \partial_t\psi\}\, dxdt \;=\; \int_{Q_T} f(x,t,\, u_2) - f(x,t,\, u_1)\psi\, dxdt \ge -\int_{Q_T} L|\hat{u}|\, \psi\, dxdt\,,$$

where $L > 0$ is the Lipschitz constant of the function $f(x,t,\cdot)$ in the set $Q_T \times [-M,M]$, uniform for $(x,t) \in \bar{Q}_T$ (see Definition 2.2.1 and assumption (H_2)). Therefore $\hat{u}$ is a subsolution of the problem

$$\left\{\begin{array}{ll}\mathcal{L}u - cu + L|u| - \partial_t u = 0 & \text{in } \Omega \times (0,T) \\ u \overset{U_0}{=} 0 & \text{in } \partial\Omega \times (0,T] \\ u = 0 & \text{in } \Omega \times \{0\} \end{array}\right. \tag{4.4.10}$$

(see Definition 4.2.4), and, in view of Remark 4.2.1, a viscosity subsolution to the differential equation in (4.4.10). Clearly, the function $\hat{u}_+ := \max\{\hat{u}, 0\}$ is a viscosity subsolution to the same equation, thus a subsolution to the problem

$$\text{(4.4.11)} \qquad \begin{cases} \mathcal{L}u + (L-c)u - \partial_t u = 0 & \text{in } \Omega \times (0,T) \\ u \overset{U_0}{=} 0 & \text{in } \partial\Omega \times (0,T] \\ u = 0 & \text{in } \Omega \times \{0\} \end{cases}$$

in the sense of Definition 4.2.4), again by Remark 4.2.1. By Theorem 4.2.5 we obtain $\hat{u}_+ \leq 0$, thus $u_2 \leq u_1$. It is similarly proved that $u_1 \leq u_2$; hence the conclusion follows. □

4.5. Further remarks

In this section we show by an explicit example that solutions to problem (4.2.11) need not exist, if condition (4.3.1) is not satisfied. To this purpose we make use of the approach to uniqueness problems developed in [**60**] - [**59**] (see also [**65**] for corresponding elliptic results). As a by-product of the following discussion, it appears that this approach is sharper than that based on the refined maximum principle.

In the general setting we are dealing with, one cannot prescribe boundary data at any point of $\partial\Omega \times [0,T]$ in the parabolic case, or of $\partial\Omega$ for elliptic problems. Therefore it is natural to think of $\partial\Omega$ as the disjoint union of the *regular boundary* $\mathcal{R}$ and the *singular boundary* $\mathcal{S} := \partial\Omega \setminus \mathcal{R}$, assuming that the coefficients a_{ij}, b_i, c are regular and the operator $\mathcal{L}$ elliptic only in the set $\Omega \cup \mathcal{R}$. Accordingly, one addresses the Dirichlet initial-boundary value problem

$$\text{(4.5.1)} \qquad \begin{cases} \mathcal{L}u - cu - \partial_t u = f(x,t,u) & \text{in } Q_T \\ u = g & \text{in } \mathcal{R} \times (0,T] \\ u = u_0 & \text{in } (\Omega \cup \mathcal{R}) \times \{0\}. \end{cases}$$

If uniqueness for problem (4.5.1) fails, it is natural to try and recover it by assigning boundary data also on some subset $\mathcal{S}_1 \times [0,T]$, for some $\mathcal{S}_1 \subseteq \mathcal{S}$. This suggests to address the problem

$$\text{(4.5.2)} \qquad \begin{cases} \mathcal{L}u - cu - \partial_t u = f(x,t,u) & \text{in } Q_T \\ u = g & \text{in } (\mathcal{R} \cup \mathcal{S}_1) \times (0,T] \\ u = u_0 & \text{in } (\Omega \cup \mathcal{R} \cup \mathcal{S}_1) \times \{0\}. \end{cases}$$

Our assumptions concerning the regular boundary $\mathcal{R}$ and the singular boundary $\mathcal{S}$ are summarized as follows:

$$(H_3) \qquad \begin{cases} (i) \quad \partial\Omega = \mathcal{R} \cup \mathcal{S},\ \mathcal{R} \cap \mathcal{S} = \emptyset,\ \mathcal{S} \neq \emptyset; \\ (ii) \quad \mathcal{R} \subseteq \partial\overline{\Omega} \text{ is open, } \Omega \text{ satisfies the outer sphere condition at } \mathcal{R}; \\ (iii) \quad \mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2,\ \ \mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset; \\ (iv) \quad \mathcal{S}_1 \text{ and } \mathcal{S}_2 \text{ have a finite number of connected components.} \end{cases}$$

The counterpart of assumptions (H_1), (H_2) are

$$(H_1') \qquad \begin{cases} (i) \quad a_{ij} = a_{ji} \in C^{1,1}(\Omega \cup \mathcal{R}),\ b_i \in C^{0,1}(\Omega \cup \mathcal{R}) \ \ (i,j = 1,\dots,n); \\ (ii) \ \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j > 0 \text{ for any } x \in \Omega \cup \mathcal{R} \text{ and any nonzero } (\xi_1,..,\xi_n); \\ (iii) \ c \in C(\Omega \cup \mathcal{R}), \end{cases}$$

respectively

$$(H_2') \quad \begin{cases} (i) \quad f \in C(\overline{Q}_T \times I\!R) \text{ Lipschitz continuous} \\ \qquad \text{with respect to } u \in I\!R, \text{ uniformly for } (x,t) \in \overline{Q}_T\,; \\ (ii) \;\; g \in C((\mathcal{R} \cup \mathcal{S}_1) \times [0,T]),\; u_0 \in C(\Omega \cup \mathcal{R} \cup \mathcal{S}_1)\,; \\ (iii)\, u_0 = g(\cdot, 0) \;\text{ on } \mathcal{R} \cup \mathcal{S}_1\,. \end{cases}$$

In the following we choose $\mathcal{R}$ as *the largest subset* of $\partial\Omega$ where ellipticity of the operator $\mathcal{L}$ holds (see assumption $(H_1') - (ii)$).

Notice that solutions to the problem (4.5.2) are meant in the sense of Definition 2.2.2.

REMARK 4.5.1. In view of assumption (H_1'), there holds $U_0 \in C(\Omega \cup \mathcal{R})$ and $U_0 = 0$ on $\mathcal{R}$; hence $\mathcal{R} \subseteq \mathcal{Z}_0$, where $\mathcal{Z}_0$ denotes the set $\mathcal{Z}$ relative to the function U_0 (see (4.2.6)). As a consequence, a solution u to problem (4.2.11) belongs to $C((\Omega \cup \mathcal{R}) \times [0,T])$ and satisfies $u = g$ in $\mathcal{R} \times (0,T]$, thus it is a solution of problem (4.5.1).

In the present approach, instead of considering the minimal positive solution U_0 of equation (4.1.1) and prolonging it to zero at the boundary, we consider the problem

$$(4.5.3) \qquad \begin{cases} \mathcal{L}U = -1 & \text{in } \Omega \\ U = 0 & \text{in } \mathcal{R}\,. \end{cases}$$

By a subsolution of problem (4.5.3) we mean any function $U \in C(\Omega \cup \mathcal{R})$, which is a subsolution of equation (4.1.1) and is nonpositive on $\mathcal{R}$. Then we have the following uniqueness result (see [**59**]).

THEOREM 4.5.2. *Let assumptions* (H_3) *and* $(H_1') - (H_2')$ *be satisfied; suppose* $\mathcal{S}_2 \neq \emptyset$, $c \geq c_1$ *for some* $c_1 \in I\!R$. *Let there exist a subsolution* $Z \leq H < 0$ *of problem (4.5.3) such that*

$$(4.5.4) \qquad \lim_{dist(x,\mathcal{S}_2) \to 0} Z(x) = -\infty\,.$$

Then there exists at most one solution $u \in L^\infty(Q_T)$ *of problem (4.5.2).*

Now we can discuss the example mentioned at the beginning of this section. Consider problem (4.2.11) with

$$(4.5.5) \qquad \mathcal{L}u = \frac{1}{(-y^2 + \frac{1}{2}y)sinx}\left[u_{xx} + y^2 u_{yy}\right],$$

$\Omega := (0,\pi) \times (0, \frac{1}{2})$, $g \in C(\partial\Omega \times [0,T])$, $u_0 \in C(\overline{\Omega}), u_0 = g$ in $\partial\Omega \times \{0\}$, $f(\cdot,\cdot,0) \in L^\infty(Q_T)$. We claim that the problem does not have a bounded solution for any g as above.

To prove the claim we need some preliminary remarks.
(i) It is immediately seen that in this case $\mathcal{R} = \emptyset$, thus $\mathcal{S} = \partial\Omega$. Choose $\mathcal{S}_1 = (\{0,\pi\} \times [0,\frac{1}{2}]) \cup [0,\pi] \times \{\frac{1}{2}\}, \mathcal{S}_2 = [0,\pi] \times \{0\}$. The function $Z(x,y) := x^2 + \log y - \pi^2$ satisfies

$$Z < 0 \text{ in } \Omega\,, \qquad \mathcal{L}Z > 0 \text{ in } \Omega\,, \qquad \lim_{y \to 0} Z(x,y) = -\infty\,.$$

Then by Theorem 4.5.2 the problem

$$(4.5.6) \qquad \begin{cases} \frac{1}{(-y^2+\frac{1}{2}y)sinx}\left[u_{xx} + y^2 u_{yy}\right] - \partial_t u = 0 & \text{in } Q_T \\ u = g & \text{in } \mathcal{S}_1 \times (0,T] \\ u = u_0 & \text{in } (\Omega \cup \mathcal{S}_1) \times \{0\}\,. \end{cases}$$

has at most one bounded solution.

(ii) In this case the function U_0 belongs to $C(\overline{\Omega})$ and satisfies $U_0 = 0$ on $\partial\Omega$ - namely, $\mathcal{Z}_0 = \partial\Omega$. In fact, the function

$$V(x,y) := y(\frac{1}{2} - y)\sin x \qquad ((x,y) \in \bar{\Omega})$$

is a positive supersolution of the problem

$$\begin{cases} \mathcal{L}U = -1 & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{4.5.7}$$

(iii) Clearly, the function $\tilde{u} := U_0 - t$ is a bounded solution of problem (4.5.6) with $g(x,y,t) = -t$ and $u_0 = U_0$, namely

$$\begin{cases} \frac{1}{(-y^2+\frac{1}{2}y)sinx}\left[u_{xx} + y^2 u_{yy}\right] - \partial_t u = 0 & \text{in } Q_T \\ u = -t & \text{in } \mathcal{S}_1 \times (0,T] \\ u = U_0 & \text{in } (\Omega \cup \mathcal{S}_1) \times \{0\}\,. \end{cases} \tag{4.5.8}$$

(observe that condition $(H_1') - (iii)$ is satisfied, due to (ii) above). In view of the above remark (i), it is the unique bounded solution of (4.5.8). There holds

$$\lim_{(x,y,t)\to(x_0,y_0,t_0)} \tilde{u}(x,y,t) = -t_0 \text{ for any } (x_0,y_0,t_0) \in \mathcal{S}_2 \times [0,T]. \tag{4.5.9}$$

Now we can prove the above claim. Consider problem (4.2.11) with $\mathcal{L}$ given in (4.5.5), $f = c = 0$ and $u_0 = U_0$. Choose $g \in C(\partial\Omega \times [0,T])$ such that $g(x,0) = 0$ for any $x \in \partial\Omega$, $g(x,y,t) = -t$ for any $(x,y,t) \in \mathcal{S}_1 \times [0,T]$, $g(x_0,y_0,t_0) \neq -t_0$ for some $(x_0,y_0,t_0) \in \mathcal{S}_2 \times [0,T]$. Then there holds

$$\begin{cases} \frac{1}{(-y^2+\frac{1}{2}y)sinx}\left[u_{xx} + y^2 u_{yy}\right] - \partial_t u = 0 & \text{in } Q_T \\ u \overset{U_0}{=} g & \text{in } \partial\Omega \times (0,T] \\ u = U_0 & \text{in } \Omega \times \{0\}\,. \end{cases}$$

Since $U_0 \in C(\overline{\Omega})$ and $U_0 = 0$ on $\partial\Omega$, this simply reads

$$\begin{cases} \frac{1}{(-y^2+\frac{1}{2}y)sinx}\left[u_{xx} + y^2 u_{yy}\right] - \partial_t u = 0 & \text{in } Q_T \\ u = g & \text{in } \partial\Omega \times (0,T] \\ u = U_0 & \text{in } \Omega \times \{0\}\,. \end{cases} \tag{4.5.10}$$

Any bounded solution of problem (4.5.10) is also a bounded solution of problem (4.5.8). As already remarked, $\tilde{u}$ is the unique solution of (4.5.8); however, in view of equality (4.5.9) and of the choice of g, it cannot satisfy the boundary condition at the point $(x_0,y_0,t_0) \in \mathcal{S}_2 \times [0,T]$. Therefore no solution of problem (4.5.10) exists, and the claim follows.

The above discussion also points out that Theorem 4.5.2 can be sharper than Theorem 4.2.5. In fact, in the above example the latter leads to assign boundary conditions on the whole of $\partial\Omega \times [0,T]$, whereas only boundary data on a proper subset are required to make the problem well-posed. Not surprisingly, prescribing boundary data on a set "too large" gives rise to nonexistence.

In this connection, let us mention that the above techniques can be combined with the ideas underlying the refined maximum principle to prove sharper results. For instance, if the sets

$$\Gamma := \{x_0 \in \mathcal{S} \mid \liminf_{x\to x_0} U_0(x) = 0\} = \mathcal{S} \cap \mathcal{Z}_0, \qquad \Gamma^* := \mathcal{S} \setminus \Gamma$$

satisfy suitable conditions, then we can prove existence of solutions to problem (4.2.11) without requiring condition (4.3.1) to be fulfilled at the regular boundary $\mathcal{R}$. This is the content of the following theorem.

THEOREM 4.5.3. *Let assumptions (H_1') and (H_2') be satisfied; suppose $c \in L^\infty(\Gamma^\varepsilon)$ for some $\varepsilon > 0$, $f(\cdot,\cdot,0) \in L^\infty(Q_T)$. In addition, let the following assumptions be satisfied:*

- *there exists $\phi \in C((\Omega \cup \mathcal{R} \cup \bar{\Gamma}) \times [0,T])$ such that $g = \phi$ in $(\Gamma \cup \mathcal{R}) \times [0,T], u_0 = \phi$ in $\bar{\Omega} \times \{0\}$;*
- *there exists $g_1 \in C([0,T])$ such that*

$$\begin{cases} g(x,t) = g(t) & \text{for any } (x,t) \in \Gamma \times [0,T]\,; \\ u_0(x) = g_1(0) & \text{for any } x \in \Gamma\,; \end{cases} \tag{4.5.11}$$

- *Γ and Γ^* have a finite number of connected components;*
- *there holds*

$$\partial\Gamma \cap \partial\Gamma^* \cap \partial\mathcal{R} = \emptyset\,. \tag{4.5.12}$$

Then there exists a bounded solution to problem (4.2.11).

To prove Theorem 4.5.3 we need some preliminary remarks.

Denote by $\mathcal{U}_1 \subseteq I\!R^n$ a neighborhood of $\partial\Gamma \cap \partial\Gamma^*$ and by $\mathcal{U}_2 \subseteq I\!R^n$ a neighborhood of $\partial\Gamma \cap \partial\mathcal{R}$. If $\partial\Gamma \cap \partial\Gamma^* = \emptyset$, we take $\mathcal{U}_1 = \emptyset, \mathcal{U}_2 \supseteq \Gamma^\varepsilon$; if $\partial\Gamma \cap \partial\mathcal{R} = \emptyset$, we take $\mathcal{U}_1 \supseteq \Gamma^\varepsilon, \mathcal{U}_2 = \emptyset$.

Observe that, if condition (4.5.12) holds true, then we can choose $\mathcal{U}_1$ and $\mathcal{U}_2$ such that $\overline{\mathcal{U}_1} \cap \overline{\mathcal{U}_2} = \emptyset$ and

$$\Gamma^{\varepsilon_0} \cap (\Gamma^*)^{\varepsilon_0} \subseteq \mathcal{U}_1,\, \Gamma^{\varepsilon_0} \cap \mathcal{R}^{\varepsilon_0} \subseteq \mathcal{U}_2,\, \Gamma^{\varepsilon_0} \setminus [\Gamma^{\varepsilon_0/2} \cup \mathcal{U}_1] \subseteq \Omega \cup \mathcal{R}\,. \tag{4.5.13}$$

for some $\varepsilon_0 > 0$.

Next proposition holds true.

PROPOSITION 4.5.4. *Let assumptions of Theorem 4.5.3 be satisfied, $\varepsilon_0 > 0$ be given by* (4.5.13) *and $\varepsilon \in (0, \varepsilon_0)$. Then there exists a nonnegative supersolution H to equation*

$$\mathcal{L}\,H = -1 \quad \text{in } \Gamma^\varepsilon \tag{4.5.14}$$

with the following properties:

(i) *$H \in C(\Gamma^\varepsilon) \cap LSC(\overline{\Gamma^\varepsilon})$;*
(ii) *$\inf_{\mathcal{A}^\varepsilon} H > 0$, where $\mathcal{A}^\varepsilon := \{x \in \bar{\Omega} \mid dist(x,\Gamma) = \varepsilon\}$;*
(iii) *$H = 0$ in Γ.*

Proof of Proposition 4.5.4. Consider $\chi \in C^2(\overline{\Gamma^\varepsilon})$ such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ in $\mathcal{A}^\varepsilon \cap \overline{\mathcal{U}_2}$, $\chi \equiv 0$ in $\Gamma^{\varepsilon/2} \cup \mathcal{U}_1$. Notice that, in view of (4.5.13) and assumption (H_2), there exists a positive constant α such that

$$\mathcal{L}\,\chi \leq \alpha \quad \text{in } \Gamma^\varepsilon \setminus [\Gamma^{\varepsilon/2} \cup \mathcal{U}_1]\,. \tag{4.5.15}$$

Moreover,

$$\mathcal{L}\,\chi = 0 \quad \text{in } \Gamma^{\varepsilon/2} \cup \mathcal{U}_1. \tag{4.5.16}$$

Define

$$H(x) := \begin{cases} (\alpha+1)U_0(x) + \chi(x) & \text{if } x \in \overline{\Gamma^\varepsilon} \setminus \partial\Omega \\ \\ (\alpha+1)\liminf_{y\to x} U_0(y) + \chi(x) & \text{if } x \in \partial\Gamma^\varepsilon \cap \partial\Omega\,. \end{cases} \tag{4.5.17}$$

Since U_0 is a solution of problem (4.5.3), from (4.5.15)-(4.5.16) we arrive to:

$$\mathcal{L}\,H \leq -(\alpha+1) + \alpha \quad \text{in } \Gamma^\varepsilon \setminus (\Gamma^{\varepsilon/2} \cup \mathcal{U}_1)\,, \tag{4.5.18}$$

$$\mathcal{L} H \leq -\alpha - 1 \quad \text{in } \Gamma^{\varepsilon/2} \cup \mathcal{U}_1 \,, \tag{4.5.19}$$

whence H is a supersolution to equation (4.5.14). It is easily seen that H is nonnegative and fulfills conditions (i) and (iii). In addition, since $\chi = 1$ in $\mathcal{A}^{\varepsilon} \cap \mathcal{U}_2$ and $\liminf_{y \to x} U_0(y) > 0$ for any $x \in \Gamma^*$, we have that property (ii) is satisfied, too. □

Proof of Theorem 4.5.3. In view of Proposition 4.5.4, we can repeat the proof of Theorem 2.22 in [**59**] (see also Chapter 4, Theorem 2.2.22) to obtain the result. □

CHAPTER 5

On the Cauchy problem for nonlinear parabolic equations with variable density

5.1. Introduction

We provide sufficient conditions for uniqueness or nonuniqueness of *bounded* solutions to the following Cauchy problem:

$$(5.1.1)\qquad \begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } I\!R^n \times (0,T] =: S_T \\ \\ u = u_0 & \text{in } I\!R^n \times \{0\}\,. \end{cases}$$

Here $n \geq 3$, $T > 0$, the density $\rho = \rho(x)$ is a positive function, u_0 is bounded, G is increasing and sufficiently smooth; precise assumptions on ρ, u_0 and G will be made in Section 5.2.

Problem (5.1.1), which arises in situations of physical interest (see [**45**]), has been the object of detailed investigation. In fact, it is well-known that it turns out to be well-posed in the class of bounded solutions when $n \leq 2$ and ρ is sufficiently smooth, or when $n \geq 3$ and ρ is constant (see [**6**], [**37**], [**42**]; see also [**11**]). Moreover, by results shown in [**43**] it follows that problem (5.1.1) with $\rho = \rho(r)$, $u_0 = u_0(r) \geq 0$, $n \geq 1$ ($r \equiv |x|$) has a unique bounded nonnegative radial solution also when

$$\int_0^\infty r\rho(r)\,dr \;=\; \infty\,.$$

On the contrary, when $n \geq 3$ and $\rho \to 0$ sufficiently fast as $r \to \infty$, some conditions at infinity are needed to restore well-posedness (see [**21**]-[**23**], [**43**], [**45**], [**68**]). Observe that conditions at infinity considered in the mentioned literature are of *Dirichlet type* and *homogeneous*, for they imply that the solution goes to zero as $r \to \infty$ in a proper sense. However, in the particular case $G(u) = u$, in [**44**] conditions at infinity of different type were considered. More precisely, existence and uniqueness of bounded classical solutions to problem (5.1.1) with $G(u) = u$, which satisfy at infinity in a suitable sense either *inhomogeneous* conditions of Dirichlet type, or conditions of *Neumann* type, were proved.

The aim of this paper is to prove the following:

(i) uniqueness of bounded solutions to problem (5.1.1), not satisfying any additional condition at infinity, when $\rho(x) \to 0$ slowly, or ρ does not go to zero, as $r \to \infty$ (see Theorem 5.2.3);

(ii) existence of bounded solutions to problem (5.1.1), satisfying at infinity possibly *inhomogeneous* conditions of Dirichlet type, when $\rho(x) \to 0$ sufficiently fast as $r \to \infty$ (see Theorems 5.2.8, 5.2.11 and 5.2.15). Observe that these results, in particular, imply nonuniqueness of bounded solutions to problem (5.1.1).

The uniqueness result outlined in (i) above generalizes those given in [**43**] in the case with radial symmetry (see Remark 5.2.5); furthermore, it extends to the nonlinear case uniqueness results established in [**37**] for general linear parabolic equations (see also [**20**]).

Besides, the results outlined in (ii) generalize the existence results given in [**23**], [**43**] and [**44**] (see Remarks 5.2.9 and 5.2.13).

5.2. Mathematical background and results

In what follows we always assume:

$$(H_0)\qquad \begin{cases} (i) & \rho \in C(\mathbb{R}^n)\,,\ \rho > 0\,;\\ (ii) & G \in C^1(\mathbb{R}),\ G(0)=0,\ G'(s)>0 \text{ for any } s \in \mathbb{R}\setminus\{0\}\,,\\ & G' \text{ decreasing in } (-\delta,0) \text{ and increasing in } (0,\delta), \text{ if } G'(0)=0\ (\delta>0);\\ (iii) & u_0 \in L^\infty(\mathbb{R}^n)\cap C(\mathbb{R}^n)\,. \end{cases}$$

Solutions, sub- and supersolutions of problem (5.1.1) are always meant in the following sense.

Definition 5.2.1. *By a* solution *of problem* (5.1.1) *we mean a function* $u \in C(S_T)\cap L^\infty(S_T)$ *such that*

$$\begin{aligned}\int_0^\tau\int_{\Omega_1}\{\rho\, u\,\partial_t\psi + G(u)\Delta\psi\}\,dxdt &= \int_{\Omega_1}\rho\left[u(x,\tau)\psi(x,\tau)-u_0(x)\psi(x,0)\right]dx+\\ &+\int_0^\tau\int_{\partial\Omega_1}G(u)\langle\nabla\psi,\,\nu\rangle d\sigma\,dt\end{aligned}\tag{5.2.1}$$

for any bounded open set $\Omega_1 \subseteq \mathbb{R}^n$ *with smooth boundary* $\partial\Omega_1$, $\tau \in (0,T]$, $\psi \in C^{2,1}(\overline{\Omega_1}\times[0,\tau])$, $\psi \geq 0$, $\psi = 0$ *in* $\partial\Omega_1\times[0,\tau]$*; here* ν *denotes the outer normal to* Ω_1 *and* $\langle\cdot,\cdot\rangle$ *the scalar product in* $\mathbb{R}^n$.

Supersolutions (subsolutions) *of* (5.1.1) *are defined replacing* " $=$ " *by* " $\leq$ " *(*" $\geq$ ", *respectively) in* (5.2.1).

Observe that, according to Definition 5.2.1, solutions of problem (5.1.1) we deal with are *bounded* in S_T.

5.2.1. No conditions at infinity.

Let us mention the following result concerning existence of solutions of (5.1.1), which can be proved by standard methods (see *e.g.* [**23**], [**43**], [**60**]).

Theorem 5.2.2. *Let assumption* (H_0) *be satisfied. Then there exists a solution of problem* (5.1.1).

Concerning uniqueness, the following result will be proved (here $B_R := \{x \in \mathbb{R}^n \mid |x| < R\}$ $(R>0)$).

Theorem 5.2.3. *Let assumption* (H_0) *be satisfied. Moreover, suppose that*

$$(H_1)\qquad \begin{cases} & \text{there exist } \hat{R}>0 \text{ and } \underline{\rho}\in C([\hat{R},\infty)) \text{ such that}\\ (i) & \rho(x) \geq \underline{\rho}(|x|) > 0 \text{ for any } x \in \mathbb{R}^n\setminus B_{\hat{R}}, \text{ and}\\ (ii) & \int_{\hat{R}}^\infty \eta\underline{\rho}(\eta)\,d\eta = \infty\,. \end{cases}$$

Then there exists at most one solution of problem (5.1.1).

Remark 5.2.4. A natural choice in Theorem 5.2.3 is $\underline{\rho}(\eta) := \eta^{-\alpha}$ $(\eta \in [\hat{R},\infty))$ for some $\alpha \in (-\infty, 2]$ and $\hat{R} > 0$.

Remark 5.2.5. Let $\rho = \rho(r)$, $u_0 = u_0(r) \geq 0$ and assumptions of Theorem 5.2.3 be satisfied. Then by Theorem 6.2 of [**43**] there exists at most one nonnegative solution $u = u(r,t)$ of problem (5.1.1). This is in agreement with Theorem 5.2.3.

5.2.2. Dirichlet conditions at infinity. If ρ satisfies the condition

$$(H_2) \qquad \Gamma * \rho \in L^\infty(\mathbb{R}^n)\,,$$

where Γ is the fundamental solution of the Laplace equation in $\mathbb{R}^n$, then any solution of problem (5.1.1) has a *trace at infinity* in a suitable sense. This is the content of the following.

THEOREM 5.2.6. *Let assumptions* $(H_0), (H_2)$ *be satisfied. Let* u *be any solution of problem* (5.1.1) *and*

$$U(x,t) := \int_0^t G(u(x,\tau))\,d\tau \quad \big((x,t)\in S_T\big)\,. \tag{5.2.2}$$

Then there exists a function $A \in Lip([0,T])$ *with* $A(0) = 0$ *such that*

$$\lim_{R\to\infty} \frac{1}{|\partial B_R|}\int_{\partial B_R} \big|U(x,t) - A(t)\big|\,d\sigma \,=\, 0 \tag{5.2.3}$$

uniformly with respect to $t \in [0,T]$.

REMARK 5.2.7. (*i*) In [**12**] it was proved that condition (H_2) is equivalent to the existence of a bounded solution to the *first exit time* equation (see [**35**])

$$\Delta U \,=\, -\rho \quad \text{in } \mathbb{R}^n\,. \tag{5.2.4}$$

(*ii*) Clearly, assumption (H_1) excludes (H_2) to be satisfied.

For any given $A \in Lip([0,T])$ equality (5.2.3) can be also regarded as an inhomogeneous Dirichlet condition at infinity. Existence of solutions to problem (5.1.1) satisfying condition (5.2.3) is dealt with by the following theorem.

THEOREM 5.2.8. *Let assumptions* $(H_0), (H_2)$ *be satisfied and* $A \in Lip([0,T])$ *with* $A(0) = 0$. *Then there exists a solution* u *of problem* (5.1.1) *satisfying condition* (5.2.3), *with* U *defined in* (5.2.2).

REMARK 5.2.9. (*i*) Theorems 5.2.6 and 5.2.8 with $G(u) = u$ were proved in [**44**], Theorems 1.1-1.2.
(*ii*) Results corresponding to Theorem 5.2.8, concerning nonnegative solutions of problem (5.1.1) with $u_0 \geq 0$ were proved in [**23**] in the case $A \equiv 0$.
(*iii*) It is known that there exists at most one solution $u \in L^\infty(Q_T)$ to problem (5.1.1) satisfying condition (5.2.3) when $G(u) = u$ (see [**44**]), or when $u \geq 0$, $u_0 \geq 0$ and $A \equiv 0$ (see [**23**]).

If assumption (H_2) is replaced by the stronger condition

$$(H_3) \qquad \begin{cases} \text{there exist } \hat{R} > 0 \text{ and } \overline{\rho} \in C([\hat{R},\infty)) \text{ such that} \\ (i) \quad \rho(x) \leq \overline{\rho}(|x|) \ \text{ for any } x \in \mathbb{R}^n \setminus B_{\hat{R}}, \text{ and} \\ (ii) \ \int_{\hat{R}}^\infty \eta\overline{\rho}(\eta)\,d\eta \,<\, \infty\,, \end{cases}$$

the following refinements of Theorems 5.2.6 and 5.2.8 are obtained.

THEOREM 5.2.10. *Let assumptions* $(H_0), (H_3)$ *be satisfied; let* u *be any solution of problem* (5.1.1). *Then there exists a function* $A \in Lip([0,T])$ *with* $A(0) = 0$ *such that*

$$\lim_{|x|\to\infty} \big|U(x,t) - A(t)\big| \,=\, 0 \tag{5.2.5}$$

uniformly with respect to $t \in [0,T]$, *with* U *defined in* (5.2.2).

THEOREM 5.2.11. *Let assumptions* $(H_0), (H_3)$ *be satisfied and* $A \in Lip([0,T])$ *with* $A(0) = 0$. *Then there exists a solution* u *of problem* (5.1.1) *satisfying condition* (5.2.5), *with* U *defined in* (5.2.2).

REMARK 5.2.12. A natural choice in Theorems 5.2.10 is $\overline{\rho}(\eta) := \eta^{-\alpha}$ $(\eta \in [\hat{R}, \infty))$ for some $\alpha \in (2, \infty]$ and $\hat{R} > 0$.

REMARK 5.2.13. (*i*) Results analogous to Theorem 5.2.11, concerning nonnegative radial solutions of problem (5.1.1) with $\rho = \rho(r)$, $u_0 = u_0(r) \geq 0$ are proved in [**43**], Theorem 6.1 when $A \equiv 0$.
(*ii*) Conditions similar to (H_1) and (H_3) have been used in [**58**] to study uniqueness of solutions to linear degenerate elliptic and parabolic equations in bounded domains.

If in Theorems 5.2.6 and 5.2.8 we replace assumption (H_2) by the weaker condition

$$(H_4) \qquad \text{there exists } x_0 \in I\!R^n \text{ such that } \{\Gamma * \rho\}(x_0) < \infty,$$

we obtain the following results.

THEOREM 5.2.14. *Let assumptions* $(H_0), (H_4)$ *be satisfied; let* u *be any solution of problem* (5.1.1). *Then there exist a function* $A \in Lip([0,T])$ *with* $A(0) = 0$ *and a sequence* $\{x_m\} \subseteq I\!R^n$, $|x_m| \to \infty$ *as* $m \to \infty$ *such that*

$$\lim_{m\to\infty} \left|U(x_m, t) - A(t)\right| = 0 \tag{5.2.6}$$

uniformly with respect to $t \in [0,T]$, *with* U *defined in* (5.2.2).

THEOREM 5.2.15. *Let assumptions* $(H_0), (H_4)$ *be satisfied and* $A \in Lip([0,T])$ *with* $A(0) = 0$. *Then there exists a sequence* $\{x_m\} \subseteq I\!R^n$, $|x_m| \to \infty$ *as* $m \to \infty$ *and a solution* u *of problem* (5.1.1) *satisfying condition* (5.2.6), *with* U *defined in* (5.2.2).

REMARK 5.2.16. If assumption (H_4) is satisfied, then $\{\Gamma * \rho\}(x) < \infty$ for any $x \in I\!R^n$ (see [**12**]).

REMARK 5.2.17. Results similar to Theorem 5.2.15 are proved for more general parabolic problems in [**60**], where also bounded domains are considered.

5.3. Proof of uniqueness result

The following lemma will play a key role in the proof of Theorem 5.2.3.

LEMMA 5.3.1. *Let* $n \geq 3$ *and* ψ_R *be the solution of the elliptic problem:*

$$\begin{cases} \Delta U = -F & \text{in } B_R \\ U = 0 & \text{on } \partial B_R, \end{cases} \tag{5.3.1}$$

where $R \geq 1$, $F \in C_0^\infty(I\!R^n)$, *supp* $F \subseteq B_1$, $F \geq 0, F \not\equiv 0$. *Then there exists a constant* $C > 0$ *(which depends only on* n *and on the function* F*) such that for any* $R \geq 1$ *the following holds:*

$$-CR^{1-n} \leq \langle \nabla\psi_R, \nu_R \rangle < 0 \quad \text{on } \partial B_R; \tag{5.3.2}$$

here ν_R *denotes the outer normal to* B_R.

Let us recall that if $n \geq 3$, the solution of the problem (5.3.1) (with $R \geq 1$) is given by (*e.g.*, see [**32**]):

$$\psi_R(x) = -\int_{B_R} G_R(x,y)\, F(y)\, dy = -\int_{B_1} G_R(x,y)\, F(y)\, dy \quad (x \in \overline{B_R}).$$

Here

$$G_R(x,y) := \begin{cases} \Gamma(|x-y|) - \Gamma\left(\frac{|x|}{R}\left|y - \frac{R^2}{|x|^2}x\right|\right) & \text{if } x, y \in I\!R^n, x \neq 0 \\ \Gamma(|y|) - \Gamma(R) & \text{if } x, y \in I\!R^n, x = 0 \end{cases} \tag{5.3.3}$$

is the *Green's function* for B_R, with

$$\Gamma(r) := \begin{cases} -\frac{1}{n(n-2)\omega_n} r^{2-n} & \text{if } r > 0 \\ \\ -\infty & \text{if } r = 0\,, \end{cases}$$

and ω_n denotes the volume of the unit ball B_1 of $I\!R^n$.

By the maximum principle we have $\psi_R \geq 0$ in B_R for any $R \geq 1$. This implies that the function $\psi_{R_1} - \psi_{R_2}$ $(1 \leq R_1 \leq R_2)$ is a subsolution of problem

$$(5.3.4)\qquad \begin{cases} \Delta U = 0 & \text{in } B_{R_1} \\ \\ U = 0 & \text{on } \partial B_{R_1}\,, \end{cases}$$

thus again by the maximum principle

$$(5.3.5)\qquad \psi_{R_1} \leq \psi_{R_2} \quad \text{in } B_{R_1} \quad (1 \leq R_1 \leq R_2)\,.$$

It is also worth observing that by (5.3.5) and the monotone convergence theorem we have

$$(5.3.6)\qquad \lim_{R\to\infty} \psi_R(x) = \psi_\infty(x) := -\int_{B_1} \Gamma(|x-y|)\, F(y)\, dy \quad (x \in I\!R^n)\,.$$

Proof of Lemma 5.3.1. Clearly, by the strong maximum principle there holds $\langle \nabla\psi_R\,,\, \nu_R\rangle < 0$ on ∂B_R for any $R \geq 1$.

Moreover, for any $R \geq 1$, $x \in \partial B_R$, $i = 1\ldots,n$ we have

$$\frac{\partial}{\partial x_i}\psi_R(x) = -\int_{B_1} \Big\{\frac{\partial}{\partial x_i}\Gamma(|x-y|) - \frac{\partial}{\partial x_i}\Gamma(\frac{|x|}{R}|y - \frac{R^2}{|x|^2}x|)\Big\}F(y)\, dy\,,$$

$$\frac{\partial}{\partial x_i}\Gamma(|x-y|) = \frac{1}{n\omega_n}(x_i - y_i)|x-y|^{-n}\,,$$

$$\frac{\partial}{\partial x_i}\Gamma\Big(\frac{|x|}{R}\Big|y - \frac{R^2}{|x|^2}x\Big|\Big) = \frac{1}{n\omega_n}R^{n-2}|x|^{-2-n}\Big|y - \frac{R^2}{|x|^2}x\Big|^{-n}\Big\{|x|^2 x_i\Big|y - \frac{R^2}{|x|^2}x\Big|^2 -$$
$$-R^2\Big[y_i|x|^2 + x_i(R^2 - 2\langle x, y\rangle)\Big]\Big\} \quad (y \in B_1)\,.$$

The previous equalities imply

$$(5.3.7)\qquad \begin{aligned} &\langle \nabla\psi_R(x), \nu_R(x)\rangle = \\ &= -\frac{1}{n\omega_n}\int_{B_1} |x-y|^{-n}\big\{2R - 2R^{-1}\langle x, y\rangle - R^{-1}|x-y|^2\big\}F(y)dy \end{aligned}$$

for any $R \geq 1$, $x \in \partial B_R$.

Observe that for any $R \geq 1, x \in \partial B_R, y \in B_1$ we have

$$(5.3.8)\qquad R - R_0 \leq |x-y| \leq R+1\,,$$

with $R_0 \in (0,1)$ such that *supp* $F \subseteq \overline{B}_{R_0}$; moreover,

$$(5.3.9)\qquad \big|\langle x, y\rangle\big| \leq R\,.$$

From (5.3.7)-(5.3.9) we obtain

$$\langle \nabla\psi_R(x), \nu_R(x)\rangle \geq -\frac{2R + 2 + R^{-1}(R+1)^2}{n\omega_n(R-R_0)^n}\int_{B_1} F(y)dy \geq -CR^{1-n}$$

for any $R \geq 1$, $x \in \partial B_R$ and for some $C > 0$ which depends only on n and on the function F. The proof is complete. □

For any $R > 0$ consider the auxiliary problem

$$\text{(5.3.10)} \qquad \begin{cases} \rho\, u_t = \Delta\big[G(u)\big] & \text{in } B_R \times (0,T] =: Q_{R,T} \\ u\, = \varphi & \text{in } \partial B_R\ \times (0,T) \\ u\, = u_0 & \text{in } \ B_R \times \{0\}\,, \end{cases}$$

where $\varphi \in L^\infty(\partial B_R \times (0,T))$.

DEFINITION 5.3.2. *By a solution of problem* (5.3.10) *we mean a function* $u \in C(Q_{R,T}) \cap L^\infty(Q_{R,T})$ *such that*

$$\int_0^\tau \int_{\Omega_1} \{\rho\, u\, \partial_t \psi + G(u)\Delta\psi\}\, dxdt = \int_{\Omega_1} \rho\left[u(x,\tau)\psi(x,\tau) - u_0(x)\psi(x,0)\right] dx +$$
$$+ \int_0^\tau \int_{\partial\Omega_1 \setminus \partial B_R} G(u)\langle \nabla\psi,\, \nu\rangle d\sigma\, dt + \int_0^\tau \int_{\partial\Omega_1 \cap \partial B_R} G(\varphi)\langle \nabla\psi,\, \nu\rangle d\sigma\, dt$$

for any bounded open set $\Omega_1 \subseteq B_R$ *with smooth boundary* $\partial\Omega_1$, $\tau \in (0,T]$, $\psi \in C^{2,1}(\overline{\Omega_1} \times [0,\tau]), \psi \geq 0, \psi = 0$ *in* $\partial\Omega_1 \times [0,\tau]$*; here* ν *denotes the outer normal to* Ω_1.

Subsolutions and supersolutions are defined accordingly.

It is well-known that existence, uniqueness and comparison results hold true for problem (5.3.10) (*e.g.* see [**23**], [**60**]).

Now we can prove Theorem 5.2.3. The proof is modelled after that given in [**43**] for the case $n = 1$.

Proof of Theorem 5.2.3. Let u_1, u_2 be any two solutions of problem (5.1.1); set

$$M := \max\{\, ||u_1||_\infty,\, ||u_2||_\infty\}\,.$$

For any $R > 0$ let u_R be the unique solution of problem (5.3.10) with $\varphi \equiv -M$. By comparison results we have:

$$\text{(5.3.11)} \qquad -M \leq u_R \leq u_1 \quad \text{and } -M \leq u_R \leq u_2 \quad \text{in } Q_{R,T}\,.$$

By usual compactness arguments there exists a subsequence $\{u_{R_m}\} \subseteq \{u_R\}$ which converges uniformly in any compact subset of S_T. Set

$$\underline{u} := \lim_{m\to\infty} u_{R_m} \quad \text{in } S_T\,.$$

The function $\underline{u}$ is a solution of problem (5.1.1); moreover, from (5.3.11) we obtain

$$\text{(5.3.12)} \qquad -M \leq \underline{u} \leq u_1 \quad \text{and } -M \leq \underline{u} \leq u_2 \quad \text{in } S_T\,.$$

The conclusion will follow, if we show that

$$\text{(5.3.13)} \qquad u_1\, =\, \underline{u}\, =\, u_2 \quad \text{in } S_T.$$

Set $w = u_1$ or $w = u_2$ for simplicity. In view of assumption $(H_0) - (ii)$, to prove (5.3.13) it is sufficient to show that

$$\text{(5.3.14)} \qquad \int_0^T \int_{I\!R^n} \big[G(w) - G(\underline{u})\big]\, F\, dx\, dt\, =\, 0$$

for any $F \in C_0^\infty(I\!R^n)$, $F \geq 0$. It is not restrictive to assume *supp* $F \subseteq B_1$, $F \not\equiv 0$, as we do in the following.

From problem (5.1.1) we have

$$\text{(5.3.15)} \qquad \begin{cases} \rho\,(\partial_t w - \partial_t \underline{u}) = \Delta\big[G(w) - G(\underline{u})\big] & \text{in } S_T \\ w - \underline{u}\, = 0 & \text{in } I\!R^n \times \{0\}\,. \end{cases}$$

Then from equality (5.2.1) with $\Omega_1 = B_R$ $(R > 1)$, $\tau \in (0, T]$ we obtain

(5.3.16)
$$\int_{B_R} \rho(x)\big[w(x,\tau) - \underline{u}(x,\tau)\big]\psi_R(x)\,dx + \int_0^\tau \int_{B_R} \big[G(w) - G(\underline{u})\big]F(x)\,dx\,dt \,=\, = -\int_0^\tau \int_{\partial B_R} \big\{G(w) - G(\underline{u})\big\}\langle \nabla\psi_R, \nu_R\rangle\,d\sigma\,dt\,,$$

ψ_R denoting the solution of (5.3.1). Since $F \geq 0$, $\psi_R \geq 0$, $w \geq \underline{u}$ and $(H_1) - (ii)$ holds true, equality (5.3.16) with $\tau = T$ gives

(5.3.17)
$$\int_0^T \int_{I\!R^n} \big[G(w) - G(\underline{u})\big]F(x)\,dx\,dt \leq \leq \liminf_{R\to\infty} \Big| \int_0^T \int_{\partial B_R} \big\{G(w) - G(\underline{u})\big\}\langle \nabla\psi_R, \nu_R\rangle\,d\sigma\,dt\Big|\,.$$

Hence (5.3.14) will follow, if we prove that

(5.3.18)
$$\liminf_{R\to\infty} \Big| \int_0^T \int_{\partial B_R} \big\{G(w) - G(\underline{u})\big\}\langle \nabla\psi_R, \nu_R\rangle\,d\sigma\,dt\Big| \,=\, 0\,.$$

Define

$$\varphi(R) := \int_0^T \int_{\partial B_R} \big\{G(w) - G(\underline{u})\big\}\,d\sigma\,dt \quad (R > 0)\,.$$

We shall prove that

(5.3.19)
$$\liminf_{R\to\infty} R^{1-n}\varphi(R) \,=\, 0\,,$$

whence the conclusion follows easily. In fact, by (5.3.19) there exists a sequence $\{R_m\} \subseteq (0,\infty)$, $R_m \to \infty$ as $m \to \infty$ such that

(5.3.20)
$$\lim_{m\to\infty} R_m^{1-n}\varphi(R_m) \,=\, \liminf_{R\to\infty} R^{1-n}\varphi(R) \,=\, 0\,.$$

Then

$$\Big| \int_0^T \int_{\partial B_{R_m}} \big\{G(w) - G(\underline{u})\big\}\langle \nabla\psi_{R_m}, \nu_{R_m}\rangle\,d\sigma\,dt\Big| \leq \leq CR_m^{1-n} \int_0^T \int_{\partial B_{R_m}} \big\{G(w) - G(\underline{u})\big\}d\sigma dt = CR_m^{1-n}\varphi(R_m) \to 0$$

as $m \to \infty$. This implies (5.3.18), whence (5.3.14) and the conclusion follow.

It remains to prove (5.3.19). To this purpose suppose by contradiction

$$\liminf_{R\to\infty} R^{1-n}\varphi(R) \geq \gamma\,;$$

for some $\gamma > 0$; then there exists $\bar{R} > 1$ such that

(5.3.21)
$$R^{1-n}\varphi(R) \,\geq\, \frac{\gamma}{2} \quad \text{for any } R > \bar{R}\,.$$

From (5.3.2) and (5.3.16) we have

(5.3.22)
$$\int_0^T \int_{B_R} \rho(x)\big[w(x,\tau) - \underline{u}(x,\tau)\big]\psi_R(x)\,dxd\tau \leq \leq \int_0^T \int_0^\tau \int_{\partial B_R} \big\{G(w) - G(\underline{u})\big\}\Big|\langle \nabla\psi_R, \nu_R\rangle\Big|d\sigma dt d\tau \leq \leq 2\max_{-M\leq r\leq M}|G(r)| \int_0^T \int_0^\tau \int_{\partial B_R} \Big|\langle \nabla\psi_R, \nu_R\rangle\Big|d\sigma dt d\tau \leq 2T^2|\partial B_1|C \max_{-M\leq r\leq M}|G(r)|\,,$$

for any $R > 1$.

Letting $R \to \infty$ in (5.3.22), by (5.3.5)-(5.3.6) and the monotone convergence theorem we have

$$\int_0^T \int_{I\!R^n} \rho(x)\big[w(x,\tau) - \underline{u}(x,\tau)\big]\psi_\infty(x)\,dxd\tau \le\; 2T^2|\partial B_1|C \max_{-M\le r\le M}|G(r)|\,. \tag{5.3.23}$$

On the other hand, we have

$$\begin{aligned}
&\int_0^T \int_{I\!R^n} \rho(x)\big[w(x,\tau) - \underline{u}(x,\tau)\big]\psi_\infty(x)\,dxd\tau \ge \\
&\ge \frac{1}{L}\int_0^T \int_{I\!R^n} \rho(x)\big[G(w) - G(\underline{u})\big]\,\psi_\infty(x)\,dx\,d\tau \ge \\
&\ge \frac{1}{L}\int_0^T \int_{I\!R^n\setminus B_{\tilde R}} \rho(x)\big[G(w) - G(\underline{u})\big]\,\psi_\infty(x)\,dx\,d\tau;
\end{aligned} \tag{5.3.24}$$

here $\tilde R := \max\{\bar R,\, \hat R\},\ L := \max_{s\in[-M,M]} G'(s)$. Observe that for any $x \in I\!R^n \setminus B_{\tilde R}$ and $y \in B_1$

$$|x-y| \le 2|x|\,. \tag{5.3.25}$$

By (H_1), (5.3.6), (5.3.21) and (5.3.25) we have

$$\begin{aligned}
&\frac{1}{L}\int_0^T \int_{I\!R^n\setminus B_{\tilde R}} \rho(x)\big[G(w) - G(\underline{u})\big]\psi_\infty(x)\,dxd\tau = \\
&= -\frac{1}{L}\int_0^T \int_{I\!R^n\setminus B_{\tilde R}} \rho(x)\big[G(w) - G(\underline{u})\big]\int_{B_1}\Gamma(|x-y|)F(y)dy\,dxd\tau \ge \\
&\ge H_n\int_0^T \int_{I\!R^n\setminus B_{\tilde R}} \underline{\rho}(|x|)\big[G(w) - G(\underline{u})\big]\int_{B_1}|x-y|^{2-n}F(y)dydxd\tau \ge \\
&\ge 2^{2-n}H_n\int_{B_1}F(y)dy\int_0^T \int_{I\!R^n\setminus B_{\tilde R}} |x|^{2-n}\underline{\rho}(|x|)\big[G(w) - G(\underline{u})\big]dxd\tau = \\
&= 2^{2-n}H_n\int_{B_1}F(y)dy\int_{\tilde R}^{+\infty} R^{2-n}\underline{\rho}(R)\Big\{\int_0^T\int_{\partial B_R}\big[G(w) - G(\underline{u})\big]d\sigma d\tau\Big\}dR = \\
&= 2^{2-n}H_n\int_{B_1}F(y)dy\int_{\tilde R}^{+\infty} R^{2-n}\underline{\rho}(R)\varphi(R)dR \ge \\
&\ge \gamma\, 2^{1-n}H_n\int_{B_1}F(y)dy\int_{\tilde R}^{+\infty} R\underline{\rho}(R)\,dR = \infty\,,
\end{aligned}$$

where $H_n := \frac{1}{L\,n(n-2)\omega_n}$. The above inequalities and (5.3.24) yield

$$\int_0^T \int_{I\!R^n} \rho(x)\big[w(x,\tau) - \underline{u}(x,\tau)\big]\psi_\infty(x)\,dx\,d\tau \;=\; \infty,$$

in contrast with (5.3.23); hence (5.3.19) follows. This completes the proof. □

5.4. Proof of nonuniqueness results

5.4.1. Proof of Theorems 5.2.6 and 5.2.8. In the sequel we deal with *bounded* solutions to elliptic problems of the following type:

$$
\begin{cases} \Delta U = -\rho f & \text{in } \Omega \\ U = g & \text{in } \partial\Omega\,, \end{cases} \tag{5.4.1}
$$

where Ω is an open subset of $I\!R^n$ with smooth boundary, $f \in C(\Omega)$ and $g \in C(\partial\Omega)$.

Let us make precise the definition of solution of problem (5.4.1).

Definition 5.4.1. *By a* solution *of problem* (5.4.1) *we mean a function* $U \in C(\Omega) \cap L^\infty(\Omega)$ *such that*

$$
\begin{aligned} \int_{\Omega_1} U\,\Delta\psi\,dx \;=\; & \int_{\partial\Omega_1\setminus\partial\Omega} U\langle\,\nabla\psi,\nu\rangle\,d\sigma + \\ + & \int_{\partial\Omega_1\cap\partial\Omega} g\,\langle\nabla\psi,\nu\rangle\,d\sigma - \int_{\Omega_1} \rho\,f\,\psi\,dx \end{aligned} \tag{5.4.2}
$$

for any open set $\Omega_1 \subseteq \Omega$ *with smooth boundary* $\partial\Omega_1$, $\psi \in C^2(\overline{\Omega_1}), \psi \geq 0, \psi = 0$ *on* $\partial\Omega_1$*; here* ν *denotes the outer normal to* Ω_1*. Subsolutions and supersolutions are defined accordingly.*

Moreover, we will consider the elliptic equation

$$
\Delta\,U \;=\; -\rho\,f \quad \text{in } I\!R^n\,, \tag{5.4.3}
$$

with $f \in C(I\!R^n)$.

Definition 5.4.2. *By a* solution *of equation* (5.4.3) *we mean a function* $u \in C(I\!R^n)$ *satisfying equality* (5.4.2) *with* $\Omega = I\!R^n$ *.*

Remark 5.4.3. (i) Let $U \in C(I\!R^n)$. It is easily seen that U is a solution of equation (5.4.3) if and only if it satisfies

$$
\int_{I\!R^n} U\Delta\psi \;=\; -\int_{I\!R^n} \rho\,f\,\psi\,dx \tag{5.4.4}
$$

for any $\psi \in C_0^2(I\!R^n)$. Then U is a solution of equation (5.4.3) if and only if U is a *viscosity* solution of the same equation (see [**?**]).
(ii) By (i) above and the standard theory of viscosity solution any bounded solution of equation (5.4.3) with $f \equiv 0$ is constant.

In the following lemma we recall the content of Lemma 2.1 in [**44**].

Lemma 5.4.4. *Let assumptions* $(H_0) - (i), (H_2)$ *be satisfied and* $f \in C(I\!R^n) \cap L^\infty(I\!R^n)$ *. Let* U *be a bounded solution of equation* (5.4.3)*. Then there exists* $A \in I\!R$ *such that*

$$
\lim_{R\to\infty} \frac{1}{|\partial B_R|} \int_{\partial B_R} |U - A|\,d\sigma \;=0\,. \tag{5.4.5}
$$

Remark 5.4.5. In connection with the above lemma, observe that in Lemma 2.1 in [**44**] U is a *classical* bounded solution of equation (5.4.3). However, in view of Remark 5.4.3 it is easily seen that the same proof also holds in the present situation.

The proofs of Theorems 5.2.6 and 5.2.8 follow by a standard adaptation of those of Theorems 1.1, respectively 1.2 in [**44**]; we limit ourselves to give only their hint.

Hint of the proof of Theorem 5.2.6. It is easily seen that the function U defined in (5.2.2) is a solution of the equation

$$\Delta U(\cdot,t) = -\rho\,[u_0 - u(\cdot,t)] \quad \text{in } I\!R^n$$

for any $t \in (0,T]$. In fact, by Definition 5.2.1 we have

$$\int_{\Omega_1} U(x,t)\,\Delta\psi(x)dx = \int_{\Omega_1} \rho(x)[u(x,t) - u_0(x)]\,\psi(x)\,dx + \int_{\partial\Omega_1} U(x,t)\langle\nabla\psi,\nu\rangle\,d\sigma$$

for any Ω_1 and $\psi = \psi(x)$ as in Definition 5.4.1.

Since assumptions $(H_0), (H_2)$ are satisfied and $u \in L^\infty(S_T)$, we can apply Lemma 5.4.4; hence there exists a function $A : (0,T] \to I\!R$ such that the equality (5.2.3) holds for any $t \in (0,T]$. By the same arguments used in the proof of Theorem 1.1 in [**44**], it is shown that

(*i*) $A \in Lip([0,T])$,

(*ii*) the convergence in (5.2.3) is uniform with respect to $t \in [0,T]$.

This proves the result. □

Observe that for any $A \in Lip([0,T])$ the derivative A' exists almost everywhere in $[0,T]$ and belongs to $L^\infty((0,T))$.

Hint of the proof of Theorem 5.2.8. For any $R > 0$ let u_R be the unique solution of the problem

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } Q_{R,T} \\ u = G^{-1}(A') & \text{in } \partial B_R \times (0,T) \\ u = u_0 & \text{in } B_R \times \{0\}\,. \end{cases} \tag{5.4.6}$$

By comparison results we have

$$|u_R| \le \|u_0\|_\infty + \max_{-\|A'\|_\infty \le r \le \|A'\|_\infty} |G^{-1}(r)| =: M \quad \text{in } Q_{R,T}\,. \tag{5.4.7}$$

By usual compactness arguments there exists a subsequence $\{u_{R_m}\} \subseteq \{u_R\}$, which converges uniformly in any compact subset of S_T to a solution u of problem (5.1.1).

Define U as in (5.2.2) and

$$U_R(x,t) := \int_0^t G(u_R(x,\tau))d\tau \quad \big((x,t) \in Q_{R,T}\big)\,. \tag{5.4.8}$$

Observe that $U_{R_m} \to U$ in S_T as $m \to \infty$.

Let us show that for any $t \in (0,T]$ the function $U_R(\cdot,t)$ satisfies the problem

$$\begin{cases} \Delta U = -\rho\,[u_0 - u_R(\cdot,t)] & \text{in } B_R \\ U = A(t) & \text{in } \partial B_R\,. \end{cases} \tag{5.4.9}$$

In fact, by Definition 5.3.2

$$\begin{aligned} \int_{\Omega_1} U_R(x,t)\Delta\psi(x)\,dx &= \int_{\Omega_1} \rho(x)\big[u_R(x,t) - u_0(x)\big]\psi(x)\,dx + \\ &+ \int_{\partial\Omega_1\setminus\partial B_R} U_R(x,t)\,\langle\nabla\psi(x),\nu\rangle\,d\sigma + \int_{\partial\Omega_1\cap\partial B_R} A(t)\,\langle\nabla\psi,\nu\rangle\,d\sigma \end{aligned} \tag{5.4.10}$$

for any $t \in (0,T]$, Ω_1 and $\psi = \psi(x)$ as in Definition 5.4.1.

It is easily seen that the function

$$V_R(x,t) = \int_{B_R} G_R(x,y)\rho(y)\big[u_0 - u_R(y,t)\big]dy + A_R(t) \;\; \big((x,t) \in Q_{R,T}\big)$$

solves problem (5.4.9) for any $t \in (0,T)$; here G_R is the Green's function defined in (5.3.3).

By uniqueness results we obtain $U_R = V_R$ in $Q_{R,T}$. Then, arguing as in the proof of Theorem 1.2 in [**44**], it is shown that

$$U(x,t) = \big\{\Gamma * \big[\rho(u_0 - u(\cdot,t))\big]\big\}(x) + A(t) \ \big((x,t) \in I\!R^n \times (0,T)\big).$$

Since $\Gamma * \rho$, u_0, $u(\cdot,t) \in L^\infty(I\!R^n)$, equality (5.2.3) can be proved arguing as in the proof of Lemma 2.1 in [**44**]. The proof is complete. □

5.4.2. Proof of Theorems 5.2.10-5.2.11. If in Lemma 5.3.1 we replace assumption (H_2) by (H_3), we obtain the following stronger result.

LEMMA 5.4.6. *Let assumptions* $(H_0)-(i),(H_3)$ *be satisfied. Suppose* $f \in C(I\!R^n) \cap L^\infty(I\!R^n)$. *Let* U *be a bounded solution of equation* (5.4.3). *Then there exists* $A \in I\!R$ *such that*

$$\lim_{|x|\to\infty} U(x) = A. \tag{5.4.1}$$

To prove Lemma 5.4.6 we need a preliminary result.

LEMMA 5.4.7. *Let assumption* $(H_0)-(i)$ *be satisfied; suppose* $f \in C(I\!R^n) \cap L^\infty(I\!R^n)$ *and* $A \in I\!R$. *Let there exist a positive supersolution* V *of equation* (5.2.4) *satisfying*

$$\lim_{|x|\to\infty} V(x) = 0. \tag{5.4.2}$$

Then there exists a solution U *of equation* (5.4.3) *such that* (5.4.1) *is satisfied.*

Lemma 5.4.7 can be proved by standard methods; we give its proof for further purposes.

Proof of Lemma 5.4.7. For any $R > 0$ denote by U_R the unique solution of the problem

$$\begin{cases} \Delta U = -\rho f & \text{in } B_R \\ U = A & \text{on } \partial B_R. \end{cases} \tag{5.4.3}$$

It is easily seen that the function $\|f\|_\infty V + A$ is a supersolution to problem (5.4.3). In fact, fix any $R > 0$, Ω_1 and ψ as in Definition 5.4.1. Since

$$\langle \nabla\psi, \nu\rangle \le 0 \quad \text{on } \partial\Omega_1, \tag{5.4.4}$$

by Definition 5.4.2 we have

$$\int_{\Omega_1} \big\{\|f\|_\infty V + A\big\}\,\Delta\psi\,dx \le \int_{\partial\Omega_1} \big\{\|f\|_\infty V + A\big\}\langle\nabla\psi,\nu\rangle\,d\sigma - \int_{\Omega_1} \|f\|_\infty\,\rho\,\psi\,dx \le$$

$$\le \int_{\partial\Omega_1\setminus\partial B_R} \big\{\|f\|_\infty V + A\big\}\langle\nabla\psi,\nu\rangle\,d\sigma + \int_{\partial\Omega_1\cap\partial B_R} A\langle\nabla\psi,\nu\rangle\,d\sigma\,;$$

here use of inequalities $V \ge 0$ and (5.4.4) has been made. Then the claim follows (see Definition 5.4.1).

It is analogously checked that the function $-\|f\|_\infty V + A$ is a subsolution to the same problem; thus by comparison principles we have

$$-\|f\|_\infty V + A \le U_R \le \|f\|_\infty V + A \quad \text{in } B_R. \tag{5.4.5}$$

By usual compactness arguments (see [**32**]) there exists a subsequence $\{U_{R_m}\} \subseteq \{U_R\}$, which converges uniformly in any compact subset of $I\!R^n$. Set

$$U := \lim_{m\to\infty} U_{R_m} \quad \text{in } I\!R^n.$$

Clearly, U is a solution to equation (5.4.3); moreover, from (5.4.5) we obtain

$$-\|f\|_\infty V + A \le U \le \|f\|_\infty V + A \quad \text{in } I\!R^n. \tag{5.4.6}$$

In view of (5.4.2) and (5.4.6) we get (5.4.1). This proves the result. □

Let us now prove Lemma 5.4.6.

Proof of Lemma 5.4.6. Define

$$\overline{\rho}_0 := \begin{cases} \bar{C}\,\overline{\rho}(\hat{R}) & \text{in} \quad [0,\hat{R}) \\ \bar{C}\,\overline{\rho} & \text{in} \quad [\hat{R},+\infty)\,; \end{cases}$$

here

$$\bar{C} := \frac{1}{\overline{\rho}(\hat{R})}\max\{\max_{\overline{B}_{\hat{R}}}\rho,\, \overline{\rho}(\hat{R})\} \in (0,\infty)\,.$$

Clearly, $\overline{\rho}_0 \in C([0,\infty))$; moreover, by assumption $(H_3)-(i)$

(5.4.7) $$\rho(x) \leq \overline{\rho}_0(|x|) \quad \text{for any } x \in I\!R^n.$$

Assumption (H_3) again implies that

$$H := \lim_{r\to\infty}\frac{1}{n-2}\big[r^{2-n}\int_0^r \eta^{n-1}\overline{\rho}_0(\eta)\,d\eta - \int_0^r \eta\overline{\rho}_0(\eta)\,d\eta\big] \leq 0\,.$$

Define

(5.4.8) $$V(x) \equiv V(r) := \frac{1}{n-2}\big[r^{2-n}\int_0^r \eta^{n-1}\overline{\rho}_0(\eta)\,d\eta - \int_0^r \eta\overline{\rho}_0(\eta)\,d\eta\big] - H \ \ (x \in I\!R^n)\,.$$

We have that $V \in C^2(I\!R^n)$ and

(5.4.9) $$\Delta V(x) \,=\, -\,\overline{\rho}_0(r) \,\leq\, -\,\rho(x) \quad (x \in I\!R^n)\,;$$

here use of inequality (5.4.7) has been made. In addition, as is easily seen,

(5.4.10) $$\lim_{r\to\infty} V(r) \,=\, 0\,,$$

(5.4.11) $$V \,>\, 0 \quad \text{in } I\!R^n\,.$$

By Lemma 5.4.7 there exist a solution U_1 to the equation

(5.4.12) $$\Delta U \,=\, -\,\rho\, f^+ \quad \text{in } I\!R^n$$

such that

(5.4.13) $$\lim_{|x|\to\infty} U_1(x) \,=\, 0$$

and a solution U_2 to the equation

(5.4.14) $$\Delta U \,=\, -\,\rho\, f^- \quad \text{in } I\!R^n$$

such that

(5.4.15) $$\lim_{|x|\to\infty} U_2(x) \,=\, 0\,;$$

here $f^\pm := \max\{\pm f,\, 0\}$. Observe that $U_1, U_2 \in L^\infty(I\!R^n)$. Then the function $U - (U_1 - U_2)$ is a bounded solution to the equation (5.4.3) with $f = 0$. Hence, in view of Remark 5.4.3,

(5.4.16) $$U \,=\, U_1 - U_2 + A \quad \text{in } I\!R^n$$

for some $A \in I\!R$. From (5.4.13) and (5.4.15)-(5.4.16) the conclusion follows. □

Proof of Theorem 5.2.10. The result can be obtained arguing as in the proof of Theorem 5.2.6, applying Lemma 5.4.6 instead of Lemma 5.3.1. □

Proof of Theorem 5.2.11. We can repeat the proof of Theorem 5.2.8, using the same notation, to construct the family $\{U_R\}$ of solutions of problem (5.4.9) such that $U_{R_m} \to U$ in S_T.

Let V be defined as in (5.4.8); hence (5.4.9)-(5.4.11) hold true.

We claim that

$$-2MV(x)+A(t)\le U(x,t)\le 2MV(x)+A(t)\qquad \big((x,t)\in I\!R^n\times(0,T]\big)\,; \tag{5.4.17}$$

here M is given by (5.4.7).

From (5.4.10) and (5.4.17) we obtain (5.2.5), hence the conclusion.

It remains to prove the claim. It is easily seen that for any fixed $t\in(0,T]$ and $R>0$ the function $2MV-U_R+A$ is a supersolution, while the function $-2MV-U_R+A$ is a subsolution of problem

$$\begin{cases} \Delta U=0 & \text{in } B_R\\ U=0 & \text{on } \partial B_R\,. \end{cases} \tag{5.4.18}$$

In fact, fix any Ω_1, ψ as in Definition 5.4.1, $R>0$ and $t\in(0,T]$. Then by (5.4.10) and Definition 5.4.2 we have

$$\begin{aligned} &\int_{\Omega_1}(2MV-U_R+A)\,\Delta\psi\,dx\;\le\;\int_{\partial\Omega_1}(2MV+A)\langle\nabla\psi,\nu\rangle\,d\sigma+\\ &-\int_{\partial\Omega_1\setminus\partial B_R}U_R\,\langle\nabla\psi,\nu\rangle d\sigma-\int_{\partial\Omega_1\cap\partial B_R}A\,\langle\nabla\psi,\nu\rangle\,d\sigma\;-\;\int_{\Omega_1}(2M+u_R-u_0)\rho\,\psi\,dx\;=\\ &=\int_{\partial\Omega_1\cap\partial B_R}2MV\,\langle\nabla\psi,\nu\rangle\,d\sigma+\int_{\partial\Omega_1\setminus\partial B_R}(2MV-U_R+A)\langle\nabla\psi,\nu\rangle\,d\sigma+\\ &-\int_{\Omega_1}(2M+u_R-u_0)\rho\,\psi\,dx\,; \end{aligned} \tag{5.4.19}$$

as before, here u_R denotes the solution of the problem (5.4.6).

From (5.4.7) and (5.4.11) we have

$$2M\ge u_0-u_R\quad\text{in } \Omega_1\,, \tag{5.4.20}$$

respectively

$$2MV\;\ge\;0\quad\text{on } \partial B_R \tag{5.4.21}$$

for any $R>0$. From (5.4.4), (5.4.19)-(5.4.21) we obtain

$$\int_{\Omega_1}(2MV-U_R+A)\,\Delta\psi\,dx\;\le\int_{\partial\Omega_1\setminus\partial B_R}(2MV-U_R+A)\langle\nabla\psi,\nu\rangle\,d\sigma$$

for any $R>0$. This shows that the function $2MV-U_R+A$ is a supersolution to problem (5.4.18) for any $R>0$ (see Definition 5.4.1). It is similarly seen that $-2MV-U_R+A$ is a subsolution of the same problem for any $R>0$.

By comparison results we obtain

$$-2MV(x)+A(t)\le U_R(x,t)\le 2MV(x)+A(t)\qquad \big((x,t)\in B_R\times(0,T]\big)$$

for any $R>0$. This implies (5.4.17), thus the proof is complete. □

5.4.3. Proof of Theorems 5.2.14-5.2.15. If in Lemma 5.4.6 we suppose that (H_4) is satisfied instead of (H_3), then we obtain the following weaker result.

LEMMA 5.4.8. *Let assumptions* $(H_0)-(i),(H_4)$ *be satisfied. Suppose* $f \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. *Then there exists a sequence* $\{x_m\} \subseteq \mathbb{R}^n, |x_m| \to \infty$ *as* $m \to \infty$ *with the following property: for any bounded solution* U *of equation* (5.4.3) *there exists* $A \in \mathbb{R}$ *such that*

$$\lim_{m\to\infty} U(x_m) = A.$$

In the proof of Lemma 5.4.8 we make use of the following auxiliary result (see [**60**], [**65**] for similar results for more general parabolic, respectively elliptic problems).

LEMMA 5.4.9. *Let assumption* $(H_0)-(i)$ *be satisfied; suppose* $f \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *and* $A \in \mathbb{R}$. *Let there exist a supersolution* V *of equation* (5.2.4) *satisfying*

$$\inf_{\mathbb{R}^n} V = 0. \tag{5.4.1}$$

Then there exist a sequence $\{x_m\} \subseteq \mathbb{R}^n, |x_m| \to \infty$ *as* $m \to \infty$ *and a solution* U *of equation* (5.4.3) *such that*

$$\lim_{m\to\infty} U(x_m) = A.$$

To prove Lemma 5.4.9 we need the following result, which is proved in [**60**], Lemma 2.6.

LEMMA 5.4.10. *Let assumption* $(H_0)-(i)$ *be satisfied,* $A \in \mathbb{R}$. *Let there exist a supersolution* V *of equation* (5.2.4) *satisfying condition* (5.4.1). *Then there exists a sequence* $\{x_m\} \subseteq \mathbb{R}^n, |x_m| \to \infty$ *as* $m \to \infty$ *such that*

$$\lim_{m\to\infty} V(x_m) = 0. \tag{5.4.2}$$

Proof of Lemma 5.4.9. By Lemma 5.4.10 there exists a sequence $\{x_m\} \subseteq \mathbb{R}^n, |x_m| \to \infty$ as $m \to \infty$ such that (5.4.2) holds. Arguing as in the proof of Lemma 5.4.7 we obtain (5.4.6), whence the conclusion follows. □

Proof of Lemma 5.4.8. In view of Remark 5.2.16 the function $\Gamma * \rho$ is well-defined in $\mathbb{R}^n$; moreover it is positive and satisfies equation (5.2.4). Hence, by Lemma 5.4.9 applied with $V = \Gamma * \rho - \inf_{\mathbb{R}^n}\big(\Gamma * \rho\big)$, there exist a solution U_1 to the equation (5.4.12) such that

$$\lim_{m\to\infty} U_1(x_m) = 0$$

and a solution U_2 to the equation (5.4.14) such that

$$\lim_{m\to\infty} U_2(x_m) = 0$$

for some sequence $\{x_m\} \subseteq \mathbb{R}^n$, $|x_m| \to \infty$ as $m \to \infty$. Then the conclusion follows arguing as in the proof of Lemma 5.4.6.

Proof of Theorem 5.2.14. The result can be obtained arguing as in the proof of Theorem 5.2.6, applying Lemma 5.4.8 instead of Lemma 5.3.1. □

Proof of Theorem 5.2.15. Observe that $V := \Gamma * \rho - \inf_{\mathbb{R}^n}\big(\Gamma * \rho\big)$ is a supersolution to equation (5.2.4) satisfying condition (5.4.1). In view of Lemma 5.4.10 there exists a sequence $\{x_m\} \subseteq \mathbb{R}^n$, $|x_m| \to \infty$ as $m \to \infty$ such that (5.4.2) is satisfied. Then by the same arguments used in the proof of Theorem 5.2.11 we get the conclusion. □

CHAPTER 6

Uniqueness and nonuniqueness of bounded solutions to singular nonlinear parabolic equations

6.1. Introduction

We address *singular* nonlinear parabolic equations of the following type:

$$\rho\,\partial_t u = \Delta\big[G(u)\big] \quad \text{in } \ \Omega \times (0,T] =: Q_T\,; \tag{6.1.1}$$

here Ω is a connected bounded open subset of $I\!R^n$, $T > 0$, ρ is a positive function of the space variables. A typical choice for the function G is $G(u) = |u|^{m-1}u$, $m \geq 1$.

An extensive literature is concerned with equation (6.1.1), which arises in various areas of science (*e.g.*, see [**45**]). Particular attention has been devoted to the Cauchy problem associated with equation (6.1.1)

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } I\!R^n \times (0,T] =: S_T \\ \\ u = u_0 & \text{in } I\!R^n \times \{0\}\,; \end{cases} \tag{6.1.2}$$

here $\rho \in C(I\!R^n)$, $u_0 \in L^\infty(I\!R^n)$. In fact, it is well-known that problem (6.1.2) is well-posed in $L^\infty(S_T)$ when $n \leq 2$; moreover, it is well-posed also when $n \geq 3$ and $\rho(x) \to 0$ "not too fast" as $|x| \to \infty$ (or does not vanish at all at infinity; see [**43**], [**62**]). On the contrary, if $n \geq 3$ and $\rho(x) \to 0$ sufficiently fast as $|x| \to \infty$, some constraints at infinity are needed to restore well-posedness (see [**23**], [**43**]-[**44**], [**62**]).

In the case of a bounded domain, inspired by [**47**] where $n = 1$ is assumed, we allow the density ρ either to vanish or to diverge, or not to have a limit as the distance $d(x, \mathcal{S})$ goes to zero, $\mathcal{S}$ being a subset of the boundary $\partial\Omega$ referred to as the *singular boundary.* On the other hand, ρ is supposed to be well-behaved both in Ω and on the *regular boundary* $\mathcal{R} := \partial\Omega \setminus \mathcal{S}$. More precisely, we always assume the following:

$$(H_0) \qquad \begin{cases} (i) & \mathcal{R} \cup \mathcal{S} = \partial\Omega\,,\ \mathcal{R} \cap \mathcal{S} = \emptyset\,; \\ (ii) & \mathcal{R} \text{ and } \mathcal{S} \text{ are } (n-1) - \text{dimensional compact submanifolds} \\ & \text{of } I\!R^n \text{ of class } C^3\,; \end{cases}$$

$$(H_1) \qquad \begin{cases} (i) & \rho \in C(\Omega \cup \mathcal{R})\,,\ \rho > 0 \text{ in } \Omega \cup \mathcal{R}\,; \\ (ii) & G \in C^1(I\!R),\ G(0) = 0,\ G'(s) > 0 \text{ for any } s \in I\!R \setminus \{0\}\,, \\ & G' \text{ decreasing in } (-\delta, 0) \text{ and increasing in } (0,\delta), \text{ if } G'(0) = 0\,; \\ (iii) & u_0 \in L^\infty(\Omega)\,. \end{cases}$$

In view of conditions $(H_0) - (H_1)$, it is natural to study the following initial-boundary value problem associated to equation (6.1.1):

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } Q_T \\ \\ u = 0 & \text{in } \mathcal{R} \times (0,T) \\ \\ u = u_0 & \text{in } \Omega \times \{0\}\,. \end{cases} \tag{6.1.3}$$

The question arises, whether problem (6.1.3) is well-posed in the class of bounded solutions not satisfying any extra condition at $\mathcal{S}$. Formally, for problem (6.1.3), the singular boundary

$\mathcal{S}$ plays the same role played by the *point at infinity* for the Cauchy problem. Thus, whereas for problem (6.1.2) the well-posedness depends on the behaviour of ρ at infinity, for problem (6.1.3) it depends on the behaviour of ρ as $d(x,\mathcal{S}) \to 0$. In fact, in [**47**] it is proved that when $n = 1, \Omega = (0,R), \mathcal{S} = \{0\}, \mathcal{R} = \{R\}, u_0 \geq 0$, there exists a unique nonnegative bounded solution to problem (6.1.3), if $\int_0^R r\rho(r)\,dr = \infty$. Instead, if $\int_0^R r\rho(r)\,dr < \infty$, to restore well-posedness some extra constraints at $\mathcal{S} = \{0\}$ are needed. Moreover, the case $G(u) = u$, $n \geq 1$ is treated in a large number of papers, using both analytical and stochastic methods (see [**26**], [**31**], [**56**], [**58**], [**70**]). In particular, in [**58**] uniqueness in $L^\infty(Q_T)$ is proven if $\rho(x) \to \infty$ fast enough as $d(x,\mathcal{S}) \to 0$, whereas nonuniqueness in $L^\infty(Q_T)$ holds otherwise. In probabilistic parlance (*e.g.*, see [**31**], [**70**]), uniqueness prevails if the singular boundary $\mathcal{S}$ is *unattainable* by Markovian particles with generator $\frac{1}{\rho}\Delta$, starting at $x_0 \in \Omega$.

The aim of this paper is to prove the following:

(*i*) uniqueness of bounded solutions to problem (6.1.3), not satisfying any additional condition at $\mathcal{S}$, when $\rho(x) \to \infty$ sufficiently fast as $d(x,\mathcal{S}) \to 0$ (see Theorem 6.2.2);

(*ii*) existence of bounded solutions to problem (6.1.3), satisfying at $\mathcal{S}$ possibly *inhomogeneous* conditions of Dirichlet type, when $\rho(x) \to \infty$ sufficiently slow as $d(x,\mathcal{S}) \to 0$, or ρ does not diverge as $d(x,\mathcal{S}) \to 0$ (see Theorems 6.2.5 and 6.2.9). Observe that these existence results, in particular, imply nonuniqueness of bounded solutions to problem (6.1.3).

Moreover, we shall prove that prescribing Dirichlet conditions at $\mathcal{S}$ implies uniqueness, when such conditions are *homogeneous,* or when $G(u) = u$ (see Theorems 6.2.11-6.2.12 and Remark 6.2.14).

The results outlined in the above $(i)-(ii)$ generalize to the case of several space dimensions those given in [**47**] in one space dimension (see Remark 6.2.4); moreover, they extend to nonlinear parabolic problem analogous results stated in [**58**] for general singular linear elliptic and parabolic problems (see Remarks 6.2.6-6.2.10, 6.2.14). Finally, they can be regarded as the counterpart for initial-boundary value problems of uniqueness and nonuniqueness results stated in [**23**], [**43**]-[**44**] and [**62**] for the Cauchy problem.

6.2. Uniqueness and nonuniqueness results

To begin with, observe that solutions, sub- and supersolutions of the problem (6.1.3) are always meant in the sense of Definition 6.3.1 below, thus they are *bounded* in Q_T.

6.2.1. No conditions at the singular boundary. Let us recall an existence result of solutions to problem (6.1.3), which follows by Theorem 2.5 in [**60**].

THEOREM 6.2.1. *Let assumptions* $(H_0) - (H_1)$ *be satisfied. Then there exists a solution of problem* (6.1.3).

Concerning uniqueness, the following result will be proved.

Set $\mathcal{S}^\varepsilon := \{x \in \Omega \,|\, d(x,\mathcal{S}) < \varepsilon\}$ ($\varepsilon > 0$ small enough).

THEOREM 6.2.2. *Let assumptions* $(H_0) - (H_1)$ *be satisfied and* $\mathcal{S} \neq \emptyset$. *Moreover, suppose that the following condition is satisfied:*

$$(H_2) \qquad \begin{cases} \textit{there exist } \hat{\varepsilon} > 0 \textit{ and } \underline{\rho} \in C((0,\hat{\varepsilon}]) \textit{ such that} \\ (i) \quad \rho(x) \geq \underline{\rho}(d(x,\mathcal{S})) > 0 \textit{ for any } x \in \mathcal{S}^{\hat{\varepsilon}}, \textit{ and} \\ (ii) \quad \int_0^{\hat{\varepsilon}} \eta\underline{\rho}(\eta)\,d\eta = \infty\,. \end{cases}$$

Then there exists at most one solution of problem (6.1.3).

REMARK 6.2.3. A natural choice in Theorem 6.2.2 is $\underline{\rho}(\eta) := \eta^{-\alpha}$ $(\eta \in (0,\hat{\varepsilon}])$ for some $\alpha \in [2,\infty)$ and $\hat{\varepsilon} > 0$.

REMARK 6.2.4. (i) Let $\Omega = (0, R)$, $u_0 \geq 0$ and assumptions of Theorem 6.2.2 be satisfied with $\mathcal{S} = \{0\}$ and $\mathcal{R} = \{R\}$. Then by Theorem 2.3 and Lemma 2.5 in [**47**] there exists at most one nonnegative bounded solution of problem (6.1.3). This is in agreement with Theorem 6.2.2.
(ii) Results similar to Theorem 6.2.2 are proved in [**58**] for more general linear degenerate elliptic and parabolic equations and in [**62**] for the Cauchy problem (see also [**43**] for the case with radial symmetry).

6.2.2. Additional conditions at the singular boundary. Concerning existence of solutions to problem (6.1.3) satisfying additional conditions at $\mathcal{S}$ we shall prove the following.

THEOREM 6.2.5. *Let assumptions* $(H_0) - (H_1)$ *be satisfied,* $\mathcal{S} \neq \emptyset$ *and* $A \in Lip([0,T])$ *with* $A(0) = 0$. *Moreover, suppose that the following condition is satisfied:*

$$(H_3) \qquad \begin{cases} \text{there exist } \hat{\varepsilon} > 0 \text{ and } \overline{\rho} \in C((0, \hat{\varepsilon}]) \text{ such that} \\ (i) \quad \rho(x) \leq \overline{\rho}(d(x, \mathcal{S})) \quad \text{for any } x \in \mathcal{S}^{\hat{\varepsilon}}, \text{ and} \\ (ii) \quad \int_0^{\hat{\varepsilon}} \eta \overline{\rho}(\eta)\, d\eta < \infty, \end{cases}$$

Then there exists a solution u *of problem* (6.1.3) *such that*

$$\lim_{d(x,\mathcal{S}) \to 0} \left| U(x,t) - A(t) \right| = 0 \tag{6.2.1}$$

uniformly with respect to $t \in [0,T]$; *here*

$$U(x,t) := \int_0^t G(u(x,\tau))\, d\tau \quad \big((x,t) \in Q_T\big). \tag{6.2.2}$$

REMARK 6.2.6. Let $\Omega = (0, R)$, $u_0 \geq 0$ and assumptions of Theorem 6.2.2 be satisfied with $\mathcal{S} = \{0\}$ and $\mathcal{R} = \{R\}, A \equiv 0$. Then by Theorem 2.4 and Lemma 2.5 in [**47**] there exists a nonnegative bounded solution of problem (6.1.3) satisfying the condition

$$\lim_{x \to 0} U(x, T) = 0$$

with U defined in (6.2.2). This is in agreement with Theorem 6.2.5.

Observe that when assumptions $(H_1) - (i)$ and (H_3) are satisfied, it could exist a point $x_0 \in \mathcal{S}$ such that

$$\lim_{m \to \infty} \rho(x_m) = 0, \quad \lim_{m \to \infty} \rho(y_m) = \infty;$$

here $\{x_m\} \subseteq \Omega$, $\{y_m\} \subseteq \Omega$ with $x_m \to x_0$, $y_m \to x_0$ as $m \to \infty$.

If we avoid that situations of this type occur (see conditions (6.2.4)-(6.2.5) below), then we obtain the following refinement of Theorem 6.2.5.

For any $x_0 \in \mathbb{R}^n$ and $R > 0$ set $B_R(x_0) := \{x \in \mathbb{R}^n \mid |x - x_0| < R\}$.

THEOREM 6.2.7. *Let assumptions* $(H_0) - (H_1), (H_3)$ *be satisfied,* $\mathcal{S} \neq \emptyset$ *and* $A \in C\big(\overline{\mathcal{S}^{\bar{\varepsilon}}} \times [0,T]\big)$, *with* $A(x, 0) = 0$ *for any* $x \in \mathcal{S}$. *Let there exist a constant* $L > 0$ *such that*

$$|A(x,t) - A(x,s)| \leq L\,|t - s| \quad \text{for any } x \in \overline{\mathcal{S}^{\bar{\varepsilon}}};\, t, s \in [0,T]. \tag{6.2.3}$$

In addition, suppose that for any $x_0 \in \mathcal{S}$ *there exists* $\bar{R} > 0$ *such that*

$$\inf_{B_{\bar{R}}(x_0) \cap \Omega} \rho > 0, \tag{6.2.4}$$

or

$$\rho \in L^\infty(B_{\bar{R}}(x_0) \cap \Omega). \tag{6.2.5}$$

Then there exists a solution u *of problem* (6.1.3) *such that for any* $x_0 \in \mathcal{S}$

$$\lim_{x \to x_0} \left| U(x,t) - A(x_0, t) \right| = 0 \tag{6.2.6}$$

uniformly with respect to $t \in [0,T]$, *with* U *defined in* (6.2.2).

REMARK 6.2.8. A natural choice in Theorem 6.2.5 is $\underline{\rho}(\eta) := \eta^{-\alpha}$ $(\eta \in (0, \hat{\varepsilon}])$ for some $\alpha \in (-\infty, 2)$ and $\hat{\varepsilon} > 0$. Moreover, the same choice can be made in Theorem 6.2.7, if in $(H_3) - (i)$ the equality sign holds true.

If in Theorem 6.2.5 instead of condition (H_3), we assume that there exists a supersolution V to the *first exit time* problem (see [**35**])

$$\begin{cases} \Delta U = -\rho & \text{in } \Omega \\ U = 0 & \text{on } \mathcal{R}, \end{cases} \tag{6.2.7}$$

satisfying the condition

$$\inf_{\Omega \cup \mathcal{R}} V = 0 < \inf_{\mathcal{R}} V, \tag{6.2.8}$$

we obtain the following weaker result.

THEOREM 6.2.9. *Let assumptions $(H_0) - (H_1)$ be satisfied, $\mathcal{S} \neq \emptyset$ and $A \in Lip([0,T])$ with $A(0) = 0$. Moreover, let there exist a supersolution V to problem* (6.2.7) *such that condition* (6.2.8) *is satisfied. Then there exist a sequence $\{x_m\} \subseteq \Omega$, $d(x_m, \mathcal{S}) \to 0$ as $m \to \infty$ and a solution u of problem* (6.1.3) *such that*

$$\lim_{m\to\infty} \left| U(x_m, t) - A(t) \right| = 0 \tag{6.2.9}$$

uniformly with respect to $t \in [0,T]$, with U defined in (6.2.2).

REMARK 6.2.10. Nonuniqueness results similar to Theorem 6.2.9 are proved in [**58**], [**60**] and [**62**].

In the particular cases $A \equiv 0$ or $G(u) = u$ we can also prove that imposing condition (6.2.1) at $\mathcal{S}$ implies uniqueness. In fact, we obtain the following.

THEOREM 6.2.11. *Let assumptions $(H_0) - (H_1), (H_3)$ be satisfied, $u_0 \geq 0$ in Ω and $\mathcal{S} \neq \emptyset$. Then there exists at most one nonnegative solution u of problem* (6.1.3) *satisfying condition* (6.2.1)*, with $A \equiv 0$ and U defined in* (6.2.2).

THEOREM 6.2.12. *Let assumptions $(H_0)-(H_1), (H_3)$ be satisfied, $\mathcal{S} \neq \emptyset$ and $A \in Lip([0,T])$ with $A(0) = 0$. In addition, suppose $G(u) = u$. Then there exists at most one solution u of problem* (6.1.3) *satisfying condition* (6.2.1)*, with U defined in* (6.2.2).

REMARK 6.2.13. Clearly, since $\mathcal{S}$ in compact (see assumption $(H_0) - (ii)$), if condition (6.2.6) is satisfied, then (6.2.1) holds true. Hence Theorems 6.2.11-6.2.12 are valid, if we replace condition (6.2.1) by (6.2.6).

REMARK 6.2.14. (i) Results corresponding to Theorem 6.2.11 are proved in [**47**] in the case of one space dimension, and in [**23**] and [**43**] for the Cauchy problem.
(ii) A similar result to Theorem 6.2.12 is obtained in [**44**] for the Cauchy problem.

6.3. Mathematical background and proofs

Let us make the following definitions.

DEFINITION 6.3.1. *By a* solution *of problem* (6.1.3) *we mean a function $u \in C\big((\Omega \cup \mathcal{R}) \times (0,T]\big) \cap L^\infty(Q_T)$ such that*

$$\begin{aligned} \int_0^\tau \int_{\Omega_1} \{\rho\, u\, \partial_t \psi + G(u) \Delta \psi\}\, dx dt = & \int_{\Omega_1} \rho \left[u(x,\tau)\psi(x,\tau) - u_0(x)\psi(x,0) \right] dx + \\ & + \int_0^\tau \int_{\partial\Omega_1 \setminus \mathcal{R}} G(u) \langle \nabla \psi,\, \nu \rangle d\sigma\, dt \end{aligned} \tag{6.3.1}$$

for any open set $\Omega_1 \subseteq \Omega$ *with smooth boundary* $\partial\Omega_1$, $\overline{\Omega_1} \subseteq \Omega \cup \mathcal{R}$, $\tau \in (0,T]$, $\psi \in C^{2,1}(\overline{\Omega_1} \times [0,\tau]), \psi \geq 0, \psi = 0$ *in* $\partial\Omega_1 \times [0,\tau]$*; here* ν *denotes the outer normal to* Ω_1 *and* $\langle\cdot,\cdot\rangle$ *the scalar product in* $\mathbb{R}^n$.

Supersolutions (subsolutions) *of* (6.1.3) *are defined replacing* " = " *by* " ≤ " (" ≥ ", *respectively) in* (6.3.1).

DEFINITION 6.3.2. *By a* supersolution *to problem* (6.2.7) *we mean a function* $U \in C(\Omega \cup \mathcal{R})$ *such that*

$$\int_{\Omega_1} U\,\Delta\psi\,dx \leq \int_{\partial\Omega_1\setminus\mathcal{R}} U\,\langle\nabla\psi,\nu\rangle\,d\sigma - \int_{\Omega_1} \rho\,\psi\,dx \tag{6.3.2}$$

for any open set $\Omega_1 \subseteq \Omega$ *with smooth boundary* $\partial\Omega_1$, $\overline{\Omega_1} \subseteq \Omega \cup \mathcal{R}$, $\psi \in C^2(\overline{\Omega_1}), \psi \geq 0, \psi = 0$ *in* $\partial\Omega_1$*; here* ν *denotes the outer normal to* Ω_1. Subsolutions *and* solutions *of problem* (6.2.7) *are defined accordingly.*

6.3.1. Proof of Theorem 6.2.2. Later on, let $\varepsilon_0 > 0$ be given by Lemma 3.4.3; set

$$\mathcal{A}^\varepsilon := \{x \in \Omega \,|\, d(x,\mathcal{S}) = \varepsilon\} \quad \big(\varepsilon \in (0,\varepsilon_0)\big)\,.$$

We prove the following

LEMMA 6.3.3. *Let assumption* (H_0) *be satisfied. Let* ψ_ε *be the solution of the elliptic problem:*

$$\begin{cases} \Delta U = -F & \text{in } \Omega \setminus \overline{\mathcal{S}^\varepsilon} \\ U = 0 & \text{on } \mathcal{R} \cup \mathcal{A}^\varepsilon\,, \end{cases} \tag{6.3.3}$$

where $\varepsilon \in (0,\varepsilon_0)$, $F \in C^\infty(\Omega)$, $F \geq 0$, $\operatorname{supp} F \subseteq \Omega \setminus \overline{\mathcal{S}^{\varepsilon_0}}$. *Then the following statements hold true:*

(i) *for any* $\varepsilon_1, \varepsilon_2 \in (0,\varepsilon_0)$, $\varepsilon_1 \geq \varepsilon_2$ *we have*

$$0 < \psi_{\varepsilon_1} \leq \psi_{\varepsilon_2} \quad \text{in } \Omega \setminus \overline{\mathcal{S}^{\varepsilon_1}}\,; \tag{6.3.4}$$

(ii) *there exists* $C > 0$ *such that for any* $\varepsilon \in (0,\varepsilon_0/2)$ *we have*

$$-C \leq \langle\nabla\psi_\varepsilon\,,\,\nu_\varepsilon\rangle < 0 \quad \text{on } \mathcal{A}^\varepsilon\,, \tag{6.3.5}$$

ν_ε *denoting the outer normal to* $\Omega \setminus \overline{\mathcal{S}^\varepsilon}$ *at* $\mathcal{A}^\varepsilon$ *;*

(iii) *there exist* $\bar\varepsilon \in (0,\varepsilon_0/2)$ *and* $\bar C > 0$ *such that*

$$\psi_0(x) \geq \bar C d(x,\mathcal{S}) \quad \text{for any } x \in \mathcal{S}^{\bar\varepsilon}\,, \tag{6.3.6}$$

where

$$\psi_0 := \lim_{m\to\infty} \psi_{\varepsilon_m} \quad \text{in } \Omega \cup \mathcal{R} \tag{6.3.7}$$

for some sequence $\{\varepsilon_m\} \subseteq (0,\varepsilon_0/2)$, $\varepsilon_m \to 0$ *as* $m \to \infty$.

Proof of Lemma 6.3.3. (i) By the strong maximum principle $\psi_\varepsilon > 0$ in $\Omega\setminus\overline{\mathcal{S}^\varepsilon}$ for any $\varepsilon \in (0,\varepsilon_0)$; hence the function $\psi_{\varepsilon_1} - \psi_{\varepsilon_2}$ $(0 < \varepsilon_2 \leq \varepsilon_1 < \varepsilon_0)$ is a subsolution of problem

$$\begin{cases} \Delta U = 0 & \text{in } \Omega \setminus \overline{\mathcal{S}^{\varepsilon_1}} \\ U = 0 & \text{on } \mathcal{R} \cup \mathcal{A}^{\varepsilon_1}\,. \end{cases} \tag{6.3.8}$$

Then again by the maximum principle we get (6.3.4).

(ii) Clearly, by the strong maximum principle it follows that

$$\langle\nabla\psi_\varepsilon\,,\,\nu_\varepsilon\rangle < 0 \quad \text{on } \mathcal{A}^\varepsilon$$

for any $\varepsilon \in (0, \varepsilon_0/2)$.

We claim that there exists $M > 0$ such that

$$0 < \psi_\varepsilon \le M \quad \text{in } \Omega \setminus \overline{\mathcal{S}^\varepsilon} \tag{6.3.9}$$

for any $\varepsilon \in (0, \varepsilon_0/2)$.

In fact, since Ω is bounded, we can suppose that Ω lies in the slab

$$\{x \equiv (x_1, \dots, x_m) \in I\!R^n \,\big|\, 0 < x_1 < d\}$$

for some $d > 0$. Then it is easily seen that the function

$$V(x) := \big(\exp\{d\} - \exp\{x_1\}\big)\|F\|_{L^\infty(\Omega)} \quad (x \in \Omega \setminus \overline{\mathcal{S}^\varepsilon}),$$

is a supersolution to problem (6.3.3) for any $\varepsilon \in (0, \varepsilon_0/2)$.

In fact, $V > 0$ in $\mathcal{R} \cup \mathcal{A}^\varepsilon$; moreover,

$$\Delta V(x) \le -\exp\{x_1\}\|F\|_{L^\infty(\Omega)} \le -F(x) \quad \text{for any } x \in \Omega \setminus \overline{\mathcal{S}^\varepsilon}.$$

By comparison principles we have for any $\varepsilon \in (0, \varepsilon_0/2)$

$$\psi_\varepsilon \le V \le M := \exp\{d\}\|F\|_{L^\infty(\Omega)} \quad \text{in } \Omega \setminus \overline{\mathcal{S}^\varepsilon}. \tag{6.3.10}$$

From (6.3.4)-(6.3.10) it follows (6.3.9).

Define

$$Z(x) := \check{C}[\exp\{-\mu\varepsilon\} - \exp\{-\mu d(x)\}] \quad (x \in \mathcal{S}^\varepsilon; \varepsilon \in (0, \varepsilon_0/2)), \tag{6.3.11}$$

where

$$\mu := (n-1)C_0, \ \check{C} := \frac{M}{\exp\{-\mu\varepsilon_0/2\} - \exp\{-\mu\varepsilon_0\}}, d(x) \equiv d(x, \mathcal{S}). \tag{6.3.12}$$

For any $\varepsilon \in (0, \varepsilon_0/2)$ and $i, j = 1, \dots, n$ we have

$$\frac{\partial Z(x)}{\partial x_i} = \check{C}\mu \frac{\partial d(x)}{\partial x_i} \exp\{-\mu d(x)\}; \tag{6.3.13}$$

$$\frac{\partial^2 Z(x)}{\partial x_i \partial x_j} = \check{C}\mu \exp\{-\mu d(x)\}\Big[-\mu \frac{\partial d(x)}{\partial x_i}\frac{\partial d(x)}{\partial x_j} + \frac{\partial^2 d(x)}{\partial x_i \partial x_j}\Big].$$

By Lemma 3.4.3 we have

$$\Delta Z(x) \le \mu\check{C}\exp\{-\mu d(x)\}[-\mu + (n-1)C_0] = 0 \quad \text{for any } x \in \mathcal{S}^\varepsilon$$

for some $C_0 > 0$ independent of x, since it is not restrictive to assume that condition (C) of Chapter 3 is satisfied at any $x \in \mathcal{S}^\varepsilon$. Hence Z is a supersolution to the problem

$$\begin{cases} \Delta U = 0 & \text{in } \mathcal{S}^{\varepsilon_0} \setminus \overline{\mathcal{S}^\varepsilon} \\ U = 0 & \text{on } \mathcal{A}^\varepsilon \\ U = M & \text{on } \mathcal{A}^{\varepsilon_0}. \end{cases} \tag{6.3.14}$$

On the other hand, the function ψ_ε $(\varepsilon \in (0, \varepsilon_0/2))$ is a subsolution to problem (6.3.14). Then by comparison principles

$$\psi_\varepsilon \le Z \quad \text{in } \mathcal{S}^{\varepsilon_0} \setminus \overline{\mathcal{S}^\varepsilon} \ \big(\varepsilon \in (0, \varepsilon_0/2)\big).$$

Observe that for any $\varepsilon \in (0, \varepsilon_0/2)$, $x \in \mathcal{A}^\varepsilon$ we have

$$\nu_\varepsilon(x) = -\nabla d_\varepsilon(x) = -\nabla d(x), \ \big|\nabla d(x)\big| = 1; \tag{6.3.15}$$

here $d_\varepsilon(x) := dist(x, \mathcal{A}^\varepsilon)$ $(x \in \Omega \setminus \mathcal{S}^\varepsilon)$.

Since $\psi_\varepsilon = Z$ on $\mathcal{A}^\varepsilon$, from (6.3.13) and (6.3.15) we have

$$\langle \nabla\psi_\varepsilon, \nu_\varepsilon\rangle \ge \langle \nabla Z, \nu_\varepsilon\rangle \ge -\langle \check{C}\mu\exp\{-\mu\varepsilon\}\nabla d(x), \nabla d(x)\rangle = -\check{C}\mu\exp\{-\mu\varepsilon\} \quad \text{on } \mathcal{A}^\varepsilon$$

for any $\varepsilon \in (0, \varepsilon_0/2)$. Hence (6.3.5) follows with $C := \mu \check{C}$.

(*iii*) From (6.3.9) and usual compactness arguments (see [**32**]) there exists a subsequence $\{\psi_{\varepsilon_m}\} \subseteq \{\psi_\varepsilon\}$ which converges, with its first and second derivatives, uniformly in any compact subset of $\Omega \cup \mathcal{R}$. Let $\psi_0 := \lim_{m\to\infty} \psi_{\varepsilon_m}$ in $\Omega \cup \mathcal{R}$.

Clearly, $\psi_0 \in C^2(\Omega \cup \mathcal{R})$ solves the problem

$$\begin{cases} \Delta U = 0 & \text{in } \Omega \\ U = 0 & \text{on } \mathcal{R}. \end{cases} \tag{6.3.16}$$

Moreover, we claim that $\psi_0 \in C^2(\overline{\Omega})$ and $\psi = 0$ on $\mathcal{S}$.

In fact, in view of assumption $(H_0) - (ii)$, Ω has the *outer sphere property* at $\mathcal{S}$; then (see [**32**]) for any $x_0 \in \mathcal{S}$ there exist $R > 0$ and a function $h \in C^2(N_R(x_0)) \cap C(\overline{N_R(x_0)})$, $h > 0$ in $\overline{N_R(x_0)} \setminus \{x_0\}$, $h(x_0) = 0$ satisfying

$$\Delta h(x) \le 0 \quad \text{for any } x \in N_R(x_0) := B_R(x_0) \cap \Omega\,; \tag{6.3.17}$$

Set

$$N_{R,\varepsilon}(x_0) := N_R(x_0) \cap (\Omega \setminus \overline{\mathcal{S}^\varepsilon})$$

for any $\varepsilon \in (0, \tilde{\varepsilon})$, where $\tilde{\varepsilon} := \min\{\varepsilon_0/2, R\}$.

Let

$$m := \min_{\partial B_R(x_0) \cap \overline{\Omega}} h > 0\,, \quad \tilde{C} := \frac{M}{m}\,. \tag{6.3.18}$$

Observe that for any $\varepsilon \in (0, \tilde{\varepsilon})$ there holds

$$\partial N_{R,\varepsilon}(x_0) = [\partial B_R(x_0) \cap (\Omega \setminus \mathcal{S}^\varepsilon)] \cup [N_R(x_0) \cap \mathcal{A}^\varepsilon]\,; \tag{6.3.19}$$

moreover, by (6.3.18) and $h > 0$ in $N_R(x_0)$ we have

$$\tilde{C}h - \psi_\varepsilon \ge M - \psi_\varepsilon \ge 0 \quad \text{on } \partial B_R(x_0) \cap (\Omega \setminus \mathcal{S}^\varepsilon)\,, \tag{6.3.20}$$

$$\tilde{C}h - \psi_\varepsilon \ge 0 \quad \text{on } N_R(x_0) \cap \mathcal{A}^\varepsilon\,. \tag{6.3.21}$$

From (6.3.17), (6.3.19)-(6.3.21) we deduce that the function $\tilde{C}h - \psi_\varepsilon$ ($\varepsilon \in (0, \tilde{\varepsilon})$) is a supersolution to problem

$$\begin{cases} \Delta U = 0 & \text{in } N_{R,\varepsilon}(x_0) \\ U = 0 & \text{on } \partial N_{R,\varepsilon}(x_0)\,. \end{cases} \tag{6.3.22}$$

It is similarly seen that the function $-\tilde{C}h - \psi_\varepsilon$ ($\varepsilon \in (0, \tilde{\varepsilon})$) is a subsolution to the same problem. Then by comparison principles we obtain

$$-\tilde{C}h \le \psi_\varepsilon \le \tilde{C}h \quad \text{in } N_{R,\varepsilon}(x_0)$$

for any $\varepsilon \in (0, \tilde{\varepsilon})$. Hence

$$-\tilde{C}h \le \psi_0 \le \tilde{C}h \quad \text{in } N_R(x_0)\,. \tag{6.3.23}$$

Letting $x \to x_0$ in (6.3.23) we deduce that $\psi_0 \in C^2(\Omega \cup \mathcal{R}) \cap C(\overline{\Omega})$ and $\psi_0 = 0$ on $\mathcal{S}$. Moreover, by usual regularity results (see [**32**]) it follows that $\psi_0 \in C^2(\overline{\Omega})$. Thus the claim has been proved.

By the strong maximum principle and assumption $(H_0) - (ii)$, there exists $\bar{C} > 0$ such that

$$\langle \nabla \psi_0, \nu \rangle \le -2\bar{C} \quad \text{on } \mathcal{S}, \tag{6.3.24}$$

ν being the outer normal to Ω at $\mathcal{S}$.

Moreover,

$$\langle \nabla[\bar{C}d(x)], \nu(x)\rangle = -\bar{C}\,\langle \nabla d(x), \nabla d(x)\rangle = -\bar{C} \quad \text{for any } x \in \mathcal{S}\,; \tag{6.3.25}$$

here use of equalities $\nabla d(x) = -\nu(x)$ and $\big|\nabla d(x)\big| = 1$ for any $x \in \mathcal{S}$ has been made.

Since $\psi_0(x) = \bar{C}d(x) = 0$ for any $x \in \mathcal{S}$, from (6.3.24)-(6.3.25) it follows (6.3.6). This concludes the proof. □

We shall prove the following.

LEMMA 6.3.4. *Let assumptions of Theorem 6.2.2 be satisfied. Let v_1, v_2 be any two solutions of problem* (6.1.3) *such that $v_1 \geq v_2$ in Q_T. Then*

$$\liminf_{\varepsilon\to 0} \int_0^T \int_{\mathcal{A}^\varepsilon} \big\{G(v_1) - G(v_2)\big\}\, d\sigma\, dt = 0\,. \tag{6.3.26}$$

Proof of Lemma 6.3.4. From problem (6.1.3) we have

$$\begin{cases} \rho\,(\partial_t v_1 - \partial_t v_2) = \Delta\big[G(v_1) - G(v_2)\big] & \text{in } Q_T \\ v_1 - v_2 = 0 & \text{in } \mathcal{R}\times(0,T) \\ v_1 - v_2 = 0 & \text{in } \Omega\times\{0\}\,. \end{cases} \tag{6.3.27}$$

Then equality (6.3.1) with $\Omega_1 = \Omega\setminus\overline{\mathcal{S}^\varepsilon}$ $(\varepsilon \in (0, \varepsilon_0/2))$, $\tau \in (0,T]$ yields

$$\begin{aligned} &\int_{\Omega\setminus\overline{\mathcal{S}^\varepsilon}} \rho(x)\big[v_1(x,\tau) - v_2(x,\tau)\big]\psi_\varepsilon(x)\,dx + \int_0^\tau \int_{\Omega\setminus\overline{\mathcal{S}^\varepsilon}} \big[G(v_1) - G(v_2)\big]F(x)\,dx\,dt = \\ &\qquad = -\int_0^\tau \int_{\mathcal{A}^\varepsilon} \big\{G(v_1) - G(v_2)\big\}\langle\nabla\psi_\varepsilon, \nu_\varepsilon\rangle\, d\sigma\, dt \end{aligned} \tag{6.3.28}$$

where ψ_ε denotes the solution of (6.3.3) .

Set

$$\varphi(\varepsilon) := \int_0^T \int_{\mathcal{A}^\varepsilon} \big\{G(v_1) - G(v_2)\big\}\, d\sigma\, dt \quad \big(\varepsilon \in (0, \varepsilon_0/2)\big)\,.$$

Suppose, by absurd, that

$$\liminf_{\varepsilon\to 0} \varphi(\varepsilon) =: \gamma > 0\,;$$

then there exists $\breve{\varepsilon} \in (0, \varepsilon_0/2)$ such that

$$\varphi(\varepsilon) \geq \frac{\gamma}{2} \quad \text{for any } \varepsilon \in (0, \breve{\varepsilon})\,. \tag{6.3.29}$$

Since $F \geq 0$, $v_1 \geq v_2$ and $(H_1) - (ii)$ holds true, from (6.3.5) and (6.3.28) we have

$$\begin{aligned} &\int_0^T \int_{\Omega\setminus\overline{\mathcal{S}^\varepsilon}} \rho(x)\big[v_1(x,\tau) - v_2(x,\tau)\big]\psi_\varepsilon(x)\,dxd\tau \leq \\ &\leq \int_0^T \int_0^\tau \int_{\mathcal{A}^\varepsilon} \big\{G(v_1) - G(v_2)\big\}\Big|\langle\nabla\psi_\varepsilon, \nu_\varepsilon\rangle\Big| d\sigma dt d\tau \leq \\ &\leq 2\max_{-M\leq r\leq M}|G(r)| \int_0^T \int_0^\tau \int_{\mathcal{A}^\varepsilon} \Big|\langle\nabla\psi_\varepsilon, \nu_\varepsilon\rangle\Big| d\sigma dt d\tau \leq 2\max_{-M\leq r\leq M}|G(r)| T^2\tilde{C}C \end{aligned} \tag{6.3.30}$$

for any $\varepsilon \in (0, \varepsilon_0/2)$; here $\tilde{C}$ is a positive constant such that $\big|\mathcal{A}^\varepsilon\big| \leq \tilde{C}$ for any $\varepsilon \in (0, \varepsilon_0)$.

Letting $\varepsilon \to 0$ in (6.3.30), by (6.3.4), (6.3.7) and the monotone convergence theorem we have

$$\int_0^T \int_\Omega \rho(x)\big[v_1(x,\tau) - v_2(x,\tau)\big]\psi_0(x)\,dxd\tau \leq 2\max_{-M\leq r\leq M}|G(r)| T^2\tilde{C}C\,. \tag{6.3.31}$$

On the other hand, we have

$$
\begin{aligned}
&\int_0^T \int_\Omega \rho(x)\big[v_1(x,\tau) - v_2(x,\tau)\big]\psi_0(x)\,dxd\tau \geq \\
&\geq \frac{1}{L}\int_0^T \int_{\mathcal{S}^{\tilde\varepsilon}} \rho(x)\big[G(v_1) - G(v_2)\big]\,\psi_0(x)\,dx\,d\tau;
\end{aligned}
\tag{6.3.32}
$$

here $\tilde\varepsilon := \min\{\hat\varepsilon, \bar\varepsilon, \check\varepsilon\}$, $L := \max_{s\in[-M,M]} G'(s)$.

By (H_2), (6.3.6) and (6.3.29) we have

$$
\begin{aligned}
&\frac{1}{L}\int_0^T \int_{\mathcal{S}^{\tilde\varepsilon}} \rho(x)\big[G(v_1)-G(v_2)\big]\psi_0(x)\,dxd\tau \geq \frac{\bar C}{L}\int_0^T \int_{\mathcal{S}^{\tilde\varepsilon}} \underline{\rho}(d(x,\mathcal{S}))\big[G(v_1)-G(v_2)\big]d(x,\mathcal{S})\,dxd\tau \geq \\
&= \frac{\bar C}{L}\int_0^T \int_0^{\tilde\varepsilon} \int_{\mathcal{A}^\varepsilon} \underline{\rho}(\varepsilon)\,\varepsilon \int_0^T \big[G(v_1) - G(v_2)\big]d\sigma d\varepsilon d\tau = \frac{\bar C}{L}\int_0^{\tilde\varepsilon} \underline{\rho}(\varepsilon)\,\varepsilon\,\varphi(\varepsilon)\,d\varepsilon \geq \\
&\geq \frac{\bar C \gamma}{2L}\int_0^{\tilde\varepsilon} \underline{\rho}(\varepsilon)\,\varepsilon\,d\varepsilon = \infty\,.
\end{aligned}
$$

The previous inequalities and (6.3.32) yield

$$
\int_0^T \int_\Omega \rho(x)\big[v_1(x,\tau) - v_2(x,\tau)\big]\psi_0(x)\,dx\,d\tau \;=\; \infty,
$$

in contrast with (6.3.31). Hence (6.3.26) follows. The proof is complete. □

For any $\varepsilon \in (0, \varepsilon_0/2)$ consider the auxiliary problem

$$
\begin{cases}
\rho\,u_t = \Delta\big[G(u)\big] & \text{in } [\Omega \setminus \overline{\mathcal{S}^\varepsilon}] \times (0,T] =: Q_{\varepsilon,T} \\
u = 0 & \text{in } \mathcal{R} \times (0,T) \\
u = \phi & \text{in } \mathcal{A}^\varepsilon \times (0,T) \\
u = u_0 & \text{in } [\Omega \setminus \overline{\mathcal{S}^\varepsilon}] \times \{0\}\,,
\end{cases}
\tag{6.3.33}
$$

where $\phi \in L^\infty(\mathcal{A}^\varepsilon \times (0,T))$.

Definition 6.3.5. *By a* supersolution *of problem* (6.3.33) *we mean a function* $u \in C\big([\overline{\Omega} \setminus \overline{\mathcal{S}^\varepsilon}] \times (0,T]\big) \cap L^\infty(Q_{\varepsilon,T})$ *such that*

$$
\begin{aligned}
&\int_0^\tau \int_{\Omega_1} \{\rho\,u\,\partial_t\psi + G(u)\Delta\psi\}\,dxdt \leq \int_{\Omega_1} \rho\,\big[u(x,\tau)\psi(x,\tau) - u_0(x)\psi(x,0)\big]\,dx + \\
&+ \int_0^\tau \int_{\partial\Omega_1 \setminus (\mathcal{R}\cup\mathcal{A}^\varepsilon)} G(u)\langle\nabla\psi,\,\nu\rangle d\sigma\,dt + \int_0^\tau \int_{\partial\Omega_1\cap\mathcal{A}^\varepsilon} G(\phi)\langle\nabla\psi,\,\nu\rangle d\sigma\,dt
\end{aligned}
$$

for any open set $\Omega_1 \subseteq \Omega \setminus \overline{\mathcal{S}^\varepsilon}$ *with smooth boundary* $\partial\Omega_1$, $\tau \in (0,T]$, $\psi \in C^{2,1}(\overline{\Omega_1} \times [0,\tau])$, $\psi \geq 0$, $\psi = 0$ *in* $\partial\Omega_1 \times [0,\tau]$*; here* ν *denotes the outer normal to* Ω_1. Solutions *and* subsolutions *are defined accordingly.*

It is well-known that existence, uniqueness and comparison results hold true for problem (6.3.33) (*e.g.* see [**60**]; see also [**23**]).

Now we can prove Theorem 6.2.2. The proof is modelled after that given in [**47**] for the case $n = 1$ (see also [**43**] and [**62**] for the Cauchy problem).

Proof of Theorem 6.2.2. Let u_1, u_2 be any two solutions of problem (6.1.3); set

$$
M := \max\{\,||u_1||_\infty,\,||u_2||_\infty\}\,.
$$

For any $\varepsilon \in (0, \varepsilon_0/2)$ let u_ε be the unique solution of problem (6.3.33) with $\phi \equiv -M$. By comparison results we have:

$$-M \le u_\varepsilon \le u_1 \quad \text{and} \quad -M \le u_\varepsilon \le u_2 \quad \text{in } Q_{\varepsilon,T}\,. \tag{6.3.34}$$

By usual compactness arguments there exists a subsequence $\{u_{\varepsilon_m}\} \subseteq \{u_\varepsilon\}$ which converges uniformly in any compact subset of $[\Omega \cup \mathcal{R}] \times (0, T]$. Set

$$\underline{u} := \lim_{m\to\infty} u_{\varepsilon_m} \quad \text{in } [\Omega \cup \mathcal{R}] \times (0, T]\,.$$

The function $\underline{u}$ is a solution of problem (6.1.3); moreover, from (6.3.34) we obtain

$$-M \le \underline{u} \le u_1 \quad \text{and} \quad -M \le \underline{u} \le u_2 \quad \text{in } Q_T\,. \tag{6.3.35}$$

Set $w = u_1$ or $w = u_2$ for simplicity. The conclusion will follow, if we show that

$$\int_0^T \int_\Omega \big[G(w) - G(\underline{u})\big]\, F\, dx\, dt \,=\, 0 \tag{6.3.36}$$

for any $F \in C_0^\infty(\Omega)$.

In fact, in view of assumption $(H_1) - (ii)$ and the arbitrariness of F, (6.3.36) implies

$$u_1 \,=\, \underline{u} \,=\, u_2 \quad \text{in } Q_T, \tag{6.3.37}$$

whence the conclusion.

Let us prove equality (6.3.36). Without loss of generality, we suppose $supp\, F \subseteq (\Omega \setminus \overline{\mathcal{S}^{\varepsilon_0}})$, $F \ge 0$, $F \not\equiv 0$.

In view of the inequality $w \ge \underline{u}$ (see (6.3.35)), arguing as in the proof of Lemma 6.3.4, we obtain

$$\begin{aligned}\int_{\Omega\setminus\overline{\mathcal{S}^\varepsilon}} \rho(x)\big[w(x,\tau) - \underline{u}(x,\tau)\big]\psi_\varepsilon(x)\, dx + \int_0^\tau \int_{\Omega\setminus\overline{\mathcal{S}^\varepsilon}} \big[G(w) - G(\underline{u})\big] F(x)\, dx\, dt \,=\\ =\, -\int_0^\tau \int_{\mathcal{A}^\varepsilon} \big\{G(w) - G(\underline{u})\big\}\langle \nabla\psi_\varepsilon, \nu_\varepsilon\rangle\, d\sigma\, dt\end{aligned} \tag{6.3.38}$$

where ψ_ε denotes the solution of (6.3.3), $\varepsilon \in (0, \varepsilon_0/2)$, $\tau \in (0, T]$. Since $F \ge 0$, $\psi_\varepsilon \ge 0$, $w \ge \underline{u}$ and $(H_1) - (ii)$ holds true, equality (6.3.38) with $\tau = T$ gives

$$\begin{aligned}\int_0^T \int_\Omega \big[G(w) - G(\underline{u})\big] F(x)\, dx\, dt \le\\ \le \liminf_{\varepsilon\to 0} \Big| \int_0^T \int_{\mathcal{A}^\varepsilon} \big\{G(w) - G(\underline{u})\big\}\langle \nabla\psi_\varepsilon, \nu_\varepsilon\rangle\, d\sigma\, dt\Big|\,.\end{aligned} \tag{6.3.39}$$

Hence, if we can prove that

$$\liminf_{\varepsilon\to 0} \Big| \int_0^T \int_{\mathcal{A}^\varepsilon} \big\{G(w) - G(\underline{u})\big\}\langle \nabla\psi_\varepsilon, \nu_\varepsilon\rangle\, d\sigma\, dt\Big| \,=\, 0\,, \tag{6.3.40}$$

the conclusion follows.

Define

$$\tilde{\varphi}(\varepsilon) := \int_0^T \int_{\mathcal{A}^\varepsilon} \big\{G(w) - G(\underline{u})\big\}\, d\sigma\, dt \quad \big(\varepsilon \in (0, \varepsilon_0/2)\big)\,.$$

By (6.3.26) with $v_1 = w$ and $v_2 = \underline{u}$ we have

$$\liminf_{\varepsilon\to 0} \tilde{\varphi}(\varepsilon) \,=\, 0\,. \tag{6.3.41}$$

From (6.3.41) we immediately deduce (6.3.40). In fact, by (6.3.41) there exists a sequence $\{\varepsilon_m\} \subseteq (0, \varepsilon_0/2)$, $\varepsilon_m \to 0$ as $m \to \infty$ such that

$$\lim_{m\to\infty} \tilde{\varphi}(\varepsilon_m) = \liminf_{\varepsilon\to 0} \tilde{\varphi}(\varepsilon) = 0 . \tag{6.3.42}$$

By (6.3.5) and (6.3.42) we obtain

$$\Big| \int_0^T \int_{\mathcal{A}^{\varepsilon_m}} \{G(w) - G(\underline{u})\} \langle \nabla\psi_{\varepsilon_m}, \nu_{\varepsilon_m} \rangle \, d\sigma \, dt \Big| \leq$$
$$\leq C \int_0^T \int_{\mathcal{A}^{\varepsilon_m}} \{G(w) - G(\underline{u})\} d\sigma \, dt = C\tilde{\varphi}(\varepsilon_m) \to 0$$

as $m \to \infty$, whence (6.3.40) and the conclusion follow. This proves the result. ☐

6.3.2. Proof of Theorems 6.2.5, 6.2.7 and 6.2.9. Observe that for any $A \in Lip([0, T])$ the derivative A' exists almost everywhere in $[0, T]$ and belongs to $L^\infty((0, T))$.

For any $\varepsilon \in (0, \varepsilon_0/2)$ we will make use of the following auxiliary problems

$$\begin{cases} \rho \, \partial_t u = \Delta\big[G(u)\big] & \text{in } Q_{\varepsilon,T} \\ u = 0 & \text{in } \mathcal{R} \times (0, T) \\ u = G^{-1}(A') & \text{in } \mathcal{A}^\varepsilon \times (0, T) \\ u = u_0 & \text{in } [\Omega \setminus \overline{\mathcal{S}^\varepsilon}] \times \{0\} \end{cases} \tag{6.3.43}$$

and

$$\begin{cases} \Delta U = f & \text{in } \Omega \setminus \overline{\mathcal{S}^\varepsilon} \\ U = 0 & \text{in } \mathcal{R} \\ U = \gamma & \text{in } \mathcal{A}^\varepsilon , \end{cases} \tag{6.3.44}$$

where $f \in C(\Omega \setminus \overline{\mathcal{S}^\varepsilon})$ and $\gamma \in C(\mathcal{A}^\varepsilon)$.

Definition 6.3.6. *By a* supersolution *of problem* (6.3.44) *we mean a function* $U \in C(\overline{\Omega} \setminus \overline{\mathcal{S}^\varepsilon}) \cap L^\infty(\Omega \setminus \overline{\mathcal{S}^\varepsilon})$ *such that*

$$\int_{\Omega_1} U \, \Delta\psi \, dx \leq \int_{\Omega_1} f \, \psi \, dx + \int_{\partial\Omega_1 \setminus (\mathcal{R} \cup \mathcal{A}^\varepsilon)} U \, \langle \nabla\psi, \, \nu \rangle d\sigma + \int_{\partial\Omega_1 \cap \mathcal{A}^\varepsilon} \gamma \, \langle \nabla\psi, \, \nu \rangle d\sigma$$

for any open set $\Omega_1 \subseteq \Omega \setminus \overline{\mathcal{S}^\varepsilon}$ *with smooth boundary* $\partial\Omega_1$, $\psi \in C^2(\overline{\Omega_1}), \psi \geq 0, \psi = 0$ *in* $\partial\Omega_1$*; here* ν *denotes the outer normal to* Ω_1. Solutions *and* subsolutions *are defined accordingly.*

Proof of Theorem 6.2.5. For any $\varepsilon \in (0, \varepsilon_0/2)$ let u_ε be the unique solution to problem (6.3.43); then by comparison results we have

$$|u_\varepsilon| \leq \|u_0\|_\infty + \max_{-\|A'\|_\infty \leq r \leq \|A'\|_\infty} |G^{-1}(r)| =: M \quad \text{in } Q_{\varepsilon,T} . \tag{6.3.45}$$

By usual compactness arguments there exists a subsequence $\{u_{\varepsilon_m}\} \subseteq \{u_\varepsilon\}$, which converges uniformly in any compact subset of $[\Omega \cup \mathcal{R}] \times (0, T]$ to a solution u of problem (6.1.3).

Define U as in (6.2.2) and

$$U_\varepsilon(x, t) := \int_0^t G(u_\varepsilon(x, \tau)) d\tau \quad \big((x, t) \in Q_{\varepsilon,T}\big) . \tag{6.3.46}$$

Observe that $U_{\varepsilon_m} \to U$ in $[\Omega \cup \mathcal{R}] \times (0, T]$ as $m \to \infty$.

It is direct to show that for any $t \in (0,T]$ the function $U_\varepsilon(\cdot,t)$ satisfies the problem

$$\begin{cases} \Delta U = -\rho\,[u_0 - u_\varepsilon(\cdot,t)] & \text{in } \Omega \setminus \overline{\mathcal{S}^\varepsilon} \\ U = 0 & \text{on } \mathcal{R} \\ U = A(t) & \text{on } \mathcal{A}^\varepsilon\,. \end{cases} \tag{6.3.47}$$

In fact, by Definition 6.3.5 we obtain

$$\begin{aligned} \int_{\Omega_1} U_\varepsilon(x,t)\Delta\psi(x)\,dx &= \int_{\Omega_1} \rho(x)\big[u_\varepsilon(x,t) - u_0(x)\big]\psi(x)\,dx + \\ + \int_{\partial\Omega_1\setminus(\mathcal{R}\cup\mathcal{A}^\varepsilon)} U_\varepsilon(x,t)\,\langle\nabla\psi(x),\nu\rangle\,d\sigma &+ \int_{\partial\Omega_1\cap\mathcal{A}^\varepsilon} A(t)\,\langle\nabla\psi(x),\nu(x)\rangle\,d\sigma \end{aligned} \tag{6.3.48}$$

for any Ω_1 and $\psi = \psi(x)$ as in Definition 6.3.6 and $t \in (0,T]$.

Arguing as in the proof of Theorem 2.18 in [**58**], we can construct a positive supersolution $V \in C^2(\mathcal{S}^{\hat\varepsilon}) \cap C(\overline{\Omega})$ to problem (6.2.7) satisfying conditions (6.2.8) and

$$V = 0 \quad \text{on } \mathcal{S}\,. \tag{6.3.49}$$

We shall prove that there exists a constant $K > 0$ such that

$$-KV(x) + A(t) \le U(x,t) \le KV(x) + A(t) \qquad \big((x,t) \in [\Omega \setminus \overline{\mathcal{S}^{\hat\varepsilon}}] \times (0,T]\big)\,. \tag{6.3.50}$$

From (6.3.49) and (6.3.50) it follows (6.2.9), whence the conclusion.

It remains to prove (6.3.50). To this purpose, if $\mathcal{R} \neq \emptyset$, set

$$m := \inf_{\mathcal{R}} V > 0\,,\ K := 2\max\{\frac{1}{m}\,\|A\|_\infty, M\}\,; \tag{6.3.51}$$

otherwise, set

$$K := 2M\,; \tag{6.3.52}$$

here M is given by (6.3.45).

Fix any Ω_1 and a ψ as in Definition 6.3.2. Let us approximate ψ by a sequence of functions $\{\psi_m\} \subseteq C^\infty(\Omega)$ such that *supp* $\psi_m \subseteq \Omega_1$ ($m \in \mathbb{N}$), $\psi_m \to \psi$ as $m \to \infty$ in $C(\overline{\Omega}_1)$ and in $C^2(\Omega_2)$ for any open subset Ω_2 with $\overline{\Omega}_2 \subseteq \Omega_1$. By Definition 6.3.2 we have for any $m \in \mathbb{N}$

$$\int_{\Omega_1} V\,\Delta\psi_m\,dx \le \int_{\partial\Omega_1} V\,\langle\nabla\psi_m,\nu\rangle\,d\sigma - \int_{\Omega_1} \rho\,\psi_m\,dx\,. \tag{6.3.53}$$

As $m \to \infty$ in (6.3.53) we obtain (see the proof of Lemma 2.6 in [**60**])

$$\int_{\Omega_1} V\,\Delta\psi\,dx \le \int_{\partial\Omega_1} V\,\langle\nabla\psi,\nu\rangle\,d\sigma - \int_{\Omega_1} \rho\,\psi\,dx\,. \tag{6.3.54}$$

It is easily seen that for any fixed $t \in (0,T]$ the function $KV - U_\varepsilon + A$ is a supersolution, while the function $-KV - U_\varepsilon + A$ is a subsolution of problem

$$\begin{cases} \Delta U = 0 & \text{in } \Omega \setminus \overline{\mathcal{S}^\varepsilon} \\ U = 0 & \text{on } \mathcal{R} \\ U = 0 & \text{on } \mathcal{A}^\varepsilon \end{cases} \tag{6.3.55}$$

for any $\varepsilon \in (0, \varepsilon_0/2)$. In fact, fix any Ω_1, ψ as in Definition 6.3.6 and $t \in (0, T]$. Then by (6.3.48) and (6.3.54) we have

(6.3.56)
$$\int_{\Omega_1} (KV - U_\varepsilon + A)\,\Delta\psi\,dx \le \int_{\partial\Omega_1\setminus(\mathcal{R}\cup\mathcal{A}^\varepsilon)} (KV - U_\varepsilon + A)\langle\nabla\psi,\nu\rangle\,d\sigma +$$
$$+ \int_{\partial\Omega_1\cap\mathcal{R}} (KV + A)\langle\nabla\psi,\nu\rangle d\sigma + \int_{\partial\Omega_1\cap\mathcal{A}^\varepsilon} KV\langle\nabla\psi,\nu\rangle\,d\sigma - \int_{\Omega_1} (K + u_\varepsilon - u_0)\rho\,\psi\,dx =$$

for any $\varepsilon \in (0, \varepsilon_0)$.

From (6.2.8), (6.3.51)-(6.3.52) we get

$$KV \ge 0 \quad \text{on } \mathcal{A}^\varepsilon, \tag{6.3.57}$$

$$KV + A \ge 0 \quad \text{on } \mathcal{R} \tag{6.3.58}$$

and

$$K \ge u_0 - u_\varepsilon \quad \text{in } \Omega_1 \tag{6.3.59}$$

for any $\varepsilon \in (0, \varepsilon_0)$. Moreover, it is easily checked that

$$\langle\nabla\psi,\nu\rangle \le 0 \quad \text{on } \partial\Omega_1. \tag{6.3.60}$$

From (6.3.56)-(6.3.60) we obtain

$$\int_{\Omega_1} (KV - U_\varepsilon + A)\,\Delta\psi\,dx \le \int_{\partial\Omega_1\setminus(\mathcal{R}\cup\mathcal{A}^\varepsilon)} (KV - U_\varepsilon + A)\langle\nabla\psi,\nu\rangle\,d\sigma \tag{6.3.61}$$

for any $\varepsilon \in (0, \varepsilon_0)$. This shows that the function $KV - U_\varepsilon + A$ is a supersolution to problem (6.3.55) for any $\varepsilon \in (0, \varepsilon_0)$ (see Definition 6.3.6). It is similarly seen that $-KV - U_\varepsilon + A$ is a subsolution of the same problem for any $\varepsilon \in (0, \varepsilon_0)$.

By comparison principles we obtain

$$-KV(x) + A(t) \le U_\varepsilon(x,t) \le KV(x) + A(t) \qquad \big((x,t) \in [\Omega \setminus \overline{\mathcal{S}^\varepsilon}] \times (0,T]\big)$$

for any $\varepsilon \in (0, \varepsilon_0)$. This implies (6.3.50), thus the proof is complete. □

To prove Theorem 6.2.7 we use arguments similar to those used to show Lemma 6.3.3-(*iii*). Observe that the same role played by problem (6.3.3) in the proof of Lemma 6.3.3 will be played by problem (6.3.47) in the proof of Theorem 6.2.7. However, we dealt with classical solutions to problem (6.3.3), whereas solutions of (6.3.47) are meant in the sense of Definition 6.3.6. This leads, as we will see later, to consider a companion problem to (6.3.22) in a domain with regular boundary (see (6.3.65) and Definition 6.3.7 below). Observe that, in general, $\partial N_{R,\varepsilon}(x_0)$ in (6.3.22) is not regular at $[\partial B_R(x_0) \cap \Omega] \cap \mathcal{A}^\varepsilon$.

More precisely, let $x_0 \in \mathcal{S}, R > 0$ arbitrarily fixed. For any $\varepsilon \in (0, R)$ we construct the domain $\tilde{N}_{R,\varepsilon}(x_0)$ taking the set $N_{R,\varepsilon}(x_0)$ and making smooth $\partial N_{R,\varepsilon}(x_0)$ at $[\partial B_R(x_0)\cap\Omega]\cap\mathcal{A}^\varepsilon$. We can suppose that the following properties are satisfied:

$$\tilde{N}_{R,\varepsilon}(x_0) \subseteq N_{R,\varepsilon}(x_0),$$
$$\partial\tilde{N}_{R,\varepsilon}(x_0) \quad \text{is smooth},$$

$$\partial\tilde{N}_{R,\varepsilon}(x_0) = \bigcup_{i=1}^{3} \gamma^i_{R,\varepsilon}(x_0),\, \gamma^i_{R,\varepsilon}(x_0) \cap \gamma^j_{R,\varepsilon}(x_0) = \emptyset \ (i,j = 1,2,3,\ i \ne j), \tag{6.3.62}$$

$$\gamma^1_{R,\varepsilon}(x_0) \subseteq \mathcal{A}^\varepsilon,\ \gamma^2_{R,\varepsilon}(x_0) \subseteq \partial B_R(x_0) \cap \Omega,\ \gamma^3_{R,\varepsilon}(x_0) \subseteq [(\Omega \setminus \overline{\mathcal{S}^\varepsilon}) \cap N_R(x_0)] \setminus [N_{\check{R}}(x_0)] \tag{6.3.63}$$

for any $\varepsilon \in (0,R)$ and for some $\check{R} \in (0,R)$ independent of ε; moreover,

$$\bigcup_{\varepsilon\in(0,R)} \tilde{N}_{R,\varepsilon}(x_0) \;=\; N_R(x_0)\,. \tag{6.3.64}$$

We will use auxiliary problems of the following type:

$$\begin{cases} \Delta\, U \;=\; f & \text{in } \tilde{N}_{R,\varepsilon}(x_0) \\ U \;=\; \gamma & \text{on } \partial\tilde{N}_{R,\varepsilon}(x_0)\,, \end{cases} \tag{6.3.65}$$

where $\varepsilon \in (0,R)$, $f \in C\big(\tilde{N}_{R,\varepsilon}(x_0)\big)$, $\gamma \in C\big(\gamma^2_{R,\varepsilon}(x_0) \cup \gamma^3_{R,\varepsilon}(x_0)\big) \cap L^\infty\big(\partial\tilde{N}_{R,\varepsilon}(x_0)\big)$.

DEFINITION 6.3.7. *By a* supersolution *of problem* (6.3.65) *we mean a function* $U \in C\big(\tilde{N}_{R,\varepsilon}(x_0)) \cup \gamma^2_{R,\varepsilon}(x_0) \cup \gamma^3_{R,\varepsilon}(x_0)\big) \cap L^\infty\big(\tilde{N}_{R,\varepsilon}(x_0)\big)$ *such that*

$$\int_{\Omega_1} U\,\Delta\psi\,dx \;\le\; \int_{\Omega_1} f\,\psi\,dx + \int_{\partial\Omega_1\cap\partial\tilde{N}_{R,\varepsilon}(x_0)} \gamma\,\langle\nabla\psi,\,\nu\rangle d\sigma$$

for any open set $\Omega_1 \subseteq \tilde{N}_{R,\varepsilon}(x_0)$ *with smooth boundary* $\partial\Omega_1$, $\psi \in C^2(\overline{\Omega_1}), \psi \ge 0, \psi = 0$ *in* $\partial\Omega_1$*; here* ν *denotes the outer normal to* Ω_1. Solutions *and* subsolutions *are defined accordingly.*

Observe that in the above definition ν is well-defined, since $\partial\tilde{N}_{R,\varepsilon}(x_0)$ is smooth for any $\varepsilon \in (0,R)$.

Let us prove Theorem 6.2.7.

Proof of Theorem 6.2.7. Condition (6.2.3) implies that for any $x \in \overline{\mathcal{S}^{\bar\varepsilon}}$ the partial derivative $\frac{\partial A(x,t)}{\partial t}$ exists almost everywhere in $[0,T]$; in addition, $\dfrac{\partial A}{\partial t} \in L^\infty(\Omega\times(0,T))$. Then we can repeat the proof of Theorem 6.2.5, replacing in problem (6.3.43) $G^{-1}(A')$ by $G^{-1}(\frac{\partial A}{\partial t})$. Hence we construct the sequence $\{U_\varepsilon\}$ $\big(\varepsilon \in (0,\min\{\varepsilon_0,\bar\varepsilon\})\big)$ of solutions of problem (6.3.47) with $A(t)$ repaced by $A(x,t)$. Thus for any $\varepsilon \in (0,\min\{\varepsilon_0,\bar\varepsilon\})$ by Definition 6.3.5 we have

$$\begin{aligned} \int_{\Omega_1} U_\varepsilon(x,t)\Delta\psi(x)\,dx &= \int_{\Omega_1} \rho(x)\big[u_\varepsilon(x,t)-u_0(x)\big]\psi(x)\,dx + \\ + \int_{\partial\Omega_1\setminus(\mathcal{R}\cup\mathcal{A}^\varepsilon)} U_\varepsilon(x,t)\,\langle\nabla\psi(x),\nu(x)\rangle\,d\sigma &+ \int_{\partial\Omega_1\cap\mathcal{A}^\varepsilon} A(x,t)\,\langle\nabla\psi(x),\nu(x)\rangle\,d\sigma \end{aligned} \tag{6.3.66}$$

for any Ω_1 and $\psi = \psi(x)$ as in Definition 6.3.6, $\varepsilon \in (0,\min\{\varepsilon_0,\bar\varepsilon\})$ and $t \in (0,T]$.

Equality (6.3.66) with Ω_1, ψ as in Definition 6.3.7 implies that the function $U_\varepsilon(\cdot,t)$ is a solution to problem

$$\begin{cases} \Delta U = -\rho[u_0 - u_\varepsilon(\cdot,t)] & \text{in } \tilde{N}_{R,\varepsilon}(x_0) \\ U \;=\; A(\cdot,t) & \text{on } \gamma^1_{R,\varepsilon}(x_0) \\ U \;=\; U_\varepsilon(\cdot,t) & \text{on } \gamma^2_{R,\varepsilon}(x_0) \cup \gamma^3_{R,\varepsilon}(x_0) \end{cases} \tag{6.3.67}$$

for any $\varepsilon \in (0,\min\{\varepsilon_0,\bar\varepsilon\})$ and $t \in (0,T]$.

(i) *Suppose that condition* (6.2.4) *is satisfied.* Let $x_0 \in \mathcal{S}$ arbitrarily fixed; hence from (6.2.4) we have $\rho_0 := \inf_{N_{\bar R}(x_0)} \rho > 0$.

Define

$$h(x) := 2\big[V(x) + \frac{\rho_0}{4}|x-x_0|^2\big] \quad \big(x \in N_R(x_0)\big)\,; \tag{6.3.68}$$

here V is the function introduced in the proof of Theorem 6.2.5 and $R \in (0,\min\{\bar R,\hat\varepsilon\})$.

Clearly, $h \in C^2\big(N_R(x_0)\big) \cap C\big(\overline{N_R(x_0)}\big)$ and satisfies the following:

$$\Delta h(x) \le -\rho(x) \quad \text{for any } x \in N_R(x_0)\,, \tag{6.3.69}$$

$$h > 0 \quad \text{in } \overline{N_R(x_0)} \setminus \{x_0\},\; h(x_0) = 0\,. \tag{6.3.70}$$

This implies that the function h is a supersolution to problem

$$\begin{cases} \Delta U = -\rho & \text{in } \tilde{N}_{R,\varepsilon}(x_0) \\ U = h & \text{on } \partial\tilde{N}_{R,\varepsilon}(x_0) \end{cases} \tag{6.3.71}$$

for any $\varepsilon \in (0, \min\{\varepsilon_0, \bar\varepsilon\})$. In fact, since $h \in C^2(N_R(x_0)) \cap C(\overline{N_R(x_0)})$, by (6.3.69) we get

$$\int_{\Omega_1} h\,\Delta\psi\,dx \le \int_{\partial\Omega_1} h\,\langle\nabla\psi, \nu\rangle\,d\sigma - \int_{\Omega_1} \rho\,\psi\,dx \tag{6.3.72}$$

for any Ω_1 and ψ as in Definition 6.3.7 and $\varepsilon \in (0, \min\{\varepsilon_0, \bar\varepsilon\})$.

We claim that there exists a constant $K > 0$ such that

$$-Kh(x) + A(x_0,t) \le U(x,t) \le Kh(x) + A(x_0,t) \quad \text{for any } (x,t) \in N_R(x_0) \times (0,T]\,. \tag{6.3.73}$$

Sending $x \to x_0$ in (6.3.73) we deduce (6.2.6), whence the conclusion.

Let us prove (6.3.73). To this purpose fix any $\sigma > 0$. Since $A \in C(\overline{\mathcal{S}^{\bar\varepsilon}} \times [0,T])$, we can find $\underline{\varepsilon} = \underline{\varepsilon}(\sigma) \in (0, \bar\varepsilon)$ such that

$$|A(x,t) - A(x_0,t)| < \sigma \quad \text{for any } (x,t) \in \gamma^1_{R,\varepsilon}(x_0) \times [0,T] \text{ and } \varepsilon \in (0, \underline{\varepsilon})\,. \tag{6.3.74}$$

Observe that in view of (6.3.63) and (6.3.70), we have for any $\varepsilon \in (0, \tilde\varepsilon)$

$$\inf_{\gamma^2_{R,\varepsilon}(x_0) \cup \gamma^3_{R,\varepsilon}(x_0)} h \ge \inf_{N_R(x_0) \setminus N_{\hat R}(x_0)} h > 0\,;$$

here $\tilde\varepsilon := \min\{\varepsilon_0, \underline{\varepsilon}, R\})$. Set

$$K := \frac{2}{m} \max\{T \max_{-M \le r \le M} |G(r)|,\; \|A\|_{L^\infty(\mathcal{S}^{\bar\varepsilon} \times (0,T))}, mM\}\,. \tag{6.3.75}$$

Let Ω_1, ψ be arbitrarily fixed as in Definition 6.3.7. Since for any $\varepsilon \in (0, \tilde\varepsilon)$ and $t \in (0,T]$, as we have already observed, the function $U_\varepsilon(\cdot, t)$ is a solution to problem (6.3.67), while the function h is a supersolution to problem (6.3.71) we arrive to:

$$\begin{aligned}
&\int_{\Omega_1} [Kh(x) + A(x_0,t) + \sigma - U_\varepsilon(x,t)]\,\Delta\psi(x)\,dx \le \\
&\le \int_{\partial\Omega_1 \setminus \gamma^1_{R,\varepsilon}(x_0)} [Kh(x) + A(x_0,t) + \sigma - U_\varepsilon(x,t)]\langle\nabla\psi(x), \nu(x)\rangle\,d\sigma + \\
&+ \int_{\partial\Omega_1 \cap \gamma^1_{R,\varepsilon}(x_0)} [Kh(x) + A(x_0,t) + \sigma - A(x,t)]\langle\nabla\psi(x), \nu(x)\rangle d\sigma - \\
&- \int_{\Omega_1} [K + u_\varepsilon(x,t) - u_0(x)]\rho(x)\,\psi(x)\,dx
\end{aligned} \tag{6.3.76}$$

for any $\varepsilon \in (0, \tilde\varepsilon)$ and $t \in (0,T]$. From (6.3.70), (6.3.74)-(6.3.75) we get for any $\varepsilon \in (0, \tilde\varepsilon)$

$$Kh(x) + A(x_0,t) + \sigma - U_\varepsilon(x,t) \ge 0 \quad \text{for any } (x,t) \in [\gamma^2_{R,\varepsilon}(x_0) \cup \gamma^3_{R,\varepsilon}(x_0)] \times (0,T]\,, \tag{6.3.77}$$

$$Kh(x) + A(x_0,t) + \sigma - A(x,t) \ge 0 \quad \text{for any } (x,t) \in \gamma^1_{R,\varepsilon}(x_0) \times (0,T] \tag{6.3.78}$$

and

$$K \ge u_0(x) - u_\varepsilon(x,t) \quad \text{for any } (x,t) \in \Omega_1 \times (0,T]\,. \tag{6.3.79}$$

Moreover, it is easily checked that

$$\langle \nabla\psi, \nu\rangle \leq 0 \quad \text{on } \partial\Omega_1 . \tag{6.3.80}$$

From (6.3.76)-(6.3.80) we obtain

$$\begin{aligned} &\int_{\Omega_1} (Kh(x) + A(x,t) + \sigma - U_\varepsilon(x,t))\, \Delta\psi(x)\, dx \;\leq \\ &\leq \int_{\partial\Omega_1 \setminus \partial\tilde N_{R,\varepsilon}(x_0)} (Kh(x) + A(x,t) + \sigma - U_\varepsilon(x,t)) \langle \nabla\psi(x), \nu(x)\rangle\, d\sigma \end{aligned} \tag{6.3.81}$$

for any $\varepsilon \in (0,\tilde\varepsilon)$ and $t \in (0,T]$.

This shows that (see Definition 6.3.7) for any $\varepsilon \in (0,\tilde\varepsilon)$ and $t \in (0,T]$ the function $Kh + A(x_0,t) + \sigma - U_\varepsilon(\cdot,t)$ is a supersolution to problem

$$\begin{cases} \Delta U = 0 & \text{in } \tilde N_{R,\varepsilon}(x_0) \\ U \;=\; 0 & \text{on } \partial\tilde N_{R,\varepsilon}(x_0). \end{cases} \tag{6.3.82}$$

It is analogously checked that the function $-Kh + A(x_0,t) - \sigma - U_\varepsilon(\cdot,t)$ ($\varepsilon \in (0,\tilde\varepsilon)$) is a subsolution of the same problem for any $\varepsilon \in (0,\tilde\varepsilon)$ and $t \in (0,T]$.

By comparison principles we obtain as $\sigma \to 0^+$

$$-Kh(x) + A(x_0,t) \leq U_\varepsilon(x,t) \leq Kh(x) + A(x_0,t) \qquad \big((x,t) \in \tilde N_{R,\varepsilon}(x_0) \times (0,T]\big)$$

for any $\varepsilon \in (0,\tilde\varepsilon)$. This combined with (6.3.64) implies, as $\varepsilon \to 0$, (6.3.73).

(*ii*) *Suppose that condition* (6.2.5) *is satisfied.* As observed in the proof of Lemma 6.3.3, Ω satisfies the *outer sphere* property at $\mathcal{S}$. Hence we can find $\tilde x \in \mathbb{R}^n \setminus \overline{\Omega}$ and $\tilde R > 0$ such that $\overline{B}_{\tilde R}(\tilde x) \cap \overline{\Omega} \,=\, \{x_0\}$.

Define

$$\tilde h(x) := \lambda_1[\exp\{-\lambda_2 \tilde R^2\} - \exp\{-\lambda_2 |x-\tilde x|^2\}] \quad (x \in N_R(x_0)) \tag{6.3.83}$$

with $\lambda_1 > 0$, $\lambda_2 > 0$, $R \in (0,\tilde R)$ to be chosen later,

Clearly, $\tilde h \in C^2(N_R(x_0)) \cap C(\overline{N_R(x_0)})$ and satisfies (6.3.70). Moreover, in view of (6.2.5) we can choose $\lambda_1 > 0, \lambda_2 > 0$ big enough and $R > 0$ small enough such that

$$\Delta\tilde h(x) \;=\; 2\lambda_1\lambda_2 \exp\{-\lambda_2|x-\tilde x|^2\}\big[n - 2\lambda_2|x-\tilde x|^2\big] \;\leq\; -\rho(x) \quad \text{for any } \; x \in N_R(x_0).$$

Then we arrive to the conclusion as well as we made in (*i*) above. The proof in complete. □

To prove Theorem 6.2.9 we need a preliminary result, which follows by Lemma 2.6 in [**60**].

LEMMA 6.3.8. *Let assumption* $(H_1) - (i)$ *be satisfied. Let there exist a supersolution to problem* (6.2.7) *such that* (6.2.8) *is satisfied. Then there exists a sequence* $\{x_m\} \subseteq \Omega$, $d(x_m,\mathcal{S}) \to 0$ *as* $m \to \infty$ *such that*

$$\lim_{m\to\infty} V(x_m) \;=\; 0 .$$

Proof of Theorem 6.2.9. By Lemma 6.3.8 there exists a sequence $\{x_m\} \subseteq \Omega$, $d(x_m,\mathcal{S}) \to 0$ as $m \to \infty$ such that $V(x_m) \to 0$ as $m \to \infty$. Then the conclusion follows repeating the proof of Theorem 6.2.5. □

6.3.3. Proof of Theorems 6.2.11-6.2.12. For further purposes observe that, when $A \equiv 0$ and $u_0 \geq 0$ the proof of Theorem 6.2.5 provides a solution $\underline{u}$ to problem (6.1.3) which satisfies at $\mathcal{S}$ the constraint (6.2.1); in addition, it is *minimal* among all nonnegative solutions to the same problem.

Proof of Theorem 6.2.11. Let w be any solution to problem (6.1.3) satisfying condition (6.2.1) and $\underline{u}$ the minimal solution to the same problem satisfying (6.2.1).

To get the conclusion we use the scheme of the proof of Theorem 6.2.2. Then the result will follow if we show that (6.3.40) holds true.

From (6.2.1) and (6.3.5) we obtain

$$\liminf_{\varepsilon\to 0}\Big|\int_0^T\int_{\mathcal{A}^\varepsilon}\big\{G(w)-G(\underline{u})\big\}\langle\nabla\psi_\varepsilon,\nu_\varepsilon\rangle\,d\sigma\,dt\Big| \leq C\lim_{\varepsilon\to 0}\int_0^T\int_{\mathcal{A}^\varepsilon}\big\{G(w)-G(\underline{u})\big\}\,d\sigma\,d\tau \;=\; 0,$$

whence (6.3.40); this completes the proof. □

Let us prove Theorem 6.2.12. We adapt to the present situation the proof of Theorem 1.3 in [**44**], concerning the Cauchy problem.

Proof of Theorem 6.2.12. Let u_1, u_2 be any two solutions to problem (6.1.3) satisfying condition (6.2.1); set $w := u_1 - u_2$. Then w is a solution to problem

$$\begin{cases} \rho\,\partial_t u = \Delta u & \text{in } Q_T \\ u = 0 & \text{in } \mathcal{R}\times(0,T) \\ u = 0 & \text{in } \Omega\times\{0\}\,. \end{cases} \tag{6.3.84}$$

Define

$$U_i(x,t) := \int_0^t u_i(x,\tau)\,d\tau \quad \big((x,t)\in Q_T;\ i=1,2\big)$$

and

$$W(x,t) := \int_0^t w(x,\tau)\,d\tau \;=\; U_1(x,t)\;-\;U_2(x,t) \quad \big((x,t)\in Q_T\big)\,.$$

Since condition (6.2.1) is satisfied, we have

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_{\mathcal{A}^\varepsilon}\big|W(x,t)\big|\,d\sigma \;=\; \lim_{\varepsilon\to 0}\int_{\mathcal{A}^\varepsilon}\big|U_1(x,t)-U_2(x,t)\big|\,d\sigma \;\leq \\ &\leq \lim_{\varepsilon\to 0}\Big\{\int_{\mathcal{A}^\varepsilon}\big|U_1(x,t)-A(t)\big|\,d\sigma+\int_{\mathcal{A}^\varepsilon}\big|U_2(x,t)-A(t)\big|\,d\sigma\Big\} \;=\; 0 \end{aligned} \tag{6.3.85}$$

uniformly with respect to $t\in[0,T]$.

The conclusion will follow if we prove that

$$\int_0^T\int_\Omega w\,F\,dx\,dt \;=\; 0 \tag{6.3.86}$$

for any $F = F(x,t) \in C_0^\infty(Q_T)$. It is not restrictive, as we do in the sequel, to assume $supp\, F \subseteq \Omega\setminus\overline{\mathcal{S}^{\varepsilon_0}}$, $F \geq 0$.

For any $\varepsilon\in(0,\varepsilon_0/2)$ consider a sequence of positive functions $\{\rho_{m,\varepsilon}\}_{m\in I\!N} \subseteq C^\infty(\overline{\Omega\setminus\mathcal{S}^\varepsilon})$ such that

$$\rho_{m,\varepsilon} \to \rho \quad \text{in } L^1(\Omega\setminus\overline{\mathcal{S}^\varepsilon}) \quad \text{as } m\to\infty\,. \tag{6.3.87}$$

For any $m \in I\!N, \varepsilon \in (0, \varepsilon_0/2)$ let $\psi_{m\,\varepsilon}$ the classical solution to the backward problem

$$(6.3.88)\qquad \begin{cases} \rho_{m,\varepsilon}\,\partial_t\psi + \Delta\psi \;=\; -\,F & \text{in } Q_{\varepsilon,T} \\ \psi \;=0 & \text{in } (\mathcal{R}\cup\mathcal{A}^\varepsilon)\times(0,T) \\ \psi \;=0 & \text{in } (\Omega\setminus\overline{\mathcal{S}^\varepsilon})\times\{T\}\,. \end{cases}$$

Observe that the regularity of $\rho_{m,\varepsilon}$ implies $\psi_{m,\varepsilon} \in C^2(\overline{Q}_{\varepsilon,T})\,(m \in I\!N, \varepsilon \in (0, \varepsilon_0/2))$.

From (6.3.84) and (6.3.1) with $\Omega_1 = \Omega\setminus\overline{\mathcal{S}^\varepsilon}$, $\tau = T$ we have

$$(6.3.89)\quad \int_0^T\int_{\Omega\setminus\overline{\mathcal{S}^\varepsilon}} w\,F\,dx\,dt \;=\; \int_0^T w\,(\rho-\rho_{m,\varepsilon})\partial_t\psi_{m,\varepsilon}\,dx\,dt \;-\; \int_0^T\int_{\mathcal{A}^\varepsilon} w\langle\nabla\psi_{m,\varepsilon},\nu_\varepsilon\rangle\,d\sigma\,dt\,,$$

where $\psi_{m,\varepsilon}$ denotes the solution of problem (6.3.88), $m \in I\!N$, $\varepsilon \in (0, \varepsilon_0/2)$.

We claim that

(i) for any $\varepsilon \in (0, \varepsilon_0/2)$

$$\lim_{m\to\infty}\Big|\int_0^T\int_{\Omega\setminus\overline{\mathcal{S}^\varepsilon}} w\,(\rho-\rho_{m,\varepsilon})\partial_t\psi_{m,\varepsilon}\,dx\,dt\Big| \;=\; 0\,;$$

(ii) there exists a positive constant C such that for any $m \in I\!N, \varepsilon \in (0, \varepsilon_0/2)$

$$\Big|\int_0^T\int_{\mathcal{A}^\varepsilon} w\,\langle\nabla\psi_{\varepsilon,m},\nu_\varepsilon\rangle\,d\sigma\,dt\Big| \;\le\; C\int_0^T\int_{\mathcal{A}^\varepsilon}\big|W\big|\,d\sigma\,dt\,.$$

Let us put-off the proof of claims $(i)-(ii)$ and get the conclusion.

From (6.3.89) and claims $(i)-(ii)$ we obtain for any $\varepsilon \in (0, \varepsilon_0/2)$

$$\Big|\int_0^T\int_{\Omega\setminus\overline{\mathcal{S}^\varepsilon}} w\,F\,dx\,dt\Big| \;\le\; C\int_0^T\int_{\mathcal{A}^\varepsilon}|W|\,d\sigma,dt\,.$$

This comblined with (6.3.85) yield, as $\varepsilon \to 0$, (6.3.86) and the conclusion.

It remains to prove claims $(i)-(ii)$. To this purpose observe that for any $m \in I\!N$, $\varepsilon \in (0, \varepsilon_0/2)$ the function $\tilde{\psi}_{m,\varepsilon} \equiv \partial_t\psi_{m,\varepsilon}$ solves the backward problem

$$(6.3.90)\qquad \begin{cases} \rho_{m,\varepsilon}\,\partial_t\psi + \Delta\psi \;=\; -\,\partial_t F & \text{in } Q_{\varepsilon,T} \\ \psi \;=0 & \text{in } (\mathcal{R}\cup\mathcal{A}^\varepsilon)\times(0,T) \\ \psi \;=0 & \text{in } (\Omega\setminus\overline{\mathcal{S}^\varepsilon})\times\{T\}\,. \end{cases}$$

Let us show that there exists $M > 0$ such that for any $m \in I\!N$, $\varepsilon \in (0, \varepsilon_0/2)$

$$(6.3.91)\qquad |\tilde{\psi}_{m,\varepsilon}| \;\le\; M \quad \text{in } Q_{\varepsilon,T}\,.$$

In fact, since Ω is bounded, we can suppose that Ω lies in the slab

$$\{x \equiv (x_1,\dots,x_n) \in I\!R^n\,\big|\,0 < x_1 < d\}$$

for some $d > 0$. Then it is easily seen that the function

$$v(x,t) := \big(\exp\{d\} - \exp\{x_1\}\big)\|\partial_t F\|_{L^\infty(Q_T)} \quad \big((x,t) \in Q_{\varepsilon,T}\big)$$

is a supersolution to problem (6.3.90) for any $\varepsilon \in (0, \varepsilon_0/2)$, while $-v$ is a subsolution to the same problem. By comparison principles we have for any $\varepsilon \in (0, \varepsilon_0/2)$

$$(6.3.92)\qquad -v \le \tilde{\psi}_{m,\varepsilon} \le v \quad \text{in } Q_{\varepsilon,T}\,.$$

Hence (6.3.91) follows with $M := \exp\{d\}\|\partial_t F\|_{L^\infty(Q_T)}$.

Then for any $\varepsilon \in (0, \varepsilon_0/2)$

$$\lim_{m\to\infty} \Big| \int_0^T \int_{\Omega\setminus\overline{\mathcal{S}^\varepsilon}} w\,(\rho - \rho_{m,\varepsilon})\tilde{\psi}_{m,\varepsilon}\, dx\, dt \Big| \leq \lim_{m\to\infty} \|w\|_\infty MT \int_{\Omega\setminus\overline{\mathcal{S}^\varepsilon}} |\rho - \rho_{m,\varepsilon}|\, dx \;=\; 0;$$

here use of the limit (6.3.87) has been made. Thus claim (i) is shown.

To prove claim (ii) consider the function Z defined in (6.3.11)-(6.3.12) with M given by (6.3.91). Hence Z is a supersolution, while $\tilde{\psi}_{m,\varepsilon}$ $(m \in I\!N, \varepsilon \in (0, \varepsilon_0/2))$ is a subsolution to the problem

$$\text{(6.3.93)} \qquad \begin{cases} \rho_{m,\varepsilon}\,\partial_t\psi + \Delta\psi \;=\; 0 & \text{in } (\mathcal{S}^{\varepsilon_0} \setminus \overline{\mathcal{S}^\varepsilon}) \times (0, T] \\ \psi \;= M & \text{in } \mathcal{A}^{\varepsilon_0} \times (0, T) \\ \psi \;= 0 & \text{in } \mathcal{A}^{\varepsilon} \times (0, T) \\ \psi \;= Z & \text{in } (\mathcal{S}^{\varepsilon_0} \setminus \overline{\mathcal{S}^\varepsilon}) \times \{T\}\,. \end{cases}$$

Then arguing as in the proof of Lemma 6.3.3 we obtain for any $m \in I\!N, \varepsilon \in (0, \varepsilon_0/2)$

$$\text{(6.3.94)} \qquad \langle \nabla\tilde{\psi}_{m,\varepsilon}, \nu_\varepsilon \rangle \;\geq\; -C \quad \text{on } \mathcal{A}^\varepsilon \times (0, T)\,,$$

where $C := \mu\hat{C}$ with μ and $\hat{C}$ given by (6.3.12). It is similarly seen that for any $m \in I\!N, \varepsilon \in (0, \varepsilon_0/2)$

$$\text{(6.3.95)} \qquad \langle \nabla\tilde{\psi}_{m,\varepsilon}, \nu_\varepsilon \rangle \;\leq\; C \quad \text{on } \mathcal{A}^\varepsilon \times (0, T)\,.$$

Integrating by parts, since $W = 0$ in $\mathcal{A}^\varepsilon \times \{0\}$ and $\langle \nabla\psi_{m,\varepsilon}, \nu_\varepsilon \rangle \;=\; 0$ in $\mathcal{A}^\varepsilon \times \{T\}$ $(m \in I\!N, \varepsilon \in (0, \varepsilon_0/2))$ we obtain

$$\int_0^T \int_{\mathcal{A}^\varepsilon} w \langle \nabla\psi_{m,\varepsilon}, \nu_\varepsilon \rangle\, d\sigma\, dt \;=\; \int_{\mathcal{A}^\varepsilon} \int_0^T \partial_t W\, \langle \nabla\psi_{m,\varepsilon}, \nu_\varepsilon \rangle\, d\sigma\, dt \;=\; -\int_{\mathcal{A}^\varepsilon} \int_0^T W\, \langle \nabla\tilde{\psi}_{m,\varepsilon}, \nu_\varepsilon \rangle\, d\sigma\, dt$$

for any $m \in I\!N, \varepsilon \in (0, \varepsilon_0/2)$. Hence by (6.3.94)-(6.3.95) it follows claim (ii). This completes the proof. □

CHAPTER 7

Phragmèn-Lindelöf principles for fully nonlinear elliptic equations with unbounded coefficients

7.1. Introduction

We deal with viscosity solutions of fully nonlinear elliptic equations of the following form:

$$F(x,u,Du,D^2u)=0 \quad \text{in } \Omega\,; \tag{7.1.1}$$

here $\Omega \subseteq I\!R^n$ is a connected open, possibly unbounded set with boundary $\partial\Omega$, F is a real-valued continuous function defined in $\Omega \times I\!R \times I\!R^n \times \Sigma^n$, Σ^n being the linear space of $n\times n$ symmetric matrices with real entries; precise assumptions will be made in Section 7.2.

We always express the boundary $\partial\Omega$ as the disjoint union of the *regular boundary* $\mathcal{R}$ and the *singular boundary* $\mathcal{S}$, for the nonlinear operator F is well-behaved in $\Omega\cup\mathcal{R}$; on the contrary, it can become hill-behaved, when $dist(x,\mathcal{S})\to 0$, or when $|x|\to\infty$, if Ω is unbounded (see assumptions $(F_1)-(F_2)$ in Section 7.2). Then it is natural to prescribe the Dirichlet boundary condition on $\mathcal{R}$; this leads to the following problem:

$$\begin{cases} F(x,u,Du,D^2u)=0 & \text{in } \Omega \\ u=g & \text{in } \mathcal{R}\,. \end{cases} \tag{7.1.2}$$

When F is a linear operator with bounded coefficients, uniqueness results for problem (7.1.2), in the class of bounded solutions, have been proved both by analytical (*e.g.*, see [**7**], [**26**], [**56**]) and probabilistic methods (*e.g.*, see [**31**]) . Such results are generalized in several respects in [**58**], [**65**](see also [**59**],[**60**], for parabolic problems), where linear degenerate operators whose coefficients can become unbounded at $\mathcal{S}$ are dealt with; moreover, uniqueness is obtained in the class of solutions satisfying a suitable growth condition at $\mathcal{S}$, as a consequence of Phragmèn-Lindelöf type results (*e.g.*, see [**61**], for classical Phragmèn-Lindelöf principles).

For fully nonlinear operators, fulfilling natural structural conditions, Phragmèn-Lindelöf principles have been proved in [**15**]-[**16**], when Ω is unbounded, $\mathcal{R}=\partial\Omega$ and Ω satisfies specific geometric conditions; roughly speaking, it is necessary that there is "enough boundary" near any point of Ω. Following the arguments of [**13**], in [**15**]-[**16**] at first a boundary weak Harnack inequality is shown, then Alexandrov-Bakelman-Pucci estimates and Phragmèn-Lindelöf principles are obtained.

On the other hand, in case of a bounded domain, in [**2**] comparison principles for problem (7.1.2) are given, if $\mathcal{S}$ is a sufficiently smooth $(n-1)-$dimensional submanifold of $I\!R^n$, $F\in C(\overline{\Omega}\times I\!R\times I\!R^n\times\Sigma^n)$ and F fulfills suitable conditions at $\mathcal{S}$, which extend to the nonlinear case those introduced in [**26**].

Our purpose is to establish Phragmèn-Lindelöf principles for problem (7.1.2), when F can diverge or need not to have a limit when $dist(x,\mathcal{S})\to 0$. Beside problem (7.1.2), we shall consider also the problems:

$$\begin{cases} F(x,u,Du,D^2u)=0 & \text{in } \Omega \\ u=g & \text{in } \mathcal{R}\cup\mathcal{S}_1 \end{cases} \tag{7.1.3}$$

and

$$
(7.1.4) \qquad \begin{cases} F(x,u,Du,D^2u)=0 & \text{in } \Omega \\ u \;=g & \text{in } \; \mathcal{R}\cup\mathcal{S}_1 \\ \lim_{|x|\to\infty} u(x) \;=\; l\,. \end{cases}
$$

since, if Phragmèn-Lindelöf principle fails for problem (7.1.2), it is natural to try and recover it by prescribing a sign condition on some subset $\mathcal{S}_1$ of the singular boundary $\mathcal{S}$, and/or a condition at infinity, if Ω is unbounded. Indeed, under certain hypotheses, we shall prove that any subsolution to problem (7.1.3) with $g=0$, which is nonpositive on $\mathcal{R}\cup\mathcal{S}_1$ and satisfies a suitable growth condition at $\mathcal{S}_2:=\mathcal{S}\setminus\mathcal{S}_1$ is necessarily nonpositive in Ω (see Theorem 7.3.1). In other words, the sign of u at $\mathcal{R}\cup\mathcal{S}_1$ propagates in the whole Ω, even if we do not require a sign condition of u on the portion $\mathcal{S}_2$ of the singular boundary $\mathcal{S}$. Moreover, observe that no regularity conditions on F at $\mathcal{S}$ are imposed (see $(F_1)-(F_2)$ in Section 7.2).

The Phragmèn-Lindelöf principle we state relies on the existence of suitable supersolutions to a companion problem of problem (7.1.2) and on their behaviour as $dist(x,\mathcal{S}_2)\to 0$ (see Subsection 7.3.1). In Subsection 7.3.2 we address the actual construction of such supersolutions, aiming to give explicit conditions for the Phragmèn-Lindelöf principle, for special classes of equations, such as semilinear degenerate equations (see Theorem 7.3.4), fully nonlinear equations related to extremal Pucci operators (see Theorem 7.3.6). Regarding this aspect, a central role will be played by geometrical properties of $\mathcal{S}_2$, in particular by its dimension. In Subsection 7.3.3 we state some comparison results for problem (7.1.2), under suitable assumptions on F and the singular boundary $\mathcal{S}$. Furthermore, in Subsection 7.3.4 we discuss some generalizations of previous results to singular fully nonlinear operators, which can not to be defined where the gradient vanishes. These general results are used in Section 7.5 to discuss a few of examples, where nonlinear operators which are unbounded at the singular boundary $\mathcal{S}$ are employed.

Finally, let us point out that the methods we use in the sequel are quite similar to those of [**58**] and [**65**], where the linear case is addressed.

7.2. Mathematical framework and auxiliary results

In what follows, concerning the regular boundary $\mathcal{R}$ and the singular boundary $\mathcal{S}$, we always assume:

$$
(H_1) \quad \begin{cases} (i) \quad \partial\Omega \,=\, \mathcal{R}\,\cup\,\mathcal{S},\; \mathcal{R}\,\cap\,\mathcal{S}\,=\,\emptyset,\; \mathcal{S}\neq\emptyset\,; \\ (ii) \quad \mathcal{R}\subseteq\partial\overline{\Omega},\; \mathcal{R} \text{ open and smooth enough}\,; \\ (iii) \quad \mathcal{S}=\mathcal{S}_1\cup\mathcal{S}_2,\;\; \mathcal{S}_1\cap\mathcal{S}_2=\emptyset\,; \\ (iv) \quad \mathcal{S}_1 \text{ and } \mathcal{S}_2 \text{ have a finite number of connected components.} \end{cases}
$$

The following assumption concerning F will be made:

$$
(F_1) \quad \begin{cases} (i) \quad F\in C((\Omega\cup\mathcal{R})\times I\!R\times I\!R^n\times\Sigma^n; I\!R)\,; \\ (ii) \; F(x,r,p,X)-F(x,s,p,X)\geq\, 0 \text{ for any } x\in\Omega, r,s,\in I\!R, r\geq s, p\in I\!R^n, X\in\Sigma^n\,; \\ (iii) \; F \text{ is \textit{(degenerate) elliptic}, that is } \; F(x,r,p,X)-F(x,r,p,X+Y)\geq 0 \\ \qquad \text{for any } x\in\Omega\cup\mathcal{R}, p\in I\!R^n, X,Y\in\Sigma^n, Y\geq 0\,; \end{cases}
$$

We say that F is *proper* if it satisfies $(F_1)-(ii),(iii)$.

Let $\Omega_1\subseteq\overline{\Omega}$. We denote by $USC(\Omega_1)$ and, respectively, by $LSC(\Omega_1)$ the sets of upper and, respectively, lower semicontinuous functions in Ω_1. Let us make the following definitions.

Definition 7.2.1. (i) *A function $u \in USC(\Omega)$ is a viscosity subsolution of equation* (7.1.1)*, provided that the following condition holds:*

$$x_0 \in \Omega, \psi \in C^2(\Omega) \text{ and } u - \psi \text{ has a local maximum at } x_0 \text{ implies}$$
$$F(x_0, u(x_0), D\psi(x_0), D^2\psi(x_0)) \leq 0.$$

(ii) *A function $u \in LSC(\Omega)$ is a viscosity supersolution of equation* (7.1.1)*, provided that the following condition holds:*

$$x_0 \in \Omega, \psi \in C^2(\Omega) \text{ and } u - \psi \text{ has a local minimum at } x_0 \text{ implies}$$
$$F(x_0, u(x_0), D\psi(x_0), D^2\psi(x_0)) \geq 0.$$

(iii) *A function $u \in C(\Omega)$ is a viscosity solution of equation* (7.1.1)*, if u is both a sub- and a supersolution of equation* (7.1.1).

Later on, we only consider *viscosity* sub-, supersolutions to equation (7.1.1).

Definition 7.2.2. *Let $\mathcal{E} \subseteq \partial\Omega$ and $g \in C(\mathcal{E})$. A function $u \in USC(\Omega \cup \mathcal{E})$ is a subsolution of problem*

$$\begin{cases} F(x, u, Du, D^2u) = 0 & \text{in } \Omega \\ u = g & \text{in } \mathcal{E}, \end{cases} \tag{7.2.1}$$

provided that

(i) *u is a subsolution of equation* (7.1.1),

(ii) *$u \leq g$ on $\mathcal{E}$.*

Supersolutions and solutions of problem (7.2.1) *are defined accordingly.*

Let Ω_1 be a subset of Ω such that $\overline{\Omega}_1 \subseteq \Omega \cup \mathcal{R}$. We say that *comparison principle* holds for F in $\Omega \cup \mathcal{R}$, if for any subsolution $\underline{u} \in USC(\overline{\Omega}_1)$ to equation

$$F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega_1, \tag{7.2.2}$$

and for any supersolution $\overline{u} \in LSC(\overline{\Omega}_1)$ to the same equation such that $\underline{u} \leq \overline{u}$ on $\partial\Omega_1$ we have

$$\underline{u} \leq \overline{u} \quad \text{in } \Omega_1 .$$

Sufficient conditions for such a comparison principle are well-known (*e.g.*, see [**5**], [**18**],[**39**]).

We always assume the following:

$$(F_2) \quad \begin{cases} \text{there exists a function } E \in C((\Omega \cup \mathcal{R}) \times I\!R \times I\!R^n \times \Sigma^n; I\!R) \text{ such that:} \\ (i) \quad F(x, r, p, X) \geq E(x, r, p, X) \text{ for any } x \in \Omega \cup \mathcal{R}, r \geq 0, p \in I\!R^n, X \in \Sigma^n ; \\ (ii) \quad E \text{ is proper} ; \\ (iii) \text{ comparison principle holds for } E \text{ in } \Omega \cup \mathcal{R} ; \\ (iv) \quad E(x, 0, 0, 0) \leq 0 \text{ for any } x \in \Omega . \end{cases}$$

In the sequel we exhibit a large class of operators F satisfying assumptions $(F_1) - (F_2)$. To this aim, consider the *Pucci extremal operators*:

$$\mathcal{P}^-_{\lambda,\Lambda}(X) := \inf_{M \in \mathcal{M}_{\lambda,\Lambda}} \{-Tr(MX)\}, \ \mathcal{P}^+_{\lambda,\Lambda}(X) := \sup_{M \in \mathcal{M}_{\lambda,\Lambda}} \{-Tr(MX)\} \ (X \in \Sigma^n); \tag{7.2.3}$$

here $0 < \lambda \leq \Lambda$, $\mathcal{M}_{\lambda,\Lambda} := \{M \in \Sigma^n \mid \lambda I \leq M \leq \Lambda I\}$.

Let us recall, in the following lemma, some properties of the operators $\mathcal{P}^{\pm}_{\lambda,\Lambda}$ (*e.g.*, see [**14**], [**49**]). As usual, we will say that F is *uniformly elliptic* (with ellipticity constants $\lambda \leq \Lambda$), if

$$\lambda tr(Y) \leq F(x, r, p, X) - F(x, r, p, X + Y) \leq \Lambda tr(Y),$$

for any $x \in \Omega \cup \mathcal{R}, r \in I\!R, p \in I\!R^n, X, Y \in \Sigma^n, Y \geq 0$.

LEMMA 7.2.3. *Let $0 < \lambda \leq \Lambda$. For any $X, Y \in \Sigma^n$ we have:*

(i) $\mathcal{P}^-_{\lambda,\Lambda}(X) = \lambda Tr(X^-) - \Lambda Tr(X^+)$, $\mathcal{P}^+_{\lambda,\Lambda}(X) = \Lambda Tr(X^-) - \lambda Tr(X^+)$, *where* $X^+ \geq 0, X^- \geq 0, X = X^+ - X^-, X^+X^- = 0$;

(ii) $\mathcal{P}^\pm_{\lambda,\Lambda}(X)$ *are uniformly elliptic operators with ellipticity constants* λ *and* Λ;

(iii) $\mathcal{P}^+_{\lambda,\Lambda}(X) = -\mathcal{P}^-_{\lambda,\Lambda}(-X)$;

(iv) $\mathcal{P}^\pm_{\lambda,\Lambda}(\alpha X) = \alpha \mathcal{P}^\pm_{\lambda,\Lambda}(X)$ *for any* $\alpha \geq 0$;

(v) $\mathcal{P}^+_{\lambda,\Lambda}$ *is convex in* X, $\mathcal{P}^-_{\lambda,\Lambda}$ *is concave in* X;

(vi)

$$\mathcal{P}^-_{\lambda,\Lambda}(X) + \mathcal{P}^-_{\lambda,\Lambda}(Y) \leq \mathcal{P}^-_{\lambda,\Lambda}(X+Y) \leq$$

$$\leq \mathcal{P}^-_{\lambda,\Lambda}(X) + \mathcal{P}^+_{\lambda,\Lambda}(Y) \leq \mathcal{P}^+_{\lambda,\Lambda}(X+Y) \leq \mathcal{P}^+_{\lambda,\Lambda}(X) + \mathcal{P}^+_{\lambda,\Lambda}(Y).$$

Observe that F is uniformly elliptic (with ellipticity constants $\lambda \leq \Lambda$), if and only if the following condition, involving the Pucci operators, holds true:

$$\mathcal{P}^-_{\lambda,\Lambda}(Y) \leq F(x, r, p, X + Y) - F(x, r, p, X) \leq \mathcal{P}^+_{\lambda,\Lambda}(Y), \tag{7.2.4}$$

for any $x \in \Omega \cup \mathcal{R}, r \in I\!R, p \in I\!R^n, X, Y \in \Sigma^n$.

We have the following

LEMMA 7.2.4. *Let assumptions* (H_1) *and* (F_1) *be satisfied. Moreover, let the following condition hold true:*

$$(F_3) \quad \begin{cases} \text{there exist } \beta, \gamma \in C(\Omega \cup \mathcal{R}; [0, \infty)) \text{ such that} \\ F(x, r, p, X) \geq \mathcal{P}^-_{\lambda,\Lambda}(X) - \beta(x)|p| + \gamma(x) r \text{ for any } x \in \Omega \cup \mathcal{R}, r \geq 0, p \in I\!R^n, X \in \Sigma^n . \end{cases}$$

Then assumption (F_2) *is satisfied.*

Proof of Lemma 7.2.4. It suffices to take

$$E(x, r, p, X) := \mathcal{P}^-_{\lambda,\Lambda}(X) - \beta(x)|p| + \gamma(x) r \ \big(x \in \Omega \cup \mathcal{R}, r \geq 0, p \in I\!R^n, X \in \Sigma^n\big). \tag{7.2.5}$$

□

Moreover, next lemma holds true.

LEMMA 7.2.5. *Let assumptions* (A_1) *and* $(F_1)-(i)$ *be satisfied. Moreover, let the following condition hold true:*

$$(F_4) \begin{cases} (i) & F \text{ is uniformly elliptic with ellipticity constants } \lambda \leq \Lambda\,; \\ (ii) & \text{there exists a function } \beta \in C(\Omega \cup \mathcal{R}; [0, \infty)) \text{ such that} \\ & F(x, 0, p, 0) \geq -\beta(x)|p|, \text{ for any } x \in \Omega \cup \mathcal{R}, p \in I\!R^n\,; \\ (iii) & \text{there exists a function } \gamma \in C(\Omega \cup \mathcal{R}; [0, \infty)) \text{ such that} \\ & F(x, r, p, X) - F(x, s, p, X) \geq \gamma(x)(r - s), \\ & \text{for any } x \in \Omega \cup \mathcal{R}, r, s \in I\!R, r \geq s, p \in I\!R^n, X \in \Sigma^n\,. \end{cases}$$

Then assumption (F_2) *is satisfied.*

Proof of Lemma 7.2.5. By (F_4) and (7.2.4) it follows that

$$F(x, r, p, X) \geq \mathcal{P}^-_{\lambda,\Lambda}(X) + F(x, r, p, 0) \geq$$

$$\geq \mathcal{P}^-_{\lambda,\Lambda}(X) + F(x, 0, p, 0) + \gamma_1(x) r \geq E(x, r, p, X)$$

for any $x \in \Omega \cup \mathcal{R}, r \geq 0, p \in I\!R^n, X \in \Sigma^n$, where E is the function defined in (7.2.5). Hence the conclusion follows by Lemma 7.2.4. □

Set $u_+ := \sup\{u, 0\}$ and $u_- := \sup\{-u, 0\}$. The following standard result holds true.

LEMMA 7.2.6. *Let assumptions* (A_1), $(F_1)-(F_2)$ *be satisfied. Let* $u \in USC(\Omega)$ *be a subsolution of equation* (7.1.1). *Then the function* u_+ *is a subsolution of equation*

$$E(x, u, Du, D^2u) = 0 \quad in\ \Omega. \tag{7.2.6}$$

Proof of Lemma 7.2.6. Clearly $u_+ \in USC(\Omega)$, since $u \in USC(\Omega)$. Take $\psi \in C^2(\Omega), x_0 \in \Omega$ such that $u_+(x_0) = \psi(x_0)$ and

$$u_+(x) - \psi(x) \le u_+(x_0) - \psi(x_0) \quad \text{for any}\, x \in B_\delta(x_0)$$

for some $\delta > 0$. If $u_+(x_0) = 0$, then $\varphi(x_0) = 0$ and

$$\psi(x) \ge u_+(x) \ge 0 \quad \text{for any}\, x \in B_\delta(x_0),$$

hence ψ has a relative minimum at x_0. Therefore $D\psi(x_0) = 0$, $D^2\psi(x_0) \ge 0$. Then by $(H_3) - (ii), (v)$ we deduce

$$E(x_0, u_+(x_0), D\psi(x_0), D^2\psi(x_0)) = E(x_0, 0, 0, D^2\psi(x_0)) \le E(x_0, 0, 0, 0) \le 0. \tag{7.2.7}$$

Now consider the case $u_+(x_0) > 0$; then $u(x_0) = u_+(x_0) = \psi(x_0)$,

$$u(x) \le u_+(x) \le \psi(x) \quad \text{for any}\, x \in B_\delta(x_0);$$

hence the function $u - \psi$ has a relative maximum in x_0. By $(H_3) - (i)$ and the fact that u is a subsolution of equation (7.1.1), we have

$$E(x_0, u_+(x_0), D\psi(x_0), D^2\psi(x_0)) \le F(x_0, u(x_0), D\psi(x_0), D^2\psi(x_0)) \le 0.$$

This inequality combined with (7.2.7) gives the conclusion. □

7.3. Results

7.3.1. Phragmèn-Lindelöf principle.

We shall prove next

THEOREM 7.3.1. *Let assumptions* $(H_1), (F_1) - (F_2)$ *be satisfied; suppose* $\mathcal{S}_2 \ne \emptyset$. *Let there exist a supersolution* $W \ge H > 0$ *of problem*

$$\begin{cases} E(x, u, Du, D^2u) = 0 & in\ \Omega \\ u = 0 & on\ \ \mathcal{R} \end{cases} \tag{7.3.1}$$

such that, for any $\alpha \in (0, \alpha_0)$, αW *is a supersolution of the same problem* $(\alpha_0 > 0)$. *If* Ω *is bounded, then any subsolution* $u \in USC(\Omega \cup \mathcal{R} \cup \mathcal{S}_1)$ *of problem* (7.1.3) *with* $g = 0$ *such that*

$$\limsup_{dist(x,\mathcal{S}_2)\to 0} \frac{u(x)}{W(x)} \le 0 \tag{7.3.2}$$

satisfies $u \le 0$ *in* Ω.
If Ω *is unbounded the same conclusion holds true under the additional condition*

$$\limsup_{|x|\to\infty} \frac{u(x)}{W(x)} \le 0\,. \tag{7.3.3}$$

In the above Theorem, the sign condition on the portion $\mathcal{S}_2$ of the singular boundary, where the subsolution u need not to be defined, is replaced by a growth rate condition with respect to a suitable supersolution to problem (7.3.1). Observe that condition (7.3.2) reduces to a sign condition for u at $\mathcal{S}_2$, whenever W is bounded in a neighbourhood of $\mathcal{S}_2$.

REMARK 7.3.2. Theorem 7.3.1 holds true also in the following cases:

(i) if condition (7.3.3) is replaced by the sign condition

$$\limsup_{|x|\to\infty} u(x) \le 0\,; \tag{7.3.4}$$

(ii) if condition (7.3.2), respectively (7.3.3), is replaced by the weaker assumption

$$\liminf_{\varepsilon \to 0} \{ \sup_{\mathcal{A}_2^\varepsilon \setminus \mathcal{S}_2} \frac{u}{W} \} \leq 0 \,, \tag{7.3.5}$$

respectively

$$\liminf_{\varepsilon \to 0} \{ \sup_{[\Omega \cup \mathcal{R}] \cap \partial B_{\frac{1}{\varepsilon}}} \frac{u}{W} \} \leq 0. \tag{7.3.6}$$

In the above inequalities and hereafter we set $B_r(\bar{x}) := \{|x - \bar{x}| < r\}$ $(\bar{x} \in I\!R^n), B_r(0) \equiv B_r$ and

$$\mathcal{A}_2^\varepsilon := \{x \in \bar{\Omega} \,|\, dist(x, \mathcal{S}_2) = \varepsilon\}$$

for any $\varepsilon \in (0, \varepsilon_0)$, (with $\varepsilon_0 > 0$ suitably small).

7.3.2. Special classes of operators. The Phragmèn-Lindelöf principle stated in Subsection 7.3.1 is implicit in character, for it relies on the actual construction of the supersolution W of problem (7.3.1). In this Subsection, for special classes of nonlinear operators F, we establish sufficient conditions for the construction of such supersolution W, aiming to give explicit criteria for the validity of Phragmèn-Lindelöf principle for problem (7.1.3).

Concerning the regular boundary and the singular boundary, beside (H_1) the following assumption will be made:

$$(H_2) \quad \begin{cases} (i) & \overline{\mathcal{R}} \cap \overline{\mathcal{S}_2} = \emptyset, \ \overline{\mathcal{S}_1} \cap \overline{\mathcal{S}_2} = \emptyset \,; \\ (ii) & \mathcal{S}_2 \text{ is a compact } k-\text{dimensional submanifold of } I\!R^n \text{ of class } C^3 \\ & \text{with } k = 0, 1, \dots, n-1. \end{cases}$$

7.3.2.1. *Semilinear degenerate equations.* Consider semilinear degenerate problems of the following type:

$$\begin{cases} -tr(A(x)D^2u) + H(x, u, Du) = 0 & \text{in } \Omega \\ \\ u = g & \text{in } \mathcal{R} \cup \mathcal{S}_1 \,, \end{cases} \tag{7.3.7}$$

where

$$(F_5) \quad \begin{cases} (i) \;\; a_{ij} = a_{ji} \in C^{0,1}(\overline{\Omega}),\, \sigma_{ij} \in C^{0,1}(\Omega \cup \mathcal{R}),\, A(x) = \sigma(x)\sigma(x)^T \, (x \in \Omega \cup \mathcal{R}) \,; \\ (ii) \;\; \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq 0 \text{ for any } x \in \Omega \text{ and } (\xi_1, .., \xi_n) \in I\!R^n \,; \\ (iii) \; H \in C((\Omega \cup \mathcal{R}) \times I\!R \times I\!R^n) \,; \\ (iv) \; \text{there exists } \beta \in C(\Omega \cup \mathcal{R}; [0, \infty)) \text{ such that} \\ \qquad H(x, 0, p) \geq -\beta(x)|p| \text{ for any } x \in \Omega \cup \mathcal{R}, p \in I\!R^n \,; \\ (v) \;\; \text{there exists } \gamma \in (\Omega \cup \mathcal{R}; [0, \infty)) \text{ such that} \\ \qquad H(x, r, p) - H(x, s, p) \geq \gamma(x)(r - s) \\ (vi) \;\; \gamma + \sum_{i=1}^n \sigma_{ji}^2 > 0 \text{ in } \Omega \cup \mathcal{R} \text{ for some } j = 1, \dots, n \,. \end{cases}$$

To state our result we need some preliminary remarks. For any fixed $y \in \mathcal{S}_2$ there exist orthonormal vectors $\eta^{(1)}(y), \dots, \eta^{(n-k)}(y) \in I\!R^n$, which are orthogonal to $\mathcal{S}_2$ at y. Consider the matrix $A_\perp(y) \equiv (\alpha_{lm}(y)) \in \Sigma^{n-k}$, where

$$\alpha_{lm}(y) := \sum_{i,j=1}^n a_{ij}(y)\eta_i^{(l)}(y)\eta_j^{(m)}(y) \qquad (l, m = 1, \dots, n-k \,; y \in \mathcal{S}_2) \,.$$

Let us make the following definition (see [**31**]).

Definition 7.3.3. *Let $y \in \mathcal{S}_2$. The rank $r(y)$ of the matrix $A_\perp(y)$ is called the* orthogonal rank of the diffusion matrix A at y.

The above definition is well posed, for $r(y)$ is independent of the choice of the set $\{\eta^{(l)}(y)\,|\,l=1,\dots,n-k\}$; observe that $r(y)\le n-k$. In view of assumption $(H_2)-(ii)$, there exist $y^1,\dots,y^N\in\mathcal{S}_2$ such that:

$$(7.3.8)\qquad \begin{cases} \mathcal{S}_2 \ \text{ is the union of the graphs } \mathcal{U}_i \ \text{ of } \ C^3 \text{ functions, say} \\ \phi^{(i)}: B_{R_i}(y_1^i,\dots,y_k^i)\subset I\!R^k\to I\!R^{n-k}, \phi^{(i)}\equiv(\phi^{(i)}_{k+1},\dots,\phi^{(i)}_n) \\ (i=1,\dots,N), \ \text{up to reorderings of the coordinates.} \end{cases}$$

We shall use the following assumption:

$$(F_6)\begin{cases} (i) \quad n\ge 2,\, dim\,\mathcal{S}_2\le n-2\,; \\ (ii)\ \ r(y)\ge 2 \text{ for any } \ y\in\mathcal{S}_2\,; \\ (iii) \text{ for any } \ y\in\mathcal{U}_i \ (i=1,\dots,N) \text{ there exist orthonormal vectors} \\ \qquad \eta^{(1)}(y),\,\dots,\,\eta^{(n-k)}(y)\in I\!R^n, \ \text{which are orthogonal to } \ \mathcal{S}_2 \text{ at } y, \\ \qquad \eta^{(l)}(\cdot)\in C^2(\mathcal{U}_i\,;\,I\!R^n) \ \ (l=1,\dots,n-k), \text{ such that the matrix} \\ \qquad A_\perp(\cdot) \ \text{ has unit eigenvectors of class } C^2(\mathcal{U}_i;I\!R^{n-k})\,. \end{cases}$$

(here the notation in (7.3.8) has been used).

Now we can state the following

THEOREM 7.3.4. *Let Ω be bounded, assumptions $(H_1)-(H_2)$ and $(F_5)-(F_6)$ be satisfied, $\mathcal{S}_2\ne\emptyset$; suppose $\gamma>0$ in Ω, or $\mathcal{S}_1=\emptyset$. Let there exist $B_0>0$ and $\tau\in[0,1)$ such that*

$$(7.3.9)\qquad \beta(x)\le\frac{B_0}{\big[dist(x,\mathcal{S}_2)\big]^\tau}\quad \textit{for any } x\in\Omega\,.$$

Let $u\in USC(\Omega\cup\mathcal{R}\cup\mathcal{S}_1)$ be a subsolution of problem (7.3.7) with $g=0$.
(i) If

$$(7.3.10)\qquad \limsup_{dist(x,\mathcal{S}_2)\to 0}\frac{u(x)}{|\log dist(x,\mathcal{S}_2)|}\le 0\,,$$

then $u\le 0$ in Ω.
(ii) Let $\alpha:=\inf_{y\in\mathcal{S}_2} r(y)-2\ge 1$. If

$$(7.3.11)\qquad \limsup_{dist(x,\mathcal{S}_2)\to 0}\frac{u(x)}{[dist(x,\mathcal{S}_2)]^{-\alpha}}\le 0\,,$$

then $u\le 0$ in Ω.

REMARK 7.3.5. For linear degenerate equations, Theorem 7.3.4 is proved in [**58**] using similar methods.

7.3.2.2. *Fully nonlinear equations related to Pucci operators.* We shall prove the following results.

THEOREM 7.3.6. *Let Ω be bounded, assumptions $(H_1)-(H_2)$, (F_1), (F_3) be satisfied, $\mathcal{S}_2\ne\emptyset$; suppose $\gamma>0$ in Ω or $\mathcal{S}_1=\emptyset$. In addition, let $n-1-\frac{\Lambda}{\lambda}\ge 0$,*

$$(7.3.12)\qquad k\le n-1-\frac{\Lambda}{\lambda}$$

and condition (7.3.9) be satisfied. Let $u\in USC(\Omega\cup\mathcal{R}\cup\mathcal{S}_1)$ be a subsolution of problem (7.1.3) with $g=0$ such that (7.3.10) is verified, then $u\le 0$ in Ω.

7.3.3. Comparison results. Let us make next assumption:

$$(F_7)\quad \begin{cases} (i)\quad F \in C(\bar\Omega \times I\!R \times I\!R^m \times \Sigma^n; I\!R)\,; \\ (ii)\ F(x,r,p,X) - F(x,s,p,X) \geq 0 \text{ for any } x \in \bar\Omega, r, s, \in I\!R, r \geq s, p \in I\!R^n, X \in \Sigma^n\,; \\ (iii)\ F \text{ is uniformly elliptic with ellipticity constants } \lambda, \Lambda\,; \\ (iv)\quad \text{for any } R > 0 \text{ there exists } L > 0 \text{ such that} \\ \qquad |F(x,r,p,X) - F(x,r,q,X)| \leq L|p-q| \\ \qquad \text{for any } x \in \bar\Omega, r \in [-R,R], p, q, \in I\!R^n, X \in \Sigma^n; \\ (v)\quad \text{comparison principle holds for } F \text{ in } \bar\Omega\,. \end{cases}$$

We can prove the following

PROPOSITION 7.3.7. *Let Ω be bounded, assumptions $(H_1) - (H_2)$ and (F_7) be satisfied; suppose $\mathcal{S} = \mathcal{S}_2 \neq \emptyset$. Let condition* (7.3.12) *be satisfied. Then problem* (7.1.2) *admits at most one bounded solution.*

Concerning semilinear degenerate equations, we have next

PROPOSITION 7.3.8. *Let Ω be bounded, assumptions $(H_1)-(H_2)$ and $(F_5)-(F_6)$ be satisfied with $H \in C(\bar\Omega \times I\!R \times I\!R^n)$, $\beta, \gamma \in C(\bar\Omega; [0,\infty))$; suppose $\mathcal{S} = \mathcal{S}_2 \neq \emptyset$. Then problem* (7.3.7) *admits at most one bounded solution.*

7.3.4. Generalizations of previous results to singular operators. Let us assume the following:

$$(F_8)\quad \begin{cases} (i)\quad F(x,r,p,X) := \hat F(x,p,X) + |p|^\beta \langle b(x), p\rangle - \hat f(r)\ \ (\beta > -1) \\ \qquad \text{for any } x \in \Omega \cup \mathcal{R}, r \in I\!R, p \in I\!R^n \setminus \{0\}, X \in \Sigma^n\,; \\ (ii)\quad \hat F \in C\big((\Omega \cup \mathcal{R}) \times I\!R \times (I\!R^n \setminus \{0\}) \times \Sigma^n; I\!R\big)\,; \\ (iii)\ \hat F \text{ is proper} \\ (iv)\quad \text{there exist } \lambda > 0,\ \Lambda > 0 \text{ with } \lambda \leq \Lambda \text{ such that } \hat F(x,p,X) \geq |p|^\beta \mathcal{P}^-_{\lambda,\Lambda}(X) \\ \qquad \text{for any } x \in \Omega \cup \mathcal{R}, p \in I\!R^n \setminus \{0\},\ X \in \Sigma^n\,; \\ (v)\quad b \in C^{0,1}(\Omega \cup \mathcal{R}; I\!R^n)\,; \\ (vi)\quad f \in C(I\!R; I\!R),\ f(0) = 0,\ f \text{ is strictly increasing in } I\!R\,. \end{cases}$$

Operators F satisfying assumption (F_8) are *singular*, in the sense that they may not be defined at $p = 0$. Some examples of such singular operators are given in the following (see [**8**]):

(i) $F(p,X) := |p|^\beta \mathcal{P}^+_{\lambda,\Lambda}(X) \quad (\beta > -1)\,;$

(ii) $F(p,X) := -tr(X) + \frac{\langle Xp,p\rangle}{|p|^2}$ (see [**24**]);

(iii) $F(p,X) := -\Delta_m \equiv -|p|^{m-2} tr(X) + (2-m)|p|^{m-4}\langle Xp,p\rangle \quad (m \geq 1)$, *i.e.* the opposite of the $m-$ Laplacian;

here $p \in I\!R^n \setminus \{0\}$, $X \in \Sigma^n$.

When F satisfies (F_8), the notion of viscosity solution of equation (7.1.1) will not be meant in the usual sense; following [**8**]-[**9**], [**24**] and [**41**] we do not take test function whose gradient is zero at the test point, where the operator F may not be defined (see $(F_8) - (i)$). To be specific, let us make next

DEFINITION 7.3.9. *A function $u \in C(\Omega)$ is a* viscosity subsolution *of equation* (7.1.1) *provided that, for any $x_0 \in \Omega$, either*

- *$u \equiv c$ in $B_\delta(x_0) \subseteq \Omega$, for some $\delta > 0$, $c \in I\!R$, and $\hat f(c) \leq 0$,*
  or
- *for any $\psi \in C^2(\Omega)$ such that $u - \psi$ has a local maximum in x_0 and $D\psi(x_0) \neq 0$ there holds*
$$F(x_0, u(x_0), D\psi(x_0), D^2\psi(x_0)) \leq 0\,.$$

A function $u \in C(\Omega)$ is a viscosity supersolution *of equation* (7.1.1) *provided that, for any $x_0 \in \Omega$, either*

- *$u \equiv c$ in $B_\delta(x_0) \subseteq \Omega$, for some $\delta > 0$, $c \in \mathbb{R}$, and $\hat{f}(c) \geq 0$,*
  or
- *for any $\psi \in C^2(\Omega)$ such that $u - \psi$ has a local minimum in x_0 and $D\psi(x_0) \neq 0$ there holds*

$$F(x_0, u(x_0), D\psi(x_0), D^2\psi(x_0)) \geq 0\,.$$

A function $u \in C(\Omega)$ is a viscosity solution of equation (7.1.1)*, if u is both a sub- and a supersolution of equation* (7.1.1)*.*

We shall prove the following

THEOREM 7.3.10. *Let Ω be bounded, assumptions $(H_1) - (H_2), (F_8)$ be satisfied and $\mathcal{S} = \mathcal{S}_2 \neq \emptyset$. Moreover, suppose that conditions* (7.3.9) *and* (7.3.12) *are verified. Let $u \in C(\Omega \cup \mathcal{R})$ be a subsolution of problem* (7.1.2) *such that condition* (7.3.10) *is satisfied. Then $u \leq 0$ in Ω.*

REMARK 7.3.11. (i) If $F \equiv -\Delta_m$ $(2 \leq m \leq n)$, then condition (7.3.12) reads $k \leq n - m$. In this case, Theorem 7.3.10 is in agreement with the nonlinear potential theory (see *e.g.* [**36**]).
(ii) If $F(p, X) = -trX + \frac{\langle Xp,p\rangle}{|p|^2}$ $(p \in \mathbb{R}^n \setminus \{0\}, X \in \Sigma^n)$, then condition (7.3.12) reads $k \leq n - 3$.

7.4. Proofs

7.4.1. Proof of Theorem 7.3.1. We adapt to the present situation proofs given in [**65**], for linear problems. We keep the same notations of Section 1.4.

Proof of Theorem 7.3.1. (a) *Let Ω be bounded.* We give the proof when condition (7.3.2) is replaced by the weaker assumption (7.3.5) (see Remark 7.3.2).

(i) In view of inequality (7.3.5), there exists a sequence $\{\varepsilon_k\} \subseteq (0, \varepsilon_0)$, $\varepsilon_k \to 0$ as $k \to \infty$, such that

$$\lim_{k\to+\infty} \left\{ \sup_{\mathcal{A}_2^{\varepsilon_k} \setminus \mathcal{S}} \frac{u_+}{W} \right\} \leq 0\,. \tag{7.4.1}$$

Then for any $\alpha \in (0, \alpha_0)$ there exists $\bar{k} = \bar{k}(\alpha) \in \mathbb{N}$ such that for any $k > \bar{k}$ there holds

$$\frac{u_+}{W} < \alpha \quad \text{in} \quad \mathcal{A}_2^{\varepsilon_k} \setminus \mathcal{S}. \tag{7.4.2}$$

(ii) Define, for any $\alpha \in (0, \alpha_0)$,

$$V_\alpha(x) := \alpha W(x) \qquad\qquad (x \in \Omega \cup \mathcal{R})\,. \tag{7.4.3}$$

Observe that

$$\alpha H \leq V_\alpha \qquad \text{in } \Omega \cup \mathcal{R}\,. \tag{7.4.4}$$

In view of (1.4.1)-(**??**), (7.4.4) the following claim is easily seen to hold.
Claim 1: For any $\alpha \in (0, \alpha_0)$, $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \frac{\varepsilon}{2})$ the function V_α defined in (7.4.3) is a supersolution of the problem

$$\begin{cases} E(x, u, Du, D^2u) = 0 & \text{in } \Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}} \\ u = 0 & \text{on } \mathcal{R}^{\varepsilon,\delta} \\ u = V_\alpha & \text{on } \mathcal{F}^{\varepsilon,\delta}\,. \end{cases} \tag{7.4.5}$$

(iii) We shall prove the following

Claim 2: For any $\alpha \in (0,\alpha_0)$ there exists $\bar{k} = \bar{k}(\alpha) \in I\!N$ with the following property: for any $k > \bar{k}$ there exists $\delta_k \in (0, \frac{\varepsilon_k}{2})$ such that the function u_+ is a subsolution of problem (7.4.5) with $\varepsilon = \varepsilon_k, \delta = \delta_k$, where $\{\varepsilon_k\}$ is the infinitesimal sequence of inequality (7.4.2).

From Claims 1 and 2 the conclusion follows immediately. In fact, by assumption $(F_2)-(iii)$ we obtain for any $\alpha \in (0,\alpha_0)$, $k > \bar{k}$

$$u_+ \le V_\alpha \qquad \text{in } \Omega \setminus \overline{\mathcal{I}^{\varepsilon_k,\delta_k}}\,.$$

Letting $\alpha \to 0$ in the latter inequality we obtain $u \le 0$ in any compact subset of Ω (observe that $\bar{k} \to \infty$, thus $\varepsilon_k \to 0$ as $\alpha \to 0$); hence the result follows.

To prove Claim 2 we use the following facts:

- for any $\alpha \in (0,\alpha_0)$, $\varepsilon \in (0,\varepsilon_0)$ there exists $\bar{\delta} \in (0,\frac{\varepsilon}{2})$ such that for any $\delta \in (0,\bar{\delta})$ there holds

$$u_+ < \alpha H \qquad \text{in } \mathcal{F}_1^{\varepsilon,\delta}\,; \tag{7.4.6}$$

- for any $\alpha \in (0,\alpha_0)$ there exists $\bar{k} = \bar{k}(\alpha) \in I\!N$ such that for any $k > \bar{k}$ and for any $\delta \in (0,\frac{\varepsilon_k}{2})$ the function V_α satisfies

$$u_+ < V_\alpha \qquad \text{in } \mathcal{F}_2^{\varepsilon_k,\delta}\,. \tag{7.4.7}$$

Let us put off the proof of (7.4.6)-(7.4.7) and complete the proof of Claim 2. Plainly, from (7.4.4) and (7.4.6)-(7.4.7) we obtain

$$u_+ < V_\alpha \qquad \text{in } \mathcal{F}^{\varepsilon_k,\delta_k} \tag{7.4.8}$$

for any $\alpha \in (0,\alpha_0)$, $k > \bar{k}$ and some $\delta_k \in (0,\frac{\varepsilon_k}{2})$. On the other hand, by Lemma 7.2.6, the function u_+ is a subsolution of the problem

$$\begin{cases} E(x,u,Du,D^2u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \mathcal{R} \cup \mathcal{S}_1\,, \end{cases} \tag{7.4.9}$$

thus in particular $u \le 0$ on $\mathcal{R}^{\varepsilon_k,\delta_k} \subseteq \mathcal{R}$. Hence Claim 2 follows.

It remains to prove inequalities (7.4.6)-(7.4.7). Concerning (7.4.6), observe that $u_+ \le 0$ on $\mathcal{S}_1$, thus in particular $u \le 0$ on $\mathcal{S}_{1,\varepsilon}$, and $u_+ \in USC(\Omega \cup \mathcal{R} \cup \mathcal{S}_1)$. As a consequence, for any $\bar{x} \in \mathcal{S}_{1,\varepsilon}$ and any $\sigma > 0$ there exists $\delta = \delta(\bar{x},\sigma) > 0$ such that

$$u_+(x) < \sigma \quad \text{for any } x \in [\Omega \cup \mathcal{R}] \cap B_\delta(\bar{x})\,.$$

It is immediately seen that $\mathcal{S}_{1,\varepsilon}$ is closed, thus compact. Hence from the covering $\{B_\delta(\bar{x})\}_{\bar{x}\in\mathcal{S}_{1,\varepsilon}}$ we can extract a finite covering $\{B_{\delta_n}(\bar{x}_n)\}_{n=1,\dots,\bar{n}}$ ($\bar{n} \in I\!N$), namely

$$\mathcal{S}_{1,\varepsilon} \subseteq \cup_{n=1}^{\bar{n}} B_{\delta_n}(\bar{x}_n) =: \mathcal{B}_{\varepsilon,\sigma}\,.$$

Set

$$\bar{\delta} := \min\{\delta_1,\dots,\delta_{\bar{n}},\frac{\varepsilon}{3}\}\,;$$

then

$$\{x \in \Omega \cup \mathcal{R} \,|\, dist(x,\mathcal{S}_{1,\varepsilon}) \le \bar{\delta}\} \subseteq [\Omega \cup \mathcal{R}] \cap \mathcal{B}_{\varepsilon,\sigma}\,,$$

thus in particular

$$\mathcal{F}_1^{\varepsilon,\delta} \subseteq [\Omega \cup \mathcal{R}] \cap \mathcal{B}_{\varepsilon,\sigma} \quad \text{for any } \delta \in (0,\bar{\delta})\,.$$

This shows that for any $\sigma > 0$, $\varepsilon \in (0,\varepsilon_0)$ and $\delta \in (0,\bar{\delta})$ there holds

$$u_+ < \sigma \qquad \text{in } \mathcal{F}_1^{\varepsilon,\delta}\,;$$

choosing $\sigma = \alpha H$ we obtain (7.4.6).

Inequality (7.4.7) follows immediately from (7.4.2), since $\mathcal{F}_2^{\varepsilon_k,\delta} \subseteq \mathcal{A}_2^{\varepsilon_k} \setminus \mathcal{S}$ for any $\delta \in (0,\frac{\varepsilon_k}{2})$. This completes the proof for bounded Ω.

(*b*) *Let* Ω *be unbounded.* We shall use the family $\Omega^{\varepsilon,\delta,\beta}$ of subsets of Ω, introduced in Section 1.4. We give the proof when condition (7.3.3) is replaced by the weaker assumption (7.3.4) (see Remark 7.3.2). (*i*) In view of inequalities (7.3.6), there exists two sequences $\{\varepsilon_k\} \subseteq (0,\varepsilon_0)$, $\varepsilon_k \to 0$ as $k \to \infty$ and $\{\beta_k\} \subseteq (0,\infty)$, $\beta_k \to 0$ as $k \to \infty$, such that

$$\lim_{k\to+\infty} \left\{ \sup_{\mathcal{A}_2^{\varepsilon_k}\setminus\mathcal{S}} \frac{u_+}{W} \right\} \le 0\,, \qquad \lim_{k\to+\infty} \left\{ \sup_{[\Omega\cup\mathcal{R}]\cap\partial B_{\frac{1}{\beta_k}}} \frac{u_+}{W} \right\} \ge 0\,. \tag{7.4.10}$$

Then for any $\alpha \in (0,\alpha_0)$ there exists $\bar{k} = \bar{k}(\alpha) \in I\!N$ such that for any $k > \bar{k}$

$$\frac{u_+}{W} < \alpha \quad \text{in} \quad \mathcal{A}_2^{\varepsilon_k} \setminus \mathcal{S}\,, \qquad \frac{u_+}{W} \le \alpha \ \text{ on } \ [\Omega\cup\mathcal{R}]\cap\partial B_{\frac{1}{\beta_k}}\,. \tag{7.4.11}$$

(*ii*) As in (*a*) , it is easily seen that the function $V_\alpha := \alpha W$ is a supersolution of the problem

$$\begin{cases} E(x,u,Du,D^2u) = 0 & \text{in } \ \Omega^{\varepsilon,\delta,\beta} \\ u = 0 & \text{on } \ \mathcal{R}^{\varepsilon,\delta} \cap \overline{B_{\frac{1}{\beta}}} \\ u = V_\alpha & \text{on } \left[\mathcal{F}^{\varepsilon,\delta} \cap \overline{B_{\frac{1}{\beta}}}\right] \cup \left[\overline{\Omega\setminus\overline{\mathcal{I}^{\varepsilon,\delta}}} \cap \partial B_{\frac{1}{\beta}}\right] \end{cases} \tag{7.4.12}$$

for any $\alpha \in (0,\alpha_0)$, $\varepsilon \in (0,\varepsilon_0)$, $\delta \in (0,\frac{\varepsilon}{2})$, $\beta > 0$.

Arguing as in (*a*) the conclusion follows from the following
Claim: For any $\alpha \in (0,\alpha_0)$ there exists $\bar{k} = \bar{k}(\alpha) \in I\!N$ with the following property: for any $k > \bar{k}$ there exists $\delta_k \in (0,\frac{\varepsilon_k}{2})$ such that the function u_+ is a subsolution of problem (7.4.12) with $\varepsilon = \varepsilon_k, \delta = \delta_k, \beta = \beta_k$, where $\{\varepsilon_k\}$ and $\{\beta_k\}$ are the infinitesimal sequences of inequalities (7.4.11).

To prove the Claim, it suffices to prove that

$$u_+ < V_\alpha \quad \text{on} \ \left[\mathcal{F}^{\varepsilon_k,\delta_k} \cap \overline{B_{\frac{1}{\beta_k}}}\right] \cup \left[\overline{\Omega\setminus\overline{\mathcal{I}^{\varepsilon_k,\delta_k}}} \cap \partial B_{\frac{1}{\beta_k}}\right] \tag{7.4.13}$$

with $\alpha, k, \varepsilon_k, \delta_k, \beta_k$ as above. Notice that (7.4.4) and (7.4.7) are still valid. Moreover, in view of the compactness of $\mathcal{S}_{1,\varepsilon} \cap \overline{B_{\frac{1}{\beta}}}$ ($\varepsilon \in (0,\varepsilon_0)$, $\beta > 0$), arguing as in the proof of (7.4.6), we get that

- for any $\alpha \in (0,\alpha_0)$, $\varepsilon \in (0,\varepsilon_0)$, $\beta > 0$ there exists $\bar{\delta} \in (0,\frac{\varepsilon}{2})$ such that for any $\delta \in (0,\bar{\delta})$ there holds

$$u_+ < \alpha H \qquad \text{in } \mathcal{F}_1^{\varepsilon,\delta} \cap \overline{B_{\frac{1}{\beta}}}\,. \tag{7.4.14}$$

Then by (7.4.4), (7.4.7), (7.4.14), the inequality

$$u_+ < V_\alpha \qquad \text{in } \ \mathcal{F}^{\varepsilon_k,\delta_k} \cap \overline{B_{\frac{1}{\beta_k}}} \tag{7.4.15}$$

follows. Concerning the inequality

$$u_+ < V_\alpha \qquad \text{in } \ \overline{\Omega\setminus\overline{\mathcal{I}^{\varepsilon_k,\delta_k}}} \cap \partial B_{\frac{1}{\beta_k}}\,, \tag{7.4.16}$$

it follows immediately from (7.4.11) since $\overline{\Omega\setminus\overline{\mathcal{I}^{\varepsilon_k,\delta}}} \subseteq \Omega\cup\mathcal{R}$ for any $\delta \in (0,\frac{\varepsilon_k}{2})$ (see (1.4.1)). Then inequality (7.4.13) and the conclusion follow. □

Proof of Remark 7.3.2. Case (*i*) holds, since (7.3.4) implies (7.3.3). Case (*ii*) has been dealt with already in the proof of Theorem 7.3.1. □

7.4.2. Proof of Theorem 7.3.4. Arguing as in the proof of Theorem 2.12 in [**58**] we obtain the following

PROPOSITION 7.4.1. *Let assumptions of Theorem 7.3.4 be satisfied. Then there exists a supersolution $W \geq H > 0$ to problem*

$$\begin{cases} -tr(A(x)D^2u) - \beta(x)|Du| + \gamma u = 0 & in\ \Omega \\ \\ u = 0 & in\ \mathcal{R} \end{cases} \tag{7.4.17}$$

such that

$$\begin{cases} W(x) \sim -\log[dist(x,\mathcal{S}_2)] \ \ as\ dist(x,\mathcal{S}_2) \to 0\,, & if\ \alpha := \inf\limits_{y\in\mathcal{S}} r(y) - 2 \geq 0\,; \\ W(x) \sim [dist(x,\mathcal{S}_2)]^{-\alpha} \ \ as\ dist(x,\mathcal{S}_2) \to 0\,, & if\ \alpha \geq 1. \end{cases}$$

Now we can prove Theorem 7.3.4.

Proof of Theorem 7.3.4. Define

$$F(x,r,p,X) := -tr(A(x)X) + H(x,r,p) \quad \big(x \in \Omega \cup \mathcal{R}, r \in I\!R, p \in I\!R^n, X \in \Sigma^n\big)\,.$$

In view of assumptions $(F_5) - (F_6)$ and comparison principles stated in [**5**], we deduce that $(F_1) - (F_2)$ are satisfied, if we choose

$$E(x,r,p,X) := -tr(A(x)X) - \beta(x)|p| + \gamma(x)r \quad \big(x \in \Omega \cup \mathcal{R}, r \geq 0, p \in I\!R^n, X \in \Sigma^n\big)\,.$$

Applying Theorem 7.3.1 and Proposition 7.4.1 the conclusion follows. □

Using Lemma 3.4.3 we can prove next

PROPOSITION 7.4.2. *Let assumptions of Theorem 7.3.6 be satisfied. Then there exists a supersolution $W \geq H > 0$ to problem*

$$\begin{cases} \mathcal{P}^-_{\lambda,\Lambda}(D^2u) - \beta(x)|Du| + \gamma u = 0 & in\ \Omega \\ \\ u = 0 & in\ \mathcal{R} \end{cases} \tag{7.4.18}$$

such that

$$W(x) \sim -\log[dist(x,\mathcal{S}_2)] \ \ as\ dist(x,\mathcal{S}_2) \to 0\,.$$

Proof of Proposition 7.4.2. Let ε_0 be given by Lemma 3.4.3. Define

$$\hat{W}(x) := -\log[d(x)] - C_1[d(x)]^{1-\tau} \quad (x \in \mathcal{S}_2^{\varepsilon_0}); \tag{7.4.19}$$

here $d(x) \equiv dist(x,\mathcal{S}_2)$ $(x \in \Omega \cup \mathcal{R})$ and C_1 is a positive constant to be fixed in the sequel. Clearly $\hat{W} \in C^2(\mathcal{S}_2^{\varepsilon})$. We claim that, for some $\varepsilon \in (0,\varepsilon_0)$, the function $\hat{W}$ satisfies

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2\hat{W}(x)) - \beta(x)|D\hat{W}(x)| + \gamma(x)\hat{W}(x) \geq 0 \quad \text{for any } \ x \in \mathcal{S}_2^{\varepsilon}\,. \tag{7.4.20}$$

In fact, let $\varepsilon \in (0,\varepsilon_0)$ and $x \in \mathcal{S}_2^{\varepsilon}$ arbitrarily fixed. In order to prove inequality (7.4.20), it is not restrictive to assume, as we will do in the following, that condition (C) of Lemma 3.4.3 is satisfied. Then, from Lemma 3.4.3, we have

$$\begin{aligned} \frac{\partial \hat{W}(x)}{\partial x_i} &= -\big\{[d(x)]^{-1} + C_1(1-\tau)[d(x)]^{-\tau}\big\}\frac{\partial d(x)}{\partial x_i} = \\ &= -\big\{[d(x)]^{-1} + C_1(1-\tau)[d(x)]^{-\tau}\big\}\delta_{in} \end{aligned} \tag{7.4.21}$$

for any $i = 1,\ldots,n$. Moreover, there holds:

$$\begin{aligned} \frac{\partial \hat{W}(x)}{\partial x_i \partial x_j} &= \big\{[d(x)]^{-2} + C_1\tau_1(1-\tau)[d(x)]^{-\tau-1}\big\}\frac{\partial d(x)}{\partial x_i}\frac{\partial d(x)}{\partial x_j} + \\ &\quad -\big\{[d(x)]^{-1} + C_1(1-\tau)[d(x)]^{-\tau}\big\}\frac{\partial d(x)}{\partial x_i \partial x_j} \end{aligned} \tag{7.4.22}$$

for any $i, j = 1, \dots, n$.

By Lemma 3.4.3, (7.4.21)-(7.4.22), we deduce that

$$D^2\hat{W}(x) \le P(x) \equiv (p_{ij}(x)) \quad (i,j = 1,\dots,n)\,, \tag{7.4.23}$$

where

$$p_{ij}(x) := \begin{cases} C_0\big\{[d(x)]^{-1} + C_1(1-\tau)[d(x)]^{-\tau}\big\} & \text{if } i=j=1,\dots,k \\ -[d(x)]^{-2} - C_1(1-\tau)[d(x)]^{-\tau-1} & \text{if } i=j=k+1,\dots,n-1 \\ [d(x)]^{-2} + C_1\tau(1-\tau)[d(x)]^{-\tau-1} & \text{if } i=j=n. \end{cases} \tag{7.4.24}$$

It is easily seen that

$$P(x) = P^+(x) - P^-(x),\ P^+(x) \ge 0,\ P^-(x) \ge 0, P^+(x)P^-(x) = 0, \tag{7.4.25}$$

if we set

$$P^+(x) \equiv (p^+_{ij}(x)),\ P^-(x) \equiv (p^-_{ij}(x)) \quad (i,j=1,\dots,n),$$

$$p^+_{ij}(x) := \begin{cases} C_0\big\{[d(x)]^{-1} + C_1(1-\tau)[d(x)]^{-\tau}\big\} & \text{if } i=j=1,\dots,k \\ [d(x)]^{-2} + C_1\tau(1-\tau)[d(x)]^{-\tau-1} & \text{if } i=j=n. \\ 0 & \text{otherwise}\,; \end{cases} \tag{7.4.26}$$

$$p^-_{ij}(x) := \begin{cases} [d(x)]^{-2} + C_1(1-\tau)[d(x)]^{-\tau-1} & \text{if } i=j=k+1,\dots,n-1 \\ 0 & \text{otherwise}\,. \end{cases} \tag{7.4.27}$$

From (i) and (ii) in Lemma 7.2.3, we obtain

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2\hat{W}(x)) \ge \mathcal{P}^-_{\lambda,\Lambda}(P(x)) = \lambda tr(P^-(x)) - \Lambda tr(P^+(x)). \tag{7.4.28}$$

By (7.3.9), (7.3.12) and (7.4.25)-(7.4.28), we have

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2\hat{W}(x)) - \beta|D\hat{W}(x)| \ge$$
$$\ge [d(x)]^{-2}\Big\{\lambda(n-k-1) - \Lambda - \Lambda k C_0 d(x) - B_0[d(x)]^{1-\tau} +$$
$$+ C_1(1-\tau)\big[\lambda(n-k-1)[d(x)]^{1-\tau} - \tau\Lambda[d(x)]^{1-\tau} - \Lambda C_0 k[d(x)]^{2-\tau} - B_0[d(x)]^{2(1-\tau)}\big]\Big\} \ge$$
$$\ge [d(x)]^{-1-\tau}\Big\{C_1(1-\tau)\big[\lambda(n-k-1) - \tau\Lambda - \Lambda k C_0 d(x) - B_0[d(x)]^{1-\tau}\big] +$$
$$-\Lambda k C_0[d(x)]^{\tau} - B_0\Big\} \ge 0,$$

provided that $\varepsilon > 0$ is small enough and $C_1 > 0$ is large enough; here use of the inequality $\lambda(N-k-1) - \tau\Lambda > 0$ has been made.

Notice that $\gamma(x)\hat{W}(x) \ge 0$, if $\varepsilon > 0$ is small enough. This fact combined with the previous inequality gives the Claim. It remains to extend $\hat{W}$ in the whole $\Omega \cup \mathcal{R}$.

(i) *Suppose* $\gamma > 0$ *in* Ω. Let us consider a function $\chi \in C^2(\overline{\Omega})$ such that $0 \le \chi \le 1$, $\chi \equiv 1$ in $\mathcal{S}_2^{\varepsilon/2}$, $\chi \equiv 0$ in $\overline{\Omega} \setminus \mathcal{S}_2^{\varepsilon}$. Define

$$W(x) := \hat{W}(x)\chi(x) + C_2 \quad (x \in \Omega \cup \mathcal{R}),$$

C_2 being a positive constant to be chosen later.

Clearly, we have that

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2W) - \beta|DW| + \gamma W \ge 0 \quad \text{in } \mathcal{S}_2^{\varepsilon/2} \cup [\Omega \setminus \mathcal{S}_2^{\varepsilon}]. \tag{7.4.29}$$

Moreover, for some constant $C_3 > 0$, there holds

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2W) - \beta|DW| + \gamma W \geq -C_3 + \gamma C_2 \geq 0 \quad \text{in } \mathcal{S}_2^\varepsilon \setminus \mathcal{S}_2^{\varepsilon/2}, \tag{7.4.30}$$

taking $C_2 \geq \frac{C_3}{\gamma_0}$, where $\gamma_0 := \min\limits_{\mathcal{S}_2^\varepsilon \setminus \mathcal{S}_2^{\varepsilon/2}} \gamma > 0$.

Observe that $W \geq 0$ on $\mathcal{R}$, if we choose $C_2 > 0$ big enough, hence W is the supersolution we had to exhibit.

(ii) *Suppose* $\mathcal{S}_1 = \emptyset$. Let us consider the solution $\tilde{W}$ of the problem

$$\begin{cases} \mathcal{P}^-_{\lambda,\Lambda}(D^2W) - B_1|DW| = 0 & \text{in } \Omega \setminus \overline{\mathcal{S}^\varepsilon} \\ W = 0 & \text{in } \mathcal{A}^\varepsilon \\ W = -1 & \text{in } \mathcal{R}\,; \end{cases} \tag{7.4.31}$$

here $B_1 := \min\limits_{\Omega \setminus \mathcal{S}^\varepsilon} \beta$. By the strong maximum principle, since $\mathcal{A}^\varepsilon$ is compact, there exists $\alpha > 0$ such that

$$\langle D\tilde{W}(x), \nu(x)\rangle \leq -\alpha \quad (x \in \mathcal{A}^\varepsilon)\,; \tag{7.4.32}$$

here ν denotes the outer normal at $\mathcal{S}^\varepsilon$. Define

$$W := \begin{cases} \hat{W} + \log\varepsilon + C_1\varepsilon^{1-\tau} + 2 & \text{in } \mathcal{S}^\varepsilon \\ H\tilde{W} + 2 & \text{in } (\Omega \cup \mathcal{R}) \setminus \mathcal{S}^\varepsilon\,, \end{cases} \tag{7.4.33}$$

with $H > 0$ to be chosen.

We claim that if $H > 0$ is sufficiently large, then for any $x_0 \in \mathcal{A}^\varepsilon$ no test functions ψ as in Definition 7.2.1 exist, or equivalently (see [**49**]),

$$J^{2,-}W(x_0) := \{(p, X) \in I\!R^n \times \Sigma^n \mid$$

$$W(x) \geq W(x_0) + \langle p, x - x_0\rangle + \frac{1}{2}\langle X(x - x_0), x - x_0\rangle + o(|x - x_0|^2)\} = \emptyset\,.$$

In fact, by absurd, suppose that $(p, X) \in J^{2,-}W(x_0)$. Then it follows that

$$\frac{W(x_0 + \delta\nu(x_0)) - W(x_0)}{\delta} \geq \langle p, \nu(x_0)\rangle + o(\delta)$$

for any $\delta > 0$ small enough. Letting $\delta \to 0^+$ in the previous inequality, we obtain

$$\langle p, \nu(x_0)\rangle \leq H\langle D\tilde{W}, \nu(x_0)\rangle \leq -\alpha H\,. \tag{7.4.34}$$

Arguing with similarly we get

$$\langle p, \nu(x_0)\rangle \geq \langle D\hat{W}, \nu(x_0)\rangle \geq -C_2\,, \tag{7.4.35}$$

for some positive constant C_2 independent on x_0 (see (7.4.19)). From (7.4.34) and (7.4.35), when $H > \frac{C_2}{\alpha}$ we obtain a contradiction. Hence the Claim follows and we can conclude that W is the desired supersolution. The proof is complete. □

Proof of Theorem 7.3.6. Lemmas 7.2.3-(iv), 7.2.4 and Proposition 7.4.2 yield the result. □

7.4.3. Proof of Propositions 7.3.7-7.3.8 . We make use of the following

DEFINITION 7.4.3. *Let* $\Omega_1 \subseteq \bar{\Omega}$.
(*i*) *A function* $u \in USC(\bar{\Omega})$ *is a viscosity subsolution of equation*

$$F(x, u, Du, D^2u) = 0 \quad in\ \Omega_1 , \tag{7.4.36}$$

provided that the following condition holds:
$x_0 \in \Omega_1, \psi\ C^2$ *in a neighbourhood of* Ω_1 *and* $u - \psi$ *has a local maximum (relative to* Ω_1*) at* x_0 *implies that*

$$F(x_0, u(x_0), D\psi(x_0), D^2\psi(x_0)) \leq 0.$$

(*ii*) *A function* $u \in LSC(\bar{\Omega})$ *is a viscosity supersolution of equation* (7.4.36)*, provided that the following condition holds:*
$x_0 \in \Omega_1, \psi\ C^2$ *in a neighbourhood of* Ω_1 *and* $u - \psi$ *has a local minimum (relative to* Ω_1*) at* x_0 *implies that*

$$F(x_0, u(x_0), D\psi(x_0), D^2\psi(x_0)) \geq 0.$$

The proof of Theorem 7.3.7 is based on next

PROPOSITION 7.4.4. *Let assumptions of Proposition 7.3.7 be satisfied. Then the following statements hold true:*

(*i*) *If* $u \in USC(\bar{\Omega})$ *is a subsolution to equation* (7.1.1)*, then* u *is a subsolution of equation*

$$F(x, u, Du, D^2u) = 0 \quad in\ \Omega \cup \mathcal{S} ; \tag{7.4.37}$$

(*ii*) *If* $u \in LSC(\bar{\Omega})$ *is a supersolution to equation* (7.1.1)*, then* u *is a supersolution of equation* (7.4.37) *.*

Proof of Proposition 7.4.4 . We adapt to the present situation the idea of the proof of Lemma 4.4 in [**2**]. Take $z \in \mathcal{S}$ and $\psi \in C^2(\Omega \cup \mathcal{S})$ such that z is a point of local maximum for $u - \psi$. Without loss of generality, we can assume that z is a point of strict local maximum for $u - \psi$, that is

$$u(z) - \psi(z) > u(x) - \psi(x) \quad \text{for any } x \in B_\delta(z) \cap [\Omega \cup \mathcal{S}] \tag{7.4.38}$$

for some $\delta \in (0, \sigma)$; here σ is defined as in Lemma 3.4.1. Define

$$Z(x) := \log[d(x)] + C_1 d(x) \quad (x \in \Omega) ,$$

$$\psi_\varepsilon(x) := u(x) - \psi(x) + \varepsilon Z(x) \quad (x \in \Omega) ;$$

here $d(x) \equiv dist(x, \mathcal{S})\ (x \in \bar{\Omega})$, $C_1 > 0$ is a constant to be fixed in the sequel and $\varepsilon > 0$. Let $x_\varepsilon \in \Omega\ (\varepsilon > 0)$ be a point of local maximum of ψ_ε (relative to $B_\delta(z) \cap \Omega$); we have

$$x_\varepsilon \to z, \quad \psi_\varepsilon(x_\varepsilon) \to u(z) - \psi(z) \quad \text{as } \varepsilon \to 0 . \tag{7.4.39}$$

Take $\bar{\varepsilon} > 0$ such that $x_\varepsilon \in \mathcal{S}^\sigma$ for any $\varepsilon \in (0, \bar{\varepsilon})$. From Lemma 3.4.3, assumptions $(F_7) - (iii), (iv)$ and condition (7.2.4) we deduce for any $\varepsilon \in (0, \bar{\varepsilon})$

$$\begin{aligned} &F(x_\varepsilon, u(x_\varepsilon), D\psi(x_\varepsilon), D^2\psi(x_\varepsilon)) + \mathcal{P}^-_{\lambda,\Lambda}(-\varepsilon D^2(Z(x_\varepsilon))) - L\varepsilon|DZ(x_\varepsilon)| \leq \\ &\leq F(x_\varepsilon, u(x_\varepsilon), D\psi(x_\varepsilon) - \varepsilon DZ(x_\varepsilon), D^2\psi(x_\varepsilon) - \varepsilon D^2 Z(x_\varepsilon)) \leq 0 . \end{aligned} \tag{7.4.40}$$

Inequality (7.4.40) and Lemma 7.2.3-(iii) yield for any $\varepsilon \in (0, \bar{\varepsilon})$

$$F(x_\varepsilon, u(x_\varepsilon), D\psi(x_\varepsilon), D^2\psi(x_\varepsilon)) \leq L\varepsilon|DZ(x_\varepsilon)| + \mathcal{P}^+_{\lambda,\Lambda}(\varepsilon D^2(Z(x_\varepsilon))) . \tag{7.4.41}$$

The same computations made in proof of Proposition 7.4.2 give, for $C_1 > 0$ big enough and $\varepsilon > 0$ small enough,

$$L\varepsilon|DZ(x_\varepsilon)| + \mathcal{P}^+_{\lambda,\Lambda}(\varepsilon D^2(Z(x_\varepsilon))) \leq 0 . \tag{7.4.42}$$

Sending $\varepsilon \to 0$ in (7.4.42), using (7.4.39), (7.4.41)-(7.4.42), we arrive to

$$F(z, u(z), D\psi(z), D^2\psi(z)) \leq 0 ,$$

thus (i) has been shown. Analogously (ii) can be deduced. □

Proof of Proposition 7.3.7 Suppose, by contradiction, that there exist two bounded solutions u_1, u_2 to problem (7.1.2). Let us prolong u_1, u_2 on $\mathcal{S}$ setting

$$u_1(x) := \limsup_{y\to x} u_1(y), \;\; u_2(x) := \liminf_{y\to x} u_2(y) \quad (x \in \mathcal{S}).$$

Proposition 7.4.4 implies that $u_1 \in USC(\bar{\Omega})$ is a subsolution to equation (7.4.36) with $\Omega_1 = \Omega \cup \mathcal{S}$, while $u_2 \in LSC(\bar{\Omega})$ is a supersolution to the same equation; in addition, $u_1 = u_2 = g$ on $\mathcal{R}$. By standard comparison principles (*e.g.*, see [**18**]) we get $u_1 \le u_2$; analogously we have $u_1 \ge u_2$, thus the result follows. □

The proof of Proposition 7.3.8 goes along the same line of the previous one; thus we omit it.

7.4.4. Proof of Theorem 7.3.10. As a particular case of results given in [**8**], the following comparison principle is valid.

LEMMA 7.4.5. *Let assumption (H_1) be satisfied, $\mathcal{R}$ be of class C^1, $\lambda > 0, \Lambda > 0, \lambda \le \Lambda$ and $\beta > -1$. Let Ω_1 an open bounded set with smooth boundary $\partial\Omega_1$ such that $\overline{\Omega_1} \subseteq \Omega \cup \mathcal{R}$; moreover, let $g \in C(\partial\Omega_1)$. If $\underline{u}$ is a subsolution and $\overline{u}$ is a supersolution of problem*

$$\text{(7.4.43)} \qquad \begin{cases} |Du|^\beta \mathcal{P}^-_{\lambda,\Lambda}(D^2u) \; + \; |Du|^\beta \langle b, Du\rangle = \hat{f}(u) & \text{in } \Omega_1 \\ \\ u \; = g & \text{on } \; \partial\Omega_1 \,, \end{cases}$$

then $\underline{u} \le \overline{u}$ in Ω_1.

Using Lemma 7.4.5, by minor changes in the proof of Theorem 7.3.1 we obtain next

PROPOSITION 7.4.6. *Let Ω be bounded and assumptions $(H_1)-(H_2)$ and (F_8) be satisfied, $\mathcal{S} = \mathcal{S}_2 \neq \emptyset$. Let there exist a supersolution $W \ge H > 0$ of problem*

$$\text{(7.4.44)} \qquad \begin{cases} \mathcal{P}^-_{\lambda,\Lambda}(D^2u) \; + \; \langle b, Du\rangle = 0 & \text{in } \Omega \\ \\ u \; = 0 & \text{on } \; \mathcal{R}\,. \end{cases}$$

Let $u \in C(\Omega \cup \mathcal{R})$ be a subsolution of problem (7.1.2) such that

$$\text{(7.4.45)} \qquad \liminf_{\varepsilon\to 0} \Big\{ \sup_{\mathcal{A}^\varepsilon} \frac{u}{W} \Big\} \le 0.$$

Then $u \le 0$ in Ω.

Proof of Theorem 7.3.10. Arguing as in the proof of Proposition 7.4.2 we can construct a supersolution W to problem (7.4.44). Thus the conclusion follows applying Proposition 7.4.6. □

7.5. Examples

Throughout this Section we always assume the following:

$$(F_7) \quad \begin{cases} (i) \;\; \tilde{F} \in C((\Omega \cup \mathcal{R}) \times I\!R \times I\!R^m \times \Sigma^m; I\!R) \; (m \in I\!N)\,; \\ (ii) \;\; \tilde{F} \text{ is proper and uniformly elliptic with ellipticity constants } \lambda \le \Lambda\,; \\ (iii) \;\; \text{there exist } \beta, \gamma \in C(\Omega \cup \mathcal{R}; [0,\infty)) \text{ such that} \\ \qquad \tilde{F}(x,r,p,X) \ge \mathcal{P}^-_{\lambda,\Lambda}(X) - \beta(x)|p| + \gamma(x)r, \\ \qquad \text{for any } x \in \Omega \cup \mathcal{R}, r \ge 0, p \in I\!R^m, \; X \in \Sigma^m\,. \end{cases}$$

EXAMPLE 7.5.1. *Application of Theorem 7.3.1.* Consider the problem

$$\begin{cases} \tilde F(x,y,u,\sigma^T Du, \sigma^T D^2 u\,\sigma) = 0 & \text{in } (1,\infty)\times(0,1) \\ \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{7.5.1}$$

where

$$\sigma(x,y) = \begin{pmatrix} x^2 & 0 \\ 0 & 1 \end{pmatrix} \quad ((x,y)\in\Omega).$$

Suppose that β in (F_7) is given by

$$\beta(x,y) = \lambda x \quad \big((x,y)\in\Omega\big).$$

Clearly,

$$F(x,y,u,Du,D^2u) := \tilde F(x,y,u,\sigma^T u, \sigma^T D^2 u\,\sigma) \geq$$
$$\geq E(x,y,u,Du,D^2u) := \mathcal{P}^-_{\lambda,\Lambda}(\sigma^T D^2 u\sigma) - \beta|\sigma^T Du| \ ((x,y)\in\Omega)\,;$$

notice that $(F_1)-(F_2)$ are satisfied (see [**5**]). The function $W(x,y) = \log x + 1$ satisfies

$$\mathcal{P}^-(\sigma^T D^2 W\sigma) - \beta|\sigma^T DW| + \gamma W \geq \lambda(x^2 - x) \geq 0 \quad \text{in } \Omega\,.$$

Moreover

$$W \geq 1\,, \quad \lim_{x\to\infty} W(x,y) = \infty \quad (y\in(0,1)).$$

Hence, if $u \in USC(\overline{\Omega}) \cap L^\infty(\Omega)$ is a subsolution of problem (7.5.2), then $u \leq 0$ in Ω, by Theorem 7.3.1.

EXAMPLE 7.5.2. *Application of Theorem 7.3.1.* Consider the problem

$$\begin{cases} \tilde F(x,y,u,\sigma^T D^2 u\,\sigma) + \langle b(x,y), \sigma^T Du\rangle = 0 & \text{in } \Omega = (0,1)\times(0,1) \\ \\ u = 0 & \text{on } \mathcal{R}\cup\mathcal{S}_1\,, \end{cases} \tag{7.5.2}$$

where

$$\sigma(x,y) = \begin{pmatrix} 1/\sqrt{x} & 0 \\ 0 & y \end{pmatrix}, \ b(x,y) = (1, -\Lambda/y) \quad ((x,y)\in\Omega),$$

$\mathcal{R} = \big((0,1]\times\{1\}\big)\cup\big(\{1\}\times(0,1)\big)$, $\mathcal{S}_1 = \{0\}\times(0,1]$, $\mathcal{S}_2 = [0,1]\times\{0\}$. Clearly,

$$F(x,y,u,Du,D^2u) := \tilde F(x,y,u,\sigma^T u, \sigma^T D^2 u\,\sigma) + \langle b(x,y), \sigma^T Du\rangle \ \geq$$
$$\geq E(x,y,u,Du,D^2u) := \mathcal{P}^-_{\lambda,\Lambda}(\sigma^T D^2 u\sigma) + \langle b(x,y), \sigma^T Du\rangle \ ((x,y)\in\Omega)\,;$$

notice that $(F_1)-(F_2)$ are satisfied (see [**5**]). The function $W(x,y) = -\log y + 1$ solves

$$\mathcal{P}^-_{\lambda,\Lambda}(\sigma^T D^2 W\sigma) + \langle b(x,y), \sigma^T DW\rangle + \gamma W \geq -\Lambda + \frac{\Lambda}{y} \geq 0 \quad \text{in } \Omega\,;$$

moreover,

$$W \geq 1, \quad \lim_{y\to 0^+} W(x,y) = +\infty.$$

Hence, if $u \in USC(\Omega\cup\mathcal{R}\cup\mathcal{S}_1)\cap L^\infty(\Omega)$ is a subsolution of problem (7.5.1), then $u \leq 0$ in Ω, by Theorem 7.3.1.

EXAMPLE 7.5.3. *Application of Theorem 7.3.1.* Consider the problem

$$\begin{cases} -\Delta_\infty u - \frac{1}{y^3} u_y + u = 0 & \text{in } \ \Omega = (0,1)\times(0,1) \\ \\ u = 0 & \text{on } \mathcal{R} = \partial\Omega\setminus\big([0,1]\times\{0\}\big)\,; \end{cases} \tag{7.5.3}$$

here $\Delta_\infty u \equiv \langle D^2 u Du, Du\rangle$ is the *infinity Laplacian*, $\mathcal{S} := [0,1]\times\{0\}$. For any $\alpha\in(0,1]$ the function $W_\alpha(x,y) := -\alpha\log y + 1$ satisfies

$$E(x,y,W_\alpha, DW_\alpha, D^2W_\alpha) := -\Delta_\infty W_\alpha - \frac{1}{y^3}\frac{\partial W_\alpha}{\partial y} + W_\alpha \geq 0 \quad ((x,y)\in\Omega);$$

moreover,

$$W_1 \geq 1\,, \qquad \lim_{y\to 0} W_1(x,y) = \infty\,.$$

Hence, if $u \in USC(\Omega\cup\mathcal{R})\cap L^\infty(\Omega)$ is a subsolution of problem (7.5.3), then $u \leq 0$ in Ω, by Theorem 7.3.1.

EXAMPLE 7.5.4. *Optimality of condition* (7.3.9) *in Theorems 7.3.4 and 7.3.6.* Consider the *linear* problem

$$\text{(7.5.4)} \qquad \begin{cases} -\Delta\, u + \frac{1}{x^2+y^2}(4xu_x + 4yu_y) + u = f & \text{in } \Omega \\ \\ u = g & \text{on } \mathcal{R}\,, \end{cases}$$

where $\Omega = B_1 \setminus \{0\} \subseteq I\!R^2$, $\mathcal{R} = \partial B_1$, $\mathcal{S} = \{0\}$, $f \in C^{0,1}(\Omega\cup\mathcal{R})\cap L^\infty(\Omega)$ and $g \in C(\mathcal{R})$. Observe that if Theorems 7.3.4, 7.3.6 apply, then problem (7.5.4) has at most one bounded solution. This is not the case, because in [**58**] it is shown that problem (7.5.4) has infinitely many bounded solutions. This is due to the fact that the condition (7.3.9) is not satisfied, since

$$|\beta(x,y)| = \frac{4}{\sqrt{x^2+y^2}} \qquad \big((x,y)\in\Omega\big).$$

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