DOTTORATO DI RICERCA IN MATEMATICA

Asymptotic analysis of discrete systems with complex interfacial interactions

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Introduction

The subject of this thesis is part of a research project dealing with the description of discrete, non-linear and non-convex variational problems defined on lattices in which the mesh size tends to zero. This family of problems has strong connections with statistical thermodynamics, continuum mechanics and numerical calculus. Technically, the analysis is performed by computing suitable Γ-limits in the continuum which approximate the discrete problems. In particular, this thesis deals with the asymptotic analysis of atomistic systems in which the limit configurations involve complex surface energies, possibly in conjunction with volume energies.

The first chapter deals with the description of the overall effect of pinning conditions in discrete systems, highlighting the analogies and differences with the corresponding continuous case. In variational problems on the continuum, pinning sites are usually modeled as small zones (or perforations) where concentrated forces or Dirichlet conditions are imposed. Their effect can be described by exhibiting suitable effective problems involving an additional "strange term" of lower-order. Despite being a volume integral, this term is indeed due to the homogenization of forces which tend to concentrate close to the perforations and can be described through capacitary formulas which account for the effect of each perforation independently. In the simplest (but already presenting most of the general features) case of *periodically-perforated domains*, one imposes homogeneous Dirichlet conditions on a periodic array $U_{\delta,R}$ of small balls of radius R and centers on a δ-periodic lattice, and considers, e.g., minimum problems of the form

$$\min\Big\{\int_\Omega \big(|Du|^p - fu\big)\,dx : u = 0 \text{ on } U_{\delta,R}\Big\}. \tag{0.0.1}$$

As $\delta, R \to 0$ these problems can be approximated by

$$\min\Big\{\int_\Omega \big(|Du|^p + C|u|^p - fu\big)\,dx\Big\}, \tag{0.0.2}$$

where the middle term replaces the constraint; the constant C depends on the mutual asymptotic behavior of the two parameters. It is suggestive to think of u as a temperature field of a mixture of water and ice, with $U_{\delta,R}$ representing the ice distribution, and the second problem as an effective approximation when the ice particles are small. Note that there is a critical ratio between R and δ below which the constant C is 0 (if the percentage of "ice" is too small then it does not influence the limit) and above which C is $+\infty$ (i.e., the percentage of ice is so high that in the limit it forces $u = 0$).

The study of problems of the form above dates back to an early work by Marchenko and Khrushlov [52]. It has been subsequently popularized by a well-known paper of Cioranescu and Murat [37] and comprises a number of generalizations which cover also non-periodic geometries and give rise to the so-called *Relaxed Dirichlet Problems* (see e.g [41, 42, 43, 45, 46, 60] and [40] for an overview on the subject). In the framework of Γ-convergence recent papers by Ansini and Braides [11] and Sigalotti [57] deal with general vector energies. At the critical scale the basis of the asymptotic description of problems (0.0.1) is a separation-of-scales argument: the contribution of the energy that "concentrates" near each of the small balls can be decoupled from the others and from the energy that is "diffused" elsewhere (this is formalized in the procedure highlighted in [11]), and can be then computed by means of suitable "capacitary formulas" that give C. It must be noted that in the subcritical case $p < n$ the contribution of each ball is of the form

$$CR^{n-p}|u|^p, \tag{0.0.3}$$

which gives the scaling $R \sim \delta^{n/n-p}$, while in the critical scale $p = n$ that contribution is

$$C|\log R|^{n-1}|u|^n, \tag{0.0.4}$$

which gives the scaling $|\log R| \sim \delta^{n/n-1}$.

In the simplest discrete case, the integrals $\int_\Omega |Du|^p$ are replaced by finite-difference energies on a cubic lattice $\varepsilon\mathbb{Z}^n$ of the form

$$\sum_{n.n.} \varepsilon^n \Big|\frac{u(i)-u(j)}{\varepsilon}\Big|^p, \tag{0.0.5}$$

where the sum ranges over all *nearest-neighbors* in $\varepsilon\mathbb{Z}^n \cap \Omega$. The continuous approximation of (0.0.5) is indeed

$$\int_\Omega \|Du\|^p \, dx, \tag{0.0.6}$$

where

$$\|Du\|_p^p = \sum_{j=1}^n \Big|\frac{\partial u}{\partial x_j}\Big|^p. \tag{0.0.7}$$

The pinning condition which replicates the perforated domain constraint is then expressed as

$$u = 0 \quad \text{ on } \delta\mathbb{Z}^n, \tag{0.0.8}$$

where of course in addition one requires $\delta/\varepsilon \in \mathbb{N}$. In a discrete setting the corresponding minimum problem can be thought as giving equilibrium configurations for an atomistic model, e.g., with hardening conditions due to the presence of transverse dislocations as in the paper by Garroni and Müller [48].

We can observe right away that the small parameter ε plays at the same time the role of both the discrete lattice scale and of the perforation size R, thus giving the critical scalings

$$\varepsilon \sim \delta^{n/n-p} \quad \text{ and } \quad |\log \varepsilon| \sim \delta^{n/n-1}. \tag{0.0.9}$$

If suitable discretizations of a forcing term are added, the choice of the critical scaling leads to limit problems of the form

$$\min\Big\{\int_\Omega \Big(\|Du\|_p^p + C|u|^p - fu\Big)\, dx\Big\}, \tag{0.0.10}$$

analogous to the ones we get in the continuous setting. The computation of the constant C presents some differences from the computation in the continuous case, even though a separation-of-scales procedure can be followed by proving a decoupling lemma (Lemma 1.6.1), which allows to analyze the single effect of each pinning site. In the critical case $p = n$ the energy "concentrating close to the pinning sites" indeed concentrates at a scale much larger than ε. In this way the capacitary computation reduces to the continuous one with a perforation of size $R = \varepsilon$ and with the anisotropic energy (0.0.6). In dimension $n = 2$ the constant is exactly the "classical" one since $\|Du\|_2$ equals the euclidean gradient norm $|Du|$. In the subcritical case $p < n$, instead, the energy concentrates at scale ε, so that the constant C is expressed by the "discrete p-capacity" of a point in the lattice $\mathbb{Z}^n$.

In the first chapter we prove the convergence result outlined above in a general setting where u can be vector-valued and the discrete energies take the form

$$E_\varepsilon(u) = \sum_{i,j} \varepsilon^n f^{(i-j)/\varepsilon}\Big(\frac{u(i)-u(j)}{\varepsilon}\Big),$$

where the interactions range over all pairs in $\Omega \cap \varepsilon\mathbb{Z}^n$, and are governed by general pair potentials depending also on the mutual distance of i and j in the reference lattice $\varepsilon\mathbb{Z}^n$. The energy densities $f^\xi(z)$, with $\xi \in \mathbb{Z}^n$, satisfy polynomial growth conditions in z of order p, and decay conditions in ξ that allow to restrict to (long-range but) finite-range interactions in $\Omega \cap \varepsilon\mathbb{Z}^n$ (following the general convergence result for unconstrained functionals by Alicandro and Cicalese [3]). The main result of the chapter is Theorem 1.3.1, where we show that the limit energies take the form

$$F(u) = \int_\Omega \Big(f_0(Du) + \Phi(u) \Big) dx,$$

where f_0 is given by the unconstrained *homogenization formula* proved in [3], and Φ is described by suitable asymptotic formulas that generalize the capacitary argument outlined above. Again, the form of Φ differs if $p = n$ or $p < n$. It must be observed that the limit in such formulas exists up to subsequences, as a consequence of the possible lack of homogeneity of degree p of the energy densities f^ξ. This non-uniqueness of the limit for the non-homogeneous case has already been observed for the continuous case (see e.g. [11]). The main technical point is the adaptation of the separation-of-scales arguments to the general long-range case. While for nearest neighbors the approach of Ansini and Braides can be easily repeated, upon adapting it to the geometry of the lattice (e.g., considering squares in the place of balls, etc.), for long-range interactions the discrete functionals are non-local and some extra care must be taken to make the procedure work.

The second chapter of this thesis deals with the description of a "defected" atomistic system: we consider a discrete system in which the interaction between the particles is given by quadratic potentials and we modify it by introducing some defects, modeled as simple nonlinear perturbations. According to the Weak-Membrane Model by Blake and Zisserman [14], a simple way to model free-discontinuity energies in a finite-difference scheme is by considering truncated quadratic energy densities (Fig. 1). The energy of such a (n-dimensional) scheme can then be

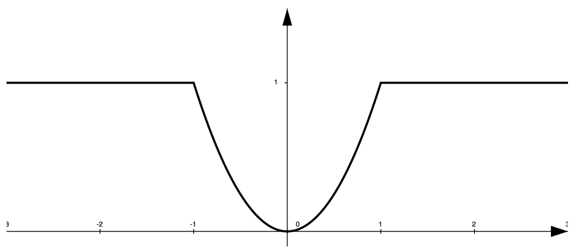

Figure 1: A truncated quadratic potential

written as

$$E(u) = \sum_{i,j} (u_i - u_j)^2 \wedge 1,$$

where u_i is a real parameter (the vertical displacement of the 'discrete membrane'), and the sum is performed over nearest neighbors in a cubic grid parameterized by $\mathbb{Z}^n$.

Thanks to a scaling argument due to Chambolle [34], which leads to the energies

$$E_\varepsilon(u) = \sum_{i,j} \varepsilon^n \Big(\Big(\frac{u_i - u_j}{\varepsilon} \Big)^2 \wedge \frac{1}{\varepsilon} \Big),$$

this discrete model can be approximated by a continuous energy defined on special functions with bounded variation. In fact, if we limit the interactions in the sum to the nearest neighbors in the portion of $\varepsilon\mathbb{Z}^n$ contained in some fixed Ω, and we interpret the values u_i as the discretization of a function defined in Ω, then these energies can be studied using the methods of Γ-convergence, and their limit is then given by a *fracture energy*

$$F(u) = \int_\Omega |\nabla u|^2\, dx + \int_{\Omega\cap S(u)} \|\nu\|_1 d\mathcal{H}^{n-1}$$

(see [34, 35, 24]), where $S(u)$ is the fracture site, ν is its normal and u is the macroscopic displacement outside the fracture site. The correct functional setting for these kinds of energies is the space $GSBV(\Omega)$ of (generalized) special functions of bounded variation in Ω introduced by Ambrosio and De Giorgi (see [18, 9]). From an atomistic standpoint, the energy $(u_i - u_j)^2 \wedge 1$ can be interpreted as that of a 'defected' quadratic spring, which breaks after reaching a critical elongation; the collective behavior of such a system gives rise to the possibility of fracture. The critical scaling in E_ε is precisely the one that allows this behavior but forbids the accumulation of 'broken springs' on sets of dimension larger than $n-1$ while keeping the energy bounded. Note that the truncated quadratic potentials are a prototypical example to which the study of more general convex-concave atomistic potentials can be often reduced such as for Lennard Jones ones (see [26, 28])

If not all springs are 'defected', but a portion of them are simple quadratic linear springs, with corresponding energy $(u_i - u_j)^2$ (for which the Γ-limit is simply the Dirichlet integral and no discontinuity is allowed for the limit u), then the problem is more complex, and a continuous description must take into account the location and 'micro-geometry' of the two types of springs. In a probabilistic setting, the location of the defected springs can be modeled in terms of realizations of i.i.d. random variables. In dimension two an analysis by Braides and Piatnitski [27] shows that the Γ-limit is deterministic and depends almost surely on the probability p of the weak springs. Its form is of 'fracture type' if p is above the *percolation threshold*, while it coincides with the Dirichlet integral for all values of p below that threshold.

A deterministic study leads necessarily to a more complex statement. In this case we look at possible Γ-limits of energies of the form

$$E_\varepsilon(u) = \sum_{i,j} \varepsilon^n f^\varepsilon_{ij}\Big(\frac{u_i - u_j}{\varepsilon}\Big),$$

where, for each ε, $f^\varepsilon_{ij}(z)$ may be chosen arbitrarily to be either z^2 or $z^2 \wedge (1/\varepsilon)$.

It must be noted beforehand that, whatever the limit percentage of weak interaction is, we can obtain in the limit both the Dirichlet integral, and the Weak-Membrane Energy above; i.e., that even if we prescribe that for every subdomain $A \subset \Omega$ we have

$$\lim_{\varepsilon\to 0} \frac{\#\{(i,j) \in A \cap \varepsilon\mathbb{Z}^n : f^\varepsilon_{ij}(z) = z^2 \wedge (1/\varepsilon)\}}{\#\{(i,j) \in A \cap \varepsilon\mathbb{Z}^n\}} = \theta$$

for any $\theta \in [0,1]$, we may obtain both such energies as Γ-limits for suitable choices of f^ε_{ij} (see [27] and Section 2.3.6 below). This is in contrast with formally similar problems where damaged springs are modeled as still quadratic with an energy density αz^2 with a constant $\alpha < 1$ (for this 'discrete G-closure' problem see Braides and Francfort [23], and Braides and Gloria [25]). This observation leads to conjecturing that indeed the possible limit energies F are (independent of the limit density and) characterized by the two inequalities deriving from the comparison with the extreme cases; i.e.,

$$F(u) \le \int_\Omega |\nabla u|^2\, dx \quad \text{ if } u \in H^1(\Omega),$$

$$F(u) \ge \int_\Omega |\nabla u|^2\, dx + \int_{S(u)} \|\nu\|_1 d\mathcal{H}^{n-1} \quad \text{ if } u \in GSBV(\Omega).$$

The two inequalities imply that indeed $F(u) = \int_\Omega |\nabla u|^2\, dx$ if $u \in H^1(\Omega)$, and suggest the conjecture that we may obtain as limits all lower-semicontinuous energies of the form

$$F(u) = \int_\Omega |\nabla u|^2\, dx + \int_{S(u)} \varphi(x, u^+ - u^-, \nu) d\mathcal{H}^{n-1} \text{ if } u \in GSBV(\Omega),$$

($u^{\pm}$ denote the traces of u on both sides of $S(u)$), where

- $\nu \mapsto \varphi(x, z, \nu)$ is even and $\varphi(x, z, \nu) \geq \|\nu\|_1$
- $z \mapsto \varphi(x, z, \nu)$ is even, and is increasing for positive z.

A complete proof of such a conjecture is not within the possibilities of the present knowledge of free-discontinuity functionals, even in the homogeneous case, i.e., with $\varphi(x, z, \nu) = \varphi(z, \nu)$. Indeed, for such energy densities the condition for lower semicontinuity is *BV-ellipticity* (see Ambrosio and Braides [8]), which is the analogue for interfacial energies of the condition of *quasiconvexity* for integral functionals (see Morrey [53]), and turns out to be necessary and sufficient if φ satisfies an inequality from above $\varphi(z, \nu) \leq C|z|$. This last growth condition is not in general satisfied by our energies, and without this assumption neither we can apply known representation results (as those by Braides and Chiadò Piat [21] or Bouchitté et al. [16]), nor we can characterize the energy density (indeed, the problem of removing growth conditions is one of the main issues also in the theory of vector energies; see Ball and Murat [13]). But even when growth assumptions from above are satisfied and the function φ is BV-elliptic this information is of little help since explicit constructions of BV-elliptic energy densities (e.g., in the spirit of the construction of quasiconvex functions by relaxation as that by Šverák [61]) or their variational approximation by simpler energies (e.g., in the spirit of appoximation of quasiconvex energies by homogenization of polyconvex functionals as by Braides [17]) are not available in general, as are not available for arbitrary quasiconvex functions.

We will then restrict our analysis to classes of simpler energy densities, proving a number of results, each of its particular interest (summarized in Theorem 2.2.2):

1) $\varphi = \varphi(\nu)$ even. In this case the condition of BV-ellipticity is equivalent to the convexity of (the one-homogeneous extension of) φ. We will prove that all such energy densities can be obtained in the limit;

2) $\varphi = \varphi(z)$. The form of the energies E_ε implies that φ is even and $z \mapsto \varphi(z)$ is increasing on $(0, +\infty)$. Moreover the growth condition gives $\varphi(z) \geq \sup_\nu \|\nu\|_1 = \sqrt{n}$. In this case the condition of BV-ellipticity is equivalent to the *subadditivity* of φ; i.e. that $\varphi(z+z') \leq \varphi(z)+\varphi(z')$ for all z, z'. This condition is rather complex, and is implied by the concavity of φ on $(0, +\infty)$. We will prove the approximation result for this restricted but important class of energy densities;

3) $\varphi = \varphi(x)$ lower semicontinuous. In this case the only condition for approximation is $\varphi(x) \geq \sqrt{n}$.

Moreover we can obtain $\varphi(x, z, \nu) = \varphi_1(\nu)\varphi_2(z)\varphi_3(x)$ by combining the approximation constructions above.

We note that other types of energies can be obtained as Γ-limits; for example, those of the form

$$F(u) = \int_\Omega |\nabla u|^2 \, dx + \int_{S(u)} \varphi(x, u^+ - u^-) d\mathcal{H}^{n-1} \text{ if } u \in SBV(\Omega),$$

with the constraint that $S(u) \subset K$ where K is a fixed $n-1$-dimensional surface. Indeed, such types of energies will be the building blocks of our approximation strategy. In fact, for case (1) above we will first use this construction with K a network of planar surfaces and φ suitable constants on each surface of the network, and then use an approximation procedure similar to the one by Ansini and Iosifescu [12] to obtain an arbitrary convex φ. Note that in particular we may obtain as φ any constant not larger than $\sqrt{n}$, so that case (3) can be derived by localizing such a construction. To obtain case (2), we first treat the case of K a single hyperplane and $\varphi(x, z) = c_1 + c_2 z^2$. This can be obtained following arguments similar to those by Ansini [10] to approximate the energy density $c(u^+ - u^-)^2$ on a surface (Neumann sieve) coupled with the description of the effect of pinning sites at the critical scaling developed by Sigalotti in [57, 59]. Note that the computation of the interfacial energy gives the same constant as in the continuous case for $n = 2$, while it highlights a more complex behavior for $n \geq 3$, where a fraction of the total contribution is actually given by the strong springs at the interface, which sums up to the contribution distributed away from the interface and summarized in a capacitary formula. By repeating this argument on more parallel surfaces concentrating to the same hyperplane we can recover an arbitrary concave function by approximation with subadditive envelopes of families of functions as above (this is the only argument where concavity is used). Finally the use of a network of hyperplanes as above allows for a radially symmetric target φ.

In the third chapter we deal with the analysis of "ternary" energies; i.e., energies depending on functions taking three values only (for simplicity, 1, 0 and -1). In our setting, it is not possible to reduce the problem to the case of binary systems (or 'spin'), due to the assumptions we make on the energies. Indeed, the phases ± 1 tend to be separated by an interface, on which the phase 0 tends to concentrate. This description corresponds to models which have been studied from the point of view of physics: it has been shown that the free energy of a system where two or more phases coexist can be altered by the presence of low concentrations of a surfactant. In other words, a surfactant (a contraction for *surface-active-agent*) is a substance which may significantly reduce the surface tension of a system by being adsorbed onto the interfaces.

In order to give a variational description of the effects caused by the presence of surfactants in phase-separation phenomena, several attempts have been made to model the physical system both as a continuum and as a discrete. Among the continuum theories, the first description of phase transitions in presence of surfactants has been developed by Laradji-Guo-Grant-Zuckermann (see [50, 51]), who suggested a variational model involving a two order parameters Ginzburg-Landau functional. Several generalizations have been later considered by Gompper and Schick in [49]. In [50] and [51] one of the two order parameters represents the local difference of density of the two phases (as in the standard Cahn-Hillard model in the gradient theory of phase transitions), while the other one represents the local surfactant density. The two order parameters are energetically coupled to favor the segregation of the surfactant at the phase interface. The coarse-graining analysis of this model has been performed through Γ-convergence methods by Fonseca, Morini and Slastikov in [47], while the mathematical analysis of more general continuum models is the subject of [1].

Many of the discrete models are variants of the one which was originally introduced by Blume, Emery and Griffiths (BEG) in [15] (see also [49] and the references therein); the third chapter of this thesis deals with its variational analysis in the framework of Γ-convergence. In their seminal paper, Blume, Emery and Griffiths introduced a simple nearest-neighbors spin-1 model as a variant of a classical Ising type spin-1/2 model, with the aim of describing a different kind of phenomena, namely He^3-He^4 λ-transitions. In the setting of phase transitions in presence of surfactants, BEG model can be briefly described as follows. On the two dimensional square lattice $\mathbb{Z}^2$, we consider a ternary system driven by an energy which is defined on functions parameterized on the points of the lattice and taking only three possible values (which we may suppose to be $-1, 0, 1$). We can identify the values of u with three different phases (in particular, the value 0 is associated with the surfactant). Omitting the chemical potentials, for a given configuration of particles, the free energy E of this system is given by

$$E(u) = -\sum_{n.n.} u(a)u(b) + \sum_{n.n.} k(u(a)u(b))^2, \tag{0.0.11}$$

where $n.n.$ means that the summations are taken over all nearest neighboring sites; i.e., the elements a, b of the lattice such that $|a-b|$ equals the lattice spacing. The constant $k > 0$ is the quotient between the so-called bi-quadratic and the quadratic exchange interaction strengths; its range will be specified later on, such as the scaling factor for the energy.

In Chapter 3 we will perform a Γ-limit analysis of these functional. As a result, we will be able to describe the behavior of the ground states of the BEG system as ε tends to 0. More precisely, let $\Omega \subset \mathbb{R}^2$ be a bounded open set and let us consider the scaled energies

$$E_\varepsilon(u) = \sum_{n.n.} \varepsilon^2(-u(a)u(b) + k(u(a)u(b))^2). \tag{0.0.12}$$

Here the array $\{u(a)\}$ can be seen as a function defined on $\varepsilon\mathbb{Z}^2 \cap \Omega$. Upon identifying such functions with their piecewise-constant interpolations, the energies E_ε can be interpreted as defined on (a subset of) $L^1(\Omega)$; we can then perform a Γ-convergence analysis in the framework of $L^1(\Omega)$. As ε tends to 0, the Γ-limit E of E_ε is particularly simple: under the trivial constraint $|u| \leq 1$, it is constantly equal to the minimum value $2|\Omega|(-1+k) \wedge 0$, corresponding to the uniform states. By choosing $k < 1$ we set the uniform states $u = \pm 1$ to be the ground states. Having fixed $k < 1$, the asymptotic behavior of E_ε implies that a sequence $(u_\varepsilon)_\varepsilon$ can arbitrarily mix the uniform phases -1 and 1 at a mesoscopic scale, though keeping its energy equal to the energy of the uniform states plus an infinitesimal function, as $\varepsilon \to 0$ (the asymptotic analysis of the bulk scaling of more general spin-type models has been performed in [5]). Thus, in order to get a better description of the ground states, in the spirit of *development by* Γ*-convergence* (see [21, 28, 4] and [2]), we select sequences which attain the minimum value with a sharper precision, meaning that

$$E_\varepsilon(u_\varepsilon) = c_\varepsilon + O(\varepsilon),$$

where c_ε is the absolute minimum of E_ε; i.e., $c_\varepsilon = \sum_{\text{n.n.}} \varepsilon^2(k-1)$. For such configurations the limit states u will take the values ± 1 only. The limit energy will be an interfacial-type energy: it can be interpreted as the surface tension of the system which undergoes a phase separation phenomenon between the phases $\{u = -1\}$ and $\{u = +1\}$. At this scaling, it is necessary to further specify the values of the parameter k, so that the phase 0 can be actually considered a surfactant phase (meaning that it contributes to lower the surface tension). In particular it can be easily shown (see Section 3.3) that, for $\frac{1}{3} < k < 1$, the energy for a transition from phase -1 phase to phase $+1$ is lowered when the surfactant is at the interface. Moreover, the measure of the phase 0 vanishes as we pass to the limit. This scaling is usually referred to as *low surfactant concentration regime*. Thus, we study the rescaled functionals

$$E_\varepsilon^{(1)}(u) := \frac{E_\varepsilon(u) - c_\varepsilon}{\varepsilon} = \sum_{\substack{a,\, b \,\in\, \varepsilon\mathbb{Z}^2 \cap \Omega \\ |a-b| = \varepsilon}} \varepsilon(1 - u(a)u(b) - k(1 - (u(a)u(b))^2).$$

Note that the interaction between two particles of the same type -1 or $+1$ has zero energy, while the interaction of a surfactant particle 0 with any other particle is repulsive and 'costs' the positive value $1-k$. For this reason, the BEG functional is also said to describe a *repulsive surfactant model*. In Theorem 3.3.2 we show that $E_\varepsilon^{(1)}$ Γ-converges (in the $L^1(\Omega)$-topology) to the interfacial-type energy functional

$$E^{(1)}(u) = \int_{S(u)} \psi(\nu_u) d\mathcal{H}^1,$$

where $u \in BV(\Omega; \{\pm 1\})$, $S(u)$ is the (essential) interface between the sets $\{u = 1\}$ and $\{u = -1\}$, ν_u is the inner normal to $S(u)$ and $\psi(\nu) = (1-k)(3|\nu_1| \vee |\nu_2| + |\nu_1| \wedge |\nu_2|)$ denotes the anisotropic surface tension of the model.

Note that in this topology the limit order parameter u does not carry any information about the surfactant phase. Actually, the role of the surfactant becomes clear when looking at the minimizing microstructure leading to the computation of the surface density ψ. In this direction, a natural further step in the analysis of the BEG model is the study of the dependence of the surface tension on the concentration of the surfactant. The literature on this subject is wide, both from the physical and the chemical point of view (see for example [49] and [54]). However, no rigorous description of the microscopic geometry of the surfactant at the interface has been developed; all the previous documented attempts to study this problem are based on numerical computations or on heuristic arguments. In order to rigorously address this problem, we need to go beyond the standard formulation of the BEG model. In particular, the functional which describes the energy of the system has to depend explicitly on the distribution of the surfactant particles. To this end, we set

$$I_0(u) = \{a \in \Omega_\varepsilon : \, u(a) = 0\}$$

and we introduce the *surfactant measure*

$$\mu(u) = \sum_{a \in I_0(u)} \varepsilon \delta_a.$$

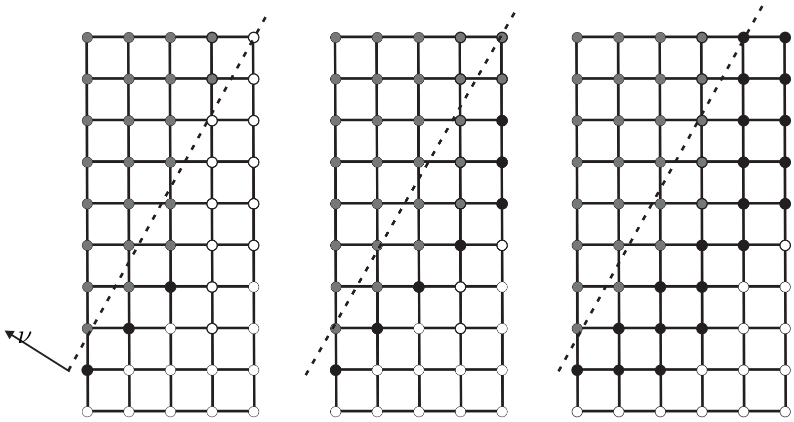

Figure 2: The local microstructure of a ground state of the BEG model at a fixed straight interface (the dashed line normal to ν) for three different values of the density of surfactants at the interface. Black, white and grey dots stand for the 0, +1 and −1 values of the spin field u, respectively.

Then, with a slight abuse of notation, we can extend $E^{(1)}_\varepsilon$ to $L^1(\Omega) \times \mathcal{M}_+(\Omega) \to [0, +\infty]$ as

$$E^{(1)}_\varepsilon(u, \mu) = \begin{cases} E^{(1)}_\varepsilon(u) & \text{if } \mu = \mu(u) \\ +\infty & \text{otherwise.} \end{cases}$$

In order to track the energy of the surfactants, we extend the functionals by decoupling the order parameter of the model. In the continuum setting, instead, the functionals were extended by introducing an additional variable (see [47] and [1]). The space $L^1(\Omega) \times \mathcal{M}_+(\Omega)$ is endowed with the topology $\tau_1 \times \tau_2$, where τ_1 denotes the strong topology in $L^1(\Omega)$ and τ_2 refers to the weak *-topology in the space of non-negative bounded Radon measures $\mathcal{M}_+(\Omega)$. In Theorem 3.3.3 we prove that $E^{(1)}_\varepsilon$ Γ-converges (with respect to $\tau_1 \times \tau_2$-topology) to the functional $E^{(1)}$: $L^1(\Omega) \times \mathcal{M}^+(\Omega) \to [0, +\infty]$ defined as

$$E^{(1)}(u, \mu) = \begin{cases} \displaystyle\int_{S(u)} \varphi\Big(\frac{d\mu}{d\mathcal{H}^1 \lfloor_{S(u)}}, \nu_u\Big) d\mathcal{H}^1 + (2k-2)|\mu^s|(\Omega) & \text{if } u \in BV(\Omega; \{\pm 1\}) \\ +\infty & \text{otherwise,} \end{cases}$$

where $\varphi : \mathbb{R} \times S^1 \to [0, +\infty)$ is computed explicitly. Looking at the graph of φ (Figure 2), we can notice that an anisotropic threshold phenomenon occurs at the phase interface. Indeed, for a fixed $\nu \in S^1$, the surface tension $\varphi(z, \nu)$ decreases up to a certain value of the density z of the surfactant, namely $z = |\nu_1| \vee |\nu_2|$. As the density of the surfactant increases further, two events can occur: if the surfactants are not absorbed onto the interface, the surface tension remains constant and the singular part of the surfactant measure increases; otherwise, the surface tension increases. As an application of Theorem 3.3.3, at the end of Section 3.3 we study an optimization problem in which the volume fractions of the different phases are prescribed.

The variety of models of phase transitions in presence of surfactants studied in the physical and chemical literature suggested that we should widen our analysis. In Section 3.4, we consider the case of a n-dimensional discrete system, driven by an energy accounting for quite general finite-range pairwise interactions, in the presence of different species of repulsive surfactant particles. For such a general system, we obtain an integral representation result for the Γ-limit, in the spirit of homogenization theory, and we study some properties of its limit densities.

Namely, given $\Omega \subset \mathbb{R}^n$ and $u : \varepsilon\mathbb{Z}^n \cap \Omega \to K$ we define the functional F_ε as

$$F_\varepsilon(u) = \sum_{\substack{a,b \in \varepsilon\mathbb{Z}^2 \cap \Omega \\ |a-b| \leq R\varepsilon}} \varepsilon^{n-1} f\left(\frac{b-a}{\varepsilon}, u(a), u(b)\right).$$

where $R > 0$ is an interaction threshold and $K = \{m_1, m_2, s_1, s_2, \dots, s_M\} \subset \mathbb{R}$ describes the finite number of phases in the system. Moreover, $f : \mathbb{Z}^n \times K^2 \to [0, +\infty)$ satisfies a 'discrete isotropy condition' (see Remark 3.4.1 and 3.4.5) and is such that $\{(m_1, m_1), (m_2, m_2)\}$ are absolute minima of $f(z, \cdot, \cdot)$. In order to study the discrete-to-continuum limit of this system, we introduce a notation which describes the subsets of Ω_ε corresponding to the different types of surfactant. For $l \in \{1, 2, \dots, M\}$ we set

$$I_l(u) := \{a \in \Omega_\varepsilon : u(a) = s_l\}, \quad I(u) := \bigcup_{l=1}^{M} I_l(u)$$

and we define

$$\mu_l(u) := \sum_{a \in I_l(u)} \varepsilon^{n-1}\delta_a, \quad \mu(u) = \{\mu_1(u), \mu_2(u), \dots, \mu_M(u)\}.$$

We then extend F_ε to $L^1(\Omega) \times (\mathcal{M}_+(\Omega))^M \to [0, +\infty]$ as

$$F_\varepsilon(u, \mu) := \begin{cases} F_\varepsilon(u) & \text{if } \mu = \mu(u) \\ +\infty & \text{otherwise.} \end{cases} \tag{0.0.13}$$

The space $L^1(\Omega) \times (\mathcal{M}_+(\Omega))^M$ is endowed with the topology $\tau_1 \times \tau_2$, where τ_1 denotes the strong topology in $L^1(\Omega)$ and τ_2 stands for the weak*-topology in $(\mathcal{M}_+(\Omega))^M$. In Theorem 3.4.4 we prove that F_ε Γ-converges to the functional

$$F(u, \mu) = \begin{cases} \int\limits_{S(u)} f_{hom}\left(\frac{d\mu}{d\mathcal{H}^{n-1}\lfloor S(u)}, \nu(u)\right) d\mathcal{H}^{n-1} + \int_\Omega g_{hom}(\mu^s) \\ \qquad \text{for } u \in BV(\Omega; \{m_1, m_2\}),\ \mu = \frac{d\mu}{d\mathcal{H}^{n-1}\lfloor S(u)}\mathcal{H}^{n-1}\lfloor S(u) + \mu^s \\ +\infty \qquad \text{otherwise.} \end{cases} \tag{0.0.14}$$

The limit densities f_{hom} and g_{hom} are given by two asymptotic homogenization formulas ((3.4.21) and (3.4.22) respectively). Whereas the formula for f_{hom} can be derived through a standard argument in homogenization theory, this is not true for g_{hom}. We will need to combine some abstract arguments of measure theory with a reflection construction, which uses the discrete isotropy assumption on the interaction densities, in order to prove that g_{hom} is well defined (see Remark 3.4.5).

It should be noted that in our models the surfactants are represented as point-like particles, with no internal structure. More general models have been developed: they describe the surfactants as polar molecules with heads and tails interacting differently with the same phase (see [36, 56, 49]). In that setting, it is known that the presence of surfactants in a mixture may lead to self-assembling and that a number of different microstructures may appear, even with non-trivial topologies. Hopefully, the analysis performed in Chapter 3 may provide the basis to address the discrete-to-continuum limit for those systems.

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Thank you, Marco. Our dreams give meaning to the wandering... Journeys end in lovers’ meeting.

This thesis is dedicated to my cousin Filippo. Hope I make you proud over the years.

Chapter 1

Homogenization of pinning conditions on networks

1.1 Introduction

This chapter deals with the description of the overall effect of pinning conditions in discrete systems, highlighting the analogies and differences with the corresponding continuous case. In variational problems on the continuum, pinning sites are usually modeled as small zones where concentrated forces or Dirichlet conditions are imposed. Their effect can be described by exhibiting suitable effective problems. In the simplest (but already presenting most of the general features) case of *periodically-perforated domains* one imposes homogeneous Dirichlet conditions on a periodic array $U_{\delta,R}$ of small balls of radius R and centers on a δ-periodic lattice, and considers, e.g., minimum problems of the form

$$\min\Big\{\int_\Omega \big(|Du|^p - fu\big)\,dx : u = 0 \text{ on } U_{\delta,R}\Big\}. \qquad (1.1.1)$$

As $\delta, R \to 0$ these problems can be approximated by

$$\min\Big\{\int_\Omega \big(|Du|^p + C|u|^p - fu\big)\,dx\Big\}, \qquad (1.1.2)$$

where the middle term replaces the constraint; the constant C depends on the mutual asymptotic behavior of the two parameters. It is suggestive to think of u as a temperature field of a mixture of water and ice, with $U_{\delta,R}$ representing the ice distribution, and the second problem as an effective approximation when the ice particles are small. Note that there is a critical ratio between R and δ below which the constant C is 0 (if the percentage of "ice" is too small then it does not influence the limit) and above which C is $+\infty$ (i.e., the percentage of ice is so high that in the limit it forces $u = 0$).

The study of problems of the form above dates back to an early work by Marchenko and Khrushlov [52]. It has been subsequently popularized by a well-known paper of Cioranescu and Murat [37] and comprises a number of generalizations which cover also non-periodic geometries and give rise to the so-called *Relaxed Dirichlet Problems* (see e.g [41],[42],[43],[45],[46],[60] and [40] for an overview on the subject). In the framework of Γ-convergence recent papers as [11] and [57] deal with general vector energies. At the critical scale the basis of the asymptotic description of problems (1.1.1) is a separation-of-scales argument: the contribution of the energy that "concentrates" near each of the small balls can be decoupled from the others and from the energy that is "diffused" elsewhere (this is formalized in the procedure highlighted in the paper by Ansini and Braides [11]), and can be then computed by means of suitable "capacitary formulas" that give C. It must be noted that in the subcritical case $p < n$ the contribution of each ball is of the form

$$CR^{n-p}|u|^p, \qquad (1.1.3)$$

which gives the scaling $R \sim \delta^{n/n-p}$, while in the critical scale $p = n$ that contribution is

$$C|\log R|^{n-1}|u|^n, \tag{1.1.4}$$

which gives the scaling $|\log R| \sim \delta^{n/n-1}$.

In the simplest discrete case, the integrals $\int_\Omega |Du|^p$ are replaced by finite-difference energies on a cubic lattice $\varepsilon\mathbb{Z}^n$ of the form

$$\sum_{NN} \varepsilon^n \Big|\frac{u(i) - u(j)}{\varepsilon}\Big|^p, \tag{1.1.5}$$

where the sum ranges over all *nearest-neighbors* in $\varepsilon\mathbb{Z}^n \cap \Omega$. The continuous approximation of (1.1.5) is indeed

$$\int_\Omega \|Du\|^p \, dx, \tag{1.1.6}$$

where

$$\|Du\|_p^p = \sum_{j=1}^n \Big|\frac{\partial u}{\partial x_j}\Big|^p. \tag{1.1.7}$$

The pinning condition which replicates the perforated domain constraint is then expressed as

$$u = 0 \quad \text{on } \delta\mathbb{Z}^n, \tag{1.1.8}$$

where of course in addition one requires $\delta/\varepsilon \in \mathbb{N}$. In a discrete setting the corresponding minimum problem can be thought as giving equilibrium configurations for an atomistic model, e.g., with hardening conditions due to the presence of transverse dislocations as in the paper by Garroni and Müller [48].

We can observe right away that the small parameter ε plays at the same time the role of both the discrete lattice scale and of the perforation size R, thus giving the critical scalings

$$\varepsilon \sim \delta^{n/n-p} \quad \text{and} \quad |\log \varepsilon| \sim \delta^{n/n-1}. \tag{1.1.9}$$

If suitable discretizations of a forcing term are added, the choice of the critical scaling leads to limit problems of the form

$$\min\Big\{\int_\Omega \Big(\|Du\|_p^p + C|u|^p - fu\Big)\, dx\Big\}, \tag{1.1.10}$$

analogous to the ones we get in the continuous setting. The computation of the constant C presents some differences from the computation in the continuous case, even though a separation-of-scales procedure can be followed by proving a decoupling lemma (Lemma 1.6.1), which allows to analyze the single effect of each pinning site. In the critical case $p = n$ the energy “concentrating close to the pinning sites” indeed concentrates at a scale much larger than ε. In this way the capacitary computation reduces to the continuous one with a perforation of size $R = \varepsilon$ and with the anisotropic energy (1.1.6). In dimension $n = 2$ the constant is exactly the “classical” one since $\|Du\|_2$ equals the euclidean gradient norm $|Du|$. In the subcritical case $p < n$, instead, the energy concentrates at scale ε, so that the constant C is expressed by the “discrete p-capacity” of a point in the lattice $\mathbb{Z}^n$.

In this chapter we prove the convergence result outlined above in a general setting where u can be vector-valued and the discrete energies take the form

$$E_\varepsilon(u) = \sum_{i,j} \varepsilon^n f^{(i-j)/\varepsilon}\Big(\frac{u(i) - u(j)}{\varepsilon}\Big),$$

where the interactions range over all pairs in $\Omega \cap \varepsilon\mathbb{Z}^n$, and are governed by general pair potentials depending also on the mutual distance of i and j in the reference lattice $\varepsilon\mathbb{Z}^n$. The energy densities $f^\xi(z)$, with $\xi \in \mathbb{Z}^n$, satisfy polynomial growth conditions in z of order p, and decay conditions in ξ that allow to restrict to (long-range but) finite-range interactions in $\Omega\cap\varepsilon\mathbb{Z}^n$ (following the general convergence result for unconstrained functionals by Alicandro and Cicalese [3]). The main result of the chapter is Theorem 1.3.1, where we show that the limit energies take the form

$$F(u)=\int_\Omega \Big(f_0(Du)+\Phi(u)\Big)dx,$$

where f_0 is given by the unconstrained *homogenization formula* proved in [3], and Φ is described by suitable asymptotic formulas that generalize the capacitary argument outlined above. Again, the form of Φ differs if $p=n$ or $p<n$. It must be observed that the limit in such formulas exists up to subsequences, as a consequence of the possible lack of homogeneity of degree p of the energy densities f^ξ. This non-uniqueness of the limit for the non-homogeneous case has already been observed for the continuous case (see e.g. [11]). The main technical point is the adaptation of the separation-of-scales arguments to the general long-range case. While for nearest neighbors the approach of Ansini and Braides can be easily repeated, upon adapting it to the geometry of the lattice (e.g., considering squares in the place of balls, etc.), for long-range interactions the discrete functionals are non-local and some extra care must be taken to make that procedure work.

This chapter is organized as follows. In Section 1.2 we introduce the necessary notation to state the main result, Theorem 1.3.1. In Section 1.4 we point out some analogies and differences between the problem we are dealing with and the corresponding continuous case, by looking at the asymptotic behavior of a family of meaningful minimum problems. In Section 1.5 we study two families of auxiliary functions; by determining their properties we highlight some of the differences between the critical case ($p=n$) and the non-critical one ($p<n$). In Section 1.6 we prove two technical lemmas. In Sections 1.7 and 1.8 we finally prove the Γ-*Liminf* inequality and the Γ-*Limsup* inequality. Section 1.9 is devoted to the description of two special cases, which show some interesting features despite having restrictive assumptions. Finally, Section 1.10 is an appendix devoted to a short proof of a discrete Poincaré inequality in our simplified context.

1.2 Notation

Let $m,n\in\mathbb{N}$ with $n\geq 2$ and $m\geq 1$. For any measurable $A\subset\mathbb{R}^n$ we denote by $|A|$ the n-dimensional Lebesgue measure of A. Let $\{e_1,\dots,e_n\}$ be the set of unit vectors along the coordinate directions. Let Ω be a bounded open subset of $\mathbb{R}^n$ with $|\partial\Omega|=0$. For fixed $\varepsilon>0$ we consider the lattice $\varepsilon\mathbb{Z}^n\cap\Omega=:\Omega_\varepsilon$; we will often write Ω_j in place of Ω_{ε_j}. We denote by $\mathcal{A}_\varepsilon(\Omega)$ the set of functions

$$\mathcal{A}_\varepsilon(\Omega)=\{u:\Omega_\varepsilon\to\mathbb{R}\}.$$

A function $u\in\mathcal{A}_\varepsilon(\Omega)$ is identified with the piecewise-constant measurable function given by $u(x)=u(z_x^\varepsilon)$, where z_x^ε is the closest point to x in $\varepsilon\mathbb{Z}^n$ (which is uniquely defined up to a set of zero measure). In this definition, we set $u(z)=0$ if $z\in\varepsilon\mathbb{Z}^n\setminus\Omega$. $\mathcal{A}_\varepsilon(\Omega)$ is then regarded as a subset of $L^1(\Omega)$.
Having fixed a constant $M>0$, we introduce the set

$$I_M=\{\xi\in\mathbb{Z}^n:|\xi|\leq M \text{ and } -\xi<^l\xi\}.$$

In the definition above $<^l$ denotes the lexicographical order: given two vectors $\xi=(\xi_1,\dots,\xi_n)$ and $\zeta=(\zeta_1,\dots,\zeta_n)$, we say that $\xi<^l\zeta$ if and only if there exists $m\in\{1,\dots,n\}$ such that $\xi_i=\zeta_i$ for all $i<m$ and $\xi_m<\zeta_m$. We introduce this notion since we decided not to count the interactions twice. Equivalently, we could have chosen to pick both ξ and $-\xi$ and add some symmetry requirement on the interaction densities. For each vector $\xi\in I_M$, we define

$$R_\varepsilon^\xi(\Omega)=\{a\in\Omega_\varepsilon:\ a+\varepsilon\xi\in\Omega_\varepsilon\}.$$

Given a function $v \in \mathcal{A}_\varepsilon(\Omega)$, we indicate by $D^\xi_\varepsilon v$ the difference quotient along ξ; i.e.,

$$D^\xi_\varepsilon v(a) = \frac{v(a) - v(a + \varepsilon\xi)}{\varepsilon|\xi|} \qquad \text{for } a \in R^\xi_\varepsilon(\Omega).$$

Sometimes it will be convenient to use a specific notation for the set of all *nearest neighbors*, defined as

$$M_\varepsilon(\Omega) = \{\{a, b\} \ : \ a, b \in \Omega_\varepsilon \text{ and } |a - b| = \varepsilon\}. \tag{1.2.11}$$

Since nearest neighbors are defined as sets containing two points, and not as pairs in $\Omega_\varepsilon \times \Omega_\varepsilon$, we will count each interaction along the coordinate directions only once.

Given $l > 0$, we denote by $[l]$ its integer part. For all $l > 0$ and $x \in \mathbb{R}^n$ we denote by $Q(l, x)$ the closed rectangle $x + [-l, l]^n$. In particular $Q_\varepsilon(l, x) = \varepsilon\mathbb{Z}^n \cap (x + [-l, l]^n)$. Moreover, for fixed $n \geq l > 0$ and $x \in \mathbb{R}^n$, we define $\mathcal{S}_\varepsilon(l, n; x) = \Omega_\varepsilon \cap (x + ([-n, n]^n \setminus (-l, l)^n))$. If $l = n$, then we write $\mathcal{S}_\varepsilon(l; x) = \mathcal{S}_\varepsilon(l, n; x) = \Omega_\varepsilon \cap \partial(x + [-l, l]^n)$. If $x = 0$ we will write $Q_\varepsilon(l)$, $\mathcal{S}_\varepsilon(l, n)$, $\mathcal{S}_\varepsilon(l)$ instead of $Q_\varepsilon(l, 0)$, $\mathcal{S}_\varepsilon(l, n; 0)$, $\mathcal{S}_\varepsilon(l; 0)$ respectively.

Given a set of points $A \subseteq \Omega_\varepsilon$, we denote by $\mathbf{A}$ the union of all the ε-cells centered in elements of A:

$$\mathbf{A} = \cup_{a \in A} \mathcal{C}(a), \quad \text{where } \mathcal{C}(a) = a + [-\varepsilon/2, \varepsilon/2]^n.$$

1.3 Main result

In this section we state the main result of the chapter.

Theorem 1.3.1 *Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$. Let $m \in \mathbb{N}$, $m \geq 1$ and $1 < p \leq n$. Let I be the set of vectors $I = \{\xi \in \mathbb{Z}^n : -\xi <^l \xi\}$. For all $\xi \in I$, we consider a function $f^\xi : \mathbb{R}^m \to [0, +\infty)$ such that $f^\xi(0) = 0$. We assume that the functions f^ξ satisfy the following conditions:*

1. *there exists a constant $c_1 > 0$ such that for all $i \in \{1, \dots, n\}$*

$$\begin{array}{ll} f^{e_i}(z) \geq c_1|z|^n \text{ for all } z \in \mathbb{R}^m & \text{if } p = n \\ f^{e_i}(z) \geq c_1(|z|^p - 1) \text{ for all } z \in \mathbb{R}^m & \text{if } p < n \end{array} \tag{1.3.12}$$

2. *there exists a sequence of constants $c_2^\xi > 0$ such that for all $\xi \in I$*

$$\begin{array}{ll} f^\xi(z) \leq c_2^\xi|z|^n \text{ for all } z \in \mathbb{R}^m & \text{if } p = n \\ f^\xi(z) \leq c_2^\xi(|z|^p + 1) \text{ for all } z \in \mathbb{R}^m & \text{if } p < n, \end{array} \tag{1.3.13}$$

   and

$$\sum_{\xi \in I} c_2^\xi < +\infty$$

3. *there exists a constant $c_3 > 0$ such that for all $\xi \in I$*

$$\begin{array}{ll} |f^\xi(z) - f^\xi(w)| \leq c_3|z - w|(|z|^{n-1} + |w|^{n-1}) \text{ for all } z, w \in \mathbb{R}^m & \text{if } p = n \\ |f^\xi(z) - f^\xi(w)| \leq c_3|z - w|(1 + |z|^{p-1} + |w|^{p-1}) \text{ for all } z, w \in \mathbb{R}^m & \text{if } p < n. \end{array} \tag{1.3.14}$$

Let (ε_j) be a sequence of positive numbers converging to zero. Let (δ_j) be a positive infinitesimal sequence such that $\delta_j/\varepsilon_j \in \mathbb{N}$ and $\lim_j \delta_j/\varepsilon_j = +\infty$. We assume that (ε_j) and (δ_j) satisfy

$$\varepsilon_j = \begin{cases} e^{-r(1+o(1))\delta_j^{n/(1-n)}} \text{ as } j \to +\infty & \text{if } p = n \\ r^{(1-n)/(n-p)}\delta_j^{n/(n-p)}(1 + o(1)) \text{ as } j \to +\infty & \text{if } p < n \end{cases} \tag{1.3.15}$$

where r is a positive constant.

- *In the case $p = n$, for all $j \in \mathbb{N}$ and $\alpha > 0$ we define the function $g_j^\alpha : \mathbb{R}^m \to [0, +\infty)$ as*

$$g_j^\alpha(z) = \inf\Big\{\sum_{\xi\in I}\sum_{A\in R_1^\xi(Q(\alpha S_j))} f^\xi\big(\varepsilon_j^{-1}D_1^\xi v(A)\big)\varepsilon_j^n : \begin{array}{l} v(0)=0 \\ v = z \text{ on } \mathcal{S}_1([\alpha S_j - M], [\alpha S_j]) \end{array}\Big\}, \tag{1.3.16}$$

where $S_j = \varepsilon_j^{-1}|\log\varepsilon_j|^{(1-n)/n}$. Then, upon possibly passing to subsequences, there exists a function $\varphi : \mathbb{R}^m \to [0, +\infty)$ such that

$$\varphi(z) = \lim_{\alpha\to 0^+}\lim_{j\to+\infty} |\log\varepsilon_j|^{n-1} g_j^\alpha(z) \text{ for all } z \in \mathbb{R}^m. \tag{1.3.17}$$

- *In the case $p < n$, for all $j \in \mathbb{N}$ and $N > 0$ we define the function $\phi_j^N : \mathbb{R}^m \to [0, +\infty)$ as*

$$\phi_j^N(z) = \inf\Big\{\sum_{\xi\in I}\sum_{A\in R_1^\xi(Q(N))} f^\xi(\varepsilon_j^{-1}D_1^\xi v(A))\varepsilon_j^p : \ v(0)=0,\ v = z \text{ on } \mathcal{S}_1([N-M],[N]\Big\}. \tag{1.3.18}$$

Then, upon possibly passing to subsequences, there exists a function $\phi : \mathbb{R}^m \to [0, +\infty)$ such that

$$\phi(z) = \lim_{N\to+\infty}\lim_{j\to+\infty} \phi_j^N(z) \text{ for all } z \in \mathbb{R}^m. \tag{1.3.19}$$

- *Moreover, for all $j \in \mathbb{N}$ we consider the functional $F_{\varepsilon_j} : \mathcal{A}_{\varepsilon_j}(\Omega) \to [0, +\infty]$ defined by*

$$F_{\varepsilon_j}(u) = \begin{cases} \displaystyle\sum_{\xi\in I}\sum_{a\in R^\xi_{\varepsilon_j}(\Omega)} f^\xi(D^\xi_{\varepsilon_j}u(a))\varepsilon_j^n & \text{if } u = 0 \text{ on } \Omega_{\delta_j} \\ +\infty & \text{otherwise.} \end{cases} \tag{1.3.20}$$

Upon extracting a subsequence such that the function

$$\Phi : \mathbb{R}^m \to [0, +\infty), \qquad \Phi(z) = \begin{cases} \varphi(z) & \text{if } p = n \\ \phi(z) & \text{if } p < n \end{cases}$$

is well defined, the family (F_{ε_j}) Γ-converges in the $L^1(\Omega; \mathbb{R}^m)$-topology to the functional $F : L^1(\Omega; \mathbb{R}^m) \to [0, +\infty)$ given by

$$F(u) = \begin{cases} \displaystyle\int_\Omega f_0(Du)\,dx + r^{1-n}\int_\Omega \Phi(u)\,dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^m) \\ +\infty & \text{otherwise,} \end{cases} \tag{1.3.21}$$

where

$$f_0(A) = \lim_{h\to+\infty}\frac{1}{h^n}\min\Big\{\sum_{\xi\in I}\sum_{a\in R_1^\xi(Q(h))} f(D_1^\xi u(a)),\ u = Ax \text{ on } \mathcal{S}_1(h)\Big\}$$

for all $A \in \mathbb{M}^{m\times n}$.

1.3.1 More notation and preliminaries

It will be convenient to introduce some additional notation. Assume that all the conditions of Theorem 1.3.1 are satisfied. For all $j \in \mathbb{N}$ we set

$$\mathcal{F}_{\varepsilon_j}(u) = \sum_{\xi\in I}\sum_{a\in R^\xi_{\varepsilon_j}(\Omega)} f^\xi(D^\xi_{\varepsilon_j}u(a))\varepsilon_j^n. \tag{1.3.22}$$

Note that $\mathcal{F}_{\varepsilon_j}$ differs from F_{ε_j} since in the latter we add the constraint $u = 0$ on Ω_{δ_j}. Namely,

$$F_{\varepsilon_j}(u) = \begin{cases} \mathcal{F}_{\varepsilon_j}(u) & \text{if } u = 0 \text{ on } \Omega_{\delta_j} \\ +\infty & \text{otherwise.} \end{cases}$$

For all $D \subseteq \Omega$ we denote by $F_{\varepsilon_j}(u; D)$ and $\mathcal{F}_{\varepsilon_j}(u; D)$ the localized functionals

$$\mathcal{F}_{\varepsilon_j}(u; D) = \sum_{\xi \in I} \sum_{a \in R^{\xi}_{\varepsilon_j}(D)} f^{\xi}(D^{\xi}_{\varepsilon_j} u(a)) \varepsilon_j^n$$

and

$$F_{\varepsilon_j}(u; D) = \begin{cases} \mathcal{F}_{\varepsilon_j}(u; D) & \text{if } u = 0 \text{ on } \Omega_{\delta_j} \cap D \\ +\infty & \text{otherwise.} \end{cases}$$

Throughout the chapter we will use a homogenization result proved by Alicandro and Cicalese in [3, Theorem 4.1]. We recall it in the form we need for our purposes.

Proposition 1.3.2 *Let f^{ξ}, $\xi \in I$, satisfy the assumptions of Theorem 1.3.1. For all $\varepsilon > 0$ we define $\mathcal{F}_{\varepsilon} : \mathcal{A}_{\varepsilon}(\Omega) \to [0, +\infty)$ as*

$$\mathcal{F}_{\varepsilon}(u) = \sum_{\xi \in I} \sum_{a \in R^{\xi}_{\varepsilon}(\Omega)} f^{\xi}(D^{\xi}_{\varepsilon} u(a)) \varepsilon^n. \tag{1.3.23}$$

Then, $(\mathcal{F}_{\varepsilon})$ Γ-converges with respect to the $L^p(\Omega; \mathbb{R}^m)$-topology to the functional $\mathcal{F}_0 : L^p(\Omega; \mathbb{R}^m) \to [0, +\infty]$ defined as

$$\mathcal{F}_0(u) = \begin{cases} \displaystyle\int_{\Omega} f_0(Du) dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^m) \\ +\infty & \text{otherwise,} \end{cases} \tag{1.3.24}$$

where $f_0 : \mathcal{M}^{m \times n} \to [0, +\infty)$ is given by the homogenization formula

$$f_0(A) = \lim_{h \to +\infty} \frac{1}{h^n} \min \Big\{ \sum_{\xi \in I} \sum_{a \in R^{\xi}_1(Q(h))} f(D^{\xi}_1 u(a)), \ u = Ax \text{ on } \mathcal{S}_1(h) \Big\}. \tag{1.3.25}$$

Remark 1.3.3 (Finite range interactions) In order not to overburden the notation, in what follows we will focus on long but finite-range interactions: we will limit out attention to a set of functions f^{ξ} with $\xi \in I_M = \{\xi \in \mathbb{Z}^n \text{ and } -\xi <^l \xi\}$. This is not restrictive thanks to the general convergence result for unconstrained functionals by Alicandro and Cicalese, recalled in Proposition 1.3.2. When no confusion can arise, we will simply write I instead of I_M. Note that, under this simplifying assumption, condition (1.3.13) can be rewritten as follows: there exists a constant $c_2 > 0$ such that for all $\xi \in I$

$$\begin{aligned} & f^{\xi}(z) \leq c_2 |z|^n \text{ for all } z \in \mathbb{R}^m && \text{if } p = n \\ & f^{\xi}(z) \leq c_2(|z|^p + 1) \text{ for all } z \in \mathbb{R}^m && \text{if } p < n. \end{aligned} \tag{1.3.26}$$

Remark 1.3.4 We write down a simple inequality which will be useful in what follows. Let the assumptions of Theorem 1.3.1 be satisfied. Let $D \subseteq \Omega$. By the growth conditions on f^{ξ} we deduce that there exists a constant $c > 0$ (independent of ξ, j and D) such that

$$\mathcal{F}_{\varepsilon}(u; D) \leq c \sum_{\{a,b\} \in M_{\varepsilon}(D)} \varepsilon^{n-p} |u(a) - u(b)|^p \tag{1.3.27}$$

for all $u \in \mathcal{A}_{\varepsilon}(D)$.

Remark 1.3.5 In some of the proofs it will be convenient to identify each function $u \in \mathcal{A}_{\varepsilon}(\Omega)$ with a piecewise affine interpolation, rather than with a piecewise constant function as explained in Section 1.2. Using the construction developed by Alicandro and Cicalese in [4, Section 4.1], we can build an interpolating function $\tilde{u}$ which is piecewise affine on a triangulation of the lattice and satisfies the following property:

$$\sum_{\{a,b\} \in M_{\varepsilon}(\Omega)} |u(a) - u(b)|^p \varepsilon^{n-p} = \int_{\Omega} \|D\tilde{u}\|^p \, dx + o(1) \text{ as } \varepsilon \to 0. \tag{1.3.28}$$

1.4 Comparison with the continuous case

In this paragraph we point out the basic difference between the critical case and the noncritical one by analyzing the asymptotic behavior of the relevant family of minimum problems $\{m_T^d : T \in \mathbb{N}\}$, defined as

$$m_T^d = \inf\Big\{ \sum_{\{a,b\}\in M_1(Q(T))} |u(a)-u(b)|^p : \begin{array}{c} u \in \mathcal{A}_1(Q(T)), \\ u(0)=0,\ u=1 \text{ on } \mathcal{S}_1(T) \end{array} \Big\}. \tag{1.4.29}$$

Note that m_T^d is the simplest version of the minimum problems which appear in (1.3.16) and (1.3.18): we consider only nearest neighbors interactions, the test functions are scalar ($m = 1$) and $f^\xi(z) = |z|^p$ for all ξ. In the proof of Theorem 1.3.1 we will use a separation-of-scales procedure: a decoupling lemma (Lemma 1.6.1) will allow to analyze the single effect of each pinning sites independently. In the simplest case, the energy "concentrating close to the pinning sites" is exactly the one we minimize in (1.4.29).

In what follows, we will determine the asymptotic behavior of m_T^d in the critical case and the noncritical one (step **1** and **2** respectively).

1. Critical case $p = n$. In the case of the critical exponent, the sequence m_T^d has the same asymptotic behavior as its continuous analogue. In the continuous setting, we consider the minimum $m_{t,T}^c$

$$m_{t,T}^c = \min\Big\{ \int_{Q(T)} \|Du\|_p^p : \ u-1 \in W_0^{1,p}(Q_T),\ u=0 \text{ on } Q(t)\Big\}, \tag{1.4.30}$$

where $\|Du\|_p = (\sum_{i=1}^n |\partial u/\partial x_i|^p)^{1/p}$. This case has been studied in the framework of Γ-convergence in [57]. In particular, we know that the sequence $(m_{t,T}^c)$ has a logarithmic behavior as T goes to $+\infty$: there exists a positive constant l_n (independent of t) such that

$$\lim_{T\to+\infty} (\log T)^{n-1} m_{t,T}^c = l_n. \tag{1.4.31}$$

We recall that this convergence can be proved by an argument based on a *telescopic construction*, as in [57], Section 5. If in particular $p = n = 2$, then the $\|Du\|_2$ norm is the same as the Euclidean norm $|Du|$ and the constant l_2 equals ω_{n-1}. We notice that the minimum in (1.4.30) is scale-invariant: if we rescale our sets by a constant $\alpha > 0$, we get $m_{\alpha t,\alpha T}^c = m_{t,T}^c$. The aim of this paragraph is to show that the discrete infima m_T^d satisfy

$$\lim_{T\to+\infty} (\log T)^{n-1} m_T^d = \lim_{T\to+\infty} (\log T)^{n-1} m_{1,T}^c = l_n. \tag{1.4.32}$$

For notational simplicity it is convenient to introduce the discrete infima

$$m_{t,T}^d = \inf\Big\{ \sum_{\{A,B\}\in M_1(Q(T))} |u(A)-u(B)|^p : \begin{array}{c} u \in \mathcal{A}_1(Q(T)), \\ u=0 \text{ on } Q_1(t),\ u=1 \text{ on } \mathcal{S}_1(T) \end{array} \Big\}. \tag{1.4.33}$$

By a two-step argument we will prove that

$$\lim_{T\to+\infty} (\log T)^{n-1} m_{1,T}^d = \lim_{T\to+\infty} (\log T)^{n-1} m_{1,T}^c = l_n$$

and then we will show that $m_{1,T}^d = m_T^d + o(1)$ as $T \to +\infty$, which implies (1.4.32).

1.1 First of all, we can identify each test function $u \in \mathcal{A}_1(Q(T))$ in the definition of $m_{1,T}^d$ with its piecewise affine interpolation on the lattice $Q_1(T)$, denoted by $\tilde{u}$, as in Remark 1.3.5. Since $u = 0$ on $Q_1(1)$ and $u = 1$ on $\mathcal{S}_1(T)$, the interpolated function $\tilde{u}$ vanishes on the cube $Q(1)$ and is in the space $1 + W_0^{1,n}(Q(T))$. Then $\tilde{u}$ is a test function for $m_{1,T}^c$. There follows that

$$m_{1,T}^c \le m_{1,T}^d.$$

1.2 In this step we want to show that the converse inequality holds, up to an infinitesimal error. Let $T \in \mathbb{N}$. Due to scale-invariance, we have $m^c_{1,T-1} = m^c_{2,2T-2}$. Let $v \in \operatorname{argmin}\{m^c_{2,2T-2}\}$; i.e., $v \in 1 + W^{1,n}_0(Q(2T-2))$, $v = 0$ on $Q(2)$ and

$$E(v) := \int_{Q(2T-2)} \|Dv\|^n_n \, dx = m^c_{2,2T-2}.$$

By (1.4.31) we deduce that $m^c_{2,2T-2} = m^c_{2,2T} + o((\log T)^{1-n})$ as $T \to +\infty$, hence $E(v) = m^c_{2,2T} + o((\log T)^{1-n})$. For all fixed $x \in [0,1)^n$ we denote by L^x the lattice $L^x = (x+\mathbb{Z}^n) \cap Q(2T+2)$ and by v^x the discretization of v over L^x. By construction we have $v^x = 0$ on $Q_1(1;x)$ and $v^x = 1$ on $\mathcal{S}_1(2T;x)$. Moreover, we indicate by $E^x(v)$ the integral of v over the family of hyperplanes parallel to the coordinate hyperplanes and passing through the points of the lattice L^x:

$$E^x(v) = \sum_{i=1}^{n} \sum_{j=-2T}^{2T} \sum_{l \neq i} \int_{P_{ij}} \Big| \frac{\partial v}{\partial y_l} \Big|^n,$$

where $P_{ij} = je_i + \{y : y \cdot e_i = 0\}$. Now, we notice that by the definition of v^x we have

$$E^x(v) \geq \sum_{i=1}^{n} \sum_{j=-2T}^{2T} \sum_{l \neq i} \Big(\int_{P_{ij}} \Big| \frac{\partial v}{\partial x_l} \Big| \Big)^n \geq \sum_{\{a,b\} \in M_1(Q(2T))} |v^x(a+x) - v^x(b+x)|^n.$$

By Fubini's Theorem we have $E(v) = \int_{[0,1)^n} E^x(v)$. Then, there exists $\overline{x} \in [0,1)^n$ such that

$$E(v) \geq E^{\overline{x}}(v) \geq \sum_{\{a,b\} \in M_1(Q(2T))} |v^{\overline{x}}(a+\overline{x}) - v^{\overline{x}}(b+\overline{x})|^n \geq m^d_{1,2T}.$$

To sum up, we got

$$m^c_{2,2T-2} = m^c_{2,2T} + o((\log T)^{1-n}) = m^c_{1,T} + o((\log T)^{1-n}) \geq m^d_{1,2T} + o((\log T)^{1-n}). \quad (1.4.34)$$

Since the limit in (1.4.31) is independent of t, we have $m^c_{1,2T} = m^c_{1,T} + o((\log T)^{1-n})$. Plugging this equation into (1.4.34), we conclude that

$$m^d_{1,2T} \leq m^c_{1,2T} + o((\log T)^{1-n}).$$

Taking step **1.1** into consideration, we finally obtain

$$\lim_{T \to +\infty} (\log T)^{n-1} m^d_{1,T} = \lim_{T \to +\infty} (\log T)^{n-1} m^c_{1,T} = l_n,$$

as desired.

2. Noncritical case $p < n$. In the subcritical case, $p < n$, we do not have the same correspondence with the continuous setting. In this scenario, the infima m^d_T converge to a positive constant C_p which can be interpreted as the discrete p-capacity of a point in $\mathbb{Z}^n$: with an abuse of notation we write

$$C_p = \inf \Big\{ \sum_{\{a,b\} \in M_1(\mathbb{Z}^n} |u(a) - u(b)|^p : \begin{array}{c} u \in \mathcal{A}_1(\mathbb{Z}^n), \\ u(0) = 0, \ u = 1 \text{ on } \mathcal{S}_1(+\infty) \end{array} \Big\}. \quad (1.4.35)$$

In fact, by definition m^d_T is a decreasing sequence of positive numbers, hence it admits a limit $C_p \geq 0$. For $N \in \mathbb{N}$ sufficiently large, we consider a function $u \in \mathcal{A}_1(Q(N))$ such that $u(0) = 0$, $u = 1$ on $\mathcal{S}_1(N)$ and

$$\sum_{\{a,b\} \in M_1(Q(N))} |u(a) - u(b)|^p < C_p + \frac{1}{N}.$$

Now, two events can occur: either $u \neq 0$ in at least one point of $Q_1(1)$, or $u = 0$ on all the points of $Q_1(1)$. In the first case, the energy of u over $Q_1(N)$ must be greater than a positive constant α, given by the non-zero interaction we certainly have in $Q_1(1)$, and then $C_p + 1/N > \alpha$. By letting $N \to +\infty$ we get $C_p \geq \alpha > 0$. In the second case, since $u = 0$ on $Q_1(1)$, by Remark 1.3.5 we can identify it with a piecewise affine function $\tilde{u}$ such that $\tilde{u} = 0$ on $Q(1)$, $\tilde{u} = 1$ on $\partial Q(N)$ and

$$\int_{Q(N)} \|D\tilde{u}\|_p^p = \sum_{\{a,b\} \in M_1(Q(N))} |u(a) - u(b)|^p < C_p + \frac{1}{N}.$$

Now,

$$\int_{Q(N)} \|D\tilde{u}\|_p^p \geq c \inf \Big\{ \int_{Q(N)} |Dv|^p \, dx : \; v = 0 \text{ on } Q(1) \Big\} \geq c \operatorname{Cap}_p(Q(1); \mathbb{R}^n),$$

where $\operatorname{Cap}_p(Q(1); \mathbb{R}^n) > 0$ is the p-capacity of the cube $Q(1)$ in $\mathbb{R}^n$. By letting $N \to +\infty$, we conclude that C_p is strictly positive.

1.5 Building blocks of the Γ-limit

In this section we study the auxiliary functions (φ_j^α) and (ϕ_j^N) we introduced in the statement of Theorem 1.3.1. We show that these families converge to some functions φ and ϕ respectively, upon possibly passing to subsequences. The limit densities φ and ϕ will reflect the contribution of the pinning sites in the Γ-limit.

1.5.1 Critical case

In this paragraph we study some properties of the auxiliary functions g_j^α we introduced in (1.3.16) for the critical case. Let all the assumptions of Theorem 1.3.1 be satisfied. It is convenient to set $T_j = \varepsilon_j^{-1}$ and $S_j = T_j (\log T_j)^{(1-n)/n}$; by construction T_j, S_j tend to $+\infty$ as $j \to +\infty$. For fixed $\alpha > 0$, $j \in \mathbb{N}$ we defined $g_j^\alpha : \mathbb{R}^m \to [0, +\infty)$ as

$$g_j^\alpha(z) = \inf \Big\{ \sum_{\xi \in I} \sum_{A \in R_1^\xi(Q(\alpha S_j))} T_j^{-n} f^\xi\big(T_j D_1^\xi v(A)\big) : \; \begin{array}{l} v(0) = 0 \\ v = z \text{ on } \mathcal{S}_1([\alpha S_j - M], [\alpha S_j]) \end{array} \Big\}.$$

Now, $\varphi_j^\alpha : \mathbb{R}^m \to [0, +\infty)$ is obtained multiplying g_j^α by a scaling factor:

$$\varphi_j^\alpha(z) = (\log T_j)^{n-1} g_j^\alpha(z). \tag{1.5.36}$$

We will apply Ascoli-Arzelà's Theorem to the family (φ_j^α) in order to prove the following result:

Proposition 1.5.1 *For all $\alpha > 0$ there exists a function $\varphi^\alpha : \mathbb{R}^m \to [0, +\infty)$ such that φ_j^α tends to φ^α as $j \to +\infty$ upon passing to subsequences, uniformly on the compact sets of $\mathbb{R}^m$.*

Proof. Firstly, we will show that the functions φ_j^α satisfy an equi-boundedness condition and then that they are locally equi-Lipschitz continuous (steps 1 and 2 respectively).

1 In this paragraph we will show that there exist two constants $C_1, C_2 > 0$ such that for all $j \in \mathbb{N}$ and $\alpha > 0$ the functions φ_j^α verify a growth condition of the form

$$C_1 |z|^n \leq \varphi_j^\alpha(z) \leq C_2 |z|^n \quad \text{for all } z \in \mathbb{R}^m. \tag{1.5.37}$$

By (1.3.26) we have

$$\varphi_j^\alpha(T_j D_1^\xi v(A)) \leq c_2 T_j^n |D_1^\xi v(A)|^n$$

for all test functions v in the infimum $g_j^\alpha(z)$; i.e., $v(0)=0$ and $v=z$ on $\mathcal{S}_1([\alpha S_j - M],[\alpha S_j])$. There follows that

$$g_j^\alpha(z) \le \inf\Big\{ \sum_{\xi\in I}\sum_{A\in R_1^\xi(Q(\alpha S_j))} c_2 T_j^{-n}|D_1^\xi v(A)|^n T_j^n : \begin{array}{l} v(0)=0 \\ v=z \text{ on } \mathcal{S}_1([\alpha S_j - M],[\alpha S_j]) \end{array} \Big\}.$$

By arguing as in Remark 1.3.4 we deduce that

$$\begin{aligned} g_j^\alpha(z) &\le c \inf\Big\{ \sum_{\{A,B\}\in M_1(Q(\alpha S_j - M))} |v(A)-v(B)|^n : \begin{array}{l} v(0)=0 \\ v=z \text{ on } \mathcal{S}_1([\alpha S_j - M]) \end{array} \Big\} \\ &\le c|z|^n \inf\Big\{ \sum_{\{A,B\}\in M_1(Q(\alpha S_j - M))} |v(A)-v(B)|^n : \begin{array}{l} v(0)=0 \\ v=e_1 \text{ on } \mathcal{S}_1([\alpha S_j - M]) \end{array} \Big\}. \end{aligned}$$

If we multiply both sides by $(\log T_j)^{n-1}$ we get

$$\begin{aligned} \varphi_j^\alpha(z) &\le c|z|^n (\log T_j)^{n-1} \\ &\quad\times \inf\Big\{ \sum_{\{A,B\}\in M_1(Q(\alpha S_j - M))} |v(A)-v(B)|^n : \begin{array}{l} v(0)=0 \\ v=e_1 \text{ on } \mathcal{S}_1([\alpha S_j - M]) \end{array} \Big\}. \end{aligned}$$

Taking into account the results of Section 1.4 we deduce that

$$\varphi_j^\alpha(z) \le c|z|^n (l_n + o(1)) \text{ as } j\to+\infty.$$

Hence there exists a positive constant C_2 such that

$$\varphi_j^\alpha(z) \le C_2|z|^n$$

for all $z\in\mathbb{R}^m$, $\alpha>0$ and $j\in\mathbb{N}$ large enough, which proves one of the inequalities in 1.5.37.

Moreover, we can show that the functions φ_j^α satisfy a growth condition of order n from below. By assumption (1.3.12) we get

$$\begin{aligned} g_j^\alpha(z) &\ge \inf\Big\{ \sum_{i=1}^n \sum_{A\in R_1^{e_i}(Q(\alpha S_j))} T_j^{-n} f^{e_i}(T_j D_1^{e_i} v(A)) : \begin{array}{l} v(0)=0 \\ v=z \text{ on } \mathcal{S}_1([\alpha S_j - M],[\alpha S_j]) \end{array} \Big\} \\ &\ge c_1 \inf\Big\{ \sum_{\{A,B\}\in M_1(Q(\alpha S_j - M))} |v(A)-v(B)|^n : \begin{array}{l} v(0)=0 \\ v=z \text{ on } \mathcal{S}_1([\alpha S_j - M]) \end{array} \Big\}. \end{aligned}$$

Arguing as for the upper bound, we conclude that there exists a positive constant C_1 such that

$$\varphi_j^\alpha(z) \ge C_1|z|^n$$

for all $z\in\mathbb{R}^m$, $\alpha>0$ and $j\in\mathbb{N}$ large enough, as desired.

2 In this paragraph we will prove that for fixed $\alpha>0$ the family (φ_j^α) is equi-Lipschitz continuous on the compact subsets of $\mathbb{R}^m$. We first fix a compact set $K\subset\mathbb{R}^m\setminus\{0\}$ and we denote by L a positive constant such that $K\subseteq B(L)$, where $B(L)$ is the m-dimensional ball of center 0 and radius L.

2.1 Let $z,z'\in K$ be such that $z'=kz$ for some $k\ne0$. Having fixed $\eta>0$, we consider a function $v\in\mathcal{A}_1(Q(\alpha S_j);\mathbb{R}^m)$ such that $v(0)=0$, $v=z$ on $\mathcal{S}_1([\alpha S_j - M],[\alpha S_j])$ and

$$(\log T_j)^{n-1}\sum_{\xi\in I}\sum_{A\in R_1^\xi(Q(\alpha S_j))} T_j^{-n} f^\xi(T_j D_1^\xi v(A)) < \varphi_j^\alpha(z) + \eta|z|^n. \qquad (1.5.38)$$

We define $w \in \mathcal{A}_1(Q(\alpha S_j);\mathbb{R}^m)$ as $w = kv$. By construction w is a test function for the infimum in $g_j^\alpha(z')$. For fixed $A \in R_1^\xi(Q(\alpha S_j))$, assumption (1.3.14) implies that

$$\begin{aligned}|f^\xi(T_j D_1^\xi v(A)) - f^\xi(T_j D_1^\xi w(A))| &= |f^\xi(T_j D_1^\xi v(A)) - f^\xi(kT_j D_1^\xi v(A))| \\ &\le c_3|T_j D_1^\xi v(A) - kT_j D_1^\xi v(A)|(|T_j D_1^\xi v(A)|^{n-1} + |kT_j D_1^\xi v(A)|^{n-1}) \\ &\le c_3|1-k|(1+|k|^{n-1})|T_j D_1^\xi v(A)|^n.\end{aligned}$$

If we multiply both sides by $(\log T_j)^{n-1}T_j^{-n}$ and sum up over $\xi \in I$ and $A \in R_1^\xi(Q(\alpha S_j))$, we get

$$\begin{aligned}&(\log T_j)^{n-1}\sum_{\xi\in I}\sum_{A\in R_1^\xi(Q(\alpha S_j))} T_j^{-n} f^\xi(T_j D_1^\xi w(A)) \\ &\le (\log T_j)^{n-1}\sum_{\xi\in I}\sum_{A\in R_1^\xi(Q(\alpha S_j))} T_j^{-n} f^\xi(T_j D_1^\xi v(A)) \\ &\quad + c|k-1|(1+|k|^{n-1})(\log T_j)^{n-1}\sum_{\xi\in I}\sum_{A\in R_1^\xi(Q(\alpha S_j))}|D_1^\xi v(A)|^n \\ &\le \varphi_j^\alpha(z) + \eta|z|^n + c|k-1|(1+|k|^{n-1})(\log T_j)^{n-1}\sum_{i=1}^n\sum_{A\in R_1^{e_i}(Q(\alpha S_j))}|D_1^{e_i} v(A)|^n.\end{aligned}$$

By (1.3.12) we have

$$\begin{aligned}&(\log T_j)^{n-1}\sum_{\xi\in I}\sum_{A\in R_1^\xi(Q(\alpha S_j))} T_j^{-n} f^\xi(T_j D_1^\xi w(A)) \\ &\le \varphi_j^\alpha(z) + \eta|z|^n + c|k-1|(1+|k|^{n-1})(\log T_j)^{n-1}\sum_{i=1}^n\sum_{A\in R_1^{e_i}(Q(\alpha S_j))} T_j^{-n} f^{e_i}(T_j D_1^{e_i} v(A)) \\ &\le \varphi_j^\alpha(z) + \eta|z|^n + c|k-1|(1+|k|^{n-1})(\log T_j)^{n-1}\sum_{\xi\in I}\sum_{A\in R_1^\xi(Q(\alpha S_j))} T_j^{-n} f^\xi(T_j D_1^\xi v(A)) \\ &\le \varphi_j^\alpha(z) + \eta|z|^n + c|k-1|(1+|k|^{n-1})(\varphi_j^\alpha(z) + \eta|z|^n).\end{aligned}$$

Since w is a test function for the infimum $g_j^\alpha(z')$ we deduce that

$$\varphi_j^\alpha(z') \le \varphi_j^\alpha(z) + \eta|z|^n + c|k-1|(|k|^{n-1}+1)(\varphi_j^\alpha(z) + \eta|z|^n).$$

Now, by the growth condition (1.5.37) we get

$$\begin{aligned}\varphi_j^\alpha(z') &\le \varphi_j^\alpha(z) + \eta|z|^n + c|k-1|(|k|^{n-1}+1)(C_2|z|^n + \eta|z|^n) \\ &\le \varphi_j^\alpha(z) + c|kz - z|(|kz|^{n-1} + |z|^{n-1}) + \eta|z|^n + \eta|kz - z|(|z|^{n-1} + |kz|^{n-1}) \\ &\le \varphi_j^\alpha(z) + c|z' - z|(|z'|^{n-1} + |z|^{n-1}) + \eta|z|^n + \eta(|z'| + |z|)(|z|^{n-1} + |z'|^{n-1}).\end{aligned}$$

Since $z, z' \in K$ we have $|z| \le L$ and $|z'| \le L$. There follows that

$$\varphi_j^\alpha(z') \le \varphi_j^\alpha(z) + c|z' - z|(|z'|^{n-1} + |z|^{n-1}) + c\eta L^n.$$

By the arbitrariness of η we get

$$\varphi_j^\alpha(z') \le \varphi_j^\alpha(z) + c|z' - z|(|z'|^{n-1} + |z|^{n-1}).$$

By symmetry reasons we can conclude that

$$|\varphi_j^\alpha(z') - \varphi_j^\alpha(z)| \le c|z - z'|(|z|^{n-1} + |z'|^{n-1}). \tag{1.5.39}$$

Note that the constant c above is independent of both j and α.

2.2 Let $z, z' \in K$ be such that $z' = \mathcal{R}z$ for some $\mathcal{R} \in SO(m)$. Arguing similarly to paragraph **2.1**, we get

$$\varphi_j^\alpha(z') \le \varphi_j^\alpha(z) + c|z' - z|(|z'|^{n-1} + |z|^{n-1}).$$

By a symmetric argument, we conclude that

$$|\varphi_j^\alpha(z') - \varphi_j^\alpha(z)| \le c|z - z'|(|z|^{n-1} + |z'|^{n-1}), \tag{1.5.40}$$

for some positive constant c independent of j and α.

2.3 We fix $z, z' \in K$ and notice that we can go from z to z' through the composition of a homothety and a rotation. By combining (1.5.39) and (1.5.40) we deduce that there exists a constant c, independent of j and α, such that

$$|\varphi_j^\alpha(z') - \varphi_j^\alpha(z)| \le c|z - z'|(|z|^{n-1} + |z'|^{n-1}) \tag{1.5.41}$$

for all $z, z' \in K$.
By (1.5.41) we can infere that the sequence (φ_j^α) satisfies an equi-Lipschitz condition on all compact subsets of $\mathbb{R}^m$.

In conclusion, we can apply Ascoli-Arzelà's Theorem: for all $\alpha > 0$ there exist a subsequence $\varphi_{j_k}^\alpha$ and a function $\varphi^\alpha : \mathbb{R}^m \to [0, +\infty)$ such that

$$\varphi^\alpha(z) = \lim_{k \to +\infty} \varphi_{j_k}^\alpha(z), \tag{1.5.42}$$

uniformly on the compact subsets of $\mathbb{R}^m$. ■

Remark 1.5.2 By construction $\varphi^\alpha(0) = \varphi_j^\alpha(0) = 0$ for all α, j. Furthermore, by passing to the limit as $j \to +\infty$ in (1.5.41) we deduce that φ^α satisfies

$$|\varphi^\alpha(z) - \varphi^\alpha(z')| \le c|z - z'|(|z|^{n-1} + |z'|^{n-1}) \text{ for all } z, z' \in \mathbb{R}^m, \tag{1.5.43}$$

for some constant $c > 0$.

1.5.2 Noncritical case

In this paragraph we analyze some properties of the functions ϕ_j^N we introduced in Theorem 1.3.1 for the noncritical case. For all $N > 0$, $j \in \mathbb{N}$ and $\xi \in I$ we define $h_j^\xi : \mathbb{R}^m \to [0, +\infty)$ as

$$h_j^\xi(z) = T_j^{-p} f^\xi(T_j z), \qquad \text{for all } z \in \mathbb{R}^m.$$

By assumptions (1.3.12)-(1.3.14) we deduce that h_j^ξ is locally Lipschitz-continuous and satisfies the following condition:

$$|h_j^\xi(z) - h_j^\xi(w)| \le c(T_j^{-p+1} + |z|^{p-1} + |w|^{p-1})|z - w| \quad \text{for all } z, w \in \mathbb{R}^m, \tag{1.5.44}$$

where the positive constant c is independent of j. Therefore, for all $\xi \in I$ there exists a function $h^\xi : \mathbb{R}^m \to [0, +\infty)$ such that h_j^ξ converges pointwise to h^ξ, upon possibly passing to subsequences. We recall that for $N, j \in \mathbb{N}$ the function $\phi_j^N : \mathbb{R}^m \to [0, +\infty)$ is defined as

$$\phi_j^N(z) = \inf\Big\{ \sum_{\xi \in I} \sum_{A \in R_1^\xi(Q(N))} h_j^\xi(D_1^\xi v(A)) : \begin{array}{l} v(0) = 0 \\ v = z \text{ on } \mathcal{S}_1([N-M],[N]) \end{array} \Big\}.$$

Moreover, for all $N \in \mathbb{N}$ we can define $\phi_N : \mathbb{R}^m \to [0, +\infty)$ as

$$\phi^N(z) = \inf\Big\{ \sum_{\xi \in I} \sum_{A \in R_1^\xi(Q(N))} h^\xi(D_1^\xi v(A)) : \begin{array}{l} v(0) = 0 \\ v = z \text{ on } \mathcal{S}_1([N-M],[N]) \end{array} \Big\}. \tag{1.5.45}$$

Finally, we set

$$\phi(z) = \lim_{N\to+\infty} \inf\Big\{ \sum_{\xi\in I} \sum_{A\in R_1^\xi(Q(N))} h^\xi(D_1^\xi v(A)) : \begin{array}{l} v(0)=0 \\ v = z \text{ on } \mathcal{S}_1([N-M],[N]) \end{array} \Big\}.$$

Note that the limit over N in the definition of ϕ coincides with the infimum over $N \in \mathbb{N}$. Let us deduce some convergence properties of the functions above.

1 By the pointwise convergence of h_j^ξ to h^ξ as $j \to +\infty$, we deduce that, for fixed $N \in \mathbb{N}$, ϕ_j^N converges pointwise to ϕ^N up to subsequences.

2 For all $N \in \mathbb{N}$ and $\eta > 0$ there exists a positive constant $c_{N,\eta} = c(N)\eta^p$ such that

$$\begin{aligned} |\phi_j^N(z) - \phi_j^N(w)| \quad &\leq \quad c_{N,\eta}\delta_j^{n(p-1)/(n-p)}|z-w|(1+|w|^{p-1}+|z|^{p-1}) \\ &\qquad +c|z-w|(|w|^{p-1}+|z|^{p-1}) \end{aligned} \tag{1.5.46}$$

for $|z|, |w| > \eta$, for all $j \in \mathbb{N}$. Taking into account (1.5.44) and the growth conditions (1.3.12)-(1.3.26), we can prove this inequality by slightly modifying the argument we followed in the critical case. For fixed N, (1.5.46) corresponds to a Lipschitz condition on the compact subsets of $\mathbb{R}^m \setminus \{0\}$, uniformly on the index j.

3 For all $N \in \mathbb{N}$ there exists a positive constant c_N such that

$$\phi_j^N(z) \leq c_N T_j^{-p} + c|z|^p \tag{1.5.47}$$

for all $z \in \mathbb{R}^m$, $j \in \mathbb{N}$. This property follows from the growth condition (1.3.26) and a comparison with the case $f^\xi(z) = |z|^p$. Note that for fixed N (1.5.47) is an equi-boundedness condition on $(\phi_j^N)_j$.

4 By (1.5.46) and (1.5.47) we can apply Ascoli-Arzelà's Theorem to the family of functions (ϕ_j^N), where N is fixed. We deduce that the convergence of ϕ_j^N to ϕ^N is uniform on the compact subsets of $\mathbb{R}^m \setminus \{0\}$, upon possibly passing to subsequences.

5 Letting $j \to +\infty$ in (1.5.47) we obtain $\phi^N(z) \leq c|z|^p$. By the growth condition from below (1.3.12), we deduce that ϕ^N satisfies the following inequality:

$$c_1 c|z|^p \leq \phi^N(z) \leq c_2 c|z|^p \text{ for all } z \in \mathbb{R}^m. \tag{1.5.48}$$

6 Arguing as in **1**, for fixed $\eta > 0$ we get a Lipschitz condition for ϕ^N in the form

$$|\phi^N(z) - \phi^N(w)| \leq c(\eta^p + |z-w|(|w|^{p-1}+|z|^{p-1})) \text{ for all } z, w \in \mathbb{R}^m. \tag{1.5.49}$$

7 By applying Ascoli-Arzelà's Theorem to (ϕ^N), we deduce that the convergence of ϕ^N to ϕ is not only pointwise but also uniform on the compact subsets of $\mathbb{R}^m$, upon passing to subsequences.

1.6 Two technical lemmas

In this section we will prove two technical lemmas which will be used in the proof of Theorem 1.3.1. The first one is a "decoupling lemma", in the spirit of [11, Lemma 3.1]. Unlike the case of periodically perforated domains, we are dealing with non-local functionals, due to presence of long-range interactions. As a consequence, the "separation of scales" procedure requires some extra care. The second lemma describes how to recombine the decoupled energies to obtain the extra term of the Γ-limit. We will prove it in a general form, which comprises both the critical and the noncritical case.

Lemma 1.6.1 *Let (u_j) be a sequence such that $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$ and $u_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$ for some $u \in W^{1,p}(\Omega;\mathbb{R}^m)$. We assume that*

$$\sup_j \sum_{\xi\in I} \sum_{a\in R^{\xi}_{\varepsilon_j}(\Omega)} \varepsilon_j^n f^{\xi}(D^{\xi}_{\varepsilon_j} u_j(a)) < +\infty. \tag{1.6.50}$$

Let (ρ_j) be a sequence of the form $\rho_j = \beta\delta_j$, with $\beta < 1/2$. We denote by Z_j the set of indices $Z_j = \{i \in \mathbb{Z}^n : \ dist(i\delta_j, \partial\Omega) > \delta_j\}$. Let $k \in \mathbb{N}$ be fixed. Then for all $i \in Z_j$ there exists $k_i \in \{0,\dots,k-1\}$ such that, having set

$$C^i_j = Q_{\varepsilon_j}\Big(\Big[\frac{\rho_j}{\varepsilon_j}2^{-k_i}\Big]\varepsilon; i\delta_j\Big) \setminus Q_{\varepsilon_j}\Big(\Big[\frac{\rho_j}{\varepsilon_j}2^{-k_i-1}\Big]\varepsilon; i\delta_j\Big), \tag{1.6.51}$$

$$\rho^i_j = \Big[\frac{3}{4}\frac{\rho_j}{\varepsilon_j}2^{-k_i}\Big]\varepsilon_j, \tag{1.6.52}$$

$$u^i_j = \frac{1}{\sharp C^i_j}\sum_{a\in C^i_j} u_j(a), \tag{1.6.53}$$

there exists a sequence $w_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$ such that $w_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$ and

$$w_j(a) = u^i_j \text{ for all } a \in \mathcal{S}_{\varepsilon_j}(\rho^i_j; i\delta_j), \tag{1.6.54}$$

$$w_j(a) = u_j(a) \text{ for all } a \in \Omega_j \setminus \bigcup_{i\in Z_j} C^i_j, \tag{1.6.55}$$

$$\Big|\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(\Omega)} \varepsilon_j^n\big(f^{\xi}\big(D^{\xi}_{\varepsilon_j}u_j(a)\big) - f^{\xi}\big(D^{\xi}_{\varepsilon_j}w_j(a)\big)\big)\Big| \le \frac{c}{k}. \tag{1.6.56}$$

Proof. We fix $i \in Z_j$ and $h \in \{0,\dots,k-1\}$. We set:

$$C^{h,i}_j = Q_{\varepsilon_j}\Big(\Big[\frac{\rho_j}{\varepsilon_j}2^{-h}\Big]\varepsilon_j; i\delta_j\Big) \setminus Q_{\varepsilon_j}\Big(\Big[\frac{\rho_j}{\varepsilon_j}2^{-h-1}\Big]\varepsilon_j; i\delta_j\Big),$$
$$\rho^{h,i}_j = \Big[\frac{3}{4}\frac{\rho_j}{\varepsilon_j}2^{-h}\Big]\varepsilon_j,$$
$$u^{h,i}_j = \frac{1}{\sharp C^{h,i}_j}\sum_{a\in C^{h,i}_j} u_j(a).$$

We denote by $C^{h,i}_{j,M}$ the following subset of $C^{h,i}_j$:

$$C^{h,i}_{j,M} = Q_{\varepsilon_j}\Big(\Big[\frac{\rho_j}{\varepsilon_j}2^{-h}\Big]\varepsilon_j - M\varepsilon_j; i\delta_j\Big) \setminus Q_{\varepsilon_j}\Big(\Big[\frac{\rho_j}{\varepsilon_j}2^{-h-1}\Big]\varepsilon_j + M\varepsilon_j; i\delta_j\Big).$$

Let $\phi^{h,i}_j \in C^\infty_0(\mathbf{C}^{h,i}_{j,M})$ be such that $\phi^{h,i}_j = 1$ on $\partial(i\delta_j + [-\rho^{h,i}_j, \rho^{h,i}_j]^n)$ and $\big|\nabla\phi^{h,i}_j\big| \le c(\rho^{h,i}_j)^{-1}$. For all $a \in C^{h,i}_j$ we set

$$w^{h,i}_j(a) := \phi^{h,i}_j(a)u^{h,i}_j + \big(1 - \phi^{h,i}_j(a)\big)u_j(a). \tag{1.6.57}$$

Note that for all $a \in C^{h,i}_j \setminus C^{h,i}_{j,M}$ we have $w^{h,i}_j(a) = u_j(a)$. Now,

$$\begin{aligned} F_{\varepsilon_j}(w^{h,i}_j; C^{h,i}_j) &= \sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)} \varepsilon_j^n f^{\xi}(D^{\xi}_{\varepsilon_j} w^{h,i}_j(a)) \\ &\le c\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)} \varepsilon_j^n(|D^{\xi}_{\varepsilon_j} w^{h,i}_j(a)|^p + 1) \\ &\le c\sum_{\{a,b\}\in M_{\varepsilon_j}(C^{h,i}_j)} \varepsilon_j^n\Big|\frac{w^{h,i}_j(a) - w^{h,i}_j(b)}{\varepsilon_j}\Big|^p + c|\mathbf{C}^{h,i}_j|. \end{aligned} \tag{1.6.58}$$

Note that the last estimate above follows from Remark 1.3.4. Let $\{a,b\}\in M_{\varepsilon_j}(C_j^{h,i})$. Then by construction

$$\frac{w_j^{h,i}(a)-w_j^{h,i}(b)}{\varepsilon_j}=\big(u_j^{h,i}-u_j(b)\big)\frac{\phi_j^{h,i}(a)-\phi_j^{h,i}(b)}{\varepsilon_j}+\big(1-\phi_{\varepsilon_j}^{h,i}(a)\big)\frac{u_j(a)-u_j(b)}{\varepsilon_j}.$$

There follows that

$$\begin{aligned}\Big|\frac{w_j^{h,i}(a)-w_j^{h,i}(b)}{\varepsilon_j}\Big|^p &\leq c|u_j^{h,i}-u_j(b)|^p\Big|\frac{\phi_j^{h,i}(a)-\phi_j^{h,i}(b)}{\varepsilon_j}\Big|^p\\ &\quad +c|1-\phi_j^{h,i}(a)|^p\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\\ &\leq c|u_j^{h,i}-u_j(b)|^p\Big|\frac{\phi_j^{h,i}(a)-\phi_j^{h,i}(b)}{\varepsilon_j}\Big|^p+c\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p.\end{aligned}$$

We want to estimate the term $\big|\phi_j^{h,i}(a)-\phi_j^{h,i}(b)\big|^p\varepsilon_j^{-p}$. Since a,b are nearest neighbors, then $a=b+\varepsilon_j e_l$ for some $l\in\{1,\dots,n\}$. We have

$$\begin{aligned}\Big|\frac{\phi_j^{h,i}(a)-\phi_j^{h,i}(b)}{\varepsilon_j}\Big|^p &= \Big|\int_0^1\frac{\partial}{\partial x_l}\big(\phi_j^{h,i}(a+(1-s)\varepsilon_j e_l)\big)\,ds\Big|^p\\ &\leq \int_0^1\Big|\frac{\partial}{\partial x_l}\big(\phi_j^{h,i}(a+(1-s)\varepsilon_j e_l)\big)\Big|^p\,ds\\ &\leq c\big|\nabla\phi_j^{h,i}\big|_\infty^p\leq c(\rho_j^{h,i})^{-p}.\end{aligned}$$

Summing up over $\{a,b\}\in M_{\varepsilon_j}(C_j^{h,i})$, we get

$$\begin{aligned}&\sum_{\{a,b\}\in M_{\varepsilon_j}(C_j^{h,i})}\Big|\frac{w_j^{h,i}(a)-w_j^{h,i}(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n\\ &\leq c\sum_{\{a,b\}\in M_{\varepsilon_j}(C_j^{h,i})}|u_j^{h,i}-u_j(b)|^p\Big|\frac{\phi_j^{h,i}(a)-\phi_j^{h,i}(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n+c\sum_{\{a,b\}\in M_{\varepsilon_j}(C_j^{h,i})}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n\\ &\leq\frac{c}{(\rho_j^{h,i})^p}\sum_{\{a,b\}\in M_{\varepsilon_j}(C_j^{h,i})}|u_j^{h,i}-u_j(b)|^p\varepsilon_j^n+c\sum_{\{a,b\}\in M_{\varepsilon_j}(C_j^{h,i})}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n.\end{aligned}$$

By Lemma 1.10.2 (a discrete version of Poincaré's inequality), we have

$$\sum_{\{a,b\}\in M_{\varepsilon_j}(C_j^{h,i})}|u_j^{h,i}-u_j(b)|^p\varepsilon_j^n\leq C(\rho_j^{h,i})^p\sum_{\{a,b\}\in M_{\varepsilon_j}(C_j^{h,i})}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n,$$

hence

$$\sum_{\{a,b\}\in M_{\varepsilon_j}(C_j^{h,i})}\Big|\frac{w_j^{h,i}(a)-w_j^{h,i}(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n\leq c\sum_{\{a,b\}\in M_{\varepsilon_j}(C_j^{h,i})}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n. \qquad (1.6.59)$$

By (1.6.58) we get

$$\sum_{\xi\in I}\sum_{a\in R_{\varepsilon_j}^{\xi}(C_j^{h,i})}\varepsilon_j^n f^{\xi}(D_{\varepsilon_j}^{\xi}w_j^{h,i}(a))\leq c\sum_{\{a,b\}\in M_{\varepsilon_j}(C_j^{h,i})}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n+c|\mathbf{C}_j^{h,i}|. \qquad (1.6.60)$$

Now, by (1.6.60) and Remark 1.3.4 we get

$$
\begin{aligned}
&\Big|\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)}\varepsilon_j^n\big(f^\xi(D^\xi_{\varepsilon_j}w^{h,i}_j(a))-f^\xi(D^\xi_{\varepsilon_j}u_j(a))\big)\Big|\\
&\quad\le\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)}\varepsilon_j^n f^\xi(D^\xi_{\varepsilon_j}w^{h,i}_j(a))+\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)}\varepsilon_j^n f^\xi(D^\xi_{\varepsilon_j}u_j(a))\\
&\quad\le c\sum_{\{a,b\}\in M_{\varepsilon_j}(C^{h,i}_j)}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n+c|\mathbf{C}^{h,i}_j|.
\end{aligned}
$$

Summing up over $h\in\{0,1,\dots,k-1\}$ we obtain

$$
\begin{aligned}
&\sum_{h=0}^{k-1}\Big|\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{h,i}_j)}\varepsilon_j^n\big(f^\xi(D^\xi_{\varepsilon_j}w^{h,i}_j(a))-f^\xi(D^\xi_{\varepsilon_j}u_j(a))\big)\Big|\\
&\quad\le c\sum_{h=0}^{k-1}\sum_{\{a,b\}\in M_{\varepsilon_j}(C^{h,i}_j)}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n+c\sum_{h=0}^{k-1}|\mathbf{C}^{h,i}_j|\\
&\quad\le c\sum_{\{a,b\}\in M_{\varepsilon_j}(Q(\rho_j;i\delta_j))}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n+c|\mathbf{Q}(\rho_j;i\delta_j)|.
\end{aligned}
$$

Therefore there exists $k_i\in\{0,1,\dots,k-1\}$ such that

$$
\begin{aligned}
&\Big|\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^{k_i,i}_j)}\varepsilon_j^n\big(f^\xi(D^\xi_{\varepsilon_j}w^{h,i}_j(a))-f^\xi(D^\xi_{\varepsilon_j}u_j(a))\big)\Big|\\
&\quad\le\frac{c}{k}\sum_{\{a,b\}\in M_{\varepsilon_j}(Q(\rho_j;i\delta_j))}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n+\frac{c}{k}|\mathbf{Q}(\rho_j;i\delta_j)|.
\end{aligned}\tag{1.6.61}
$$

With this choice of k_i for all $i\in Z_j$, conditions (1.6.54)-(1.6.56) are satisfied by picking $h=k_i$ in the definitions above; i.e.,

$$
\begin{aligned}
&C^i_j=C^{k_i,i}_j,\qquad u^i_j=u^{k_i,i}_j,\qquad \rho^i_j=\rho^{k_i,i}_j,\\
&\text{and}\quad w_j(a)=\begin{cases}u^i_j\phi^{i,k_i}_j(a)+\big(1-\phi^{i,k_i}_j(a)\big)u_j(a) & \text{for } a\in C^i_j,\ i\in Z_j,\\ u_j(a) & \text{otherwise.}\end{cases}
\end{aligned}\tag{1.6.62}
$$

In fact by (1.6.61), (1.6.62) and the fact that $u_j=w_j$ on $\Omega_j\setminus\bigcup_{i\in Z_j}C^{k_i,i}_{j,M}$ we get:

$$
\begin{aligned}
&\Big|\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(\Omega)}\varepsilon_j^n\big(f^\xi(D^\xi_{\varepsilon_j}w_j(a))-f^\xi(D^\xi_{\varepsilon_j}u_j(a))\big)\Big|\\
&\quad\le\sum_{i\in Z_j}\Big|\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^i_j)}\varepsilon_j^n\big(f^\xi(D^\xi_{\varepsilon_j}w_j(a))-f^\xi(D^\xi_{\varepsilon_j}u_j(a))\big)\Big|\\
&\quad\le\frac{c}{k}\sum_{i\in Z_j}\Big(\sum_{\{a,b\}\in M_{\varepsilon_j}(Q(\rho_j;i\delta_j))}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n+|\mathbf{Q}(\rho_j;i\delta_j)|\Big)\\
&\quad\le\frac{c}{k}\Big(\sum_{\{a,b\}\in M_{\varepsilon_j}(\Omega)}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n+|\Omega|\Big)\le\frac{c}{k},
\end{aligned}
$$

where the latter inequality follows from (1.6.50). Finally, we prove that $w_j\to u$ in $L^1(\Omega;\mathbb{R}^m)$. By construction

$$
\int_\Omega|w_j-u|\,dx=\int_{\Omega\setminus\bigcup_{i\in Z_j}\mathbf{C}^i_j}|u_j-u|\,dx+\sum_{i\in Z_j}\int_{\mathbf{C}^i_j}|w_j-u|\,dx.\tag{1.6.63}
$$

Now, the first term in (1.6.63) is infinitesimal:

$$\int_{\Omega\setminus\bigcup_{i\in Z_j}\mathbf{C}_j^i}|u_j-u|\,dx\leq\int_\Omega|u_j-u|\,dx\to 0\qquad\text{as } j\to+\infty.$$

By (1.6.62) the second term in (1.6.63) can be estimated as follows:

$$\begin{aligned}\sum_{i\in Z_j}\int_{\mathbf{C}_j^i}|w_j-u|\,dx &\leq c\sum_{i\in Z_j}\Big(\int_{\mathbf{C}_j^i}|u_j^i-u_j|\,dx+\int_{\mathbf{C}_j^i}|u_j-u|\,dx\Big)\\ &\leq c\sum_{i\in Z_j}\sum_{a\in C_j^i}|u_j(a)-u_j^i|\varepsilon_j^n+\int_\Omega|u_j-u|\,dx\end{aligned}$$

Now, by discrete Hölder's inequality, Lemma 1.10.2 and the concavity of $y\mapsto y^{\frac{1}{p}}$, we get

$$\begin{aligned}\sum_{i\in Z_j}\sum_{a\in C_j^i}|u_j(a)-u_j^i|\varepsilon_j^n &\leq \sum_{i\in Z_j}\varepsilon_j^n\Big(\sum_{a\in C_j^i}|u_j(a)-u_j^i|^p\Big)^{\frac{1}{p}}\Big(\sharp C_j^i\Big)^{1-\frac{1}{p}}\\ &\leq c\varepsilon_j^n\varepsilon_j^{-n/p}\Big(\sum_{a\in C_j^i}|u_j(a)-u_j^i|^p\varepsilon_j^n\Big)^{\frac{1}{p}}\Big(\frac{\delta_j^n}{\varepsilon_j^n}\Big)^{1-\frac{1}{p}}\\ &\leq c\delta_j^{n-\frac{n}{p}}\Big(\delta_j^p\sum_{\{a,b\}\in M_{\varepsilon_j}(C_j^i)}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n\Big)^{\frac{1}{p}}\\ &\leq c\delta_j^{n-\frac{n}{p}}\delta_j(\sharp Z_j)^{1-\frac{1}{p}}\Big(\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(C_j^i)}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n\Big)^{\frac{1}{p}}\\ &\leq c\delta_j.\end{aligned}$$

In conclusion,

$$\sum_{i\in Z_j}\int_{\mathbf{C}_j^i}|w_j-u|\,dx\leq c\delta_j+\int_\Omega|u-u_j|\,dx\to 0\qquad\text{as } j\to+\infty.$$

■

Lemma 1.6.2 *Let $1<p\leq n$. Let (ε_j) and (δ_j) be as in (1.3.15). Let (u_j) be a sequence such that $u_j\in\mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$. Assume that $u_j\to u$ in $L^1(\Omega;\mathbb{R}^m)$ for some $u\in W^{1,p}(\Omega;\mathbb{R}^m)$ and that (u_j) is bounded in $L^\infty(\Omega;\mathbb{R}^m)$. Let $k\in\mathbb{N}$ be fixed. Let (ρ_j) be a sequence of the form $\rho_j=\beta\delta_j$, with $\beta<1/2$. For all $i\in Z_j$ we define the set*

$$C_j^i=Q_{\varepsilon_j}\Big(\Big[\frac{\rho_j}{\varepsilon_j}2^{-k_i}\Big]\varepsilon_j;i\delta_j\Big)\setminus Q_{\varepsilon_j}\Big(\Big[\frac{\rho_j}{\varepsilon_j}2^{-k_i-1}\Big]\varepsilon_j;i\delta_j\Big),$$

where k_i is arbitrarily chosen in $\{0,1,\dots,k-1\}$. Let

$$u_j^i=\frac{1}{\sharp C_j^i}\sum_{a\in C_j^i}u_j(a)\qquad\text{and}\qquad Q_j^i=Q_{\varepsilon_j}(\delta_j;i\delta_j).$$

For all $N,j\in\mathbb{N}$ we consider two families of functions $r_{N,j},r_N:\mathbb{M}^{m\times n}\to[0,+\infty)$ such that the following assumptions hold:

1. *$r_{N,j}\to r_N$ as $j\to+\infty$, uniformly on the compact sets of $\mathbb{R}^m\setminus\{0\}$, for all $N\in\mathbb{N}$;*
2. *there exist a positive infinitesimal sequence ν_j and a constant $c>0$ such that*

$$r_{N,j}(z)\leq\nu_j+c|z|^p\quad\text{for all } z\in\mathbb{R}^m;\tag{1.6.64}$$

3. for fixed $\eta > 0$ there exists a constant $c > 0$ such that for all $w, z \in \mathbb{R}^m$ we have

$$|r_N(z) - r_N(w)| \leq c(\eta^p + |z-w|(|w|^{p-1} + |z|^{p-1})); \qquad (1.6.65)$$

4. for $z = 0$ we have

$$r_N(0) = r_{N,j}(0) = 0. \qquad (1.6.66)$$

We define $\psi_j^N \in \mathcal{A}_{\varepsilon_j}(\Omega; \mathbb{R}^m)$ as

$$\psi_j^N(a) = \sum_{i \in Z_j} r_{N,j}(u_j^i)\chi_{Q_j^i}(a), \quad a \in \Omega_j,$$

where χ indicates the characteristic function. Then,

$$\lim_{j\to+\infty} \sum_{a\in\Omega_j} \psi_j^N(a)\varepsilon_j^n = \lim_{j\to+\infty} \sum_{i\in Z_j} r_{N,j}(u_j^i)\delta_j^n = \int_\Omega r_N(u)\,dx. \qquad (1.6.67)$$

Proof. Let $\eta > 0$ be fixed. For $\eta \leq |z| \leq \sup_j \|u_j\|_\infty$ we have $|r_{N,j}(z) - r_N(z)| \to 0$ as $j \to +\infty$ by assumption 1. For all $|z| < \eta$ conditions (1.6.64)-(1.6.66) imply that

$$|r_{N,j}(z) - r_N(z)| \leq \nu_j + c\eta^p.$$

Since $u_j \to u$ in $L^1(\Omega; \mathbb{R}^m)$, we get

$$\begin{aligned}\limsup_{j\to+\infty}\Big|\sum_{i\in Z_j} r_{N,j}(u_j^i)\delta_j^n - \int_\Omega r_N(u)\,dx\Big| &\leq \limsup_{j\to+\infty}\Big|\sum_{i\in Z_j} r_N(u_j^i)\delta_j^n - \int_\Omega r_N(u)\,dx\Big| + c\eta^p\\ &\leq \limsup_{j\to+\infty}\sum_{i\in Z_j}\int_{\mathbf{Q}_j^i}\big|r_N(u_j^i) - r_N(u)\big|\,dx + c\eta^p\\ &= \limsup_{j\to+\infty}\sum_{i\in Z_j}\sum_{a\in Q_j^i}\big|r_N(u_j^i) - r_N(u_j(a))\big|\varepsilon_j^n + c\eta^p.\end{aligned}$$

By (1.6.65) and the boundedness of (u_j), we obtain

$$\big|r_N(u_j^i) - r_N(u_j(a))\big| \leq c(|u_j^i - u_j(a)|(|u_j^i|^{p-1} + |u_j(a)|^{p-1})\big| + \eta^p) \leq c(|u_j^i - u_j(a)| + \eta^p),$$

where the constant c is independent of j. There follows that

$$\limsup_{j\to+\infty}\Big|\sum_{i\in Z_j} r_{N,j}(u_j^i)\delta_j^n - \int_\Omega r_N(u)\,dx\Big| \leq c\limsup_{j\to+\infty}\sum_{i\in Z_j}\sum_{a\in Q_j^i}|u_j^i - u_j(a)|\varepsilon_j^n + c\eta^p. \qquad (1.6.68)$$

By the discrete version of Hölder's inequality we get

$$\begin{aligned}\sum_{i\in Z_j}\sum_{a\in Q_j^i}|u_j^i - u_j(a)|\varepsilon_j^n &\leq \varepsilon_j^n\sum_{i\in Z_j}\Big(\sum_{a\in Q_j^i}|u_j^i - u_j(a)|^p\Big)^{\frac{1}{p}}\big(\sharp Q_j^i\big)^{1-\frac{1}{p}}\\ &\leq c\delta_j^n\delta_j^{-n/p}\sum_{i\in Z_j}\Big(\sum_{a\in Q_j^i}|u_j^i - u_j(a)|^p\Big)^{\frac{1}{p}}.\end{aligned}$$

By Lemma 1.10.2, we deduce that

$$\sum_{a\in Q_j^i}|u_j^i - u_j(a)|^p \leq c\delta_j^p\sum_{\{a,b\}\in M_{\varepsilon_j}(Q_j^i)}\Big|\frac{u_j(a) - u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n.$$

Note that in the inequality above the constant c can be chosen to be independent of i, since for fixed j the family $\{C_j^i, i \in Z_j\}$ is s finite collection of homothetic sets. Therefore,

$$\sum_{i\in Z_j}\sum_{a\in Q_j^i}|u_j^i - u_j(a)|\varepsilon_j^n \le c\delta_j^n\delta_j^{-n/p}\sum_{i\in Z_j}\delta_j\Big(\sum_{\{a,b\}\in M_{\varepsilon_j}(Q_j^i)}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n\Big)^{\frac{1}{p}}.$$

Taking into account the concavity of the real function $x \mapsto x^{\frac{1}{p}}$, we get

$$\begin{aligned}\sum_{i\in Z_j}\sum_{a\in Q_j^i}|u_j^i - u_j(a)|\varepsilon_j^n &\le c\delta_j^n\delta_j^{-n/p}\delta_j\big(\sharp Z_j^i\big)^{1-\frac{1}{p}}\Big(\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(Q_j^i)}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n\Big)^{\frac{1}{p}}\\ &\le c\delta_j\Big(\sum_{\{a,b\}\in M_{\varepsilon_j}(\Omega)}\Big|\frac{u_j(a)-u_j(b)}{\varepsilon_j}\Big|^p\varepsilon_j^n\Big)^{\frac{1}{p}} \le c\delta_j. \qquad (1.6.69)\end{aligned}$$

By (1.6.68), (1.6.69) and the arbitrariness of η we conclude that

$$\limsup_{j\to+\infty}\Big|\sum_{i\in Z_j} r_{N,j}(u_j^i)\delta_j^n - \int_\Omega r_N(u)\,dx\Big| \le \limsup_{j\to+\infty} c\delta_j = 0.$$

■

Remark 1.6.3 In the noncritical case $p < n$, we will apply Lemma 1.6.2 with $r_{N,j} = \phi_j^N$ and $r_N = \phi^N$. Then

$$\psi_j^N(a) = \sum_{i\in Z_j}\phi_j^N(u_j^i)\chi_{Q_j^i}(a), \quad a \in \Omega_j. \qquad (1.6.70)$$

Remark 1.6.4 In the critical case $n = p$, we will apply Lemma 1.6.2 with $r_{N,j} = \varphi_j^{1/N}$ and $r_N = \varphi^{1/N}$. Setting $\alpha = N^{-1}$ and writing ψ_j^α in place of $\psi_j^{\alpha^{-1}}$, we will have

$$\psi_j^\alpha(a) = \sum_{i\in Z_j}\varphi_j^\alpha(u_j^i)\chi_{Q_j^i}(a), \quad a \in \Omega_j. \qquad (1.6.71)$$

1.7 Γ-lim inf inequality

Proposition 1.7.1 (Γ-lim inf inequality) *Let $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$ be such that $u_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$. Then*

$$\liminf_{j\to+\infty} F_{\varepsilon_j}(u_j) \ge F(u). \qquad (1.7.72)$$

In the proof we will use the following truncation Lemma, which is a discrete version of [22], Lemma 3.5, and can be proved by adjusting to the discrete setting the arguments used in [22].

Lemma 1.7.2 *Let (u_j) be a sequence such that $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$, (u_j) is bounded in $L^1(\Omega;\mathbb{R}^m)$ and $\sup_j \mathcal{F}_{\varepsilon_j}(u_j) < +\infty$. Then, for all $L \in \mathbb{N}$ and $\eta > 0$ there exist a subsequence ε_j (not relabeled), a constant $R_L > L$ and a Lipschitz function $t_L : \mathbb{R}^m \to \mathbb{R}^m$ of Lipschitz constant 1 such that*

$$t_L(z) = z \text{ if } |z| < R_L$$
$$t_L(z) = 0 \text{ if } |z| > 2R_L$$

and $\lim_j \mathcal{F}_{\varepsilon_j}(u_j) \ge \liminf_j \mathcal{F}_{\varepsilon_j}(t_L(u_j)) - \eta$.

Proof of Proposition 1.7.1. With no loss of generality we assume that $\liminf_j F_{\varepsilon_j}(u_j) < +\infty$. We will first derive the lim inf inequality under a boundedness assumption, and then we will deal with the general case (step **A** and **B** respectively).

A We assume that (u_j) is bounded in $L^\infty(\Omega;\mathbb{R}^m)$ (we will remove this assumption through a truncation argument). We fix $k \in \mathbb{N}$ and we consider a sequence (ρ_j) of the form $\rho_j = \beta\delta_j$, with $\beta < 1/2$. We apply Lemma 1.6.1 to (u_j) in order to get a new sequence $w_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$ satisfying (1.6.54)-(1.6.56). We denote by E_j the discrete set

$$E_j = \bigcup_{i\in Z_j} Q_{\varepsilon_j}(\rho_j^i; i\delta_j).$$

By construction

$$\liminf_j F_{\varepsilon_j}(u_j) \geq \liminf_j F_{\varepsilon_j}(u_j; E_j) + \liminf_j F_{\varepsilon_j}(u_j; \Omega_j \setminus E_j).$$

First of all, we want to find a lower bound for the contribution of (u_j) on $\Omega_j \setminus E_j$ and then we will estimate the energy on E_j (steps **A.1** and **A.2** respectively).

A.1 In this step we will find a lower bound for the contribution of the energy far from the pinning sites; i.e., the term $\liminf_j F_{\varepsilon_j}(u_j; \Omega_j \setminus E_j)$. The proof of this estimate is formally the same for the critical case $p = n$ and the non-critical one, $p < n$; note that the formula defining the bulk term of the Γ-limit has the same structure for any order of growth. However, the critical scaling for δ_j (and hence ρ_j) as a function of ε_j is obviously different, so the set E_j has a different "size" in the two cases.

We define a new sequence $v_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$ by modifying w_j as follows:

$$v_j(a) = \begin{cases} u_j^i & \text{for } \ a \in Q_j^i := Q_{\varepsilon_j}(\rho_j^i; i\delta_j), \quad i \in Z_j \\ w_j(a) & \text{otherwise.} \end{cases} \tag{1.7.73}$$

Note that $v_j(a) = u_j(a)$ for all $a \in \Omega_j \setminus \bigcup_{i\in Z_j} Q([2^{-k_i}\rho_j/\varepsilon_j]\varepsilon_j; i\delta_j)$, since w_j is such that $u_j = w_j$ on $\Omega_j \setminus \bigcup_{i\in Z_j} C_j^i$. Note moreover that $v_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$. In fact

$$\begin{aligned}
\lim_j \int_\Omega |v_j - u|\,dx &\leq \lim_j \int_\Omega |u_j - v_j|\,dx + \lim_j \int_\Omega |u_j - u|\,dx \\
&= \lim_j \sum_{a\in\Omega_j} |u_j(a) - v_j(a)|\varepsilon_j^n \\
&\leq \lim_j \sum_{a\in\Omega_j\setminus E_j} |u_j(a) - v_j(a)|\varepsilon_j^n + \lim_j \sum_{a\in E_j} |u_j(a) - v_j(a)|\varepsilon_j^n \\
&\leq \lim_j \sum_{a\in\Omega_j\setminus E_j} |u_j(a) - w_j(a)|\varepsilon_j^n + \lim_j \sum_{a\in E_j} |u_j(a) - v_j(a)|\varepsilon_j^n \\
&\leq \lim_j \Big(\int_\Omega |u_j - u|\,dx + \int_\Omega |w_j - u|\,dx\Big) + \lim_j \sum_{i\in Z_j}\sum_{i\in Q_j^i} |u_j^i - u_j(a)|\varepsilon_j^n \\
&= \lim_j \sum_{i\in Z_j}\sum_{i\in Q_j^i} |u_j^i - u_j(a)|\varepsilon_j^n.
\end{aligned}$$

Arguing as in Lemma 1.6.1 we get

$$\lim_j \sum_{i\in Z_j}\sum_{i\in Q_j^i} |u_j^i - u_j(a)|\varepsilon_j^n \leq \lim_j c\delta_j \Big(\sum_{\{a,b\}\in M_{\varepsilon_j}(\Omega)} \varepsilon_j^{n-p}|u_j(a) - u_j(b)|^p\Big)^{1/p} \leq \lim_j c\delta_j = 0.$$

Now, Lemma 1.6.1 implies that

$$\liminf_j F_{\varepsilon_j}(u_j; \Omega_j \setminus E_j) + \frac{c}{k} \geq \liminf_j F_{\varepsilon_j}(w_j; \Omega_j \setminus E_j).$$

We can write

$$F_{\varepsilon_j}(w_j; \Omega_j \setminus E_j) = \mathcal{F}_{\varepsilon_j}(v_j) - R_j, \tag{1.7.74}$$

where

$$R_j = \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in\mathcal{Y}^{\xi}_{\varepsilon_j}(\rho^i_j;i\delta_j)} \varepsilon_j^n f^{\xi}(D^{\xi}_{\varepsilon_j} v_j(a))$$

and $\mathcal{Y}^{\xi}_{\varepsilon_j}(l;c) = \{a\in\Omega_{\varepsilon_j}:\ a\in Q_{\varepsilon_j}(l;c),\ a+\varepsilon_j\xi\in\Omega_j\setminus Q_{\varepsilon_j}(l;c)\}$ accounts for the interactions across $\partial(c+[-l,l]^n)$ for all $c\in\mathbb{R}^n$ and $l>0$.

We want to show that R_j is negligible. Note that for each $a\in\mathcal{Y}^{\xi}_{\varepsilon_j}(\rho^i_j;i\delta_j)$ we have $a, a+\varepsilon_j\xi\in C^i_j$, since $\mathrm{dist}(a;\mathcal{S}_{\varepsilon_j}(\rho^i_j;i\delta_j))\le M\varepsilon_j < ([2^{-k_i}\rho_j/\varepsilon_j]\varepsilon_j - [2^{-k_i-1}\rho_j/\varepsilon_j]\varepsilon_j)/2$ (and the same holds for $a+\varepsilon_j\xi$). Hence

$$\begin{aligned}
R_j &\le \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(C^i_j)} \varepsilon_j^n f^{\xi}(D^{\xi}_{\varepsilon_j} v_j(a))\\
&\le c\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(C^i_j)} \varepsilon_j^n\Big(1+\Big|\frac{v_j(a)-v_j(b)}{\varepsilon_j}\Big|^p\Big)\\
&\le c\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(C^i_j)} \varepsilon_j^n + c\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(C^i_j\setminus Q(\rho^i_j;i\delta_j))} \varepsilon_j^n\Big|\frac{w_j(a)-w_j(b)}{\varepsilon_j}\Big|^p\\
&\le c\varepsilon_j^n(\sharp Z_j)(\sharp C^i_j) + c\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(C^i_j)} \varepsilon_j^n\Big|\frac{w_j(a)-w_j(b)}{\varepsilon_j}\Big|^p\\
&\le c\beta^n + c\sum_{i\in Z_j} F_{\varepsilon_j}(w_j;C^i_j).
\end{aligned}$$

By Lemma 1.6.1 we deduce that

$$R_j \le c\beta^n + \frac{c}{k} \qquad \text{for } j \text{ large enough.} \tag{1.7.75}$$

By (1.7.74) and (1.7.75) we get

$$\liminf_j F_{\varepsilon_j}(w_j;\Omega_j\setminus E_j) \ge \liminf_j \mathcal{F}_{\varepsilon_j}(v_j) - c\beta^n - \frac{c}{k}.$$

Since $v_j\to u$ in $L^1(\Omega;\mathbb{R}^m)$, by Proposition 1.3.2 we have

$$\liminf_j F_{\varepsilon_j}(w_j;\Omega_j\setminus E_j) \ge \liminf_j \mathcal{F}_{\varepsilon_j}(v_j) - c\beta^n - \frac{c}{k} \ge \int_\Omega f_0(Du)\,dx - c\beta^n - \frac{c}{k}, \tag{1.7.76}$$

where $f_0:\mathbb{M}^{m\times n}\to[0,+\infty)$ is given by the homogenization formula in (1.3.25).

A.2 In this paragraph we focus our attention on the contribution of u_j on E_j; i.e., close to the pinning sites. By Lemma 1.6.1 we have

$$\liminf_j F_{\varepsilon_j}(u_j;E_j) + \frac{c}{k} \ge \liminf_j F_{\varepsilon_j}(w_j;E_j) \ge \liminf_j \sum_{i\in Z_j} F_{\varepsilon_j}(w_j;Q(\rho^i_j;i\delta_j)).$$

For fixed $j\in\mathbb{N}$ and $i\in Z_j$ we define the function $w_{i,j}\in\mathcal{A}_{\varepsilon_j}(\mathbb{R}^n;\mathbb{R}^m)$ as

$$w_{i,j}(a) = \begin{cases} w_j(a+i\delta_j) & \text{if } a\in Q_{\varepsilon_j}(\rho^i_j)\\ u^i_j & \text{if } a\in\varepsilon_j\mathbb{Z}^n\setminus Q_{\varepsilon_j}(\rho^i_j).\end{cases}$$

We will deal separately with the case $p=n$ and the case $p<n$ (steps **2.1** and **2.2** respectively), since the asymptotic behavior of the energies close to the pinning sites is determined by the growth exponent p.

A.2.1 Critical exponent $p = n$. Let $j \in \mathbb{N}$ and $i \in Z_j$ be fixed. By a rescaling argument on the space variable we define $\zeta^i_j \in \mathcal{A}_1(\mathbb{Z}^n;\mathbb{R}^m)$ as $\zeta^i_j(A) = w_{i,j}(A\varepsilon_j)$. By construction $\zeta^i_j(0) = 0$ and $\zeta^i_j = u^i_j$ on $\mathbb{Z}^n \setminus Q_1(\rho^i_j T_j - 1)$. In particular, we notice that $\zeta^i_j = u^i_j$ on $\mathcal{S}_1([\beta\delta_j T_j - M], [\beta\delta_j T_j])$ (provided that j is large enough). Now,

$$\begin{aligned} F_{\varepsilon_j}(w_j; Q(\rho^i_j, i\delta_j)) &= F_{\varepsilon_j}(w_{i,j}; Q(\rho^i_j)) \\ &= \sum_{\xi\in I}\sum_{A\in R^\xi_1(Q_1(\rho^i_j T_j))} T_j^{-n} f^\xi(D^\xi_1\zeta^i_j(A)T_j) \\ &= \sum_{\xi\in I}\sum_{A\in R^\xi_1(Q_1(\beta\delta_j T_j))} T_j^{-n} f^\xi(D^\xi_1\zeta^i_j(A)T_j) - R^i_j, \end{aligned}$$

where

$$R^i_j = \sum_{\xi\in I}\sum_{A\in\mathcal{Y}^\xi_1(\rho^i_j T_j)} \varepsilon^n_j f^\xi(T_j D^\xi_1\zeta^i_j(A)).$$

Summing up over $i \in Z_j$ we have

$$\sum_{i\in Z_j} F_{\varepsilon_j}(w_j; Q(\rho^i_j; i\delta_j)) \geq \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{A\in R^\xi_1(Q_1(\beta\delta_j T_j))} \varepsilon^n_j f^\xi(D^\xi_1\zeta^i_j(A)T_j) - \sum_{i\in Z_j} R^i_j.$$

Taking into account Lemma 1.6.1 we can show that $\sum_{i\in Z_j} R^i_j$ is negligible. In fact by a change of variables we get:

$$\begin{aligned} \sum_{i\in Z_j} R^i_j &\leq \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in R^\xi_{\varepsilon_j}(C^i_j)} \varepsilon^n_j f^\xi(D^\xi_{\varepsilon_j} w_{i,j}(a - i\delta_j)) \\ &\leq c\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(C^i_j)} |w_{i,j}(a - i\delta_j) - w_{i,j}(b - i\delta_j)|^n \\ &\leq c\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(C^i_j\cap Q(\rho^i_j, i\delta_j))} |w_j(a) - w_j(b)|^n \\ &\leq c\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(C^i_j)} |w_j(a) - w_j(b)|^n \leq c\sum_{i\in Z_j} F_{\varepsilon_j}(w_j; C^i_j) \leq \frac{c}{k}. \end{aligned}$$

There follows that

$$\begin{aligned} &\liminf_j \sum_{i\in Z_j} F_{\varepsilon_j}(w_j; Q(\rho^i_j, i\delta_j)) \\ &\quad\geq \liminf_j \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{A\in R^\xi_1(Q_1(\beta\delta_j T_j))} \varepsilon^n_j f^\xi(T_j D^\xi_1\zeta^i_j(A)) - \frac{c}{k} \\ &\quad\geq \liminf_j \sum_{i\in Z_j} \inf\Big\{\sum_{\xi\in I}\sum_{A\in R^\xi_1(Q_1(\beta\delta_j T_j))} \varepsilon^n_j f^\xi(T_j D^\xi_1\zeta(A)) :\ \zeta(0) = 0, \\ &\qquad\qquad \zeta = u^i_j \text{ on } \mathcal{S}_1([\beta\delta_j T_j - M], [\beta\delta_j T_j])\Big\} - \frac{c}{k}. \end{aligned}$$

Recalling that we set $S_j = T_j(\log T_j)^{(1-n)/n}$, we can write $\beta\delta_j T_j = \beta r^{(n-1)/n} S_j$. Letting $\alpha = \beta r^{(n-1)/n}$, we can re-write the inequality above as follows:

$$\begin{aligned} \liminf_j \sum_{i\in Z_j} F_{\varepsilon_j}(w_j; Q(\rho^i_j, i\delta_j)) &\geq \liminf_j \sum_{i\in Z_j} \frac{1}{(\log T_j)^{n-1}}\varphi^\alpha_j(u^i_j) - \frac{c}{k} \\ &= r^{1-n}\liminf_j \sum_{i\in Z_j} \delta^n_j\varphi^\alpha_j(u^i_j) - \frac{c}{k}. \end{aligned}$$

By Lemma 1.6.2 and Remark 1.6.3 we know that there exists the limit

$$\lim_j \sum_{i\in Z_j} \delta_j^n \varphi_j^\alpha(u_j^i) = \int_\Omega \varphi^\alpha(u)\,dx,$$

provided that we extract a suitable subsequence (not relabeled). Hence

$$\liminf_j F_{\varepsilon_j}(u_j;E_j) \geq \liminf_j F_{\varepsilon_j}(w_j;E_j) - \frac{c}{k} \geq r^{1-n}\int_\Omega \varphi^\alpha(u)\,dx - \frac{c}{k}, \qquad (1.7.77)$$

with $\alpha = \beta r^{(n-1)/n}$. By (1.7.76) and (1.7.77) we can conclude that in the case $n = p$

$$\liminf_j F_{\varepsilon_j}(u_j) \geq \int_\Omega f_0(Du)\,dx + r^{1-n}\int_\Omega \varphi^\alpha(u)\,dx - \frac{c}{k} - c\beta^n.$$

By letting first $\beta \to 0^+$ and then $k \to +\infty$ we finally obtain the desired inequality:

$$\liminf_j F_{\varepsilon_j}(u_j) \geq \int_\Omega f_0(Du)\,dx + r^{1-n}\int_\Omega \varphi(u)\,dx = F(u).$$

A.2.2 Noncritical exponent $p < n$. Let $j \in \mathbb{N}$ and $i \in Z_j$ be fixed. By rescaling $w_{i,j}$ we define the function $\zeta_j^i \in \mathcal{A}_1(\mathbb{Z}^n, \mathbb{R}^m)$ as

$$\zeta_j^i(A) = \begin{cases} w_{i,j}(\varepsilon_j A) & \text{for } A \in Q_1(\rho_j^i T_j) \\ u_j^i & \text{for } A \in \mathbb{Z}^n \setminus Q_1(\rho_j^i T_j). \end{cases}$$

Note that $\zeta_j^i(0) = 0$ and $\zeta_j^i = u_j^i$ on $\mathcal{S}_1([\delta_j T_j - M], [\delta_j T_j])$. By a change of variables we have

$$F_{\varepsilon_j}(w_j; Q(\rho_j^i, i\delta_j)) = F_{\varepsilon_j}(w_{i,j}, Q(\rho_j^i)) = \varepsilon_j^{n-p}\sum_{\xi\in I}\sum_{A\in R_1^\xi(Q(\delta_j T_j))} h_j^\xi(D_1^\xi \zeta_j^i(A)) - R_j^i, \qquad (1.7.78)$$

where $h_j^\xi(x) = T_j^{-p} f^\xi(T_j x)$ and the term R_j^i corresponds to the interactions across $\partial([-[\rho_j^i T_j], [\rho_j^i T_j]]^n)$:

$$R_j^i = \varepsilon_j^{n-p}\sum_{\xi\in I}\sum_{a\in\mathcal{Y}_1^\xi([\rho_j^i T_j])} h_j^\xi(D_1^\xi \zeta_j^i(A)).$$

By construction the function ζ_j^i satisfies

$$\begin{aligned}
&\sum_{\xi\in I}\sum_{A\in R_1^\xi(Q(\delta_j T_j))} h_j^\xi(D_1^\xi \zeta_j^i(A)) \\
&\quad \geq \inf\Big\{\sum_{\xi\in I}\sum_{A\in R_1^\xi(Q(\delta_j T_j))} h_j^\xi(D_1^\xi v(A)) : \begin{array}{l} v(0)=0 \\ v = u_j^i \text{ on } \mathcal{S}_1([\delta_j T_j - M], [\delta_j T_j]) \end{array}\Big\} \\
&\quad \geq \inf_N \inf\Big\{\sum_{\xi\in I}\sum_{A\in R_1^\xi(Q(N))} h_j^\xi(D_1^\xi v(A)) : \begin{array}{l} v(0)=0 \\ v = u_j^i \text{ on } \mathcal{S}_1([N-M], [N]) \end{array}\Big\} \\
&\quad = \inf_N \phi_j^N(u_j^i). \qquad (1.7.79)
\end{aligned}$$

Summing up over the pinning sites $i \in Z_j$ and taking into account (1.7.78) and (1.7.79), we get

$$F_{\varepsilon_j}(w_j;E_j) \geq \sum_{i\in Z_j} F_{\varepsilon_j}(w_j; Q(\rho_j^i; i\delta_j)) \geq \inf_N \varepsilon_j^{n-p}\sum_{i\in Z_j}\phi_j^N(u_j^i) - \sum_{i\in Z_j} R_j^i.$$

The term $\sum_{i\in Z_j} R^i_j$ is negligible; in fact

$$\begin{aligned}
\sum_{i\in Z_j} R^i_j &= \varepsilon_j^{n-p}\sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in\mathcal{Y}_1^\xi([\rho^i_j T_j])} h^\xi_j(D^\xi_1\zeta^i_j(A))\\
&\le c\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(C^i_j)} \varepsilon^n_j\Big(\Big|\frac{w_{i,j}(a-i\delta_j)-w_{i,j}(b-i\delta_j)}{\varepsilon_j}\Big|^p+1\Big)\\
&\le c\sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(C^i_j\cap Q(\rho^i_j,i\delta_j))} \varepsilon_j^{n-p}|w_j(a)-w_j(b)|^p+c\beta\\
&\le c\sum_{i\in Z_j} F_{\varepsilon_j}(w_j;C^i_j)+c\beta\le\frac{c}{k}+c\beta.
\end{aligned}$$

Moreover, by Lemma 1.6.2 and Remark 1.6.4 we get that for fixed N there exists the limit

$$\lim_j\sum_{i\in Z_j}\varepsilon_j^{n-p}\phi^N_j(u^i_j)=\lim_j r^{1-n}\sum_{i\in Z_j}\delta^n_j\phi^N_j(u^i_j)=r^{1-n}\int_\Omega\phi^N(u)\,dx,$$

upon extracting a suitable subsequence. There follows that

$$\begin{aligned}
\liminf_j F_{\varepsilon_j}(u_j;E_j)&\ge\liminf_j F_{\varepsilon_j}(w_j;E_j)-\frac{c}{k}\\
&\ge r^{1-n}\inf_N\int_\Omega\phi^N(u)\,dx-\frac{c}{k}-c\beta=r^{1-n}\int_\Omega\phi(u)\,dx-\frac{c}{k}-c\beta. \qquad (1.7.80)
\end{aligned}$$

By (1.7.76) and (1.7.80) we have

$$\liminf_j F_{\varepsilon_j}(u_j)\ge\int_\Omega f_0(Du)\,dx+r^{1-n}\int_\Omega\phi(u)\,dx-\frac{c}{k}-c\beta.$$

By letting $\beta\to0^+$ and $k\to+\infty$ we conclude that

$$\liminf_j F_{\varepsilon_j}(u_j)\ge\int_\Omega f_0(Du)\,dx+r^{1-n}\int_\Omega\phi(u)\,dx. \qquad (1.7.81)$$

B It remains to show that the Γ-liminf inequality holds even if we remove the boundedness assumption on the sequence (u_j). For all $L\in\mathbb{N}$ and $\eta>0$ we apply the previous arguments to the truncated sequence $t_L(u_j)$, where t_L is as in the statement of Lemma 1.7.2; i.e.,

$$\begin{gathered}
t_L(u_j)=z \text{ if } |u_j|<R_L\\
t_l(u_j)=0 \text{ if } |u_j|>2R_L\\
\text{and } \liminf_j F_{\varepsilon_j}(u_j)\ge\liminf_j F_{\varepsilon_j}(t_L(u_j))-\eta.
\end{gathered} \qquad (1.7.82)$$

By step **A** we get

$$\liminf_j F_{\varepsilon_j}(t_L(u_j))\ge\int_\Omega f_0(Dt_L(u))\,dx+r^{1-n}\int_\Omega\varphi(t_L(u))\,dx$$

if $n=p$, and

$$\liminf_j F_{\varepsilon_j}(t_L(u_j))\ge\int_\Omega f_0(Dt_L(u))\,dx+r^{1-n}\int_\Omega\phi(t_L(u))\,dx$$

if $n>p$. Note that $t_L(u)\to u$ as $L\to+\infty$, with respect to the weak convergence of $W^{1,p}(\Omega;\mathbb{R}^m)$. By (1.7.82) and the arbitrariness of η, we can pass to the limit as $L\to+\infty$ and finally deduce that

$$\liminf_j F_{\varepsilon_j}(u_j)\ge F_0(u).$$

■

1.8 Γ-lim sup inequality

Proposition 1.8.1 (Limsup inequality) *For all $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ there exists a sequence (v_j) such that $v_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$, $v_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$ and*

$$\limsup_j F_{\varepsilon_j}(v_j) \leq F(u).$$

Proof. First of all we will prove that the Γ-lim sup inequality holds for all piecewise affine functions and then we will obtain the general case through a density argument (step **A** and **B** respectively).

A Let $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ be a piecewise affine function. Let $\eta > 0$ be fixed. By carefully applying the construction of Proposition 1.3.2 to the function u, we can prove that there exists a sequence $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$ such that $u_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$,

$$\limsup_j \mathcal{F}_{\varepsilon_j}(u_j) = \limsup_j \sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(\Omega)} \varepsilon_j^n f^{\xi}(D^{\xi}_{\varepsilon_j}u_j(a)) \leq \int_{\Omega} f_0(Du)\,dx + \eta \tag{1.8.83}$$

and

$$\frac{|u_j(a)-u_j(b)|}{\varepsilon_j} \leq C \text{ for some } C>0, \quad \text{for all } \{a,b\} \in M_{\varepsilon_j}(\Omega). \tag{1.8.84}$$

Moreover, the sequence (u_j) can be chosen to be bounded in $L^{\infty}(\Omega;\mathbb{R}^m)$. Note that the two boundedness requirements (on the sequence (u_j) and on the difference quotients along the coordinate directions) can be fulfilled by applying a smoothing procedure on the sequence built in the proof of Proposition 1.3.2.

In order to construct an approximate recovery sequence for u (for any value of the parameter η), we will deal separately with the case $p = n$ and the case $p < n$ (steps **A.1** and **A.2** respectively).

A.1 Critical exponent $p = n$. We want to modify (u_j) in order to get an approximate recovery sequence for u. We fix $k \in \mathbb{N}$ and $\beta > 0$ such that $2^{k+1}\beta < 1/2$. Let $\rho_j = 2^{k+1}\beta\delta_j$. Given this choice of ρ_j, we apply Lemma 1.6.1 to (u_j) and we get a sequence $w_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$ satisfying (1.6.54)-(1.6.56). We denote by Z'_j the set of indices $Z'_j = \{i \in \mathbb{Z}^n \setminus Z_j : i\delta_j \in \Omega\}$, corresponding to the pinning sites close to the boundary of Ω. We define the sets

$$E_j = \bigcup_{i\in Z_j} Q_{\varepsilon_j}(\rho^i_j; i\delta_j), \qquad E'_j = \bigcup_{i\in Z'_j} Q_{\varepsilon_j}(\rho_j; i\delta_j) \cap \Omega.$$

By suitably modifying w_j on $E_j \cup E'_j$ we will get an approximate recovery sequence for u.

A.1.1 Firstly we deal with E_j. By construction we have $\rho^i_j \geq \beta\delta_j$ for all $i \in Z_j$. We set $T_j = \varepsilon_j^{-1}$ and $S_j = T_j(\log T_j)^{(1-n)/n}$. For fixed $j \in \mathbb{N}$ and $i \in Z_j$ we consider a function $\zeta^i_j \in \mathcal{A}_1(Q(r^{(n-1)/n}\beta S_j);\mathbb{R}^m)$ such that

$$\zeta^i_j(0) = 0, \qquad \zeta^i_j = u^i_j \text{ on } \mathcal{S}_1([r^{(n-1)/n}\beta S_j - M], [r^{(n-1)/n}\beta S_j])$$

and

$$(\log T_j)^{n-1} \sum_{\xi\in I}\sum_{A\in R^{\xi}_1(Q(\beta r^{(n-1)/n}S_j))} f^{\xi}(T_j D^{\xi}_1 \zeta^i_j(A))T_j^{-n} < \varphi_j^{\beta r^{(n-1)/n}}(u^i_j) + \eta. \tag{1.8.85}$$

We define $v_j : E_j \to \mathbb{R}^m$ as follows:

$$v_j(a) = \begin{cases} \zeta^i_j\Big(\dfrac{a - i\delta_j}{\varepsilon_j}\Big) & \text{if } a \in Q_{\varepsilon_j}(\beta\delta_j; i\delta_j),\ i \in Z_j \\ u^i_j & \text{if } a \in Q_{\varepsilon_j}(\rho^i_j; i\delta_j) \setminus Q_{\varepsilon_j}(\beta\delta_j; i\delta_j),\ i \in Z_j \end{cases}$$

A.1.2 Now we focus our attention on the set E'_j. Let $\gamma_j \in \mathcal{A}_1(Q(\rho_j/\varepsilon_j);\mathbb{R})$ be a function such that $\gamma_j(0)=0$, $\gamma_j=1$ on $\mathcal{S}_1([\rho_j/\varepsilon_j])$ and

$$\sum_{\{A,B\}\in M_1(Q(\rho_j/\varepsilon_j))} |\gamma_j(A)-\gamma_j(B)|^n$$
$$< \inf\Big\{ \sum_{\{A,B\}\in M_1(Q(\rho_j/\varepsilon_j))} |v(A)-v(B)|^n : \begin{array}{l} v\in\mathcal{A}_1(Q(\rho_j/\varepsilon_j)) \\ v(0)=0,\ v=1 \text{ on } \mathcal{S}_1([\rho_j/\varepsilon_j]) \end{array} \Big\} + \eta.$$

By the computations of Section 1.4 we know that the infimum above satisfies

$$\Big|\log\Big(\frac{\rho_j}{\varepsilon_j}\Big)\Big|^{n-1} \inf\Big\{ \sum_{\{A,B\}\in M_1(Q(\rho_j/\varepsilon_j))} |v(A)-v(B)|^n : \begin{array}{l} v\in\mathcal{A}_1(Q(\rho_j/\varepsilon_j)) \\ v(0)=0,\ v=1 \text{ on } \mathcal{S}_1([\rho_j/\varepsilon_j]) \end{array} \Big\} \to l_n.$$

We define $v_j : E'_j \to \mathbb{R}^m$ as

$$v_j(a) = \gamma_j\Big(\frac{a-i\delta_j}{\varepsilon_j}\Big)u_j(a) \qquad \text{for } a\in Q_{\varepsilon_j}\Big(\frac{\rho_j}{\varepsilon_j}; i\delta_j\Big)\cap\Omega,\ i\in Z'_j.$$

A.1.3 Finally, we define $v_j(a)=w_j(a)$ for all $a\in\Omega_j\setminus(E_j\cup E'_j)$. To sum up, we set

$$v_j(a) = \begin{cases} \zeta_j^i\Big(\dfrac{a-i\delta_j}{\varepsilon_j}\Big) & \text{if } a\in Q_{\varepsilon_j}(\beta\delta_j; i\delta_j),\ i\in Z_j \\ u_j^i & \text{if } a\in Q_{\varepsilon_j}(\rho_j^i; i\delta_j)\setminus Q_{\varepsilon_j}(\beta\delta_j; i\delta_j),\ i\in Z_j \\ \gamma_j\Big(\dfrac{a-i\delta_j}{\varepsilon_j}\Big)u_j(a) & \text{for } a\in Q_{\varepsilon_j}(\rho_j/\varepsilon_j; i\delta_j)\cap\Omega,\ i\in Z'_j \\ w_j(a) & \text{if } a\in\Omega_j\setminus(E_j\cup E'_j). \end{cases}$$

Now we can prove that (v_j) is an approximate recovery sequence for u. By construction we have

$$\limsup_j F_{\varepsilon_j}(v_j) \le \limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(v_j; Q(\beta\delta_j, i\delta_j)) \tag{1.8.86}$$
$$+\limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(v_j; Q_{\varepsilon_j}(\rho_j^i, i\delta_j)\setminus Q_{\varepsilon_j}(\beta\delta_j, i\delta_j)) \tag{1.8.87}$$
$$+\limsup_j F_{\varepsilon_j}(v_j; \Omega_j\setminus(E_j\cup E'_j)) \tag{1.8.88}$$
$$+\limsup_j \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in\mathcal{Y}^\xi_{\varepsilon_j}(\beta\delta_j;i\delta_j)} f^\xi(D^\xi_{\varepsilon_j}v_j(a))\varepsilon_j^n \tag{1.8.89}$$
$$+\limsup_j \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in\mathcal{Y}^\xi_{\varepsilon_j}(\rho_j^i;i\delta_j)} f^\xi(D^\xi_{\varepsilon_j}v_j(a))\varepsilon_j^n \tag{1.8.90}$$
$$+\limsup_j \sum_{i\in Z'_j} F_{\varepsilon_j}(v_j; Q_{\varepsilon_j}(\rho_j, i\delta_j)\cap\Omega) \tag{1.8.91}$$
$$+\limsup_j \sum_{i\in Z'_j}\sum_{\xi\in I}\sum_{a\in\mathcal{Y}^\xi_{\varepsilon_j}(\rho_j;i\delta_j)} f^\xi(D^\xi_{\varepsilon_j}v_j(a))\varepsilon_j^n. \tag{1.8.92}$$

The terms above can be estimated separately. First of all we focus our attention on (1.8.86) and we notice that by a change of variables

$$\begin{aligned}\limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(v_j; Q(\beta\delta_j, i\delta_j)) &= \limsup_j \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in R^{\xi}_{\varepsilon_j}(Q(\beta\delta_j,i\delta_j))} \varepsilon_j^n f^{\xi}(D^{\xi}_{\varepsilon_j} v_j(a))\\
&= \limsup_j \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{A\in R^{\xi}_1(Q(\beta r^{(n-1)/n}S_j))} T_j^{-n} f^{\xi}(T_j D^{\xi}_1 \zeta^i_j(A))\\
&\leq \limsup_j \sum_{i\in Z_j} r^{1-n}\delta_j^n\Big(\varphi_j^{\beta r^{(n-1)/n}}(u^i_j)+\eta\Big)\\
&\leq \limsup_j \sum_{i\in Z_j} r^{1-n}\delta_j^n\varphi_j^{\beta r^{(n-1)/n}}(u^i_j)+c\eta.\end{aligned}$$

Taking into account Lemma 1.6.2 and Remark 1.6.4 we get

$$\limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(v_j; Q(\beta\delta_j, i\delta_j)) \leq r^{1-n}\int_{\Omega}\varphi^{\beta r^{(n-1)/n}}(u)\,dx + c\eta. \tag{1.8.93}$$

As far as (1.8.87) is concerned, by construction for all $i\in Z_j$ we have $v_j\equiv u^i_j$ on $Q_{\varepsilon_j}(\rho^i_j; i\delta_j)\setminus Q_{\varepsilon_j}(\beta\delta_j; i\delta_j)$. Since $f^{\xi}(0)=0$ for all $\xi\in I$, we get

$$\limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(v_j; Q(\rho^i_j; i\delta_j)\setminus Q(\beta\delta_j; i\delta_j)) = 0. \tag{1.8.94}$$

Now we focus our attention on (1.8.88); i.e.,

$$\limsup_j F_{\varepsilon_j}(v_j;\Omega_j\setminus(E_j\cup E'_j)) = \limsup_j F_{\varepsilon_j}(w_j;\Omega_j\setminus(E_j\cup E'_j)).$$

By Lemma 1.6.1 and (1.8.83) we get

$$\begin{aligned}\limsup_j F_{\varepsilon_j}(w_j;\Omega_j\setminus(E_j\cup E'_j)) &\leq \limsup_j F_{\varepsilon_j}(u_j;\Omega_j\setminus(E_j\cup E'_j)) + \frac{c}{k}\\
&\leq \limsup_j \mathcal{F}_{\varepsilon_j}(u_j) + \frac{c}{k} \leq \int_{\Omega} f_0(Du)\,dx + \frac{c}{k}.\end{aligned}$$

Now we consider (1.8.89). By construction $\zeta^i_j = u^i_j$ on $\mathcal{S}_1([r^{(n-1)/n}\beta S_j - M],[r^{(n-1)/n}\beta S_j])$, hence $v_j = u^i_j$ on $Q_{\varepsilon_j}(\beta\delta_j, i\delta_j)\setminus Q_{\varepsilon_j}(\beta\delta_j - M\varepsilon_j - \varepsilon_j, i\delta_j)$. There follows that

$$\limsup_j \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in\mathcal{Y}^{\xi}_{\varepsilon_j}(\beta\delta_j,i\delta_j)} \varepsilon_j^n f^{\xi}(D^{\xi}_{\varepsilon_j} v_j(a)) = 0. \tag{1.8.95}$$

Moreover, we show that (1.8.90) is negligible. We have:

$$\begin{aligned}&\limsup_j \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in\mathcal{Y}^{\xi}_{\varepsilon_j}(\rho^i_j,i\delta_j)} \varepsilon_j^n f^{\xi}(D^{\xi}_{\varepsilon_j} v_j(a))\\
&\quad\leq c\limsup_j \sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(Q(\rho^i_j+\varepsilon_j M, i\delta_j)\setminus Q(\rho^i_j-\varepsilon_j M, i\delta_j))} |v_j(a)-v_j(b)|^n\\
&\quad\leq c\limsup_j \sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(Q(\rho^i_j+\varepsilon_j M, i\delta_j)\setminus Q(\rho^i_j, i\delta_j))} |w_j(a)-w_j(b)|^n\\
&\quad\leq c\limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(w_j; C^j_i).\end{aligned}$$

We recall that the computations in the proof of Lemma 1.6.1 imply that

$$\limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(w_j; C_i^j) \le \frac{c}{k},$$

hence

$$\limsup_j \sum_{i\in Z_j} \sum_{\xi\in I} \sum_{a\in \mathcal{Y}^\xi_{\varepsilon_j}(\rho_j^i, i\delta_j)} \varepsilon_j^n f^\xi(D^\xi_{\varepsilon_j} v_j(a)) \le \frac{c}{k}. \tag{1.8.96}$$

Finally, we deal with (1.8.91). By construction

$$\begin{aligned}\limsup_j \sum_{i\in Z'_j} F_{\varepsilon_j}(v_j; Q_{\varepsilon_j}(\rho_j, i\delta_j)\cap\Omega) &\le c\limsup_j \sum_{i\in Z'_j} \sum_{\{a,b\}\in M_{\varepsilon_j}(Q(\rho_j,i\delta_j)\cap\Omega)} |v_j(a)-v_j(b)|^n \\ &\le c\limsup_j \sum_{i\in Z'_j} \sum_{\{a,b\}\in M_{\varepsilon_j}(Q(\rho_j,i\delta_j)\cap\Omega)} (|u_j(a)-u_j(b)|^n|\gamma_j(a-i\delta_j)|^n \\ &\qquad +|u_j(b)|^n|\gamma_j(a-i\delta_j)-\gamma_j(b-i\delta_j)|^n).\end{aligned}$$

Since (u_j) is bounded in $L^\infty(\Omega;\mathbb{R}^m)$ and (1.8.84) holds, we get

$$\begin{aligned}&\limsup_j \sum_{i\in Z'_j} F_{\varepsilon_j}(v_j; Q(\rho_j, i\delta_j)\cap\Omega) \\ &\quad\le c\limsup_j \sum_{i\in Z'_j} \sum_{\{a,b\}\in M_{\varepsilon_j}(Q(\rho_j,i\delta_j)\cap\Omega)} (\varepsilon_j^n|\gamma_j(a-i\delta_j)|^n + |\gamma_j(a-i\delta_j)-\gamma_j(b-i\delta_j)|^n).\end{aligned}$$

By construction (γ_j) is bounded in $L^\infty(\Omega)$ and satisfies

$$(\log(\rho_j T_j))^{n-1} \sum_{\{a,b\}\in M_{\varepsilon_j}(Q(\rho_j,i\delta_j)\cap\Omega)} |\gamma_j(a-i\delta_j)-\gamma_j(b-i\delta_j)|^n \le c+\eta\log(\rho_j T_j))^{n-1}.$$

Since $(\log(\rho_j T_j))^{n-1}/(\log(T_j))^{n-1} \to 1$ as $j\to+\infty$ and $(\log T_j)^{n-1} = r^{n-1}\delta_j^{-n} + o(1)$, we get

$$\begin{aligned}\limsup_j \sum_{i\in Z'_j} F_{\varepsilon_j}(v_j; Q(\rho_j, i\delta_j)\cap\Omega) &\le \limsup_j c\sum_{i\in Z'_j}\delta_j^n + \eta \\ &\le \limsup_j |\Omega'_j| + \eta|\Omega| = \eta|\Omega|,\end{aligned}$$

where $\Omega'_j = \cup_{i\in Z'_j}\mathbf{Q}_{\varepsilon_j}(\rho_j; i\delta_j)\cap\Omega$.

To sum up the estimates we got so far, we have

$$\limsup_j F_{\varepsilon_j}(v_j) \le \int_\Omega f_0(Du)\,dx + r^{1-n}\int_\Omega \varphi^{\beta r^{(n-1)/n}}(u)\,dx + \frac{c}{k} + c\eta. \tag{1.8.97}$$

It remains to show that $v_j\to u$ in $L^1(\Omega)$. By construction $|\{u_j\ne v_j\}|\to 0$ and $u_j\to u$ in $L^1(\Omega)$. Since (Du_j) and (Dv_j) are bounded in $L^1(\Omega)$, by a compactness argument we deduce that $u_j - v_j\to 0$ in $L^1(\Omega)$ and then $v_j\to u$ in $L^1(\Omega)$.

Finally, we let $\beta\to 0^+$ and $k\to+\infty$ in (1.8.97) and we obtain

$$\limsup_j F_{\varepsilon_j}(v_j) \le \int_\Omega f_0(Du)\,dx + r^{1-n}\int_\Omega \varphi(u)\,dx,$$

as desired.

A.2 Noncritical exponent $p < n$**.** We want to modify (u_j) in order to get an approximate recovery sequence for u. Let $k \in \mathbb{N}$ be equal to $[1/\eta]$. Let $\rho_j = \beta\delta_j$, with $\beta < 1/2$. By applying Lemma 1.6.1 to the sequence (u_j), we get a modified sequence $w_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$ such that conditions (1.6.54)-(1.6.56) are satisfied. We build an approximate recovery sequence v_j by carefully modifying w_j close to the pinning sites. To this purpose we define the sets

$$E_j = \bigcup_{i\in Z_j} Q_{\varepsilon_j}(\rho_j^i, i\delta_j) \qquad \text{and } E_j' = \bigcup_{i\in Z_j'} Q_{\varepsilon_j}(\rho_j, i\delta_j)\cap\Omega,$$

where $Z_j' = \{i \in \mathbb{Z}^n \setminus Z_j : i\delta_j \in \Omega\}$ indexes the pinning sites which are close to the boundary of Ω. We will deal separately with E_j, E_j' and $\Omega_j \setminus (E_j \cup E_j')$ (steps **A.2.1**, **A.2.2** and **A.2.3** respectively). Let $N > 0$ be fixed.

A.2.1 Firstly, we deal with E_j. For all $i \in Z_j$ we consider a function $\mu_{i,j}^N \in \mathcal{A}_1(Q(N);\mathbb{R}^m)$ such that $\mu_{i,j}^N(0) = 0$, $\mu_{i,j}^N = u_j^i$ on $\mathcal{S}_1([N-M],[N])$ and

$$\sum_{\xi\in I}\sum_{A\in R_1^\xi(Q(N))} T_j^{-p} f^\xi(T_j D_1^\xi \mu_{i,j}^N(A)) < \phi_j^N(u_j^i) + \eta.$$

We define $v_j : E_j \to \mathbb{R}^m$ as

$$v_j(a) = \begin{cases} \mu_{i,j}^N\Big(\dfrac{a-i\delta_j}{\varepsilon_j}\Big) & \text{for } a \in Q_{\varepsilon_j}(N\varepsilon_j, i\delta_j),\ i\in Z_j \\ u_j^i & \text{for } a\in Q_{\varepsilon_j}(\rho_j^i, i\delta_j)\setminus Q_{\varepsilon_j}(N\varepsilon_j, i\delta_j),\ i\in Z_j. \end{cases} \tag{1.8.98}$$

A.2.2 In this step we focus on E_j' and the pinning sites which are close to the boundary of Ω. For N as in the previous step, we consider a scalar function $\mu^N \in \mathcal{A}_1(Q(N))$ such that $\mu^N(0) = 0$, $\mu^N = 1$ on $\mathcal{S}_1([N-M],[N])$ and $0 \le \mu^N \le 1$. We define $v_j : E_j' \to \mathbb{R}^m$ as

$$v_j(a) = u_j(a)\mu^N(a),\ \text{for } a \in E_j'. \tag{1.8.99}$$

A.2.3 Finally we set $v_j(a) = w_j(a)$ for all $a \in \Omega_j \setminus (E_j \cup E_j')$.
We then have:

$$\limsup_j F_{\varepsilon_j}(v_j) \le \limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(v_j; Q(N\varepsilon_j, i\delta_j)) \tag{1.8.100}$$

$$+\limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(v_j; Q_{\varepsilon_j}(\rho_j^i, i\delta_j)\setminus Q_{\varepsilon_j}(N\varepsilon_j, i\delta_j)) \tag{1.8.101}$$

$$+\limsup_j F_{\varepsilon_j}(v_j; \Omega_j \setminus (E_j\cup E_j')) \tag{1.8.102}$$

$$+\limsup_j \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in\mathcal{Y}_{\varepsilon_j}^\xi(N\varepsilon_j; i\delta_j)} f^\xi(D_{\varepsilon_j}^\xi v_j(a))\varepsilon_j^n \tag{1.8.103}$$

$$+\limsup_j \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in\mathcal{Y}_{\varepsilon_j}^\xi(\rho_j^i; i\delta_j)} f^\xi(D_{\varepsilon_j}^\xi v_j(a))\varepsilon_j^n \tag{1.8.104}$$

$$+\limsup_j \sum_{i\in Z_j'} F_{\varepsilon_j}(v_j; Q_{\varepsilon_j}(\rho_j, i\delta_j)\cap\Omega) \tag{1.8.105}$$

$$+\limsup_j \sum_{i\in Z_j'}\sum_{\xi\in I}\sum_{a\in\mathcal{Y}_{\varepsilon_j}^\xi(\rho_j; i\delta_j)\cap\Omega} f^\xi(D_{\varepsilon_j}^\xi v_j(a))\varepsilon_j^n. \tag{1.8.106}$$

Arguing similarly to paragraph **A.1**, we deduce that (1.8.101), (1.8.103) and (1.8.104) are infinitesimal. As far as (1.8.100) is concerned, by construction we have

$$\begin{aligned}\limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(v_j; Q(N\varepsilon_j, i\delta_j)) &= \limsup_j \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{a\in R^\xi_{\varepsilon_j}(Q(N\varepsilon_j,i\delta_j))} f^\xi(D^\xi_{\varepsilon_j} v_j(a))\varepsilon_j^n \\ &= \limsup_j \sum_{i\in Z_j}\sum_{\xi\in I}\sum_{A\in R^\xi_1(Q(N))} f^\xi(T_j D^\xi_1 \mu^N_{i,j}(A))T_j^{-n} \\ &\leq \limsup_j \sum_{i\in Z_j} r^{1-n}\delta_j^n(\phi^N_j(u^i_j)+\eta) \leq \limsup_j \sum_{i\in Z_j} r^{1-n}\delta_j^n \phi^N_j(u^i_j) + c\eta|\Omega|.\end{aligned}$$

By Lemma 1.6.2 and Remark 1.6.3 we get

$$\limsup_j \sum_{i\in Z_j} F_{\varepsilon_j}(v_j; Q(N\varepsilon_j, i\delta_j)) \leq r^{1-n}\int_\Omega \phi^N(u)\,dx + c\eta. \tag{1.8.107}$$

In order to estimate (1.8.102) we note that Lemma 1.6.1 implies

$$\begin{aligned}\limsup_j F_{\varepsilon_j}(v_j; \Omega_j\setminus(E_j\cup E'_j)) &= \limsup_j F_{\varepsilon_j}(w_j; \Omega_j\setminus(E_j\cup E'_j)) \\ &\leq \limsup_j F_{\varepsilon_j}(u_j; \Omega_j\setminus(E_j\cup E'_j)) + \frac{c}{k} \\ &\leq \limsup_j \mathcal{F}_{\varepsilon_j}(u_j) + \frac{c}{k} \leq \int_\Omega f_0(Du)\,dx + \frac{c}{k}.\end{aligned} \tag{1.8.108}$$

It remains to show that (1.8.105) and (1.8.106) are negligible. By the definition of v_j on E'_j and the equiboundedness of (u_j) we get

$$\begin{aligned}&\limsup_j \sum_{i\in Z'_j} F_{\varepsilon_j}(v_j; Q_{\varepsilon_j}(\rho_j, i\delta_j)\cap\Omega) \\ &\leq c\limsup_j \sum_{i\in Z_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(Q(\rho_j,i\delta_j)\cap\Omega)} \varepsilon_j^{n-p}(|u_j(a)\mu^N(a)-u_j(b)\mu^N(b)|^p+\varepsilon_j^p) \\ &\leq c\limsup_j \sum_{i\in Z'_j}\sum_{\{a,b\}\in M_{\varepsilon_j}(Q(\rho_j,i\delta_j)\cap\Omega)} \varepsilon_j^{n-p}(|u_j(a)-u_j(b)|^p+|\mu^N(a)-\mu^N(b)|^p+\varepsilon_j^p).\end{aligned}$$

By (1.8.84) we deduce that

$$\limsup_j \sum_{i\in Z'_j} F_{\varepsilon_j}(v_j; Q_{\varepsilon_j}(\rho_j, i\delta_j)\cap\Omega) \leq c\limsup_j \sum_{i\in Z'_j}\delta_j^n = c\limsup_j |\mathbf{E}'_j| = |\partial\Omega| = 0.$$

Finally, we can prove that (1.8.106) is infinitesimal in a similar way, using the equiboundedness of (u_j) and the fact that $|\mathbf{E}'_j|$ tends to zero.

To sum up, we proved that

$$\limsup_j F_j(v_j) \leq \int_\Omega f_0(Du)\,dx + r^{1-n}\int_\Omega \phi^N(u)\,dx + \frac{c}{k} + c\eta.$$

Note that the sequence v_j we built converges to u strongly in $L^1(\Omega;\mathbb{R}^m)$. This follows from $|\{u_j\neq v_j\}|\to 0$ and a compactness argument. Passing to the limit as $N\to+\infty$ we have

$$\limsup_j F_j(v_j) \leq \int_\Omega f_0(Du)\,dx + r^{1-n}\int_\Omega \phi(u)\,dx + \frac{c}{k} + c\eta,$$

which proves the existence of an approximate recovery sequence for u for each value of the parameter η. Hence, for all piecewise affine functions in $W^{1,p}(\Omega;\mathbb{R}^m)$ there exists a recovery sequence.

B We can finally prove the Γ-lim sup inequality by using a density argument. For any $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ there exists a sequence (u_k) of piecewise affine functions such that $u_k \to u$ strongly in $W^{1,p}(\Omega;\mathbb{R}^m)$. In step **A** we proved that for all $k \in \mathbb{N}$ the Γ-lim sup $F''(u_k)$ satisfies

$$F''(u_k) \leq F(u_k).$$

By the semicontinuity of F'' with respect to the strong convergence in $L^p(\Omega;\mathbb{R}^m)$ and the continuity of F with respect to $W^{1,p}(\Omega;\mathbb{R}^m)$-convergence, we get

$$F''(u) \leq \liminf_k F''(u_k) \leq \liminf_k F(u_k) = F(u),$$

as desired. ■

1.9 Special cases

In this section we highlight two particular cases. Despite requiring restrictive assumptions, they provide explicit formulas for the densities of the Γ-limit.

1.9.1 Convex energy densities

If for all $\xi \in I$, $f^\xi : \mathbb{R}^m \to [0,+\infty)$ is a convex function, then the density function in the bulk term of the Γ-limit can be expressed through an explicit formula:

$$f_0(A) = \sum_{\xi\in I} f^\xi\Big(A \cdot \frac{\xi}{|\xi|}\Big) \quad \text{for all } A \in \mathbb{M}^{m\times n}. \tag{1.9.109}$$

In fact, under the convexity condition we can use [3, Remark 5.3], which states that in this case Proposition 1.3.2 holds with f_0 as in (1.9.109). Then the Γ-limit is

$$F(u) = \sum_{\xi\in I} \int_\Omega f^\xi\Big(Du \cdot \frac{\xi}{|\xi|}\Big)\,dx + r^{1-n}\int_\Omega \Phi(u)\,dx.$$

1.9.2 Nearest neighbors interactions and homogeneous density functions in the critical case

In this paragraph we consider a special case which is of some interest on its own, despite being very specific. We are in the critical case $p = n$ and we consider nearest neighbors interactions only. Moreover, we assume that the functions f^ξ, $\xi \in I = \{e_1,\dots,e_n\}$, are all equal to the same function f, which is positively homogeneous of degree n and convex. In particular, these assumptions encompass the case $f(z) = \|z\|_n^n$, which has been analyzed in Section 1.4.

In this case the Γ-convergence result holds for the whole sequence F_{ε_j} and the limit functional F is given by

$$F(u) = \sum_{i=1}^n \int_\Omega f\Big(\frac{\partial u}{\partial x_i}\Big)\,dx + \int_\Omega d(u)\,dx,$$

where $d : \mathbb{R}^m \to [0,+\infty)$ equals

$$d(z) = \lim_{T\to+\infty} (\log T)^{n-1} \inf\Big\{ \sum_{\{A,B\}\in M_1(Q(T))} f(v(A)-v(B)) : \begin{array}{l} v \in \mathcal{A}_1(Q(T);\mathbb{R}^m) \\ v(0)=0,\ v=z \text{ on } \mathcal{S}_1([T]) \end{array} \Big\}.$$

Let us prove that the function d is well defined.

Lemma 1.9.1 *Let $f : \mathbb{R}^m \to [0,+\infty)$ be a convex function which is positively homogeneous of degree n and such that $f(0) = 0$. We assume that there exist two constants $c_1, c_2 > 0$ such that $c_1|z|^n \leq f(z) \leq c_2|z|^n$ for all $z \in \mathbb{R}^m$. Then for all $z \in \mathbb{R}^m$ there exists the limit*

$$d(z) = \lim_{T\to+\infty} (\log T)^{n-1} \inf\Big\{ \sum_{\{A,B\}\in M_1(Q(T))} f(v(A)-v(B)) : \begin{array}{l} v \in \mathcal{A}_1(Q(T);\mathbb{R}^m) \\ v(0)=0,\ v=z \textit{ on } \mathcal{S}_1([T]) \end{array} \Big\}.$$

Proof. By the homogeneity of f, it suffices to prove the existence of $d(\nu)$, with $\nu \in \mathbb{R}^m$ and $|\nu| = 1$. We denote by μ_T the infimum which appears in the definition of $d(\nu)$:

$$\mu_T = \inf\Big\{ \sum_{\{A,B\}\in M_1(Q(T))} f(v(A)-v(B)) : \begin{array}{l} v \in \mathcal{A}_1(Q(T);\mathbb{R}^m) \\ v(0)=0,\ v=\nu \text{ on } \mathcal{S}_1([T]) \end{array} \Big\}.$$

It is convenient to introduce a new family of infima $\tilde{\mu}_T$, defined as

$$\tilde{\mu}_T = \min\Big\{ \sum_{\{A,B\}\in M_1(Q(T))} f(v(A)-v(B)) : \begin{array}{l} v \in \mathcal{A}_1(Q(T);\mathbb{R}^m) \\ v=0 \text{ on } Q_1(1),\ v=\nu \text{ on } \mathcal{S}_1([T]) \end{array} \Big\}.$$

The test functions for $\tilde{\mu}_T$ vanish on the whole set $Q_1(1)$ (not only on 0 as for μ_T). The proof is made of two steps: firstly we show that there exists the limit

$$\lim_{T\to+\infty} (\log T)^{n-1}\tilde{\mu}_T \in [0,+\infty);$$

and then we prove that the limit above equals

$$\lim_{T\to+\infty} (\log T)^{n-1}\mu_T = d(\nu).$$

1. Let $S \gg T$. Let $u_T \in \mathcal{A}_1(Q(T);\mathbb{R}^m)$ be such that $u_T = 0$ on $Q_1(1)$, $u_T(A) = \nu$ for all $A \in \mathcal{S}_1([T])$ and

$$\sum_{\{A,B\}\in M_1(Q(T))} f(u_T(A)-u_T(B)) \le \tilde{\mu}_T + \frac{1}{T}. \tag{1.9.110}$$

We will define a convenient test function for m_S by suitably modifying u_T and we will deduce an inequality of the form

$$(\log S)^{n-1}\tilde{\mu}_S \le (\log T)^{n-1}\tilde{\mu}_T + r(S,T) \quad \text{with } \liminf_{T\to+\infty}\limsup_{S\to+\infty} r(S,T) = 0. \tag{1.9.111}$$

Let $k \in \mathbb{N}$ be such that $[T]^k \le [S] < [T]^{k+1}$; i.e., $k = [\log([S])/\log([T])]$. We consider the set $Q_1(S)$ and we denote by C_h its subsets

$$C_h = Q_1([T]^{h+1}) \setminus Q_1([T]^h - 1) \qquad h = 0,\dots,k-1.$$

In each C_h we consider an additional *meso-lattice* $C_h \cap [T]^h\mathbb{Z}^n$ and we use it to define a convenient test function u_S for $\tilde{\mu}_S$. For all $A \in C_h \cap [T]^h\mathbb{Z}^n$ we set

$$u_{S,h}(A) = \frac{\log([S])}{\log([T])}\Big(u_T\Big(\frac{A}{[T]^h}\Big) + h\nu\Big).$$

We denote by $\tilde{u}_{S,h}$ an interpolating function for $u_{S,h}$ which is piecewise affine on a triangulation defined by the lattice $C_h \cap [T]^h\mathbb{Z}^n$ and satisfies

$$\sum_{\{A,B\}\in M_{[T]^h}(C_h)} f(\tilde{u}_S(A)-\tilde{u}_S(B)) = \sum_{l=1}^n \int_{\mathbf{C}_h} f\Big(\frac{\partial \tilde{u}_S}{\partial x_l}(x)\Big)dx.$$

The existence of $\tilde{u}_{S,h}$ follows from Remark 1.3.5; in particular, we can choose our triangulations of $C_h \cap [T]^h\mathbb{Z}^n$ to be homothetic to each other (since the sets are obtained by rescaling). The test function $u_S \in \mathcal{A}_1(Q(S);\mathbb{R}^m)$ is defined as follows:

$$u_S(A) = \begin{cases} \tilde{u}_{S,h}(A) & \text{for } A \in C_h,\ h=0,\dots,k-1 \\[2ex] \nu\dfrac{\log([T]^k+q)}{\log([S])} & \text{for } A \in \mathcal{S}_1([T]^k+q),\ q=1,\dots,[S]-[T]^k. \end{cases} \tag{1.9.112}$$

Then u_S is an admissible test function for $\tilde{\mu}_S$; in fact $u_S = 0$ on $Q_1(1)$ and $u_S = \nu$ on $\mathcal{S}_1([S])$. Now we want to estimate the energy of u_S on $Q(S)$:

$$\begin{aligned}
\sum_{\{A,B\}\in M_1(Q(S))} f(u_S(A)-u_S(B)) &\leq \sum_{h=0}^{k-1}\sum_{\{A,B\}\in M_1(C_h)} f(u_S(A)-u_S(B)) \\
&\quad + \sum_{\{A,B\}\in M_1(Q(S)\setminus Q([T]^k))} f(u_S(A)-u_S(B)) \\
&\leq \sum_{h=0}^{k-1}\sum_{l=1}^{n}\int_{\mathbf{C}_h} f\Big(\frac{\partial \tilde{u}_{S,h}}{\partial x_l}(x)\Big)dx \\
&\quad + \sum_{\{A,B\}\in M_1(Q(S)\setminus Q([T]^k))} f(u_S(A)-u_S(B)).
\end{aligned}$$

If we set $y = [T]^{-h}x$ and we denote by $\tilde{u}_T$ the piecewise affine interpolation of u_T on the lattice $Q_1(T)$ (built on a triangulation that is homothetic to the one on which we constructed $\tilde{u}_S$), we obtain

$$\begin{aligned}
\int_{\mathbf{C}_h} f\Big(\frac{\partial \tilde{u}_{S,h}}{\partial x_l}(x)\Big)dx &= \Big(\frac{\log([T])}{\log([S])}\Big)^n \int_{\mathbf{C}_1} f\Big(\frac{\partial \tilde{u}_T}{\partial y_l}(y)[T]^{-h}\Big)[T]^{hn}dy \\
&= \Big(\frac{\log([T])}{\log([S])}\Big)^n \sum_{A\in R_1^{e_l}(Q(T))} f(u_T(A+e_l)-u_T(A))
\end{aligned}$$

for all $l \in \{1,\dots,n\}$. There follows that

$$\begin{aligned}
&\sum_{h=0}^{k-1}\sum_{\{A,B\}\in M_1(C_h)} f(u_S(A)-u_S(B)) \\
&\quad = \Big(\frac{\log([T])}{\log([S])}\Big)^n \sum_{h=0}^{k-1}\sum_{l=1}^{n}\sum_{A\in R_1^{e_l}(\overline{Q}(T))} f(u_T(A+e_l)-u_T(A)) \\
&\quad = k\Big(\frac{\log([T])}{\log([S])}\Big)^n \sum_{\{A,B\}\in M_1(Q(T))} f(u_T(A)-u_T(B)) \\
&\quad \leq \Big[\frac{\log([S])}{\log([T])}\Big]\Big(\frac{\log([T])}{\log([S])}\Big)^n\Big(m_T+\frac{1}{T}\Big). \qquad (1.9.113)
\end{aligned}$$

Finally we consider the contribution of u_S on the set $Q_1(S)\setminus Q_1([T]^k)$. By construction

$$\begin{aligned}
&\sum_{\{A,B\}\in M_1(Q(S)\setminus Q([T]^k))} f(u_S(A)-u_S(B)) \\
&\quad \leq cn^2 \sum_{q=0}^{[S]-[T]^k-1} f\Big(\nu\frac{\log([T]^k+q+1)-\log([T]^k+q)}{\log([S])}\Big) \\
&\quad \leq \frac{c}{(\log([S]))^n}([S]-[T]^k)\Big|\log\Big(1+\frac{1}{[T]^k}\Big)\Big|^n \\
&\quad \leq \frac{c}{(\log([S]))^n}([S]-[T]^k)\frac{1}{[T]^{kn}}. \qquad (1.9.114)
\end{aligned}$$

By combining (1.9.113) and (1.9.114) we get

$$
\begin{aligned}
(\log S)^{n-1}\tilde{\mu}_S &\le (\log S)^{n-1}\sum_{\{A,B\}\in M_1(Q(S))} f(u_S(A)-u_S(B))\\
&\le \Big[\frac{\log([S])}{\log([T])}\Big]\frac{\log([T])}{\log([S])}\Big((\log([T]))^{n-1}\tilde{\mu}_T+\frac{(\log([T]))^{n-1}}{T}\Big)\\
&\quad +\frac{c}{\log([S])}([S]-[T]^k)\frac{1}{[T]^{kn}}.
\end{aligned}
$$

Passing to the lim sup as $S\to+\infty$ we obtain

$$
\limsup_{S\to+\infty}(\log S)^{n-1}\tilde{\mu}_S \le (\log([T]))^{n-1}\tilde{\mu}_T+\frac{(\log([T]))^{n-1}}{T},
$$

since $k=[\log([S])/\log([T])]$. Finally, we take the lim inf as $T\to+\infty$ and we get

$$
\begin{aligned}
\limsup_{S\to+\infty}(\log S)^{n-1}\tilde{\mu}_S &\le \liminf_{T\to+\infty}(\log([T]))^{n-1}\tilde{\mu}_T+\lim_{T\to+\infty}\frac{(\log([T]))^{n-1}}{T}\\
&= \liminf_{T\to+\infty}(\log([T]))^{n-1}\tilde{\mu}_T.
\end{aligned}
$$

Hence, there exists the limit

$$
\lim_{T\to+\infty}(\log([T]))^{n-1}\tilde{\mu}_T. \tag{1.9.115}
$$

Note that for all $\nu\in\mathbb{R}^m$, $|\nu|=1$, the limit above is in $(0,+\infty)$. In fact, by the growth conditions on f there exist two constants $\tilde{c}_1$ and $\tilde{c}_2$ such that

$$
\tilde{c}_1 m^d_{1,T}\le\tilde{\mu}_T\le\tilde{c}_2 m^d_{1,T},
$$

where $m^d_{1,T}$ is as in (1.4.33). In Section 1.4 we proved that

$$
\lim_{T\to+\infty}(\log T)^{n-1}m^d_{1,T}=l_n\in(0,+\infty).
$$

By comparison, $\lim_T(\log T)^{n-1}\tilde{\mu}_T\in(0,+\infty)$.

2. It remains to show that the limit in (1.9.115) equals $d(\nu)$. First of all, we note that $\mu_T\le\tilde{\mu}_T$ by construction. Let $v_T\in\mathcal{A}_1(Q(T);\mathbb{R}^m)$ be such that $v_T(0)=0$, $v_T=\nu$ on $\mathcal{S}_1([T])$ and

$$
\sum_{\{A,B\}\in M_1(Q(T))} f(v_T(A)-v_T(B))\le\mu_T+\frac{1}{T}.
$$

Let $\eta>0$ be a fixed constant. Then, for all T large enough we have $|v_T|\le\eta$ on $Q_1(1)$. In fact: if $|v_T(a)|>\eta$ for some $a\in Q_1(1)\setminus\{0\}$, then we have

$$
\tilde{\mu}_T+\frac{1}{T}\ge\mu_T+\frac{1}{T}\ge\sum_{\{A,B\}\in M_1(Q(T))} f(v_T(A)-v_T(B))>c\eta^n.
$$

By (1.9.115) we know that

$$
\lim_{T\to+\infty}\tilde{\mu}_T+\frac{1}{T}=0,
$$

which leads to a contradiction. Therefore we have

$$\begin{aligned}
\mu_T + \frac{1}{T} &\geq \sum_{\{A,B\}\in M_1(Q(T))} f(v_T(A) - v_T(B)) \\
&\geq \inf\Big\{ \sum_{\{A,B\}\in M_1(Q(T))} f(v(A)-v(B)) : \begin{array}{l} v \in \mathcal{A}_1(Q(T)) \\ |v| \leq \eta \text{ on } Q_1(1),\ v = 1 \text{ on } \mathcal{S}_1([T]) \end{array} \Big\} \\
&= \inf\Big\{ \sum_{\{A,B\}\in M_1(Q(T))} f(v(A)-v(B)) : \begin{array}{l} v \in \mathcal{A}_1(Q(T)) \\ v = \eta \text{ on } Q_1(1),\ v = 1 \text{ on } \mathcal{S}_1([T]) \end{array} \Big\} \\
&= |1-\eta|^n \inf\Big\{ \sum_{\{A,B\}\in M_1(Q(T))} f(w(A)-w(B)) : \begin{array}{l} w \in \mathcal{A}_1(Q(T)) \\ w = 0 \text{ on } Q_1(1),\ w = 1 \text{ on } \mathcal{S}_1([T]) \end{array} \Big\} \\
&= |1-\eta|^n \tilde{\mu}_T.
\end{aligned}$$

To sum up, we got

$$\tilde{\mu}_T + \frac{1}{T} \geq \mu_T + \frac{1}{T} \geq |1-\eta|^n \tilde{\mu}_T.$$

If we multiply by $(\log T)^{n-1}$, pass to the limit as $T \to +\infty$ and take into consideration the arbitrariness of η, we deduce that the limit in (1.9.115) equals $d(\nu)$.

Finally, we notice that d can by extended to any vector in $\mathbb{R}^m$ by n-homogeneity:

$$d(z) = \begin{cases} 0 & \text{if } z = 0 \\ |z|^n d\Big(\dfrac{z}{|z|}\Big) & \text{otherwise.} \end{cases}$$

■

In conclusion, we can state and prove the Γ-convergence result in this particular case.

Proposition 1.9.2 *Let $m, n \in \mathbb{N}$ with $m \geq 1$ and $n \geq 2$. Let Ω be a bounded open subset of $\mathbb{R}^n$ with $|\partial\Omega| = 0$. Let $f : \mathbb{R}^m \to [0, +\infty)$ be a convex function which is positively homogeneous of degree n and such that $f(0) = 0$. We assume that there exist two constants $c_1, c_2 > 0$ such that $c_1|z|^n \leq f(z) \leq c_2|z|^n$ for all $z \in \mathbb{R}^m$. Let (ε_j) be a positive infinitesimal sequence. We consider an additional sequence (δ_j) such that $\delta_j/\varepsilon_j \in \mathbb{N}$, $\delta_j \gg \varepsilon_j$, $\delta_j \to 0$ and*

$$\varepsilon_j = e^{-r(1+o(1))\delta_j^{n/(1-n)}}, \qquad \textit{for some constant } r > 0.$$

For all $j \in \mathbb{N}$ we define the functional $F_{\varepsilon_j} : \mathcal{A}_{\varepsilon_j}(\Omega) \to [0, +\infty]$ as

$$F_{\varepsilon_j}(u) = \begin{cases} \displaystyle\sum_{\{a,b\}\in M_{\varepsilon_j}(\Omega)} f(u(a)-u(b)) & \textit{if } u = 0 \textit{ on } \Omega_{\delta_j} \\ +\infty & \textit{otherwise.} \end{cases} \tag{1.9.116}$$

Then F_{ε_j} Γ-converges, with respect to $L^1(\Omega;\mathbb{R}^m)$-convergence, to the limit functional $F : W^{1,n}(\Omega;\mathbb{R}^m) \to [0, +\infty)$ given by

$$F(u) = \sum_{i=1}^n \int_\Omega f\Big(\frac{\partial u}{\partial x_i}\Big)\,dx + \int_\Omega d(u)\,dx,$$

where $d : \mathbb{R}^m \to [0, +\infty)$ is obtained as

$$d(z) = \lim_{T\to+\infty} (\log T)^{n-1} \inf\Big\{ \sum_{\{A,B\}\in M_1(Q(T))} f(v(A)-v(B)) : \begin{array}{l} v \in \mathcal{A}_1(Q(T);\mathbb{R}^m) \\ v(0) = 0,\ v = z \textit{ on } \mathcal{S}_1([T]) \end{array} \Big\}.$$

Proof. The proof follows immediately from Theorem 1.3.1, Lemma 1.9.1 and (1.9.109). By Theorem 1.3.1 and (1.9.109) we deduce that there exists a subsequence (ε_{j_k}) such that $F_{\varepsilon_{j_k}}$ Γ-converges to

$$F(u) = \sum_{i=1}^{n} \int_{\Omega} f\Big(\frac{\partial u}{\partial x_i}\Big)\, dx + \int_{\Omega} \varphi(u)\, dx,$$

where

$$\varphi(z) = \lim_{\alpha\to 0^+} \lim_{k\to+\infty} |\log \varepsilon_{j_k}|^{n-1} \inf\Big\{ \sum_{\{A,B\}\in M_1(Q(\alpha S_{j_k}))} f(v(A)-v(B)) : \begin{array}{l} v(0)=0 \\ v = z \text{ on } \mathcal{S}_1([\alpha S_{j_k}]) \end{array} \Big\}$$

and $S_{j_k} = \varepsilon_{j_k}^{-1} |\log \varepsilon_{j_k}|^{(1-n)/n}$. Note that $|\log \varepsilon_{j_k}|/\log(\alpha S_{j_k}) \to 1$ for any value of $\alpha > 0$. Then

$$\varphi(z) = \lim_{\alpha,k} |\log \alpha S_{j_k}|^{n-1} \inf\Big\{ \sum_{\{A,B\}\in M_1(Q(\alpha S_{j_k}))} f(v(A) - v(B)) : \begin{array}{l} v(0)=0 \\ v = z \text{ on } \mathcal{S}_1([\alpha S_{j_k}]) \end{array} \Big\}.$$

By Lemma 1.9.2 we can deduce that $\varphi(z) = d(z)$ for all $z \in \mathbb{R}^m$, and d is independent of the subsequence ε_{j_k}, as desired. ■

1.10 Appendix: discrete Poincarè's inequalities

We give a simple proof of the discrete version of Poincarè's inequality in the simplified situations of the chapter.

Lemma 1.10.1 (A discrete version of Poincarè-Wirtinger's Lemma) *Let $\Omega \subset \mathbb{R}^n$ be a finite union of rectangles, and $p > 1$. There exists ε_0 and a constant $C = C(p,\Omega)$ such that for all $\varepsilon < \varepsilon_0$ and $u : \Omega_\varepsilon \to \mathbb{R}^m$, having set*

$$\widetilde{u} = \frac{1}{\sharp\Omega_\varepsilon} \sum_{a\in\Omega_\varepsilon} u(a),$$

we have

$$\sum_{a\in\Omega_\varepsilon} |u(a) - \widetilde{u}|^p \varepsilon^n \le C \sum_{\{a,b\}\in M_\varepsilon} \Big|\frac{u(a)-u(b)}{\varepsilon}\Big|^p \varepsilon^n. \qquad (1.10.117)$$

Proof. By construction we have

$$\begin{aligned}\sum_{a\in\Omega_\varepsilon} |u(a) - \widetilde{u}|^p \varepsilon^n &= \sum_{a\in\Omega_\varepsilon} \Big|u(a) - \frac{1}{\sharp\Omega_\varepsilon} \sum_{b\in\Omega_\varepsilon} u(b)\Big|^p \varepsilon^n \\ &= \sum_{a\in\Omega_\varepsilon} \Big|\frac{1}{\sharp\Omega_\varepsilon} \sum_{b\in\Omega_\varepsilon} (u(a)-u(b))\Big|^p \le \sum_{a\in\Omega_\varepsilon} \frac{1}{\sharp\Omega_\varepsilon} \sum_{b\in\Omega_\varepsilon} |u(a)-u(b)|^p \varepsilon^n.\end{aligned}$$

We want to estimate the term

$$\frac{1}{\sharp\Omega_\varepsilon} \sum_{a,b\in\Omega_\varepsilon} |u(a)-u(b)|^p$$

by comparing it with the sum of all nearest neighbors interactions.

Consider the case when Ω is a single rectangle of side-lengths $L_1,\dots,L_n$. Then we may consider a path connecting a and b composed of n segments in the directions $e_1,\dots,e_n$ in that order, and the points a_i on that path, so that (by Jensen's inequality) for ε small

$$|u(a)-u(b)|^p \le \frac{\max\{L_1,\dots,L_n\}^{p-1}}{\varepsilon^{p-1}} \sum_i |u(a_i) - u(a_{i-1})|^p.$$

Since each pair of nearest neighbors belongs at most to $\max\{L_1,\dots,L_n\}/\varepsilon$ such paths and $\sharp\Omega_\varepsilon$ is approximately $L_1\cdots L_n/\varepsilon^n$, we obtain

$$\frac{1}{\sharp\Omega_\varepsilon}\sum_{a,b\in\Omega_\varepsilon}|u(a)-u(b)|^p\le c\frac{(\max\{L_1,\dots,L_n\})^p}{L_1\cdots L_n}\sum_{\{a,b\}\in M_\varepsilon}\Big|\frac{u(a)-u(b)}{\varepsilon}\Big|^p\varepsilon^n$$

If Ω is a union of N rectangles of side-lenghts $L_1^j,\dots,L_n^j$ then we obtain the thesis with a constant

$$C=c\frac{(\sum_j\max\{L_1^j,\dots,L_n^j\})^p}{\sum_j L_1^j\cdots L_n^j} \tag{1.10.118}$$

by following the same reasoning, but joining points a,b in Ω by a path through possibly all the rectangles. ∎

Lemma 1.10.2 (Rescaled version of Lemma 1.10.1) *Let Ω,ε,p be as in Lemma* 1.10.1 *and let C be the constant in* (1.10.117). *We fix $\delta>0$. We denote by Ω^δ the rescaled set $\Omega^\delta=\{x\in\mathbb{R}^n:x/\delta\in\Omega\}$ and by $\Omega^\delta_\varepsilon$ the lattice $\Omega^\delta_\varepsilon=\Omega^\delta\cap\varepsilon\mathbb{Z}^n$ (and accordingly, the set of nearest neighbors M^δ_ε). Then, for $\varepsilon<\varepsilon_0$ and for all $u:\Omega^\delta_\varepsilon\to\mathbb{R}^m$ we have:*

$$\sum_{a\in\Omega^\delta_\varepsilon}|u(a)-\widetilde{u}|\varepsilon^n\le C\delta^p\sum_{\{a,b\}\in M^\delta_\varepsilon}\Big|\frac{u(a)-u(b)}{\varepsilon}\Big|^p\varepsilon^n, \tag{1.10.119}$$

where $\widetilde{u}=(\sharp\Omega^\delta_\varepsilon)^{-1}\sum_{a\in\Omega^\delta_\varepsilon}u(a)$.

Proof. By applying Lemma 1.10.1 to the function $v:\Omega_{\varepsilon/\delta}\to\mathbb{R}^m$ defined as $v(A)=u(\delta A)$ for $A\in\Omega_{\varepsilon/\delta}$, we get:

$$\sum_{A\in\Omega_{\varepsilon/\delta}}\Big|v(A)-\frac{1}{\sharp(\Omega_{\varepsilon/\delta})}\sum_{B\in\Omega_{\varepsilon/\delta}}v(B)\Big|\frac{\varepsilon^n}{\delta^n}\le C\sum_{\{A,B\}\in M_{\varepsilon/\delta}}\Big|\frac{v(A)-v(B)}{\varepsilon/\delta}\Big|^p\frac{\varepsilon^n}{\delta^n}.$$

Scaling back the space variable we obtain

$$\sum_{a\in\Omega^\delta_\varepsilon}\Big|u(a)-\frac{1}{\sharp(\Omega^\delta_\varepsilon)}\sum_{b\in\Omega^\delta_\varepsilon}\Big|\varepsilon^n\le C\delta^p\sum_{\{a,b\}\in M^\delta_\varepsilon}\Big|\frac{u(a)-u(b)}{\varepsilon}\Big|^p\varepsilon^n,$$

with C as in (1.10.118) as desired. ∎

Chapter 2

Models of defects in atomistic systems

2.1 Introduction

According to the Weak-Membrane Model by Blake and Zisserman [14], a simple way to model free-discontinuity energies in a finite-difference scheme is by considering truncated quadratic energy densities (Fig. 2.1). The energy of such a (n-dimensional) scheme can then be written

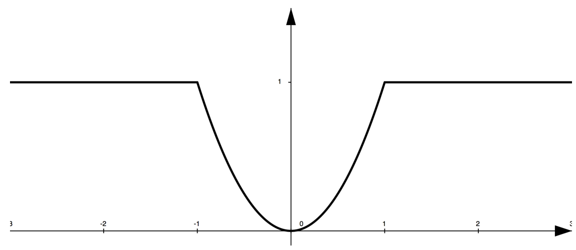

Figure 2.1: A truncated quadratic potential

as

$$E(u) = \sum_{i,j} (u_i - u_j)^2 \wedge 1,$$

where u_i is a real parameter (the vertical displacement of the 'discrete membrane'), and the sum is performed over nearest neighbors in a cubic grid parameterized by $\mathbb{Z}^n$.

Thanks to a scaling argument due to Chambolle [34], which leads to the energies

$$E_\varepsilon(u) = \sum_{i,j} \varepsilon^n \Big(\Big(\frac{u_i - u_j}{\varepsilon} \Big)^2 \wedge \frac{1}{\varepsilon} \Big),$$

this discrete model can be approximated by a continuous energy defined on special functions with bounded variation. In fact, if we limit the interactions in the sum to the nearest neighbors in the portion of $\varepsilon\mathbb{Z}^n$ contained in some fixed Ω, and we interpret the values u_i as the discretization of a function defined in Ω, then these energies can be studied using the methods of Γ-convergence, and their limit is then given by a *fracture energy*

$$F(u) = \int_\Omega |\nabla u|^2 \, dx + \int_{\Omega \cap S(u)} \|\nu\|_1 d\mathcal{H}^{n-1}$$

(see [34, 35, 24]), where $S(u)$ is the fracture site, ν is its normal and u is the macroscopic displacement outside the fracture site. The correct functional setting for these kinds of energies is the space $GSBV(\Omega)$ of (generalized) special functions of bounded variation in Ω introduced by Ambrosio and De Giorgi (see [18, 9]). From an atomistic standpoint, the energy $(u_i-u_j)^2\wedge 1$ can be interpreted as that of a 'defected' quadratic spring, which breaks after reaching a critical elongation; the collective behavior of such a system gives rise to the possibility of fracture. The critical scaling in E_ε is precisely the one that allows this behavior but forbids the accumulation of 'broken springs' on sets of dimension larger than $n-1$ while keeping the energy bounded. Note that the truncated quadratic potentials are a prototypical example to which the study of more general convex-concave atomistic potentials can be often reduced such as for Lennard Jones ones (see [26, 28])

If not all springs are 'defected', but a portion of them are simple quadratic linear springs, with corresponding energy $(u_i-u_j)^2$ (for which the Γ-limit is simply the Dirichlet integral and no discontinuity is allowed for the limit u), then the problem is more complex, and a continuous description must take into account the location and 'micro-geometry' of the two types of springs. In a probabilistic setting the location of the defected springs can be modeled in terms of realizations of i.i.d. random variables. In dimension two an analysis by Braides and Piatnitski [27] shows that the Γ-limit is deterministic and depends almost surely on the probability p of the weak springs. Its form is of 'fracture type' if p is above the *percolation threshold*, while it coincides with the Dirichlet integral for all values of p below that threshold.

A deterministic study leads necessarily to a more complex statement. In this case we look at possible Γ-limits of energies of the form

$$E_\varepsilon(u)=\sum_{i,j}\varepsilon^n f^\varepsilon_{ij}\Big(\frac{u_i-u_j}{\varepsilon}\Big),$$

where, for each ε, $f^\varepsilon_{ij}(z)$ may be chosen arbitrarily to be either z^2 or $z^2\wedge(1/\varepsilon)$.

It must be noted beforehand that, whatever the limit percentage of weak interaction is, we can obtain in the limit both the Dirichlet integral, and the Weak-Membrane Energy above; i.e., that even if we prescribe that for every subdomain $A\subset\Omega$ we have

$$\lim_{\varepsilon\to 0}\frac{\#\{(i,j)\in A\cap\varepsilon\mathbb{Z}^n: f^\varepsilon_{ij}(z)=z^2\wedge(1/\varepsilon)\}}{\#\{(i,j)\in A\cap\varepsilon\mathbb{Z}^n\}}=\theta$$

for any $\theta\in[0,1]$, we may obtain both such energies as Γ-limits for suitable choices of f^ε_{ij} (see [27] and Section 2.3.6 below). This is in contrast with formally similar problems where damaged springs are modeled as still quadratic with an energy density αz^2 with a constant $\alpha<1$ (for this 'discrete G-closure' problem see Braides and Francfort [23], and Braides and Gloria [25]). This observation leads to conjecturing that indeed the possible limit energies F are (independent of the limit density and) characterized by the two inequalities deriving from the comparison with the extreme cases; i.e.,

$$F(u)\le\int_\Omega|\nabla u|^2\,dx\qquad\text{if } u\in H^1(\Omega),$$

$$F(u)\ge\int_\Omega|\nabla u|^2\,dx+\int_{S(u)}\|\nu\|_1 d\mathcal{H}^{n-1}\qquad\text{if } u\in GSBV(\Omega).$$

The two inequalities imply that indeed $F(u)=\int_\Omega|\nabla u|^2\,dx$ if $u\in H^1(\Omega)$, and suggest the conjecture that we may obtain as limits all lower-semicontinuous energies of the form

$$F(u)=\int_\Omega|\nabla u|^2\,dx+\int_{S(u)}\varphi(x,u^+-u^-,\nu)d\mathcal{H}^{n-1}\text{ if } u\in GSBV(\Omega),$$

($u^\pm$ denote the traces of u on both sides of $S(u)$), where

- $\nu\mapsto\varphi(x,z,\nu)$ is even and $\varphi(x,z,\nu)\ge\|\nu\|_1$
- $z\mapsto\varphi(x,z,\nu)$ is even, and is increasing for positive z.

A complete proof of such a conjecture is not within the possibilities of the present knowledge of free-discontinuity functionals, even in the homogeneous case, i.e., with $\varphi(x,z,\nu)=\varphi(z,\nu)$.

Indeed, for such energy densities the condition for lower semicontinuity is *BV-ellipticity* (see Ambrosio and Braides [8]), which is the analog for interfacial energies of the condition of *quasiconvexity* for integral functionals (see Morrey [53]), and turns out to be necessary and sufficient if φ satisfies an inequality from above $\varphi(z,\nu) \le C|z|$. This last growth condition is not in general satisfied by our energies, and without this assumption neither we can apply known representation results (as those by Braides and Chiadò Piat [21] or Bouchitté et al. [16]), nor we can characterize the energy density (indeed, the problem of removing growth conditions is one of the main issues also in the theory of vector energies; see Ball and Murat [13]). But even when growth assumptions from above are satisfied and the function φ is BV-elliptic this information is of little help since explicit constructions of BV-elliptic energy densities (e.g., in the spirit of the construction of quasiconvex functions by relaxation as that by Šverák [61]) or their variational approximation by simpler energies (e.g., in the spirit of appoximation of quasiconvex energies by homogenization of polyconvex functionals as by Braides [17]) are not available in general, as are not available for arbitrary quasiconvex functions.

We will then restrict our analysis to classes of simpler energy densities, proving a number of results, each of its particular interest (summarized in Theorem 2.2.2)

1) $\varphi = \varphi(\nu)$ even. In this case the condition of BV-ellipticity is equivalent to the convexity of (the one-homogneous extension of) φ. We will prove that all such energy densities can be obtained in the limit;

2) $\varphi = \varphi(z)$. The form of the energies E_ε implies that φ is even and $z \mapsto \varphi(z)$ is increasing on $(0,+\infty)$. Moreover the growth condition gives $\varphi(z) \ge \sup_\nu \|\nu\|_1 = \sqrt{n}$. In this case the condition of BV-ellipticity is equivalent to the *subadditivity* of φ; i.e. that $\varphi(z+z') \le \varphi(z)+\varphi(z')$ for all z, z'. This condition is rather complex, and is implied by the concavity of φ on $(0,+\infty)$. We will prove the approximation result for this restricted but important class of energy densities;

3) $\varphi = \varphi(x)$ lower semicontinuous. In this case the only condition for approximation is $\varphi(x) \ge \sqrt{n}$.

Moreover we can obtain $\varphi(x,z,\nu) = \varphi_1(\nu)\varphi_2(z)\varphi_3(x)$ by combining the approximation constructions above.

We note that other types of energies can be obtained as Γ-limits; for example, those of the form

$$F(u) = \int_\Omega |\nabla u|^2\,dx + \int_{S(u)} \varphi(x, u^+ - u^-) d\mathcal{H}^{n-1} \text{ if } u \in SBV(\Omega),$$

with the constraint that $S(u) \subset K$ where K is a fixed $n-1$-dimensional surface. Indeed, such types of energies will be the building blocks of our approximation strategy. In fact, for case (1) above we will first use this construction with K a network of planar surfaces and φ suitable constants on each surface of the network, and then use an approximation procedure similar to the one by Ansini and Iosifescu [12] to obtain an arbitrary convex φ. Note that in particular we may obtain as φ any constant not larger than $\sqrt{n}$, so that case (3) can be derived by localizing such a construction. To obtain case (2), we first treat the case of K a single hyperplane and $\varphi(x,z) = c_1 + c_2 z^2$. This can be obtained following arguments similar to those by Ansini [10] to approximate the energy density $c(u^+ - u^-)^2$ on a surface (Neumann sieve) coupled with the description of the effect of pinning sites at the critical scaling developed in [57, 59]. Note that the computation of the interfacial energy gives the same constant as in the continuous case for $n = 2$, while it highlights a more complex behavior for $n \ge 3$, where a fraction of the total contribution is actually given by the strong springs at the interface, which sums up to the contribution distributed away from the interface and summarized in a capacitary formula. By repeating this argument on more parallel surfaces concentrating to the same hyperplane we can recover an arbitrary concave function by approximation with subadditive envelopes of families of functions as above (this is the only argument where concavity is used). Finally the use of a network of hyperplanes as above allows for a radially symmetric target φ.

This chapter is organized as follows. In Section 2.2 we introduce the necessary notation to state the main result (Theorem 2.2.2). In Section 2.3 we treat discrete energies with defects on coordinate hyperplanes. Its main result is Theorem 2.3.1, where we describe the effect of a (small) percentage of strong springs distributed on a planar interface, and which exhibits an interesting separation of scales effect. Another result of independent interest is Theorem 2.3.11, which treats the case when weak springs in that interface are substituted by voids (in other words, we consider two quadratic discrete media weakly connected through a hypersurface). In Section 2.4 we prove a number of Γ-convergence results for functionals defined on $GSBV$ starting from the energies obtained in Theorem 2.3.1, eventually proving Theorem 2.2.2 by successive constructions.

2.2 Setting of the problem. Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^n$ with $|\partial\Omega| = 0$. For fixed $\varepsilon > 0$ we consider the lattice $\varepsilon\mathbb{Z}^n \cap \Omega =: \Omega_\varepsilon$ and we denote by $\mathcal{A}_\varepsilon(\Omega)$ the set of functions

$$\mathcal{A}_\varepsilon(\Omega) = \{u : \varepsilon\mathbb{Z}^n \cap \Omega \to \mathbb{R}\}.$$

We define the set of all *nearest neighbors* (using a terminology borrowed from Mechanics we will also call such sets the *springs* in Ω).

$$M_\varepsilon(\Omega) = \{\{a, b\} : a, b \in \varepsilon\mathbb{Z}^n \cap \Omega \text{ and } |a - b| = \varepsilon\}. \tag{2.2.1}$$

We will simply write M_ε if Ω is fixed and no confusion is possible. In order not to count the interactions twice, nearest neighbors are defined as sets containing two points, and not as pairs in $(\varepsilon\mathbb{Z}^n \cap \Omega) \times (\varepsilon\mathbb{Z}^n \cap \Omega)$. We can equivalently state our results in the latter notation, in which case we must take care in considering symmetric subsets of $(\varepsilon\mathbb{Z}^n \cap \Omega) \times (\varepsilon\mathbb{Z}^n \cap \Omega)$ only.

With fixed a subset $W_\varepsilon \subseteq M_\varepsilon$, we define the functional $F^{W_\varepsilon} : \mathcal{A}_\varepsilon(\Omega) \to [0, +\infty)$ as

$$\begin{aligned} F^{W_\varepsilon}(u) &= \sum_{\{a,b\} \in M_\varepsilon \setminus W_\varepsilon} \varepsilon^n \Big(\frac{u(a) - u(b)}{\varepsilon}\Big)^2 + \sum_{\{a,b\} \in W_\varepsilon} \varepsilon^n \Big(\Big(\frac{u(a) - u(b)}{\varepsilon}\Big)^2 \wedge \frac{1}{\varepsilon}\Big), \\ &= \sum_{\{a,b\} \in M_\varepsilon} \varepsilon^n f^\varepsilon_{a,b}(u(a) - u(b)), \end{aligned} \tag{2.2.2}$$

where

$$f^\varepsilon_{a,b}(z) = \begin{cases} \Big(\dfrac{z}{\varepsilon}\Big)^2 & \text{if } \{a, b\} \in M_\varepsilon \setminus W_\varepsilon \\ \Big(\dfrac{z}{\varepsilon}\Big)^2 \wedge \dfrac{1}{\varepsilon} & \text{if } \{a, b\} \in W_\varepsilon. \end{cases}$$

Remark 2.2.1 A function $u \in \mathcal{A}_\varepsilon(\Omega)$ will be identified with the piecewise-constant measurable function given by $u(x) = u(z^\varepsilon_x)$, where z^ε_x is the closest point to x in $\varepsilon\mathbb{Z}^n$ (which is uniquely defined up to a set of zero measure). In this definition, we set $u(z) = 0$ if $z \in \varepsilon\mathbb{Z}^n \setminus \Omega$. In this way $A_\varepsilon(\Omega)$ will be regarded as a subset of $L^1(\Omega)$.

With the identification above, and a slight abuse of notation, we can extend F^{W_ε} to a functional $F^{W_\varepsilon} : L^1(\Omega) \to [0, +\infty]$ as

$$F^{W_\varepsilon}(u) = \begin{cases} \displaystyle\sum_{\{a,b\} \in M_\varepsilon} \varepsilon^n f^\varepsilon_{a,b}(u(a) - u(b)), & \text{if } u \in \mathcal{A}_\varepsilon(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.2.3}$$

The notation of (2.2.3) will be “localized” to subsets A of Ω by setting

$$F^{W_\varepsilon}(u; A) = \begin{cases} \displaystyle\sum_{\{a,b\} \in M_\varepsilon(A)} \varepsilon^n f^\varepsilon_{a,b}(u(a) - u(b)), & \text{if } u \in \mathcal{A}_\varepsilon(\Omega), \\ +\infty & \text{otherwise} \end{cases} \tag{2.2.4}$$

(and accordingly for other functionals).

We will study the Γ-convergence of families of such functionals with varying W_ε with respect to the L^1-convergence.

Given an arbitrary distribution of weak springs $W_j = W_{\varepsilon_j}$, we may define the limit density in Ω of the weak springs W_j. This can be done after identifying each weak spring with a scaled Dirac delta; i.e., when W_j is identified with the measure

$$\lambda^{W_j} = \frac{\varepsilon_j^n}{n} \sum_{\{a,b\}\in W_j} \delta_{(a+b)/2}.$$

Upon passing to a subsequence, λ^{W_j} has a weak* limit λ in the sense of measures. Furthermore, since this limit is simply the Lebesgue measure if $W_j = M_{\varepsilon_j}(\mathbb{R}^n)$, then λ is absolutely continuous with respect to $\mathcal{L}^n$, so that we can write $\lambda = \theta\,\mathcal{L}^n$, with $0 \le \theta \le 1$. We will then simply write

$$W_j \to \theta\,. \tag{2.2.5}$$

We will show that all results that we obtain can also be obtained by prescribing θ

As an important preliminary step in Section 2.3 we will consider the case of W_ε concentrating on coordinate hyperplanes, and in Section 2.4 we will use that result to obtain a wide class of limit energies. Before stating the main result of that section we introduce the necessary function setting.

2.2.1 Special functions of bounded variations

Our limit energies will be defined on the Ambrosio-De Giorgi space of *generalized special functions with bounded variation* $GSBV(\Omega)$ (for all the definitions in this section see e.g. [9, 18]).

We recall that the space $SBV(\Omega)$ is defined as the set of functions u in $BV(\Omega)$ such that their measure distributional derivative Du admits the representation

$$Du = \nabla u\,\mathcal{L}^n + (u^+ - u^-)\nu(u)\mathcal{H}^{n-1} \llcorner S(u),$$

where

- ∇u is the *approximate differential* of u
- $S(u)$ is the set of essential discontinuity points or *jump set* of u
- $\nu(u)$ is the measure theoretical *normal* to $S(u)$, which is defined $\mathcal{H}^{n-1}$ on $S(u)$
- $u^\pm$ are the *traces* of u on both sides of $S(u)$.

$\mathcal{L}^n$ and $\mathcal{H}^{n-1}$ denote the Lebesgue measure in $\mathbb{R}^n$ and the $n-1$-dimensional Hausdorff measure, respectively. $\lambda \llcorner B$ denotes the restriction of the measure λ to B; i.e., $(\lambda \llcorner B)(A) = \lambda(A \cap B)$.

A function u belongs to $GSBV(\Omega)$ if for all $T > 0$ its truncations $u_T := (u \wedge T) \vee (-T)$ belong to $SBV(\Omega)$.

2.2.2 Statement of the main result

The results of the final section are (partly) summarized in the following theorem, which is the main result in the chapter.

Theorem 2.2.2 *Let $\varphi : \mathbb{R}^n \to [0, +\infty)$ be any convex, even and positively homogeneous function of degree one with*

$$\varphi(w) \ge \|w\|_1 := \sum_{j=1}^n |w_j|,$$

$\psi : (0,+\infty) \to [1,+\infty)$ be any concave function, $a : \Omega \to [1,+\infty)$ be any lower semicontinuous function, and let $F : L^1(\Omega) \to [0,+\infty]$ be given by

$$F(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\,dx + \int_{\Omega\cap S(u)} a(x)\varphi(\nu(u))\psi(|u^+ - u^-|)\,d\mathcal{H}^{n-1} & \\ \qquad\qquad \text{if } u \in GSBV(\Omega) \text{ and } \mathcal{H}^{n-1}(S(u)\cap\Omega) < +\infty & \\ +\infty & \text{otherwise.} \end{cases} \tag{2.2.6}$$

Then there exists a family W_ε such that functionals F^{W_ε} given by (2.2.2) *Γ-converge to F above in the L^1-topology. Furthermore, for any $\theta \in L^\infty(\Omega)$ with $0 \le \theta \le 1$ we can additionally choose W_ε with $W_\varepsilon \to \theta$ in the sense of* (2.2.5)

2.2.3 Preliminary results

The case when $W_\varepsilon = M_\varepsilon$ in (2.2.2) is described by the following result by Chambolle [35] (see also Braides and Gelli [24])

Theorem 2.2.3 (Blake-Zisserman weak membrane) *The functionals defined by*

$$F_\varepsilon(u) = \sum_{\{a,b\}\in M_\varepsilon(\Omega)} \varepsilon^n \Big(\Big(\frac{u(a)-u(b)}{\varepsilon}\Big)^2 \wedge \frac{1}{\varepsilon}\Big) \tag{2.2.7}$$

on $\mathcal{A}_\varepsilon(\Omega)$ and extended to $L^1(\Omega)$ by $+\infty$ as in (2.2.3) *Γ-converge with respect to the $L^1(\Omega)$-convergence to the functional*

$$F(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\,dx + \int_{\Omega\cap S(u)} \|\nu(u)\|_1\,d\mathcal{H}^{n-1} & \\ \qquad\qquad \text{if } u \in GSBV(\Omega) \text{ and } \mathcal{H}^{n-1}(S(u)\cap\Omega) < +\infty & \\ +\infty & \text{otherwise.} \end{cases}$$

Furthermore, if (u_ε) is a bounded sequence in $L^\infty(\Omega)$ such that $\sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon) < +\infty$ then, up to extraction of a subsequence, it converges to a function in $SBV(\Omega)$.

Note that F is an anisotropic version of the Mumford-Shah functional, and enjoys all the coerciveness and lower-semicontinuity properties of that functional (see [18]).

Remark 2.2.4 (1) Since for general W_ε we have

$$F_\varepsilon \le F^{W_\varepsilon}$$

(F_ε as in (2.2.7)), the previous result provides a lower bound for all our Γ-limits, and in particular it implies that their domain will always be contained in

$$\{u \in GSBV(\Omega) : \mathcal{H}^{n-1}(S(u)\cap\Omega) < +\infty\}.$$

(2) Since all energies are decreasing by truncation (i.e., $F^{W_\varepsilon}(u_T) \le F^{W_\varepsilon}(u)$), it will suffice to characterize Γ-limits on

$$\{u \in SBV(\Omega)\cap L^\infty(\Omega) : \mathcal{H}^{n-1}(S(u)\cap\Omega) < +\infty\}.$$

In fact, on one hand, given a sequence u_ε converging to u, once a Γ-limit F is characterized on bounded functions, we have a lower bound

$$F(u_T) \le \liminf_{\varepsilon\to 0} F^{W_\varepsilon}((u_\varepsilon)_T) \le \liminf_{\varepsilon\to 0} F^{W_\varepsilon}(u_\varepsilon),$$

from which the liminf inequality

$$F(u) \leq \liminf_{\varepsilon\to 0} F^{W_\varepsilon}(u_\varepsilon)$$

will follow by Beppo-Levi's Theorem. On the other hand, since the Γ-limsup

$$F''(u) = \Gamma\text{-}\limsup_{\varepsilon\to 0} F^{W_\varepsilon}(u)$$

defines a lower-semicontinuous functional, from the equality $F'' = F$ for bounded functions, and the convergence $F(u_T) \to F(u)$, we have

$$F(u) = \lim_{T\to+\infty} F(u_T) = \liminf_{T\to+\infty} F''(u_T) \geq F''(u)$$

for a general u, which is the limsup inequality in the definition of Γ-limit.

(3) If $W_\varepsilon = \emptyset$ (all springs are strong), then if $\sup_j F^{W_{\varepsilon_j}}(u_j) < +\infty$, then, upon extraction of subsequences and addition of constants, u_j converge to some $u \in H^1(\Omega)$. As a consequence, the Γ-limit is simply the Dirichlet integral.

(4) For general W_ε, by comparison with the cases above, the Γ-limit always exists and is equal to the Dirichlet integral on functions $u \in H^1(\Omega)$.

We will use well-known results on $GSBV$-functions, referring to the monographs [9, 18] above mentioned when needed. We only recall the following approximation result, since it will be crucial to understand our strategy. To that end we introduce the following set of "piecewise-Lipschitz functions".

Definition 2.2.5 *We denote by* PC(Ω) *the set of all functions* $u \in SBV(\Omega)$ *such that* $\overline{S(u)}$ *is a finite union of* $(n-1)$*-dimensional simplices with disjoint closures, and* $u \in W^{1,\infty}(\Omega \setminus \overline{S(u)})$.

The set PC(Ω) is "strongly dense" in $SBV(\Omega)$ as implied by the following theorem.

Theorem 2.2.6 (Cortesani-Toader [38]) *For all* $u \in SBV(\Omega) \cap L^\infty(\Omega)$ *there exists a sequence* (u_j) *in* PC(Ω) *such that*

$$\begin{aligned}\int_\Omega |\nabla u|^2\,dx &+ \int_{\Omega\cap S(u)} f(\nu(u), u^+ - u^-)\,d\mathcal{H}^{n-1} \\ &= \lim_j \Big(\int_\Omega |\nabla u_j|^2\,dx + \int_{\Omega\cap S(u_j)} f(\nu(u_j), u_j^+ - u_j^-)\,d\mathcal{H}^{n-1}\Big)\end{aligned}$$

for all continuous f. *Moreover we can take* $u_j \in C^\infty(\Omega \setminus \overline{S(u_j)}) \cap W^{k,\infty}(\Omega \setminus \overline{S(u_j)})$ *for all* k.

Remark 2.2.7 For functions $u \in$ PC(Ω) it is easily seen that the Γ-limit in Theorem 2.2.3 is actually a pointwise limit (see [35, 24]).

The approximation result above will guarantee that it is sufficient to prove the Γ-limsup inequality for functions in $SBV(\Omega)\cap L^\infty(\Omega)$, whose jump set is a finite union of $n-1$-dimensional simplices and are smooth outside that jump set. In particular, for those functions the jump set is contained in a finite union of hyperplanes. It will be crucial then first to construct limit energies whose domain implies the constraint that the jump set be (a union of simplices) contained in a given finite union of hyperplanes, and then remove that constraint through a homogenization procedure by considering an "invading" family of hyperplanes.

2.3 Discrete energies with defects on coordinate hyperplanes

This section will be the cornerstone of our approximation procedure. We will analyze the case when the 'defected springs' W_ε are located across a coordinate hyperplane, that we can assume being $x_n = 0$. More precisely, pairs in W_ε have their middle point on the set $\mathbb{Z}^{n-1} \times \varepsilon/2$, contained in the hyperplane $\{x_n = \varepsilon/2\}$ (essentially, the hyperplane $\{x_n = 0\}$).

2.3.1 Notation

In this section we will use a notation suitable to the discrete setting. The subscript ε will indicate the intersection with $\varepsilon\mathbb{Z}^n$, so that in particular

$$\Omega_\varepsilon = \Omega \cap \varepsilon\mathbb{Z}^n.$$

The closed cube centered in x and with side length $2L$ will be denoted by

$$Q(L;x) = x + [-L, L]^n.$$

If $x = 0$ then $Q_L = Q(L, 0)$. Accordingly, we will write

$$Q_\varepsilon(L;x) = \Omega_\varepsilon \cap \big(x + [-L, L]^n\big).$$

Intersections of boundary of cubes with $\varepsilon\mathbb{Z}^n$ will be denoted by

$$\mathcal{L}(L;x) = \Omega_\varepsilon \cap \partial(x + [-L, L]^n),$$

and we write $\mathcal{L}(L) = \mathcal{L}(L, 0)$ when $x = 0$.

A subset A of $\varepsilon\mathbb{Z}^n$ is identified with the measurable set in $\mathbb{R}^n$ obtained as the union of the ε-cubes centered in A. We will highlight this identification with boldface cases:

$$\mathbf{A} = \bigcup_{a\in A} \mathcal{C}(a),$$

where

$$\mathcal{C}(a) = \mathcal{C}_\varepsilon(a) = Q(\varepsilon/2; a) = a + \Big[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\Big]^n$$

.

Finally, $[t]$ stands for the integer part of t.

2.3.2 Statement of the result

The following theorem describes the situation when all springs (parameterized by $\varepsilon_j\mathbb{Z}^{n-1}$) across the coordinate hyperplane are defected except those on a lattice $\delta_j\mathbb{Z}^{n-1}$ with $\delta_j >> \varepsilon_j$.

We denote by K the set $K = \Omega \cap \{x \in \mathbb{R}^n : \ x_n = 0\}$. If necessary, a vector $a \in \mathbb{R}^n$ will be written as $a = (a', a_n)$, with $a' = (a_1, \ldots, a_{n-1})$.

Theorem 2.3.1 *Let (ε_j) be a positive sequence such that $\varepsilon_j \to 0$ as $j \to +\infty$. Let (δ_j) be a positive infinitesimal sequence such that $\delta_j/\varepsilon_j \in \mathbb{N}$ and $\lim_j \delta_j/\varepsilon_j = +\infty$. We assume that (ε_j) and (δ_j) are such that*

$$\varepsilon_j = \begin{cases} e^{-\beta(1+o(1))/\delta_j} \ \text{as } j \to +\infty & \text{if } n = 2 \\ \beta^{2-n}\delta_j^{(n-1)/(n-2)}(1 + o(1)) \ \text{as } j \to +\infty & \text{if } n > 2 \end{cases} \tag{2.3.8}$$

where β is a positive constant. For all $j \in \mathbb{N}$ we set

$$W_{\varepsilon_j} = \{\{a, b\} \in M_j : \ a' = b', a_n = 0, b_n = \varepsilon_j, a' \in \varepsilon_j\mathbb{Z}^{n-1} \setminus \delta_j\mathbb{Z}^{n-1}\}. \tag{2.3.9}$$

Let C_n be defined as follows:

$$C_n = \begin{cases} \dfrac{\pi}{2} & \text{if } n = 2 \\ \dfrac{l_n}{4 + l_n} & \text{if } n > 2, \end{cases} \tag{2.3.10}$$

where

$$l_n = \lim_{T\to+\infty} \min\Big\{ \sum_{\{a,b\}\in M_1(Q_T)} (v(a)-v(b))^2 : \ v=1 \ on \ \partial Q_T, \ v(0)=v(e_n)=0\Big\}$$

(recall that $M_1(\Omega)$ denotes the set of nearest neighbors of $\mathbb{Z}^n$ inside Ω) is a positive constant for all $n>2$.

Let $F : L^1(\Omega) \to [0,+\infty]$ be given by

$$F(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 dx + \int_{S(u)} \Big(1+\frac{C_n}{\beta}|u^+-u^-|^2\Big) d\mathcal{H}^{n-1} & \text{if } u\in SBV(\Omega), \ S(u)\subseteq K \\ +\infty & \text{otherwise.}\end{cases}$$

Then we have

(i) (coerciveness) *for any sequence $(u_j)_j$ bounded in $L^1(\Omega)$ such that $\sup_j F^{W_{\varepsilon_j}}(u_j)<+\infty$ there exist a subsequence $(u_{j_h})_h$ and a function $u\in SBV(\Omega)$ with $S(u)\subseteq K$ such that $u_{j_h}\to u$ as $h\to+\infty$ in $L^1(\Omega)$;*

(ii) (lower bound) *for all $u\in L^1(\Omega)$ and $u_j\to u$ in $L^1(\Omega)$ we have*

$$F(u) \le \liminf_j F^{W_{\varepsilon_j}}(u_j); \tag{2.3.11}$$

(iii) (upper bound) *for all $u\in \mathrm{PC}(\Omega)$ there exists $u_j\to u$ in $L^1(\Omega)$ such that*

$$F(u) = \lim_j F^{W_{\varepsilon_j}}(u_j). \tag{2.3.12}$$

Note that the coerciveness is an immediate consequence of Remark 2.2.4 (1) and (3) (the latter applied on all open sets not intersecting K). The rest of the theorem will be proven throughout this section, separately proving the upper and lower bounds.

Remark 2.3.2 1) Note that, since we have the constraint $S(u)\subset K$ for the jump set, the domain of F is actually contained in $H^1(\Omega\setminus K)$;

2) It is worth noting the two different definitions of the constant C_n in the cases $n=2$, which are connected to capacitary issues due to the presence of a portion of strong springs on the interface. In particular,

• in the case $n=2$ the constant is the same as the one for the Neumann sieve for continuous problems. This highlights that capacitary potentials in dimension 2 are logarithmic, and their contribution is at a scale much larger than the lattice;

• in the case $n\ge 3$ a scaling argument leads to a *discrete capacitary problem* involving the capacity l_n of a *discrete dipole* in $\mathbb{Z}^n$;

3) Hypothesis (2.3.8) can be restated in terms of the percentage $p_j=(\varepsilon_j/\delta_j)^{n-1}$ of strong springs at the interface, which now reads

$$p_j = \begin{cases} \dfrac{\varepsilon_j|\log\varepsilon_j|}{\beta}(1+o(1)) & \text{if } n=2\\ \dfrac{\varepsilon_j}{\beta}(1+o(1)) & \text{if } n>2.\end{cases}$$

Remark 2.3.3 The constraint in the theorem can be generalized to $S(u)\subset K$ up to sets of $\mathcal{H}^{n-1}$-measure zero, where K is the closure of a relatively open subset A of a coordinate hyperplane with $\mathcal{H}^{n-1}(K)=\mathcal{H}^{n-1}(A)$, or more in general $S(u)\subset K$ up to sets of $\mathcal{H}^{n-1}$-measure zero, where K is the closure of a relatively open subset A of a union of coordinate hyperplanes. with $\mathcal{H}^{n-1}(K)=\mathcal{H}^{n-1}(A)$. The proof is exactly the same, upon noticing that the constraint $S(u)\subset K$ up to sets of $\mathcal{H}^{n-1}$-measure zero is closed, thus compatible with the lower bound, and that the proof of the upper bound only involves a local argument.

For notational simplicity we will often write F_j in place of $F^{W_{\varepsilon_j}}$.

2.3.3 Two technical lemmas

In this section we state and prove the discrete analogs of the key propositions that allow the treatment of perforated domains as envisaged by Ansini and Braides [11] and restated for transmission problems across an interface by Ansini [10]. These results will be used to reduce to the case when the competing functions are constant on the upper and lower parts of the boundary of suitable squares centered in the strong springs of the interface. This reduction will then allow to estimate the contribution close to the strong springs at the interface with suitable discrete capacitary problems.

The set Z_j will be defined as

$$Z_j = \{(z',0) : z' \in \delta_j \mathbb{Z}^{n-1} : Q(\delta_j/2; (z',0)) \subset \Omega\}. \tag{2.3.13}$$

Recall that $\delta_j/\varepsilon_j \in \mathbb{N}$, so that $\delta_j\mathbb{Z}^{n-1} \subset \varepsilon_j\mathbb{Z}^{n-1}$.

Lemma 2.3.4 *Let $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega)$ and $u \in SBV(\Omega)$ with $S(u) \subset K$. We assume that $u_j \to u$ in $L^1(\Omega)$ and $\sup_j F_j(u_j) < +\infty$. With fixed $\alpha < 1/2$, let $\rho_j = \alpha\delta_j$. Let $k \in \mathbb{N}$ be fixed. Then, for all $l \in Z_j$ there exists $k_l \in \{0, 1, \dots, k-1\}$ such that, having set*

$$C_j^l = Q_{\varepsilon_j}\Big(\Big[\frac{1}{2^{k_l}}\frac{\rho_j}{\varepsilon_j}\Big]\varepsilon_j, l\Big) \setminus Q_{\varepsilon_j}\Big(\Big[\frac{1}{2^{k_l+1}}\frac{\rho_j}{\varepsilon_j}\Big]\varepsilon_j, l\Big) \tag{2.3.14}$$

$$C_j^{l\pm} = C_j^l \cap \{\pm(x_n - \varepsilon_j/2) \geq 0\} \tag{2.3.15}$$

$$u_j^{l\pm} = \frac{1}{\sharp C_j^{l\pm}} \sum_{a \in C_j^{l\pm}} u_j(a) \tag{2.3.16}$$

$$\rho_j^l = \Big[\frac{3}{4}\frac{1}{2^{k_l}}\frac{\rho_j}{\varepsilon_j}\Big]\varepsilon_j, \tag{2.3.17}$$

there exists a sequence $w_j \in \mathcal{A}_{\varepsilon_j}(\Omega)$ such that $w_j \to u$ in $L^1(\Omega)$ satisfying the following conditions:

$$w_j = u_j \ \text{ on } \Omega_j \setminus \bigcup_{l \in Z_j} C_j^l \tag{2.3.18}$$

$$w_j = u_j^{l\pm} \ \text{ on } \mathcal{L}_{\varepsilon_j}\big(l, \rho_j^l\big) \cap \{\pm(x_n - \varepsilon_j/2) \geq 0\} \tag{2.3.19}$$

$$\Big|\sum_{l \in Z_j}\Big(F_j(u_j; C_j^{l+}) + F_j(u_j; C_j^{l-}) - \big(F_j(w_j; C_j^{l+}) + F_j(w_j; C_j^{l-})\big)\Big)\Big| \leq \frac{c}{k}. \tag{2.3.20}$$

Proof. For all $h \in \{0, 1, \dots, k-1\}$ we define

$$C_{j,h}^l = Q_{\varepsilon_j}\Big(\Big[\frac{1}{2^h}\frac{\rho_j}{\varepsilon_j}\Big]\varepsilon_j, l\Big) \setminus Q_{\varepsilon_j}\Big(\Big[\frac{1}{2^{h+1}}\frac{\rho_j}{\varepsilon_j}\Big]\varepsilon_j, l\Big)$$

$$C_{j,h}^{l\pm} = C_{j,h}^l \cap \{\pm(x_n - \varepsilon_j/2) \geq 0\}$$

$$u_{j,h}^{l\pm} = \frac{1}{\sharp C_{j,h}^{l\pm}} \sum_{a \in C_{j,h}^{l\pm}} u_j(a)$$

$$\rho_{j,h}^l = \Big[\frac{3}{4}\frac{1}{2^h}\frac{\rho_j}{\varepsilon_j}\Big]\varepsilon_j.$$

For fixed $h \in \{0, 1, \dots, k-1\}$ we consider a function $\phi = \phi_{j,h}^l \in C_c^\infty(\mathbf{C}_{j,h}^l)$ such that $\phi = 1$ on $\partial Q(\rho_{j,h}^l, l)$ and $\|\nabla\phi\|_\infty \leq c/\delta_j$. We define a sequence $w_{l,h}^j \in \mathcal{A}_{\varepsilon_j}(\Omega)$ as follows:

$$w_{l,h}^j(a) = \begin{cases} \phi(a)u_{j,h}^{l+} + (1-\phi(a))u_j(a) & \text{if } a_n \geq \varepsilon_j \\ \\ \phi(a)u_{j,h}^{l-} + (1-\phi(a))u_j(a) & \text{if } a_n \leq 0. \end{cases} \tag{2.3.21}$$

We focus our attention on $C^{l+}_{j,h}$ and notice that for all $\{a,b\} \in M_{\varepsilon_j}(C^{l+}_{j,h})$ we have:

$$w^j_{l,h}(a) - w^j_{l,h}(b) = (1-\phi(b))(u_j(a)-u_j(b)) + (\phi(a)-\phi(b))(u^{l+}_{j,h} - u_j(a)).$$

Moreover, by Jensen's inequality and the assumptions on ϕ we get that

$$|\phi(a)-\phi(b)|^2 \le \varepsilon_j^2 \|\nabla\phi\|^2_\infty \le c\varepsilon_j^2\delta_j^{-2} \qquad \text{for all } \{a,b\} \in M_{\varepsilon_j}(C^{l+}_{j,h})$$

There follows that

$$\begin{aligned}
&\sum_{\{a,b\}\in M_{\varepsilon_j}(C^{l+}_{j,h})} |w^j_{l,h}(a) - w^j_{l,h}(b)|^2 \varepsilon_j^{n-2} \\
&\le c \sum_{\{a,b\}\in M_{\varepsilon_j}(C^{l+}_{j,h})} \big(|1-\phi(b)|^2|u_j(a)-u_j(b)|^2 + |\phi(a)-\phi(b)|^2|u^{l+}_{j,h}-u_j(a)|^2\big)\varepsilon_j^{n-2} \\
&\le c \sum_{\{a,b\}\in M_{\varepsilon_j}(C^{l+}_{j,h})} |u_j(a)-u_j(b)|^2\varepsilon_j^{n-2} + \frac{c}{\delta_j^2}\varepsilon_j^n \sum_{\{a,b\}\in M_{\varepsilon_j}(C^{l+}_{j,h})} |u^{l+}_{j,h} - u_j(a)|^2.
\end{aligned}$$

By Lemma 1.10.2 we deduce that the last term above can be estimated as follows:

$$\frac{c}{\delta_j^2}\varepsilon_j^n \sum_{a\in C^{l+}_{j,h}} |u^{l+}_{j,h} - u_j(a)|^2 \le c \sum_{\{a,b\}\in M_{\varepsilon_j}(C^{l+}_{j,h})} |u_j(a)-u_j(b)|^2\varepsilon_j^{n-2},$$

hence

$$\sum_{\{a,b\}\in M_{\varepsilon_j}(C^{l+}_{j,h})} |w^j_{l,h}(a) - w^j_{l,h}(b)|^2 \varepsilon_j^{n-2} \le c \sum_{\{a,b\}\in M_{\varepsilon_j}(C^{l+}_{j,h})} |u_j(a)-u_j(b)|^2\varepsilon_j^{n-2}.$$

Note that the constant c in the right-hand side can be chosen such that it is independent of $h \in \{0,\dots,k-1\}$, thanks to the fact that the sets $C^{l+}_{j,h}$ are obtained from homothetic sets. Arguing similarly on $C^{l-}_{j,h}$ we deduce that

$$\sum_{\{a,b\}\in M_{\varepsilon_j}(C^{l-}_{j,h})} |w^j_{l,h}(a) - w^j_{l,h}(b)|^2 \varepsilon_j^{n-2} \le c \sum_{\{a,b\}\in M_{\varepsilon_j}(C^{l-}_{j,h})} |u_j(a)-u_j(b)|^2\varepsilon_j^{n-2}.$$

Therefore there exists $c > 0$ (independent of h) such that

$$F_j(w^j_{l,h}; C^{l+}_{j,h}) + F_j(u_j; C^{l+}_{j,h}) \le c\, F_j(u_j; C^{l+}_{j,h})$$

and

$$F_j(w^j_{l,h}; C^{l-}_{j,h}) + F_j(u_j; C^{l-}_{j,h}) \le c\, F_j(u_j; C^{l-}_{j,h}).$$

Summing up over $h \in \{0,1,\dots,k-1\}$ we get

$$\sum_{h=0}^{k-1} \big(F_j(w^j_{l,h}; C^{l+}_{j,h}) + F_j(u_j; C^{l+}_{j,h}) + F_j(w^j_{l,h}; C^{l-}_{j,h}) + F_j(u_j; C^{l-}_{j,h})\big) \le cF_j(u_j; Q_{\varepsilon_j}(\rho_j, l)).$$

Hence there exists $k_l \in \{0,1,\dots,k-1\}$ such that

$$F_j(w^j_{l,h}; C^{l+}_{j,k_l}) + F_j(u_j; C^{l+}_{j,k_l}) + F_j(w^j_{l,h}; C^{l-}_{j,k_l}) + F_j(u_j; C^{l-}_{j,k_l}) \le \frac{c}{k} F_j(u_j; Q_{\varepsilon_j}(\rho_j, l)).$$

Now, we set

$$C^l_j = C^l_{j,k_l}, \qquad C^{l\pm}_j = C^{l\pm}_{j,k_l}, \qquad u^{l\pm}_j = u^{l\pm}_{j,k_l}, \qquad \rho^l_j = \rho^l_{j,k_l}$$

and we define $w_j : \Omega_j \to \mathbb{R}$ as follows:

$$w_j(a) = \begin{cases} w^j_{l,k_l}(a) & \text{if } a \in C^l_j, \qquad l \in Z_j \\ u_j(a) & \text{if } a \in \Omega_j \setminus \bigcup_{l\in Z_j} C^l_j. \end{cases} \tag{2.3.22}$$

The sequence (w_j) defined in (2.3.22) satisfies all the required conditions. In fact, $w_j \equiv u_j$ on $\Omega_j \setminus \bigcup_{l\in Z_j} C_j^l$, $w_j = u_j^{l\pm}$ on $\partial Q(\rho_j^l, l) \cap \{\pm(x_n - \varepsilon_j/2) \geq 0\}$ and

$$\begin{aligned}
&\Big| \sum_{l\in Z_j} \Big(F_j(u_j; C_j^{l+}) + F_j(u_j; C_j^{l-}) - \big(F_j(w_j; C_j^{l+}) + F_j(w_j; C_j^{l-})\big)\Big)\Big| \\
&\quad \leq \sum_{l\in Z_j} \big(F_j(u_j; C_j^{l+}) + F_j(u_j; C_j^{l-}) + F_j(w_j; C_j^{l+}) + F_j(w_j; C_j^{l-})\big) \\
&\quad \leq \frac{c}{k} \sum_{l\in Z_j} F_j(u_j; Q_{\varepsilon_j}(\rho_j, l)) \leq \frac{c}{k} F_j(u_j) \leq \frac{c}{k}.
\end{aligned}$$

Moreover, we show that $w_j \to u$ in $L^1(\Omega)$:

$$\begin{aligned}
\int_\Omega |w_j - u| dx &\leq \int_{\Omega \setminus \bigcup_{l\in Z_j} \mathbf{C}_j^l} |w_j - u| dx + \sum_{l\in Z_j} \int_{\mathbf{C}_j^l} |w_j - u| dx \\
&\leq \int_{\Omega \setminus \bigcup_{l\in Z_j} \mathbf{C}_j^l} |u_j - u| dx + \sum_{l\in Z_j} \int_{\mathbf{C}_j^l} |w_j - u_j| dx + \sum_{l\in Z_j} \int_{\mathbf{C}_j^l} |u_j - u| dx \\
&\leq \int_\Omega |u_j - u| dx + \sum_{l\in Z_j} \Big(\sum_{a\in C_j^{l+}} |u_j(a) - u_j^{l+}| \varepsilon_j^n + \sum_{a\in C_j^{l-}} |u_j(a) - u_j^{l-}| \varepsilon_j^n \Big).
\end{aligned}$$

By Hölder's inequality and Lemma 1.10.2 we have

$$\begin{aligned}
\sum_{l\in Z_j} \sum_{a\in \mathbf{C}_j^{l\pm}} |u_j(a) - u_j^{l\pm}| \varepsilon_j^n &= \sum_{l\in Z_j} \sum_{a\in C_j^{l\pm}} |u_j(a) - u_j^{l\pm}| \varepsilon_j^n \\
&\leq \sum_{l\in Z_j} \varepsilon_j^{n/2} \Big(\sum_{a\in C_j^{l\pm}} |u_j(a) - u_j^{l\pm}|^2 \varepsilon_j^n \Big)^{1/2} \big(\sharp C_j^{l\pm}\big)^{1/2} \\
&\leq c\delta_j^{n/2} \sum_{l\in Z_j} \Big(c\delta_j^2 \sum_{\{a,b\}\in M_{\varepsilon_j}(C_j^{l\pm})} |u_j(a) - u_j(b)|^2 \varepsilon_j^{n-2} \Big)^{1/2} \\
&\leq c\delta_j^{n/2} \delta_j (\sharp Z_j)^{1/2} \big(F_j(u_j)\big)^{1/2} \leq c\delta_j^{3/2}.
\end{aligned}$$

In conclusion

$$\limsup_j \int_\Omega |w_j - u| dx \leq \limsup_j \Big(\int_\Omega |u_j - u| dx + c\delta_j^{3/2} \Big) = 0$$

as desired. ■

Proposition 2.3.5 *Let (u_j) be a sequence such that $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega)$ and $u_j \to u$ in $L^1(\Omega)$ for some $u \in SBV(\Omega)$ with $S(u) \subset K$. Assume that (u_j) is bounded in $L^\infty(\Omega)$. We fix $k \in \mathbb{N}$ and consider a positive infinitesimal sequence $\rho_j = \alpha\delta_j$, with $\alpha < 1/2$. Following the notation of Lemma 2.3.4, we fix (arbitrarily) $k_l \in \{0, 1 \ldots, k-1\}$ and we denote by $u_j^{l\pm}$ the discrete average of u_j on $C_{j,k_l}^{l\pm}$. Then we have*

$$\lim_j \sum_{l\in Z_j} |u_j^{l+} - u_j^{l-}|^2 \delta_j^{n-1} = \int_{\Omega\cap S(u)} |u^+ - u^-|^2 d\mathcal{H}^{n-1}. \tag{2.3.23}$$

Proof. For all $l \in Z_j$ we define $I_j^l = l + [-\delta_j/2, \delta_j/2)^{n-1} \subseteq K$. Let $\psi_j : \Omega \cap K \to [0, +\infty)$ be given by

$$\psi_j(x', 0) = \sum_{l\in Z_j} |u_j^{l+} - u_j^{l-}|^2 \chi_{I_j^l}(x') = \begin{cases} |u_j^{l+} - u_j^{l-}|^2 & \text{for } x' \in I_j^l, \quad l \in Z_j, \\ 0 & \text{otherwise.} \end{cases}$$

We have to prove that

$$\lim_j \int_{\Omega\cap K} \psi_j(x',0)dx' = \int_{\Omega\cap S(u)} |u^+ - u^-|^2 d\mathcal{H}^{n-1}. \tag{2.3.24}$$

Since (u_j) is bounded in $L^\infty(\Omega)$, the following inequalities hold:

$$\begin{aligned}
\Big| \int_{\Omega\cap K} \Big(\sum_{l\in Z_j} |u_j^{l+} - u_j^{l-}|^2 \chi_{I_j^l}(x') - |u^+(x',0) - u^-(x',0)|^2 \Big) dx' \Big| \\
\leq \sum_{l\in Z_j} \int_{I_j^l} \big| |u_j^{l+} - u_j^{l-}|^2 - |u^+(x',0) - u^-(x',0)|^2 \big| dx' \\
\leq c \sum_{l\in Z_j} \int_{I_j^l} \big(|u_j^{l+} - u^+(x',0)| + |u_j^{l-} - u^-(x',0)| \big) dx'.
\end{aligned}$$

We want to prove that

$$\limsup_j \sum_{l\in Z_j} \int_{I_j^l} |u_j^{l\pm} - u^\pm(x',0)| dx' = 0. \tag{2.3.25}$$

We notice that

$$\sum_{l\in Z_j} \int_{I_j^l} |u_j^{l+} - u^+(x',0)| dx' \leq \sum_{l\in Z_j} \int_{I_j^l} |u_j^{l+} - u_j(x',\varepsilon_j)| dx' + \int_{I_j^l} |u^+(x',0) - u_j(x',\varepsilon_j)| dx'$$

and

$$\sum_{l\in Z_j} \int_{I_j^l} |u_j^{l-} - u^-(x',0)| dx' \leq \sum_{l\in Z_j} \int_{I_j^l} |u_j^{l-} - u_j(x',0)| dx' + \int_{I_j^l} |u^-(x',0) - u_j(x',0)| dx'.$$

We focus our attention on $\Omega^+ = \Omega \cap \{x_n > 0\}$ and we prove that

$$\limsup_j \sum_{l\in Z_j} \int_{I_j^l} |u_j^{l+} - u_j(x',\varepsilon_j)| dx' = 0. \tag{2.3.26}$$

By Hölder's and Jensen's inequalities we get

$$\begin{aligned}
\sum_{l\in Z_j} \int_{I_j^l} |u_j^{l+} - u_j(x',\varepsilon_j)| dx' &\leq c \sum_{l\in Z_j} \delta_j^{(n-1)/2} \Big(\int_{I_j^l} |u_j^{l+} - u_j(x',\varepsilon_j)|^2 dx' \Big)^{1/2} \\
&\leq c \big(\sharp Z_j\big)^{1/2} \delta_j^{(n-1)/2} \Big(\sum_{l\in Z_j} \int_{I_j^l} |u_j^{l+} - u_j(x',\varepsilon_j)|^2 dx' \Big)^{1/2} \\
&\leq c \Big(\sum_{l\in Z_j} \int_{I_j^l} |u_j^{l+} - u_j(x',\varepsilon_j)|^2 dx' \Big)^{1/2}.
\end{aligned}$$

By construction

$$\int_{I_j^l} |u_j^{l+} - u_j(x',\varepsilon_j)|^2 dx' = \sum_{(a',\varepsilon_j)\in\Omega_j:\ a'\in I_j^l} |u_j^{l+} - u_j(a',\varepsilon_j)|^2 \varepsilon_j^{n-1}.$$

We claim that the following inequality holds:

$$\begin{aligned}
&\sum_{(a',\varepsilon_j)\in\Omega_j:\ a'\in I_j^l} |u_j^{l+} - u_j(a',\varepsilon_j)|^2 \varepsilon_j^{n-1} \\
&\leq c \Big(\frac{1}{\delta_j} \sum_{a\in R_j^{l+}} |u_j^{l+} - u_j(a)|^2 \varepsilon_j^n + \delta_j \sum_{\{a,b\}\in M_{\varepsilon_j}(R_j^{l+})} |u_j(a) - u_j(b)|^2 \varepsilon_j^{n-2} \Big),
\end{aligned} \tag{2.3.27}$$

where $R_j^{l+} = \Omega_j \cap \{I_j^l \times [\varepsilon_j, \delta_j/2]\}$ and the constant c is independent of j and l. By applying Lemma 1.10.2 to the first term in (2.3.27) we deduce that

$$\sum_{(a',\varepsilon_j)\in\Omega_j:\ a'\in I_j^l} |u_j^{l+} - u_j(a',\varepsilon_j)|^2 \varepsilon_j^{n-1} \le c\delta_j \sum_{\{a,b\}\in M_{\varepsilon_j}(R_j^{l+})} |u_j(a) - u_j(b)|^2 \varepsilon_j^{n-2}.$$

By summing over $l \in Z_j$ we get

$$\begin{aligned}\sum_{l\in Z_j} \int_{I_j^l} |u_j^{l+} - u_j(x',\varepsilon_j)|^2 dx' &\le c\delta_j \sum_{l\in Z_j} \sum_{\{a,b\}\in M_{\varepsilon_j}(R_j^{l+})} |u_j(a) - u_j(b)|^2 \varepsilon_j^{n-2} \\ &\le c\delta_j \sup_j F_j(u_j) \le c\delta_j\end{aligned}$$

and this implies that (2.3.26) holds. Moreover, by the definition of the trace of a function in $SBV(\Omega)$ (actually in $H^1(\Omega \setminus K)$), we deduce that

$$\limsup_j \sum_{l\in Z_j} \int_{I_j^l} |u^+(x',0) - u_j(x',\varepsilon_j)| dx' = 0. \tag{2.3.28}$$

By (2.3.26) and (2.3.28) we deduce that

$$\limsup_j \sum_{l\in Z_j} \int_{I_j^l} |u_j^{l+} - u^+(x',0)| dx' = 0,$$

as desired. By arguing similarly on Ω^-, we can conclude that (2.3.25) holds. It remains to show that inequality (2.3.27) holds. We have:

$$\begin{aligned}&\sum_{(a',\varepsilon_j)\in\Omega_j:\ a'\in I_j^l} |u_j^{l+} - u_j(a',\varepsilon_j)|^2 \varepsilon_j^{n-1} \\
&\le c\frac{\varepsilon_j^n}{\delta_j^n} \sum_{b\in R_j^{l+}} \sum_{(a',\varepsilon_j)\in\Omega_j:\ a'\in I_j^l} |u_j^{l+} - u_j(a',\varepsilon_j)|^2 \varepsilon_j^{n-1} \\
&\le c\frac{\varepsilon_j^n}{\delta_j^n} \sum_{b\in R_j^{l+}} \sum_{(a',\varepsilon_j)\in\Omega_j:\ a'\in I_j^l} \big(|u_j^{l+} - u_j(b)|^2 + |u_j(b) - u_j(a',\varepsilon_j)|^2\big) \varepsilon_j^{n-1} \\
&\le c\frac{\varepsilon_j}{\delta_j} \sum_{b\in R_j^{l+}} |u_j^{l+} - u_j(b)|^2 \varepsilon_j^{n-1} + c\frac{\varepsilon_j^n}{\delta_j^n} \sum_{b\in R_j^{l+}} \sum_{(a',\varepsilon_j)\in\Omega_j:\ a'\in I_j^l} |u_j(b) - u_j(a',\varepsilon_j)|^2 \varepsilon_j^{n-1} \\
&\le \frac{c}{\delta_j} \sum_{b\in R_j^{l+}} |u_j^{l+} - u_j(b)|^2 \varepsilon_j^n + c\frac{\varepsilon_j^n}{\delta_j^n}\frac{\delta_j^{n+1}}{\varepsilon_j^{n+1}} \sum_{\{a,b\}\in M_{\varepsilon_j}(R_j^{l+})} |u_j(a) - u_j(b)|^2 \varepsilon_j^{n-1} \\
&\le \frac{c}{\delta_j} \sum_{b\in R_j^{l+}} |u_j^{l+} - u_j(b)|^2 \varepsilon_j^n + c\delta_j \sum_{\{a,b\}\in M_{\varepsilon_j}(R_j^{l+})} |u_j(a) - u_j(b)|^2 \varepsilon_j^{n-2}\end{aligned}$$

as desired. ■

2.3.4 Lower bound

In this section we prove the lower bound for the sequence F_j by combining a scale-separation and a capacitary argument. The energy of a sequence $F_j(u_j)$, with $u_j \to u$, can be decomposed in

- a bulk energy away from the interface;

• an interfacial term due to the presence of the weak springs at the interface. This term corresponds to the surface term of the Blake-Zisserman weak membrane;

• an additional interfacial term decoupled from the previous one due to the presence of a (small) percentage of strong springs at the interface. This is a quadratic term on the interface depending on the discontinuity $|u^+ - u^-|^2$.

The key argument is the separation of scales at the interface. By using Lemma 2.3.4 we can separately examine the energy contributions on cubes of side length $\alpha\delta_j$ (with α small) and centered on the strong springs, and the energy elsewhere. Outside those cubes a lower bound is given by the Blake-Zisserman weak membrane (with an error vanishing with α). Again by Lemma 2.3.4 it is not restrictive to suppose that on the (upper/lower) boundary of the cubes the value of the functions u_j is exactly $u^\pm$, and a capacitary argument then allows to compare the contribution on each cube by a term $\delta_j^{n-1} c_n |u^+ - u^-|^2$, which gives a Riemann sum converging to the correct interfacial energy (the precise statement uses Proposition 2.3.5).

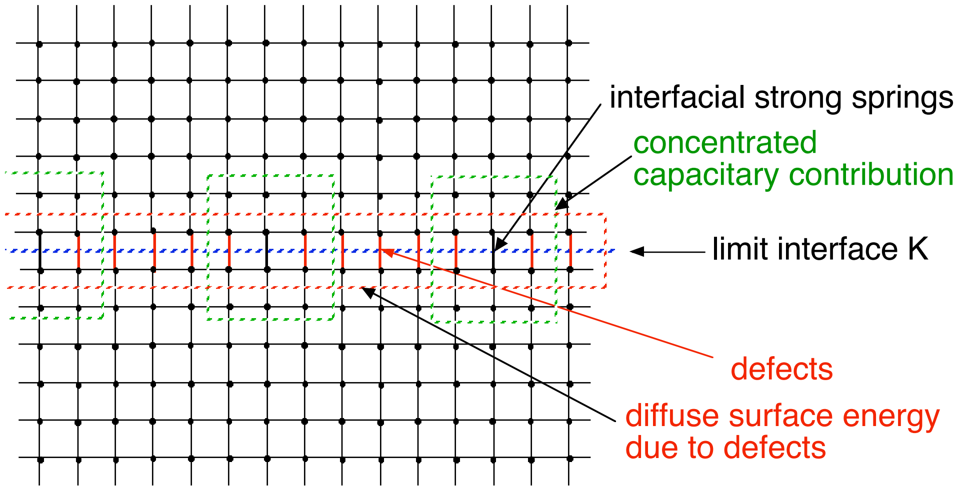

Figure 2.2: Scale and concentration effects on the interface

Proposition 2.3.6 (lower bound) *Let $u_j \to u$ in $L^1(\Omega)$, with $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega)$ and $u \in SBV(\Omega) \cap H^1(\Omega \setminus K)$. Then*

$$\liminf_j F_j(u_j) \geq F(u).$$

Remark 2.3.7 In our computation of the lower bound, we have found it convenient to deal with the contribution due to the quadratic strong springs (both on the interface and elsewhere) separately from that of the weak springs. To that end we introduce the energies (in "localized form" on subsets A of Ω)

$$G_j(v; A) = F_j(v; A) - \sum_{a \in \Omega_j \cap K,\ a \notin Z_j} \varepsilon_j^{n-2}((v(a) - v(a', \varepsilon_j))^2 \wedge \varepsilon_j) \tag{2.3.29}$$

for any $v \in \mathcal{A}_{\varepsilon_j}(A)$, and $G_j(v) = G_j(v, \Omega)$. The Γ-limit of G_j is of interest in itself (see Section 2.3.7).

Proof. Let $k \in \mathbb{N}$ and let $\alpha < 1/2$. By applying Lemma 2.3.4 to (u_j) we build a sequence $w_j \to u$ in $L^1(\Omega)$ satisfying conditions (2.3.18)–(2.3.20). We define the set $E_j \subset \Omega_j$ as

$$E_j = \bigcup_{l \in Z_j} Q_{\varepsilon_j}(\rho_j^l; l).$$

Since

$$\liminf_j F_j(u_j) \geq \liminf_j F_j(u_j;E_j) + \liminf_j F_j(u_j;\Omega_j \setminus E_j)$$

we will estimate the contributions of u_j on E_j and $\Omega_j\backslash E_j$ separately (step **A** and **B** respectively).

A. We want to prove that

$$\liminf_j F_j(u_j;E_j) \geq \frac{C_n}{\beta}\int_{S(u)} |u^+ - u^-|^2 d\mathcal{H}^{n-1} - \frac{c}{k}. \tag{2.3.30}$$

Lemma 2.3.4 implies that

$$\begin{aligned}
\liminf_j F_j(u_j;E_j) \quad &\geq \quad \liminf_j \sum_{l\in Z_j} \Big(F_j(u_j;Q_{\varepsilon_j}(\rho_j^l,l)\cap\{x_n \geq \varepsilon_j\}) \\
&\qquad + F_j(u_j;Q_{\varepsilon_j}(\rho_j^l,l)\cap\{x_n\leq 0\}) \\
&\qquad + (u_j(l',\varepsilon_j) - u_j(l',0))^2\varepsilon_j^{n-2}\Big) - \frac{c}{k} \\
&\geq \quad \liminf_j \sum_{l\in Z_j} \Big(F_j(w_j;Q_{\varepsilon_j}(\rho_j^l,l)\cap\{x_n \geq \varepsilon_j\}) \\
&\qquad + F_j(w_j;Q_{\varepsilon_j}(\rho_j^l,l)\cap\{x_n\leq 0\}) \\
&\qquad + (u_j(l',\varepsilon_j) - u_j(l',0))^2\varepsilon_j^{n-2}\Big) - \frac{c}{k} \\
&= \quad \liminf_j \sum_{l\in Z_j} G_j(w_j;Q_{\varepsilon_j}(\rho_j^l,l)) - \frac{c}{k},
\end{aligned} \tag{2.3.31}$$

where G_j in defined in (2.3.29).

Having fixed $l \in Z_j$, we look for an estimate from below for $G_j(w_j;Q_{\varepsilon_j}(\rho_j^l,l))$. Let $\widetilde{w}_j \in \mathcal{A}_{\varepsilon_j}(Q_{\varepsilon_j}(\rho_j,l))$ be defined as

$$\widetilde{w}_j(a) = \begin{cases} u_j^{l+} & \text{if } a \in (Q_{\varepsilon_j}(\rho_j,l)\setminus Q_{\varepsilon_j}(\rho_j^l,l))\cap\{x_n\geq\varepsilon_j\} \\ w_j(a) & \text{if } a \in Q_{\varepsilon_j}(\rho_j^l,l) \\ u_j^{l-} & \text{if } a \in (Q_{\varepsilon_j}(\rho_j,l)\setminus Q_{\varepsilon_j}(\rho_j^l,l))\cap\{x_n\leq 0\}. \end{cases}$$

By construction

$$G_j(w_j;Q_{\varepsilon_j}(\rho_j^l,l)) = G_j(\widetilde{w}_j;Q_{\varepsilon_j}(\rho_j,l))$$

and, minimizing over all v subject to the boundary conditions satisfied by $\widetilde{w}_j$,

$$G_j(\widetilde{w}_j;Q_{\varepsilon_j}(\rho_j,l)) \geq \inf\big\{G_j(v,Q_{\varepsilon_j}(\rho_j,l)):\ v = u_j^{l\pm} \text{ on } \mathcal{L}^{\pm}_{\varepsilon_j}(\rho_j,l)\big\}.$$

After writing

$$\widetilde{w}_j = \frac{(u_j^{l+}+u_j^{l-})}{2} + \frac{(u_j^{l+}-u_j^{l-})}{2}\, v_j,$$

by a translation and a scaling argument we get

$$G_j(\widetilde{w}_j;Q_{\varepsilon_j}(\rho_j,l)) \geq \frac{(u_j^{l+}-u_j^{l-})^2}{4} \inf\big\{G_j(v,Q_{\varepsilon_j}(\rho_j)) : v = \pm 1 \text{ on } \mathcal{L}^{\pm}_{\varepsilon_j}(\rho_j)\big\}. \tag{2.3.32}$$

We denote by m_j the rescaled infimum

$$m_j = \varepsilon_j^{2-n} \inf\big\{G_j(v;Q_{\varepsilon_j}(\rho_j)) : v = \pm 1 \text{ on } \mathcal{L}^{\pm}_{\varepsilon_j}(\rho_j)\big\}.$$

We want to study the asymptotic behavior of m_j by comparing it with the infimum

$$\mu_j = \inf\Big\{ \sum_{\{a,b\}\in M_{\varepsilon_j}(Q(\rho_j))} (v(a)-v(b))^2 :\ v = 1 \text{ on } \mathcal{L}_{\varepsilon_j}(\rho_j),\ v(0) = v(\varepsilon_j e_n) = 0\Big\}.$$

The limit behavior of μ_j has been studied in details in [59]; to our purposes we recall that

$$\begin{aligned} &\text{for } n \geq 3 \qquad \lim_{j\to+\infty} \mu_j = l_n \in (0,+\infty) \\ &\text{for } n = 2 \qquad \lim_{j\to+\infty} \beta\,\frac{\mu_j}{\delta_j} = l_2 \in (0,+\infty), \end{aligned} \tag{2.3.33}$$

where

$$l_n = \lim_{T\to+\infty} \min\Big\{ \sum_{\{a,b\}\in M_1(Q_T)} (v(a)-v(b))^2 :\ v = 1 \text{ on } \partial Q_T,\ v(0)=v(e_n)=0 \Big\} > 0 \tag{2.3.34}$$

for $n \geq 3$, and

$$l_2 = \lim_{T\to+\infty} \log T \,\min\Big\{ \sum_{\{a,b\}\in M_1(Q_T)} (v(a)-v(b))^2 :\ v = 1 \text{ on } \partial Q_T,\ v(0)=v(e_n)=0 \Big\} = 2\pi \tag{2.3.35}$$

(T is understood to be integer).

We now focus our attention on m_j. It is not restrictive to substitute the cube $Q_{\varepsilon_j}(\rho_j)$ with a cube with center in $(0',\varepsilon/2)$ (for which we use the same symbol, with abuse of notation), so that we may use a symmetry argument. First, we note that if $z_j \in \mathcal{A}_{\varepsilon_j}(Q_{\varepsilon_j}(\rho_j))$ is a minimizer for m_j, then it satisfies the following condition:

$$z_j(x',x_n) = -z_j(x',-x_n+\varepsilon_j). \tag{2.3.36}$$

In fact, let $\overline{z}_j$ be the function $\overline{z}_j(x',x_n) := -z_j(x_1,\dots,-x_n+\varepsilon_j)$. Then $\overline{z}_j$ is a minimizer for m_j by construction. The function $(z_j+\overline{z}_j)/2$ is a test function for m_j; by the strict convexity of G_j we get that if $z_j \neq \overline{z}_j$ then

$$G_j\Big(\frac{z_j+\overline{z}_j}{2}; Q_{\varepsilon_j}(\rho_j)\Big) < \frac{1}{2} G_j(z_j; Q_{\varepsilon_j}(\rho_j)) + \frac{1}{2} G_j(\overline{z}_j; Q_{\varepsilon_j}(\rho_j)) = m_j.$$

There follows that $z_j = \overline{z}_j$; i.e., condition (2.3.36) holds.

Now, let

$$\gamma = z_j(0,\dots,0,\varepsilon_j) \tag{2.3.37}$$

denote the 'half-elongation of the strong spring at the interface' (hence, $z_j(0) = -\gamma$). We note that

$$\begin{aligned} &G_j(z_j; Q_{\varepsilon_j}(\rho_j)\cap\{x_n \geq \varepsilon_j\}) \\ &\quad = \min\big\{G_j(v; Q_{\varepsilon_j}(\rho_j)\cap\{x_n\geq\varepsilon_j\}) :\ v = 1 \text{ on } \mathcal{L}^+_{\varepsilon_j}(\rho_j), v(0,\dots,0,\varepsilon_j) = \gamma\big\} \\ &\quad = (1-\gamma)^2 \min\big\{G_j(v; Q_{\varepsilon_j}(\rho_j)\cap\{x_n\geq\varepsilon_j\}) :\ v = 1 \text{ on } \mathcal{L}^+_{\varepsilon_j}(\rho_j), v(0,\dots,0,\varepsilon_j) = 0\big\} \\ &\quad = \frac{1}{2}\mu_j(1-\gamma)^2. \end{aligned}$$

By (2.3.36) we deduce that

$$G_j(z_j; Q_{\varepsilon_j}(\rho_j)) = 2\times\frac{1}{2}\mu_j(1-\gamma)^2 + 4\gamma^2.$$

The quantity above attains its minimum for $\gamma = \mu_j/(\mu_j+4)$, hence

$$m_j = \frac{4\mu_j}{\mu_j+4}. \tag{2.3.38}$$

The asymptotic behavior of m_j can be specified as follows:

$$\text{for } n \geq 3 \qquad \lim_{j\to+\infty} m_j = \frac{4l_n}{l_n+4} =: C_n \tag{2.3.39}$$

$$\text{for } n = 2 \qquad \lim_{j\to+\infty} \beta\,\frac{m_j}{\delta_j} = l_2 =: C_2. \tag{2.3.40}$$

By (2.3.31) and (2.3.32) we get

$$\liminf_j F_j(u_j; E_j) \geq \liminf_j \sum_{l \in Z_j} \frac{(u_j^{l+} - u_j^{l-})^2}{4} \varepsilon_j^{n-2} m_j - \frac{c}{k}.$$

Note that by (2.3.39), (2.3.40) if $n \geq 3$ we have

$$\varepsilon_j^{n-2} m_j = \frac{1}{\beta} \delta_j^{n-1} m_j (1 + o(1)) = \frac{1}{\beta} \delta_j^{n-1} C_n (1 + o(1)), \tag{2.3.41}$$

while if $n = 2$ we have

$$\varepsilon_j^{n-2} m_j = \delta_j \cdot \frac{m_j}{\delta_j} = \frac{\delta_j}{\beta} C_2 (1 + o(1)). \tag{2.3.42}$$

In both cases, taking into account Proposition 2.3.5, we then deduce that

$$\liminf_j F_j(u_j; E_j) \geq \frac{C_n}{\beta} \int_{S(u)} |u^+ - u^-|^2 d\mathcal{H}^{n-1} - \frac{c}{k}.$$

B. We want to prove that

$$\liminf_j F_j(u_j; \Omega_j \setminus E_j) \geq \int_\Omega |\nabla u|^2 dx + (1 - \alpha) \mathcal{H}^{n-1}(S(u)). \tag{2.3.43}$$

Having fixed a parameter $s > 0$, we consider the "s-neighborhood of K" defined by

$$P_j^s = \{a \in \Omega_j \setminus E_j : \ |a_n| \leq s\}$$

and its complement $R_j^s = (\Omega_j \setminus E_j) \setminus P_j^s = \{a \in \Omega_j \setminus E_j : \ |a_n| > s\}$. Since

$$\liminf_j F_j(u_j; \Omega_j \setminus E_j) \geq \liminf_j F_j(u_j; R_j^s) + \liminf_j F_j(u_j; P_j^s),$$

we can estimate the contribution of u_j separately near the (hyperplane containing the) jump set and far from it (steps **B.1** and **B.2** below, respectively). By letting $s \to 0^+$, we will finally get the desired inequality (2.3.43).

B.1 First, we focus our attention on P_j^s and we prove that

$$\liminf_j F_j(u_j; P_j^s) \geq (1 - \alpha) \mathcal{H}^{n-1}(S(u)). \tag{2.3.44}$$

The proof of (2.3.44) will be performed through the blow-up technique. For all $A \in \mathcal{B}(\Omega)$, we set

$$\lambda_j(A) = F_j(u_j; P_j^s \cap A),$$

which defines a family of measures. The family (λ_j) is equi-bounded; i.e., $\sup_j |\lambda_j|(\Omega) \leq \sup_j F_j(u_j) < +\infty$. Hence, there exists $\mu \in \mathcal{M}^+(\Omega)$ such that λ_j converges weakly* to λ up to subsequences. We consider the Radon-Nykodim decomposition of λ with respect to $\mathcal{H}^{n-1} \llcorner S(u)$: there exists a non-negative function $g \in L^1(\Omega)$ such that

$$\lambda = g \mathcal{H}^{n-1} \llcorner S(u) + \lambda^s,$$

where $\lambda^s \in \mathcal{M}^+(\Omega)$ is such that $\lambda^s \perp (\mathcal{H}^{n-1} \llcorner S(u))$. We want to prove that

$$g(x_0) \geq (1 - \alpha) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x_0 \in S(u). \tag{2.3.45}$$

To this end we follow an argument by contradiction: we assume that

$$g(x_0) < (1 - \alpha) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x_0 \in S(u).$$

Let $x_0 \in S(u)$. We denote by Q the open cube $Q = (-1/2, 1/2)^n$. We can assume that

$$g(x_0) = \lim_{\rho \to 0^+} \frac{\lambda(x_0 + \rho Q)}{\mathcal{H}^{n-1}(S(u) \cap (x_0 + \rho Q))}, \tag{2.3.46}$$

$$\lim_{\rho \to 0^+} \frac{\mathcal{H}^{n-1}(S(u) \cap (x_0 + \rho Q))}{\rho^{n-1}} = 1 \tag{2.3.47}$$

$$\text{and} \qquad \lim_{\rho \to 0^\pm} \frac{1}{\rho^n} \int_{x_0 + \rho Q^\pm} |u(x) - u^\pm(x_0)| dx = 0 \tag{2.3.48}$$

since these properties are satisfied up to a set of zero-$\mathcal{H}^{n-1}$ measure. Moreover, up to a countable set of values of ρ, we can assume that

$$\lambda(\partial(x_0 + \rho Q)) = 0. \tag{2.3.49}$$

By (2.3.46), (2.3.47) and (2.3.49) we get

$$\begin{aligned} g(x_0) &= \lim_{\rho \to 0^+} \lim_{j \to +\infty} \frac{\lambda_j(x_0 + \rho Q)}{\mathcal{H}^{n-1}(S(u) \cap (x_0 + \rho Q))} \\ &= \lim_{\rho \to 0^+} \frac{1}{\rho^{n-1}} \lim_{j \to +\infty} \lambda_j(x_0 + \rho Q) \\ &= \lim_{\rho \to 0^+} \frac{1}{\rho^{n-1}} \lim_{j \to +\infty} F_j(u_j; P_j^s \cap (x_0 + \rho Q)). \end{aligned}$$

By a diagonal argument we can find a sequence $\rho_j \to 0$ such that

$$g(x_0) = \lim_{j \to +\infty} \frac{1}{\rho_j^{n-1}} F_j(u_j; P_j^s \cap (x_0 + \rho_j Q)).$$

We now rescale the space variable by defining

$$A = \frac{\varepsilon_j}{\rho_j} \Big[\frac{a - x_0}{\varepsilon_j} \Big], \quad \text{for all } a \in (x_0 + \rho_j Q) \cap P_j^s,$$

and we set

$$v_j(A) = u_j(\rho_j A + x_0) = u_j(a), \quad \text{for all } a \in (x_0 + \rho_j Q) \cap P_j^s.$$

Up to a further diagonalization we can find a subsequence v_j (not relabelled) such that

$$v_j \to u_0 \text{ in } L^1(Q) \;\text{ and }\; g(x_0) = \lim_j \frac{1}{\rho_j^{n-1}} G_j(v_j; Q),$$

where

$$u_0(x) = \begin{cases} u^+(x_0) & \text{for } x > 0 \\ u^-(x_0) & \text{for } x \leq 0 \end{cases}$$

and

$$G_j(v_j; Q) = F_j(u_j; P_j^s \cap (x_0 + \rho_j Q)).$$

By the modification of De Giorgi's method for matching boundary conditions adapted to the discrete setting (see e.g. [3]), we can build a sequence $\widetilde{v}_j \in \mathcal{A}_{\varepsilon_j/\rho_j}(Q)$ such that $v_j \to u_0$ in $L^1(Q)$,

$$\frac{1}{\rho_j^{n-1}} G_j(\widetilde{v}_j; Q) \leq \frac{1}{\rho_j^{n-1}} G_j(v_j; Q) + o(1)$$

and

$$\widetilde{v}_j(a) = u^\pm(x_0) \text{ for } a \in \frac{\varepsilon_j}{\rho_j} \mathbb{Z}^n \cap Q: \; a_n \in \pm[1 - \varepsilon_j/\rho_j, 1]. \tag{2.3.50}$$

Let I_j be the set

$$I_j = \Big\{A \in Q \cap \Big[\frac{P_j^s - x_0}{\varepsilon_j}\Big]\frac{\varepsilon_j}{\rho_j} : A_n = 0,\ (\widetilde{v}_j(A) - \widetilde{v}_j(A', \varepsilon_j/\rho_j))^2 < \varepsilon_j\Big\}$$

of springs (both weak and strong) where the elongation is below the 'fracture threshold'. We claim that the cardinality of I_j satisfies the following condition:

$$\frac{\sharp I_j}{\rho_j^{n-1}\varepsilon_j^{1-n}(1-\alpha^{n-1})} \geq \beta \qquad \text{for } j \geq j_0, \tag{2.3.51}$$

for some constant $\beta \in (0, 1]$ and $j_0 \in \mathbb{N}$. This can be proved through an argument by contradiction: we assume that for all $\beta > 0$ there exists $j_0 \in \mathbb{N}$ such that

$$\sharp I_j < \beta\rho_j^{n-1}\varepsilon_j^{1-n}(1-\alpha^{n-1}) \ \text{ for } j \geq j_0.$$

Having set

$$I_j^c = \Big\{A \in Q \cap \Big[\frac{P_j^s - x_0}{\varepsilon_j}\Big]\frac{\varepsilon_j}{\rho_j} : A_n = 0,\ (\widetilde{v}_j(A) - \widetilde{v}_j(A', \varepsilon_j/\rho_j))^2 \geq \varepsilon_j\Big\},$$

there follows that

$$\sharp I_j^c \geq (1-\beta)\rho_j^{n-1}\varepsilon_j^{1-n}(1-\alpha^{n-1}) \ \text{ for } j \geq j_0.$$

Hence for all $j \geq j_0$ we have

$$\frac{1}{\rho_j^{n-1}}G_j(\widetilde{v}_j; Q) \geq \frac{1}{\rho_j^{n-1}}\sharp I_j^c\varepsilon_j^{n-1} \geq \frac{1}{\rho_j^{n-1}}(1-\beta)\rho_j^{n-1}\varepsilon_j^{1-n}(1-\alpha^{n-1})\varepsilon_j^{n-1}.$$

Since $\rho_j^{1-n}G_j(\widetilde{v}_j; Q) < 1-\alpha$ by assumption (for j large enough), we get

$$(1-\beta)(1-\alpha^{n-1}) < 1-\alpha.$$

By letting $\beta \to 0^+$ we get $1-\alpha^{n-1} < 1-\alpha$, which is in contrast with the assumption $\alpha < 1/2$.

Let $A \in I_j$. Note that by Hölder's inequality

$$\begin{aligned}
(u^+(x_0) - u^-(x_0))^2 &\leq \Big(\sum_{B\in Q:\ B'=A'}(\widetilde{v}_j(B) - \widetilde{v}_j(B', B_n + \varepsilon_j/\rho_j))\Big)^2 \\
&\leq \sharp\{B \in Q :\ B' = A'\}\sum_{B\in Q:\ B'=A'}(\widetilde{v}_j(B) - \widetilde{v}_j(B', B_n + \varepsilon_j/\rho_j))^2 \\
&\leq \frac{\rho_j}{\varepsilon_j}\sum_{B\in Q:\ B'=A'}(\widetilde{v}_j(B) - \widetilde{v}_j(B', B_n + \varepsilon_j/\rho_j))^2.
\end{aligned}$$

By summing up over $A \in I_j$ we get

$$\begin{aligned}
\sum_{A\in I_j}(u^+(x_0) - u^-(x_0))^2 &\leq \frac{\rho_j}{\varepsilon_j}\sum_{A\in I_j}\sum_{B\in Q:\ B'=A'}(\widetilde{v}_j(B) - \widetilde{v}_j(B', B_n + \varepsilon_j/\rho_j))^2 \\
&\leq \frac{\rho_j}{\varepsilon_j^{n-1}}\sum_{A\in I_j}\sum_{B\in Q:\ B'=A'}(\widetilde{v}_j(B) - \widetilde{v}_j(B', B_n + \varepsilon_j/\rho_j))^2\varepsilon_j^{n-2}
\end{aligned}$$

hence (for $j \geq j_0$)

$$\begin{aligned}
(u^+(x_0) - u^-(x_0))^2 &\leq \frac{1}{\sharp I_j}\frac{\rho_j}{\varepsilon_j^{n-1}}G_j(\widetilde{v}_j; Q) \\
&\leq \frac{1}{\beta\rho_j^{n-1}\varepsilon_j^{1-n}(1-\alpha^{n-1})}\frac{\rho_j}{\varepsilon_j^{n-1}}G_j(\widetilde{v}_j; Q) \leq c\rho_j.
\end{aligned}$$

By letting $j \to +\infty$, we get $u^+(x_0) = u^-(x_0)$, which is in contradiction with the assumption $x_0 \in S(u)$. In conclusion, our arguments imply that

$$g(x_0) \geq (1-\alpha) \ \text{ for } \ \mathcal{H}^{n-1}\text{-a.e. } x_0 \in S(u).$$

Finally, by (2.3.45) we deduce that

$$\lambda(\Omega) \geq \int_\Omega g\, d(\mathcal{H}^{n-1} \llcorner S(u)) \geq (1-\alpha)\mathcal{H}^{n-1}(S(u)),$$

which implies the desired inequality (2.3.44):

$$\liminf_j F_j(u_j; P_j^s) = \liminf_j \lambda_j(\Omega) \geq (1-\alpha)\mathcal{H}^{n-1}(S(u)).$$

B.2 To estimate the contribution of u_j on $R_j^s = (\Omega \setminus E_j) \cap \{x \in \mathbb{R}^n : \ |x_n| > s\}$ is suffices to recall that the weak-membrane functional is always a lower bound, from which we obtain

$$\begin{aligned}\liminf_j F_j(u_j; R_j^s) &\geq \int_{\Omega \cap \{|x_n|>s\}} |\nabla u|^2 dx + \mathcal{H}^{n-1}(S(u) \cap \{x \in \Omega : \ |x_n| > s\}) \\ &= \int_{\Omega \cap \{|x_n|>s\}} |\nabla u|^2 dx \qquad (2.3.52)\end{aligned}$$

since $\mathcal{H}^{n-1}(S(u) \cap \{x \in \Omega : \ |x_n| > s\}) = 0$.

Taking into account (2.3.44) and (2.3.52) and letting $s \to 0^+$, we deduce that the contribution of u_j outside E_j can be estimated as follows:

$$\liminf_j F_j(u_j; \Omega_j \setminus E_j) \geq (1-\alpha)\mathcal{H}^{n-1}(S(u)) + \int_\Omega |\nabla u|^2 dx,$$

as desired. ■

Remark 2.3.8 From the characterization of m_j in (2.3.38), and the limit behavior of μ_j (described in (2.3.33) we deduce that the 'elongation of the strong springs' (scaled by ε_j) at the interface $2\gamma_j$ (defined in (2.3.37)) asymptotically vanishes in the case $n = 2$, while it is finite and given by (2.3.39) if $n \geq 3$.

2.3.5 Upper bound

We now prove the upper bound for our energies.

Proposition 2.3.9 (upper bound) *For all $u \in \mathrm{PC}(\Omega)$ such that $S(u) \subseteq K$, there exists a sequence (v_j) such that $v_j \in \mathcal{A}_{\varepsilon_j}(\Omega)$, $v_j \to u$ in $L^1(\Omega)$ and*

$$\limsup_j F_j(v_j) \leq F(u). \qquad (2.3.53)$$

Proof. For all $j \in \mathbb{N}$ we denote by u_j the function $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega)$ defined as the discretization of u on the lattice $\Omega_j = \varepsilon_j\mathbb{Z}^n \cap \Omega$:

$$u_j(a) = u(a) \text{ for } a \in \Omega_j.$$

(If $a \in \Omega_j \cap S(u)$ we set $u_j(a) = u^-(x_0)$). Let $k \in \mathbb{N}$ and $\alpha > 0$ be such that $2^{k+1}\alpha < 1/2$. Let (ρ_j) be a positive infinitesimal sequence of the form $\rho_j = 2^{k+1}\alpha\delta_j$. By applying Lemma 2.3.4 to the sequence $u_j \to u$, we get a new sequence $w_j \to u$ satisfying conditions (2.3.18)–(2.3.20).

We want to modify the functions w_j on the cubes $Q_{\varepsilon_j}(\rho_j^l, l)$, $l \in Z_j$, in order to get a recovery sequence for u. Following the notation of the lemma, we note that

$$\rho_j^l \geq \Big[\frac{\alpha\delta_j}{\varepsilon_j}\Big]\varepsilon_j \qquad \text{for all } l \in Z_j.$$

For fixed $l \in Z_j$, we denote by $z_j^l \in \mathcal{A}_{\varepsilon_j}(\mathbb{R}^n)$ the minimizer of the following minimum problem:

$$\min\Big\{G_j(v; Q_{\varepsilon_j}(\alpha\delta_j)) : \; v = u_j^{l\pm} \text{ on } \mathcal{L}^{\pm}_{\varepsilon_j}\Big(\Big[\frac{\alpha\delta_j}{\varepsilon_j}\Big]\varepsilon_j\Big)\Big\}, \tag{2.3.54}$$

where G_j is defined by (2.3.29). We define the sequence $v_j \in \mathcal{A}_{\varepsilon_j}(\Omega)$ as follows:

$$v_j(a) = \begin{cases} z_j^l(a-l) & \text{if } a \in Q_{\varepsilon_j}\Big(\Big[\frac{\alpha\delta_j}{\varepsilon_j}\Big]\varepsilon_j, l\Big), \; l \in Z_j \\ u_j^{l\pm} & \text{if } a \in \Omega^{\pm} \cap \Big(Q_{\varepsilon_j}(\rho_j^l, l) \setminus Q_{\varepsilon_j}\Big(\Big[\frac{\alpha\delta_j}{\varepsilon_j}\Big]\varepsilon_j\Big), l\Big), \; l \in Z_j \\ w_j(a) & \text{if } \Omega_j \setminus \bigcup_{l \in Z_j} Q_{\varepsilon_j}(\rho_j^l, l). \end{cases}$$

We want to prove that (v_j) is a recovery sequence for u. By construction

$$F_j(v_j) \quad \leq \quad \sum_{l \in Z_j} F_j(v_j; Q_{\varepsilon_j}(\rho_j^l, l)) + F_j\Big(v_j; \Omega_j \setminus \bigcup_{l \in Z_j} Q_{\varepsilon_j}(\rho_j^l, l)\Big).$$

Having fixed $l \in Z_j$, we focus our attention on $F_j(v_j; Q_{\varepsilon_j}(\rho_j^l, l))$. By construction:

$$\begin{aligned} F_j(v_j; Q_{\varepsilon_j}(\rho_j^l, l)) &\leq G_j(z_j; Q_{\varepsilon_j}(\alpha\delta_j)) + \varepsilon_j^{n-1}\sharp\big(K \cap Q_{\varepsilon_j}(\rho_j^l, l)\big) \\ &\leq G_j(z_j; Q_{\varepsilon_j}(\alpha\delta_j)) + \varepsilon_j^{n-1}(2^{k+1}\alpha)^{n-1}\frac{\delta_j^{n-1}}{\varepsilon_j^{n-1}}. \end{aligned}$$

By summing over $l \in Z_j$ we get

$$\begin{aligned} \sum_{l \in Z_j} F_j(v_j; Q_{\varepsilon_j}(\rho_j^l, l)) &\leq \sum_{l \in Z_j} G_j(z_j; Q_{\varepsilon_j}(\alpha\delta_j)) + c(2^{k+1}\alpha)^{n-1} \\ &= \sum_{l \in Z_j} \min\Big\{G_j(v; Q_{\varepsilon_j}(\alpha\delta_j)) : \; v = u_j^{l\pm} \text{ on } \mathcal{L}^{\pm}_{\varepsilon_j}\Big(\Big[\frac{\alpha\delta_j}{\varepsilon_j}\Big]\varepsilon_j\Big)\Big\} \\ &\qquad + c(2^{k+1}\alpha)^{n-1} \\ &= \min\Big\{G_j(v; Q_{\varepsilon_j}(\alpha\delta_j)) : \; v = \pm 1 \text{ on } \mathcal{L}^{\pm}_{\varepsilon_j}\Big(\Big[\frac{\alpha\delta_j}{\varepsilon_j}\Big]\varepsilon_j\Big)\Big\} \times \\ &\qquad \times \sum_{l \in Z_j} \frac{(u_j^{l+} - u_j^{l-})^2}{4} + c(2^{k+1}\alpha)^{n-1}. \end{aligned}$$

Having defined the scaled minimum problems

$$m_j = \varepsilon_j^{2-n} \min\Big\{G_j(v; Q_{\varepsilon_j}(\alpha\delta_j)) : \; v = \pm 1 \text{ on } \mathcal{L}^{\pm}_{\varepsilon_j}\Big(\Big[\frac{\alpha\delta_j}{\varepsilon_j}\Big]\varepsilon_j\Big)\Big\},$$

their asymptotic behavior is given by (2.3.41) and (2.3.42). By Proposition 2.3.5 we then get

$$\limsup_j \sum_{l \in Z_j} F_j(v_j; Q_{\varepsilon_j}(\rho_j^l, l)) \leq \frac{C_n}{\beta} \int_{S(u)} |u^+ - u^-|^2 d\mathcal{H}^{n-1} + c(2^{k+1}\alpha)^{n-1}. \tag{2.3.55}$$

Finally, we estimate the contribution of v_j on $\Omega_j \setminus \bigcup_{l\in Z_j} Q_{\varepsilon_j}(\rho_j^l, l)$. By Lemma 2.3.4 we have

$$\begin{aligned}
F_j(v_j;\Omega_j\setminus \cup_{l\in Z_j} Q_{\varepsilon_j}(\rho_j^l,l)) &= F_j\Big(w_j;(\Omega_j\setminus \bigcup_{l\in Z_j} Q_{\varepsilon_j}(\rho_j^l,l))\cap\Omega^+\Big)\\
&\quad + F_j\Big(w_j;(\Omega_j\setminus \bigcup_{l\in Z_j} Q_{\varepsilon_j}(\rho_j^l,l))\cap\Omega^-\Big)\\
&\quad + \sum_{a\in K\cap(\Omega_j\setminus\bigcup_{l\in Z_j} Q_{\varepsilon_j}(\rho_j^l,l))} \varepsilon_j^{n-2}(w_j(a)-w_j(a',\varepsilon_j))^2\wedge\varepsilon_j^{n-1}\\
&\le F_j\Big(u_j;(\Omega_j\setminus \bigcup_{l\in Z_j} Q_{\varepsilon_j}(\rho_j^l,l))\cap\Omega^+\Big)\\
&\quad + F_j\Big(u_j;(\Omega_j\setminus \bigcup_{l\in Z_j} Q_{\varepsilon_j}(\rho_j^l,l))\cap\Omega^-\Big)\\
&\quad + \sum_{a\in K\cap(\Omega_j\setminus\bigcup_{l\in Z_j} Q_{\varepsilon_j}(\alpha 2^{k+1}\delta_j,l))} \varepsilon_j^{n-2}(u_j(a)-u_j(a',\varepsilon_j))^2\wedge\varepsilon_j^{n-1}\\
&\quad + c(\alpha 2^{k+1})^{n-1}+\frac{c}{k}\\
&\le F_j^{V_j}(u_j)+c(\alpha 2^{k+1})^{n-1}+\frac{c}{k}.
\end{aligned}$$

where $V_j := \{\{a,b\}\in M_j:\ a\in K\cap\Omega_j,\ b=a+\varepsilon_j e_n\}\subseteq M_j$. Taking Remark 2.2.7 into accountwe get

$$\limsup_j F^{V_j}(u_j)\le \int_\Omega |\nabla u|^2dx+\mathcal{H}^{n-1}(S(u)).$$

There follows that

$$\limsup_j F_j\Big(v_j;\Omega_j\setminus \bigcup_{l\in Z_j} Q_{\varepsilon_j}(\rho_j^l,l)\Big)\le \frac{c}{k}+\int_\Omega |\nabla u|^2dx+\mathcal{H}^{n-1}(S(u)). \tag{2.3.56}$$

By (2.3.55) and (2.3.56) we deduce that

$$\begin{aligned}
\liminf_j F_j(v_j) &\le \frac{C_n}{\beta}\int_{S(u)} |u^+-u^-|^2 d\mathcal{H}^{n-1}\\
&\quad +\int_\Omega |\nabla u|^2dx+\mathcal{H}^{n-1}(\Omega)+\frac{c}{k}+c(2^{k+1}\alpha)^{n-1}.
\end{aligned}$$

By letting first $\alpha\to 0^+$ and then $k\to+\infty$, we get (2.3.53). ■

2.3.6 Limits with prescribed density of weak springs

We will show that all the constructions throughout the chapter can be repeated also with *any* prescribed limit density $\theta\in L^\infty(\Omega;[0,1])$.

Note that in the construction considered in this section, the weak springs are until now concentrated on a $(n-1)$-dimensional hyperplane, so that θ is identically 0 a.e. We now show that for all given θ our construction can be repeated with a different choice of W_j such that (2.2.5) holds.

Proposition 2.3.10 (Prescribed density of weak springs) *Let $F: SBV(\Omega)\to[0,+\infty]$ be the functional*

$$F(u)=\begin{cases}\displaystyle\int_\Omega |\nabla u|^2dx+\int_{S(u)}\Big(\frac{C_n}{\beta}|u^+-u^-|^2+1\Big)d\mathcal{H}^{n-1} & \text{if } S(u)\subseteq K\\ +\infty & \text{otherwise.}\end{cases} \tag{2.3.57}$$

For all $\theta\in L^\infty(\Omega;[0,1])$ there exists a sequence of arrangements (W_j) such that $W_j\to\theta$, $F\le\Gamma\text{-}\liminf_j F_j$, and $\Gamma\text{-}\lim_j F_j=F$ on $\mathrm{PC}(\Omega)$

Proof. It is sufficient to prove the thesis with a choice of W_j with limit density 1. The result will then follow by comparison by taking $W'_j \subset W_j$, W'_j containing the weak springs in Theorem 2.3.1 and $W'_j \to \theta$. Note that, by the compactness of Γ-convergence ([19]), we can always suppose that Γ-limits exist when needed (even though we characterize them only on PC(Ω)).

We fix $\eta > 0$ and $N \in \mathbb{N}$, and define $W_j = W^{\eta,N}_{\varepsilon_j} = W^1_j \cup W^2_j$ as follows:
- W^1_j defined as the set W_{ε_j} in (2.3.9) (weak springs at the interface)
- W^2_j defined by (here we use the notation $\widehat{a}_j = (a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n)$)

$$W^2_j = \{\{a, b\} \in M_1 : |a_n| \geq \eta \text{ and } |b_n| \geq \eta\} \setminus \{\{a, b\} : \widehat{a}_j = \widehat{b}_j \in \varepsilon N \mathbb{Z}^{n-1}\} \tag{2.3.58}$$

is the set of all springs outside the η-tubular neighborhood of the interface and not lying between any two neighboring points of the lattice $\varepsilon N \mathbb{Z}^n$. In other words, the strong springs outside the η-tubular neighborhood of the interface are exactly those on straight segments nearest neighbors of the lattice $\varepsilon N \mathbb{Z}^n$.

Note that for this choice of W_j the limit density of weak springs is $\theta_{\eta,N}$ given by

$$\theta_{\eta,N}(x) = \begin{cases} 0 & \text{if } |x_n| \leq \eta \\ \frac{1}{N^{n-1}} & \text{if } |x_n| > \eta. \end{cases} \tag{2.3.59}$$

We will denote by $F^{\eta,N}_j$ the functional with W_j as set of weak springs, and we will use the usual notation for its localized version. To prove that the Γ-limit is given by F it suffices to show the lower bound, since the upper bound follows by Theorem 2.3.1 by comparison since the set W_j contains the one in that theorem.

We consider now a sequence $u_j \to u$ such that $\liminf_j F^{\eta,N}_j(u_j) < +\infty$. It is not restrictive to suppose that indeed $\sup_j F^{\eta,N}_j(u_j) < +\infty$. Note that we can apply Theorem 2.3.1 to $F^{\eta,N}_j(\cdot; \Omega_\eta)$, where $\Omega_\eta = \Omega \cap \{|x_n| < \eta\}$, since $F^{\eta,N}_j$ coincides with the energy therein on Ω_η. In particular we deduce that the Γ-limit is finite only on functions $u \in H^1(\Omega_\eta \setminus K) \cap SBV(\Omega_\eta)$, and we have

$$\liminf_j F^{\eta,N}_j(u_j; \Omega_\eta) \geq \int_{\Omega_\eta} |\nabla u|^2 \, dx + \int_{S(u)} \Big(\frac{C_n}{\beta} |u^+ - u^-|^2 + 1 \Big) d\mathcal{H}^{n-1} \tag{2.3.60}$$

We now focus our attention on $\Omega \cap \{x_n > \eta/2\}$ and we define the sequence $v_j : (\Omega \cap \{x_n > \eta/2\}) \cap \varepsilon_j N \mathbb{Z}^n \to \mathbb{R}$ as

$$v_j(a) = u_j(a) \text{ for } a \in (\Omega \cap \{x_n > \eta\}) \cap \varepsilon_j N \mathbb{Z}^n.$$

By construction (v_j) satisfies

$$\sum_{|a-b| = N\varepsilon_j} \Big(\frac{v_j(a) - v_j(b)}{N \varepsilon_j} \Big)^2 (\varepsilon_j N)^n \leq c N^{n-2} F^{\eta,N}_j(u_j).$$

By Remark 2.2.4(3) (applied to $\varepsilon N \mathbb{Z}^n$ in place of $\varepsilon \mathbb{Z}^n$) up to subsequences, $v_j \to v \in H^1(\Omega \cap \{x_n > \eta/2\})$. We now denote by χ_j the characteristic function of the set

$$\bigcup_{k \in \varepsilon_j N \mathbb{Z}^n} (k + (-\varepsilon_j/2, \varepsilon_j/2)^n),$$

which converge weakly* in $L^\infty(\Omega)$ to the constant N^{-n}. This implies that $u_j \chi_j \to N^{-n} u$ in $L^1(\Omega \cap \{x_n > \eta/2\})$ and $v_j \chi_j \to N^{-n} v$ in $L^1(\Omega \cap \{x_n > \eta/2\})$. After noticing that $\chi_j u_j \equiv \chi_j v_j$, we conclude that u coincides on $\Omega \cap \{x_n > \eta\}$ with a function $v \in H^1(\Omega \cap \{x_n > \eta/2\})$. By a similar argument on $\Omega \cap \{x_n < -\eta/2\}$ we conclude that $u \in H^1(\Omega \cap \{|x_n| > \eta/2\})$.

As a result, we have $u \in H^1(\Omega \setminus K)$. By Remark 2.2.4(4), applied to $\Omega \cap \{|x_n| > \eta\}$ we have

$$\liminf_j F^{\eta,N}_j(u_j; \Omega \cap \{|x_n| > \eta\}) \geq \int_{\Omega \cap \{|x_n| > \eta\}} |\nabla u|^2 \, dx. \tag{2.3.61}$$

Taking into account (2.3.60) and (2.3.61) we conclude that

$$\liminf_j F_j^{\eta,N}(u_j) \geq \int_\Omega |\nabla u|^2\, dx + \int_{S(u)} \Big(\frac{C_n}{\beta}|u^+ - u^-|^2 + 1\Big) d\mathcal{H}^{n-1}$$

as desired.

Since $\theta_{\eta,N}$ to 1 as $\eta \to 0$ and $N \to +\infty$, we can choose $\eta_j \to 0$ and $N_j \to +\infty$ such that, having redefined $W_j = W_{\varepsilon_j}^{\eta_j,N_j}$ the corresponding Γ-limit still satisfied the thesis, thus obtaining the desired result. Note that this last argument uses a diagonalization procedure, which is possible by the metrizability properties of Γ-convergence (see [39] Theorem 10.22, which requires a common lower bound for all functionals with a coercive energy. In our case that energy is the Mumford-Shah functional, after identification of the functions in $\mathcal{A}(\Omega)$ with suitable functions in $SBV(\Omega)$ – see e.g. [35]). ■

2.3.7 The discrete Neumann sieve problem

We consider the energies

$$G_j(u) = F_j(u) - \sum_{a \in \Omega_j \cap K,\ a \notin Z_j} \varepsilon_j^{n-2}((u(a) - u(a', \varepsilon_j))^2 \wedge \varepsilon_j) \tag{2.3.62}$$

for any $v \in \mathcal{A}_{\varepsilon_j}(A)$, introduced (in a local form) in (2.3.29) and used in the proof of Theorem 2.3.1

The energies G_j do not take the weak springs into account, which are replaced by 'voids', and are the discrete analog of the energy of a "Neumann sieve" [10], where the interface is now free (i.e., we have Neumann boundary conditions at the interface) except for the strong springs (see Fig. 2.3). The Γ-limit of G_j consists of the quadratic part of the limit of F_j and is

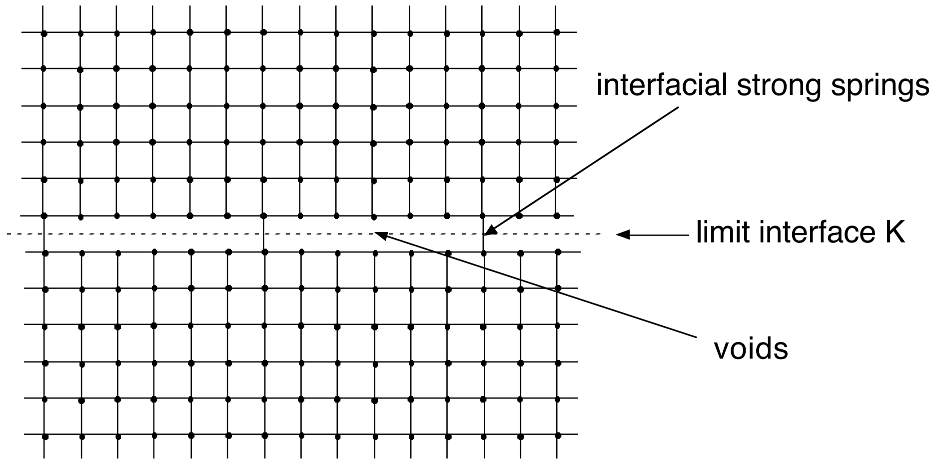

Figure 2.3: The discrete Neumann sieve

described as follows.

Theorem 2.3.11 *The functionals G_j defined by* (2.3.62) Γ*-converge, with respect to the strong convergence in $L^1(\Omega)$, to the functional $G : L^1(\Omega) \to [0, +\infty]$ given by*

$$G(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 dx + \frac{C_n}{\beta} \int_{S(u)} |u^+ - u^-|^2 d\mathcal{H}^{n-1} & \text{if } u \in SBV(\Omega),\ S(u) \subseteq K \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. The proof of this theorem is contained in that of Theorem 2.3.1, whose separation of scale argument is precisely to consider quadratic and non-quadratic interactions separately. Note that in this case we can prove the Γ-limsup inequality for all functions in $H^1(\Omega \setminus K) \cap SBV(\Omega)$, since a mollification argument easily shows the density in energy of the set $\mathrm{PC}(\Omega)$. ■

2.4 Closure results for a class of free-discontinuity fracture energies

In the previous section we have obtained as limits of discrete energies (in the sense of Theorem 2.3.1) functionals of the form

$$F_{K,b}(u,\Omega) = \int_\Omega |\nabla u|^2\,dx + \int_{S(u)\cap\Omega} (1 + b|u^+ - u^-|^2)\,d\mathcal{H}^{n-1}, \tag{2.4.63}$$

with the constraint that $S(u) \subset K$ up to $\mathcal{H}^{n-1}$-negligible sets, where K is the closure of an open set A of the union of coordinate hyperplanes with $\mathcal{H}^{n-1}(K) = \mathcal{H}^{n-1}(A)$, and $b \geq 0$ is any positive constant (by the arbitrariness of β in Theorem 2.3.1).

Scope of this section is to describe a wide class of $GSBV$ energies obtained as Γ-limits of energies of the form (2.4.63) with varying K and b. In order not to overburden the notation, in the definition of the energies u is understood to be in $GSBV(\Omega)$, and the energies are extended to $+\infty$ where not explicitly defined.

Even though applying some new arguments, in this section we will use well-known techniques in Geometric Measure Theory, so we will feel free to drop some details in order to lighten the presentation.

Notation. We now deal with energies on the continuum, for which we find it convenient to change the notation used in a discrete setting. In particular note that, with a slight abuse of notation, in this section cubes will be open and not closed.

2.4.1 Energies with the constraint $S(u) \subset K$

In this section we consider varying K_j still converging to some K, and examine the class of energy densities that can be obtained in this way.

1. Limit energies of the type

$$F_{K,a,b}(u,\Omega) = \int_\Omega |\nabla u|^2\,dx + \int_{S(u)\cap\Omega} (a + b|u^+ - u^-|^2)\,d\mathcal{H}^{n-1}, \tag{2.4.64}$$

with the constraint that $S(u) \subset K$ up to $\mathcal{H}^{n-1}$-negligible sets, where K is the closure of an open set A of a union of coordinate hyperplanes with $\mathcal{H}^{n-1}(K) = \mathcal{H}^{n-1}(A)$, $a \geq 1$ and $b \geq 0$

We define the approximating energies as

$$F_j(u) = F_{K_j, b/a}(u,\Omega), \tag{2.4.65}$$

where K_j are *oscillating fracture sites* defined as follows. For the sake of simplicity it is not restrictive to suppose that

$$K \subset \{x_n = 0\}.$$

For all $i \in \mathbb{Z}^{n-1}$ we consider the coordinate open cube $Q^{n-1}_{1/2}(i)$ of centre i and side length $1/2$ in $\mathbb{R}^{n-1}$ and correspondingly the coordinate parallelepiped

$$R^a_i = \overline{Q^{n-1}_{1/2}(i)} \times \Big[0, \frac{a-1}{(n-1)\vee 2}\Big]$$

in $\mathbb{R}^n$. We then set

$$K_j = \bigcup\Big\{\Big(K\cup\frac{1}{j}\partial R_i^a\Big)\setminus\Big(K\cap\frac{1}{j}\partial R_i^a\Big) : \frac{1}{j}Q^{n-1}_{1/2}(i)\subset K\Big\}$$
$$\cup\bigcup\Big\{\partial R_i^a : \frac{1}{j}Q^{n-1}_{1/2}(i)\cap\subset K\neq\emptyset,\ \frac{1}{j}Q^{n-1}_{1/2}(i)\not\subset K\Big\} \qquad (2.4.66)$$

(see Fig. 2.4).

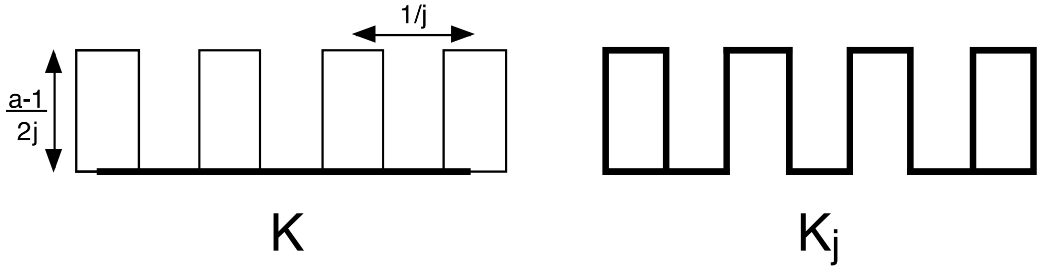

Figure 2.4: Construction of oscillating fracture sites (two-dimensional picture)

Note in particular that we have

$$\mathcal{H}^{n-1}\llcorner K_j \rightharpoonup^* a\mathcal{H}^{n-1}\llcorner K.$$

Theorem 2.4.1 *Let F_j be the functionals in* (2.4.65) *and $F_{K,a,b}$ that in* (2.4.64)*. Then*
(i) *for all $u\in SBV(\Omega)$ and $u_j\to u$ we have*

$$\liminf_j F_j(u_j)\geq F_{K,a,b}(u);$$

(i) *for all $u\in \mathrm{PC}(\Omega)$ there exist $u_j\in\mathrm{PC}(\Omega)$ with $u_j\to u$ and*

$$\limsup_j F_j(u_j)\leq F_{K,a,b}(u).$$

PROOF. Let $u_j\to u$ in $L^1(\Omega)$ with $\sup_j F_j(u_j)<+\infty$. Then we have $u_j\to u$ in $SBV(\Omega)$ and $u_j\rightharpoonup u$ in $H^1_{\rm loc}(\Omega\setminus\overline{K})$. Hence, $u\in SBV(\Omega)\cap H^1(\Omega\setminus\overline{K})$; i.e., $S(u)\subset K$ $\mathcal{H}^{n-1}$-a.e., and u belongs to the domain of K.

For $\mathcal{H}^{n-1}$-a.e. $x_0\in S(u)$, by a blow up argument around x_0 we can find (up to a relabeling of the indices j and possible extraction of subsequences) a sequence v_j converging in $L^1(Q_1(0))$ (we use the notation $Q_1(0)=Q_1^n(0)$) to the function

$$u^{x_0}(x)=\begin{cases}u^+(x_0) & \text{if } x_n>0\\ u^-(x_0) & \text{if } x_n<0,\end{cases} \qquad (2.4.67)$$

and $\sup_j F_{K_j,a/b}(v_j,Q_1(0))<+\infty$. Note that we then have $v_j\rightharpoonup u^{x_0}$ in $H^1_{\rm loc}(Q_1(0)\setminus\overline{K})$, and that $v_j-u^{x_0}\to 0$ in $L^2(Q_1(0)\cap\{x_n=t\})$ for a.a. $-1/2<t<1/2$. Note that by the blow up argument around x_0 we can suppose that $Q_1^{n-1}(0)\subset K$.

We first prove that $\liminf_j\mathcal{H}^{n-1}(S(v_j)\cap Q_1(0))\geq a$. Suppose otherwise that $\mathcal{H}^{n-1}(S(v_j)\cap Q_1(0))<a$; i.e., that

$$\mathcal{H}^{n-1}(K_j\setminus S(v_j))\geq c>0 \qquad (2.4.68)$$

for j sufficiently large. We can find disjoint smooth one-dimensional paths γ_y^j in $Q_1(0)$ indexed by $y\in Q_1(0)\cap K_j$ with the two endpoints in $Q_1(0)\cap\{x_n=\pm1/2\}$, respectively, such that

$$Q_1(0)=\bigcup\Big\{\gamma_y^j : y\in K_j\Big\}$$

and $\mathcal{H}^1(\gamma_y^j) = 1 + o(1)$ as $j \to +\infty$. For $\mathcal{H}^{n-1}$-a.a. $y \in K_j \setminus S(v_j)$ the functions v_j belong to $H^1(\gamma_y^j)$. For fixed $\delta > 0$ such that

$$v_j - u^{x_0} \to 0 \text{ in } L^2(Q_1(0) \cap \{x_n = \pm\delta\}) \tag{2.4.69}$$

we set

$$x_{y,j}^{\delta,\pm} = \gamma_y^j \cap \{x_n = \pm\delta\}$$

and estimate

$$\begin{aligned} |v_j(x_{y,j}^{\delta,+}) - v_j(x_{y,j}^{\delta,+})| &\leq \int_{\gamma_y^j \cap \{-\delta < x_n < \delta\}} |\nabla v_j| d\mathcal{H}^{n-1} \\ &\leq c\sqrt{\delta}\Big(\int_{\gamma_y^j} |\nabla v_j|^2 d\mathcal{H}^{n-1}\Big)^{1/2}. \end{aligned}$$

Integrating this inequality for $y \in K_j \setminus S(v_j)$ by (2.4.68), (2.4.69) and (2.4.67) we then obtain

$$|u^+(x_0) - u^-(x_0)|^2 \leq c \liminf_j \delta \int_{Q_1(0)} |\nabla v_j|^2 \, dx \leq c\delta,$$

contradicting that $x_0 \in S(u)$ by the arbitrariness of δ.

The same type of argument, used by comparing v_j on $\{x_n = \pm\delta\}$ and on K_j shows that

$$\liminf_j \int_{K_j} |v_j^\pm - u^\pm(x_0)|^2 \, d\mathcal{H}^{n-1} \leq c\delta$$

so that indeed

$$\begin{aligned} \lim_j \int_{S(v_j) \cap Q_1(0)} |v_j^+ - v_j^-|^2 \, d\mathcal{H}^{n-1} &= \lim_j \int_{K_j \cap Q_1(0)} |u^+(x_0) - u^-(x_0)|^2 \, d\mathcal{H}^{n-1} \\ &= a|u^+(x_0) - u^-(x_0)|^2. \end{aligned}$$

The blow-up method of Fonseca and Müller allows then to conclude that

$$\liminf_j F_j(u) \geq \int_\Omega |\nabla u|^2 \, dx + \int_{S(u) \cap \Omega} (a + b|u^+ - u^-|^2) \, d\mathcal{H}^{n-1}$$

since the inequality of the bulk part follows trivially from the lower semicontinuity of the Dirichlet integral.

Let $u \in \mathrm{PC}(\Omega)$ with $F_{K,a,b}(u) < +\infty$. To check the limsup inequality we simply extend the restriction of u to $\{x_n < 0\}$ by reflexion to a neighborhood of K and denote it by $\widetilde{u}$. The sequence u_j is simply given by

$$u_j(x) = \begin{cases} \widetilde{u}(x) & \text{if } x \in \bigcup\{\frac{1}{j} R_i^a : \frac{1}{j} Q_{1/2}^{n-1}(i) \cap K \neq \emptyset\} \\ u(x) & \text{otherwise,} \end{cases} \tag{2.4.70}$$

which satisfies the constraint $S(u_j) \subset K_j$, and for which the desidered inequality immediately follows. Note that $u \in PC(\Omega)$. ■

2. Limit energies of the type

$$F_{K,a,b}(u, \Omega) = \int_\Omega |\nabla u|^2 \, dx + \int_{S(u) \cap \Omega} (a + b|u^+ - u^-|^2) \|\nu(u)\|_1 \, d\mathcal{H}^{n-1}, \tag{2.4.71}$$

with the constraint that $S(u) \subset K$ up to $\mathcal{H}^{n-1}$-negligible sets, where K is the closure of a relatively open set A of a union of (not necessarily coordinate) hyperplanes

$$\Pi_m = \{\langle x - x_m, \nu_m \rangle = 0\}, \qquad m \in M,$$

with $\mathcal{H}^{n-1}(K) = \mathcal{H}^{n-1}(A)$, $a \geq 1$ and $b \geq 0$.

The approximating functionals will be of the same form

$$F_j = F_{K_j,a,b} \tag{2.4.72}$$

with K_j subsets of coordinate hyperplanes. It is sufficient to consider the case of a single hyperplane

$$K \subset \Pi_0 = \{\langle x - x_0, \nu_0 \rangle = 0\}.$$

We then consider the sets of indices

$$I_1^j = \Big\{ i \in \mathbb{Z}^n : \frac{1}{j} Q_1(i) \cap K \neq \emptyset, \overline{\frac{1}{j} Q_1(i)} \cap (\Pi_0 \setminus K) = \emptyset \Big\} \tag{2.4.73}$$

$$I_2^j = \Big\{ i \in \mathbb{Z}^n : \frac{1}{j} Q_1(i) \cap K \neq \emptyset, \overline{\frac{1}{j} Q_1(i)} \cap (\Pi_0 \setminus K) \neq \emptyset \Big\}, \tag{2.4.74}$$

and define

$$K_j = \Big(\bigcup_{i \in I_2^j} \frac{1}{j} \partial Q_1(i) \Big) \cup \partial \Big(\{ x : x - x_0, \nu_0 \rangle > 0 \} \cap \bigcup_{i \in I_1^j} \frac{1}{j} Q_1(i) \Big) \tag{2.4.75}$$

(see Fig. 2.5).

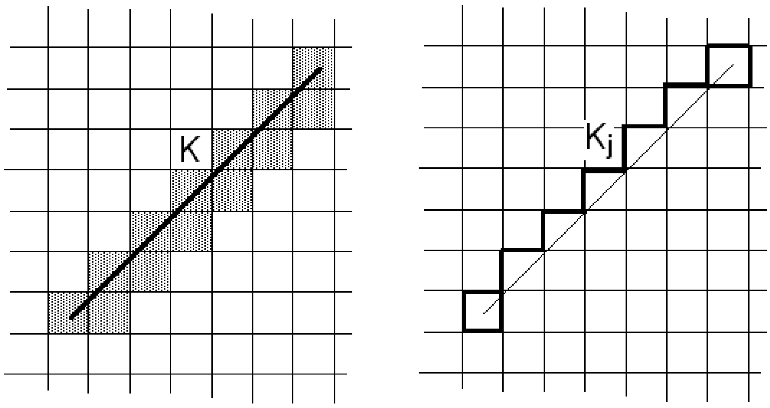

Figure 2.5: Construction of oscillating fracture sites for non-coordinate planes

Theorem 2.4.2 *Let F_j be the functionals in* (2.4.72) *and $F_{K,a,b}$ that in* (2.4.71). *Then*
(i) *for all $u \in SBV(\Omega)$ and $u_j \to u$ we have*

$$\liminf_j F_j(u_j) \geq F_{K,a,b}(u);$$

(i) *for all $u \in \mathrm{PC}(\Omega)$ there exist $u_j \in \mathrm{PC}(\Omega)$ with $u_j \to u$ and*

$$\limsup_j F_j(u_j) \leq F_{K,a,b}(u).$$

Proof. After noting that

$$\mathcal{H}^{n-1} \llcorner K_j \rightharpoonup^* \|\nu_0\|_1 \mathcal{H}^{n-1} \llcorner K$$

the proof of the liminf inequality follows word for word that of Theorem 2.4.1, with the hyperplane $\{x_n = 0\}$ substituted by Π_0.

As for the limsup inequality, the sequence u_j is simply given by

$$u_j(x) = \begin{cases} \widetilde{u}(x) & \text{if } x \in \bigcup\{\frac{1}{j}Q_1(i) : \frac{1}{j}Q_1(i) \cap K \neq \emptyset\} \\ u(x) & \text{otherwise,} \end{cases} \tag{2.4.76}$$

where $\widetilde{u}$ is an extension by symmetry of the restriction of u to $\Omega \cap \{x : x - x_0, \nu_0\rangle < 0\}$. This sequence belongs to $\mathrm{PC}(\Omega)$, satisfies the constraint $S(u_j) \subset K_j$, and the desidered inequality immediately follows. Again, the general case is obtained by the usual localization arguments. ■

Remark 2.4.3 (generalizations) (i) The construction works exactly in the same way when K is a subset of a smooth hypersurface, in which case ν stands for the normal to that surface;

(ii) Since the construction is independent on each Π_m we can choose b and a depending on the particular index m, so that we also obtain energies of the form

$$F_{K,a,b}(u,\Omega) = \int_\Omega |\nabla u|^2\,dx + \sum_m \int_{S(u)\cap\Pi_m\cap\Omega} (a_m + b_m|u^+ - u^-|^2)\|\nu_m\|_1\,d\mathcal{H}^{n-1} \tag{2.4.77}$$

(ν_m the normal to Π_m), always with the constraint $S(u) \subset K$;

(iii) By localizing the construction we also may choose lower-semicontinuous piecewise constant a and b, and then by approximation all lower semicontinuous coefficients a and b, with $a \geq 1$ and $b \geq 0$, thus approximating

$$F_{K,\{a_m\},\{b_m\}}(u,\Omega) = \int_\Omega |\nabla u|^2\,dx + \int_{S(u)\cap\Omega} (a(x) + b(x)|u^+ - u^-|^2)\|\nu(u)\|_1\,d\mathcal{H}^{n-1}, \tag{2.4.78}$$

always with the constraint $S(u) \subset K$.

3. Limit energies of the type

$$F_{K,\psi}(u,\Omega) = \int_\Omega |\nabla u|^2\,dx + \int_{S(u)\cap\Omega} \psi(|u^+ - u^-|)\|\nu(u)\|_1\,d\mathcal{H}^{n-1}, \tag{2.4.79}$$

with the constraint that $S(u) \subset K$up to $\mathcal{H}^{n-1}$-negligible sets, where K is the closure of a relatively open set A of a locally finite union of hyperplanes as above with $\mathcal{H}^{n-1}(K) = \mathcal{H}^{n-1}(A)$, and $\psi : (0, +\infty) \to \mathbb{R}$ is of the form

$$\psi(z) = \min\Big\{\sum_{m\in J}(a_m + b_m z_m^2) : J \subset \{0,\ldots,M\},\ J \neq \emptyset,\ \sum_{m\in J} z_m = z\Big\}, \tag{2.4.80}$$

where $M \in \mathbb{N}$ is fixed, and $a_0,\ldots,a_M$ and $b_0,\ldots,b_M$ are given numbers with $a_m \geq 1$ and $b_m \geq 0$.

By reasoning locally, it is not restrictive to suppose that K is a subset of a single hyperplane $\{\langle x - x_0, \nu_0\rangle = 0\}$. The approximating energies will be obtained by *piling up* $M+1$ copies of K, on which energies of the form (2.4.77) are considered. More precisely, we define

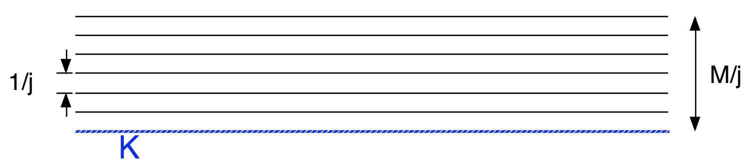

Figure 2.6: 'Micro-cracks' piling up to a 'macro-crack'

$$F_j(u) = \int_\Omega |\nabla u|^2\,dx + \sum_{m=0}^{M} \int_{S(u)\cap(K+\frac{m}{j}\nu_0)} (a_m + b_m|u^+ - u^-|^2)\|\nu_m\|_1\,d\mathcal{H}^{n-1}, \qquad (2.4.81)$$

with the constraint that $S(u) \subset \bigcup_m (K + \frac{m}{j}\nu_0)$. For the sake of simplicity we may suppose that

$$K \subset \{x_n = 0\}.$$

Theorem 2.4.4 *Let F_j be the functionals in* (2.4.81) *and $F_{K,\psi}$ that in* (2.4.79). *Then*
(i) *for all $u \in SBV(\Omega)$ and $u_j \to u$ we have*

$$\liminf_j F_j(u_j) \geq F_{K,a,b}(u);$$

(i) *for all $u \in \mathrm{PC}(\Omega)$ there exist $u_j \in \mathrm{PC}(\Omega)$ with $u_j \to u$ and*

$$\limsup_j F_j(u_j) \leq F_{K,a,b}(u).$$

Proof. Let $u_j \to u$ in $L^1(\Omega)$ and in SBV with equibounded energy. It suffices to check the liminf inequality on the interfacial part. Note that

$$u_j^+\Big(y, \frac{M}{j}\Big) \to u^+(y,0), \qquad u_j^-(y,0) \to u^-(y,0)$$

and that by Poincaré's inequality and the equi-boundedness of the L^2 norms of ∇u_j, for a.a. $y \in K$ we have, upon extraction of subsequences,

$$u_j^+\Big(y, \frac{m-1}{j}\Big) - u_j^-\Big(y, \frac{m}{j}\Big) \to 0$$

for all $m = 1, \dots, M$.
For all $y \in K$ we set

$$J_j(y) = \Big\{m \in \{0, \dots, M\} : u^+\Big(y, \frac{m}{j}\Big) \neq u^-\Big(y, \frac{m}{j}\Big)\Big\}.$$

We then have, by Fatou's Lemma,

$$\begin{aligned}
&\liminf_j \sum_{m=0}^{M} \int_{S(u)\cap(K+\frac{m}{j}e_n)} (a_m + b_m|u^+ - u^-|^2)\,d\mathcal{H}^{n-1}\\
&\quad= \liminf_j \int_K \sum_{m\in J_j(y)} \Big(a_m + b_m\Big|u^+(y,\frac{m}{j}) - u^-(y,\frac{m}{j})\Big|^2\Big)\,d\mathcal{H}^{n-1}\\
&\quad\geq \int_K \liminf_j \sum_{m\in J_j(y)} \Big(a_m + b_m\Big|u^+(y,\frac{m}{j}) - u^-(y,\frac{m}{j})\Big|^2\Big)\,d\mathcal{H}^{n-1}\\
&\quad\geq \int_K \psi(|u^+ - u^-|)\,d\mathcal{H}^{n-1}\\
&\quad\geq \int_{S(u)} \psi(|u^+ - u^-|)\,d\mathcal{H}^{n-1},
\end{aligned}$$

as desired.
As for the limsup inequality, we can perform the proof in the case $M = 1$, the general case following by induction. By the Lipschitz continuity of u outside $\overline{S(u)}$ and the continuity of the functions $z \mapsto a_m + b_m z^2$, for fixed $\eta > 0$ we can find a function $v_\eta : S(u) \to \mathbb{R}$ such that v is constant on each cube $\eta Q_1^{n-1}(i) \cap S(u)$ for all $i \in \mathbb{Z}^{n-1}$, and for almost all $y \in S(u)$

$$\begin{aligned}
&\chi_{\{v\neq u^-\}}(y)(a_0 + b_0|v(y) - u^-(y)|^2) + \chi_{\{v\neq u^+\}}(y)(a_1 + b_1|v(y) - u^+(y)|^2)\\
&\quad\leq \psi(|u^+(y) - u^-(y)|) + r_\eta,
\end{aligned}$$

with $r_\eta \to 0$ as $\eta \to 0$. We then fix δ_j with

$$1 >> \delta_j^2 >> \frac{1}{j}$$

and define functions $v_j^\eta \in W^{1,\infty}(\{x_n = 0\})$ as functions with minimal Lipschitz constant satisfying

$$v_j^\eta(y) = v_\eta(y) \text{ if } y \in \eta Q_1^{n-1}(j) \cap S(u) \text{ and } \operatorname{dist}\Big(y, (\{x_n = 0\} \setminus S(u)) \cup \bigcup_{i \neq j} \eta Q_1^{n-1}(j)\Big) > \delta,$$

$$v_j^\eta(y) = u(y, 0) \text{ if } y \in \{x_n = 0\} \setminus S(u).$$

By construction we have

$$|\nabla v_j^\eta| \le \frac{C}{\delta_j}.$$

If we define u_j as

$$u_j = \begin{cases} u(x) & \text{if } x_n < 0 \\ v_j^\eta(y) & \text{if } x = (y, t) \text{ with } 0 \le t \le 1/j \\ u(x - \frac{1}{j}e_n) & \text{if } x_n > 1/j, \end{cases}$$

then we have $u_j \to u$ and

$$\limsup_j F_j(u) \le \int_\Omega |\nabla u|^2 \, dx + \int_{S(u)} \psi(|u^+ - u^-|) \, d\mathcal{H}^{n-1} + r_\eta \mathcal{H}^{n-1}(S(u)).$$

The thesis then follows by the arbitrariness of η. ■

Remark 2.4.5 If K is the closure of a relatively open subset of a locally finite union of hyperplanes

$$\Pi_l = \{\langle x - x_l, \nu_l \rangle = 0\}, \qquad l \in L,$$

then we can localize the argument above. The general form of the limit is then

$$F_{K,\{\psi_l\}}(u, \Omega) = \int_\Omega |\nabla u|^2 \, dx + \sum_l \int_{S(u)\cap\Pi_l\cap\Omega} \psi_l(|u^+ - u^-|) \|\nu_l\|_1 \, d\mathcal{H}^{n-1}, \tag{2.4.82}$$

where each ψ_l is of the form (2.4.80).

2.4.2 Homogenized energies

In this section we consider sequences of planar systems invading the space $\mathbb{R}^n$. As a consequence the constraint $S(u) \subset K$ will be lost in the limit, and will appear only through inequalities on the limit energy densities. Moreover, by a density argument of the functions in PC(Ω) the Γ-limit will be characterized on the whole $GSBV(\Omega)$.

4. Limit energies of the type

$$F_\varphi(u, \Omega) = \int_\Omega |\nabla u|^2 \, dx + \int_{S(u)\cap\Omega} \varphi(\nu) \, d\mathcal{H}^{n-1}, \tag{2.4.83}$$

where φ is any even convex function positively homogeneous of degree one with

$$\varphi(\nu) \ge \|\nu\|_1. \tag{2.4.84}$$

Note that in particular we can obtain the Mumford-Shah functionals

$$F_c(u, \Omega) = \int_\Omega |\nabla u|^2 \, dx + c\,\mathcal{H}^{n-1}(S(u) \cap \Omega), \tag{2.4.85}$$

corresponding to $\psi(\nu) = c\|\nu\|_2$ (the Euclidean norm), provided that $c \geq \sqrt{n}$.

To define the approximating functionals we consider a dense sequence $\{\nu_k : k \in \mathbb{N}\}$ in S^{n-1}, set

$$\Pi_k = \{x : \langle x, \nu_k \rangle = 0\}, \tag{2.4.86}$$

and consider the family of hyperplanes (see Fig. 2.7)

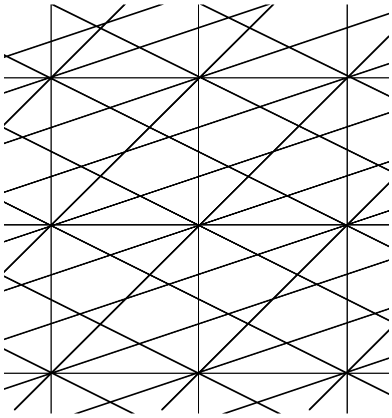

Figure 2.7: A system of hyperplanes

$$\Big\{\frac{1}{j}\mathbb{Z}^n + \Pi_k : k = 1, \dots, j\Big\} =: \{\Pi^j_m : m \in M_j\}, \tag{2.4.87}$$

where M_j is a set of indices such that $\Pi^j_m \neq \Pi^j_{m'}$ if $m \neq m'$ (we can directly take $M_j = \mathbb{Z}^n \times \{1, \dots, j\}$ if all ν_k are irrational directions: i.e., if $t\nu_k \in \mathbb{Z}^n$ only if $t = 0$).

We then take

$$K_j = \bigcup_{m \in M_j} \Pi^j_m \tag{2.4.88}$$

and F_j defined as in Remark 2.4.3(ii) by $F_{K_j, \{a^j_m\}, \{b^j_m\}}(u, \Omega)$, where

$$a^j_m = \frac{1}{\|\nu^j_m\|_1} \varphi(\nu^j_m) \quad \text{and} \quad b^j_m = 0;$$

i.e.,

$$F_j(u) = \int_\Omega |\nabla u|^2 \, dx + \sum_{m \in M_j} a^j_m \, \mathcal{H}^{n-1}(S(u) \cap \Pi^j_m \cap \Omega) \tag{2.4.89}$$

always with the constraint $S(u) \subset K_j$.

Theorem 2.4.6 *The functionals F_j in* (2.4.89) *Γ-converge to the functional F_φ in* (2.4.83). *Moreover recovery sequences can be constructed in* PC(Ω).

Proof. To prove the liminf inequality it suffices to remark that if $F_j(u_j) < +\infty$ then

$$F_j(u_j) = \int_\Omega |\nabla u_j|^2 \, dx + \int_{S(u_j) \cap \Omega} \varphi(\nu(u_j)) \, d\mathcal{H}^{n-1} = F_\varphi(u_j),$$

so that the desired inequality immediately follows from the lower semicontinuity of F_φ.

To prove the limsup inequality by approximation it suffices to treat the case when $u \in PC(\Omega)$; in particular $S(u)$ is a finite union of $n-1$-dimensional simplexes with disjoint closures.

If $S(u) = K_0$ is a single simplex in

$$\Pi_0 = \{\langle x - x_0, \nu_0 \rangle\} \text{ with } x_0 \in K_0,$$

then we find a neighborhood U of $\overline{K}_0$, choose ν_{k_j} with $k_j \leq j$ converging to ν_0 and $x_j \in \frac{1}{j}\mathbb{Z}^n$ converging to x_0, and smooth invertible $\Phi_j \in \text{Id} + C_0^\infty(U; U)$ with

$$\Phi_j \to \text{Id in } W^{1,\infty}(U; \mathbb{R}^m)$$

and

$$\Phi_j(K_0) = x_j + R_j(K_0 + (x_j - x_0)),$$

where R_j is a rotation such that $R_j \nu_0 = \nu_j$. Then, we set

$$u_j(x) = u(\Phi_j^{-1} x),$$

so that $S(u_j) \subset \{\langle x - x_j, \nu_{k_j} \rangle\}$ (note that this hyperplane is of the form Π^j_m), and

$$\limsup_j F_j(u_j) = \int_\Omega |\nabla u|^2 \, dx + \varphi(\nu_0)\mathcal{H}^{n-1}(K_0) = F(u).$$

If $S(u)$ is composed of more than one simplex then the same construction must be repeated locally, taking care of choosing disjoint neighborhoods. ■

5. Limit energies of the type

$$F_\psi(u, \Omega) = \int_\Omega |\nabla u|^2 \, dx + \int_{S(u)\cap\Omega} \psi(|u^+ - u^-|) \, d\mathcal{H}^{n-1}, \tag{2.4.90}$$

where ψ is any concave function on $(0, +\infty)$ with

$$\inf \psi \geq \sqrt{n}. \tag{2.4.91}$$

Note that this constraint is optimal, and derives from the inequality

$$\psi(z) \geq \|\nu\|_1,$$

which must hold for all $z > 0$ and $\nu \in S^{n-1}$.

Since ψ is concave, we can find two sequences $\{a_j\}$ and $\{b_j\}$ such that

$$\psi(z) = \inf\{a_j + b_j z^2 : j = 0, 1, \ldots\}$$

for all $z > 0$ (see Fig. 2.8). Moreover, the convergence is uniform on bounded subsets of $(0, +\infty)$.

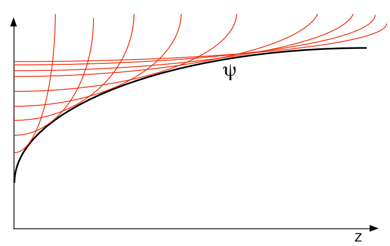

Figure 2.8: Approximation of a concave function

Note that $a_j \geq \sqrt{n}$ and $b_j \geq 0$ for all j. We define ψ_j as in (2.4.80) by

$$\psi_j(z) = \min\Big\{\sum_{l\in J}(a_l + b_l z_l^2) : J \subset \{0,\dots,j\},\ J \neq \emptyset,\ \sum_{l\in J} z_l = z\Big\}. \tag{2.4.92}$$

We have $\psi_j \geq \psi$ and $\psi_j \to \psi$ uniformly on bounded sets of $(0,+\infty)$.

We consider the hyperplanar networks $\{\Pi^j_m\}_{m\in M_j}$ as in (2.4.87), and their union as the corresponding K_j defined in (2.4.88). Denoted by ν^j_m the normal to Π^j_m,

$$\begin{aligned}\psi^j_m(z) &= \frac{1}{\|\nu^j_m\|_1}\psi_j(z) \qquad (2.4.93)\\ &= \min\Big\{\sum_{l\in J}\Big(\frac{a_l}{\|\nu^j_m\|_1} + \frac{b_l}{\|\nu^j_m\|_1}z_l^2\Big) : J \subset \{0,\dots,j\},\ J \neq \emptyset,\ \sum_{l\in J} z_l = z\Big\}.\end{aligned}$$

Since $a_l \geq \sqrt{n} \geq \|\nu\|_1$ for all $\nu \in S^{n-1}$ the functions ψ^j_m satisfy the hypotheses of Remark 2.4.5 (with the system of planes $\{\Pi^j_m\}$ in place of $\{\Pi_l\}$). The functionals F_j are then defined by $F_{K_j,\{\psi^j_m\}}(u,\Omega)$ in (2.4.82); namely,

$$\begin{aligned}F_j(u) &= \int_\Omega |\nabla u|^2\,dx + \sum_{m\in M_j}\int_{S(u)\cap\Pi^j_m\cap\Omega}\psi^j_m(|u^+-u^-|)\|\nu^j_m\|_1\,d\mathcal{H}^{n-1} \qquad (2.4.94)\\ &= \int_\Omega |\nabla u|^2\,dx + \sum_{m\in M_j}\int_{S(u)\cap\Pi^j_m\cap\Omega}\psi_j(|u^+-u^-|)\,d\mathcal{H}^{n-1},\end{aligned}$$

with the constraint $S(u) \subset K_j$.

Theorem 2.4.7 *The functionals F_j in* (2.4.94) *Γ-converge to the functional F_ψ in* (2.4.90). *Moreover recovery sequences can be constructed in* PC(Ω).

Proof. To prove the liminf inequality it suffices to remark that, since $\psi_j \geq \psi$, if $F_j(u_j) < +\infty$ then

$$F_j(u_j) \geq \int_\Omega |\nabla u_j|^2\,dx + \int_{S(u_j)\cap\Omega}\psi(|u_j^+ - u_j^-|)\,d\mathcal{H}^{n-1} = F_\psi(u_j),$$

so that the desired inequality immediately follows from the lower semicontinuity of F_ψ.

To prove the converse inequality, we can follow the same construction of Theorem 2.4.6. For the functions u_j defined therein we have

$$\begin{aligned}\limsup_j F_j(u_j) &\leq \lim_j\Big(\int_\Omega |\nabla u|^2\,dx + \int_{S(u)}\psi_j(|u^+-u^-|)\,d\mathcal{H}^{n-1}\Big)\\ &= \int_\Omega |\nabla u|^2\,dx + \int_{S(u)}\psi(|u^+-u^-|)\,d\mathcal{H}^{n-1}\end{aligned}$$

by the uniform convergence of ψ_j to ψ on bounded sets of $(0,+\infty)$ (recall that we can always assume u in L^∞ by a truncation argument). ■

Remark 2.4.8 (a wider class of surface energy densities) Theorem 2.4.7 is sharp on the set of concave target functions ψ. The same proof holds for a wider class, namely that of non-decreasing lower-semicontinuous subadditive functions that can be written as an infimum of functions ψ_j as in (2.4.80). It is not clear if there is a more transparent characterization of this class, which is strictly larger than the class of concave functions, containing for example

$$\psi_1(z) = \sqrt{n}\,\min\Big\{j + \frac{1}{j}z^2 : j = 1,2,\dots\Big\}$$

(the subadditive envelope of $\sqrt{n}(1+z^2)$), and

$$\psi_2(z) = \sqrt{n}\,\min\{1+z^2, 2\},$$

or if these are all the accessible energy densities.

Note that not all subadditive non-decresing functions are in this class, as for example

$$\psi_3(z) = \begin{cases} \sqrt{n} & \text{if } z \le 1 \\ 2\sqrt{n} & \text{if } z > 1 \end{cases}$$

(all functions ψ such that $\sup \psi \le 2 \inf \psi$ are subadditive), which does not seem to be an accessible target function.

2.4.3 Locally inhomogeneous energies

We can reach all energies of the form

$$F(u) = \int_\Omega |\nabla u|^2 \, dx + \int_{S(u)\cap\Omega} a(x)\psi(|u^+ - u^-|)\varphi(\nu(u)) \, d\mathcal{H}^{n-1} \tag{2.4.95}$$

with a lower semicontinuous with $a \ge 1$, ψ concave with $\psi \ge 1$, and φ convex and $\varphi(\nu) \ge \|\nu\|_1$ on S^{n-1}. The approximating energies can be easily constructed by localizing the arguments in the previous sections.

2.4.4 Proof of Theorem 2.2.2

We are eventually in the position to prove Theorem 2.2.2 using a diagonal procedure. Since all functionals we considered have the weak-membrane functional as a lower bound, we can use the metrizability of Γ-convergence ([39] Theorem 10.22), and a diagonal argument to deduce that there exists a sequence of sets K_i such that the energies $F_{K_i,b}$ defined as in (2.4.63) Γ-converge to the energy F in (2.4.95).

We call that if $u \in \mathrm{PC}(\Omega)$ then the recovery sequences constructed in Section 2.4.1 again belong to $\mathrm{PC}(\Omega)$, while again we have used a density argument with that set in Section 2.4.2. As a consequence, also the functionals

$$H_{K_i,b}(u) = \begin{cases} F_{K_i,b} & \text{if } u \in \mathrm{PC}(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

Γ-converge to the same F.

On the other hand, Theorem 2.3.1 ensures that for all i there exist a family $W^i_{\varepsilon_j}$ such that the Γ-limit F^i of $F^{W^i_{\varepsilon_j}}_{\varepsilon_j}$, which we can always suppose exists up to subsequences, satisfies

$$F_{K_i,b} \le F^i \le H_{K_i,b}, \tag{2.4.96}$$

so that also F^i Γ-converges to F.

We then conclude the existence of $W_{\varepsilon_j} = W^{i_j}_{\varepsilon_j}$ satisfying the thesis of Theorem 2.2.2 by a diagonal argument. Finally, note that again by Theorem 2.3.1, for given θ we can always suppose that the limit density of $W^i_{\varepsilon_j}$ is θ for all i, and then that this holds also for W_{ε_j}. ■

Chapter 3

Phase transition in presence of surfactants: from discrete to continuum

3.1 Introduction

The free energy of a system where two or more phases coexist can be altered by the presence of low concentrations of a surfactant (a contraction for *surface-active-agent*), a substance which, by being adsorbed onto the interfaces, may significantly reduce the surface tension of the system.

In order to give a variational description of the effects caused by the presence of surfactants in phase-separation phenomena, several attempts have been made to model the physical system both as a continuum and as a discrete. Among the continuum theories, the first description of phase transitions in presence of surfactants has been developed by Laradji-Guo-Grant-Zuckermann (see [50, 51]), who suggested a variational model involving a two order parameters Ginzburg-Landau functional. Several generalizations have been later considered by Gompper and Schick in [49]. In [50] and [51] one of the two order parameters represents the local difference of density of the two phases (as in the standard Cahn-Hillard model in the gradient theory of phase transitions), while the other one represents the local surfactant density. The two order parameters are energetically coupled to favor the segregation of the surfactant at the phase interface. The coarse-graining analysis of this model has been performed through Γ-convergence methods by Fonseca, Morini and Slastikov in [47], while the mathematical analysis of more general continuum models is the subject of [1].

Many of the discrete models are variants of the one which was originally introduced by Blume, Emery and Griffiths (BEG) in [15] (see also [49] and the references therein); this chapter deals with its variational analysis in the framework of Γ-convergence. In their seminal paper, Blume, Emery and Griffiths introduced a simple nearest-neighbors spin-1 model as a variant of a classical Ising type spin-1/2 model, with the aim of describing a different kind of phenomena, namely He^3-He^4 λ-transitions. In the setting of phase transitions in presence of surfactants, BEG model can be briefly described as follows. On the two dimensional square lattice $\mathbb{Z}^2$, we consider a ternary system driven by an energy which is defined on functions parameterized on the points of the lattice and taking only three possible values (which we may suppose to be $-1, 0, 1$). We can identify the values of u with three different phases (in particular, the value 0 is associated with the surfactant). Omitting the chemical potentials, for a given configuration of particles, the free energy E of this system is given by

$$E(u) = -\sum_{n.n.} u(a)u(b) + \sum_{n.n.} k(u(a)u(b))^2, \tag{3.1.1}$$

where $n.n.$ means that the summations are taken over all nearest neighboring sites; i.e., the elements $a,\ b$ of the lattice such that $|a-b|$ equals the lattice spacing. The constant $k>0$ is the quotient between the so-called bi-quadratic and the quadratic exchange interaction strengths; its range will be specified later on, such as the scaling factor for the energy.

In this chapter we will perform a Γ-limit analysis of these functional. As a result, we will be able to describe the behavior of the ground states of the BEG system as ε tends to 0. More precisely, let $\Omega \subset \mathbb{R}^2$ be a bounded open set and let us consider the scaled energies

$$E_\varepsilon(u) = \sum_{n.n.} \varepsilon^2(-u(a)u(b) + k(u(a)u(b))^2). \tag{3.1.2}$$

Here the array $\{u(a)\}$ can be seen as a function defined on $\varepsilon\mathbb{Z}^2 \cap \Omega$. Upon identifying such functions with their piecewise-constant interpolations, the energies E_ε can be interpreted as defined on (a subset of) $L^1(\Omega)$; we can then develop a Γ-convergence analysis in the framework of $L^1(\Omega)$. As ε tends to 0, the Γ-limit E of E_ε is particularly simple: under the trivial constraint $|u| \le 1$, it is constantly equal to the minimum value $2|\Omega|(-1+k) \wedge 0$, corresponding to the uniform states. By choosing $k<1$ we set the uniform states $u=\pm 1$ to be the ground states. Having fixed $k<1$, the asymptotic behavior of E_ε implies that a sequence $(u_\varepsilon)_\varepsilon$ can arbitrarily mix the uniform phases -1 and 1 at a mesoscopic scale, though keeping its energy equal to the energy of the uniform states plus an infinitesimal function, as $\varepsilon \to 0$ (the asymptotic analysis of the bulk scaling of more general spin-type models has been performed in [5]). Thus, in order to get a better description of the ground states, in the spirit of *development by Γ-convergence* (see [21], [28], [4] and [2]), we select sequences which attain the minimum value with a sharper precision, meaning that

$$E_\varepsilon(u_\varepsilon) = c_\varepsilon + O(\varepsilon),$$

where c_ε is the absolute minimum of E_ε; i.e., $c_\varepsilon = \sum_{\text{n.n.}} \varepsilon^2(k-1)$. For such configurations the limit states u will take the values ± 1 only. The limit energy will be an interfacial-type energy: it can be interpreted as the surface tension of the system which undergoes a phase separation phenomenon between the phases $\{u=-1\}$ and $\{u=+1\}$. At this scaling, it is necessary to further specify the values of the parameter k, so that the phase 0 can be actually considered a surfactant phase (meaning that it contributes to lower the surface tension). In particular it can be easily shown (see Section 3.3) that, for $\frac{1}{3} < k < 1$, the energy for a transition from phase -1 phase to phase $+1$ is lowered when the surfactant is at the interface. Moreover, the measure of the phase 0 vanishes as we pass to the limit. This scaling is usually referred to as *low surfactant concentration regime*. Thus, we study the rescaled functionals

$$E^{(1)}_\varepsilon(u) := \frac{E_\varepsilon(u) - c_\varepsilon}{\varepsilon} = \sum_{\substack{a,b\,\in\,\varepsilon\mathbb{Z}^2\cap\Omega\\ |a-b|=\varepsilon}} \varepsilon(1 - u(a)u(b) - k(1-(u(a)u(b))^2).$$

Note that the interaction between to particles of the same type -1 or $+1$ has zero energy, while the interaction of a surfactant particle 0 with any other particle is repulsive and 'costs' the positive value $1-k$. For this reason, the BEG functional is also said to describe a *repulsive surfactant model*. In Theorem 3.3.2 we show that $E^{(1)}_\varepsilon$ Γ-converges (in the $L^1(\Omega)$-topology) to the interfacial-type energy functional

$$E^{(1)}(u) = \int_{S(u)} \psi(\nu_u)\,d\mathcal{H}^1,$$

where $u \in BV(\Omega;\{\pm 1\})$, $S(u)$ is the (essential) interface between the sets $\{u=1\}$ and $\{u=-1\}$, ν_u is the inner normal to $S(u)$ and $\psi(\nu) = (1-k)(3|\nu_1| \vee |\nu_2| + |\nu_1| \wedge |\nu_2|)$ denotes the anisotropic surface tension of the model.

Note that in this topology the limit order parameter u does not carry any information about the surfactant phase. Actually, the role of the surfactant becomes clear when looking at the minimizing microstructure leading to the computation of the surface density ψ. In this direction, a natural further step in the analysis of the BEG model is the study of the dependence of the surface tension on the concentration of the surfactant. The literature on this subject is wide, both from the physical and the chemical point of view (see for example [49] and [54]). However, no rigorous description of the microscopic geometry of the surfactant at the interface is present in literature; all the previous documented attempts to study this problem are based on numerical computations or on heuristic arguments. In order to rigorously address this problem, we need to go beyond the standard formulation of the BEG model. In particular, the functional which describes the energy of the system has to depend explicitly on the distribution of the surfactant particles. To this end, we set

$$I_0(u) = \{a \in \Omega_\varepsilon : u(a) = 0\}$$

and we introduce the *surfactant measure*

$$\mu(u) = \sum_{a \in I_0(u)} \varepsilon \delta_a.$$

Then, with a slight abuse of notation, we can extend $E_\varepsilon^{(1)}$ to $L^1(\Omega) \times \mathcal{M}_+(\Omega) \to [0, +\infty]$ as

$$E_\varepsilon^{(1)}(u, \mu) = \begin{cases} E_\varepsilon^{(1)}(u) & \text{if } \mu = \mu(u) \\ +\infty & \text{otherwise.} \end{cases}$$

In order to track the energy of the surfactants, we extend the functionals by decoupling the order parameter of the model. In the continuum setting, instead, the functionals were extended by introducing an additional variable (see [47] and [1]). The space $L^1(\Omega) \times \mathcal{M}_+(\Omega)$ is endowed with the topology $\tau_1 \times \tau_2$, where τ_1 denotes the strong topology in $L^1(\Omega)$ and τ_2 refers to the weak *-topology in the space of non-negative bounded Radon measures $\mathcal{M}_+(\Omega)$. In Theorem 3.3.3 we prove that $E_\varepsilon^{(1)}$ Γ-converges (with respect to $\tau_1 \times \tau_2$-topology) to the functional $E^{(1)}$: $L^1(\Omega) \times \mathcal{M}^+(\Omega) \to [0, +\infty]$ defined as

$$E^{(1)}(u, \mu) = \begin{cases} \displaystyle\int_{S(u)} \varphi\Big(\frac{d\mu}{d\mathcal{H}^1 \lfloor_{S(u)}}, \nu_u\Big) d\mathcal{H}^1 + (2k-2)|\mu^s|(\Omega) & \text{if } u \in BV(\Omega; \{\pm 1\}) \\ +\infty & \text{otherwise,} \end{cases}$$

where $\varphi : \mathbb{R} \times S^1 \to [0, +\infty)$ is computed explicitly. Looking at the graph of φ (Figure 3.2), it stands out that an anisotropic threshold phenomenon occurs at the phase interface. Indeed, for a fixed $\nu \in S^1$, the surface tension $\varphi(z, \nu)$ decreases up to a certain value of the density z of the surfactant, namely $z = |\nu_1| \vee |\nu_2|$. As the density of the surfactant increases further, two events can happen: if the surfactants are not absorbed onto the interface, the surface tension remains constant and the singular part of the surfactant measure increases; otherwise, the surface tension increases. As an application of Theorem 3.3.3, at the end of Section 3.3 we study an optimization problem in which the volume fractions of the different phases are prescribed.

The variety of models of phase transitions in presence of surfactants studied in the physical/chemical literature suggested that we should widen our analysis. In Section 3.4, we consider the case of a n-dimensional discrete system, driven by an energy accounting for quite general finite-range pairwise interactions, in the presence of different species of repulsive surfactant particles. For such a general system, we obtain an integral representation result for the Γ-limit, in the spirit of homogenization theory, and we study some properties of its limit densities. Namely, given $\Omega \subset \mathbb{R}^n$ and $u : \varepsilon\mathbb{Z}^n \cap \Omega \to K$ we define the functional F_ε as

$$F_\varepsilon(u) = \sum_{\substack{a, b \in \varepsilon\mathbb{Z}^2 \cap \Omega \\ |a-b| \le R\varepsilon}} \varepsilon^{n-1} f\left(\frac{b-a}{\varepsilon}, u(a), u(b)\right).$$

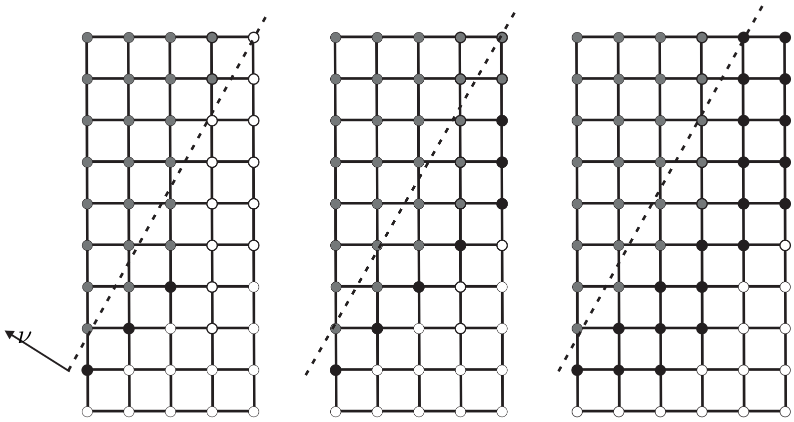

Figure 3.1: The local microstructure of a ground state of the BEG model at a fixed straight interface (the dashed line normal to ν) for three different values of the density of surfactants at the interface. Black, white and grey dots stand for the 0, +1 and −1 values of the spin field u, respectively.

where $R > 0$ is an interaction threshold and $K = \{m_1, m_2, s_1, s_2, \dots, s_M\} \subset \mathbb{R}$ describes the finite number of phases in the system. Moreover, $f : \mathbb{Z}^n \times K^2 \to [0, +\infty)$ satisfies some sort of discrete isotropy condition (see Remark 3.4.1 and 3.4.5) and is such that $\{(m_1, m_1), (m_2, m_2)\}$ are absolute minima of $f(z, \cdot, \cdot)$. In order to study the discrete-to-continuum limit of this system, we introduce a notation which describes the subsets of Ω_ε corresponding to the different types of surfactant. For $l \in \{1, 2, \dots, M\}$ we set

$$I_l(u) := \{a \in \Omega_\varepsilon : u(a) = s_l\}, \quad I(u) := \bigcup_{l=1}^{M} I_l(u)$$

and we define

$$\mu_l(u) := \sum_{a \in I_l(u)} \varepsilon^{n-1} \delta_a, \quad \mu(u) = \{\mu_1(u), \mu_2(u), \dots, \mu_M(u)\}.$$

We then extend F_ε to $L^1(\Omega) \times (\mathcal{M}_+(\Omega))^M \to [0, +\infty]$ as

$$F_\varepsilon(u, \mu) := \begin{cases} F_\varepsilon(u) & \text{if } \mu = \mu(u) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1.3}$$

The space $L^1(\Omega) \times (\mathcal{M}_+(\Omega))^M$ is endowed with the topology $\tau_1 \times \tau_2$, where τ_1 denotes the strong topology in $L^1(\Omega)$ and τ_2 stands for the weak*-topology in $(\mathcal{M}_+(\Omega))^M$. In Theorem 3.4.4 we prove that F_ε Γ-converges to the functional

$$F(u, \mu) = \begin{cases} \int\limits_{S(u)} f_{hom}\left(\frac{d\mu}{d\mathcal{H}^{n-1}\lfloor S(u)}, \nu(u)\right) d\mathcal{H}^{n-1} + \int_\Omega g_{hom}(\mu^s) \\ \qquad \text{for } u \in BV(\Omega; \{m_1, m_2\}),\ \mu = \frac{d\mu}{d\mathcal{H}^{n-1}\lfloor S(u)} \mathcal{H}^{n-1}\lfloor S(u) + \mu^s \\ +\infty \qquad \text{otherwise.} \end{cases} \tag{3.1.4}$$

The limit densities f_{hom} and g_{hom} are given by two asymptotic homogenization formulas, stated in (3.4.21) and (3.4.22). Whereas the formula for f_{hom} can be derived through a standard argument in homogenization theory, this is not true for g_{hom}. We will need to combine some abstract arguments of measure theory with a reflection construction, which uses the discrete isotropy assumption on the interaction densities, in order to prove that g_{hom} is well defined (see Remark 3.4.5).

It should be noted that in our models the surfactants are represented as point-like particles, with no internal structure. More general models have been developed: they describe the surfactants as polar molecules with heads and tails interacting differently with the same phase (see [36], [56] and [49]). In that setting, it is known that the presence of surfactants in a mixture may lead to self-assembling and that a number of different microstructures may appear, even with non-trivial topologies. Hopefully, the analysis performed in this chapter may provide the basis to address the discrete-to-continuum limit for those systems.

3.2 Notation and preliminaries

In what follows, given $x, y \in \mathbb{R}^n$ we denote by (x, y) the usual scalar product in $\mathbb{R}^n$ and we set $|x| = \sqrt{(x,x)}$. Moreover we denote by $\|\cdot\|_1$ the l_1-norm in $\mathbb{R}^n$ defined as $\|x\|_1 = |x_1| + \cdots + |x_n|$. Given $t > 0$, we write $[t]$ for the integer part of t. For any measurable $A \subset \mathbb{R}^n$ we denote by $|A|$ the n-dimensional Lebesgue measure of A. Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary. For fixed $\varepsilon > 0$ we consider the lattice $\varepsilon\mathbb{Z}^n \cap \Omega =: \Omega_\varepsilon$. Given $K \subset \mathbb{R}$ we denote by $\mathcal{A}_\varepsilon(\Omega; K)$ the set of functions

$$\mathcal{A}_\varepsilon(\Omega; K) := \{u : \Omega_\varepsilon \to K\}.$$

Remark 3.2.1 A function $u \in \mathcal{A}_\varepsilon(\Omega; K)$ will be identified with its piecewise-constant interpolation still denoted by u and given by $u(x) = u(z^\varepsilon_x)$, where $z^\varepsilon_x \in \mathbb{Z}^n$ is the closest point to x (which is uniquely defined up to a set of zero measure). In this definition, we set $u(z) = 0$ if $z \in \varepsilon\mathbb{Z}^n \setminus \Omega$. In such a way $\mathcal{A}_\varepsilon(\Omega; K)$ will be regarded as a subset in $L^1(\Omega)$.

We denote by $\mathcal{H}^{n-1}$ the $n-1$-dimensional Hausdorff measure. Given $\nu = (\nu_1, \ldots, \nu_n) \in S^{n-1}$ we set

$$Q_\nu := (-r_\nu, r_\nu)^n,$$

where $r_\nu > 0$ is such that

$$\mathcal{H}^{n-1}(Q_\nu \cap \Pi_\nu) = 1,$$

with $\Pi_\nu := \{x \in \mathbb{R}^n : (x, \nu) = 0\}$. We drop the dependence on ν whenever $\nu = e_i$ for $i \in \{1, 2, \ldots, n\}$ and we set $Q := Q_{e_i} = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$.
For any $T > 0$ we set

$$\partial^\pm(TQ_\nu) := \{x \in \partial(TQ_\nu) : \pm(x, \nu) \geq 0\}$$

and then we introduce the discrete boundary of TQ_ν as

$$\partial^\pm_\varepsilon(TQ_\nu) := \{a \in \varepsilon\mathbb{Z}^n \cap TQ_\nu : (a + [-R\varepsilon, R\varepsilon]^n) \cap \partial^\pm(TQ_\nu) \neq \emptyset\}.$$

Next we recall some basic properties of BV functions with values in a finite set (see [8] for a general description of the subject). Let A be an open subset of $\mathbb{R}^n$ and let J be a finite subset of $\mathbb{R}$. We denote by $BV(A; J)$ the set of measurable function $u : A \to J$ whose distributional derivative Du is a measure with bounded total variation. We denote by $S(u)$ the jump set of u and by $\nu_u(x)$ the measure theoretic inner normal to $S(u)$ at x, which is defined for $\mathcal{H}^{n-1}$ a.e. $x \in S(u)$.

We now recall a compactness result in BV (see [8]).

Theorem 3.2.2 *Let $u_k \in BV(A; J)$ such that*

$$\sup_n \mathcal{H}^{n-1}(S(u_k)) < +\infty.$$

Then there exists a subsequence (not relabelled) and $u \in BV(A; J)$ such that $u_k \to u$ in the L^1 convergence.

If Q is a cube we will denote by $BV^\#(Q; J)$ the set of Q-periodic functions belonging to $BV_{\mathrm{loc}}(\mathbb{R}^n; J)$.

3.3 The Blume-Emery-Griffiths model

In this section we briefly introduce the Blume-Emery-Griffiths model for phase transitions in presence of surfactants.

3.3.1 A brief description of the model: from bulk to surface scaling

In its standard formulation the Blume-Emery-Griffiths model can be described as follows. Given a bounded open set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary, we consider the set $\mathcal{A}_\varepsilon(\Omega; \{\pm 1, 0\}) := \{u : \Omega_\varepsilon \to \{\pm 1, 0\}\}$, where $\Omega_\varepsilon = \varepsilon\mathbb{Z}^2 \cap \Omega$. The sites on which $u = \pm 1$ corresponds to particles of water (or oil), while the sites on which $u = 0$ correspond to particles of surfactant (in this framework the scale is not precisely specified and the term *particle* means generically a molecule or an aggregate of molecules). We then introduce the family of energies $E_\varepsilon^{latt}(u) : \mathcal{A}_\varepsilon(\Omega; \{\pm 1, 0\}) \mapsto \mathbb{R}$

$$E_\varepsilon^{latt}(u) = \sum_{\substack{a, b \in \Omega_\varepsilon \\ |a-b| = \varepsilon}} \varepsilon^2(-u(a)u(b) + k((u(a)u(b))^2), \tag{3.3.5}$$

where $k > 0$ is a parameter which measures the relative strength of the quadratic vs the bi-quadratic interactions. The asymptotic analysis of such a family of energies, as ε tends to 0, is particularly simple and can be obtained through a dual lattice approach as in [2]. Indeed, by identifying the functions $u \in \mathcal{A}_\varepsilon(\Omega; \{\pm 1, 0\})$ with their piecewise constant interpolations (see Remark 3.2.1), we first extend the energies E_ε^{latt} in (3.3.5) to a functional $E_\varepsilon : L^1(\Omega) \to [0, +\infty]$ as

$$E_\varepsilon(u) = \begin{cases} E_\varepsilon^{latt}(u) & \text{if } \mathcal{A}_\varepsilon(\Omega; \{\pm 1, 0\}) \\ +\infty & \text{otherwise,} \end{cases}$$

and then compute the Γ-limit of (E_ε) with respect to the weak topology in $L^1(\Omega)$.
As a result one obtains that the following Theorem holds true:

Theorem 3.3.1 *The family (E_ε) Γ-converges with respect to the $L^1(\Omega)$-weak topology to the functional $E : L^1(\Omega) \to \mathbb{R} \cap \{+\infty\}$ defined as*

$$E(u) = \begin{cases} 2|\Omega|(k-1) \wedge 0 & \textit{if } u \in L^1(\Omega; [-1, 1]) \\ +\infty & \textit{otherwise.} \end{cases}$$

Let us comment the previous result in the interesting case when $k < 1$. In this regime the lattice energy is minimized by the two pure states $u = \pm 1$ and all the deviations of the order parameter from these states count at order ε^2 in the discrete energy. This implies that, in the continuum limit, it is possible to obtain, with finite energy, any value of the order parameter u in $[-1, +1]$ by arbitrarily mixing the two ground states on a mesoscopic scale $\varepsilon \ll \delta \ll 1$. In particular this makes the energy of a phase separation negligible. More precisely, a phase transition from a bulk -1 phase to a bulk $+1$ phase, separated by an interface of finite length, has an energy of order ε. This suggests the correct scaling to track the energetic behavior of a phase separation phenomenon. We observe that the absolute minimum value at scale ε is given by

$$m_\varepsilon = \sum_{\substack{a, b \in \Omega_\varepsilon \\ |a-b| = \varepsilon}} \varepsilon^2(k-1).$$

In order to get a richer description of the ground states we select the configurations corresponding to functions u_ε which attain the minimum value with a sharper precision; i.e., such that

$$E_\varepsilon(u_\varepsilon) = m_\varepsilon + O(\varepsilon).$$

In other words this amounts to study the family of discrete energies $E^{(1)}_\varepsilon : L^1(\Omega) \to [0,+\infty]$ defined as

$$E^{(1)}_\varepsilon(u) := \frac{E_\varepsilon(u) - m_\varepsilon}{\varepsilon};$$

i.e.,

$$E^{(1)}_\varepsilon(u) = \begin{cases} \displaystyle\sum_{\substack{a,b\in\Omega_\varepsilon \\ |a-b|=\varepsilon}} \varepsilon(1 - u(a)u(b) - k(1-(u(a)u(b))^2) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega;\{0,\pm1\}), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.3.6}$$

Having picked this scaling, the measure of the surfactant phase has to be negligible in the continuum limits. Namely, it is easy to see that, since each interaction with a surfactant particle pays a positive energy $1-k$, the following estimate

$$E^{(1)}_\varepsilon(u) \geq \#\{a \in \Omega_\varepsilon : u(a) = 0\}\varepsilon(1-k) \geq \frac{C}{\varepsilon}|\{x \in \Omega : u(x) = 0\}|$$

implies that the measure of the surfactant phase scales as ε. As a result, the finite energy states u will only take the values ±1.

Moreover, it is possible to further specify the values of the parameter k in such a way that the phase 0 can be actually considered a surfactant phase, meaning that it lowers the surface tension in the continuum limit. To obtain an estimate on the values of k, we can proceed by computing the energy for a transition from -1 to $+1$ in the simple case in which the interface is a straight line parallel to one of the directions of the lattice (say e_1). Suppose for simplicity that $\Omega = Q$ and that the interface is the set $\{x \in Q : (x, e_2) = 0\}$. Our estimate is obtained by comparing the energy for such a macroscopic transition when the microscopic structure is given by either u_ε or by v_ε, where

$$u_\varepsilon(a) = \begin{cases} +1 & \text{if } (a,e_2) \geq 0 \\ -1 & \text{otherwise.} \end{cases}$$

and

$$v_\varepsilon(a) = \begin{cases} +1 & \text{if } (a,e_2) > 0 \\ 0 & \text{if } (a,e_2) = 0 \\ -1 & \text{otherwise.} \end{cases}$$

We have:

$$\begin{aligned} E^{(1)}_\varepsilon(u_\varepsilon) &= 2 + o(1) \\ E^{(1)}_\varepsilon(v_\varepsilon) &= 3(1-k) + o(1). \end{aligned}$$

If we require that $E^{(1)}_\varepsilon(v_\varepsilon) < E^{(1)}_\varepsilon(u_\varepsilon)$, then the interface energy is lower for the the microstructure with the surfactant. This turns into the condition $k > \frac{1}{3}$. In Theorem 3.3.2, we will see that such an estimate is sufficient to deal with the case of a more general interface.

Finally, we remark that the previous heuristic derivation of the range of parameter k leads us to refer to this scaling as *low surfactant concentration regime.*

Theorem 3.3.2 *Let $\frac{1}{3} < k < 1$ and let $(E^{(1)}_\varepsilon)_\varepsilon$ be the family of functionals defined as in* (3.3.6)*. Then we have*

(i) for any sequence $(u_\varepsilon) \subseteq L^1(\Omega)$ such that

$$\sup_\varepsilon E^{(1)}_\varepsilon(u_\varepsilon) \leq C < +\infty$$

there exist $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ *and* $u \in BV(\Omega;\{\pm 1\})$ *such that*

$$u_{\varepsilon_k} \to u \text{ for } k \to +\infty$$

with respect to the $L^1(\Omega)$*-topology;*

(ii) the family of functionals $(E_\varepsilon^{(1)})$ Γ*-converges with respect to the* $L^1(\Omega)$*-topology to the functional* $E^{(1)} : L^1(\Omega) \to [0,+\infty]$ *defined by*

$$E^{(1)}(u) = \begin{cases} \displaystyle\int_{S(u)} \psi(\nu_u) d\mathcal{H}^1 & \text{if } u \in BV(\Omega;\{\pm 1\}) \\ +\infty & \text{otherwise,} \end{cases} \tag{3.3.7}$$

where $\psi : S^1 \to [0,+\infty)$ *is given by*

$$\psi(\nu) = (1-k)(3|\nu_1| \vee |\nu_2| + |\nu_1| \wedge |\nu_2|).$$

Proof. For the sake of simplicity we will derive the proof as a consequence of Theorem 3.3.3. The compactness result in (i) is a straightforward consequence of the analogous result stated in Theorem 3.3.3(i). In order to prove the Γ-lim inf inequality, let us first note that the function $\varphi(\cdot,\cdot)$ defined in (3.3.10) satisfies

$$\min\{\varphi(z,\nu) : \ z \in \mathbb{R}_+\} = \varphi(|\nu_1| \vee |\nu_2|, \nu) = \psi(\nu).$$

Hence the functionals $E^{(1)}(\cdot,\cdot)$ defined in (3.3.9) verify

$$E^{(1)}(u,\mu) \geq E^{(1)}(u), \qquad \text{for all } (u,\mu) \in BV(\Omega;\{\pm 1\}) \times \mathcal{M}^+(\Omega)$$

Let $u_\varepsilon \to u$ in $L^1(\Omega)$. By Theorem 3.3.3(i), we may assume that $\mu(u_\varepsilon) \to \mu$ weakly in the sense of measures. Then

$$\liminf_\varepsilon E_\varepsilon^{(1)}(u_\varepsilon) \geq E^{(1)}(u,\mu) \geq E^{(1)}(u).$$

By a density argument it suffices to prove the Γ-lim sup inequality for a function u with a polyhedral jump set. Since the construction is local it is enough to consider $u = u_\nu$, where u_ν is defined in (3.5.76). For such a function the optimizing sequence is given by $v_{z,\nu}(\frac{\cdot}{\varepsilon})$, where $v_{z,\nu}$ is defined in (3.5.77), with $z = |\nu_1| \vee |\nu_2|$. ∎

3.3.2 Low concentration of surfactants: discrete-to-continuum limit

As seen in the previous section, in the topology we chose the limit order parameter u does not carry any information about the surfactant phase. Actually, the role of the surfactant becomes clear when one looks at the minimizing microstructure leading to the computation of the limiting surface density ψ. In this direction, a natural further step in the analysis of the BEG model is the study of the dependence of the surface tension on the concentration of the surfactant. To address this problem, we need to go beyond the standard formulation of the BEG model and let the energy functional of the system depend explicitly on the distribution of the surfactant particles. To this end, for all $u \in \mathcal{A}_\varepsilon(\Omega;\{0,\pm 1\})$ we set

$$I_0(u) = \{a \in \Omega_\varepsilon : \ u(a) = 0\},$$

and we introduce the following *surfactant measure*

$$\mu(u) = \sum_{a \in I_0(u)} \varepsilon \delta_a.$$

Then, with a slight abuse of notation, we can extend $E_\varepsilon^{(1)}$ to a functional $E_\varepsilon^{(1)} : L^1(\Omega) \times \mathcal{M}_+(\Omega) \to [0,+\infty]$ as

$$E_\varepsilon^{(1)}(u,\mu) = \begin{cases} E_\varepsilon^{(1)}(u) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega;\{0,\pm 1\}), \ \mu = \mu(u) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.3.8}$$

We endow the space $L^1(\Omega) \times \mathcal{M}_+(\Omega)$ with the topology $\tau_1 \times \tau_2$ where τ_1 denotes the strong topology in $L^1(\Omega)$ and τ_2 denotes the weak*-topology in the space of non-negative bounded Radon measures $\mathcal{M}_+(\Omega)$.

The following Theorem holds.

Theorem 3.3.3 *Let $E_\varepsilon^{(1)}$ be defined as in (3.3.8). We have:*

(i) let $\varepsilon_k \to 0$ and let $(u_k, \mu_k) \in L^1(\Omega) \times \mathcal{M}_+(\Omega)$ be such that

$$\sup_k E_{\varepsilon_k}^{(1)}(u_k, \mu_k) < +\infty.$$

Then there exist a subsequence (not relabeled) such that $(u_k, \mu_k) \to (u, \mu)$ with respect to the $\tau_1 \times \tau_2$ topology, for some $(u, \mu) \in L^1(\Omega) \times \mathcal{M}_+(\Omega)$;

(ii) the family $(E_\varepsilon^{(1)})$ Γ-converges with respect to the $\tau_1 \times \tau_2$ topology to the functional $E^{(1)} : L^1(\Omega) \times \mathcal{M}_+(\Omega) \to [0, +\infty]$ defined by

$$E^{(1)}(u,\mu) = \begin{cases} \displaystyle\int_{S(u)} \varphi\Big(\frac{d\mu}{d\mathcal{H}^1 \lfloor S(u)}, \nu_u\Big) d\mathcal{H}^1 + (2k-2)|\mu^s|(\Omega) & \text{if } u \in BV(\Omega; \{\pm 1\}) \\ +\infty & \text{otherwise,} \end{cases} \tag{3.3.9}$$

where, for any $\mu \in \mathcal{M}_+(\Omega)$, μ^s indicates $\mu^s := \mu - \frac{d\mu}{d\mathcal{H}^1 \lfloor S(u)} \mathcal{H}^1 \lfloor S(u)$ and the function $\varphi : \mathbb{R}_+ \times S^1 \to [0, +\infty)$ is given by

$$\varphi(z, \nu) = \max\{\varphi_1(z,\nu), \varphi_2(z,\nu), \varphi_3(z,\nu)\}, \tag{3.3.10}$$

with

$$\begin{aligned} \varphi_1(z,\nu) &= -4kz + 2(|\nu_1| + |\nu_2|), \\ \varphi_2(z,\nu) &= (1-3k)z + 2(|\nu_1| \vee |\nu_2|) + (1-k)(|\nu_1| \wedge |\nu_2|), \\ \varphi_3(z,\nu) &= 2(1-k)z + (1-k)(|\nu_1| + |\nu_2|). \end{aligned}$$

We postpone the proof of this Theorem to Section 3.5, since it makes use of the integral representation result stated in Theorem 3.4.4.

Remark 3.3.4 Looking at the graph of φ (see Figure 3.2), it is clear that an anisotropic threshold phenomenon occurs at the phase interface. For fixed $\nu \in S^1$ the surface tension $\varphi(z, \nu)$ of the system may decrease only up to a certain value of the density z of the surfactant, namely $z = |\nu_1| \vee |\nu_2|$. If the density of the surfactant increases further, then two cases can occur: either the surfactant is not absorbed onto the interface and then surface tension remains constant, or the surfactant is absorbed by the interface and the surface tension increases. In the first case, the singular part of the surfactant measure increases.

As an application of the previous result, one may study the asymptotic behavior, as $\varepsilon \to 0$, of the following constrained optimization problem:

$$m_\varepsilon := \left\{ E_\varepsilon^{(1)}(u),\ \varepsilon \# I_0(u) = \alpha_\varepsilon,\ \varepsilon^2 \# I_1(u) = \beta_\varepsilon \right\}, \tag{3.3.11}$$

where

$$I_1(u) = \{a \in \Omega_\varepsilon : u(a) = 1\},$$

$\lim_\varepsilon \alpha_\varepsilon = \alpha > 0$ and $\lim_\varepsilon \beta_\varepsilon = \beta > 0$. Since we are not interested in boundary layer effects, we consider the case in which Ω is a torus and we identify it with the semi-open cube $\mathcal{Q} := [0, 1)^2$, assuming that the admissible functions u in (3.3.11) are $\mathcal{Q}$-periodic. In addition, we let $\varepsilon = 1/k$, $k \in \mathbb{N}$. The solution to this problem is a particular case of the result stated in Corollary 3.4.11 (see Remark 3.4.12).

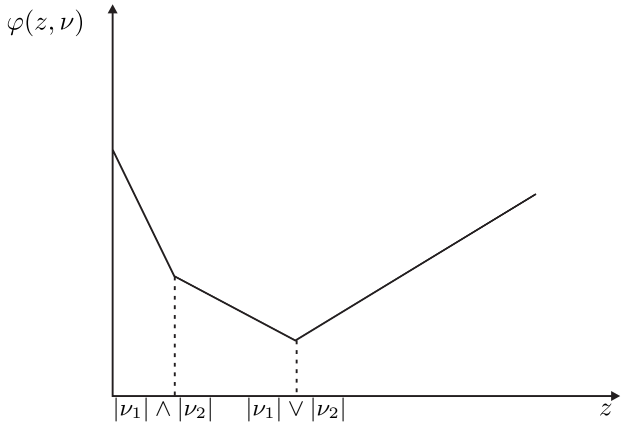

Figure 3.2: The graph of the surface tension density $\varphi(z,\nu)$ as a function of the density z of surfactant at the phase interface.

3.4 More general models

In this section we consider a class of energies that generalizes those involved in the BEG model and in which long range interactions and different types of surfactant are taken into account.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. We consider the family of functionals $F_\varepsilon : \mathcal{A}_\varepsilon(\Omega; K) \to [0,+\infty)$ defined by

$$F_\varepsilon(u) = \sum_{a,b\in\Omega_\varepsilon,\ |a-b|\le R\varepsilon} \varepsilon^{n-1} f\left(\frac{b-a}{\varepsilon}, u(a), u(b)\right) \tag{3.4.12}$$

where $R>0$, $K=\{m_1, m_2, s_1, s_2, \dots, s_M\} \subset \mathbb{R}$, $M\in\mathbb{N}$, and $f : \mathbb{Z}^n \times K^2 \to [0,+\infty)$ satisfies the following conditions:

$$f^{-1}(0) = \mathbb{Z}^n \times \{(m_1,m_1),(m_2,m_2)\}, \tag{3.4.13}$$

$$f(R^i\xi, u, v) = f(\xi,u,v) \quad \text{for all } i\in\{1,2,\dots,n\}, \tag{3.4.14}$$

where $R^i(\xi_1,\xi_2,\dots,\xi_i,\dots,\xi_n) = (\xi_1,\xi_2,\dots,-\xi_i,\dots,\xi_n)$ is the reflection with respect to the i-th coordinate axis. Moreover we define a localized energy for every $A\subset\Omega$ as

$$F_\varepsilon(u,A) = \sum_{a,b\in A\cap\varepsilon\mathbb{Z}^n,\ |a-b|\le R\varepsilon} \varepsilon^{n-1} f\left(\frac{b-a}{\varepsilon}, u(a), u(b)\right). \tag{3.4.15}$$

Remark 3.4.1 *We remark that* (3.4.13) *implies that the pure phases* $u\equiv m_i$, $i=1,2$, *are the ground states of the energy* F_ε. *Assumption* (3.4.14) *is a sort of discrete isotropy condition on the energy density; in particular, it is satisfied if* $f(\xi,u,v)=f(|\xi|,u,v)$.

Remark 3.4.2 *We observe that the functional* $E_\varepsilon^{(1)}$ *defined in* (3.3.6) *is a special case of* (3.4.12), *with*

$$f(\xi,u,v) := \begin{cases} -uv - k(1-uv)^2 & \text{if } \xi=\pm e_i,\ i\in\{1,2\} \\ 0 & \text{otherwise.} \end{cases} \tag{3.4.16}$$

It satisfies assumptions (3.4.13) *and* (3.4.14) *with* $K=\{\pm1,0\}$, $m_1=-1$ *and* $m_2=1$.

We now set a notation to describe the sets of points in Ω_ε corresponding to the different types of surfactant and we introduce suitable measures associated to them. For $l \in \{1, 2, \dots, M\}$ and $A \subset \Omega$, we set

$$I_l(u, A) = \{a \in A \cap \Omega_\varepsilon : \, u(a) = s_l\},$$

$$I(u, A) = \bigcup_{l=1}^{M} I_l(u, A).$$

For the sake of simplicity, let $I_l(u, \Omega) = I_l(u)$ and $I(u, \Omega) = I(u)$. Moreover we define

$$\mu_l(u) = \sum_{a \in I_l(u)} \varepsilon^{n-1} \delta_a$$
$$\mu(u) = \{\mu_1(u), \mu_2(u), \dots, \mu_M(u)\}. \tag{3.4.17}$$

With the identification given in Remark 3.2.1 and a slight abuse of notation, we can extend F_ε to a functional $F_\varepsilon : L^1(\Omega) \times (\mathcal{M}_+(\Omega))^M \to [0, +\infty]$ as

$$F_\varepsilon(u, \mu) = \begin{cases} F_\varepsilon(u) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega; K), \; \mu = \mu(u) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.4.18}$$

We endow the space $L^1(\Omega) \times (\mathcal{M}_+(\Omega))^M$ with the topology $\tau_1 \times \tau_2$ where τ_1 denotes the strong topology in $L^1(\Omega)$ and τ_2 denotes the weak*-topology in $(\mathcal{M}_+(\Omega))^M$. The choice of this topology is suggested by the following compactness result.

Proposition 3.4.3 *Let $\varepsilon_k \to 0$ and let (u_k, μ_k) be such that*

$$\sup_k F_{\varepsilon_k}(u_k, \mu_k) < +\infty.$$

Then there exists a subsequence (not relabeled) such that $(u_k, \mu_k) \to (u, \mu)$ with respect to the $\tau_1 \times \tau_2$-topology, for some $(u, \mu) \in BV(\Omega; \{m_1, m_2\}) \times (\mathcal{M}_+(\Omega))^M$.

Proof. Firstly, note that

$$\mathcal{H}^{n-1}(S(u_k)) + \mu_k(\Omega) \leq C F_{\varepsilon_k}(u_k, \mu_k). \tag{3.4.19}$$

By Theorem 3.2.2 and the estimate

$$|\{x \in \Omega : \, u_k(x) \notin \{m_1, m_2\}\}| \leq C \varepsilon_k \mu_k(\Omega) \to 0,$$

we easily get the conclusion. ■

3.4.1 Main result

In this section we state and prove an integral representation result for the Γ-limit of the family F_ε. To this end, we introduce for any $\varepsilon > 0$ and $\nu \in S^{n-1}$ the class of discrete functions

$$\mathcal{B}_\varepsilon(TQ_\nu; K) = \{u \in \mathcal{A}_\varepsilon(TQ_\nu; K) : \, u(a) = m_1 \text{ for all } a \in \partial^+_\varepsilon(TQ_\nu), \, u(a) = m_2 \text{ for all } a \in \partial^-_\varepsilon(TQ_\nu)\}.$$

Theorem 3.4.4 *The family (F_ε) Γ-converges with respect to the $\tau_1 \times \tau_2$-topology to the functional $F : L^1(\Omega) \times (\mathcal{M}_+(\Omega))^M \to [0, +\infty]$ defined by*

$$F(u, \mu) = \begin{cases} \int\limits_{S(u)} f_{hom}\left(\frac{d\mu}{d\mathcal{H}^{n-1}\lfloor S(u)}, \nu(u)\right) d\mathcal{H}^{n-1} + \int_\Omega g_{hom}(\mu^s) & \text{if } u \in BV(\Omega; \{m_1, m_2\}) \\ +\infty & \text{otherwise,} \end{cases} \tag{3.4.20}$$

where, for $\mu \in (\mathcal{M}_+(\Omega))^M$*, we set* $\mu^s := \mu - \frac{d\mu}{d\mathcal{H}^{n-1}\lfloor S(u)}\mathcal{H}^{n-1}\lfloor S(u)$*. Here,* $f_{hom} : (\mathbb{R}_+)^M \times S^{n-1} \to [0,,+\infty)$ *is defined as*

$$f_{hom}(z,\nu) := \lim_{\delta\to 0^+}\lim_{T\to+\infty}\frac{1}{T^{n-1}}\inf\Big\{F_1(u,TQ_\nu):\ u\in\mathcal{B}_1(TQ_\nu;K),\ \max_{l\in\{1,\dots,M\}}\Big|\frac{\#I_l(u,TQ_\nu)}{T^{n-1}}-z_l\Big|<\delta\Big\}, \tag{3.4.21}$$

while $g_{hom} : (\mathbb{R}_+)^M \to [0,+\infty)$ *is 1-homogeneous and, for any* $\zeta \in (\mathbb{R}_+)^M$ *such that* $\|\zeta\|_1 = 1$*, is defined as*

$$g_{hom}(\zeta) := \lim_{\delta\to 0^+}\liminf_{T\to+\infty}\inf\Big\{\frac{F_1(u,TQ)}{\#I(u,TQ)}:\ u\in\mathcal{A}_1(TQ;K),\ \frac{F_1(u,TQ\setminus(T-R)Q)}{\#I(u,TQ)}<\delta,\ \max_{l\in\{1,\dots,M\}}\Big|\frac{\#I_l(u,TQ)}{\#I(u,TQ)}-\zeta_l\Big|<\delta\Big\}. \tag{3.4.22}$$

Remark 3.4.5 We observe that, while the formula for f_{hom} can be proved by using standard arguments in homogenization theory, the same does not hold for g_{hom}. In particular, as it will be clear in the proof of Theorem 3.4.4, optimizing sequences for $f_{hom}(z,\nu)$ can be constructed, as it is usual in this framework, by "periodically gluing" a solution of the minimum problem on TQ_ν given in (3.4.21). In such a construction, the energy due to the interactions which cross the boundary of the periodicity cell is asymptotically negligible thanks to the Dirichlet type condition we are allowed to impose by using a De Giorgi's cut-off construction (see Lemma 3.4.1). The same arguments do not apply to g_{hom}. In fact, for this term, we cannot be sure that, imposing the same type of boundary conditions, we do not modify too much the energy of minimal configurations in (3.4.22) since the distribution of the phases m_1 and m_2 for such configurations is not known. This fact rules out the standard "periodic gluing" construction. Instead, we first make use of an abstract argument from measure theory which allows us to prove that the minimal configurations do not concentrate energy at the boundary of the periodic cell and then, by using hypothesis (3.4.14), we construct optimizing sequences for $g_{hom}(\zeta)$ through a reflection argument (see the proof of Proposition 3.4.9).

Before proving Theorem 3.4.4, we point out some properties verified by f_{hom} and g_{hom}. In the next Proposition we prove that the homogenization formula for f_{hom} is well defined.

Proposition 3.4.6 *For any* $z \in \mathbb{R}^M$*,* $\nu \in S^{n-1}$ *and* $\delta > 0$ *there exists the limit*

$$\lim_{T\to+\infty}\frac{1}{T^{n-1}}\inf\Big\{F_1(u,TQ_\nu):\ u\in\mathcal{B}_1(TQ_\nu;K),\ \max_{l\in\{1,\dots,M\}}\Big|\frac{\#I_l(u,TQ_\nu)}{T^{n-1}}-z_l\Big|<\delta\Big\}.$$

Proof. For simplicity of notation we develop the proof in the case $\nu = e_n$, but the argument obviously applies to the general case. We recall that we set $Q := Q_{e_n} = (-\frac{1}{2},\frac{1}{2})^n$. Let us define

$$\mathcal{I}_T(z,\delta) := \inf\Big\{F_1(u,TQ):\ u\in\mathcal{B}_1(TQ;K),\ \max_{l\in\{1,\dots,M\}}\Big|\frac{\#I_l(u,TQ)}{T^{n-1}}-z_l\Big|<\delta\Big\}. \tag{3.4.23}$$

Given $\eta > 0$, let $u_T \in \mathcal{B}_1(TQ;K)$ be such that $\max_{l\in\{1,\dots,M\}}\Big|\frac{\#I_l(u,TQ)}{T^{n-1}}-z_l\Big| < \delta$ and

$$F_1(u_T,TQ) \le \mathcal{I}_T(z,\delta) + \eta.$$

Let u_T be extended on the stripe $\big((-\frac{T}{2},\frac{T}{2})^{n-1}\times\mathbb{R}\big)\cap\mathbb{Z}^n$ by setting

$$u_T(a) = m_1 \text{ if } a_n \ge \frac{T}{2}, \quad u_T(a) = m_2 \text{ if } a_n < \frac{T}{2}.$$

Let us set $\hat{T} := 2\left[\frac{T}{2}\right]$. For $S > T$, let $v_S \in \mathcal{B}_1(SQ;K)$ be defined as

$$v_S(a) = u_T(a) \quad \text{if } a \in \Big\{\frac{-\hat{T}}{2},\dots,\frac{\hat{T}}{2}\Big\}^{n-1}\times\mathbb{Z},$$

and then extended by periodicity in the $e_1, \dots, e_{n-1}$ directions as

$$v_S(a + j\hat{T}e_i) = v_S(a) \quad \text{for all } j \in \mathbb{Z},\ i \in \{1, \dots, n-1\}.$$

Note that, for S large enough, we have

$$\left| \frac{\#I_l(v_S, SQ)}{S^{n-1}} - z_l \right| < \delta.$$

Hence, splitting the energy into two terms, the first one accounting for the interactions inside each periodic cell of the type $\left[-\frac{\hat{T}}{2}, \frac{\hat{T}}{2}\right]^n + j\hat{T}e_i$, and the second one accounting for the interactions which cross the boundary of the same cells, we get

$$\begin{aligned}
\frac{1}{S^{n-1}}\mathcal{I}_S(z,\delta) \le \frac{1}{S^{n-1}}F_1(v_S, SQ) &\le \frac{1}{S^{n-1}}\left[\frac{S}{\hat{T}}\right]^{n-1}(F_1(u_T, TQ) + C\hat{T}^{n-2}) \\
&\le \frac{T^{n-1}}{S^{n-1}}\left[\frac{S}{\hat{T}}\right]^{n-1}\left(\frac{1}{T^{n-1}}\mathcal{I}_T(z,\delta) + \frac{C\hat{T}^{n-2} + \eta}{T^{n-1}}\right).
\end{aligned}$$

By letting first S and then T go to $+\infty$, by the arbitrariness of η we finally get

$$\limsup_{S\to+\infty} \frac{1}{S^{n-1}}\mathcal{I}_S(z,\delta) \le \liminf_{T\to+\infty} \frac{1}{T^{n-1}}\mathcal{I}_T(z,\delta).$$

■

In the next Proposition we prove growth and convexity properties of the functions f_{hom} and g_{hom}. We remark that, in the proof the Γ-convergence result, we will only use the continuity of f_{hom} and g_{hom} and that their convexity will be a consequence of the lower semicontinuity of the Γ-limit. However, the proof of the continuity will rely on the same argument we use here with no relevant simplification.

Proposition 3.4.7 *Let f_{hom} and g_{hom} be defined as in* (3.4.21) *and* (3.4.22). *We have:*

(i) the 1*-homogeneous extension of f_{hom} in $(\mathbb{R}_+)^M \times \mathbb{R}^n$ is convex. Moreover there exists $C > 0$ such that*

$$f_{hom}(z,\nu) \le C(|z| + 1), \quad \text{for all } (z,\nu) \in (\mathbb{R}_+)^M \times S^{n-1}. \tag{3.4.24}$$

(ii) g_{hom} is convex.

Proof of (i). By the 1-homogeneity of f_{hom} the proof of its convexity reduces to prove that, given $\nu, \nu^{(1)}, \nu^{(2)} \in S^{n-1}$ and $z, z^{(1)}, z^{(2)} \in (\mathbb{R}_+)^M$ such that $(z,\nu) = l_1(z^{(1)},\nu^{(1)}) + l_2(z^{(2)},\nu^{(2)})$, for $l_1, l_2 \in \mathbb{R}$, we have

$$f_{hom}(z,\nu) \le l_1 f_{hom}(z^{(1)},\nu^{(1)}) + l_2 f_{hom}(z^{(2)},\nu^{(2)}). \tag{3.4.25}$$

Without loss of generality, for simplicity of notation we prove (3.4.25) under the assumption that $l_1, l_2 > 0$, $\nu = e_n$, $\nu^{(1)}, \nu^{(2)} \in \{x \in \mathbb{R}^n : x_1 = \dots = x_{n-2} = 0\} =: \Pi$, $(\nu^{(1)}, \nu^{(2)}) \ge 0$ and the ordered base $\{\nu^{(1)}, \nu^{(2)}\}$ has the same orientation as $\{e_n, e_{n-1}\}$. Given $\eta > 0$, let $\delta > 0$, $T > \frac{1}{\eta}$, $u_1 \in \mathcal{B}_1(TQ_{\nu^{(1)}}; K)$, $u_2 \in \mathcal{B}_1(TQ_{\nu^{(2)}}; K)$ such that, for $i \in \{1,2\}$ it holds

$$\begin{aligned}
\frac{1}{T^{n-1}}F_1(u_i, TQ_{\nu^{(i)}}) &\le f_{hom}(z^{(i)},\nu^{(i)}) + \eta \\
\max_{l\in\{1,\dots,M\}} &\left| \frac{\#I_l(u_i, TQ_{\nu^{(i)}})}{T^{n-1}} - z_l^{(i)} \right| < \delta.
\end{aligned}$$

For $i \in \{1,2\}$ we set $r_i := r_{\nu^{(i)}}$, so that $Q_{\nu^{(i)}} = [-\frac{r_i}{2}, \frac{r_i}{2}]^2$. Let u_i be identified with its piecewise-constant interpolation and extended to $\mathbb{R}^{n-2} \times T[-\frac{r_i}{2}, \frac{r_i}{2}]^2$ by periodicity in the $\{e_1, \dots, e_{n-2}\}$

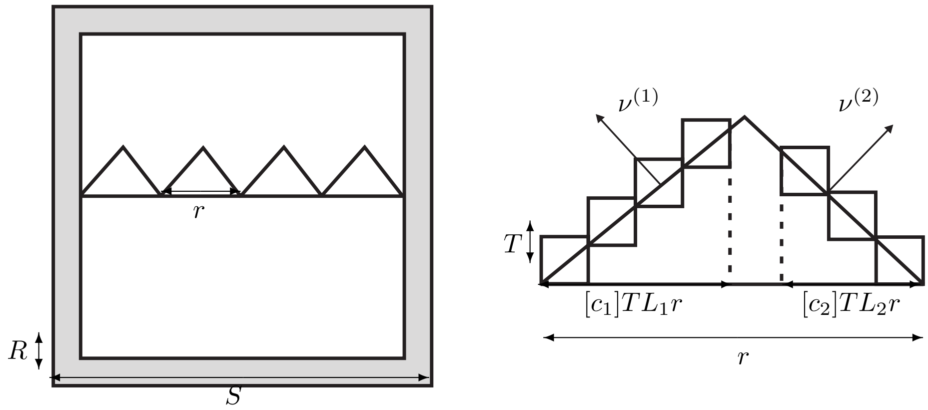

Figure 3.3: The *zig-zag* construction leading to u_S in the proof of Proposition 3.4.7. On the right, a zoom in on one of the triangles on the left.

directions. Moreover we set $L_i := \mathcal{H}^1(\{x \in \Pi : \; x = t(\nu^{(i)})^\perp, \; t \in \mathbb{R}\} \cap Q_{\nu^{(i)}})$ and $(\nu^{(i)})^\perp = (0, \dots, 0, \nu_n^{(i)}, -\nu_{n-1}^{(i)})$. We can then further extend u_i by periodicity in the direction $(\nu^{(i)})^\perp$ as

$$u_i(x + L_i(\nu^{(i)})^\perp) = u_i(x), \qquad x \in \mathbb{R}^{n-2} \times \bigcup_{j \in \mathbb{Z}} \Big(T[-\frac{r_i}{2}, \frac{r_i}{2}]^2 + jL_i(\nu^{(i)})^\perp \Big).$$

We now use a *zig-zag* construction to define a suitable test function $u_S \in \mathcal{B}_1(SQ; K)$, with $S \gg T$, for the minimum problem occurring in the definition of $f_{hom}(z, \nu)$ (see Figure 3.3). Let $V := \{x \in \mathbb{R}^n : \; 0 \le x_{n-1} \le 1\}$, $\Pi_\nu^\pm := \{x \in \mathbb{R}^n : \; \pm(x, \nu) \ge 0\}$, $\nu \in S^{n-1}$, and let $w : V \to \{m_1, m_2\}$ be defined as

$$w(x) = \begin{cases} m_1 & \text{if } x \in \Pi^+_{\nu^{(1)}} \cup (\Pi^+_{\nu^{(2)}} + l_1(\nu^{(1)})^\perp) \\ m_2 & \text{otherwise.} \end{cases}$$

Let $c_1, c_2 > 0$ be such that $c_1 \frac{TL_1}{l_1} = c_2 \frac{TL_2}{l_2} =: r$. Let $u : \mathbb{R}^n \to \mathbb{R}$ be the re_{n-1}-periodic function defined in rV as

$$u(x) = \begin{cases} u_1(x) & \text{if } 0 \le x_{n-1} \le [c_1]TL_1(\nu^{(1)}, e_n) \\ u_2(x - re_{n-1}) & \text{if } r - [c_2]TL_2(\nu^{(2)}, e_n) \le x_{n-1} \le r \\ w(\frac{x}{r}) & \text{otherwise in } rV, \end{cases}$$

extended by periodicity in the direction e_{n-1}. We define $u_S \in \mathcal{B}_1(SQ_\nu; K)$ as follows:

$$u_S(a) = \begin{cases} u(a) & \text{if } a \in (S - R)Q_\nu \cap \mathbb{Z}^n \\ u_\nu(a) & \text{otherwise in } SQ_\nu \cap \mathbb{Z}^n. \end{cases}$$

Now it is possible to estimate the energy of u_S:

$$\begin{aligned} \frac{1}{S^{n-1}} F_1(u_S, SQ_\nu) \quad \le \quad & \frac{1}{S^{n-1}} \sum_{i=1}^{2} \left[\frac{S}{r}\right] [c_i] \left[\frac{S}{Tr_i}\right]^{n-2} \big(F_1(u_i, TQ_{\nu^{(i)}}) + CT^{n-2}\big) \\ & + \frac{1}{S^{n-1}} CS^{n-2}, \end{aligned} \tag{3.4.26}$$

where the term of the type CT^{n-2} is the energetic contribution due to the interactions near each set of the type $\partial(TQ_{\nu^{(i)}}) \cap \Pi_{\nu^{(i)}}$, while the term of the type CS^{n-2} accounts for the interactions near $\partial(SQ_\nu) \cap \Pi_\nu$. By construction we have that for $l \in \{1, \dots, M\}$

$$\frac{\# I_l(u_S, SQ_\nu)}{S^{n-1}} = \frac{1}{S^{n-1}} \sum_{i=1}^{2} \left[\frac{S}{r}\right] [c_i] \left[\frac{S}{Tr_i}\right]^{n-2} \# I_l(u_i, TQ_{\nu^{(i)}}) + o(1). \tag{3.4.27}$$

Taking into account the definition of c_i and the fact that $r_i^{n-2} L_i = 1$, we get

$$\frac{1}{S^{n-1}} \left[\frac{S}{r}\right] [c_i] \left[\frac{S}{Tr_i}\right]^{n-2} \leq \frac{l_1 + C\eta}{T^{n-1}}, \tag{3.4.28}$$

provided that T and S are large enough. Hence, by (3.4.27) u_S is a good test function for the minimum problem defining $f_{hom}(z, \nu)$. By (3.4.26) we obtain

$$f_{hom}(z, \nu) \leq l_1 f_{hom}(z^{(1)}, \nu^{(1)}) + l_2 f_{hom}(z^{(2)}, \nu^{(2)}) + C\eta,$$

which implies (3.4.25) since η is arbitrary.

Proof of (ii). In order to prove (3.4.24), we observe that the energy $F_1(u, TQ_\nu)$ can be rewritten as

$$F_1(u, TQ_\nu) = \sum_{(a,b) \in D(u)} f(b-a, u(a), u(b)) + F_1^{sur}(u, TQ_\nu), \tag{3.4.29}$$

where

$$D(u) := \{(a, b) \in (\mathbb{Z}^n \cap TQ_\nu)^2 : 0 < |b - a| \leq R,\ u(a) \neq u(b),\ \{u(a), u(b)\} = \{m_1 m_2\}\}.$$

Here F_1^{sur} is the energy accounting only for the contribution due to the interactions of surfactant particles. Since f is bounded, the first term on the right hand side of (3.4.29) is proportional to $\# D(u)$. Hence, since each particle has only an equi-bounded number of interactions, $F_1^{sur}(u, TQ_\nu)$ is proportional to $\# I(u, TQ_\nu)$. Then the estimate (3.4.24) is proved by choosing, in the problem defining $f_{hom}(z, \nu)$, any test function u such that $\# D(u) \simeq CT^{n-1}$.

In order to prove (ii), by the 1-homogeneity of g_{hom}, it suffices to show that, given $\zeta^{(1)}, \zeta^{(2)} \in (\mathbb{R}_+)^M$ with $\|\zeta^{(1)}\|_1 = \|\zeta^{(2)}\|_1 = 1$ and $t \in (0, 1)$ we have

$$g_{hom}(t\zeta^{(1)} + (1-t)\zeta^{(2)}) \leq t g_{hom}(\zeta^{(1)}) + (1-t) g_{hom}(\zeta^{(2)}). \tag{3.4.30}$$

For any $\delta > 0$ and $\zeta \in (\mathbb{R}_+)^M$ with $\|\zeta\|_1 = 1$, we set

$$\begin{aligned} g(\delta, \zeta) \quad := \quad & \liminf_{T \to +\infty} \inf \left\{ \frac{F_1(u, TQ)}{\# I(u, TQ)} : u \in \mathcal{A}_1(TQ; K), \right. \\ & \left. \frac{F_1(u, TQ \setminus (T-R)Q)}{\# I(u, TQ)} < \delta,\ \max_{l \in \{1, \dots, M\}} \left| \frac{\# I_l(u, TQ)}{\# I(u, TQ)} - \zeta_l \right| < \delta \right\}. \end{aligned} \tag{3.4.31}$$

Given $\delta > 0$ and $i \in \{1, 2\}$, let $u_i \in \mathcal{A}(TQ; K)$ be such that

$$\begin{aligned} & \frac{F_1(u_i, TQ)}{\# I(u_i, TQ)} < g(\delta, \zeta^{(i)}) + \delta, \qquad \frac{F_1(u_i, TQ \setminus (T-R)Q)}{\# I(u_i, TQ)} < \delta \\ & \text{and} \max_{l \in \{1, \dots, M\}} \left| \frac{\# I_l(u_i, TQ)}{\# I(u_i, TQ)} - \zeta_l^{(i)} \right| < \delta. \end{aligned} \tag{3.4.32}$$

Having set $\tilde{\zeta} = t\zeta^{(1)} + (1-t)\zeta^{(2)}$, we build a suitable test function for the minimum problem defining $g(\delta, \tilde{\zeta})$. Let $S = kT$ with $1 \ll k \in \mathbb{N}$. Let $h \in \mathbb{N}$ be such that $h < k$ and

$$|t - \lambda(h)| < \delta, \tag{3.4.33}$$

where

$$\lambda(h) = \frac{h\#I(u_1, TQ)}{h\#I(u_1, TQ) + (k-h)\#I(u_2, TQ)}.$$

Without loss of generality we can assume that T is an even number. We extend u_i by reflection with respect to the coordinate axes. Namely, we set $u_{i,0} = u_i$ and we define $u_{i,j}$ by recursion for all $j \in \{1, \dots, n\}$: $u_{i,j}$ is the extension of $u_{i,j-1}$ on $\mathbb{R}^j \times \left[-\frac{T}{2}, \frac{T}{2}\right]^{n-j}$ satisfying

$$u_{i,j}(a) = u_{i,j}(a + (T - 2a_j)e_j), \quad \text{for all } a \in \mathbb{R}^{j-1} \times \left[-\frac{T}{2}, \frac{T}{2}\right]^{n-(j-1)} + mTe_j, \quad m \in \mathbb{Z}.$$

Thus the function $u_{i,n}$ extends u_i to the whole $\mathbb{Z}^n$. By the symmetry condition in (3.4.13), we have

$$F_1(u_{i,n}, TQ + m) = F_1(u_i, TQ) \quad \text{for all } m \in \mathbb{Z}^n. \tag{3.4.34}$$

Let $u : \mathbb{Z}^n \cap SQ \to K$ be defined as

$$u(a) = \begin{cases} u_{1,n}(a) & \text{if } -S/2 \le a_n \le -S/2 + h \\ u_{2,n}(a) & \text{otherwise.} \end{cases}$$

By (3.4.32) and (3.4.33) we get:

$$\begin{aligned}
\left| \frac{\#I_l(u, SQ)}{\#I(u, SQ)} - \tilde{\zeta}_l \right| &= \left| \frac{k^{n-1}h\#I_l(u_1, TQ) + k^{n-1}(k-h)\#I_l(u_2, TQ)}{k^{n-1}h\#I(u_1, TQ) + k^{n-1}(k-h)\#I(u_2, TQ)} - \tilde{\zeta}_l \right| \\
&\le \lambda(h) \left| \frac{\#I_l(u_1, TQ)}{\#I(u_1, TQ)} - \zeta_l^{(1)} \right| + (1 - \lambda(h)) \left| \frac{\#I_l(u_2, TQ)}{\#I(u_2, TQ)} - \zeta_l^{(2)} \right| \\
&\quad + (\|\zeta^{(1)}\|_1 \vee \|\zeta^{(2)}\|_1)\delta \\
&\le (1 + \|\zeta^{(1)}\|_1 \vee \|\zeta^{(2)}\|_1)\delta.
\end{aligned}$$

We now estimate the energy $\frac{F_1(u, SQ)}{\#I(u, SQ)}$:

$$\begin{aligned}
\frac{F_1(u, SQ)}{\#I(u, SQ)} &\le \frac{k^{n-1}hF_1(u_1, TQ) + k^{n-1}(k-h)F_1(u_2, TQ)}{\#I(u, SQ)} \\
&\quad + R_1 + R_2.
\end{aligned} \tag{3.4.35}$$

Here the first term on the right hand side derives from (3.4.34). Moreover R_1 is the energy due to the interactions which cross the set $SQ \cap \{x_n = -\frac{S}{2} + h\}$, where in the construction of u we switch from u_1 to u_2, while R_2 accounts for all the other interactions which cross the boundary of the cubes of type $TQ + m$ with $m \in \mathbb{Z}$. An easy computation shows that

$$R_1 \le C \frac{k^{n-1}T}{\#I(u, SQ)} \le C \frac{T}{k}.$$

In addition, by (3.4.32) we get

$$\begin{aligned}
R_2 &\le \frac{k^{n-1}hF_1(u_1, TQ \setminus (T-R)Q) + k^{n-1}(k-h)F_1(u_2, TQ \setminus (T-R)Q)}{\#I(u, SQ)} \\
&= \frac{k^{n-1}hF_1(u_1, TQ \setminus (T-R)Q) + k^{n-1}(k-h)F_1(u_2, TQ \setminus (T-R)Q)}{k^{n-1}h\#I(u_1, TQ) + k^{n-1}(k-h)\#I(u_2, TQ)} \\
&\le \lambda(h) \frac{F_1(u_1, TQ \setminus (T-R)Q)}{\#I(u_1, TQ)} + (1 - \lambda(h)) \frac{F_1(u_2, TQ \setminus (T-R)Q)}{\#I(u_2, TQ)} \\
&\le \delta.
\end{aligned}$$

The first term in the right hand side of (3.4.35) can be estimated analogously. Then, by (3.4.32), (3.4.33) and the upper bounds for R_1 and R_2, we get

$$\begin{aligned}\frac{F_1(u,SQ)}{\#I(u,SQ)} &= \lambda(h)\frac{F_1(u_1,TQ)}{\#I(u_1,TQ)}+(1-\lambda(h))\frac{F_1(u_2,TQ)}{\#I(u_2,TQ)}+R_1+R_2\\ &\leq tg(\delta,\zeta^{(1)})+(1-t)g(\delta,\zeta^{(2)})+C(\delta+\frac{T}{k}).\end{aligned}$$

Letting k tend to $+\infty$ we get

$$g(C\delta,\zeta)\leq tg(\delta,\zeta^{(1)})+(1-t)g(\delta,\zeta^{(2)})+C\delta.$$

As $\delta\to 0$, we obtain the desired inequality. ■

The proof of Theorem 3.4.4 will follow by Propositions 3.4.8 and 3.4.9 in which we prove the Γ-lim inf and the Γ-lim sup inequality, respectively.

Proposition 3.4.8 (Γ-lim inf inequality) *We have*

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u,\mu)\geq F(u,\mu) \tag{3.4.36}$$

Proof. Let $\varepsilon_k\to 0$. Up to subsequences, it suffices to consider $(u_k,\mu_k)\to(u,\mu)$ with respect to the $\tau_1\times\tau_2$-topology such that

$$\liminf_k F_{\varepsilon_k}(u_k,\mu_k)=\lim_k F_{\varepsilon_k}(u_k,\mu_k)<+\infty. \tag{3.4.37}$$

By Proposition 3.4.3 we know that $u\in BV(\Omega;\{m_1,m_2\})$. We now consider the family of measures $(\lambda_k)_k\subset\mathcal{M}_+(\Omega)$ defined as

$$\lambda_k:=\sum_{a\in\Omega_{\varepsilon_k}}\sum_{b\in\Omega_{\varepsilon_k}:\,|a-b|\leq R\varepsilon_k}\varepsilon_k^{n-1}f\left(\frac{b-a}{\varepsilon_k},u(a),u(b)\right)\delta_a. \tag{3.4.38}$$

Note that $\lambda_k(\Omega)=F_{\varepsilon_k}(u_k,\mu_k)$. Then, by (3.4.37) there exist $\lambda\in\mathcal{M}_+(\Omega)$ such that $\lambda_k\rightharpoonup\lambda$ upon passing to a subsequence (not relabeled). We now use a blow-up argument. By the Radon-Nikodym Theorem, μ can be decomposed into two mutually singular measures in $(\mathcal{M}_+(\Omega))^M$

$$\mu=z\mathcal{H}^{n-1}\lfloor S(u)+\mu^s=z\mathcal{H}^{n-1}\lfloor S(u)+\zeta\|\mu^s\|_1$$

and λ can be decomposed into three mutually singular non-negative measures

$$\lambda=\xi\mathcal{H}^{n-1}\lfloor S(u)+\eta\|\mu^s\|_1+\lambda^s,$$

with $\|\mu^s\|_1:=\sum_{l=1}^M\mu_l^s$. The proof is complete once we show that

$$\xi(x_0)\geq f_{hom}(z(x_0),\nu_u(x_0)),\quad\text{for }\mathcal{H}^{n-1}\text{- }a.e.\,x_0\in S(u) \tag{3.4.39}$$

and

$$\eta(x_0)\geq g_{hom}(\zeta(x_0)),\quad\text{for }|\mu^s|\text{- }a.e.\,x_0\in\Omega. \tag{3.4.40}$$

The proof of (3.4.39) and (3.4.40) will be performed in two steps.
Step 1. Proof of (3.4.39).
By the properties of BV functions (see [9]), for $\mathcal{H}^{n-1}$-$a.e.\,x_0\in S(u)$ we have

(i) $\displaystyle\lim_{\rho\to 0^+}\frac{1}{\rho^n}\int_{x_0+\rho Q^{\pm}_{\nu_u(x_0)}}|u(x)-u^{\pm}(x_0)|\,dx=0,$

(ii) $\lim_{\rho\to 0^+} \frac{1}{\rho^{n-1}}\mathcal{H}^{n-1}(S(u)\cap\{x_0+\rho Q_{\nu_u(x_0)}\})=1$

(iii) $\xi(x_0)=\lim_{\rho\to 0^+}\frac{1}{\rho^{n-1}}\lambda(\{x_0+\rho Q_{\nu_u(x_0)}\})$

(iv) $z(x_0)=\lim_{\rho\to 0^+}\frac{1}{\rho^{n-1}}\mu(\{x_0+\rho Q_{\nu_u(x_0)}\})$.

For such a $x_0\in S(u)$, let (ρ_m) be a positive infinitesimal such that

$$\lambda(\partial\{x_0+\rho_m Q_{\nu_u(x_0)}\})=0 \quad \text{and } |\mu|(\partial\{x_0+\rho_m Q_{\nu_u(x_0)}\})=0.$$

By (ii) and (iii) we get

$$\xi(x_0)=\lim_m \frac{1}{\rho_m^{n-1}}\lambda(\{x_0+\rho_m Q_{\nu_u(x_0)}\})\geq \lim_m\lim_k \frac{1}{\rho_m^{n-1}}F_{\varepsilon_k}(u_k,\{x_0+\rho_m Q_{\nu_u(x_0)}\}).$$

By (iv) we have

$$\lim_m\lim_k \frac{1}{\rho_m^{n-1}}\mu(u_k)(\{x_0+\rho_m Q_{\nu_u(x_0)}\})=z(x_0). \tag{3.4.41}$$

Note that, for every m and k we can find $\rho_{m,k}$ and $x_0^k\in\varepsilon_k\mathbb{Z}^n$ such that $\lim_k\rho_{m,k}=\rho_m$, $\lim_k x_0^k=x_0$ and

$$\varepsilon_k\mathbb{Z}^n\cap(\{x_0^k+\rho_{m,k}Q_{\nu_u(x_0)}\})=\varepsilon_k\mathbb{Z}^n\cap\{x_0+\rho_m Q_{\nu_u(x_0)}\},$$

which implies

$$F_{\varepsilon_k}(u_k,\{x_0+\rho_m Q_{\nu_u(x_0)}\})=F_{\varepsilon_k}(u_k,\{x_0^k+\rho_{m,k}Q_{\nu_u(x_0)}\}).$$

Then,

$$\xi(x_0)\geq \lim_m\lim_k \frac{1}{\rho_{m,k}^{n-1}}F_{\varepsilon_k}(u_k,\{x_0^k+\rho_{m,k}Q_{\nu_u(x_0)}\}). \tag{3.4.42}$$

Let

$$u_{m,k}(a)=u_k(x_0^k+\rho_{m,k}a),\qquad a\in\frac{\varepsilon_k}{\rho_{m,k}}\mathbb{Z}^n\cap Q_{\nu_u(x_0)}$$

and

$$u_0(x)=\begin{cases}u^+(x_0) & \text{if } (x,\nu_u(x_0))>0\\ u^-(x_0) & \text{if } (x,\nu_u(x_0))\leq 0.\end{cases}$$

Since $u_k\to u$ in $L^1(\Omega)$, by (i) we get

$$\lim_m\lim_k\int_{Q_{\nu_u(x_0)}}|u_{m,k}(x)-u_0(x)|\,dx=0. \tag{3.4.43}$$

Moreover, by (3.4.41) we have

$$\lim_m\lim_k\mu(u_{m,k})(Q_{\nu_u(x_0)})=z(x_0) \tag{3.4.44}$$

Note that (3.4.42) can be rewritten as

$$\xi(x_0)\geq\lim_m\lim_k F_{\frac{\varepsilon_k}{\rho_{m,k}}}(u_{m,k},Q_{\nu_u(x_0)}). \tag{3.4.45}$$

Let us show that the mass of $\mu(u_{m,k})$ does not concentrate near $\partial Q_{\nu_u(x_0)}$ for m and k large enough. Given $\delta > 0$, by (iv) there exists $\rho(\delta)$ such that for all $\rho < \rho(\delta)$ and for all $l \in \{1, \ldots, M\}$ we have

$$\rho^{n-1}(z_l(x_0) - \frac{\delta}{4}) < \mu_l(x_0 + \rho Q_{\nu_u(x_0)}) < \rho^{n-1}(z_l(x_0) + \frac{\delta}{4}).$$

Let $m(\delta)$ be such that $\rho_m < \rho(\delta)$ for all $m > m(\delta)$. Then, for every $t \in (0,1]$ we get:

$$(t\rho_m)^{n-1}(z_l(x_0) - \frac{\delta}{4}) < \mu_l(x_0 + t\rho_m Q_{\nu_u(x_0)}) < (t\rho_m)^{n-1}(z_l(x_0) + \frac{\delta}{4}). \tag{3.4.46}$$

Thus, for all t such that $|\mu|(x_0 + \partial(t\rho_m Q_{\nu_u(x_0)})) = 0$, by (3.4.46) we get

$$\begin{aligned} \mu_l(x_0 + (\rho_m Q_{\nu_u(x_0)} \setminus t\rho_m Q_{\nu_u(x_0)})) &= \mu_l(x_0 + \rho_m Q_{\nu_u(x_0)}) - \mu_l(x_0 + t\rho_m Q_{\nu_u(x_0)}) \\ &< \rho_m^{n-1}\left(z_l(x_0)(1 - t^{n-1}) + \frac{\delta}{2}\right) \end{aligned} \tag{3.4.47}$$

Let $t(\delta) < 1$ be such that for every $l \in \{1, \ldots, M\}$ we have $z_l(0)(1 - (t(\delta))^{n-1}) < \frac{\delta}{2}$. Then, by (3.4.47), for every $t \in (t(\delta), 1))$ we get

$$\mu_l(x_0 + (\rho_m Q_{\nu_u(x_0)} \setminus t\rho_m Q_{\nu_u(x_0)})) < \delta\rho_m^{n-1} \tag{3.4.48}$$

Set $\bar{t}(\delta) := \frac{1+t(\delta)}{2}$ and let $t_m \in \big(t(\delta), \bar{t}(\delta)\big)$ be such that $|\mu|(x_0 + \partial(t_m\rho_m Q_{\nu_u(x_0)})) = 0$. Thus

$$\lim_k \mu(u_k)(x_0 + (\rho_m Q_{\nu_u(x_0)} \setminus t_m\rho_m Q_{\nu_u(x_0)})) = \mu(x_0 + (\rho_m Q_{\nu_u(x_0)} \setminus t_m\rho_m Q_{\nu_u(x_0)}))$$

and, by (3.4.48), we may conclude that for any $m > m(\delta)$ there exists $k(m)$ such that for every $l \in \{1, \ldots, M\}$ and $k > k(m)$ the following condition holds:

$$\mu_l(u_k)(x_0 + (\rho_m Q_{\nu_u(x_0)} \setminus \bar{t}(\delta)\rho_m Q_{\nu_u(x_0)})) < \delta\rho_m^{n-1}. \tag{3.4.49}$$

Hence, by (3.4.49) we infere that for m and k large enough

$$\mu_l(u_{m,k})(Q_{\nu_u(x_0)} \setminus \bar{t}(\delta)Q_{\nu_u(x_0)}) < \delta. \tag{3.4.50}$$

Taking into account (3.4.43), (3.4.44), (3.4.45) and (3.4.50), by a standard diagonalization procedure we can then find a sequence of positive numbers $s_j \to 0$ and a sequence $w_j \in \mathcal{A}_{s_j}(Q_{\nu_u(x_0)}; K)$ such that $w_j \to u_0$ in $L^1(Q_{\nu_u(x_0)})$ and

$$\lim_j \mu(w_j)(Q_{\nu_u(x_0)}) = z(x_0), \tag{3.4.51}$$

$$\mu_l(w_j)(Q_{\nu_u(x_0)} \setminus \bar{t}(\delta)Q_{\nu_u(x_0)}) < \delta \text{ for all } l \in \{1, \ldots, M\}, \tag{3.4.52}$$

$$\xi(x_0) \geq \lim_j F_{s_j}(w_j, Q_{\nu_u(x_0)}).$$

Then, by Lemma 3.4.1, we can find a sequence $(v_j) \subset B_{s_j}(Q_{\nu_u(x_0)}; K)$ such that (3.4.64) holds and

$$\xi(x_0) \geq \lim_j F_{s_j}(v_j, Q_{\nu_u(x_0)}). \tag{3.4.53}$$

Moreover, by (3.4.51), (3.4.52) and (3.4.64) we have that for j large enough

$$|\mu_l(v_j)(Q_{\nu_u(x_0)}) - z_l(x_0)| < \delta \text{ for all } l \in \{1, \ldots, M\}. \tag{3.4.54}$$

Set $T_j := \left[\frac{1}{s_j}\right]$ and let $\hat{v}_j \in B_1(T_j Q_{\nu_u(x_0)}; K)$ defined by

$$\hat{v}_j(a) := v(s_j a) \quad a \in \mathbb{Z}^n \cap T_j Q_{\nu_u(x_0)}.$$

Then (3.4.54) implies that for j large enough

$$\left| \frac{\# I_l(\hat{v}_j)}{T_j^{n-1}} - z_l \right| < \delta \qquad \text{for all } \in \{1, \dots, M\}$$

and (3.4.53) reads

$$\xi(x_0) \geq \lim_j \frac{1}{T_j{}^{n-1}} F_1(\hat{v}_j, T_j Q_{\nu_u(x_0)}).$$

Hence (3.4.39) immediately follows by the definition of f_{hom} and by Proposition 3.4.6.
Step 2. Proof of (3.4.40).
For $|\mu^s|$-$a.e$ $x_0 \in \Omega$ we have

(v) $\eta(x_0) = \lim_{\rho \to 0} \dfrac{\lambda(x_0 + \rho Q)}{\|\mu\|_1(x_0 + \rho Q)}$;

(vi) $\zeta(x_0) = \lim_{\rho \to 0} \dfrac{\mu(x_0 + \rho Q)}{\|\mu\|_1(x_0 + \rho Q)}$

Fix such a $x_0 \in \Omega$ and let (ρ_m) be a sequence of positive numbers converging to zero such that

$$\lambda(\partial\{x_0 + \rho_m Q_{\nu_u(x_0)}\}) = 0, \quad |\mu|(\partial\{x_0 + \rho_m Q_{\nu_u(x_0)}\}) = 0$$

By (v) and (vi) we get

$$\eta(x_0) = \lim_m \lim_k \frac{\lambda_k(x_0 + \rho_m Q)}{\|\mu(u_k)\|_1(x_0 + \rho_m Q)}, \tag{3.4.55}$$

$$\zeta(x_0) = \lim_m \lim_k \frac{\mu(u_k)(x_0 + \rho_m Q)}{\|\mu(u_k)\|_1(x_0 + \rho_m Q)}. \tag{3.4.56}$$

We now show that for a suitable sequence $k_m \in \mathbb{N}$ the mass of λ_{k_m} does not concentrate near $\partial(x_0 + \rho_m Q)$.

By the inner regularity of λ, given $\delta > 0$, for any $\rho > 0$ with $\lambda(\partial(x_0 + \rho Q)) = 0$ there exists $t(\rho)$ such that for all $t \in [t(\rho), 1]$ we have

$$0 < \lambda(x_0 + \rho Q) - \lambda(x_0 + t\rho Q) < \delta \|\mu\|_1(x_0 + \rho Q).$$

Let $t_m \in [t(\rho_m), 1]$ be such that $\lambda(\partial(x_0 + t_m \rho_m Q)) = 0$. Then

$$\lambda(x_0 + (\rho_m Q \setminus t_m \rho_m Q)) = \lambda(x_0 + \rho_m Q) - \lambda(x_0 + t_m \rho_m Q) < \delta \|\mu\|_1(x_0 + \rho_m Q)$$

In particular, since $\lim_k \lambda_k(x_0 + (\rho_m Q \setminus t_m \rho_m Q)) = \lambda(x_0 + (\rho_m Q \setminus t_m \rho_m Q))$ and $\lim_k \|\mu(u_k)\|_1(x_0 + \rho_m Q) = \|\mu\|_1(x_0 + \rho_m Q)$, we have that for k large enough

$$\lambda_k(x_0 + (\rho_m Q \setminus t_m \rho_m Q)) < \delta \|\mu(u_k)\|_1(x_0 + \rho_m Q).$$

Hence, by the previous inequality and by (3.4.55) and (3.4.56), we can find a sequence k_m such that $\varepsilon_{k_m} \ll \rho_m$ and

$$\eta(x_0) = \lim_m \frac{\lambda_{k_m}(x_0 + \rho_m Q)}{\|\mu(u_{k_m})\|_1(x_0 + \rho_m Q)} \geq \lim_m \frac{F_{\varepsilon_{k_m}}(u_{k_m}, x_0 + \rho_m Q)}{\|\mu(u_{k_m})\|_1(x_0 + \rho_m Q)}, \tag{3.4.57}$$

$$\zeta(x_0) = \lim_m \frac{\mu(u_{k_m})(x_0 + \rho_m Q)}{\|\mu(u_{k_m})\|_1(x_0 + \rho_m Q)}, \tag{3.4.58}$$

$$\lambda_{k_m}(x_0 + (\rho_m Q \setminus t_m \rho_m Q)) < \delta \|\mu(u_{k_m})\|_1 (x_0 + \rho_m Q),$$
$$R\varepsilon_{k_m} < \rho_m(1 - t_m).$$

Note that the two last inequalities imply that

$$F_{\varepsilon_{k_m}}(u_{k_m}, x_0 + (\rho_m Q \setminus (\rho_m - R\varepsilon_{k_m})Q)) < \delta \|\mu(u_{k_m})\|_1 (x_0 + \rho_m Q) \tag{3.4.59}$$

Observe that, for every m we can find $\tilde{\rho}_m$ with $\lim_m \frac{\tilde{\rho}_m}{\rho_m} = 1$ and $x_0^m \in \varepsilon_{k_m}\mathbb{Z}^n$ with $\lim_m x_0^m = x_0$, such that

$$\varepsilon_{k_m}\mathbb{Z}^n \cap (x_0^m + \tilde{\rho}_m Q) = \varepsilon_{k_m}\mathbb{Z}^n \cap (x_0 + \rho_m Q)$$

which implies

$$F_{\varepsilon_{k_m}}(u_{k_m}, x_0 + \rho_m Q) = F_{\varepsilon_{k_m}}(u_{k_m}, x_0^m + \tilde{\rho}_m Q).$$
$$\mu(u_{k_m})(x_0 + \rho_m Q) = \mu(u_{k_m})(x_0^m + \tilde{\rho}_m Q).$$

Set $T_m := \frac{\tilde{\rho}_m}{\varepsilon_{k_m}}$ and let $\hat{u}_m \in \mathcal{A}_1(T_m Q; K)$ be defined by

$$\hat{u}_m(a) = u_{k_m}(x_0^m + \varepsilon_{k_m} a) \quad a \in \mathbb{Z}^n \cap T_m Q$$

Note that, by definition, $\|\mu(\hat{u}_m)\|_1(A) = \#I(\hat{u}_m, A)$ for any $A \subset \mathbb{R}^n$. Then (3.4.57) and (3.4.58) read

$$\eta(x_0) \geq \lim_m \frac{F_1(\hat{u}_m, T_m Q)}{\#I(\hat{u}_m, T_m Q)}. \tag{3.4.60}$$

$$\zeta(x_0) = \lim_m \frac{\mu(\tilde{u}_m)(Q)}{\#I(\hat{u}_m, T_m Q)}. \tag{3.4.61}$$

Moreover, by (3.4.59), we get

$$F_1(\hat{u}_m, T_m Q \setminus (T_m - R)Q) < \delta \#I(\hat{u}_m, T_m Q). \tag{3.4.62}$$

Hence, (3.4.40) immediately follows by the definition of g_{hom}, taking into account (3.4.60), (3.4.61) and (3.4.62).

■

Set, for $\nu \in S^{n-1}$,

$$u_\nu = \begin{cases} m_1 & \text{if } (x, \nu) > 0 \\ m_2 & \text{if } (x, \nu) \leq 0. \end{cases} \tag{3.4.63}$$

Moreover we recall that we have set $Q_\nu = (-r_\nu, r_\nu)^n$.

Lemma 3.4.1 *Let $s_j \to 0^+$, $\nu \in S^{n-1}$ and let $w_j \in \mathcal{A}_{s_j}(Q_\nu; K)$ be such that $w_j \to u_\nu$ in $L^1(Q_\nu)$. Then there exist $v_j \in B_{s_j}(Q_\nu; K)$ such that*

$$v_j \equiv u_\nu \text{ on } Q_\nu \setminus Q_j, \tag{3.4.64}$$

where $Q_j := (-r_j, r_j)^n \subset Q_\nu$, for some $r_j > 0$ such that $\lim_j r_j = r_\nu$, and

$$\liminf_j F_{s_j}(w_j, Q_\nu) \geq \liminf_j F_{s_j}(v_j, Q_\nu). \tag{3.4.65}$$

Proof. Set

$$\delta_j := \int_{Q_\nu} |w_j - u_\nu| \, dx$$

Let $k_j \in \mathbb{N}$ be such that

$$\frac{\delta_j}{s_j} << k_j << \frac{1}{s_j} \tag{3.4.66}$$

and set, for $i \in \{0, \ldots, k_j\}$,

$$r_j^i := \left[\frac{r_\nu}{s_j}\right] + (i - k_j)M$$
$$Q_j^i := (-r_j^i s_j, r_j^i s_j).$$

Then we get

$$\delta_j \geq \int_{Q_\nu \setminus Q_j^0} |w_j - u_\nu|\, dx \geq \sum_{i=0}^{k_j - 1} \int_{Q_j^{i+1} \setminus Q_j^i} |w_j - u_\nu|\, dx.$$

Hence, there exists $i_j \in \{0, \ldots, k_j - 1\}$ such that, set $S_j := Q_j^{i_j+1} \setminus Q_j^{i_j}$, we have

$$\delta_j \geq k_j \int_{S_j} |w_j - u_\nu|\, dx \geq C k_j s_j^n \#\{a \in s_j\mathbb{Z}^n \cap S_j : \; w_j(a) \neq u_\nu(a)\}.$$

By (3.4.66), there follows that

$$s_j^{n-1} \#\{a \in \lambda_j\mathbb{Z}^n \cap S_j : \; w_j(a) \neq u_\nu(a)\} \to 0. \tag{3.4.67}$$

Let, then, $v_j \in B_{s_j}(Q_\nu; K)$ defined by

$$v_j(a) := \begin{cases} w_j(a) & \text{if } a \in s_j\mathbb{Z}^n \cap Q_j^{i_j} \\ u_\nu(a) & \text{otherwise.} \end{cases}$$

Thus, by (3.4.67), we get

$$\begin{aligned} F_{s_j}(v_j, Q_\nu) \quad &\leq \quad F_{s_j}(w_j, Q_j^{i_j}) + F_{s_j}(u_\nu, Q_\nu \setminus Q_j^{i_j}) \\ &\quad + C s_j^{n-1} \#\{a \in s_j\mathbb{Z}^n \cap S_j : \; w_j(a) \neq u_\nu(a)\} \leq F_j(w_j, Q) + o(1), \end{aligned}$$

from which we get the conclusion. ■

Proposition 3.4.9 (Γ-lim sup inequality) *We have*

$$\Gamma\text{-}\limsup_{\varepsilon \to 0} F_\varepsilon(u, \mu) \leq F(u, \mu). \tag{3.4.68}$$

Proof. We will use the notation $F'' := \Gamma\text{-}\limsup_{\varepsilon \to 0} F_\varepsilon$.We split the proof in several steps.

Step 1. Claim: (3.4.68) holds for every $(u, \mu) \in BV(\Omega; \{m_1, m_2\}) \times (\mathcal{M}_+(\Omega))^M$ such that $S(u)$ is a polyhedral set and μ is of the form $\mu = \varphi \mathcal{H}^{n-1} \lfloor S(u) + \sum_{j=1}^N w_j \delta_{x_j}$, where $\varphi : \Omega \to \mathbb{R}^M$ is a piecewise-constant function, $N \in \mathbb{N}$ and, for all $j \in \{1, 2, \ldots, N\}$, $w_j \in (\mathbb{R}_+)^M$ and $x_j \in \Omega$.

Since the construction we provide is local, without loss of generality, we prove the claim in the particular case $u = u_\nu$, and $\mu = z\mathcal{H}^{n-1}_{\lfloor S(u)} + w\delta_0$ with $\nu \in S^{n-1}$, $z, w \in (\mathbb{R}_+)^M$. Here, without loss of generality, we also suppose $0 \in \Omega$. Note that

$$F(u, \mu) = f_{hom}(z, \nu)\mathcal{H}^{n-1}(S(u)) + g_{hom}(\zeta)\|w\|_1,$$

where $\zeta = \frac{w}{\|w\|_1}$. By the lower semicontinuity of F'', in order to show (3.4.68), it suffices to prove that there exists $(\mu_j)_j \subset (\mathcal{M}_+(\Omega))^M$ weakly converging to μ as $j \to +\infty$ such that, for every $j \in \mathbb{N}$, there exists $u_\varepsilon \in \mathcal{A}_\varepsilon(\Omega; K)$ such that $(u_\varepsilon, \mu(u_\varepsilon)) \to (u, \mu_j)$ with respect to the $\tau_1 \times \tau_2$-convergence and

$$\limsup_\varepsilon F_\varepsilon(u_\varepsilon) \leq F(u,\mu) + \frac{C}{j}. \tag{3.4.69}$$

For simplicity of notation we provide the construction of such u_ε in the case $\nu = e_n$, the same argument applying to the general case. Such a u_ε will be obtained by scaling the periodic extension of an optimal function for the problem defining f_{hom} in a neighborhood of $S(u)$ and a proper extension of an optimal function for the problem defining g_{hom} in a suitable neighborhood of 0.

Let $0 < \delta_j < \frac{1}{j}$, $T_j > 0$, $u_j \in \mathcal{B}_1(T_jQ; K)$ and $v_j \in \mathcal{A}_1(T_jQ; K)$ be such that

$$\max_{l\in\{1,\dots,M\}} \left| \frac{\#I_l(u_j, T_jQ)}{T_j^{n-1}} - z_l \right| \leq \delta_j, \quad \max_{l\in\{1,\dots,M\}} \left| \frac{\#I_l(v_j, T_jQ)}{\#I(v_j, T_jQ)} - \zeta_l \right| \leq \delta_j$$

$$\frac{1}{T_j^{n-1}} F_1(u_j, T_jQ) \leq f_{hom}(z, e_n) + \frac{1}{j},$$

$$\frac{F_1(v_j, T_jQ)}{\#I(v_j, T_jQ)} \leq g_{hom}(\zeta) + \frac{1}{j},$$

$$\frac{F_1(v_j, T_jQ \setminus (T_j - R)Q)}{\#I(v_j, T_jQ)} \leq \delta_j.$$

Without loss of generality we choose T_j to be an even number. With a slight abuse of notation we consider u_j to be extended by periodicity to $\mathbb{Z}^n$. Moreover we extend v_j by reflection with respect to the coordinate axes. More precisely we set $v_{j,0} = v_j$ and, for all $k \in \{1, \dots, n\}$, we define $v_{j,k}$ recursively as follows: $v_{j,k}$ is the extension of $v_{j,k-1}$ on $\mathbb{R}^k \times \left[-\frac{T_j}{2}, \frac{T_j}{2}\right]^{n-k}$ satisfying the following property

$$v_{j,k}(a) = v_{j,k}(a + (T_j - 2a_k)e_k), \quad \text{for all } a \in \mathbb{R}^{k-1} \times \left[-\frac{T_j}{2}, \frac{T_j}{2}\right]^{n-(k-1)} + hT_je_k, \quad h \in \mathbb{Z}.$$

We have then obtained that the function $v_{j,n}$ extends v_j on all $\mathbb{Z}^n$. Let us observe that, by the symmetry hypothesis in (3.4.13), we have that

$$F_1(v_{j,n}, T_jQ + h) = F_1(v_j, T_jQ) \quad \text{for all } h \in \mathbb{Z}^n. \tag{3.4.70}$$

Let $\tilde{u}_\varepsilon : \varepsilon\mathbb{Z}^n \to K$ be defined as

$$\tilde{u}_\varepsilon(a) = \begin{cases} m_1 & \text{if } a_n \geq \varepsilon\frac{T}{2} \\ u_j\left(\frac{a}{\varepsilon}\right) & \text{if } |a_n| \leq \varepsilon\frac{T}{2} \\ m_2 & \text{if } a_n \leq -\varepsilon\frac{T}{2} \end{cases}$$

and set

$$k_\varepsilon = \left[\left(\frac{\|w\|_1}{\varepsilon^{n-1}\#I(v_j, T_jQ)} \right)^{\frac{1}{n}} \right]. \tag{3.4.71}$$

Note that $\tilde{u}_\varepsilon \to u_\nu$ in $L^1(\Omega)$ and that $\varepsilon k_\varepsilon \to 0$. We now define $u_\varepsilon : \varepsilon\mathbb{Z}^n \to K$ as

$$u_\varepsilon(a) = \begin{cases} v_{j,n}\left(\frac{a}{\varepsilon}\right) & \text{if } a \in \varepsilon k_\varepsilon T_jQ \\ \tilde{u}_\varepsilon(a) & \text{otherwise.} \end{cases}$$

Then u_ε still converges to u_ν in $L^1(\Omega)$. Moreover, by construction, we have that $\mu(u_\varepsilon) \to \mu^j \in (\mathcal{M}_+(\Omega))^M$ where, for $l \in \{1, 2, \dots, M\}$,

$$\mu_l^j = \frac{\#I_l(u_j, T_jQ)}{T_j^{n-1}} \mathcal{H}^{n-1} \lfloor S(u) + \frac{\#I_l(v_j, T_jQ)}{\#I(v_j, T_jQ)} \|w\|_1 \delta_0.$$

We can now estimate the energy of u_ε. Taking into account the invariance of the energy under integer translations and (3.4.70), we get

$$\begin{aligned} F_\varepsilon(u_\varepsilon) \leq \quad & \left[\frac{\mathcal{H}^{n-1}(S(u))}{(T_j\varepsilon)^{n-1}}\right]\varepsilon^{n-1}F_1(u_j,T_jQ) \\ + \quad & k_\varepsilon^n\varepsilon^{n-1}F_1(v_j,T_jQ)+Ck_\varepsilon^n\varepsilon^{n-1}F_1(v_j,T_jQ\setminus(T_j-R)Q)+o(1), \end{aligned}$$

where the third term in the right-hand-side is obtained by estimating the energy due to the interactions that cross the boundary of each cube of size εT_j contained in $\varepsilon K_\varepsilon T_j Q$. By (3.4.70) we eventually have

$$\begin{aligned} F_\varepsilon(u_\varepsilon) \leq \quad & \left[\frac{\mathcal{H}^{n-1}(S(u))}{(T_j\varepsilon)^{n-1}}\right]\varepsilon^{n-1}T_j^{n-1}\left(f_{hom}(z,e_n)+\frac{1}{j}\right) \\ + \quad & k_\varepsilon^n\varepsilon^{n-1}\#I(v_j,T_jQ)\left(g_{hom}(\zeta)+C\delta_j+\frac{1}{j}\right)+o(1) \end{aligned}$$

The conclusion follows passing to the limsup as ε tends to 0, taking into account (3.4.71).
Step 2. Claim: (3.4.68) holds for every (u,μ) as in Step 1 but with $\varphi\in C(\Omega;\mathbb{R}^M)$.

Let φ_k be a sequence of piecewise constant functions such that $\varphi_k\to\varphi$ with respect to the $L^1(S(u);\mathcal{H}^{n-1})$ and let $\mu_k=\varphi_k\mathcal{H}^{n-1}\lfloor S(u)+\sum_{j=1}^N w_j\delta_{x_j}$. Then $\mu_k\rightharpoonup\mu$, and by the convexity and growth properties of $f_{hom}(\cdot,\nu)$ stated in Proposition 3.4.7, $F(u,\mu_k)\to F(u,\mu)$. Eventually, by the lower semicontinuity of $F''(u,\mu)$ and by Step 1, we have

$$F''(u,\mu)\leq\liminf_k F''(u,\mu_k)\leq\liminf_k F(u,\mu_k)=F(u,\mu).$$

Step 3. Claim: (3.4.68) holds for every $(u,\mu)\in BV(\Omega;\{m_1,m_2\})\times(\mathcal{M}_+(\Omega))^M$ such that $\mu=\varphi\mathcal{H}^{n-1}\lfloor S(u)+\sum_{j=1}^N w_j\delta_{x_j}$ with $\varphi\in C(\Omega;\mathbb{R}^M)$.

Let $u_k\in BV(\Omega;\{m_1,m_2\})$ be such that $u_k\to u$ in $L^1(\Omega;\{m_1,m_2\})$, $S(u_k)$ is a polyhedral set and $\mathcal{H}^{n-1}(S(u_k))\to\mathcal{H}^{n-1}(S(u))$. Let $\mu_k=\varphi\mathcal{H}^{n-1}\lfloor S(u_k)+\sum_{j=1}^N w_j\delta_{x_j}$. Then we have that $\mu_k\rightharpoonup\mu$ and $|\varphi|\mathcal{H}^{n-1}\lfloor S(u_k)(\Omega)\to|\varphi|\mathcal{H}^{n-1}\lfloor S(u)(\Omega)$. Then, by the convexity of f_{hom} stated in Proposition 3.4.7 and by Reshetnyak's theorem we have that $F(u_k,\mu_k)\to F(u,\mu)$. Hence we conclude as in Step 2.

Step 4. Claim: (3.4.68) holds for every $(u,\mu)\in BV(\Omega;\{m_1,m_2\})\times(\mathcal{M}_+(\Omega))^M$.

Let $\varphi_k\in C(\Omega;\mathbb{R}^M)$ be such that $\varphi_k\to\varphi$ in $L^1(S(u);\mathcal{H}^{n-1}))$ and let $\mu_k^s=\sum_{j=1}^N w_j\delta_{x_j}$ be such that $\mu_k^s\rightharpoonup\mu^s$ and $|\mu_k^s|(\Omega)=|\mu^s|(\Omega)$. Let then $\mu_k=\varphi_k\mathcal{H}^{n-1}\lfloor S(u)+\mu_k^s$. We have that $\mu_k\rightharpoonup\mu$ and, by the convexity and growth properties of $f_{hom}(\cdot,\nu)$ and the convexity of g_{hom} stated in Proposition 3.4.7, applying Reshetnyak's theorem we get that $F(u_k,\mu_k)\to F(u,\mu)$. Hence we conclude as in Step 2. ■

3.4.2 Prescribed volume-fractions

In this Section we study a generalization of the constrained minimum problems introduce at the end of Section 3.3.2 in the case of the BEG model.
In what follows we set $\mathcal{Q}=[0,1)^n$, $\varepsilon_k=\frac{1}{k}$, $k\in\mathbb{N}$ (for simplicity of notation we will drop the k and write ε instead of ε_k),

$$\mathcal{A}_\varepsilon^\#(\mathcal{Q};K):=\{u:\varepsilon\mathbb{Z}^n\to K:\ u \text{ is } \mathcal{Q}\text{-periodic}\}$$

and define $F_\varepsilon^\#:\mathcal{A}_\varepsilon^\#(\mathcal{Q};K)\to\mathbb{R}$ as

$$F_\varepsilon^\#(u)=\sum_{|\xi|\leq R}\sum_{a\in\mathcal{Q}_\varepsilon}\varepsilon^{n-1}f(\xi,u(a),u(a+\varepsilon\xi)).$$

Given $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_M) \in \mathbb{R}^M$ and $\beta \in \mathbb{R}$, let $\alpha_\varepsilon = (\alpha_{1,\varepsilon}, \alpha_{2,\varepsilon}, \dots, \alpha_{M,\varepsilon}) \to \alpha$ and $\beta_\varepsilon \to \beta$ as $\varepsilon \to 0$. We define the set of admissible functions $\mathcal{A}_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon}(\mathcal{Q};K)$ as

$$\mathcal{A}_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon}(\mathcal{Q};K) := \big\{u \in \mathcal{A}_\varepsilon^{\#}(\mathcal{Q};K): \ \varepsilon^{n-1}\#I_l(u,\mathcal{Q}) = \alpha_{l,\varepsilon} \text{ for all } l \in \{1,\dots,M\}, \\ \varepsilon^n \# I_{m_1}(u,\mathcal{Q}) = \beta_\varepsilon\big\},$$

where we have set $I_{m_1}(u,\mathcal{Q}) := \{a \in \mathcal{Q}_\varepsilon : \ u(a) = m_1\}$, and consider the family of minimum problems

$$m_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon} := \min\{F_\varepsilon^{\#}(u) : \ u \in \mathcal{A}_\varepsilon^{\alpha,\beta}(\mathcal{Q};K)\}.$$

Note that if $u \in \mathcal{A}_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon}(\mathcal{Q};K)$ then $\mu(u_\varepsilon)(\mathcal{Q}) = \alpha_\varepsilon$. We are interested in studying the limit, as $\varepsilon \to 0$, of $m_\varepsilon^{\alpha,\beta}$. To this end, we introduce the family of functionals $F_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon} : L^1_{loc}(\mathbb{R}^n;K) \times (\mathcal{M}_+^{\#}(\mathbb{R}^n))^M \to [0,+\infty]$ defined as

$$F_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon}(u,\mu) = \begin{cases} F_\varepsilon^{\#}(u) & \text{if } u \in \mathcal{A}_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon}(\mathcal{Q};K), \ \mu = \mu(u), \\ +\infty & \text{otherwise,} \end{cases}$$

where $u \in \mathcal{A}_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon}(\mathcal{Q};K)$ is identified with its piecewise-constant interpolation on the cells of the lattice $\varepsilon\mathbb{Z}^n$, $\mu(u)$ is the surfactant measure defined in (3.4.17) and $\mathcal{M}_+^{\#}(\mathbb{R}^n)$ is the space of $\mathcal{Q}$-periodic non negative Radon measures. We endow the space $L^1_{loc}(\mathbb{R}^n;K)\times(\mathcal{M}_+^{\#}(\mathbb{R}^n))^M$ with the convergence $\tau_1 \times \tau_2$ where τ_1 denotes the strong convergence in $L^1_{loc}(\mathbb{R}^n)$ and τ_2 denotes the weak*-convergence in $(\mathcal{M}_+(\mathbb{R}^n))^M$. Then the following Theorem holds.

Theorem 3.4.10 *The family $(F_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon})$ Γ-converges with respect to the $\tau_1 \times \tau_2$ convergence to the functional $F^{\alpha,\beta} : L^1_{loc}(\mathbb{R}^n) \times (\mathcal{M}_+^{\#}(\mathbb{R}^n))^M \to [0,+\infty]$ defined by*

$$F^{\alpha,\beta}(u,\mu) = \begin{cases} \overline{F}(u,\mu) & \text{if } \ u \in BV^{\#}(\mathbb{R}^n;\{m_1,m_2\}),\ |\{x \in \mathcal{Q} : \ u(x) = m_1\}| = \beta, \\ & \qquad\qquad \text{and } \mu(\bar{\mathcal{Q}}) = \alpha, \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\overline{F}(u,\mu) = \int_{S(u)\cap\bar{\mathcal{Q}}} f_{hom}\left(\frac{d\mu}{d\mathcal{H}^{n-1}\lfloor S(u)}, \nu(u)\right) d\mathcal{H}^{n-1} + \int_{\bar{\mathcal{Q}}} g_{hom}(\mu^s),$$

$\mu^s = \mu - \frac{d\mu}{d\mathcal{H}^{n-1}\lfloor S(u)}\mathcal{H}^{n-1}\lfloor S(u)$, and the densities f_{hom} and g_{hom} are defined in (3.4.21) and in (3.4.22).

Proof. It is easy to show that if $u_\varepsilon \in \mathcal{A}_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon}(\mathcal{Q};K)$ and $(u_\varepsilon, \mu(u_\varepsilon)) \to (u,\mu)$ with respect to the $\tau_1 \times \tau_2$ convergence, then $(u,\mu) \in BV^{\#}(\mathbb{R}^n;\{m_1,m_2\}) \times (\mathcal{M}_+^{\#}(\mathbb{R}^n))^M$, $|\{x \in \mathcal{Q} : \ u(x) = m_1\}| = \beta$ and $\mu(\bar{\mathcal{Q}}) = \alpha$. The Γ-lim inf inequality follows by Theorem 3.4.4.
The proof of the opposite inequality can be obtained by following the lines of the proof of the Γ-lim sup inequality of Theorem 3.4.4, with some extra care to show that the recovery sequence u_ε for $(u,\mu) \in BV^{\#}(\mathbb{R}^n;\{m_1,m_2\}) \times (\mathcal{M}_+^{\#}(\mathbb{R}^n))^M$ such that $\mu(\bar{\mathcal{Q}}) = \alpha$ and $|\{x \in Q : \ u(x) = m_1\}| = \beta$ can be slightly modified so that $u_\varepsilon \in \mathcal{A}_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon}(\mathcal{Q};K)$. ∎

As a consequence of the previous Theorem, by the standard properties of Γ-convergence (see e.g. [19] and [39]), we derive the following result about the convergence of the family of minimum problems defined above.

Corollary 3.4.11 *We have:*

$$\lim_\varepsilon m_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon} = \min\{\overline{F}(u,\mu),\ |\{x \in \mathcal{Q} : \ u(x) = m_1\}| = \beta,\ \mu(\bar{\mathcal{Q}}) = \alpha\}.$$

Moreover if $(u_\varepsilon) \subset \mathcal{A}_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon}(\mathcal{Q};K)$ is such that

$$\lim_\varepsilon F_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon}(u_\varepsilon) = \lim_\varepsilon m_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon},$$

then any cluster point $(\overline{u},\overline{\mu})$ of $(u_\varepsilon, \mu(u_\varepsilon))$ with respect to the $\tau_1 \times \tau_2$ convergence is a minimizer for $\min\{\overline{F}(u,\mu),\ |\{x \in \mathcal{Q} : \ u(x) = m_1\}| = \beta,\ \mu(\bar{\mathcal{Q}}) = \alpha\}$.

Remark 3.4.12 The previous Corollary applies to the case of the BEG model, $f(\xi,u,v)$ being defined in (3.4.16) and the limit energy densities f_{hom} and g_{hom} being given by

$$f_{hom}(z,\nu)=\varphi(z,\nu),\quad g_{hom}(1)=2(1-k),$$

with φ defined in (3.3.10).

3.5 The Blume-Emery-Griffiths model: proof of Theorem 3.3.3

By Remark 3.4.2, the functionals $E_\varepsilon^{(1)}$ satisfy all the hypotheses of Proposition 3.4.3 and Theorem 3.4.4. Hence the compactness result asserted in (i) follows by Proposition 3.4.3. Moreover the integral representation result stated in Theorem 3.4.4 holds true for the Γ-limit of $E_\varepsilon^{(1)}$. Thus, in order to conclude, it is only left to prove that, for all $(z,\nu)\in\mathbb{R}^+\times S^1$,

$$\begin{aligned} f_{hom}(z,\nu)&=\varphi(z,\nu),\\ g_{hom}(1)&=2(1-k). \end{aligned}\tag{3.5.72}$$

Without loss of generality we prove (3.5.72) for $\nu_1,\nu_2>0$.
Step 1 (lower bounds). In this Step we prove the following two inequalities:

$$\begin{aligned} f_{hom}(z,\nu)&\geq\varphi(z,\nu),\\ g_{hom}(1)&\geq 2(1-k). \end{aligned}\tag{3.5.73}$$

Let us first prove the lower bound for f_{hom}. Without loss of generality we consider T to be an even number. Let $J_{1,T}$, $J_{2,T}$ be the following sets of integers:

$$J_{1,T}=\Big\{-\Big[\frac{T\nu_2}{2}\Big],-\Big[\frac{T\nu_2}{2}\Big]+1,\dots,\Big[\frac{T\nu_2}{2}\Big]-1,\Big[\frac{T\nu_2}{2}\Big]\Big\}$$

$$J_{2,T}=\Big\{-\Big[\frac{T\nu_1}{2}\Big],-\Big[\frac{T\nu_1}{2}\Big]+1,\dots,\Big[\frac{T\nu_1}{2}\Big]-1,\Big[\frac{T\nu_1}{2}\Big]\Big\}.$$

Let u be an admissible test function in the problem defining $f_{hom}(z,\nu)$ in (3.4.21); that is $u\in\mathcal{B}_\varepsilon(TQ_\nu;\{\pm1,0\})$ and

$$\left|\frac{\#I_0(u,TQ_\nu)}{T}-z\right|<\delta.\tag{3.5.74}$$

We define

$$J^0_{1,T}(u)=\{i\in J_{1,T}:\ \exists\, j\in\mathbb{Z}\text{ such that }(i,j)\in I_0(u)\}$$

and

$$J^0_{2,T}(u)=\{j\in J_{2,T}:\ \exists\, i\in\mathbb{Z}\text{ such that }(i,j)\in I_0(u)\}.$$

Note that, by (3.5.74), for $i\in\{1,2\}$

$$\#J^0_{i,T}(u)\leq\#I_0(u,TQ_\nu)\leq(z+\delta)T.$$

The proof will be the result of the following three estimates.

Estimate (i). By a slicing argument, splitting the energy into the contribution of the horizontal and the vertical interactions, we get

$$\begin{aligned} E_1^{(1)}(u)&\geq 2(1-k)\#J^0_{1,T}(u)+2\big(\#J_{1,T}-\#J^0_{1,T}(u)\big)\\ &\quad+2(1-k)\#J^0_{2,T}(u)+2\big(\#J_{2,T}-\#J^0_{2,T}(u)\big)\\ &=-2k\#J^0_{1,T}(u)-2k\#J^0_{2,T}(u)+2\big(\#J_{1,T}+\#J_{2,T}\big)\\ &\geq-4k(z+\delta)T+2\left(2\left[\frac{T\nu_2}{2}\right]+2\left[\frac{T\nu_1}{2}\right]\right). \end{aligned}$$

There follows that:

$$
\begin{aligned}
f_{hom}(z,\nu) &\geq \lim_{\delta\to 0}\lim_{T\to+\infty}\frac{1}{T}\left(-4k(z+\delta)T+2\left(2\left[\frac{T\nu_2}{2}\right]+2\left[\frac{T\nu_1}{2}\right]\right)\right)\\
&= -4kz+2(\nu_1+\nu_2).
\end{aligned}
$$

Estimate (ii). We observe that we may split the energy of the surfactants into two terms. The first term accounts for 2 vertical and 1 horizontal interaction for each column, while the second term accounts for the remaining interaction in each row. By counting the energy due to the non-surfactant particles as in estimate (i), we get

$$
\begin{aligned}
E_1^{(1)}(u) &\geq 3(1-k)\#J^0_{1,T}(u)+2(\#J_{1,T}-\#J^0_{1,T}(u))\\
&\quad +(1-k)\#J^0_{2,T}(u)+2(\#J_{2,T}-\#J^0_{2,T}(u))\\
&\geq (1-3k)\#J^0_{1,T}(u)+2\#J_{1,T}\\
&\quad +(1-k)\#J^0_{2,T}(u)+(1-k)(\#J_{2,T}-\#J^0_{2,T}(u))\\
&\geq (1-3k)(z+\delta)T+2\#J_{1,T}+(1-k)\#J_{2,T}(u)\\
&\geq (1-3k)(z+\delta)T+4\left[\frac{T\nu_2}{2}\right]+2(1-k)\left[\frac{T\nu_1}{2}\right]
\end{aligned}
$$

By exchanging the role of $J^0_{1,T}(u)$ with that of $J^0_{2,T}(u)$ in the previous estimate we have

$$
E_1^{(1)}(u)\geq(1-3k)(z+\delta)T+4\left[\frac{T(\nu_1\vee\nu_2)}{2}\right]+2(1-k)\left[\frac{T(\nu_1\wedge\nu_2)}{2}\right]
$$

Therefore we get

$$
\begin{aligned}
f_{hom}(z,\nu) &\geq \lim_{\delta\to 0}\lim_{T\to+\infty}\frac{1}{T}(1-3k)(z+\delta)T+4\left[\frac{T(\nu_1\vee\nu_2)}{2}\right]+2(1-k)\left[\frac{T(\nu_1\wedge\nu_2)}{2}\right]\\
&\geq (1-3k)z+2(\nu_1\vee\nu_2)+(1-k)(\nu_1\wedge\nu_2).
\end{aligned}
$$

Estimate (iii). We observe that the we may split the energy of the surfactant into three terms. The first term takes into account two interactions for each surfactant particle. The other two terms take into account in each row and column containing a surfactant at least one interaction between a surfactant and a non-surfactant particle. Counting as in the previous estimates the remaining interactions, we have

$$
\begin{aligned}
E_1^{(1)}(u) &\geq 2(1-k)\#I_0(u,TQ_\nu)+2(\#J_{1,T}-\#J^0_{1,T})+(1-k)\#J^0_{1,T}\\
&\quad +2(\#J_{2,T}-\#J^0_{2,T})+(1-k)\#J^0_{2,T}\\
&\geq 2(1-k)(z-\delta)T+(1-k)(\#J_{1,T}+\#J_{2,T})\\
&\geq 2(1-k)(z-\delta)T+(1-k)\left(2\left[\frac{T\nu_1}{2}\right]+2\left[\frac{T\nu_2}{2}\right]\right).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
f_{hom}(z,\nu) &\geq \lim_{\delta\to 0}\lim_{T\to+\infty}\frac{1}{T}\left(2(1-k)(z-\delta)T+(1-k)\left(2\left[\frac{T\nu_1}{2}\right]+2\left[\frac{T\nu_2}{2}\right]\right)\right)\\
&\geq 2(1-k)z+(1-k)(\nu_1+\nu_2).
\end{aligned}
$$

The lower bound in (3.5.73) for g_{hom} is straightforward. In fact, let us observe that, except for a negligible error due to the interactions at the boundary of TQ, the energy accounts for at least two interactions of each surfactant particle. Hence, for any test function u in the minimum problem defining $g_{hom}(1)$ we have

$$
E_1^{(1)}(u,TQ)\geq 2(1-k)\#I_0(u,TQ)+o(1)
$$

Then

$$g_{hom}(1) \geq \lim_{\delta} \liminf_{T} \frac{E_1^{(1)}(u, TQ)}{\#I_0(u, TQ)} \geq 2(1-k).$$

Step 2 (upper bounds). In this Step we conclude the proof by showing that the inequalities

$$\begin{aligned} f_{hom}(z,\nu) &\leq \varphi(z,\nu), \\ g_{hom}(1) &\leq 2(1-k). \end{aligned} \tag{3.5.75}$$

hold true. With a slight abuse of notation, for $\nu \in S^1$ we denote by $u_\nu : \mathbb{R}^2 \to \{\pm 1\}$ the function defined in (3.4.63), with -1 and 1 in place of m_1 and m_2, that is

$$u_\nu(x) := \begin{cases} -1 & \text{if } (x,\nu) > 0 \\ +1 & \text{if } (x,\nu) \leq 0, \end{cases} \tag{3.5.76}$$

and, for $z \in \mathbb{R}^+$, we set

$$\mu_{z,\nu} := z\mathcal{H}^1 \lfloor S(u_\nu).$$

In order to prove the first inequality in (3.5.75), by Theorem 3.4.4 it is enough to construct $u_\varepsilon \in \mathcal{A}_\varepsilon(\Omega; \{0, \pm 1\})$ such that $(u_\varepsilon, \mu(u_\varepsilon)) \to (u_\nu, \mu_{z,\nu})$ with respect to the $\tau_1 \times \tau_2$-convergence and

$$\limsup_{\varepsilon} E_\varepsilon^{(1)}(u_\varepsilon) \leq \varphi(z,\nu)\mathcal{H}^1(S(u_\nu) \cap \Omega).$$

To this end we find it useful to rewrite φ as

$$\varphi(z,\nu) = \begin{cases} \varphi_1(z,\nu) & \text{if } 0 \leq z < |\nu_1| \wedge |\nu_2| \\ \varphi_2(z,\nu) & \text{if } |\nu_1| \wedge |\nu_2| \leq z \leq |\nu_1| \vee |\nu_2| \\ \varphi_3(z,\nu) & \text{if } z > |\nu_1| \vee |\nu_2|. \end{cases}$$

The construction of u_ε differs in the three cases $0 < z < |\nu_1| \wedge |\nu_2|$, $|\nu_1| \wedge |\nu_2| \leq z \leq |\nu_1| \vee |\nu_2|$ or $z > |\nu_1| \vee |\nu_2|$. Without loss of generality, for simplicity of exposition, we may suppose that $-\nu_1 \geq \nu_2 > 0$. Moreover, by the continuity of $f_{hom}(z, \cdot)$ and $\varphi(z, \cdot)$, by a density argument we may assume that $\frac{-\nu_1}{\nu_2} \in \mathbb{Q}$. Let $p, q \in \mathbb{N}$ be such that $\frac{-\nu_1}{\nu_2} = \frac{p}{q}$. By the continuity of $f_{hom}(\cdot, \nu)$ and $\varphi(\cdot, \nu)$ we may further assume that $z' := z\sqrt{p^2+q^2} \in \mathbb{Q}$. Hence, by possibly replacing (p, q) by (mp, mq) for some $m \in \mathbb{N}$, we may reduce to the case $z' \in \mathbb{N}$.

Let us set $\overline{\nu} = \frac{e_1+e_2}{\sqrt{2}}$ and let $u^0 : \{1, 2, \ldots, q\} \times \mathbb{Z} \to \{\pm 1, 0\}$ be defined as

$$u^0(a) = \begin{cases} 0 & \text{if } a_1 = a_2 \leq q \text{ or } a_1 = q,\ q < a_2 \leq p \\ u_{\overline{\nu}}(a) & \text{otherwise.} \end{cases}$$

where $a = (a_1, a_2)$.

Case 1: $0 < z < |\nu_1| \wedge |\nu_2|$. By the assumptions on ν_1, ν_2 this case corresponds to $z' < q$. Let $u_{z,\nu} : \mathbb{Z}^2 \to \{\pm 1, 0\}$ be such that

$$u_{z,\nu}(a + (p,q)) = u_{z,\nu}(a), \qquad \text{for all } a \in \mathbb{Z}^2$$

and, on $\{1, 2, \ldots, q\} \times \mathbb{Z} \to \{\pm 1, 0\}$ is defined as (see Figure 4)

$$u_{z,\nu}(a) = \begin{cases} u^0(a) & \text{if } 0 < a_1 \leq z' \\ u_\nu & \text{otherwise.} \end{cases}$$

Let then $u_\varepsilon : \varepsilon\mathbb{Z}^2 \to \{\pm 1, 0\}$ be such that $u_\varepsilon(a) = u_{z,\nu}(\frac{a}{\varepsilon})$. It holds that $(u_\varepsilon, \mu(u_\varepsilon)) \to (u_\nu, \mu_{z,\nu})$

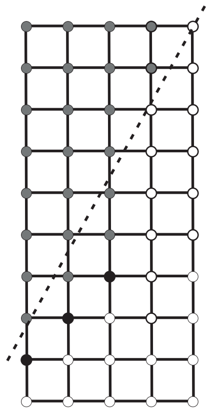

Figure 3.4: $u_{z,\nu}$ in the periodicity cell $\{1,2,\dots,q\}\times\{1,2,\dots,p\}$ with $q=5$, $p=9$ and $z'=3$.

with respect to the $\tau_1\times\tau_2$-convergence. In order to estimate the energy of u_ε we observe that it concentrates on each rectangle of the type $R_{\varepsilon,j}:=(0,\varepsilon q]\times(0,\varepsilon p]+\varepsilon j(p,q)$, $j\in\mathbb{Z}$ where, by periodicity, it takes the constant value

$$E^{(1)}_\varepsilon(u_\varepsilon,R_{\varepsilon,j})=\varepsilon\left(4(1-k)z'+2(p-z')+2(q-z')\right)=\varepsilon\left(-4kz'+2(p+q)\right).$$

Then

$$\begin{aligned}E^{(1)}_\varepsilon(u_\varepsilon) &\le E^{(1)}_\varepsilon(u_\varepsilon,R_{\varepsilon,0})(\#\{j\in\mathbb{Z}:\ R_{\varepsilon,j}\subset\Omega\}+2)\\ &\le \varepsilon\left(-4kz'+2(p+q)\right)\left(\left[\frac{\mathcal{H}^1(S(u_\nu)\cap\Omega)}{\varepsilon\sqrt{p^2+q^2}}\right]+2\right)\end{aligned}$$

Eventually, letting ε tend to 0 we obtain

$$\limsup_\varepsilon E^{(1)}_\varepsilon(u_\varepsilon)\le\varphi_1(z,\nu)\mathcal{H}^1(S(u_\nu)\cap\Omega).$$

Case 2: $|\nu_1|\wedge|\nu_2|\le z\le|\nu_1|\vee|\nu_2|$. This case corresponds to $q\le z'\le p$. Let $v_{z,\nu}:\mathbb{Z}^2\to\{\pm1,0\}$ be such that

$$v_{z,\nu}(a+(p,q))=v_{z,\nu}(a),\qquad\text{for all } a\in\mathbb{Z}^2$$

and, on $\{1,2,\dots,q\}\times\mathbb{Z}\to\{\pm1,0\}$ is defined as (see Figure 5)

$$v_{z,\nu}(a)=\begin{cases}-1 & \text{if } a_1=q \text{ and } a_2>z'\\ u^0 & \text{otherwise.}\end{cases}\tag{3.5.77}$$

Let then $v_\varepsilon:\varepsilon\mathbb{Z}^2\to\{\pm1,0\}$ be such that $v_\varepsilon(a)=v_{z,\nu}(\frac{a}{\varepsilon})$. It holds that $(v_\varepsilon,\mu(v_\varepsilon))\to(u_\nu,\mu_{z,\nu})$ with respect to the $\tau_1\times\tau_2$-convergence. By arguing as in Case 1, taking into account that

$$\begin{aligned}E^{(1)}_\varepsilon(v_\varepsilon,R_{\varepsilon,j}) &= \varepsilon\left(4(1-k)q+3(1-k)(z'-q)+2(p-z')\right)\\ &= \varepsilon\left((1-3k)z'+(1-k)q+2p\right),\end{aligned}$$

we get

$$E^{(1)}_\varepsilon(u_\varepsilon)\quad\le\quad\varepsilon\left((1-3k)z'+(1-k)q+2p\right)\left(\left[\frac{\mathcal{H}^1(S(u_\nu)\cap\Omega)}{\varepsilon\sqrt{p^2+q^2}}\right]+2\right).$$

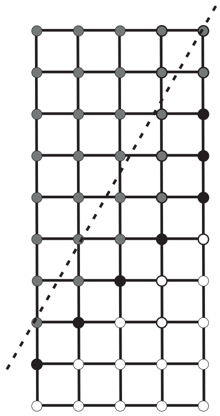

Figure 3.5: $u_{z,\nu}$ in the periodicity cell $\{1,2,\dots,q\}\times\{1,2,\dots,p\}$ with $q=5$, $p=9$ and $z'=7$.

Eventually, letting ε tend to 0 we obtain

$$\limsup_{\varepsilon} E^{(1)}_\varepsilon(v_\varepsilon)\le \varphi_2(z,\nu)\mathcal{H}^1(S(u_\nu)\cap\Omega).$$

Case 3: $z>|\nu_1|\vee|\nu_2|$. This case corresponds to $z'>p$. Let us extend the function u^0 to $\mathbb{Z}^2$ in such a way that

$$u^0(a+(p,q))=u^0(a),\qquad \text{for all } a\in\mathbb{Z}^2.$$

We now construct $w_{z,\nu}:\mathbb{Z}^2\to\{\pm1,0\}$ by modifying the function u^0 suitably increasing the numbers of its zeros in order to match the density constraint on the surfactant phase. More precisely we set $z'':=z'-[\frac{z'}{p}]p$ and

$$\tilde I_0:=\bigcup_{m=0}^{[z'/p]}(I_0(u^0)+me_1)\cap\bigcup_{j\in\mathbb{Z}}\{a\in\mathbb{Z}^2:\ jp+1\le a_2\le jp+z''\}$$

$$\hat I_0:=\bigcup_{m=0}^{[z'/p]-1}(I_0(u^0)+me_1)\cap\bigcup_{j\in\mathbb{Z}}\{a\in\mathbb{Z}^2:\ jp+z''+1\le a_2\le (j+1)p\}$$

we define (see Figure 6)

$$w_{z,\nu}(a)=\begin{cases}0 & \text{if } a\in\tilde I_0\cup\hat I_0\\ u^0 & \text{otherwise.}\end{cases}$$

Let then $w_\varepsilon:\varepsilon\mathbb{Z}^2\to\{\pm1,0\}$ be such that $w_\varepsilon(a)=w_{z,\nu}(\frac{a}{\varepsilon})$. It holds that $(w_\varepsilon,\mu(w_\varepsilon))\to(u_\nu,\mu_{z,\nu})$ with respect to the $\tau_1\times\tau_2$-convergence. An easy computation shows that, the energy of each stripe $S_{\varepsilon,j}:=\mathbb{R}\times(jp,(j+1)p]$, $j\in\mathbb{Z}$, is

$$\begin{aligned}E^{(1)}_\varepsilon(w_\varepsilon,S_{\varepsilon,j}) &= \varepsilon\left((1-3k)p+(1-k)q+2p+2(1-k)(z'-p)\right)\\ &= \varepsilon\left((1-k)(p+q)+(1-k)z'\right).\end{aligned}$$

Then

$$\begin{aligned}E^{(1)}_\varepsilon(w_\varepsilon) &\le E^{(1)}_\varepsilon(w_\varepsilon,S_{\varepsilon,0})(\#\{j\in\mathbb{Z}:\ S_{\varepsilon,j}\subset\Omega\}+2)\\ &\le \varepsilon\left((1-k)(p+q)+(1-k)z'\right)\left(\left[\frac{\mathcal{H}^1(S(u_\nu)\cap\Omega)}{\varepsilon\sqrt{p^2+q^2}}\right]+2\right)\end{aligned}$$

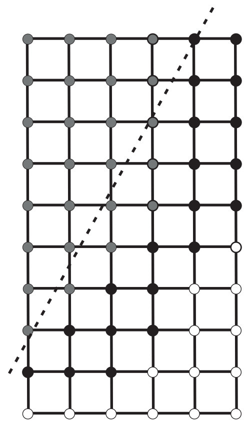

Figure 3.6: $u_{z,\nu}$ in the periodicity cell $\{1, 2, \dots, q\} \times \{1, 2, \dots, p\}$ with $q = 5$, $p = 9$ and $z' = 20$.

Eventually, letting ε tend to 0 we obtain

$$\limsup_{\varepsilon} E^{(1)}_{\varepsilon}(w_\varepsilon) \le \varphi_3(z,\nu)\mathcal{H}^1(S(u_\nu) \cap \Omega).$$

We now prove the second inequality in (3.5.75). To this end, by Theorem 3.4.4, given $x_0 \in \Omega$, it is enough to construct a sequence of functions $u_\varepsilon \in \mathcal{A}_\varepsilon(\Omega; \{0, \pm 1\})$ such that $(u_\varepsilon, \mu(u_\varepsilon)) \to (1, \delta_{x_0})$ with respect to the $\tau_1 \times \tau_2$-convergence and that

$$\limsup_{\varepsilon} E^{(1)}_{\varepsilon}(u_\varepsilon) \le 2(1-k).$$

We set

$$u_\varepsilon(a) = \begin{cases} 0 & \text{if } a \in \left(x_0 + \left(-\frac{\sqrt{\varepsilon}}{2}, \frac{\sqrt{\varepsilon}}{2}\right)^2\right) \cap \varepsilon\mathbb{Z}^2 \\ 1 & \text{otherwise.} \end{cases}$$

Let us observe that $\varepsilon \# I_0(u_\varepsilon) = 1 + o(1)$ and that, each surfactant particle whose interactions do not cross the boundary of $x_0 + \left(-\frac{\sqrt{\varepsilon}}{2}, \frac{\sqrt{\varepsilon}}{2}\right)^2$ gives a contribution to the energy which is equal to $2(1-k)$. Moreover, since the number of the remaining surfactants scales as $\frac{1}{\sqrt{\varepsilon}}$, we have

$$E^{(1)}_{\varepsilon}(u_\varepsilon) = \varepsilon\, 2(1-k)\# I_0(u_\varepsilon) + o(1) = 2(1-k) + o(1).$$

Letting ε tend to 0 we get the conclusion. ∎

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