UNIVERSITÀ DI ROMA LA SAPIENZA

Doctoral Thesis

ASYMPTOTIC BEHAVIOUR OF SINGULARLY PERTURBED CONTROL SYSTEMS IN THE NON-PERIODIC SETTING

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A thesis submitted in fulfilment of the requirements for the degree of Doctor of Philosophy

in the

Departimento di Matematica "Guido Castelnuovo"
Università di Roma La Sapienza

Rome, June 2015

Declaration of Authorship

I, Nguyen Ngoc Quoc Thuong, declare that this thesis titled, *Asymptotic behaviour of singularly perturbed control system in the non-periodic setting* and the original results presented in it are my own.

This thesis is written at Department of Mathematics "Guido Castelnuovo", University of Rome "La Sapienza", Rome, under supervision of Prof. Antonio Siconolfi.

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Abstract

The main scope of the thesis is to investigate asymptotic behaviour of value function in singularly perturbed optimal control problem in the non-periodic (non-compact) setting. The approach is PDE and we aim at characterizing its weak semi-limits as viscosity sub- and supersolution of the so-called effective equation. This PDE approach is extensively studied by Alvarez and Bardi in the periodic setting as well as in the case where fast trajectories are constrained in a compact set, see [AB01], [AB03], [AB10].

Our contribution is to extend the results of Alvarez and Bardi to the non-periodic (non-compact) case. The key idea is to replace the periodicity (in fast variable) on the datum by coercivity (in fast variable) on the running cost. We also require (strong) controllability on the fast system in order for the Bellman-Hamiltonian is coercive in the fast momentum. The remarkable novelty of our work is to employ the techniques of Weak KAM theory (see [FS05]) to construct suitable correctors and then apply Evans' perturbed test function method with some modifications.

Acknowledgements

I wish to express my deep gratitude to my supervisor, Prof. Antonio Siconolfi, for very helpful discussions on the subject, for his constant and precious support in the research as well as in the preparation of the present thesis.

The author would like to thank Department of Mathematics, University of Rome "La Sapienza", Rome, Italy for the training and support during the last three years of the PhD programme. I also would like to thank ENSTA ParisTech, Paris, France, where a partial thesis was written, for the invitation and hospitality. I am grateful to Prof. Hasnaa Zidani for her supervision when I studied at ENSTA ParisTech as a secondment organization (January-June 2014).

The author wishes to thank SADCO project "Sensitivity Analysis for Deterministic Controller Design" (SADCO is funded by the European Union under the "FP7-People-ITN" program) for the financial support. I wish also to express my sincere gratitude to Prof. Maurizio Falcone, Prof. Hasnaa Zidani and Ms. Estelle Bouzat for their help in the framework of SADCO project.

My thanks also go to Department of Mathematics, Quy Nhon University, Vietnam for the support so that I can pursue and complete my graduate studies.

Finally, I am greatly indebted to my family, especially my wife and my daughter, for the valuable support and encouragement during my abroad studies.

I dedicate this thesis to my wife, **Vo Thi Thu Hang**, *and my daughter,* **Nguyen Ngoc Cat Tien**.

Contents

Basic Notations

$\mathbb{N}$	the set of natural numbers, i.e., $\mathbb{N} = \{0, 1, 2, ...\}$
$\mathbb{N}^*$	the set of positive natural numbers, i.e., $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$
$\mathbb{Z}$	the set of integer numbers, i.e., $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^*$	the set of real numbers and different from zero, i.e., $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$
$\mathbb{R}^+$	the set of positive real numbers
$\mathbb{R}^-$	the set of negative real numbers
$\overline{\mathbb{R}}$	the set of complete real numbers, i.e., $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$
$\mathbb{R}^N$	the euclidean N-dimensional space
$\mathbb{T}^N$	the flat torus, i.e., $\mathbb{T}^N = \mathbb{R}^N / \mathbb{Z}^N$
$x \cdot y$	the scalar product of vectors $x = (x_1, ..., x_N)$ and $y = (y_1, ..., y_N)$, $x \cdot y = \sum_{i=1}^{N} x_i y_i$
$\|x\|$	the euclidean norm of $x \in \mathbb{R}^N$, $\|x\| = \sqrt{x \cdot x}$
$B(x_0, r)$	the open ball centered at x_0 of radius r
$\overline{B}(x_0, r)$	the closed ball centered at x_0 of radius r
∂E	the boundary of the set E
$\overline{E}$	the closure of the set E
$\operatorname{int} E$	the interior of the set E
$\mathrm{co}E$	the convex hull of the set E
$\|E\| = \operatorname{meas} E$	the Lebesgue measure of the set E
$d(x, E)$	the distance from x to E, i.e., $d(x, E) = \inf_{y \in E} \|x - y\|$
$d^{\#}(x, E)$	the signed distance from x to E, i.e., $d^{\#}(x, E) := 2d(x, E) - d(x, \partial E)$
$\operatorname{diam} E$	the diameter of the set E, i.e., $\operatorname{diam} E = \sup \big\{ \|x - y\| \, : \, x, y \in E \big\}$

$\|u\|_\infty$	the supremum norm of a function $u : E \to \mathbb{R}$, i.e., $\|u\|_\infty = \sup_{x\in E} \|u(x)\|$
$\arg\min_E u$	the set of minimizers (minimum points) of $u : E \to \mathbb{R}$
$\arg\max_E u$	the set of maximizers (maximum points) of $u : E \to \mathbb{R}$
$Du(x)$	the gradient of the function u at x, i.e., $Du(x) = \left(\frac{\partial u}{\partial x_1}(x), ..., \frac{\partial u}{\partial x_N}(x)\right)$
$D^+u(x), D^-u(x)$	the super- and sub differential of u at x
$\partial u(x)$	the Clarke's gradient of u at x
$\mathcal{C}$	the set of controls, i.e., (Lebesgue) measurable functions $\alpha : [0, +\infty) \to A$
$\mathcal{A}$	the Aubry set
$\mathrm{B}\,(E)$	the space of bounded functions $u : E \to \mathbb{R}$
$\mathrm{C}\,(E)$	the space of continuous functions $u : E \to \mathbb{R}$
$\mathrm{C}^1(E)$	the space of continuously differentiable functions $u : E \to \mathbb{R}$
$\mathrm{UC}\,(E)$	the space of uniformly continuous function $u : E \to \mathbb{R}$
$\mathrm{BC}\,(E)$	the space of bounded, continuous functions $u : E \to \mathbb{R}$
$\mathrm{BUC}\,(E)$	the space of bounded, uniformly continuous functions $u : E \to \mathbb{R}$
$\mathrm{USC}\,(E)$	the space of upper semicontinuous functions $u : E \to \mathbb{R}$
$\mathrm{LSC}\,(E)$	the space of lower semicontinuous functions $u : E \to \mathbb{R}$
$\mathrm{Lip}\,(E)$	the space of Lipschitz continuous functions $u : E \to \mathbb{R}$
$\mathrm{Lip}_{\mathrm{loc}}\,(E)$	the space of locally Lipschitz continuous functions $u : E \to \mathbb{R}$
$\mathrm{Lip}_{\,x,y}\big([0,1];\mathbb{R}^N\big)$	the set of Lipschitz continuous curves $\xi : [0,1] \to \mathbb{R}^N$ joining x to y, i.e., $\xi(0) = x$, $\xi(1) = y$

Introduction

The main purpose of the present thesis is to investigate asymptotic behaviour of singularly perturbed control system as the perturbed term vanishes, where there is no periodic condition (in fast variable) on the datum, as well as we do not require any compact constraint on the fast trajectory. More precisely, we are interested in the singularly perturbed control system of deterministic type

$$\begin{cases} X'(s) = f\big(X(s), Y(s), \alpha(s)\big), \quad s > 0 \\ Y'(s) = \dfrac{1}{\varepsilon}\, g\big(X(s), Y(s), \alpha(s)\big), \quad s > 0 \\ X(0) = x, \ Y(0) = y, \end{cases} \tag{1}$$

here, the slow variable X and fast variable Y are taken in $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively; α varies in the set of admissible control functions, say $\mathcal{C}$, and get values in a compact subset A of some Euclidean space; ε is a small positive parameter which represents the perturbed term. This is the model of deterministic control system where the state variable Y evolves at a much faster time scale than the variable X, that explains the names "fast variable" and "slow variable". It is worth noticing that, if we interpret the original system (1) as a ε-perturbation of some limiting system (also called a nominal system) with $\varepsilon = 0$, then the fast equation in (1) has the form

$$0 = g\big(X(s), Y(s), \alpha(s)\big) \tag{2}$$

which is no longer a differential equation, and hence a limiting system must have some qualitative properties which are different from those of original perturbed system. This is why the original system (1) is called the singularly perturbed one.

The goal in singular perturbation problem is to pass to the limit when the perturbed term ε goes to zero. The result is the elimination of the state variable Y, and the system is then reduced from $N + M$ to N dimensions. That means the limiting system involves only the slow variable but still keeps some information on the fast part of the dynamics.

The singularly perturbed control system is motivated by many problems in engineering, chemistry, physics and it has been received a lot of interests for a long time. We refer the reader to [KKO86] for more details on the topic. We will only mention in this thesis some mathematical literature related to our subject.

The first approach to singular perturbation problem is to describe explicit form of a limiting system in the sense that the trajectory of perturbed system converges to the corresponding trajectory of limiting system. This is indeed called *dynamical system* approach. The first method in this direction is referred to as *order reduction*, initiated by Levinson and Tichonov in the context of ordinary differential equations (ODE). This method is then extended to control system by several authors, see Kokotovíc, Khalil & O'Reilly [KKO86], Bensoussan [B88], Dontchev & Zolezzi [DZ93], Veliov [Ve97], and the references therein. A prime condition in order for the order reduction model works is that the fast system

$$Y'(s) = g\big(X(s), Y(s), \alpha(s)\big)$$

has some stability property and then we get limiting system by replacing formally ε by zero in the second equation of system (1). Namely, the limiting system is of differential-algebraic form

$$\begin{cases} X'(s) = f\big(X(s), Y(s), \alpha(s)\big), \ s > 0 \\ 0 = g\big(X(s), Y(s), \alpha(s)\big), \ s > 0 \\ X(0) = x, \ Y(0) = y. \end{cases} \tag{3}$$

If the fast variable has more general asymptotic behaviour, the classical averaging method for ODE of Krylov and Bogolyubov was developed to the theory of limit occupational measures for control systems by Artstein, Gaitsgory, Leizarowitz, and others, see for instance [G92], [AG97], [Art99], [GL99], [QW03], [G04], [A04]. In the averaging method via limit occupational measures, the limiting system is described by a differential inclusion

$$\begin{cases} X'(s) \in \mathcal{F}\big(X(s)\big) \\ X(0) = 0, \end{cases} \tag{4}$$

where $\mathcal{F}(\cdot)$ is the set of relaxed dynamics obtained by averaging slow dynamic f with respect to limit occupational measures, which are probability measures generated by solutions of the fast system

$$\begin{cases} Y'(\tau) = g\big(x, Y(\tau), \alpha(\tau)\big) \\ Y(0) = y, \quad x \text{ is fixed in } \mathbb{R}^N. \end{cases} \tag{5}$$

It is important to notice that, in contrast with the order reduction method where it is possible to prove that the trajectory $\big(X^\varepsilon(\cdot), Y^\varepsilon(\cdot)\big)$ of (1) converges (locally uniformly), as $\varepsilon \to 0$, to some solution $\big(X(\cdot), Y(\cdot)\big)$ of the differential-algebraic system (3), while using averaging method we only know asymptotic behaviour of the slow trajectory $X^\varepsilon(\cdot)$, which converges (locally uniformly) to some solution $X(\cdot)$ of the differential inclusion (4). In fact, although the fast motion disappears by averaging, it influences, of course, on the structure of the differential inclusion (4) via relaxation procedure with limit occupational measures.

In the context of singular perturbation problem, it includes also the optimal control problem that aims at minimizing the cost functional

$$J^\varepsilon(t, x, y, \alpha) := \int_0^t \ell\big(X^\varepsilon(s), Y^\varepsilon(s), \alpha(s)\big)\, ds + u_0\big(X^\varepsilon(t), Y^\varepsilon(t)\big), \tag{6}$$

here, $\ell : \mathbb{R}^N \times \mathbb{R}^M \times A \to \mathbb{R}$ and $u_0 : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ are given functions, called *running cost* and *final cost*, respectively. It is evident that the so-called *limiting optimal control problem* is well understood if the form of the limiting system (4) is described explicitly by averaging method via limit occupational measures.

Another approach to the singular perturbation problem consists of investigating asymptotic behaviour, as $\varepsilon \to 0$, of the corresponding value function

$$V^\varepsilon(t, x, y) := \inf_{\alpha \in \mathcal{C}} J^\varepsilon(t, x, y, \alpha), \tag{7}$$

and characterizing its limits (weak limit or uniform limit) as viscosity sub- and supersolution of a certain limiting equation (also called effective equation). This PDE approach is extensively studied in a series of papers by Alvarez and Bardi in the periodic setting ([AB03], [AB10]) as well as in the case of compact constraint on the fast trajectories [AB01]. We also refer the readers to some earlier papers in this direction, see for instance [G92], [BB98], [AG00]. Following the PDE approach proposed in this context by Alvarez and Bardi, we first note that, under suitable assumptions the value function V^ε, for each $\varepsilon > 0$, solves in viscosity sense the Hamilton-Jacobi-Bellman equation (HJB equation) in $[0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^M$

$$\begin{cases} V_t^\varepsilon + H\big(x, y, D_x V^\varepsilon, \frac{D_y V^\varepsilon}{\varepsilon}\big) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^M, \\ V^\varepsilon(0, x, y) = u_0(x, y) & \text{in } \mathbb{R}^N \times \mathbb{R}^M, \end{cases} \tag{8}$$

where H is the Bellman-Hamiltonian given by

$$H(x, y, p, q) := \max_{a \in A} \big\{ -p \cdot f(x, y, a) - q \cdot g(x, y, a) - \ell(x, y, a) \big\}.$$

Similar to the averaging method mentioned above, when passing to the limit as $\varepsilon \to 0$, the so-called limiting value function $V(t, x)$ does not depend on the fast variable y, and it solves in viscosity sense a limiting HJB equation in $(0, +\infty) \times \mathbb{R}^N$, which is governed by an effective Hamiltonian $\overline{H}$, namely

$$V_t + \overline{H}(x, DV) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^N, \tag{9}$$

The aforementioned result is due to Alvarez and Bardi in the periodic setting and under a suitable controllability on the fast system (5). A crucial step in the analysis is to consider a family of stationary equations in the fast variable, for given $(x_0, p_0) \in \mathbb{R}^N \times \mathbb{R}^N$,

$$H\big(x_0, y, p_0, Du(y)\big) = b \quad \text{in } \mathbb{R}^M, \tag{10}$$

where b is a real parameter. A key property linking to asymptotic analysis is *ergodicity* (due to Alvarez and Bardi) of the Hamiltonian H. More precisely, the Hamiltonian H is said to be *ergodic* if for each $(x_0, p_0) \in \mathbb{R}^N \times \mathbb{R}^N$, there exists a unique real value $b = \overline{H}(x_0, p_0)$ for which the stationary equation

$$H\big(x_0, y, p_0, Du(y)\big) = \overline{H}(x_0, p_0) \quad \text{in } \mathbb{R}^M \tag{11}$$

admits a periodic viscosity subsolution and a periodic viscosity supersolution. Such a pair $(u, \overline{H}\,)$ is referred to as solution of cell problem (also called corrector problem) in periodic homogenization, and the equation (11) is called *critical equation*. Note also that the ergodicity of H can be characterized in terms of cell δ-problem and cell t-problem, which are related to ergodic control problems (see Definitions 2.2 and 2.3 in Chapter 2). The result of Alvarez and Bardi is inspired from the ideas and methods of periodic homogenization for Hamilton-Jacobi equation, pioneered by Lions, Papanicolau, Varadhan in the unpublished but classic paper [LPV86], and then revisited by Evans [E92] who introduced perturbed test function method. Notice that, the periodic sub- and supersolution of (11) play the roles of correctors allowing to adapt Evans' perturbed test function method. In the periodic setting, as showed in the recent paper [AB10], the bounded time controllability (see Definition 2.8) on the fast system (5) is sufficient for ergodicity of the Hamiltonian H. The controllability condition can be viewed as a weak form of coercivity which is needed for homogenization problem.

It is important to emphasize that, under the $\mathbb{Z}^M$-periodic condition (in fast variable) on the datum, the stationary equation (10) can be equivalently defined in the flat torus $\mathbb{T}^M = \mathbb{R}^M/\mathbb{Z}^M$, which is a compact set.

In this thesis, we are interested in asymptotic behaviour of the value function v^ε, as $\varepsilon \to 0$, following the PDE approach proposed in this context by Alvarez and Bardi. The new point is that our setting is non-periodic (non-compact). Namely, we do not assume $\mathbb{Z}^M$-periodic condition (in fast variables) on the datum. We also do not ask any compact constraint on the fast trajectory of (5) as in [AB01].

To clarify the explanation, we point out here two key assumptions we use in the thesis (the set of assumptions will be stated in detail in standing assumptions of Chapter 3):

(i) the running cost ℓ is coercive with respect to the fast variable, namely

$$\min_{a\in A} \ell(x,y,a) \to +\infty, \quad \text{as } |y| \to +\infty, \text{ uniformly in } x \in \mathbb{R}^N;$$

(ii) the fast system (5) is strong controllable in the sense that, for any compact subset $K \subset \mathbb{R}^N \times \mathbb{R}^M$ there exists $r = r(K) > 0$ such that

$$B(0,r) \subset \overline{\text{co}}\left\{g(x,y,a) : a \in A\right\}, \quad \text{for any } (x,y) \in K.$$

Due to the lack of periodicity (compactness) on the fast variable, the techniques and arguments in periodic (compact) framework are not valid any more. In fact, this drawback is counterbalanced by requiring coercivity condition (i) on the running cost. The strong controllability condition (ii) is standard in control theory and used in [AB01]. This condition implies (local) bounded-time controllability mentioned in [AB10]. We still keep the above (strong) controllability in this thesis in order for the Bellman-Hamiltonian $H_0(y,q) := H(x_0,y,p_0,q)$, for fixed (x_0,p_0), is coercive with respect to the fast momentum, in the following sense (see Remark 3.4)

$$\lim_{|q|\to+\infty} \min_{y\in K} H_0(y,q) = +\infty \quad \text{for any compact subset } K \subset \mathbb{R}^M.$$

For reader's convenience, let us explain briefly some main results obtained in this thesis as well as the ideas and methods to prove them. All these results are presented in Chapter 3 and in the paper we insert in the Appendix.

In contrast with periodic (compact) setting, there is no hope to have bounded supersolution to the equation (10) for some distinguished real value b.

The first main result of the thesis is Theorem 4.1 (in the Appendix: the paper). This theorem plays a crucial role in asymptotic analysis later on. Let us explain how to achieve this result as well as the ideas of the proof. We first note that, for given $(x_0,p_0) \in \mathbb{R}^N \times \mathbb{R}^N$, the Bellman-Hamiltonian $(y,q) \mapsto H_0(y,q) = H(x_0,y,p_0,q)$ is convex and coercive in q. We are thus in the setting of convex, coercive Hamiltonian. The idea is to adapt the techniques of Weak KAM theory (see [FS05]) to define the so-called *effective Hamiltonian* $\overline{H}(x_0,p_0)$ for the Hamiltonian $H(x_0,y,p_0,q)$ and hereafter to construct a bounded Lipschitz-continuous subsolution and locally Lipschitz-continuous supersolution to the corresponding critical equation

$$H\big(x_0,y,p_0,Du(y)\big) = \overline{H}(x_0,p_0) \quad \text{in } \mathbb{R}^M. \tag{12}$$

We emphasize that semidistance (intrinsic distance) and Aubry set associated to the Hamiltonian $H(x_0,y,p_0,q)$ play important roles in the qualitative analysis of the critical equation (12) as well as the construction of sub- and supersolution with the desired properties. These sub- and supersolution play the roles of correctors allowing us to apply Evans's perturbed test function method with some modifications.

We proceed discussing about asymptotic analysis of the value function which is the main goal of the present work. Under standing assumptions (see in Chapter 3), the family of value functions V^ε, for $\varepsilon > 0$, are locally equibounded in $[0,+\infty) \times \mathbb{R}^N \times \mathbb{R}^M$ (see Proposition 2.6 in the paper). Therefore we can define its weak semi-limits, as $\varepsilon \to 0$ (see Definition 1.8, Chapter 1). These weak semi-limits are not only finite in $[0,+\infty)\times\mathbb{R}^N\times\mathbb{R}^M$ but also independent of the fast variable y. Namely, they define functions on $[0,+\infty)\times\mathbb{R}^N$ (see Proposition 2.7 in the paper).

We now state the second main result characterizing weak semi-limits of value function as viscosity sub-and supersolution of a limiting equation (effective equation) (see Theorem 4.3 and Theorem 4.4 in the paper).

Theorem 0.1. *The weak sup-semilimit $\overline{V}$ and weak inf-semilimit $\underline{V}$ of the value functions V^ε, as $\varepsilon\to 0$, are viscosity subsolution and viscosity supersolution, respectively, to the effective equation*

$$V_t + \overline{H}(x, DV) = 0 \quad \textit{in } (0,+\infty)\times\mathbb{R}^N,$$

and satisfy

$$\limsup_{\substack{(t,x)\to(0,x_0)\\ t>0}} \overline{V}(t,x) \le \overline{u}_0(x_0) \qquad \textit{for any } x_0\in\mathbb{R}^N,$$

$$\liminf_{\substack{(t,x)\to(0,x_0)\\ t>0}} \underline{V}(t,x) \ge (\overline{u}_0)_\#(x_0) \qquad \textit{for any } x_0\in\mathbb{R}^N,$$

where, $\overline{H}$ is the effective Hamiltonian defined by

$$\overline{H}(x_0,p_0) := \inf\left\{b\in\mathbb{R} \,:\, (10) \textit{ admits a subsolution}\right\}, \tag{13}$$

$$\overline{u}_0(x) := \inf_{y\in\mathbb{R}^M} u_0(x,y) \qquad \textit{for any } x\in\mathbb{R}^N,$$

and $(\overline{u}_0)_\#$ is the lower semicontinuous envelope of $\overline{u}_0$.

It is important to notice that the proof of supersolution property of $\underline{V}$ is an easy part, since we are able to take a bounded corrector (bounded critical subsolution) u to define an appropriate perturbed test function. While, the part of subsolution of $\overline{V}$ is much more difficult than the former one, due to the lack of boundedness of a corrector which is a critical supersolution. To overcome this drawback, some additional technicalities are needed and these constructions are settled in subsections 3.2, 3.3 of the paper (in the Appendix).

The thesis is organized as follows:

In Chapter 1, we present some basic materials we need in the whole thesis, where the viscosity solution theory for first order Hamilton-Jacobi equations and optimal control problems are used throughout the thesis.

Chapter 2 is devoted to the study of asymptotic behaviour of singularly perturbed control system in the periodic setting. Some main results in classical periodic homogenization for Hamilton-Jacobi equation are also reviewed in this chapter.

The last chapter (and also the paper added in the Appendix) is the main contribution of the author, where we extend the results in chapter 2 to the non-periodic (non-compact) setting.

Chapter 1

Viscosity solutions and optimal control problems

1.1 Some basic tools

We briefly review in this section some basic tools from convex and nonsmooth analysis. For more details about the proofs, we refer the readers to [BCD97], [CS04], [C83], [C13] and references therein.

1.1.1 Semiconcavity, weak semilimits

Definition 1.1. Let $\Omega \subset \mathbb{R}^N$ be an open convex subset. A function $u : \Omega \longrightarrow \mathbb{R}$ is called *semiconcave* in Ω if one of the following equivalent conditions is valid, for some constant $C \geq 0$:

a) $x \longmapsto u(x) - C|x|^2$ is concave[1];

b) $x \longmapsto u(x) - C|x - x_0|^2$ is concave, for some $x_0 \in \Omega$;

c) $u\big[\lambda x + (1-\lambda)y\big] \geq \lambda u(x) + (1-\lambda)u(y) - C\lambda(1-\lambda)|x-y|^2$, for any $x, y \in \Omega$ and any $\lambda \in [0,1]$.

[1] recall that a function $w : \Omega \longrightarrow \mathbb{R}$ is called concave in the open convex subset $\Omega \subset \mathbb{R}^N$ if

$$w\big[\lambda x + (1-\lambda)y\big] \geq \lambda w(x) + (1-\lambda)w(y) \quad \text{for any } x, y \in \Omega \text{ and any } \lambda \in [0,1].$$

We refer to C as a *semiconcavity constant* for u in Ω. The function u is called *semiconvex* in Ω if $-u$ is semiconcave in Ω.

Some main properties of locally Lipschitz continuous function, semiconcave (or semiconvex) function which will be useful in the sequel are summarized in Proposition 1.2, Theorem 1.3, see for instance, [BCD97], [CS04] and references therein.

Proposition 1.2. *Let $\Omega \subset \mathbb{R}^N$ be an open convex subset. If u is semiconcave (or semiconvex) in Ω, then u is locally Lipschitz continuous in Ω.*

Theorem 1.3 (Rademacher' theorem)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open subset. If $u : \Omega \longrightarrow \mathbb{R}$ is locally Lipschitz continuos in Ω, then u is differentiable almost everywhere in Ω.*

We are going to introduce a notion of weak semilimits (also called relaxed semilimits) which will be useful in some asymptotic problems we will deal with in the next chapters.

Definition 1.4. Let $E \subset \mathbb{R}^N$ be a subset and consider the family of functions $u^\varepsilon : E \to \mathbb{R}$, for $\varepsilon > 0$.

a) The *upper semilimit* in E as $\varepsilon \to 0$ at the point $x \in E$ is defined by

$$u^*(x) \equiv \limsup{}^* u^\varepsilon(x) := \limsup_{(y,\varepsilon)\to(x,0)} u^\varepsilon(y);$$

b) The *lower semilimit* in E as $\varepsilon \to 0$ at the point $x \in E$ is defined by

$$u_*(x) \equiv \liminf{}_* u^\varepsilon(x) := \liminf_{(y,\varepsilon)\to(x,0)} u^\varepsilon(y).$$

Upper semilimit and lower semilimit are called weak semilimits.

Remark 1.5. Following the definition of lim sup and lim inf, we have

$$u^*(x) = \limsup_{(y,\varepsilon)\to(x,0^+)} u^\varepsilon(y) = \inf_{\delta>0} \sup\{u^\varepsilon(y) : \; y \in E, \, |x-y| < \delta \text{ and } 0 < \varepsilon < \delta\};$$

$$u_*(x) = \liminf_{(y,\varepsilon)\to(x,0^+)} u^\varepsilon(y) = \sup_{\delta>0} \inf\{u^\varepsilon(y) : \; y \in E, \, |x-y| < \delta \text{ and } 0 < \varepsilon < \delta\}.$$

If the sequence u^ε, $\varepsilon > 0$, is locally equibounded in E, that is for any compact subset $K \subset E$, there is a constant C_K (independent of ε) such that

$$\big|u^\varepsilon(x)\big| \leq C_K \quad \forall x \in K, \forall \varepsilon > 0, \tag{1.1}$$

then the weak semilimits are finite and define two functions $u^*, u_* : E \to \mathbb{R}$. The main properties of these functions are stated in the following lemmata (see [BCD97]).

Lemma 1.6. *The upper semilimit u^* and lower semilimit u_* are, respectively, upper semicontinuous and lower semicontinuous in E.*

Lemma 1.7. *Assume u^ε satisfies (1.1) on a compact set K and*

$$u^* = u_* := u \quad \textit{in } K.$$

Then u^ε uniformly converges to u in K, as $\varepsilon \to 0$.

We can also define weak semilimits of a sequence of functions in an alternate way which are useful in some asymptotic problems, as follows:

Definition 1.8. Assume $u^\varepsilon : E \to \mathbb{R}$, $\varepsilon > 0$, be a sequence of locally equibounded functions. The weak sup-semilimit and weak inf-semilimit of u^ε, as $\varepsilon \to 0$, are defined, respectively, by

$$\overline{u}(x) \equiv \limsup{}^{\#} u^\varepsilon(x) := \sup \big\{ \limsup_{\varepsilon \to 0} u^\varepsilon(x_\varepsilon) \,:\, x_\varepsilon \to x \big\},$$
$$\underline{u}(x) \equiv \liminf{}_{\#}\, u^\varepsilon(x) := \inf \big\{ \liminf_{\varepsilon \to 0} u^\varepsilon(x_\varepsilon) \,:\, x_\varepsilon \to x \big\}.$$

Note that, the supremum and infimum in the above definition are taken overall possible sequences x_ε converging to x, as $\varepsilon \to 0$.

Similar to Lemma 1.6, it is possible to prove that the weak sup-semilimit $\overline{u}$ and weak inf-semilimit $\underline{u}$ are, respectively, upper semicontinuous and lower semicontinuous in E.

Remark 1.9. If u is a locally bounded function and we take in the above formula the sequence u^ε constantly equal to u then we get through upper (resp. lower) weak semilimit the upper (resp. lower) semicontinuous envelope of u, denoted by $u^\#$ (resp. $u_\#$). It is minimal (resp. maximal) upper (resp. lower) semicontinuous function greater (resp. less) than or equal to u.

1.1.2 Semidifferentials

We now introduce the *semidifferentials* of a function which will be important in our later analysis.

Definition 1.10. Given a function $u : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^N$ be an open subset and $x \in \Omega$.

a) The *superdifferential* of u at x is defined by

$$D^+u(x) := \Big\{ p \in \mathbb{R}^N : \limsup_{y \to x,\, y \in \Omega} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \le 0 \Big\}.$$

b) The *subdifferential* of u at x is defined by

$$D^-u(x) := \Big\{ p \in \mathbb{R}^N : \liminf_{y \to x,\, y \in \Omega} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \ge 0 \Big\}.$$

The superdifferential and subdifferential are called *semidifferentials.*

From the definition it follows that, for any $x \in \Omega$

$$D^-(-u)(x) = -D^+u(x).$$

Some basic properties of superdifferential and subdifferential are collected in the following lemma.

Lemma 1.11. *Let $u \in \mathrm{C}(\Omega)$ and $x \in \Omega$. Then,*

a) *$D^+u(x)$ and $D^-u(x)$ are closed convex (possibly empty) subsets of $\mathbb{R}^N$;*

b) *if u is differentiable at x, then $\{Du(x)\} = D^+u(x) = D^-u(x)$;*

c) *if for some x both $D^+u(x)$ and $D^-u(x)$ are nonempty, then u is differentiable at x, and*

$$D^+u(x) = D^-u(x) = \{Du(x)\};$$

d) *the following sets are dense in Ω*

$$A^+ = \big\{ x \in \Omega : D^+u(x) \ne \emptyset \big\},$$

$$A^- = \big\{ x \in \Omega : D^-u(x) \ne \emptyset \big\}.$$

1.1.3 Generalized gradient

One of the most important concepts in nonsmooth analysis is *generalized gradient* (also called *Clarke's gradient*). This is a very useful tool in optimal control theory and PDEs. These materials are well-known now, so we omit the proofs. We refer the reader to the books by Clarke [C83], [C13], and by Bardi & Capuzzo-Dolcetta [BCD97] for more details.

We first present some fundamental facts which motivate the notion of generalized gradient.

Definition 1.12 (support function)**.** Let $C \subset \mathbb{R}^N$ be a nonempty subset. The *support function* of C is defined by

$$\sigma_C(v) := \sup \{ p\,v \,:\, p \in C \}, \quad v \in \mathbb{R}^N.$$

The following result is well-known:

Proposition 1.13. *Let C, D be nonempty, closed, convex subsets of $\mathbb{R}^N$. Then, $C \subset D$ if and only if $\sigma_C(v) \le \sigma_D(v)$ for any $v \in \mathbb{R}^N$.*

The above proposition claims an important fact that closed convex sets are characterized by their support functions.

We recall that a function $\sigma : \mathbb{R}^N \to \mathbb{R}$ is called *positive homogeneous* if

$$\sigma(\lambda x) = \lambda \sigma(x), \quad \forall \lambda \ge 0, \forall x \in \mathbb{R}^N;$$

and is called *subadditive* if

$$\sigma(x+y) \le \sigma(x) + \sigma(y), \quad \forall x, y \in \mathbb{R}^N.$$

Proposition 1.14. *Given any positive homogeneous, subadditive function $\sigma : \mathbb{R}^N \to \mathbb{R}$, there exists a unique compact convex subset C of $\mathbb{R}^N$ such that σ is a support function of C, that is $\sigma = \sigma_C$.*

Definition 1.15 (generalized directional derivative)**.** Let $\Omega \subset \mathbb{R}^N$ be an open subset and $u : \Omega \to \mathbb{R}$ be a locally Lipschitz continuous function. The *generalized* *directional derivative* of u at $x \in \Omega$ in the direction $v \in \mathbb{R}^N$, denoted $u^0(x; v)$, is defined by

$$u^0(x; v) := \limsup_{y \to x,\, t \to 0} \frac{u(y + tv) - u(y)}{t},$$

where $y \in \Omega$ and t is a positive real number.

Proposition 1.16. *Let $\Omega \subset \mathbb{R}^N$ be an open subset and $u : \Omega \to \mathbb{R}$ be locally Lipschitz continuous. Then, for any $x \in \Omega$, the function $v \mapsto u^0(x; v)$ is positive homogeneous, subadditive in $\mathbb{R}^N$.*

We see that, for any $x \in \Omega$, the function $v \mapsto u^0(x; v)$ is positive homogeneous, subadditive in $\mathbb{R}^N$, hence, by Proposition 1.14, there exists a unique compact convex subset of $\mathbb{R}^N$ for which it is a support function. We call this compact convex subset is the *generalized gradient* of u at x. More precisely, we have the following definition.

Definition 1.17 (generalized gradient)**.** Let $\Omega \subset \mathbb{R}^N$ be an open subset and $u : \Omega \to \mathbb{R}$ be locally Lipschitz continuous. The *generalized gradient* (also called *Clarke's gradient*) of u at $x \in \Omega$, is defined by

$$\partial u(x) := \big\{\, p \in \mathbb{R}^N : \, u^0(x; v) \geq p \cdot v, \; \forall v \in \mathbb{R}^N \big\}.$$

Since $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, by classical Rademacher's theorem, u is differentiable almost everywhere with locally bounded gradient. Hence, the set

$$D^* u(x) := \big\{\, p \in \mathbb{R}^N : \, p = \lim_{n \to \infty} Du(x_n), \, x_n \to x, \, u \text{ is differentiable at } x_n \big\}$$

is nonempty and closed for any $x \in \Omega$. This set is referred to as *the set of reachable gradients* of u at x. Denote by $\mathrm{co}\, D^* u(x)$ its convex hull. It is a well-known result in nonsmooth analysis that

$$\mathrm{co}\, D^* u(x) = \partial u(x), \quad \forall x \in \Omega,$$

thus

$$\partial u(x) = \mathrm{co} \big\{\, p \in \mathbb{R}^N : \, p = \lim_{n \to \infty} Du(x_n), \, x_n \to x, \, u \text{ is differentiable at } x_n \big\}.$$

Some basic properties of *generalized gradient* are collected in the following proposition.

Proposition 1.18. *Let $u : \Omega \to \mathbb{R}$ be locally Lipschitz continuous in the open set $\Omega \subset \mathbb{R}^N$. The following hold*

a) *if $\partial u(x) = \{p\}$, a singleton, then u is strictly differentiable at x, in the sense that u is differentiable and Du is continuous at x. In this case, $Du(x) = p$.*

b) *let $x_n \in \Omega$ and $p_n \in \mathbb{R}^N$ be sequences such that $p_n \in \partial u(x_n)$ for any $n \in \mathbb{N}^*$. Suppose that $x_n \to x$ and $p_n \to p$, as $n \to \infty$. Then one has $p \in \partial u(x)$, that is the set-valued map $\partial u(\cdot)$ is closed, in the sense that its graph is closed in $\Omega \times \mathbb{R}^N$;*

c) *the set-valued map $\partial u(\cdot)$ is upper semicontinuous in $\mathbb{R}^N$.*

In Lemma 1.11 we have known some basic properties of semidifferentials of continuous function. In addition, for locally Lipschitz continuous function, its semidifferentials are bounded and are subsets of generalized gradient, as follows.

Proposition 1.19. *Let $u : \Omega \to \mathbb{R}$ be locally Lipschitz continuous in the open set $\Omega \subset \mathbb{R}^N$. Then for all $x \in \Omega$*

a) *$D^+u(x)$ and $D^-u(x)$ are bounded in $\mathbb{R}^N$;*

b) *$D^+u(x) \subset \partial u(x)$ and $D^-u(x) \subset \partial u(x)$.*

Remark 1.20. i) Note that if x is a local maximum of u then $0 \in D^+u(x)$, and if x is a local minimum of u then $0 \in D^-u(x)$. Hence, by the above proposition, we get the variational property for generalized gradient:

$$0 \in \partial u(x) \quad \text{at any local maximum or minimum } x \text{ of } u.$$

ii) If $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and $\varphi \in \mathrm{C}^1(\Omega)$ then

$$\partial(u - \varphi)(x) = \partial u(x) - D\varphi(x), \quad \forall x \in \Omega.$$

We have already known that $D^+u(x)$ and $D^-u(x)$ are subsets of $\partial u(x)$ for any $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$. If, in addition, u is semiconcave (respectively, semiconvex), it turns out that $D^+u(x) = \partial u(x)$ (respectively, $D^-u(x) = \partial u(x)$). These important facts are stated in the following propositions:

Proposition 1.21. *Let u be semiconcave in the open set Ω. Then for all $x \in \Omega$*

a) *$D^+u(x) = \partial u(x)$;*

b) *either $D^-u(x) = \emptyset$ or u is differentiable at x;*

Proposition 1.22. *Let u be semiconvex in the open set Ω. Then for all $x \in \Omega$*

a) *$D^-u(x) = \partial u(x)$;*

b) *either $D^+u(x) = \emptyset$ or u is differentiable at x;*

We conclude the section by presenting the following extensions of the classical mean value theorem and chain rule.

Theorem 1.23 (mean value theorem)**.** *Let $x, z \in \mathbb{R}^N$ and suppose that $u : \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous in a neighbourhood of the line segment $[x, z]$. Then there exists a point ξ in (x, z) such that*

$$u(z) - u(x) \in \partial u(\xi) \cdot (z - x), \tag{1.2}$$

equivalently, there exists $p \in \partial u(\xi)$ such that

$$u(z) - u(x) = p \cdot (z - x).$$

Theorem 1.24 (chain rule)**.** *Assume $u : \mathbb{R}^N \to \mathbb{R}$ is locally Lipschitz continuous, and $\xi : [0, T] \to \mathbb{R}^N$ is Lipschitz continuous in $[0, T]$, for any given $T > 0$. Then, the function $u \circ \xi$ is Lipschitz continuous in $[0, T]$, and for almost everywhere $t \in [0, T]$, we have*

$$\partial(u \circ \xi)(t) \subset \partial u\big(\xi(t)\big) \cdot \xi'(t). \tag{1.3}$$

1.1.4 Sup and inf convolutions

A convenient way to approximate a given function by a semiconvex function (respectively, semiconcave function) is provided by the method of sup-convolution (respectively, inf-convolution). This kind of procedure was introduced by J. M. Lasry and P. L. Lions [LL86] in the context of Hamilton-Jacobi equations and has become a very useful tool for regularizing or analyzing viscosity solutions. The results we prove in this subsection will be of crucial important in asymptotic analysis of chapter 3.

Definition 1.25. Let $u : \mathbb{R}^N \longrightarrow \mathbb{R}$ be bounded from above and $v : \mathbb{R}^N \longrightarrow \mathbb{R}$ be bounded from below. We define, for any $\varepsilon > 0$,

$$u^\varepsilon(x) := \sup_{y\in\mathbb{R}^N} \Big\{ u(y) - \frac{1}{2\varepsilon}|x-y|^2 \Big\}, \quad x \in \mathbb{R}^N, \tag{1.4}$$

$$v_\varepsilon(x) := \inf_{y\in\mathbb{R}^N} \Big\{ v(y) + \frac{1}{2\varepsilon}|x-y|^2 \Big\}, \quad x \in \mathbb{R}^N. \tag{1.5}$$

The functions u^ε and v_ε are called, respectively, sup-convolution of u and inf-convolution of v.

Note that, for any $\varepsilon > 0$ and $x \in \mathbb{R}^N$

$$v_\varepsilon(x) = -\sup_{y\in\mathbb{R}^N} \Big\{ -v(y) - \frac{1}{2\varepsilon}|x-y|^2 \Big\} = -(-v)^\varepsilon(x).$$

This relation implies that the property of inf-convolution can be obtained from the corresponding property of sup-convolution. Thus we only need to deal with sup-convolution.

It is clear that $u^\varepsilon \geq u$ for any $\varepsilon > 0$, and u^ε is a nonincresing sequence in ε.

In what follows we assume in addition that u is upper semicontinuous in $\mathbb{R}^N$. The supremum in the formula (1.4) is therefore actually a maximum, namely

$$u^\varepsilon(x) = \max_{y\in\mathbb{R}^N} \Big\{ u(y) - \frac{1}{2\varepsilon}|x-y|^2 \Big\}, \quad x \in \mathbb{R}^N.$$

In deed, the maximum does exist in force of upper semicontinuity and boundedness from above of u.

For given $x \in \mathbb{R}^N$ and $\varepsilon > 0$. An element $y \in \mathbb{R}^N$ such that

$$u^\varepsilon(x) = u(y) - \frac{1}{2\varepsilon}|x-y|^2$$

is called u^ε-optimal for x. The set of such u^ε-optimal points is denoted by

$$\begin{aligned} \mathcal{M}^\varepsilon(x) &= \big\{ y \mid y \text{ is } u^\varepsilon\text{-optimal for } x \big\} \\ &= \arg\max \Big\{ u(y) - \frac{1}{2\varepsilon}|x-y|^2 \ : \ y \in \mathbb{R}^N \Big\}. \end{aligned}$$

Some basic properties of sup-convolution are following.

Lemma 1.26. *Let $u : \mathbb{R}^N \longrightarrow \mathbb{R}$ be upper semicontinuous and bounded. Then*

a) *the sup-convolutions* u^ε*, for any* $\varepsilon > 0$*, are semiconvex in* $\mathbb{R}^N$*;*

b) *the sup-convolutions* u^ε*, for any* $\varepsilon > 0$*, are equi-bounded in* $\mathbb{R}^N$*.*

c) $u^\varepsilon \to u$*, as* $\varepsilon \to 0$*, in the upper semilimit sense, that is*

$$u^*(x) := \limsup{}^* u^\varepsilon(x) = u(x), \quad \textit{for any } x \in \mathbb{R}^N.$$

In addition, $u^\varepsilon \to u$*, as* $\varepsilon \to 0$*, pointwise in* $\mathbb{R}^N$*, and the convergence will be locally uniform if* $u \in \mathrm{C}(\mathbb{R}^N)$*;*

Proof. a) We have,

$$u^\varepsilon(x) = \max_{y \in \mathbb{R}^N} \big\{ u(y) - \frac{1}{2\varepsilon}\, x^2 - \frac{1}{2\varepsilon}\, y^2 + \frac{1}{\varepsilon}\, x \cdot y \big\},$$

this yields

$$u^\varepsilon(x) + \frac{1}{2\varepsilon}\, x^2 = \max_{y \in \mathbb{R}^N} \big\{ u(y) - \frac{1}{2\varepsilon}\, y^2 + \frac{1}{\varepsilon}\, x \cdot y \big\}.$$

For each fix $y \in \mathbb{R}^N$, the function

$$x \longmapsto u(y) - \frac{1}{2\varepsilon}\, y^2 + \frac{1}{\varepsilon}\, x \cdot y$$

is affine, thus convex. Hence the function $u^\varepsilon(x) + \frac{1}{2\varepsilon}\, x^2$ is convex. This implies $u^\varepsilon(x)$ is semiconvex.

b) For any $x \in \mathbb{R}^N$ and $\varepsilon > 0$, let y be u^ε-optimal for x, then

$$-R \le u(x) \le u^\varepsilon(x) = u(y) - \frac{1}{2\varepsilon}|x - y|^2 \le u(y) \le R,$$

where $R > 0$ be constant such that

$$|u(x)| \le R, \quad \text{for any } x \in \mathbb{R}^N.$$

This shows the equi-boundedness of u^ε in $\mathbb{R}^N$.

c) Fix $x_0 \in \mathbb{R}^N$, we will show that

$$u^*(x_0) := \limsup{}^* u^\varepsilon(x_0) = u(x_0).$$

By the definition of sup-convolution,

$$u^\varepsilon(x_0) \geq u(x_0), \quad \text{for any } \varepsilon > 0,$$

hence

$$u^*(x_0) := \limsup{}^* u^\varepsilon(x_0) = \limsup_{(y,\varepsilon)\to(x_0,0)} u^\varepsilon(y) \geq \limsup_{\varepsilon\to 0} u^\varepsilon(x_0) \geq u(x_0).$$

We are left to prove $u^*(x_0) \leq u(x_0)$. Let $x_\varepsilon \to x_0$ as $\varepsilon \to 0$ such that

$$u^*(x_0) = \lim_{\varepsilon\to 0} u^\varepsilon(x_\varepsilon).$$

We take $y_\varepsilon \in \mathbb{R}^N$ be u^ε-optimal for x_ε, then

$$u^\varepsilon(x_\varepsilon) = u(y_\varepsilon) - \frac{1}{2\varepsilon}|x_\varepsilon - y_\varepsilon|^2 \leq u(y_\varepsilon).$$

We can check that $|x_\varepsilon - y_\varepsilon| \to 0$ as $\varepsilon \to 0$ (see Lemma 1.30 (c) below), so that $y_\varepsilon \to x_0$ as $\varepsilon \to 0$.

By the above estimates and the upper semicontinuity of u, we obtain

$$u^*(x_0) = \lim_{\varepsilon\to 0^+} u^\varepsilon(x_\varepsilon) \leq \lim_{\varepsilon\to 0^+} u(y_\varepsilon) \leq \limsup_{\varepsilon\to 0^+} u(y_\varepsilon) \leq u(x_0).$$

In order to prove the pointwise convergence, we take $\{x_\varepsilon\} \equiv \{x_0\}$ for all $\varepsilon > 0$. Then by the previous estimates, we get

$$\limsup_{\varepsilon\to 0} u^\varepsilon(x_0) \leq u^*(x_0) \leq u(x_0),$$

and

$$\liminf_{\varepsilon\to 0^+} u^\varepsilon(x_0) \geq u(x_0).$$

This implies

$$\liminf_{\varepsilon\to 0} u^\varepsilon(x_0) = \limsup_{\varepsilon\to 0} u^\varepsilon(x_0) = \lim_{\varepsilon\to 0} u^\varepsilon(x_0) = u(x_0).$$

Note that for each fixed $x_0 \in \mathbb{R}^N$, the sequence $u^\varepsilon(x_0)$ is nonincreasing. Hence, if $u \in \mathrm{C}(\mathbb{R}^N)$ the locally uniform convergence follows from classical Dini's theorem.

□

Remark 1.27. For the case of weak sup-semilimit $\overline{u}$ (see Definition 1.8), we get the similar statement as in item (c) of the above lemma. Namely, $u^\varepsilon \to u$, as $\varepsilon \to 0$, in the weak sup-semilimit sense, i.e.,

$$\overline{u}(x) := \limsup{}^{\#} u^\varepsilon(x) = u(x), \quad x \in \mathbb{R}^N.$$

In addition, $u^\varepsilon \to u$, as $\varepsilon \to 0$, pointwise in $\mathbb{R}^N$, and the convergence will be locally uniform if $u \in \mathrm{C}(\mathbb{R}^N)$;

We are going to show a continuity property for u^ε-optimal points.

Lemma 1.28. *Assume x_n be a sequence converging to x_0, as $n \to +\infty$. If, for any $n \in \mathbb{N}^*$, $y_n \in \mathcal{M}^\varepsilon(x_n)$ and $y_n \to y_0$, as $n \to +\infty$. Then $y_0 \in \mathcal{M}^\varepsilon(x_0)$ and $u(y_n) \to u(y_0)$, as $n \to +\infty$.*

Proof. We have, for any $n \in \mathbb{N}^*$,

$$u^\varepsilon(x_n) = u(y_n) - \frac{1}{2\varepsilon}|x_n - y_n|^2.$$

Passing to the limit as $n \to +\infty$, and taking into account that u^ε is continuous and u is upper semi-continuous, we get

$$u^\varepsilon(x_0) \le u(y_0) - \frac{1}{2\varepsilon}|x_0 - y_0|^2.$$

Moreover,

$$u^\varepsilon(x_0) = \sup_y \left\{u(y) - \frac{1}{2\varepsilon}|x_0 - y|^2\right\} \ge u(y_0) - \frac{1}{2\varepsilon}|x_0 - y_0|^2.$$

Conclusion,

$$u^\varepsilon(x_0) = u(y_0) - \frac{1}{2\varepsilon}|x_0 - y_0|^2,$$

that means $y_0 \in \mathcal{M}^\varepsilon(x_0)$. The convergence of $u(y_n)$ to $u(y_0)$ is then clearly. □

Lemma 1.29. *Fix $\varepsilon > 0$, then for any x_0, we have*

$$\partial u^\varepsilon(x_0) = \mathrm{co}\left\{\frac{y_0 - x_0}{\varepsilon} \mid y_0 \in \mathcal{M}^\varepsilon(x_0)\right\}.$$

Consequently, u^ε is differentiable at x_0 if and only if $\mathcal{M}^\varepsilon(x_0) = \{y_0\}$, and in this case $Du^\varepsilon(x_0) = \dfrac{y_0 - x_0}{\varepsilon}$.

Proof. Let us fix x_0 and take $y_0 \in \mathcal{M}^\varepsilon(x_0)$. Note that, by definition of u^ε, the function

$$\varphi(x) := u(y_0) - \frac{1}{2\varepsilon}|x - y_0|^2$$

is subtangent to u^ε at x_0, thus

$$\frac{y_0 - x_0}{\varepsilon} = D\varphi(x_0) \in D^- u^\varepsilon(x_0) = \partial u^\varepsilon(x_0).$$

Since the generalized gradient is a convex set, then

$$\partial u^\varepsilon(x_0) \supset \text{co}\left\{\frac{y_0 - x_0}{\varepsilon} \mid y_0 \in M^\varepsilon(x_0)\right\}.$$

Take a sequence $x_n \to x_0$ so that u^ε is differentiable at x_n and $Du^\varepsilon(x_n)$ is convergent, then

$$Du^\varepsilon(x_n) = \frac{1}{\varepsilon}(y_n - x_n), \quad \forall n \in \mathbb{N}^*,$$

where $y_n \in \mathcal{M}^\varepsilon(x_n)$. By Lemma 1.28, we get

$$\lim_{n\to\infty} Du^\varepsilon(x_n) = \frac{y_0 - x_0}{\varepsilon}$$

for some y_0 which is u^ε-optimal for x_0. By the definition of generalized gradient, we get

$$\partial u^\varepsilon(x_0) \subset \text{co}\left\{\frac{y_0 - x_0}{\varepsilon} \mid y_0 \in \mathcal{M}^\varepsilon(x_0)\right\}.$$

The proof is complete. □

Lemma 1.30. *Let $u : \mathbb{R}^N \longrightarrow \mathbb{R}$ be upper semicontinuous and bounded. We set $R := \sup_{\mathbb{R}^N} |u|$. Then the following statements hold true*

a) $D^- u^\varepsilon(x) = \partial u^\varepsilon(x)$;

b) *either* $D^+ u^\varepsilon(x) = \emptyset$ *or* $D^+ u^\varepsilon(x) = \{\frac{y_\varepsilon - x}{\varepsilon}\}$, *where* $\{y_\varepsilon\} = \mathcal{M}^\varepsilon(x)$;

c) *for any* $y_\varepsilon \in \mathcal{M}^\varepsilon(x)$, *we have*

   i) $|y_\varepsilon - x| \leq 2\sqrt{R\varepsilon}$;

   ii) $\frac{|y_\varepsilon - x|^2}{\varepsilon} \to 0$, *as* $\varepsilon \to 0$ *pointwise in* $\mathbb{R}^N$. *If, in addition* $u \in \mathrm{C}(\mathbb{R}^N)$, *the convergence will be locally uniform in* $\mathbb{R}^N$;

   iii) $\frac{y_\varepsilon - x}{\varepsilon} \in D^+ u(y_\varepsilon)$.

Proof. Since u^ε is semiconvex in $\mathbb{R}^N$, then the statements (a) and (b) follow from Proposition 1.22 and Lemma 1.29.

We are going to prove (c). Let $x \in \mathbb{R}^N$ and $y_\varepsilon \in \mathcal{M}^\varepsilon(x)$, we have

$$u(x) \le u^\varepsilon(x) = u(y_\varepsilon) - \frac{1}{2\varepsilon}|y_\varepsilon - x|^2.$$

This yields

$$\frac{1}{2\varepsilon}|y_\varepsilon - x|^2 \le u(y_\varepsilon) - u(x). \tag{1.6}$$

Taking into account $R := \sup\limits_{\mathbb{R}^N} |u|$, one has

$$|y_\varepsilon - x|^2 \le 4R\varepsilon,$$

and then we get (i).

Again, from (1.6), one has

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}|y_\varepsilon - x|^2 \le \limsup_{\varepsilon\to 0} \frac{1}{2\varepsilon}|y_\varepsilon - x|^2 \le \limsup_{\varepsilon\to 0} u(y_\varepsilon) - u(x).$$

Since $y_\varepsilon \to x$, as $\varepsilon \to 0$, by the above item (i), and $u \in \mathrm{USC}\,(\mathbb{R}^N)$, then

$$\limsup_{\varepsilon\to 0} u(y_\varepsilon) \le u(x).$$

Therefore, $\frac{|y_\varepsilon - x|^2}{\varepsilon} \to 0$, as $\varepsilon \to 0$ pointwise in $\mathbb{R}^N$.

If $u \in \mathrm{C}(\mathbb{R}^N)$, then $u \in \mathrm{UC}\,(K)$ for any compact subset $K \subset \mathbb{R}^N$. Denote by ω an uniform modulus of u in K. From (1.6), one has

$$\frac{1}{2\varepsilon}|y_\varepsilon - x|^2 \le \omega\big(|y_\varepsilon - x|\big) \le \omega\big(2\sqrt{R\varepsilon}\big), \quad \forall x \in K.$$

That means $\frac{|y_\varepsilon - x|^2}{\varepsilon} \to 0$, as $\varepsilon \to 0^+$ uniformly on compact subsets of $\mathbb{R}^N$. Conclusion, (ii) is proved.

About the statement (iii), consider the function of two variables

$$g(x,y) := u(y) - \frac{1}{2\varepsilon}|y - x|^2.$$

Since $g(x,\cdot)$ attains a maximum at y_ε, then

$$0 \in D_y^+ g(x,y_\varepsilon) = D^+u(y_\varepsilon) - \frac{1}{\varepsilon}(y_\varepsilon - x).$$

This yields

$$\frac{y_\varepsilon - x}{\varepsilon} \in D^+u(y_\varepsilon).$$

□

Remark 1.31. From Lemma 1.29 and Lemma 1.30 (c), we deduce an important fact, as follows: fix $\varepsilon > 0$ and assume x_0 is a differentiability point of u^ε. Let y_0 be the unique u^ε-optimal point for x_0, then

$$Du^\varepsilon(x_0) \in D^+u(y_0).$$

The above relation shows how viscosity test information is transferred from u to the sup-convolution u^ε.

We end this subsection by proving the following result which will be useful for comparison result in chapter 3.

Proposition 1.32. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded subset, u and w be an bounded upper semicontinuous and a lower semicontinuous function defined in $\overline{\Omega}$, respectively. We denote, for any $\varepsilon > 0$, by u^ε the sup-convolution of u in $\overline{\Omega}$, namely,*

$$u^\varepsilon(x) := \sup_{y\in\overline{\Omega}} \left\{ u(y) - \frac{1}{2\varepsilon}|x-y|^2 \right\}, \quad x \in \overline{\Omega}.$$

Assume, for each $\varepsilon > 0$, x_ε be a maximizer of $u^\varepsilon - w$ in $\overline{\Omega}$, and y_ε be an u^ε-optimal point for x_ε. We set

$$M_\varepsilon := \max_{\overline{\Omega}}(u^\varepsilon - w), \quad M_0 := \max_{\overline{\Omega}}(u - w).$$

Then

(a) $M_\varepsilon \to M_0$, *as* $\varepsilon \to 0$;

(b) $\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} \to 0$, *as* $\varepsilon \to 0$.

Proof. (a) The sequence u^ε decreases with respect to ε, and $u^\varepsilon \geq u$, for any ε. Therefore, by monotonicity, $M_\varepsilon := \max_{\overline{\Omega}}(u^\varepsilon - w)$ does converges, as $\varepsilon \to 0$, and we have

$$\lim_{\varepsilon\to 0} M_\varepsilon \geq M_0 := \max_{\overline{\Omega}}(u-w).$$

We can assume, without loss of generality, that x_ε converges to x_0, as $\varepsilon \to 0$, for some $x_0 \in \overline{\Omega}$. Taking into account that w is lower semicontinuous and the relation

$$\limsup{}^{\#} u^\varepsilon(x) = u(x), \quad x \in \overline{\Omega},$$

we get

$$\begin{aligned} M_0 \leq \limsup_{\varepsilon\to 0} M_\varepsilon &= \limsup_{\varepsilon\to 0} \left[u^\varepsilon(x_\varepsilon) - w(x_\varepsilon)\right] \\ &\leq \limsup_{\varepsilon\to 0} u^\varepsilon(x_\varepsilon) - \liminf_{\varepsilon\to 0} w(x_\varepsilon) \\ &\leq \limsup{}^{\#} u^\varepsilon(x_0) - w(x_0) \\ &= u(x_0) - w(x_0). \end{aligned}$$

This show, at the same time, that x_0 is a maximizer of $u-w$ in $\overline{\Omega}$ and that $M_\varepsilon \to M_0$, as $\varepsilon \to 0$, as claimed.

(b) We have

$$M_\varepsilon = u^\varepsilon(x_\varepsilon) - w(x_\varepsilon) = u(y_\varepsilon) - \frac{1}{2\varepsilon}|x_\varepsilon - y_\varepsilon|^2 - w(x_\varepsilon).$$

Therefore,

$$\limsup_{\varepsilon\to 0} \frac{1}{2\varepsilon}|x_\varepsilon - y_\varepsilon|^2 = \limsup_{\varepsilon\to 0} \left[u(y_\varepsilon) - w(x_\varepsilon) - M_\varepsilon\right].$$

We know by Lemma 1.30 (c), that x_ε and y_ε have the same limit points, and moreover $u-w$ is upper semicontinuous, hence

$$\limsup_{\varepsilon\to 0} \left[u(y_\varepsilon) - w(x_\varepsilon)\right] \leq M_0.$$

We also know, by item (a), that $M_\varepsilon \to M_0$, as $\varepsilon \to 0$. Hence we can conclude that $\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} \to 0$, as $\varepsilon \to 0$. □

1.2 Viscosity solutions

This section is devoted to the basic theory of viscosity solution. The theory of viscosity solutions first appeared in the 80's by the works of M. G. Crandall, P. L. Lions [CL83], M. G. Crandall, L. C. Evans [CEL84]. It provides a very convenient PDE framework for dealing with the lack of smoothness of the value functions arising in dynamic optimization problems. Connections between optimal control problems and viscosity solutions of Hamilton-Jacobi-Bellman equations are examined in detail in the books of Bardi and Capuzzo Dolcetta [BCD97], Barles [B94], Fleming and Soner [FS06]. In this section, we present some important facts on the theory of viscosity solution that we need in this thesis.

We will be concerned with the first order Hamilton-Jacobi equation of the form

$$H\big(x, u(x), Du(x)\big) = 0 \quad \text{in } \Omega, \tag{HJ}$$

where $\Omega \subset \mathbb{R}^N$ be an open domain, and the Hamiltonian $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function.

We begin with some notations for the spaces of semicontinuous functions on Ω.

$$\begin{aligned} \mathrm{USC}\,(\Omega) &:= \big\{u : \Omega \to \mathbb{R} \mid u \text{ is upper semicontinuous}\big\}, \\ \mathrm{LSC}\,(\Omega) &:= \big\{u : \Omega \to \mathbb{R} \mid u \text{ is lower semicontinuous}\big\}. \end{aligned}$$

1.2.1 Definitions

Definition 1.33. a) A function $u \in \mathrm{USC}\,(\Omega)$ is called *viscosity subsolution* of (HJ) if, for any $\varphi \in \mathrm{C}^1(\Omega)$,

$$H\big(x_0, u(x_0), D\varphi(x_0)\big) \le 0 \tag{1.7}$$

at any local maximum point $x_0 \in \Omega$ of $u - \varphi$.

b) A function $u \in \mathrm{LSC}\,(\Omega)$ is called *viscosity supersolution* of (HJ) if, for any $\varphi \in \mathrm{C}^1(\Omega)$,

$$H\big(x_1, u(x_1), D\varphi(x_1)\big) \ge 0 \tag{1.8}$$

at any local minimum point $x_1 \in \Omega$ of $u - \varphi$.

c) A function $u \in \mathrm{C}(\Omega)$ is called *viscosity solution* of (HJ) if it is simultaneously viscosity sub- and supersolution.

Let us note that the above definition applies well to evolutionary Hamilton-Jacobi equation of the form

$$u_t(t,y) + H\big(t,y,u(t,y),D_y u(t,y)\big) = 0, \quad (t,y) \in (0,T) \times D.$$

Indeed, this equation is reduced to the form (HJ) by changing of variables

$$x = (t,y) \in \Omega = (0,T) \times D \subset \mathbb{R}^{N+1}, \quad \widetilde{H}(x,r,q) = q_{N+1} + H(x,r,q_1,...,q_N)$$

with $q = (q_1,...,q_N,q_{N+1}) \in \mathbb{R}^{N+1}$.

Remark 1.34. In the definition of viscosity subsolution we can always assume that x_0 is a local strict maximum point for $u - \varphi$, otherwise, replace $\varphi(x)$ by $\varphi(x) + |x - x_0|^2$. Moreover, since (1.7) depends only on the value of $D\varphi$ at x_0, it is not restrictive to assume that $u(x_0) = \varphi(x_0)$. Geometrically, this means that the validity of the viscosity subsolution condition (1.7) for u is tested on smooth functions φ "touching from above" the graph of u at x_0. In this case, φ is called *supertangent* to u at x_0. Similar remarks apply of course to the definition of viscosity supersolution, and in this case, φ is called *subtangent* to u at x_1.

The following elementary fact is very useful in many situations in viscosity theory (see Lemma 2.4 in [BCD97]):

Lemma 1.35. *Let $v \in \mathrm{C}(\Omega)$. Suppose that v has a strict local maximum (or minimum) at a point $x_0 \in \Omega$. If $v_n \in \mathrm{C}(\Omega)$ converges locally uniformly to v in Ω, then there exist a sequence $\{x_n\}$ such that $x_n \to x_0$, as $n \to +\infty$, and v_n has a local maximum (or minimum) at x_n.*

The meaning of the above lemma is that, whenever v has a strict local maximum (or minimum) at some point x_0 and $v_n \to v$ locally uniformly, then v_n has a local maximum (or minimum) at a nearby point x_n. In practice, we often consider the case $v := u - \varphi$, where u is a viscosity solution of some Hamilton-Jacobi equation and $\varphi \in \mathrm{C}^1$ is a test function.

The following proposition explains the local character of the notion of viscosity solution and its consistency with the classical pointwise definition.

Proposition 1.36. a) *If $u \in \mathrm{C}(\Omega)$ is a viscosity solution of* (HJ) *in Ω, then u is a viscosity solution of* (HJ) *in Ω', for any open subset $\Omega' \subset \Omega$;*
b) *$u \in \mathrm{C}^1(\Omega)$ is a classical solution of* (HJ) *in Ω if and only if it is a viscosity solution of* (HJ) *in Ω;*

Remark 1.37. In above proposition, statement (a) says that the notion of viscosity solution is a local one. Hence, we can take the test functions in (1.7) and (1.8) in $\mathrm{C}^1(\Omega)$ or in any sufficient small ball $B(x,r)$ centered at $x \in \Omega$.

Before giving an alternative way of defining viscosity solution, we show that the semidifferentials $D^+u(x)$ and $D^-u(x)$ can be described in terms of test functions, as follows.

Lemma 1.38. *Let $u \in \mathrm{C}(\Omega)$. Then*

a) *$p \in D^+u(x)$ if and only if there exists $\psi \in \mathrm{C}^1(\Omega)$ such that $D\psi(x) = p$ and $u - \psi$ has a local maximum at x;*

b) *$p \in D^-u(x)$ if and only if there exists $\varphi \in \mathrm{C}^1(\Omega)$ such that $D\varphi(x) = p$ and $u - \varphi$ has a local minimum at x;*

As a direct consequence of the above lemma, we can give here an equivalent definition of viscosity solution which is sometimes easier to handle than the previous one.

Definition 1.39. a) A function $u \in \mathrm{USC}(\Omega)$ is called *viscosity subsolution* of (HJ) in Ω if

$$H(x, u(x), p) \le 0, \quad \forall x \in \Omega,\ \forall p \in D^+u(x); \tag{1.9}$$

b) A function $u \in \mathrm{LSC}(\Omega)$ is called *viscosity supersolution* of (HJ) in Ω if

$$H(x, u(x), p) \ge 0, \quad \forall x \in \Omega,\ \forall p \in D^-u(x). \tag{1.10}$$

c) A function $u \in \mathrm{C}(\Omega)$ is called *viscosity solution* of (HJ) in Ω if (1.9) and (1.10) hold simultaneously.

Remark 1.40. i) From Remark 1.34 and Lemma 1.38, we can describe the sets $D^+u(x)$ and $D^-u(x)$ in terms of *supertangent* and *subtangent*, as follows.

$$\begin{aligned} D^+u(x) &= \{ q = D\psi(x) \, : \, \psi \text{ is supertangent to } u \text{ at } x \}, \\ D^-u(x) &= \{ p = D\varphi(x) \, : \, \varphi \text{ is subtangent to } u \text{ at } x \}. \end{aligned}$$

ii) Let $u \in \mathrm{C}(\Omega)$ and $\chi \in \mathrm{C}^1(\Omega)$. A function $\psi \in \mathrm{C}^1(\Omega)$ is supertangent (respectively, subtangent) to u at some point x_0 if and only if $\psi - \chi$ is supertangent (respectively, subtangent) to $u - \chi$ at x_0.

1.2.2 Stability, existence and standard comparison results

One of the most useful properties of viscosity solution is stability which allows us to pass to limits even when the Hamilton-Jacobi equation is fully nonlinear. The following stability result plays an important role in viscosity solution theory.

Theorem 1.41. *Let $H_\varepsilon : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $\varepsilon > 0$, be a sequence of continuous functions and H_ε locally uniformly converges to some function $H \in \mathrm{C}\left(\Omega \times \mathbb{R} \times \mathbb{R}^N\right)$. Assume u^ε be a viscosity solution of*

$$H_\varepsilon\big(x, u^\varepsilon, Du^\varepsilon(x)\big) = 0 \quad \text{in } \Omega. \tag{HJ$_\varepsilon$}$$

If u^ε locally uniformly converges on Ω, as $\varepsilon \to 0$, to some function u, then u is a viscosity solution of (HJ) in Ω.

The notion of viscosity solutions were defined through the “vanishing viscosity method”, and this explains the name “viscosity solutions”. The following theorem (in [CEL84]) gives the existence of a viscosity solution via the “vanishing viscosity method”.

Theorem 1.42. *Let $H \in \mathrm{C}(\Omega \times \mathbb{R} \times \mathbb{R}^N)$ and assume for each $\varepsilon > 0$, $u^\varepsilon \in \mathrm{C}^2(\Omega)$ is a classical solution of*

$$-\varepsilon \Delta u^\varepsilon + H(x, u^\varepsilon, Du^\varepsilon) = 0 \quad \text{in } \Omega.$$

If u^ε locally uniformly converges on Ω, as $\varepsilon \to 0$, to some function u, then u is a viscosity solution of

$$H(x, u, Du) = 0 \quad \text{in } \Omega.$$

As stated in Theorem 1.41, the notion of continuous viscosity solution is stabe with respect to the uniform convergence. Now we generalize this stability property to semicontinuous viscosity sub- and supersolutions with respect to *weak semilimits* (Definitions 1.4 and 1.8). These weak semilimits are used extensively to study the convergence of approximation schemes and several asymptotic problems.

Theorem 1.43. *Let $H_\varepsilon : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $\varepsilon > 0$, be a sequence of continuous functions and H_ε locally uniformly converges to some function $H \in \mathrm{C}(\Omega \times \mathbb{R} \times \mathbb{R}^N)$. Assume $u^\varepsilon : \Omega \to \mathbb{R}$ is locally equibounded in Ω.*

a) *If $u^\varepsilon \in \mathrm{USC}(\Omega)$ is a viscosity subsolution of* (HJ_ε) *in Ω, then the upper semilimit $u^* \in \mathrm{USC}(\Omega)$ is a viscosity subsolution of* (HJ) *in Ω;*

b) *if $u^\varepsilon \in \mathrm{LSC}(\Omega)$ is a viscosity supersolution of* (HJ_ε) *in Ω, then the lower semilimit $u_* \in \mathrm{LSC}(\Omega)$ is a viscosity supersolution of* (HJ) *in Ω.*

Remark 1.44. The above stability properties also true for weak sup-semilimit $\overline{u}$ and weak inf-semilimit $\underline{u}$ (see Definition 1.8). More precisely, under the assumptions as in the above theorem, then $\overline{u} \in \mathrm{USC}(\Omega)$ is a viscosity subsolution and $\underline{u} \in \mathrm{LSC}(\Omega)$ is a viscosity supersolution of (HJ) in Ω.

We record here existence and comparison results we will use in the sequel. These results are due to M. G. Grandall and P. L. Lions (in [CL86]).

Let $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a Hamiltonian satisfying the following conditions:

(H1) H is continuous in $\mathbb{R}^N \times \mathbb{R}^N$;

(H2) H is uniformly continuous in $\mathbb{R}^N \times B(0,R)$, for any $R > 0$, i.e., there exists a uniform modulus[2] ω_1 such that

$$\big|H(x,p) - H(y,q)\big| \le \omega_1\big(|x-y| + |p-q|\big) \quad \forall (x,p), (y,q) \in \mathbb{R}^N \times B(0,R);$$

(H3) there exists a uniform modulus ω_2 such that

$$\big|H(x,p) - H(y,p)\big| \le \omega_2\big(|x-y|(1+|p|)\big) \quad \forall x,y,p \in \mathbb{R}^N.$$

Theorem 1.45. *Assume that H satisfies the conditions* (H1)-(H3)*.*

Comparison. *if $u, v \in \mathrm{UC}([0,T] \times \mathbb{R}^N)$ are, respectively, sub- and supersolution of*

$$u_t + H(x, Du) = 0 \quad \textit{in } (0,T) \times \mathbb{R}^N,$$

and

$$u \le v \quad \textit{on } \{t = 0\} \times \mathbb{R}^N,$$

[2] by a uniform modulus we mean a function $\omega : [0, +\infty) \to [0, +\infty)$ such that ω is continuous, nondecreasing and $\omega(0) = 0$.

then

$$u \leq v \quad \textit{in } [0,T] \times \mathbb{R}^N.$$

Existence. *if* $u_0 \in \mathrm{UC}\,(\mathbb{R}^N)$*, then the Cauchy problem*

$$\begin{cases} u_t + H(x, Du) = 0 & \textit{in } (0,T) \times \mathbb{R}^N \\ u(x) = u_0(x) & \textit{on } \mathbb{R}^N. \end{cases} \tag{1.11}$$

has a unique viscosity solution u*, which is uniformly continuous in* $[0,T] \times \mathbb{R}^N$.

Remark 1.46. In the above theorem, if we assume in addition that $u_0 \in \mathrm{BUC}\,(\mathbb{R}^N)$, then the problem (1.11) has a unique viscosity solution $u \in \mathrm{BUC}\,(\mathbb{R}^N)$. We refer the readers to the Lecture notes by G. Barles [B13] for detailed proof of this result.

Theorem 1.47. *Assume that* H *satisfies the conditions* (H1)-(H3).

Comparison. *Let* $\Omega \subset \mathbb{R}^N$ *be an open subset. If* $u, v \in \mathrm{UC}\,(\overline{\Omega})$ *are, respectively, sub- and supersolution of*

$$\lambda u + H(x, Du) = 0, \quad x \in \Omega \quad (\lambda > 0),$$

and

$$u \leq v \quad \textit{on } \partial\Omega,$$

then

$$u \leq v \quad \textit{in } \Omega.$$

Existence. *the stationary equation*

$$\lambda u + H(x, Du) = 0, \quad x \in \mathbb{R}^N \quad (\lambda > 0)$$

has a viscosity solution u *which is uniformly continuous in* $\mathbb{R}^N$.

Remark 1.48. In the above theorem, if $\Omega = \mathbb{R}^N$, the comparison result are then stated as follows: if $u, v \in \mathrm{UC}\,(\mathbb{R}^N)$ are, respectively, sub- and supersolution of

$$\lambda u + H(x, Du) = 0 \quad \text{in } \mathbb{R}^N \quad (\lambda > 0),$$

then

$$u \leq v \quad \text{in } \mathbb{R}^N.$$

1.2.3 Comparison result for boundary value problem

This subsection deals with comparison result for boundary value problem of an evolutionary equation in an unbounded domain we need in the last chapter (Theorem **??**). The Hamilton-Jacobi-Bellman equation we consider is of the evolutionary form

$$u_t + H(x, y, D_x u, D_y u) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^M, \tag{1.12}$$

where the Hamiltonian $H : \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ is given by

$$H(x, y, p, q) = \max_{a \in A} \big\{ -p \cdot f(x, y, a) - q \cdot g(x, y, a) - \ell(x, y, a) \big\}.$$

Assume that H is continuous in $\mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^N \times \mathbb{R}^M$ and there exist some uniform modulus ω and positive constant Q_0 such that

$$|H(x, y, p, q) - H(x', y', p, q)| \leq \omega\big(|(x, y) - (x', y')|(1 + |(p, q)|)\big) \tag{1.13}$$

for any $(x, y, p, q), (x', y', p, q) \in \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^N \times \mathbb{R}^M$; and

$$|H(x, y, p, q) - H(x, y, p', q')| \leq Q_0 \big|(p, q) - (p', q')\big| \tag{1.14}$$

for any $(x, y, p, q), (x, y, p', q') \in \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^N \times \mathbb{R}^M$.

The goal is to provide a comparison result for boundary value problem of (1.12) in an unbounded domain of $(0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^M$. To this end we first prove a comparison result in an open bounded subset $\Omega \subset (0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^M$, and then derive a comparison result in an unbounded domain as a generalized version later on.

Theorem 1.49. *Assume $u \in \mathrm{USC}\,(\overline{\Omega})$ and $v \in \mathrm{LSC}\,(\overline{\Omega})$ be a subsolution and a supersolution, respectively, to the equation*

$$u_t + H(x, y, D_x u, D_y u) = 0 \quad \textit{in } \Omega. \tag{1.15}$$

Assume in addition that u is bounded and $u - v$ is bounded from above in $\overline{\Omega}$. Then

$$u \leq v \quad \textit{on } \partial\Omega \quad \Longrightarrow \quad u \leq v \quad \textit{in } \overline{\Omega}.$$

Proof. The goal is to show that $M_0 := \max_{\overline{\Omega}}(u - v) \le 0$. We assume by contradiction that

$$M_0 := \max_{\overline{\Omega}}(u - v) > 0.$$

Then, it is evident that

$$\operatorname{argmax}(u - v) \subset \Omega.$$

Step 1: We may assume without loss of generality that u is a strict subsolution of (1.15). Indeed, by setting $u_\delta(t, x, y) := u(t, x, y) - \delta t$ for some $\delta > 0$ (small), then u_δ is a subsolution of

$$\frac{\partial u_\delta}{\partial t} + H(x, y, D_x u_\delta, D_y u_\delta) \le -\delta < 0 \quad \text{in } \Omega,$$

and notice also that we still have $u_\delta \le v$ on $\partial\Omega$, u_δ is bounded and $u_\delta - v$ is bounded from above in $\overline{\Omega}$. If we can show that $u_\delta \le v$ on $\overline{\Omega}$, then by letting δ tends to 0, we get $u \le v$ on $\overline{\Omega}$.

Step 2: For $\varepsilon > 0$, we define sup-convolution of u on $\overline{\Omega}$, given by

$$u^\varepsilon(t, x, y) := \max_{(s,\xi,\eta)\in\overline{\Omega}} \Big\{ u(s, \xi, \eta) - \frac{1}{2\varepsilon}\big|(t, x, y) - (s, \xi, \eta)\big|^2 \Big\}.$$

Assume, for each $\varepsilon > 0$, $(t_\varepsilon, x_\varepsilon, y_\varepsilon)$ to be a maximizer of $u^\varepsilon - v$ on $\overline{\Omega}$, and $(s_\varepsilon, \xi_\varepsilon, \eta_\varepsilon)$ be an u^ε-optimal point for $(t_\varepsilon, x_\varepsilon, y_\varepsilon)$. The following facts are due to basic properties of sup-convolution (see Lemma 1.29, Lemma 1.30, Remark 1.31 and Proposition 1.32, chapter 1):

(a) $\big|(t_\varepsilon, x_\varepsilon, y_\varepsilon) - (s_\varepsilon, \xi_\varepsilon, \eta_\varepsilon)\big| \le 2\sqrt{R\varepsilon}, \quad$ where $R := \sup_{\overline{\Omega}} |u|$;

(b) $\dfrac{\big|(t_\varepsilon, x_\varepsilon, y_\varepsilon) - (s_\varepsilon, \xi_\varepsilon, \eta_\varepsilon)\big|^2}{\varepsilon} \to 0, \quad$ as $\varepsilon \to 0$;

(c) fix $\varepsilon > 0$, then for any (t, x, y)

$$\partial u^\varepsilon(t, x, y) = \operatorname{co}\Big\{ \Big(\frac{s - t}{\varepsilon}, \frac{\xi - x}{\varepsilon}, \frac{\eta - y}{\varepsilon}\Big) \;\Big|\; (s, \xi, \eta) \text{ is } u^\varepsilon\text{-optimal for } (t, x, y) \Big\}.$$

Consequently, u^ε is differentiable at (t, x, y) if and only if it admits the unique u^ε-optimal point (s, ξ, η), and in this case

$$Du^\varepsilon(t, x, y) = \Big(\frac{s - t}{\varepsilon}, \frac{\xi - x}{\varepsilon}, \frac{\eta - y}{\varepsilon}\Big);$$

(d) fix $\varepsilon > 0$, and assume (t, x, y) to be a differentiability point of u^ε. Let (s, ξ, η) be the unique u^ε-optimal point for (t, x, y), then $Du^\varepsilon(t, x, y) \in D^+u(s, \xi, \eta)$.

Step 3: (using supersolution property). We have $D^-u^\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) = \partial u^\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon)$, being u^ε is semiconvex. Note that u^ε is subtangent to v at $(t_\varepsilon, x_\varepsilon, y_\varepsilon)$, hence $D^-u^\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) \subset D^-v(t_\varepsilon, x_\varepsilon, y_\varepsilon)$.

Since v is supersolution and taking into account that

$$\Big(\frac{s_\varepsilon - t_\varepsilon}{\varepsilon}, \frac{\xi_\varepsilon - x_\varepsilon}{\varepsilon}, \frac{\eta_\varepsilon - y_\varepsilon}{\varepsilon}\Big) \in \partial u^\varepsilon(t_\varepsilon, x_\varepsilon, x_\varepsilon)$$

we get

$$\frac{s_\varepsilon - t_\varepsilon}{\varepsilon} + H\Big(x_\varepsilon, y_\varepsilon, \frac{\xi_\varepsilon - x_\varepsilon}{\varepsilon}, \frac{\eta_\varepsilon - y_\varepsilon}{\varepsilon}\Big) \geq 0. \tag{1.16}$$

Step 4: (using strict subsolution property). Fix ε and assume (t_n, x_n, y_n), $n \in \mathbb{N}^*$, be a sequence of points where u^ε is differentiable. We denote by (s_n, ξ_n, η_n), for each n, an u^ε-optimal point for (t_n, x_n, y_n). Such the (s_n, ξ_n, η_n) is univocally determined since (t_n, x_n, y_n), for each n, is a differentiability point of u^ε. We have the relations (see the above step 2)

$$Du^\varepsilon(t_n, x_n, y_n) \in D^+u(s_n, \xi_n, \eta_n), \quad \text{and}$$

$$Du^\varepsilon(t_n, x_n, y_n) = \Big(\frac{s_n - t_n}{\varepsilon}, \frac{\xi_n - x_n}{\varepsilon}, \frac{\eta_n - y_n}{\varepsilon}\Big).$$

Since u is strict subsolution, then for some $\delta > 0$ one has

$$\frac{s_n - t_n}{\varepsilon} + H\Big(\xi_n, \eta_n, \frac{\xi_n - x_n}{\varepsilon}, \frac{\eta_n - y_n}{\varepsilon}\Big) \leq -\delta < 0.$$

Passing to the limit as $n \to +\infty$, the sequence (s_n, ξ_n, η_n) converges, up to a subsequence, to some $(s_\varepsilon, \xi_\varepsilon, \eta_\varepsilon)$ which is u^ε-optimal for $(t_\varepsilon, x_\varepsilon, y_\varepsilon)$, thanks to Lemma 1.28. We also know that $(t_n, x_n, y_n) \to (t_\varepsilon, x_\varepsilon, y_\varepsilon)$, as $n \to +\infty$ owing to Lemma 1.30 (c). Therefore we get

$$\frac{s_\varepsilon - t_\varepsilon}{\varepsilon} + H\Big(\xi_\varepsilon, \eta_\varepsilon, \frac{\xi_\varepsilon - x_\varepsilon}{\varepsilon}, \frac{\eta_\varepsilon - y_\varepsilon}{\varepsilon}\Big) \leq -\delta < 0. \tag{1.17}$$

Step 5: (get contradiction). By subtracting (1.17) to (1.16) and using inequality (1.14), we get

$$\begin{aligned}
\delta &\leq H\Big(x_\varepsilon, y_\varepsilon, \frac{\xi_\varepsilon - x_\varepsilon}{\varepsilon}, \frac{\eta_\varepsilon - y_\varepsilon}{\varepsilon}\Big) - H\Big(\xi_\varepsilon, \eta_\varepsilon, \frac{\xi_\varepsilon - x_\varepsilon}{\varepsilon}, \frac{\eta_\varepsilon - y_\varepsilon}{\varepsilon}\Big) \\
&\leq \omega\Big[\big|(x_\varepsilon, y_\varepsilon) - (\xi_\varepsilon, \eta_\varepsilon)\big|\Big(1 + \frac{\big|(x_\varepsilon, y_\varepsilon) - (\xi_\varepsilon, \eta_\varepsilon)\big|}{\varepsilon}\Big)\Big] \\
&= \omega\Big[\big|(x_\varepsilon, y_\varepsilon) - (\xi_\varepsilon, \eta_\varepsilon)\big| + \frac{\big|(x_\varepsilon, y_\varepsilon) - (\xi_\varepsilon, \eta_\varepsilon)\big|^2}{\varepsilon}\Big].
\end{aligned}$$

By the properties (a), (b) in step 2, we get that

$$\omega\Big[\big|(x_\varepsilon, y_\varepsilon) - (\xi_\varepsilon, \eta_\varepsilon)\big| + \frac{\big|(x_\varepsilon, y_\varepsilon) - (\xi_\varepsilon, \eta_\varepsilon)\big|^2}{\varepsilon}\Big] \to 0 \quad \text{as } \varepsilon \to 0,$$

which shows the contradiction. □

Remark 1.50. The above comparison result reduces to standard comparison result for boundary value problem between bounded sub- and supersolution as follows: Assume $D \subset \mathbb{R}^N$ be an open bounded subset and $F : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a continuous Hamiltonian satisfying

- $|F(x,p) - F(x',p)| \leq \omega\big(|x - x'|(1 + |p|)\big)$ for any $x, x', p \in \mathbb{R}^N$;
- $|F(x,p) - F(x,p')| \leq Q_0\big|p - p'\big|$ for any $x, p, p' \in \mathbb{R}^N$,

for some uniform modulus ω and positive constant Q_0. If $u \in \mathrm{BC}\,(\overline{D})$ and $v \in \mathrm{BC}\,(\overline{D})$ be a subsolution and a supersolution, respectively, to the equation

$$u_t + F(x, Du,) = 0 \quad \text{in } D,$$

then

$$u \leq v \quad \text{on } \partial D \quad \Longrightarrow \quad u \leq v \quad \text{in } \overline{D}.$$

We proceed proving a comparison result in an unbounded domain $I \times B \times \mathbb{R}^M \subset (0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^M$, where $I \subset (0, +\infty)$ be an open interval and $B \subset \mathbb{R}^N$ be an open bounded subset. We set $C := I \times B$ for simplicity. We get the following

Theorem 1.51. *Assume* $u \in \mathrm{USC}\,(\overline{C} \times \mathbb{R}^M)$ *and* $v \in \mathrm{LSC}\,(\overline{C} \times \mathbb{R}^M)$ *be a subsolution and a supersolution, respectively, to the equation*

$$u_t + H(x, y, D_x u, D_y u) = 0 \quad \textit{in } C \times \mathbb{R}^M. \tag{1.18}$$

Assume in addition that u is locally bounded and $u - v$ is bounded from above in $\overline{C} \times \mathbb{R}^M$. Then

$$u \leq v \quad \text{on } \partial C \times \mathbb{R}^M \quad \Longrightarrow \quad u \leq v \quad \text{in } \overline{C} \times \mathbb{R}^M.$$

Proof. We assume by contradiction that

$$u(t_0, x_0, y_0) - v(t_0, x_0, y_0) > 0$$

at some point $(t_0, x_0, y_0) \in C \times \mathbb{R}^M$.

Step 1: We introduce a differentiable function $\varphi : \mathbb{R}^M \to \mathbb{R}$ which is equal to $|y - y_0|$, apart a slight adjustment in a small neighbourhood of y_0 to make it C^1 in the whole space $\mathbb{R}^M$. We can assume in addition that

$$\varphi \geq 0 \quad \text{and} \quad |D\varphi| \leq 2 \quad \text{in } \mathbb{R}^M. \tag{1.19}$$

We choose $\delta > 0$ to be small enough so that

$$u(t_0, x_0, y_0) - v(t_0, x_0, y_0) - \delta\varphi(y_0) - 2\delta Q_0 t_0 > 0. \tag{1.20}$$

Step 2: Define a new function

$$\overline{u}(t, x, y) := u(t, x, y) - \delta\varphi(y) - 2\delta Q_0 t, \quad (t, x, y) \in \overline{C} \times \mathbb{R}^M.$$

Note that, any element $\left(\overline{p}_t, \overline{p}_x, \overline{p}_y\right)$ of $D^+\,\overline{u}(t, x, y)$, for any (t, x, y), is of the form

$$\left(\overline{p}_t, \overline{p}_x, \overline{p}_y\right) = \left(p_t - 2\delta Q_0, p_x, p_y - \delta D\varphi(y)\right),$$

where $(p_t, p_x, p_y) \in D^+ u(t, x, y)$.

Since u is a subsolution to (1.18) and taking into account (1.14), (1.19), we get that

$$\begin{aligned} \overline{p}_t + H\left(x, y, \overline{p}_x, \overline{p}_y\right) &= p_t - 2\delta Q_0 + H\left(x, y, p_x, p_y - \delta D\varphi(y)\right) \\ &\leq p_t - 2\delta Q_0 + H\left(x, y, p_x, p_y\right) + \delta Q_0 |D\varphi(y)| \leq 0, \end{aligned}$$

which shows $\overline{u}$ is a subsolution of (1.18).

Step 3: We have

$$\overline{u} - v = (u - v) - \delta\varphi(y) - 2\delta Q_0 t, \quad (t, x, y) \in \overline{C} \times \mathbb{R}^M.$$

Note that, $\varphi(y) \to +\infty$, as $|y| \to +\infty$, and $u - v$ is bounded from above in $\overline{C} \times \mathbb{R}^M$, hence

$$\overline{u}(t, x, y) - v(t, x, y) \to -\infty \quad \text{as } |y| \to +\infty, \, (t, x) \in C.$$

This yields

$$\overline{u} \leq v \quad \text{on } C \times \partial B(y_0, R) \tag{1.21}$$

for some $R > 0$ sufficiently large.

Moreover, by the assumption $u \leq v$ on $\partial C \times \mathbb{R}^M$ and notice that $\overline{u} \leq u$ in $\overline{C} \times \mathbb{R}^M$, we get that

$$\overline{u} \leq v \quad \text{on } \partial C \times \mathbb{R}^M. \tag{1.22}$$

From (1.21), (1.22) and notice that

$$\partial\big(C \times B(y_0, R)\big) = \partial C \times B(y_0, R) \cup C \times \partial B(y_0, R),$$

we get

$$\overline{u} \leq v \quad \text{on } \partial\big(C \times B(y_0, R)\big).$$

By setting $\Omega := C \times B(y_0, R)$, we know that $\overline{u}$ is bounded and $\overline{u} - v$ is bounded from above in the bounded domain $\overline{\Omega}$, therefore by applying Theorem 1.49, we get that

$$\overline{u} \leq v \quad \text{in } C \times B(y_0, R).$$

This yields

$$\overline{u}(t_0, x_0, y_0) - v(t_0, x_0, y_0) \leq 0$$

which contradicts (1.20), because

$$\overline{u}(t_0, x_0, y_0) - v(t_0, x_0, y_0) = u(t_0, x_0, y_0) - v(t_0, x_0, y_0) - \delta\varphi(y_0) - 2\delta Q_0 t_0.$$

□

1.2.4 Viscosity solutions and almost everywhere solutions

Note that, by Rademacher's theorem, a locally Lipschitz continuous function is differentiable almost everywhere. It thus seems natural to introduce a concept of "generalized solution" for (HJ).

Definition 1.52. A function u is called an *almost everywhere solution* (or *locally Lipschitz generalized solution*) of (HJ) if it is locally Lipschitz continuous in Ω and satisfies

$$H\big(x, u(x), Du(x)\big) = 0$$

almost everywhere in Ω.

The first relation between *viscosity solution* and *almost everywhere solution* is stated in the following proposition.

Proposition 1.53. a) *If $u \in \mathrm{C}(\Omega)$ is a viscosity solution of* (HJ)*, then*

$$H\big(x, u(x), Du(x)\big) = 0$$

at any point $x \in \Omega$ where u is differentiable;
b) *if $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is a viscosity solution of* (HJ)*, then*

$$H\big(x, u(x), Du(x)\big) = 0 \quad \textit{almost everywhere in } \Omega.$$

Proof. a) Assume x is a point of differentiability for u, then by Lemma 1.11 we have

$$D^+u(x) = D^-u(x) = \{Du(x)\}.$$

By definition 1.39, we get

$$0 \leq H\big(x, u(x), Du(x)\big) \leq 0,$$

which proves (a).

b) Note that, by Rademacher Theorem, a locally Lipschitz continuous function is differentiable almost everywhere, hence the statement (b) follows immediately from (a). □

Remark 1.54. Part (b) of the above proposition says that any locally Lipschitz continuous function which is a viscosity solution of (HJ) is also an almost everywhere. The converse is false in general. Indeed, there are many locally Lipschitz generalized solutions which are not viscosity solutions. For example, the function $u(x) = |x|$ satisfies

$$|u'(x)| - 1 = 0 \quad \text{in } (-1, 1) \setminus \{0\},$$

but it is not a viscosity supersolution of the same equation in $(-1, 1)$.

If we assume in addition that the Hamiltonian is convex in the gradient variable, then the *viscosity subsolution* and *almost everywhere subsolution* are equivalent. This indeed gives a partial converse to statement (b) in Proposition 1.53.

Proposition 1.55. *Assume for each $x \in \Omega$, $r \in \mathbb{R}$, the Hamiltonian $H(x, r, p)$ is convex in p, and $u : \Omega \to \mathbb{R}^N$ be a locally Lipschitz continuous function. Then, the following statements are equivalent:*

a) *u is a viscosity subsolution to (HJ) in Ω;*

b) *u is an almost everywhere subsolution to (HJ) in Ω;*

c) *$H(x, u(x), p) \leq 0$, for any $x \in \Omega$ and any $p \in \partial u(x)$.*

Proof. (a) $\Rightarrow$ (b): This is evident by the statement (b) in Proposition 1.53.

(b) $\Rightarrow$ (c): Assume u be an almost everywhere subsolution to (HJ) and take any point $x \in \Omega$ and let $p \in \partial u(x)$. By the very definition of generalized gradient,

$$p = \sum_i \lambda_i p_i,$$

where

$$\lambda_i \geq 0, \ \sum_i \lambda_i = 1, \ p_i = \lim_n Du(x_n^i), \ \lim_n x_n^i = x, \quad \text{for any } i.$$

Since H and u are continuous, we obtain

$$H(x, u(x), p_i) = \lim_n H(x_n^i, u(x_n^i), Du(x_n^i)) \leq 0 \quad \text{for any } i.$$

By the convexity assumption on H, we get

$$H(x, u(x), p) = H(x, u(x), \sum_i \lambda_i p_i) \leq \sum_i \lambda_i H(x, u(x), p_i) \leq 0.$$

(c) $\Rightarrow$ (a): The implication directly comes from the fact that $D^+u(x) \subset \partial u(x)$ for any $x \in \Omega$ (Proposition 1.19), and taking into account the definition of viscosity subsolution (Definition 1.39). □

If we ask some more on the differential structure of u, we can state a more general result without the assumption of convexity for H.

Proposition 1.56. *Let u be a locally Lipschitz continuous function in Ω.*

a) *If $D^+u(x) = \partial u(x)$, $\forall x \in \Omega$, then u is a viscosity supersolution to (HJ) if and only if it is an almost everywhere supersolution;*

b) *If $D^-u(x) = \partial u(x)$, $\forall x \in \Omega$, then u is a viscosity subsolution to (HJ) if and only if it is an almost everywhere subsolution.*

Remark 1.57. There are two special classes of locally Lipschitz continuous functions for which the properties assumed in Proposition 1.56 hold true, they are semiconcave and semiconvex functions, as in Proposition 1.21 and Proposition 1.22. Namely, we get the following facts: *semiconcave generalized supersolution* is equivalent to *viscosity supersolution* , and *semiconvex generalized subsolution* is equivalent to *viscosity subsolution.*

It is important to notice that a viscosity subsolution is not necessarily locally Lipschitz continuous. However, this will be the case if the Hamiltonian $H(x, r, p)$ is assumed to satisfy the coercive condition in p, i.e.,

$$H(x, r, p) \longrightarrow +\infty \quad \text{as } |p| \to +\infty, \text{ uniformly in } x \text{ and in } r. \tag{1.23}$$

Proposition 1.58. *Assume that the Hamiltonian H satisfies the coercive condition (1.23). If $u \in \mathrm{BC}(\mathbb{R}^N)$ is a viscosity subsolution of (HJ), then u is Lipschitz continuous in $\mathbb{R}^N$.*

Proof. Fix $x \in \mathbb{R}^N$. Consider the function

$$\phi(y) = u(y) - C|y - x|,$$

where $C > 0$ is a suitable constant to be chosen later. Since $u \in \mathrm{BC}(\mathbb{R}^N)$, then $\phi \in \mathrm{C}(\mathbb{R}^N)$ and $\phi(y) \to -\infty$ as $|y| \to +\infty$, thus there exists $y_0 \in \mathbb{R}^N$ such that

$$\phi(y_0) = \max_{y \in \mathbb{R}^N} \phi(y).$$

We will prove that $y_0 \equiv x$ for C large. Indeed, if $y_0 \neq x$, the function $\varphi(y) := C|y - x|$ is differentiable at y_0. Moreover, $u(y) - \varphi(y)$ has a maximum at y_0, and since u is a viscosity subsolution of (HJ), we get

$$H\big(y_0, u(y_0), D\varphi(y_0)\big) = H\Big(y_0, u(y_0), C\frac{y_0 - x}{|y_0 - x|}\Big) \leq 0.$$

For sufficiently large C, independent of x and y_0, above inequality is in contradiction to the coercivity condition (1.23). Therefore, for such $C > 0$, we must have

$$u(y) - C|y - x| \leq u(y_0) - C|y_0 - x| \equiv u(x), \quad \forall y \in \mathbb{R}^N.$$

By interchanging the roles of x and y, we get $u \in \mathrm{Lip}\,(\mathbb{R}^N)$. □

From above proposition, we easily get the following important corollary.

Corollary 1.59. *Assume the Hamitonian H satisfies the coercivity condition (1.23) and $u \in \mathrm{C}(\mathbb{R}^N)$ is a viscosity subsolution of (HJ), then u is locally Lipschitz continuous in $\mathbb{R}^N$.*

The following lemma is useful in periodic homogenization we will deal with in the next chapter.

Lemma 1.60. *Assume u is the unique viscosity solution of the stationary equation*

$$\lambda u(x) + H(x, Du) = 0, \quad x \in \mathbb{R}^N, \quad \lambda > 0, \tag{1.24}$$

and $H(x,p)$ is $\mathbb{Z}^N$-periodic in x, for any $p \in \mathbb{R}^N$. Then u is $\mathbb{Z}^N$-periodic.

Proof. Fix $k \in \mathbb{Z}^N$ and let $\widetilde{u}(x) := u(x + k)$, where u is the unique viscosity solution of (1.24). We need to prove $\widetilde{u} = u$. To this end we will show that $\widetilde{u}$ is also a viscosity solution of (1.24).

Assume $\varphi \in \mathrm{C}^1(\mathbb{R}^N)$ and $\widetilde{u} - \varphi$ has a maximum at x_0. Then

$$\begin{aligned} & \widetilde{u}(x) - \varphi(x) \leq \widetilde{u}(x_0) - \varphi(x_0), \quad \forall x \in \mathbb{R}^N, \\ \Leftrightarrow \quad & u(x+k) - \varphi(x) \leq u(x_0 + k) - \varphi(x_0), \quad \forall x \in \mathbb{R}^N, \\ \Leftrightarrow \quad & u(x) - \varphi(x - k) \leq u(x_0 + k) - \varphi(x_0 + k - k), \quad \forall x \in \mathbb{R}^N. \end{aligned}$$

By setting $\psi(x) := \varphi(x-k)$, then $u-\psi$ has a maximum at x_0+k. Since u is a viscosity subsolution of (1.24) and H is $\mathbb{Z}^N$-periodic in the state variable, we obtain

$$\begin{aligned} \lambda u(x_0+k) + H\big(x_0+k, D\psi(x_0+k)\big) &\leq 0 \\ \Leftrightarrow \qquad \lambda\widetilde{u}(x_0) + H\big(x_0, D\varphi(x_0)\big) &\leq 0. \end{aligned}$$

Therefore, $\widetilde{u}$ is also a viscosity subsolution of (1.24). Similarly, we can prove that $\widetilde{u}$ is also a viscosity supersolution of (1.24). Conclusion, $\widetilde{u}$ is a viscosity solution of (1.24), and by the uniqueness, we get $\widetilde{u} = u$. □

1.3 Optimal control problems

In this section we review some main results in optimal control problems and its connection with viscosity solutions theory. More precisely, we consider the infinite horizon optimal control problem and finite horizon optimal control problem whose values functions are defined and continuous on the whole space $\mathbb{R}^N$. In such a context, we establish the Dynamic Programming Principle and derive from it the appropriate Hamilton-Jacobi-Bellman equations for the value functions. This topic is examined in detail in the book by Bardi and Capuzzo Dolcetta [BCD97]. All materials we present in this section are from [BCD97].

1.3.1 The controlled dynamical systems

Consider a nonlinear control system of the form

$$\begin{cases} y'(t) = f\big(y(t), \alpha(t)\big), & t > 0 \\ y(0) = x, \end{cases} \tag{1.25}$$

where, the *control* α is any measurable function defined on $[0, +\infty)$ with values in a compact subset A of some Euclidean space, say *control set*. We denote this class of controls by

$$\mathcal{C} = \big\{\alpha : [0,+\infty) \to A \,|\, \alpha(t) \text{ measurable}\big\}.$$

Assume that the *dynamic* $f : \mathbb{R}^N \times A \longrightarrow \mathbb{R}^N$ is such that, for any choice of the control $\alpha \in \mathcal{C}$ and of the initial position $x \in \mathbb{R}^N$, the system (1.25) has a unique solution defined for all $t \in [0, +\infty)$.

The basic assumptions on the dynamic f are as follows

(A1) f is continuous in $\mathbb{R}^N \times A$;

(A2) f is Lipschitz continuous in the state variables, uniformly in the control variables, i.e., there exists some constant $L := L_f > 0$ such that

$$|f(x,a) - f(y,a)| \leq L|x-y|, \quad \forall x, y \in \mathbb{R}^N, \forall a \in A.$$

We look at trajectory (or solution) of the system (1.25) for control function $\alpha \in \mathcal{C}$. This means that $y(\cdot)$ solves the integral equation

$$y(t) = x + \int_0^t f\big(y(s), \alpha(s)\big)\, ds, \quad t > 0,$$

so that, in particular, $y(\cdot)$ is absolutely continuous on compact intervals of $[0, +\infty)$ and it solves (1.25) almost everywhere.

Under the assumptions (A1)-(A2), follows from the standard theory of ordinary differential equations, for given $x \in \mathbb{R}^N$ and $\alpha \in \mathcal{C}$, the system (1.25) has a unique global solution, denoted by $y_x(t, \alpha)$, or briefly by $y(t)$.

The following estimates on dynamics and trajectories are useful in our later analysis. See [BCD97] for the proofs.

Lemma 1.61. *Assume* (A1), (A2)*. Then there exist some constant* C *such that*

a) $|f(x,a)| \leq C(1 + |x|)$ *for any* $x \in \mathbb{R}^N$ *and* $a \in A$*;*

b) $|y_x(t, \alpha)| \leq (|x| + Ct)e^{Ct}$ *for any* $\alpha \in \mathcal{C}$ *and* $t \geq 0$*.*
Consequently, for any $r > 0$*, there exists* $R > 0$ *such that for any control* $\alpha : [0,T] \to A$ *and any* $x \in \overline{B}(0,r)$*, one has*

$$y_x(t, \alpha) \in \overline{B}(0, R) \quad \textit{for any } t \in [0, T].$$

The model includes also some *running cost* and *terminal cost* associated with this control system and then consider the optimization problems. We will examine, in the next two subsections, two basic models in optimal control problems, they are infinite horizon problem and finite horizon problem.

1.3.2 The infinite horizon problem

Consider the optimization problem: to minimize the cost functional, overall $\alpha \in \mathcal{A}$,

$$I(x, \alpha) := \int_0^{+\infty} \ell\big(y_x(t), \alpha(t)\big) e^{-\lambda t}\, dt,$$

where $\lambda > 0$ represents a *discount factor* (or *constant interest rate*), the function $\ell : \mathbb{R}^N \times A \to \mathbb{R}$ is a *running cost* and satisfies

(A3) ℓ is continuous in $\mathbb{R}^N \times A$; there is a constant M_ℓ and a uniform modulus ω_ℓ such that

$$|\ell(x, a)| \leq M_\ell \quad \text{and} \quad |\ell(x, a) - \ell(y, a)| \leq \omega_\ell(|x - y|) \quad \forall x, y \in \mathbb{R}^N,\ \forall a \in A.$$

Let us introduce the *value function* for this problem, defined by

$$u(x) := \inf_{\alpha \in \mathcal{C}} I(x, \alpha).$$

The value function is a function of initial positions and it plays an important role in the study of optimal control problem via dynamic programming approach. In this subsection, we will state that, the value function satisfies a functional equation, called *dynamic programming principle*, and then its infinitesimal version, *Hamilton-Jacobi-Bellman equation.*

We first look at the regularity of the value function.

Proposition 1.62. *Assume* (A1), (A2) *and* (A3). *Then,*

a) *u is bounded, uniformly continuous in $\mathbb{R}^N$, that is, $u \in \mathrm{BUC}\,(\mathbb{R}^N)$;*

b) *assume, in addition, $\omega_\ell(r) = Lr$ and $\lambda > L$, that is, ℓ is Lipschitz continuous in the state variables, uniformly in the control variables, then u is Lipschitz continuous in $\mathbb{R}^N$.*

Instead of considering the constant interest rate λ in the item (b) of the above proposition, we can obtain the Lipschitz continuity of the value function u by assuming some controllability on the control system, as follows

Definition 1.63. We say that the system (1.25) is *local Lipschitz controllable* if, for given compact set $K \subset \mathbb{R}^N$, for any $x, z \in K$, there exists $L_K > 0$ and $\overline{\alpha} \in \mathcal{C}$ such that $y_x(\overline{t}, \overline{\alpha}) = z$ for some $\overline{t} \leq L_K|x - z|$.

In geometrically, the local Lipschitz controllability means that the system can reach any point of a compact set K in a time proportional to the distance from the initial point.

Proposition 1.64. *Assume* (A1), (A2), (A3) *and the system* (1.25) *is local Lipschitz controllable. Then, the value function u is locally Lipschitz continuous in $\mathbb{R}^N$, more precisely, for any compact subset $K \subset \mathbb{R}^N$,*

$$\big|u(x) - u(z)\big| \leq M_\ell L_K |x - z|, \quad \forall x, z \in K.$$

The aim of the theory is to define a partial differential equation solved by the value function. To this end, we first present the so-called *Dynamic Programming Principle*:

Theorem 1.65 (Dynamic Programming Principle)**.** *Assume* (A1), (A2) *and* (A3). *Then, for all $x \in \mathbb{R}^N$ and $t > 0$, we have*

$$u(x) = \inf_{\alpha \in \mathcal{C}} \left\{ \int_0^t \ell\big(y_x(s,\alpha), \alpha(s)\big) e^{-\lambda s}\, ds + u\big(y_x(t,\alpha)\big) e^{-\lambda t} \right\} \tag{DPP}$$

Remark 1.66. This principle express the intuitive remark that the minimum cost is achieved if one behaves as follows:

1. let the system evolve for a small amount of time choosing an arbitrary control α on the interval $[0, t]$ ($\Rightarrow y_x(s, \alpha)$, for $s \in [0, t]$);
2. pay the corresponding cost ($\Rightarrow \int_0^t \ell\big(y_x(s,\alpha), \alpha(s)\big) e^{-\lambda s}\, ds$);
3. pay what remains to pay after time t with the best possible control ($\Rightarrow u\big(y_x(t,\alpha)\big) e^{-\lambda t}$);
4. minimize the sum of these two costs over all the possible controls α ($\Rightarrow$ (DPP)).

We want to derive now an infinitesimal version of the Dynamic Programming Principle, the so called *Hamilton-Jacobi-Bellman Equation* (or *Dynamic Programming Equation*). The Hamiltonian in the HJB equation for the infinite horizon problem is $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, defined by

$$H(x,p) := \sup_{a\in A} \big\{ -p\cdot f(x,a) - \ell(x,a) \big\}.$$

Theorem 1.67 (HJB Equation)**.** *Assume* (A1)*,* (A2) *and* (A3)*. Then, the value function u is the unique viscosity solution in the class* $\mathrm{BUC}(\mathbb{R}^N)$ *of the Hamilton-Jacobi-Bellman equation*

$$\lambda u + H(x, Du) = 0 \quad \text{in } \mathbb{R}^N. \tag{HJB}$$

1.3.3 The finite horizon problem

In this subsection, the cost to minimize is

$$J(t,x,\alpha) := \int_0^t \ell\big(y_x(s,\alpha),\alpha(s)\big)\,ds + h\big(y_x(t,\alpha)\big),$$

where, the running cost $\ell : \mathbb{R}^N \times A \to \mathbb{R}$ satisfies the assumption (A3), and the terminal cost (also called final cost) $h : \mathbb{R}^N \to \mathbb{R}$ satisfies

(A4) h is bounded uniform continuous in $\mathbb{R}^N$, namely, there exists a constant M_h and a uniform modulus ω_h such that

$$\begin{aligned} |h(x)| &\leq M_h \quad \text{for any } x\in\mathbb{R}^N, \\ |h(x)-h(y)| &\leq \omega_h(|x-y|) \quad \text{for any } x,y\in\mathbb{R}^N. \end{aligned}$$

The value function for the finite horizon problem is

$$v(t,x) := \inf_{\alpha\in\mathcal{C}} J(t,x,\alpha).$$

Following the terminology of the calculus of variations, the above problem is called the *Bolza problem*. In the case $\ell \equiv 0$, the problem is referred to as the *Mayer problem*. Of course, we can regard the Mayer form as a special case of the Bolza form when $\ell \equiv 0$. Moreover, the Bolza problem can be converted into a Mayer problem by adding a scalar state variable y_{N+1} to the vector y, with dynamic

$$y'_{N+1}(s) = \ell\big(\alpha(s), y(s)\big), \quad y_{N+1}(0) = 0,$$

and the cost to minimize now becomes

$$y_{N+1}(t) + h\big(y(t)\big).$$

We state now a regularity result for the value function v.

Proposition 1.68. *Assume* (A1), (A2), (A3) *and* (A4). *Then*

a) *v is bounded continuous in $[0,T] \times \mathbb{R}^N$, for any $T > 0$;*

b) *assume, in addition, h is Lipschitz continuous, and $\omega_\ell(r) = L_\ell r$, i.e., ℓ is Lipschitz continuous in state variables, uniformly in control variables, then v is Lipschitz continuos in $[0,T] \times K$, for any compact subset $K \subset \mathbb{R}^N$ and any $T > 0$.*

Similar to the infinite horizon case, we record here Dynamic Programming Principle and HJB equation (also called Dynamic Programming equation) for the value function in the finite horizon problem.

Theorem 1.69 (Dynamic Programming Principle)**.** *Assume* (A1), (A2), (A3) *and* (A4). *Then, for all $x \in \mathbb{R}^N$ and $0 < \tau \le t$, we have*

$$v(t,x) = \inf_{\alpha \in \mathcal{C}} \left\{ \int_0^\tau \ell\big(y_x(s), \alpha(s)\big)\, ds + v\big(t - \tau, y_x(\tau)\big) \right\}. \tag{1.26}$$

Theorem 1.70 (HJB Equation)**.** *Assume* (A1), (A2), (A3) *and* (A4). *Then, the value function v is the unique viscosity solution of the Cauchy problem*

$$\begin{cases} v_t + H(x, Dv) = 0 & \text{in } (0,T) \times \mathbb{R}^N \\ v(0,x) = h(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.27}$$

where

$$H(x,p) := \sup_{a \in A} \big\{ -p \cdot f(x,a) - \ell(x,a) \big\}.$$

Chapter 2

Asymptotic behaviour of singularly perturbed control system: periodic setting

This chapter is devoted to the study of asymptotic behaviour of singularly perturbed control system in the periodic setting. We first review some main results in the classical periodic homogenization for Hamilton-Jacobi equation, which was initiated by Lions, Papanicolau and Varadhan [LPV86] and revisited by Evans [E89], [E89]. We next concern asymptotic behaviour of singularly perturbed control system in the periodic setting via PDE approach, obtained by Alvarez and Bardi in the series of papers [AB03], [AB10].

2.1 Review of periodic homogenization for Hamilton-Jacobi equation

Homogenization theory for partial differential equations studies the effects upon solutions when coefficients of the equations have high-frequency oscillations. In typical application, these rapid oscillations represent the small-scale, "microscopic" structure of a material. The goal of homogenization theory is to study the asymptotic behaviour as the oscillations become more and more rapid. The idea is that in this limit the high-frequency effects will "average out" and we hope to obtain a simpler, "macroscopic" equations. In other words, starting from a microscopic description of a problem, we look for a macroscopic (or effective) description.

In this section, we briefly review the main results on homogenization of Hamilton-Jacobi equation in periodic media, pioneered by Lions, Papanicolau & Varadhan [LPV86], and then refined and generalized by Evans [E89], [E92], who introduced the method of perturbed test function for viscosity solutions.

2.1.1 Introduction to periodic homogenization

We consider the Hamilton-Jacobi equation of the evolutionary form

$$v_t + H(x, Dv) = 0,$$

where the Hamiltonian $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is periodic in the first variable. Suppose that the period, denoted by ε, is very small and we want to describe the limit behaviour of the solution when ε tends to zero. By rescaling on the state variable, $H(\frac{x}{\varepsilon}, p)$ is $\mathbb{Z}^N$-periodic in the first variable, and we arrive at the following problem:

To study asymptotic behaviour, as $\varepsilon \to 0$, of the solution v^ε to the Hamilton-Jacobi equation

$$\begin{cases} v_t^\varepsilon + H\big(\frac{x}{\varepsilon}, Dv^\varepsilon\big) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N, \\ v^\varepsilon(0, x) = v_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{2.1}$$

where, the Hamiltonian $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function and satisfies the $\mathbb{Z}^N$-periodic condition in the state variable. The initial data $v_0 : \mathbb{R}^N \to \mathbb{R}$ in the above problem is given, and satisfies some suitable assumptions.

We expect that v^ε (locally uniformly) converges, as $\varepsilon \to 0$, to a function v which is viscosity solution of an evolution problem of the form

$$\begin{cases} v_t + \overline{H}(Dv) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N, \\ v(0, x) = v_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{2.2}$$

for a suitable Hamiltonian $\overline{H}$. If this is satisfied and $\overline{H}$ does not depend on v_0, $\overline{H}$ is called the *effective Hamiltonian* (or *limiting Hamiltonian*).

Some basic assumptions on the Hamiltonian and initial data are as follows.

(H1) H is uniformly continuous on $\mathbb{R}^N \times B(0,R)$ for any $R>0$, and there exists a uniform modulus[1] ω such that

$$\big|H(x,p)-H(y,p)\big| \le \omega\big(|x-y|(1+|p|)\big) \quad \text{for any } x,y\in\mathbb{R}^N,\ p\in\mathbb{R}^N.$$

(H2) $x\mapsto H(x,p)$ is $\mathbb{Z}^N$-periodic, for any $p\in\mathbb{R}^N$, i.e., for each $p\in\mathbb{R}^N$

$$H(x+k,p)=H(x,p) \quad \text{for any } x\in\mathbb{R}^N, \text{ any } k\in\mathbb{Z}^N;$$

(H3) $H(x,p)$ is coercive in p, uniformly for $x\in\mathbb{R}^N$, i.e.,

$$\lim_{|p|\to+\infty}\ \inf_{x\in\mathbb{R}^N} H(x,p)=+\infty;$$

(H4) $v_0\in \mathrm{BUC}\,(\mathbb{R}^N)$ and $Dv_0\in L^\infty(\mathbb{R}^N)$.

Under the assumptions (H1) and (H4), by Theorem 1.45 and Remark 1.46, for each $\varepsilon>0$ the problem (2.1) has a unique viscosity solution v^ε which is bounded uniformly continuous in $[0,T]\times\mathbb{R}^N$, for any given $T>0$.

The existence of $\overline{H}$ and the convergence of v^ε, as $\varepsilon\to 0$, to the viscosity solution v of (2.2) were proved by Lions, Papanicolaou, Varadhan [LPV86], and also by Evans [E92].

2.1.2 Cell problem, effective Hamiltonian and convergence result

A key step in homogenization problem is to identify the effective Hamiltonian $\overline{H}$. For this purpose, a formal asymptotic expansion of v^ε is useful.

A formal calculation: In order to guess $\overline{H}$, we assume that $v^\varepsilon\to v$, as $\varepsilon\to 0$, and using the formal two-scale asymptotic expansion with respect to ε of v^ε:

$$v^\varepsilon(t,x)=v^0\big(t,x,\frac{x}{\varepsilon}\big)+\varepsilon v^1\big(t,x,\frac{x}{\varepsilon}\big)+\varepsilon^2 v^2\big(t,x,\frac{x}{\varepsilon}\big)+\cdots, \tag{2.3}$$

[1] by a uniform modulus we mean a function $\omega:[0,+\infty)\to[0,+\infty)$ such that ω is continuous, nondecreasing and $\omega(0)=0$.

where v^0, v^1, v^2, ... are continuous in all variables and $\mathbb{Z}^N$-periodic in the second variable. Substituting (2.3) into equation (2.1), we obtain

$$v_t^0\big(t,x,\frac{x}{\varepsilon}\big)+O(\varepsilon)+H\Big(\frac{x}{\varepsilon},D_2v^0(t,x,\frac{x}{\varepsilon})+\frac{1}{\varepsilon}D_3v^0(t,x,\frac{x}{\varepsilon})+D_3v^1(t,x,\frac{x}{\varepsilon})+O(\varepsilon)\Big)=0, \tag{2.4}$$

where D_2 and D_3 represent the derivatives with respect to the second and third variables, and $O(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Fix t,x and let $\varepsilon\in(0,1)$. By periodicity in the third variable, the derivatives of v^0, v^1 are bounded, and by coercivity assumption, it is easy to check that $D_3v^0=0$. This implies $v^0(t,x,\frac{x}{\varepsilon})=v(t,x)$, and then (2.4) becomes

$$v_t(t,x)+O(\varepsilon)+H\Big(\frac{x}{\varepsilon},Dv(t,x)+D_3v^1\big(t,x,\frac{x}{\varepsilon}\big)+O(\varepsilon)\Big)=0. \tag{2.5}$$

From the above equation, it follows that

$$H\Big(\frac{x}{\varepsilon},Dv(t,x)+D_3v^1\big(t,x,\frac{x}{\varepsilon}\big)\Big)\to -v_t(t,x),\quad \text{as } \varepsilon\to 0.$$

Moreover, the function

$$\frac{x}{\varepsilon}\mapsto H\Big(\frac{x}{\varepsilon},Dv(t,x)+D_3v^1(t,x,\frac{x}{\varepsilon})\Big)$$

is periodic, hence the following equality must hold true:

$$H\big(y,Dv(t,x)+D_3v^1(t,x,y)\big)=-v_t(t,x),\quad \forall y.$$

Note that when t and x are fixed, $\lambda:=-v_t(t,x)$ is a given real number, $p:=Dv(t,x)$ is a given vector in $\mathbb{R}^N$, and $y\mapsto v^1(t,x,y)$ is a $\mathbb{Z}^N$-periodic function. We arrive at therefore the following problem:

Cell problem. For each $p\in\mathbb{R}^N$, find $\lambda=\lambda(p)\in\mathbb{R}$ such that the equation

$$H\big(y,p+Du(y)\big)=\lambda\quad \text{in } \mathbb{R}^N \tag{2.6}$$

has a periodic viscosity solution u.

The following result is well-known ([LPV86], [E92]):

Theorem 2.1. *Assume* (H1)-(H3). *Then,*

a) *for each* $p \in \mathbb{R}^N$*, there exists a unique real value* $\lambda = \lambda(p)$ *for which the equation* (2.6) *has a periodic viscosity solution* u*.*

b) *Define* $\overline{H}$ *by setting* $\overline{H}(p) = \lambda$ *as above, then* $\overline{H}$ *satisfies* (H1), (H3)*. Moreover, if* $H(x,p)$ *is convex in* p*, then* $\overline{H}$ *is convex in* p *as well.*

c) *Assume in addition* (H4)*, then the viscosity solution* v^ε *of* (2.1) *locally uniformly converges on* $[0,+\infty) \times \mathbb{R}^N$*, as* $\varepsilon \to 0$*, to the viscosity solution* v *of the effective problem*

$$\begin{cases} v_t + \overline{H}(Dv) = 0 & \text{in } (0,+\infty) \times \mathbb{R}^N, \\ v(0,x) = v_0(x) & \text{in } \mathbb{R}^N \end{cases} \tag{2.7}$$

We refer the readers to [LPV86], [E92] for the proof. The proof is also presented very in detail in the PhD thesis of Concordel [C95], so we do not present again the proof in this thesis. We instead mention here some main ideas on the existence and uniqueness of the critical value λ in the cell problem that defined the effective Hamiltonian, and then recall Evans's perturbed test function method which is useful in the proof of convergence result. Note also that the techniques and methods of cell problem and Evans's perturbed test function are repeated several times throughout this thesis.

Some main ideas: (on existence and uniqueness of critical value $\lambda = \overline{H}(p)$ and Evans's perturbed test function method)

The key step in the proof of item (a) is to consider, for each $0 < \delta < 1$ the approximating equation

$$\delta w_\delta(y) + H\big(y, p + Dw_\delta(y)\big) = 0 \quad \text{in } \mathbb{R}^N \tag{2.8}$$

where p is fixed in $\mathbb{R}^N$. By Theorem 1.47 and Lemma 1.60, for each $0 < \delta < 1$, the above equation has a unique periodic viscosity solution $w_\delta \in \mathrm{BUC}\,(\mathbb{R}^N)$.

By the periodicity and coercivity on H, we can show that the sequence $\{\delta w^\delta\}$ is equibounded and equi-Lipschitz continuous in $\mathbb{R}^N$, hence by Ascoli-Arzelà theorem, we can conclude that

$$\delta_i w_{\delta_i} \longrightarrow -\lambda \quad \text{locally uniformly in } \mathbb{R}^N.$$

for some sequence $\delta_i \to 0$. Note that, by the uniqueness of λ (will be showed later), the whole sequence δw_δ converges to $-\lambda$, as $\delta \to 0$, locally uniformly in $\mathbb{R}^N$.

Define

$$u_\delta(y) := w_\delta(y) - w_\delta(0),$$

then the sequence $\{u_\delta\}$ is also equibounded and equi-Lipschitz continuous in $\mathbb{R}^N$. Again, by Ascoli-Arzelà theorem, there is a function $u \in \mathrm{Lip}\,(\mathbb{R}^N)$ and a sequence (which is still denoted by δ_i) $\delta_i \to 0$ such that

$$u_{\delta_i} \longrightarrow u \quad \text{locally uniformly in } \mathbb{R}^N.$$

Note that u is periodic owing to the periodicity of u_δ.

Since w_δ is a viscosity solution of (2.8), it follows that u_{δ_i} is a viscosity solution of

$$\delta_i u_{\delta_i}(y) + \delta_i w_{\delta_i}(0) + H\big(y, p + Du_{\delta_i}(y)\big) = 0 \quad \text{in } \mathbb{R}^N.$$

Note that, $\delta_i u_{\delta_i} \to 0$ and $\delta_i w_{\delta_i}(0) \to -\lambda$, as $\delta_i \to 0$. By stability of viscosity solution (Theorem 1.41), we deduce that u is a viscosity solution of

$$H\big(y, p + Du(y)\big) = \lambda \quad \text{in } \mathbb{R}^N.$$

The uniqueness of λ then follows by standard comparison result, since there exist bounded solutions of the cell problem.

In order to emphasize the perturbed test function method developed by Evans, we proceed assuming that v^ε locally uniformly converges on $[0, +\infty) \times \mathbb{R}^N$ to some function v, and we will show that v is a viscosity solution of (2.7).

We only need to check v is a viscosity subsolution of

$$v_t + \overline{H}(Dv) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^N.$$

Fix $(t_0, x_0) \in (0, +\infty) \times \mathbb{R}^N$ and assume $\psi \in \mathrm{C}^1\big((0, +\infty) \times \mathbb{R}^N\big)$ be a strict supertangent to v at (t_0, x_0) with $\psi(t_0, x_0) = v(t_0, x_0)$. We want to prove that

$$\psi_t(t_0, x_0) + \overline{H}\big(D\psi(t_0, x_0)\big) \le 0.$$

We assume by contradiction that there exists some $\eta > 0$ such that

$$\psi_t(t_0, x_0) + \overline{H}(D\psi(t_0, x_0)) \geq \eta. \tag{2.9}$$

Let $p := D\psi(t_0, x_0) \in \mathbb{R}^N$, and consider u the periodic viscosity solution to the cell problem

$$H\big(y, p + Du(y)\big) = \overline{H}(p) \quad \text{in } \mathbb{R}^N. \tag{2.10}$$

Define the perturbed test function, for $\varepsilon > 0$,

$$\psi^\varepsilon(t, x) := \psi(t, x) + \varepsilon\, u\Big(\frac{x}{\varepsilon}\Big), \quad (t, x) \in (0, +\infty) \times \mathbb{R}^N.$$

Note that, since u is continuous and bounded in $\mathbb{R}^N$, $\psi^\varepsilon \in \mathrm{C}\big(\mathbb{R}^N \times (0, +\infty)\big)$ and

$$\psi^\varepsilon \to \psi \quad \text{uniformly in } (0, +\infty) \times \mathbb{R}^N, \quad \text{as } \varepsilon \to 0.$$

We claim that ψ^ε satisfies in the viscosity sense

$$\psi^\varepsilon_t + H\Big(\frac{x}{\varepsilon}, D\psi^\varepsilon\Big) \geq \frac{\eta}{2}$$

in a ball $B\big((t_0, x_0), r\big) \subset \mathbb{R}^{N+1}$ for r small enough.

Moreover, v^ε is a viscosity subsolution of this equation. Standard comparison result (see Theorem 1.49 and Remark 1.50) yields that

$$v^\varepsilon(t_0, x_0) - \psi^\varepsilon(t_0, x_0) \leq \max_{\partial B((t_0, x_0), r)} (v^\varepsilon - \psi^\varepsilon).$$

Taking the limit as $\varepsilon \to 0$, we obtain

$$v(t_0, x_0) - \psi(t_0, x_0) \leq \max_{\partial B((t_0, x_0), r)} (u - \psi),$$

and this contradicts the assumption that $v - \psi$ has a strict local maximum at (t_0, x_0). Conclusion,

$$\psi_t(t_0, x_0) + \overline{H}(D\psi(t_0, x_0)) \geq \eta$$

is impossible, and thus v is a viscosity subsolution of (2.7).

2.2 Asymptotic behaviour of singularly perturbed control system

This section is devoted to the study of asymptotic behaviour of singularly perturbed control system via PDE approach. This approach aims at studying limit behaviour of the value function and characterizing its limit as the unique viscosity solution of the so-called effective HJB equation. All the results we present here are due to Alvarez and Bardi, in the series of papers [AB03], [AB10].

2.2.1 Singularly perturbed control system

Consider the singularly perturbed control system

$$\begin{cases} X'(s) = f\big(X(s), Y(s), \alpha(s)\big), \ s > 0 \\ Y'(s) = \dfrac{1}{\varepsilon} g\big(X(s), Y(s), \alpha(s)\big), \ s > 0 \\ X(0) = x, \ Y(0) = y, \end{cases} \tag{2.11}$$

Here, the control $\alpha : [0, +\infty) \to A$ is any measurable function, A be a compact subset of some Euclidean space; $f : \mathbb{R}^N \times \mathbb{R}^M \times A \to \mathbb{R}^N$, $g : \mathbb{R}^N \times \mathbb{R}^M \times A \to \mathbb{R}^M$ are the dynamics; $(x, y) \in \mathbb{R}^N \times \mathbb{R}^M$ is initial position; $\varepsilon > 0$ is a small parameter. We denote by $\mathcal{C}$ the set of controls,

$$\mathcal{C} = \big\{\alpha : [0, +\infty) \to A \ \big| \ \alpha(\cdot) \text{ measurable}\big\}.$$

Because of the presence of the small parameter $\varepsilon > 0$, the state variable X is called the *slow variable*, and the state variable Y is referred to as the *fast variable*.

We consider the following optimal control problem: for any $x \in \mathbb{R}^N$, $y \in \mathbb{R}^M$, $t > 0$, to minimize the following cost functional, subject to (2.11),

$$J^\varepsilon(t, x, y, \alpha) := \int_0^t \ell\big(X^\varepsilon(s), Y^\varepsilon(s), \alpha(s)\big)\, ds + h\big(X^\varepsilon(t)\big),$$

where $\big(X^\varepsilon(s), Y^\varepsilon(s)\big) := \big(X^\varepsilon(s; x, y, \alpha), Y^\varepsilon(s; x, y, \alpha)\big)$ is the solution of (2.11), $\ell : \mathbb{R}^N \times \mathbb{R}^M \times A \to \mathbb{R}$ and $h : \mathbb{R}^N \to \mathbb{R}$ are given functions, called *running cost* and *final cost*, respectively.

The value function for this problem is

$$V^\varepsilon(t,x,y) := \inf_{\alpha\in\mathcal{C}} J^\varepsilon(t,x,y,\alpha). \tag{2.12}$$

The goal is to study asymptotic behaviour of the value function v^ε when the perturbed parameter ε vanishes.

The following assumptions will hold throughout this section, called *basic assumptions*.

Basic assumptions

(B1) f and g are bounded uniformly continuous in $\mathbb{R}^N\times\mathbb{R}^M\times A$;

(B2) $f(x,y,a)$ and $g(x,y,a)$ are Lipschitz continuous in (x,y), uniformly in a;

(B3) ℓ is bounded uniformly continuous in $\mathbb{R}^N\times\mathbb{R}^M\times A$;

(B4) h is bounded uniformly continuous in $\mathbb{R}^N$;

(B5) all the datum f, g, ℓ are $\mathbb{Z}^M$-periodic in the fast variable y, namely, for any $x\in\mathbb{R}^N$, $y\in\mathbb{R}^M$, $a\in A$

$$f(x,y+k,a) = f(x,y,a) \quad \text{for any } k\in\mathbb{Z}^M.$$

Similar argument for g and ℓ.

2.2.2 Ergodicity, effective Hamiltonian and convergence result

It is well-known that (see [BCD97]), under the basic assumptions, the value function $V^\varepsilon(t,x,y)$ is the unique bounded continuous viscosity solution of the Hamilton-Jacobi-Bellman equation (HJB equation)

$$\begin{cases} V_t^\varepsilon + H\big(x,y,D_xV^\varepsilon,\frac{D_yV^\varepsilon}{\varepsilon}\big) = 0 & \text{in } (0,+\infty)\times\mathbb{R}^N\times\mathbb{R}^M \\ V^\varepsilon(0,x,y) = h(x) & \text{in } \mathbb{R}^N\times\mathbb{R}^M. \end{cases} \tag{2.13}$$

where

$$H(x,y,p,q) := \max_{a\in A}\big\{-p\cdot f(x,y,a) - q\cdot g(x,y,a) - \ell(x,y,a)\big\}. \tag{2.14}$$

Note that, by the periodic condition (B5), the Hamiltonian H is Z^M-periodic in the fast variable y. The study of limit behaviour of V^ε, as $\varepsilon \to 0$, is hence translated into the periodic homogenization for the HJB equation (2.13).

We expect that the value function $V^\varepsilon(t,x,y)$ converges, as $\varepsilon \to 0$, to a function $V(t,x)$ where the fast variable y disappears, and $V(t,x)$ solves in viscosity sense an effective equation (also called limiting equation)

$$\begin{cases} V_t + \overline{H}(x, DV) = 0 & \text{in } (0,+\infty) \times \mathbb{R}^N \\ V(x,0) = h(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{2.15}$$

The Hamiltonian $\overline{H}$ is referred to as the *effective Hamiltonian*. The study of the existence, and possibly explicit formula of such an effective Hamiltonian is the main goal in homogenization problem. This approach was initiated by Lions, Papanicolaou and Vradhan [LPV86], then refined and generalized by Evans [E89], [E92] who introduced the perturbed test function method, and it has become a wide direction of research.

To deal with periodic homogenization for HJB equation (2.13), the so-called "ergodicity" property of the Hamiltonian H has been extensively studied by Alvarez and Bardi (see [AB03], [AB10]). This ergodicity property allow us to define the effective Hamiltonian $\overline{H}$, a crucial step in periodic homogenization.

We review here three equivalent definitions of ergodicity of H, and then claims that bounded time controllability (see Definition 2.8) on the fast system is sufficient for ergodicity of H. The convergence result is then stated first for weak semilimits, and then for locally uniform convergence with some stronger assumptions. Evans's perturbed test function method is partially useful in the proof of convergence result. All these results are due to Alvarez and Bardi, see for instance [AB10] and references therein.

Notice that in [AB10], the authors deal with asymptotic behaviour even for Bellman-Isaacs equations (in the stochastic case) and the final cost depends also on fast variable, i.e. $h = h(x,y)$. In this case, the so-called stabilization (to a constant) of the pair (H,h) is studied and this permits to define the effective initial data $\overline{h}$ in the effective Cauchy problem. In the context of the present chapter, we will be concerned with deterministic case and we restrict the discussion to the case where the final cost h depends only on slow variable x.

The *ergodicity* of H can be expressed in several equivalent ways. The first definition of ergodicity is based on the approximating equation for $\delta > 0$, called *cell δ-problem*, as follows:

For each $(x_0, p_0) \in \mathbb{R}^N \times \mathbb{R}^N$, consider the stationary equation, for $\delta > 0$,

$$\delta w_\delta + H\big(x_0, y, p_0, Dw_\delta(y)\big) = 0 \quad \text{in } \mathbb{R}^M, \quad w_\delta \text{ periodic.} \tag{2.16}$$

Under basic assumptions, the equation (2.16) has a unique periodic viscosity solution, denoted by $w_\delta(y) := w_\delta(y; x_0, p_0)$ so as to display its dependence on the frozen slow variables.

It is well-known that (see [BCD97]), w_δ can be represented as the value function of an infinite horizon optimal control problem with discounted factor $\delta > 0$, namely,

$$w_\delta(y; x_0, p_0) = \inf_{\alpha \in \mathcal{C}} \int_0^{+\infty} \ell_0\big(Y(s), \alpha(s)\big) e^{-\delta s}\, ds \tag{2.17}$$

where, $Y(\cdot) := Y_y(\cdot, \alpha)$ is solution of the fast system

$$\begin{cases} Y'(s) = g_0\big(Y(s), \alpha(s)\big) \\ Y(0) = y, \end{cases} \tag{2.18}$$

here we use the notations, for simplicity

$$\begin{aligned} \ell_0(Y, \alpha) &:= p_0 \cdot f(x_0, Y, \alpha) + \ell(x_0, Y, \alpha), \\ g_0(Y, \alpha) &:= g(x_0, Y, \alpha). \end{aligned}$$

Definition 2.2 (cell δ-problem)**.** The Hamiltonian H is said *ergodic* in the fast variable at (x_0, p_0) if

$$\lim_{\delta \to 0} \delta w_\delta(y; x_0, p_0) = \text{const}, \quad \text{uniformly in } y. \tag{2.19}$$

We say that it is ergodic at x_0 if it is ergodic at (x_0, p_0) for every p_0, and that it is ergodic if it is ergodic at every $x_0 \in \mathbb{R}^N$. In this case we set

$$\overline{H}(x_0, p_0) := -\text{const},$$

and it is called the *effective Hamiltonian*.

Note that the constant "const" in (2.19) of course depends on (x_0, p_0), and the effective Hamiltonian $\overline{H}$ therefore can be represented by the formula

$$\overline{H}(x_0, p_0) = -\lim_{\delta \to 0} \inf_{\alpha \in \mathcal{A}} \delta \int_0^{+\infty} \ell_0\big(Y(s), \alpha(s)\big) e^{-\delta s}\, ds$$

for any initial position y of the fast system (2.18).

The second definition of ergodicity is based on the evolutionary problem, called cell t-problem:

For each $(x_0, p_0) \in \mathbb{R}^N \times \mathbb{R}^N$, consider the evolutionary problem

$$\begin{cases} w_t + H(x_0, y, p_0, D_y w) = 0, & (t, y) \in (0, +\infty) \times \mathbb{R}^M, \\ w(0, y) = 0, \quad w \text{ periodic in } y, & y \in \mathbb{R}^M. \end{cases} \tag{2.20}$$

Under basic assumptions, the problem (2.20) has a unique viscosity solution, denoted by $w(t, y) := w(t, y; x_0, p_0)$ so as to display its dependence on the frozen slow variables.

It is well known that (see [BCD97]), $w(t, y)$ can be represented as the value function of a finite horizon optimal control problem subject to (2.18), namely,

$$w(t, y; x_0, p_0) = \inf_{\alpha \in \mathcal{A}} \int_0^t \ell_0\big(Y(s), \alpha(s)\big)\, ds. \tag{2.21}$$

Definition 2.3 (cell t-problem)**.** The Hamiltonian H is said *ergodic* at (x_0, p_0) if

$$\lim_{t \to +\infty} \frac{w(t, y; x_0, p_0)}{t} = \text{const}, \quad \text{uniformly in } y. \tag{2.22}$$

In this case we set

$$\overline{H}(x_0, p_0) := -\text{const}$$

and it is called the *effective Hamiltonian.*

Note that the constant "const" in (2.22) of course depends on (x_0, p_0), and the effective Hamiltonian $\overline{H}$ therefore can be represented by the formula

$$\overline{H}(x_0, p_0) = -\lim_{t \to +\infty} \inf_{\alpha \in \mathcal{A}} \frac{1}{t} \int_0^t \ell_0\big(Y(s), \alpha(s)\big)\, ds$$

for any initial position y of the fast system (2.18).

Remark 2.4. The existence of the limits in (2.19) and (2.22) and that these limits are independent of the initial positions y of the fast system (2.18) are referred to as the "ergodic control problems" in the literature, see for instance [Ari97], [Ari98]. This constant is also called *ergodic constant* (or *critical value*).

The third characterization of the ergodicity of H is given in terms of the *true cell problem*, as follows:

For each $(x_0, p_0) \in \mathbb{R}^N \times \mathbb{R}^N$, consider the family of stationary equations

$$H\big(x_0, y, p_0, Du(y)\big) = b \quad \text{in } \mathbb{R}^M, \tag{2.23}$$

where b is a real parameter.

Definition 2.5 (true cell problem)**.** The Hamiltonian H is said *ergodic* at (x_0, p_0) if there exists a unique real value $c_0 := c_0(x_0, p_0)$ for which the equation (2.23), with $b = c_0$, has a periodic viscosity subsolution and a periodic viscosity supersolution. In this case, we set

$$\overline{H}(x_0, p_0) := c_0$$

and it is called the *effective Hamiltonian.*

The following theorem states that the above three definitions are equivalent. This result is due to Alvarez and Bardi, see Theorem 4 in [AB03].

Theorem 2.6. *The three* Definitions 2.2, 2.3 and 2.5 *are equivalent, and when H is ergodic at (x_0, p_0), one has*

$$\overline{H}(x_0, p_0) = c_0 = -\text{const}.$$

Remark 2.7. Note that the effective Hamiltonian $\overline{H}$ defined in the above definitions is automatically continuous in $\mathbb{R}^N \times \mathbb{R}^N$ (see Proposition 3 in [AB03]).

As showed in [AB10], the bounded time controllability on the fast system (2.18) is sufficient for the ergodicity of H. We first recall the condition of bounded time controllability.

Definition 2.8. The fast system (2.18) is *bounded time controllable* if for fixed $x_0 \in \mathbb{R}^N$, for any y, z in $\mathbb{R}^M$, there exist $T = T(x_0) > 0$ and a control $\overline{\alpha} \in \mathcal{C}$ such that

$$Y_y(\bar{t}, \overline{\alpha}) = z \quad \text{for some } \bar{t} \leq T.$$

Theorem 2.9. *Under basic assumptions and bounded time controllability on the fast system (2.18), the Hamiltonian H is ergodic at (x_0, p_0).*

Proof. We fix $y, z \in \mathbb{R}^M$ and $\overline{t} \le T$, $\overline{\alpha} \in \mathcal{A}$ such that

$$Y_y(\overline{t}, \overline{\alpha}) = z.$$

By Dynamic Programming Principle, the solution w_δ of (2.16) satisfies

$$w_\delta(y) = \inf_{\alpha \in \mathcal{C}} \left\{ \int_0^{\overline{t}} \ell_0\big(Y_y(s,\alpha), \alpha(s)\big) e^{-\delta s}\, ds + w_\delta\big(Y_y(\overline{t}, \alpha)\big) e^{-\delta \overline{t}} \right\}.$$

In particular, at $\overline{\alpha} \in \mathcal{C}$, one has

$$w_\delta(y) \le \int_0^{\overline{t}} \ell_0\big(Y_y(s,\overline{\alpha}), \overline{\alpha}(s)\big) e^{-\delta s}\, ds + w_\delta(z) e^{-\delta \overline{t}}.$$

By the boundedness of ℓ_0, it is easy to check that δw_δ are equi-bounded in $\mathbb{R}^M$, hence we can find some constant $L > 0$ such that

$$\delta w_\delta(y) - \delta w_\delta(z) \le L(1 - e^{-\delta T}).$$

Similarly, by exchanging the roles of y and z, we get

$$\big|\delta w_\delta(y) - \delta w_\delta(z)\big| \le L(1 - e^{-\delta T}),$$

and hence

$$\lim_{\delta \to 0} \big|\delta w_\delta(y) - \delta w_\delta(z)\big| = 0, \quad \text{uniformly in } y, z \in \mathbb{R}^M.$$

If we fix $z \in \mathbb{R}^M$ and choose a sequence $\delta_k \to 0$ such that $\delta_k w_{\delta_k}(z) \to \theta$, we get the uniform convergence of $\delta_k w_{\delta_k}$ to θ.

We next prove that θ is independent of the choice of the sequence δ_k, which shows the uniform convergence of the whole net δw_δ to θ, as desired. To this end, we consider the true cell problem (2.23),

$$H\big(x_0, y, p_0, Du(y)\big) = b \quad \text{in } \mathbb{R}^M, \quad u \text{ periodic.} \tag{2.24}$$

Let

$$b_1 := \inf\big\{ b \in \mathbb{R} \,:\, (2.24) \text{ has a subsolution} \big\},$$

$$b_2 := \sup \{ b \in \mathbb{R} \, : \, (2.24) \text{ has a supersolution} \}.$$

We are going to prove that $b_1 \geq b_2$. Indeed, let u_1 be a subsolution of (2.24) with $b = b_1$, and u_2 be a supersolution of (2.24) with $b = b_2$. We assume by contradiction that $b_1 < b_2$. Note that, by adding a suitable constant to u_1, we can assume that

$$u_1 > u_2 \quad \text{in } \mathbb{R}^M. \tag{2.25}$$

We have

$$H\big(x_0, y, p_0, Du_1(y)\big) \leq b_1 < b_2 \leq H\big(x_0, y, p_0, Du_2(y)\big)$$

in viscosity sense.

Since u_1 and u_2 are bounded, we can choose $\varepsilon > 0$ sufficiently small such that

$$\varepsilon u_1 + H\big(x_0, y, p_0, Du_1(y)\big) \leq \varepsilon u_2 + H\big(x_0, y, p_0, Du_2(y)\big).$$

We therefore get, by comparison principle

$$u_1 \leq u_2 \quad \text{in } \mathbb{R}^M$$

which contradicts (2.25). We thus conclude that $b_1 \geq b_2$.

Note that, w_δ is solution of the equation

$$\delta w_\delta + H\big(x_0, y, p_0, Dw_\delta(y)\big) = 0 \quad \text{in } \mathbb{R}^M,$$

we observe that for $b > -\theta$, $v = w_{\delta_k}$ is a subsolution of (2.24) for k large enough. This yields $b_1 \leq -\theta$. By a similar argument, we get $b_2 \geq -\theta$, and hence

$$b_1 = b_2 = -\theta.$$

Conclusion, H is ergodic at (x_0, p_0) and

$$\overline{H}(x_0, p_0) = b_1 = b_2 = -\theta = -\lim_{\delta \to 0} \delta w_\delta(y), \quad \text{uniformly in } y \in \mathbb{R}^M.$$

□

We now prove that whenever the Hamiltonian H is ergodic, the value function v^ε converges to viscosity solution v of the effective Cauchy problem. Note that, since the effective Hamiltonian is, in general, not Lipschitz continuous, the comparison principle for effective equation may not hold. Then the convergence result is first proved for upper and lower semilimits of v^ε.

Under basic assumptions, the family of value functions v^ε, $\varepsilon > 0$, is equi-bounded in $[0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^M$, hence we can define upper semilimit and lower semilimit of v^ε, as $\varepsilon \to 0$, respectively, by

$$\begin{aligned} V^*(t,x) &:= \limsup_{\varepsilon\to 0,\,(t',x')\to(t,x)} \sup_{y\in\mathbb{R}^M} V^\varepsilon(t',x',y), \\ V_*(t,x) &:= \liminf_{\varepsilon\to 0,\,(t',x')\to(t,x)} \inf_{y\in\mathbb{R}^M} V^\varepsilon(t',x',y). \end{aligned}$$

Note that, the upper semilimit $V^*(t,x)$ and lower semilimit $V_*(t,x)$ are, respectively, bounded upper semicontinuous and bounded lower semicontinuous in $[0, +\infty) \times \mathbb{R}^N$.

The following convergence result is due to Alvarez and Bardi, see [AB10] and references therein.

Theorem 2.10. *Assume that the Hamiltonian H is ergodic. Then the upper semilimit V^* and lower semilimit V_* are subsolution and supersolution, respectively, of the effective Cauchy problem*

$$\begin{cases} V_t + \overline{H}(x, DV) = 0 & \text{in } (0,+\infty)\times\mathbb{R}^N \\ V(0,x) = h(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{2.26}$$

Proof. We only prove the result for subsolution. The case of supersolution can be argued in a similar way. It is evident that the upper semilimit V^* satisfies the initial condition

$$V^*(0,x) = h(x) \quad \text{for any } x \in \mathbb{R}^N.$$

Fix $(t_0, x_0) \in (0, +\infty) \times \mathbb{R}^N$ and assume $\psi \in \mathrm{C}^1\big((0,+\infty)\times\mathbb{R}^N\big)$ be a strict supertangent to V^* at (t_0, x_0) with $V^*(t_0,x_0) = \psi(t_0,x_0)$. We need to prove that

$$\psi_t(t_0,x_0) + \overline{H}\big(x_0, D\psi(t_0,x_0)\big) \le 0.$$

We assume for contradiction that there exists some $\eta > 0$ such that

$$\psi_t(t_0,x_0) + \overline{H}\big(x_0, D\psi(t_0,x_0)\big) \ge 3\eta. \tag{2.27}$$

For $r > 0$ we define

$$H_r(y,q) := \min\big\{H\big(x,y,D\psi(x,t),q\big)\ :\ |t-t_0|\le r,\ |x-x_0|\le r\big\}. \tag{2.28}$$

We claim that, for $r > 0$ small enough there exists a periodic viscosity solution $w(y)$ of

$$H_r\big(y,Dw(y)\big) \ge \overline{H}(x_0,p_0) - 2\eta \quad \text{in } \mathbb{R}^M, \tag{2.29}$$

here $p_0 := D\psi(t_0,x_0) \in \mathbb{R}^N$.

Notice that, in view of the cell δ-problem and the definition of the effective Hamiltonian $\overline{H}$, we can find δ such that the solution w_δ of (2.16) with $p_0 = D\psi(t_0,x_0)$ verifies

$$\big\|\delta w_\delta + \overline{H}(x_0,p_0)\big\|_{L^\infty(\mathbb{R}^M)} \le \eta.$$

We next consider the approximating equation

$$\delta w_{\delta,r} + H_r(y,Dw_{\delta,r}) = 0 \quad \text{in } \mathbb{R}^M, \quad w_{\delta,r} \text{ periodic.} \tag{2.30}$$

We claim that the above equation has a unique solution (see Lemma 2.11 below). Since

$$H_r(y,q) \to H(x_0,y,p_0,q) \quad \text{locally uniformly in } \mathbb{R}^M\times\mathbb{R}^M, \text{ as } r\to 0,$$

and moreover the cell problem (2.16) has a unique solution, we deduce that, by the stability property of viscosity solution,

$$w_{\delta,r} \to w_\delta \quad \text{locally uniformly in } \mathbb{R}^M, \text{ as } r \to 0.$$

In particular, we can choose $r > 0$ small enough so that

$$\big\|\delta w_{\delta,r} + \overline{H}(x_0,p_0)\big\|_{L^\infty(\mathbb{R}^M)} \le 2\eta.$$

The function $w := w_{\delta,r}$ is viscosity solution to (2.29) as claimed.

We now define the perturbed test function, for $\varepsilon > 0$,

$$\psi^\varepsilon(t,x,y) := \psi(t,x) + \varepsilon\, w(y), \quad (t,x,y) \in (0,+\infty)\times\mathbb{R}^N\times\mathbb{R}^M.$$

We fix $r > 0$ as above so that

$$\left|\psi_t(t,x) - \psi_t(t_0,x_0)\right| \leq \eta \quad \text{for } |t-t_0| < r,\ |x-x_0| < r. \tag{2.31}$$

We consider the cylinder

$$Q(r) := \left\{(t,x,y) \in (0,+\infty) \times \mathbb{R}^N \times \mathbb{R}^M : |t-t_0| < r,\ |x-x_0| < r,\ y \in \mathbb{R}^M\right\}.$$

We will prove in the last part of the proof that ψ^ε is a viscosity supersolution of

$$\psi^\varepsilon_t + H\Big(x, y, D_x\psi^\varepsilon, \frac{D_y\psi^\varepsilon}{\varepsilon}\Big) = 0 \quad \text{in } Q(r) \tag{2.32}$$

Since ψ^ε converges uniformly to ψ on $\overline{Q}(r)$, as $\varepsilon \to 0$, it follows that

$$\limsup_{\varepsilon\to 0,\,(t',x')\to(t,x)} \sup_{y\in\mathbb{R}^M} (V^\varepsilon - \psi^\varepsilon)(t',x',y) = V^*(t,x) - \psi(t,x).$$

Note that (t_0,x_0) is a strict maximum point of $V^* - \psi$, hence

$$V^*(t,x) - \psi(t,x) < V^*(t_0,x_0) - \psi(t_0,x_0) = 0 \quad \text{on } \partial Q(r).$$

This yields the above upper semilimit is strict negative on $\partial Q(r)$. Note that V^ε and ψ^ε are $\mathbb{Z}^M$-periodic in y, hence we can find some constant $\eta' > 0$ such that

$$V^\varepsilon - \psi^\varepsilon \leq -\eta' \quad \text{on } \partial Q(r),$$

for $\varepsilon > 0$ small enough. That means

$$\psi^\varepsilon \geq V^\varepsilon + \eta' \quad \text{on } \partial Q(r).$$

Moreover, V^ε and ψ^ε are, respectively, subsolution and supersolution of (2.13) in $Q(r)$, then by comparison principle, one has

$$\psi^\varepsilon \geq V^\varepsilon + \eta' \quad \text{in } Q(r), \quad \text{for } \varepsilon \text{ small.}$$

Taking the upper semilimit, as $\varepsilon \to 0$, we get

$$\psi \geq V^* + \eta' \quad \text{in } (t_0 - r, t_0 + r) \times B(x_0, r).$$

In particular, one has

$$\psi(t_0, x_0) \geq V^*(t_0, x_0) + \eta'.$$

This contradicts the fact that $\psi(t_0, x_0) = V^*(t_0, x_0)$.

To finish the proof, it is left to prove the claim that ψ^ε is viscosity supersolution of (2.32) in $Q(r)$. Let $(t_1, x_1, y_1) \in Q(r)$ and $\varphi \in \mathrm{C}^1\big((0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^M\big)$ be a subtangent to ψ^ε at (t_1, x_1, y_1). This implies

$$\phi(t, x, y) := \frac{1}{\varepsilon}\big[\varphi(t, x, y) - \psi(t, x)\big]$$

is a subtangent to $w(y)$ at (t_1, x_1, y_1).

By (2.29), we have

$$H_r\big(y_1, D_y\phi(t_1, x_1, y_1)\big) \geq \overline{H}\big(x_0, p_0\big) - 2\eta.$$

Using (2.27) and note that

$$D_y\phi(t_1, x_1, y_1) = \frac{1}{\varepsilon} D_y\varphi(t_1, x_1, y_1),$$

we get

$$\psi_t(t_0, x_0) + H_r\Big(y_1, \frac{D_y\varphi(t_1, x_1, y_1)}{\varepsilon}\Big) \geq \eta,$$

and thus, by (2.31),

$$\psi_t(t_1, x_1) + H_r\Big(y_1, \frac{D_y\varphi(t_1, x_1, y_1)}{\varepsilon}\Big) \geq 0.$$

Notice that $w - \phi$ has a minimum at (t_1, x_1, y_1) and w is independent of (t, x), hence one has

$$\begin{aligned} \frac{\partial}{\partial t}(w - \phi)(t_1, x_1, y_1) &= 0, \\ D_x(w - \phi)(t_1, x_1, y_1) &= 0. \end{aligned}$$

This yields,

$$\begin{aligned} \frac{1}{\varepsilon}\big[\varphi_t(t_1, x_1, y_1) - \psi_t(t_1, x_1)\big] &= 0, \\ \frac{1}{\varepsilon}\big[D_x\varphi(t_1, x_1, y_1) - D\psi(t_1, x_1)\big] &= 0, \end{aligned}$$

and thus

$$\varphi_t(t_1,x_1,y_1)=\psi_t(t_1,x_1),\quad D_x\varphi(t_1,x_1,y_1)=D\psi(t_1,x_1).$$

Taking into account the definition of H_r we get

$$\varphi_t(t_1,x_1,y_1)+H\Big(x_1,y_1,D_x\varphi(t_1,x_1,y_1),\frac{D_y\varphi(t_1,x_1,y_1)}{\varepsilon}\Big)\geq 0.$$

The proof of the claim (2.32) is complete. □

The following lemma used in the above proof, see Lemma 1 [AB03]:

Lemma 2.11. *For a given test function ψ and $r>0$, consider the Hamiltonian H_r defined by* (2.28)*. Then, for each $\delta>0$, there exists a unique viscosity solution to the equation* (2.30)*.*

Remark 2.12. The above convergence result is stated in a general form by using weak semilimits. If we assume, in addition, that V^ε locally uniformly converges on $[0,+\infty)\times\mathbb{R}^N\times\mathbb{R}^M$, as $\varepsilon\to 0$, to some function V, then we have

$$V^*=V_*=V\quad\text{in } [0,+\infty)\times\mathbb{R}^N,$$

hence V is continuous in $[0,+\infty)\times\mathbb{R}^N$, and it is a viscosity solution of (2.26). However, in order to claim the locally uniform convergence of V^ε (or a subsequence), it often requires an equi-continuity of V^ε, a delicate question in singular perturbation problems.

Instead of asking the locally uniform convergence of V^ε, a simpler way is to ask that the comparison principle holds for the limiting equation (2.26) in the sense that every usc viscosity subsolution must be smaller than every lsc viscosity supersolution. From the above theorem, we get that

$$V^*\leq V_*\quad\text{in } [0,+\infty)\times\mathbb{R}^N.$$

The reverse inequality is obvious by the very definition of weak semilimits, hence we actually have

$$V^*=V_*:=V\quad\text{in } [0,+\infty)\times\mathbb{R}^N.$$

This yields that V^ε locally uniformly converges on $[0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^M$, as $\varepsilon \to 0$, to the function V (see Lemma 1.7), and V is the unique solution of (2.26). This useful result is stated in the following corollary:

Corollary 2.13. *Assume that, in addition to the assumptions of the* Theorem 2.10, $\overline{H}$ *satisfies the usual regularity assumptions so as to get comparison principle for* (2.26)*. Then* V^ε *locally uniformly converges on* $[0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^M$*, as* $\varepsilon \to 0$*, to the unique continuous viscosity* V *of the effective problem* (2.26)*.*

Chapter 3

Asymptotic behaviour of singularly perturbed control system: non-periodic setting

3.1 Setting of problem

Consider the singularly perturbed control system of deterministic type

$$\begin{cases} X'(s) = f(X(s), Y(s), \alpha(s)), \ s > 0 \\ Y'(s) = \dfrac{1}{\varepsilon}\, g(X(s), Y(s), \alpha(s)), \ s > 0 \\ X(0) = x, \ Y(0) = y, \end{cases} \tag{S_ε}$$

where ε is a small positive parameter which presents the perturbed term, the control $\alpha : [0, +\infty) \to A$ is any measurable function, A be a compact subset of some Euclidean space; $f : \mathbb{R}^N \times \mathbb{R}^M \times A \to \mathbb{R}^N$, $g : \mathbb{R}^N \times \mathbb{R}^M \times A \to \mathbb{R}^M$ are the dynamics; $(x, y) \in \mathbb{R}^N \times \mathbb{R}^M$ is initial position. We denote by $\mathcal{C}$ the set of controls,

$$\mathcal{C} = \big\{\alpha : [0, +\infty) \to A \mid \alpha(\cdot) \text{ measurable}\big\}.$$

For each $\varepsilon > 0$, $(x, y) \in \mathbb{R}^N \times \mathbb{R}^M$ and $t > 0$, consider the finite horizon optimal control problem (Bolza form) subject to (S_ε): to minimize the following cost functional

$$J^\varepsilon(t,x,y,\alpha) := \int_0^t \ell\big(X^\varepsilon(s), Y^\varepsilon(s), \alpha(s)\big)\, ds + u_0\big(X^\varepsilon(t), Y^\varepsilon(t)\big), \tag{3.1}$$

where $\big(X^\varepsilon(s), Y^\varepsilon(s)\big) := \big(X^\varepsilon(s;x,y,\alpha), Y^\varepsilon(s;x,y,\alpha)\big)$ is solution (also called trajectory) of (S_ε), the functions $\ell : \mathbb{R}^N \times \mathbb{R}^M \times A \to \mathbb{R}$ and $u_0 : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ are given and called *running cost* and *final cost*, respectively.

The classical singular perturbation problem is to pass to the limit when the perturbed parameter ε goes to zero. Its solution leads to the elimination of the state variables Y and the reduction of the dimensions of the system from $N+M$ to N.

As mentioned in the introduction part, there are two main approaches for the above singular perturbation problem. The first one is dynamical system approach which aims at deriving directly an explicit description of the limiting system, see for instance ([AG97], [Art99], [GL99], [AG00], [A04], [G04]).

The second approach to the singular perturbation problem consists of investigating the limit, as $\varepsilon \to 0$, of the corresponding value function

$$V^\varepsilon(t,x,y) := \inf_{\alpha \in \mathcal{C}} J^\varepsilon(t,x,y,\alpha), \tag{3.2}$$

and characterizing its limit, say V, as the unique viscosity solution of a limiting Hamilton-Jacobi-Bellman equation. This approach is extensively studied in a series of papers by Alvarez and Bardi in the periodic framework as well as in the case of compact constraint on the fast trajectory (see [AB01], [AB03], [AB10]).

In this chapter, we will follow the second line of approach, namely PDE approach, and our setting is non-periodic (non-compact). More precisely, we will replace the periodicity on the datum (in fast variable) by coercivity on the running cost (in fast variable). We still keep the strong controllability on the fast system in order for the Bellman-Hamiltonian is coercive in the fast momentum. We are thus in the framework of convex, coercive Hamiltonian with non-compact ground space, namely $\mathbb{R}^M$. The peculiarity of our work is to employ the techniques of Weak KAM theory to define a critical level of the Hamiltonian for which the critical Hamilton-Jacobi equation admits a bounded subsolution and a locally bounded coercive supersolution. These sub- and supersolutions play an important role in asymptotic analysis.

It is worth mentioning that in [AB10], the authors deal with asymptotic behaviour even for Bellman-Isaacs equations (in the stochastic case). In the scope of the present thesis, we will be concerned with deterministic case.

Throughout the chapter we will suppose the following assumptions on the datum that we will refer to as the *standing assumptions*:

Standing assumptions:

(C1) $f(x, y, a)$, $g(x, y, a)$ are bounded continuous in $\mathbb{R}^N \times \mathbb{R}^M \times A$;

(C2) $f(x, y, a)$, $g(x, y, a)$ are Lipschitz continuous in (x, y), uniformly in a;

(C3) $\ell(x, y, a)$ is continuous in $\mathbb{R}^N \times \mathbb{R}^M \times A$; there is a uniform modulus ω_ℓ such that

$$|\ell(x_1, y_1, a) - \ell(x_2, y_2, a)| \leq \omega_\ell\big(|x_1 - x_2| + |y_1 - y_2|\big)$$

for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^N \times \mathbb{R}^M$, uniformly in $a \in A$; and ℓ satisfies the coercive condition in the fast variable, i.e.,

$$\min_{a \in A} \ell(x, y, a) \to +\infty, \quad \text{as } |y| \to +\infty, \text{ uniformly in } x \in \mathbb{R}^N;$$

(C4) u_0 is continuous and bounded from below in $\mathbb{R}^N \times \mathbb{R}^M$;

(C5) for given compact subset $K \subset \mathbb{R}^N \times \mathbb{R}^M$, there exists $r = r(K) > 0$ such that

$$B(0, r) \subset \overline{\text{co}}\, g(x, y, A), \quad \text{for any } (x, y) \in K,$$

where, $B(0, r) \subset \mathbb{R}^M$ denotes the open ball of radius r centered at the origin,

$$g(x, y, A) := \big\{g(x, y, a) : a \in A\big\} \subset \mathbb{R}^M,$$

and $\overline{\text{co}}\, g(x, y, A)$ is the closed convex hull of the set $g(x, y, A)$.

Remark 3.1. (i) Notice that the datum f, g, ℓ in the present setting do not satisfy the periodicity in the fast variable, and moreover the running cost ℓ is no longer bounded in $\mathbb{R}^N \times \mathbb{R}^M \times A$.

(ii) For simplicity of the presentation, we can assume Q_0, L_0 are positive constants such that

$$\begin{aligned}
|f(x,y,a)| &\le Q_0 \quad \text{for any } (x,y,a)\in\mathbb{R}^N\times\mathbb{R}^M\times A;\\
|g(x,y,a)| &\le Q_0 \quad \text{for any } (x,y,a)\in\mathbb{R}^N\times\mathbb{R}^M\times A;\\
u_0(x,y) &\ge -Q_0 \quad \text{for any } (x,y)\in\mathbb{R}^N\times\mathbb{R}^M;\\
|f(x_1,y_1,a)-f(x_2,y_2,a)| &\le L_0\big(|x_1-x_2|+|y_1-y_2|\big),\\
|g(x_1,y_1,a)-g(x_2,y_2,a)| &\le L_0\big(|x_1-x_2|+|y_1-y_2|\big)
\end{aligned}$$

for any $(x_1,y_1,a),(x_2,y_2,a)\in\mathbb{R}^N\times\mathbb{R}^M\times A$. Note that the Lipschitz constant L_0 is independent of control variable a.

(iii) The condition (C5) is referred to as *strong controllability* on the fast part of the dynamic (also called fast system). It is well-known that the *strong controllability* condition (C5) implies the local bounded time controllability on the fast system (see Definition 3.2).

(iv) Taking into account assumption (C4), we define

$$\overline{u}_0(x) := \inf_{y\in\mathbb{R}^M} u_0(x,y), \quad x\in\mathbb{R}^N. \tag{3.3}$$

This function is apparently upper semicontinuous, and will play the role of initial condition in the limit equation we get in the asymptotic procedure.

Note that, under the assumptions (C1)-(C2), for each $\varepsilon>0$, $(x,y)\in\mathbb{R}^N\times\mathbb{R}^M$ and $\alpha\in\mathcal{C}$, the system (S_ε) has a unique solution $\big(X^\varepsilon(s),Y^\varepsilon(s)\big)$, defined for all $s\ge 0$. The components $X^\varepsilon(\cdot)$ and $Y^\varepsilon(\cdot)$ are called, respectively, the *slow trajectory* and *fast trajectory* of the system (S_ε).

Let us rewrite the system (S_ε) in the new time scale $\tau=\dfrac{s}{\varepsilon}$,

$$\begin{cases}
X'(\tau)=\varepsilon f\big(X(\tau),Y(\tau),\alpha(\tau)\big), & \tau>0\\
Y'(\tau)=g\big(X(\tau),Y(\tau),\alpha(\tau)\big), & \tau>0\\
X(0)=x,\ Y(0)=y.
\end{cases} \tag{$\widetilde{\mathrm{S}}_\varepsilon$}$$

Since f is bounded, when letting $\varepsilon \to 0$, we get the so-called *fast system* (also called *associated system*) of (S_ε), as follows

$$\begin{cases} Y'(\tau) = g\big(x, Y(\tau), \alpha(\tau)\big), & \tau > 0 \\ Y(0) = y, \ x \text{ is fixed in } \mathbb{R}^N. \end{cases} \tag{FS}$$

The underlying idea of the above fast system is that when the parameter ε is very small, and tends to zero, the time scale of the variable Y is so fast relative to the variable X. In other words, we can regard the slow variable X is frozen at its initial position $X(0) = x$, while the fast variable Y evolves in time, and is governed by the fast system (FS).

It is worth remarking that in order to solve the cell problem in periodic homogenization, the authors in [LPV86] (and also [E92]) require the coercivity on the Hamiltonian. In the context of singularly perturbed optimal control problem whose value function satisfies in viscosity sense a Hamilton-Jacobi-Bellman equation, Alvarez and Bardi use the bounded-time controllability on the fast system (see [AB10], Chapter 6). This controllability condition is sufficient for the ergodicity on the Hamiltonian allowing to define the effective Hamiltonian. The bounded-time controllability thus can be viewed as a weak form of coercivity, which is of crucial importance in homogenization problem.

We record here a local version of bounded time controllability which is useful in our later analysis.

Definition 3.2. The fast system (FS) is called *local bounded time controllable* if, for given compact subsets $K_1 \subset \mathbb{R}^N$, $K_2 \subset \mathbb{R}^M$, fix $x \in K_1$ and let any $y, z \in K_2$, there exist $T_0 := T_0(K_1, K_2) > 0$ and $\alpha \in \mathcal{C}$ such that

$$Y_y(t; \alpha, x) = z \quad \text{for some} \quad t \leq T_0,$$

where $Y_y(\cdot; \alpha, x)$ is the solution of (FS) with control α, initial position y and fixed parameter x.

In geometrically, the local bounded time controllability means that the fast system can reach any point of a compact subset $K_2 \subset \mathbb{R}^M$ in a bounded time. The strong controllability condition (C5) we set in the standing assumptions implies, of course, the above local bounded time controllability.

Remark 3.3. Notice that if $\big(X^\varepsilon(s), Y^\varepsilon(s)\big)$ is solution of (S_ε) then

$$\big(\widetilde{X}^\varepsilon(s), \widetilde{Y}^\varepsilon(s)\big) := \big(X^\varepsilon(\varepsilon s), Y^\varepsilon(\varepsilon s)\big), \quad s \geq 0$$

is solution of ($\widetilde{\mathrm{S}}_\varepsilon$). Moreover, the minimization problem (3.2) can be rewritten as an equivalent form

$$V^\varepsilon(t,x,y) = \inf_{\alpha \in \mathcal{C}} \left\{ \varepsilon \int_0^{t/\varepsilon} \ell\big(\widetilde{X}^\varepsilon(\tau), \widetilde{Y}^\varepsilon(\tau), \alpha(\tau)\big)\, d\tau + u_0\big(\widetilde{X}^\varepsilon(t/\varepsilon), \widetilde{Y}^\varepsilon(t/\varepsilon)\big) \right\} \quad (3.4)$$

where $\big(\widetilde{X}^\varepsilon(\cdot), \widetilde{Y}^\varepsilon(\cdot)\big)$ is solution of ($\widetilde{\mathrm{S}}_\varepsilon$).

3.2 Some facts of Weak KAM theory

This section contains some basic materials from Weak KAM theory for convex, coercive Hamiltonians. This topic is extensively studied in the compact setting, in particular on the flat torus $\mathbb{T}^M = \mathbb{R}^M/\mathbb{Z}^M$, by Fathi and Siconolfi [FS05]. The aim is to record here some similar results that are true also in the whole space $\mathbb{R}^M$, and more importantly give some new facts holding only in the noncompact setting, namely in $\mathbb{R}^M$. The results we present in this section will be used in the next sections.

Here and through this chapter, we simply refer viscosity (sub/super) solutions as (sub/super) solutions.

Consider an abstract Hamiltonian $F : \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}$ satisfying the following assumptions:

(H1) $(y,q) \mapsto F(y,q)$ is continuous;

(H2) $q \mapsto F(y,q)$ is convex on $\mathbb{R}^M$, for any $y \in \mathbb{R}^M$;

(H3) $q \mapsto F(y,q)$ is coercive in the following sense

$$\lim_{|q|\to+\infty} \min_{y\in K} F(y,q) = +\infty \quad \text{for any compact subset } K \subset \mathbb{R}^M;$$

(H4) $y \mapsto F(y,0)$ is bounded from above in $\mathbb{R}^M$.

Remark 3.4. (i) The assumptions (H1)-(H3) are standard in the setting of Weak KAM theory (see for instance [FS05], [F12]), while the additional condition (H4) ensures that critical value in the non-compact setting differs from $+\infty$ (see Lemma 3.5). In fact, for given $(x,p) \in \mathbb{R}^N \times \mathbb{R}^N$, by setting $F(y,q) := H(x,y,p,q)$, where $H : \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ is the Bellman-Hamiltonian given by (??), then by the standing assumptions, we get that

$$\begin{aligned} F(y,0) &= H(x,y,p,0) \\ &= \max_{a \in A} \big\{ -p \cdot f(x,y,a) - \ell(x,y,a) \big\} \to -\infty, \quad \text{as } |y| \to +\infty, \end{aligned}$$

which shows that F satisfies (H4). The coercive condition (H3) is satisfied by F as well owing to strong controllability condition (C5), see Remark 3.1 (iv).

(ii) Note that, if the ground space is a compact set, for instance the flat torus $\mathbb{T}^M = \mathbb{R}^M/\mathbb{Z}^M$, the condition (H4) is automatically satisfied.

All the curves considered in this section will be Lipschitz continuous. Given such a curve ξ, $L(\xi)$ indicates its length.

We denote by $B(y,r)$ the open ball centered at $y \in \mathbb{R}^M$ with radius a positive constant r, and $\overline{B}(y,r)$ stands for closed ball. For every subset $C \subset \mathbb{R}^M$, $d(\cdot)$ stands for the distance from it. The signed distance is defined by

$$d^{\#}(\cdot, C) = 2d(\cdot, C) - d(\cdot, \partial C).$$

Note that, if C is closed, then

$$d^{\#}(y, C) \leq 0 \iff y \in C. \tag{3.5}$$

It is also well-known that $d^{\#}(\cdot, C)$ is convex if C is itself convex.

For a given closed, convex set C, we set for every $v \in \mathbb{R}^M$

$$\sigma_C(v) = \sup\{\, q \cdot v \,:\, q \in C \,\},$$

and it is referred to as the support function of the set C.

We consider the family of Hamilton-Jacobi equations

$$F(y, Du) = b \quad \text{in } \mathbb{R}^M, \tag{HJ$_\text{b}$}$$

with b real parameter.

We denote by $\mathcal{F}_b$ (possibly empty) the family of viscosity subsolutions of (HJ$_b$) in $\mathbb{R}^M$. Thanks to the coercivity of F, every subsolution of (HJ$_b$) is locally Lipschitz continuous. Moreover by the convexity assumption, the notion of viscosity subsolution and almost everywhere subsolution are equivalent, and furthermore a function u is subsolution of (HJ$_b$) if and only if

$$F(y,q) \leq b \quad \text{for any } y \in \mathbb{R}^M, \text{ and any } q \in \partial u(y),$$

where $\partial u(y)$ is the Clarke's gradient of u at y.
(see Propositions 1.55, 1.58, Chapter 1).

We set for every $b \in \mathbb{R}$, and $y, q, v \in \mathbb{R}^M$

$$Z_b(y) := \big\{ q \in \mathbb{R}^M \ : \ F(y,q) \leq b \big\}.$$

This set is referred to as the b-sub level of $F(y,\cdot)$. Owing to the convexity and coercivity conditions (H2)-(H3), if $Z_b(y) \neq \emptyset$ for every $y \in \mathbb{R}^M$, the set-valued map $y \mapsto Z_b(y)$ is convex and compact in $\mathbb{R}^M$.

We further set

$$\begin{aligned} \sigma_b(y,v) &\equiv \sigma_{Z_b(y)}(v) \\ &= \sup \big\{ q \cdot v \ : \ q \in Z_b(y) \big\} \\ &= \sup \big\{ q \cdot v \ : \ \mathcal{H}(y,q) \leq b \big\}. \end{aligned}$$

Note that, given $b \in \mathbb{R}$ with $Z_b(y) \neq \emptyset$ for every y, in view of (3.5), a function u is a viscosity subsolution (resp. viscosity supersolution) of (HJ$_b$) if and only if $d^{\#}\big(D\psi(y), Z_b(y)\big) \leq 0$ (resp. ≥ 0) for every $y \in \mathbb{R}^M$ and every ψ supertangent (resp. subtangent) to u at y.

In the qualitative analysis of the family of Hamilton-Jacobi equations (HJ$_b$) in $\mathbb{R}^M$, a special value of b is relevant and is defined by

$$c := \inf \big\{ b \in \mathbb{R} \ : \ (\text{HJ}_b) \text{ admits a subsolution} \big\}.$$

Note that, this value in general may be infinite. However, with the above assumptions (H1)-(H5), the value c is in fact finite.

Lemma 3.5. *Under the assumptions* (H1)-(H4)*, the value* c *is finite.*

Proof. By (H4), there exists

$$\sup_{y\in\mathbb{R}^M} F(y,0) := b_0 \in \mathbb{R}.$$

Therefore, for any $b \geq b_0$, the null function is a subsolution to ($\mathrm{HJ_b}$). In other words,

$$\mathcal{F}_b \neq \emptyset, \quad \text{for any } b \geq b_0,$$

which shows $c < +\infty$.

Assume u is a subsolution to ($\mathrm{HJ_b}$) for some $b \in \mathbb{R}$. Since H is convex and coercive in the momentum, we know that

$$F(y,q) \leq b \quad \text{for any } y \in \mathbb{R}^M, \text{ and any } q \in \partial u(y).$$

In particular, for fixed $y_0 \in \mathbb{R}^M$, one has

$$-\infty < \min_{q\in\mathbb{R}^M} F(y_0,q) \leq b.$$

Taking the infimum over all $b \in \mathbb{R}$ such that ($\mathrm{HJ_b}$) admits a subsolution, we conclude that

$$-\infty < \min_{q\in\mathbb{R}^M} F(y_0,q) \leq c,$$

which completes the proof. □

We will refer to c as the *critical value* of the Hamiltonian F, and the corresponding equation

$$F(y, Du) = c \quad \text{in } \mathbb{R}^M, \tag{$\mathrm{HJ_c}$}$$

is called *critical equation*. Its (sub/super) solutions will be also qualified as critical.

Following the metric method which has revealed to be a powerful tool for the analysis of Hamilton-Jacobi equations (see for instance [S03], [FS05]), we first introduce the semidistance related to the Hamiltonian F. Namely, for every $a \in \mathbb{R}$ and $x, y \in \mathbb{R}^M$, we set

$$S_b(y,x) := \inf\left\{ \int_0^1 \sigma_b\big(\xi(t), \xi'(t)\big)\, dt \;:\; \xi \in \mathrm{Lip}_{\,y,x}\big([0,1];\mathbb{R}^M\big)\right\},$$

here, $\text{Lip}_{y,x}\big([0,1];\mathbb{R}^M\big)$ stands for the family of Lipschitz continuous curves ξ defined in $[0,1]$, joining y to x, i.e., $\xi(0) = y$, $\xi(1) = x$.

The semidistance S_b plays a crucial role in the representation formulae for (sub) solutions of (HJ_b). We first recall some basic properties of the semidistance S_b and then provide a class of fundamental (sub) solutions to (HJ_b), at supercritical value $b \geq c$, by using semidistance S_b (see [S03], [FS05]).

Lemma 3.6. *For any $b \in \mathbb{R}$ and any $x, y, z \in \mathbb{R}^M$, one has*

(i) $S_b(y,x) \leq S_b(y,z) + S_b(z,x)$

(ii) $S_b(y,x) \leq R_b|y-x|$ *for some* $R_b > 0$.

Proposition 3.7. *For any $b \geq c$, the followings hold true*

(i) *for any $y \in \mathbb{R}^M$, the functions $x \mapsto S_b(y,x)$ and $x \mapsto -S_b(x,y)$ are both subsolutions of (HJ_b) in $\mathbb{R}^M$ and solutions of (HJ_b) in $\mathbb{R}^M \backslash \{y\}$. In addition*

$$S_b(y,x) = \max\big\{u(x) \,:\, u \textit{ subsolution of } (\text{HJ}_\text{b}) \textit{ with } u(y) = 0\big\};$$

(ii) *a function u is subsolution of (HJ_b) if and only if*

$$u(x) - u(y) \leq S_b(y,x) \quad \textit{for any } x, y \in \mathbb{R}^M.$$

Proposition 3.8. *Given $b \geq c$. Let $C \subset \mathbb{R}^M$ be a closed subset and $u_0 : C \to \mathbb{R}$ be a continuous function such that*

$$u_0(x) - u_0(y) \leq S_b(y,x) \quad \textit{for any } x, y \in C. \tag{3.6}$$

Then the function

$$u(\cdot) := \min\big\{u_0(y) + S_b(y,\cdot) \,:\, y \in C\big\}$$

satisfies

(i) *$u = u_0$ on C;*

(ii) *u is subsolution of (HJ_b) in $\mathbb{R}^M$;*

(iii) *u is solution of (HJ_b) in $\mathbb{R}^M \setminus C$.*

Definition 3.9. The function u_0 satisfying the estimate (3.6) in the above proposition is called *admissible trace* for subsolutions to the equation ($\mathrm{HJ_b}$) on C .

It is well-known that, if the ground space is the flat torus $\mathbb{T}^M$ (a compact set), the equation ($\mathrm{HJ_b}$) has solutions in $\mathbb{T}^M$ if and only if $b = c$. That means the critical value c is characterized by the property of being the unique value such that ($\mathrm{HJ_b}$), with $b = c$, has solutions in $\mathbb{T}^M$ (see [LPV86] and [FS05]). We next discuss the issue that in the whole space $\mathbb{R}^M$ (non-compact case) a solution does exist at the critical value as well as at any supercritical value level. We get the following result:

Proposition 3.10. *The equation* ($\mathrm{HJ_b}$) *has solutions in* $\mathbb{R}^M$ *if and only if* $b \geq c$.

Proof. Given $b \geq c$, and for any $x, y \in \mathbb{R}^M$, we set

$$u(x) := S_b(y, x).$$

Then, by Proposition 3.7, u is subsolution of ($\mathrm{HJ_b}$) in $\mathbb{R}^M$ and solution in $\mathbb{R}^M \setminus \{y\}$.

Let y_k, $k \in \mathbb{N}^*$, be a sequence in $\mathbb{R}^M$ with $|y_k| \to +\infty$ as $k \to +\infty$, and we set

$$u_k(x) := S_b(y_k, x),$$

then u_k is a sequence of subsolutions of ($\mathrm{HJ_b}$) in $\mathbb{R}^M$ and solutions in $\mathbb{R}^M \setminus \{y_k\}$.

Define, for each $k \in \mathbb{N}^*$,

$$\begin{aligned} \widetilde{u}_k(x) &:= u_k(x) - u_k(0) \\ &= S_b(y_k, x) - S_b(y_k, 0). \end{aligned}$$

Clearly, $\widetilde{u}_k$ are also subsolutions of ($\mathrm{HJ_b}$) in $\mathbb{R}^M$ and solutions in $\mathbb{R}^M \setminus \{y_k\}$, and moreover

$$\widetilde{u}_k(0) = 0, \quad \text{for any } k \in \mathbb{N}^*.$$

We next prove that the sequence $\widetilde{u}_k$ is equi-Lipschitz continuous and locally equi-bounded in $\mathbb{R}^M$. In deed, for any $k \in \mathbb{N}^*$ and $x, z \in \mathbb{R}^M$, one has

$$\begin{aligned} \widetilde{u}_k(x) - \widetilde{u}_k(z) &= u_k(x) - u_k(z) \\ &= S_b(y_k, x) - S_b(y_k, z) \\ &\leq S_b(z, x) \\ &\leq R_b|x - z|, \end{aligned}$$

for some $R_b > 0$, where the inequalities follows from the triangle inequality of semidistance S_b (see Lemma 3.6).

Similarly, by exchanging the roles of x and z, we also have

$$\widetilde{u}_k(z) - \widetilde{u}_k(x) \leq S_b(x, z) \leq R_b|z - x|.$$

We thus get

$$|\widetilde{u}_k(x) - \widetilde{u}_k(z)| \leq R_b|x - z|, \quad \text{for any } x, z \in \mathbb{R}^M, \text{ any } k \in \mathbb{N}^*,$$

which shows the equi-Lipschitz continuity of $\widetilde{u}_k$ in $\mathbb{R}^M$. The local equi-boundedness of $\widetilde{u}_k$ in $\mathbb{R}^M$ is easily obtained from the following estimate

$$|\widetilde{u}_k(x)| = |\widetilde{u}_k(x) - \widetilde{u}_k(0)| \leq R_b|x|, \quad \forall x \in \mathbb{R}^M, \forall k \in \mathbb{N}^*.$$

We now apply the Ascoli-Arzelà theorem, there exists a continuous function u_0 such that

$$\widetilde{u}_k \to u_0, \quad \text{locally uniformly in } \mathbb{R}^M, \quad \text{as } k \to +\infty.$$

In the rest of the proof we verify that u_0 is a solution of ($\mathrm{HJ_b}$) in $\mathbb{R}^M$. Clearly, u_0 is a subsolution of ($\mathrm{HJ_b}$) in $\mathbb{R}^M$ by using subsolution property of $\widetilde{u}_k$ and basic stability property of viscosity solution theory. We are left to prove the supersolution property of u_0 in $\mathbb{R}^M$. To this end, we take any $x_0 \in \mathbb{R}^M$ and assume $\varphi \in \mathrm{C}^1$ be a strict subtangent to u_0 at x_0. Since $\widetilde{u}_k$ locally uniformly converges to u_0 in $\mathbb{R}^M$, by Lemma 1.35 (Chapter 1), there exists a sequence x_k, $x_k \to x_0$, as $k \to +\infty$, such that φ is subtangent to $\widetilde{u}_k$ at x_k. Note that, $x_k \neq y_k$ for k large enough, hence by the supersolution property of $\widetilde{u}_k$ in $\mathbb{R}^M \setminus \{y_k\}$, one has

$$F\big(x_k, D\varphi(x_k)\big) \geq b \quad \text{for } k \text{ large enough.}$$

Passing to the limit as $k \to +\infty$, and by the continuity of $\mathcal{H}$ and $\varphi \in \mathrm{C}^1$, we get

$$F\big(x_0, D\varphi(x_0)\big) \geq b.$$

This yields u_0 is supersolution to ($\mathrm{HJ_b}$) in $\mathbb{R}^M$. □

In the analysis of the properties of critical subsolutions, a special role is played by a set $\mathcal{A}$, which has been called in [FS05] the (projected) Aubry set, defined as the collection of points $y \in \mathbb{R}^M$ such that

$$\inf\left\{ \int_0^1 \sigma_c(\xi, \xi')\, dt \,:\, \xi \in \operatorname{Lip}_{y,y}\big([0,1];\mathbb{R}^M\big),\ L(\xi) \geq \delta \right\} = 0 \quad \text{for some } \delta > 0, \tag{3.7}$$

or, equivalently (cf. [FS05], Lemma 5.1),

$$\inf\left\{ \int_0^1 \sigma_c(\xi, \xi')\, dt \,:\, \xi \in \operatorname{Lip}_{y,y}\big([0,1];\mathbb{R}^M\big),\ L(\xi) \geq \delta \right\} = 0 \quad \text{for any } \delta > 0, \tag{3.8}$$

where $L(\xi)$ indicates the length of the curve ξ.

We recall that a subsolution u of ($\mathrm{HJ_b}$) is said to be *strict* in some open subset $\Omega \subset \mathbb{R}^M$ if there exists a constant $b_1 < b$ such that u is a subsolution of

$$F\big(y, Du(y)\big) = b_1 \quad \text{in } \Omega.$$

Equivalently, a subsolution u of ($\mathrm{HJ_b}$) is strict in Ω if

$$\partial u(y) \subset \operatorname{int} Z_b(y), \quad \text{for any } y \in \Omega,$$

where int stands for interior.

We also say that a subsolution u is strict at $y_0 \in \mathbb{R}^M$ if it is strict in some open neighbourhood $U(y_0)$ of y_0.

It is apparent that there is no strict subsolution to ($\mathrm{HJ_c}$) in $\mathbb{R}^M$. However, critical subsolution can be strict in some neighbourhood of any point which is outside the Aubry set $\mathcal{A}$.

The following result shows the link between the Aubry set $\mathcal{A}$ and critical equation ($\mathrm{HJ_c}$) (see [FS05]).

Proposition 3.11. *The following statements hold true*

(i) $y_0 \in \mathcal{A}$ *if and only if* $y \mapsto S_c(y_0, y)$ *is a solution of* ($\mathrm{HJ_c}$)*;*

(ii) $y_0 \notin \mathcal{A}$ *if and only if there exists a critical subsolution which is* C^1 *and strict in some neighbourhood of* y_0*.*

Remark 3.12. From the above proposition, the Aubry set $\mathcal{A}$ can be defined as the set of $y_0 \in \mathbb{R}^M$ such that $y \mapsto S_c(y_0, y)$ is a solution of ($\mathrm{HJ_c}$). The points of $\mathcal{A}$ are also characterized by the fact that no critical subsolution is strict around them.

Starting from the material we have so far illustrated in the chapter the full convergence result is contained in the following paper in collaboration with the supervisor (see in the Appendix, at the end of the thesis).

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Appendix: the paper

SINGULARLY PERTURBED CONTROL SYSTEMS WITH NONCOMPACT FAST VARIABLE

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Abstract. We deal with a singularly perturbed optimal control problem with slow and fast variable depending on a parameter ε. We study the asymptotic, as ε goes to 0, of the corresponding value functions, and show convergence, in the sense of weak semilimits, to sub and supersolution of a suitable limit equation containing the effective Hamiltonian.

The novelty of our contribution is that no compactness condition are assumed on the fast variable. This generalization requires, in order to perform the asymptotic procedure, an accurate qualitative analysis of some auxiliary equations posed on the space of fast variable. The task is accomplished using some tools of Weak KAM theory, and in particular the notion of Aubry set.

1. Introduction

We study a singularly perturbed optimal control problem with a slow variable, say x, and a fast one, denoted by y, with dynamics depending on a parameter ε devoted to become infinitesimal. We are interested in the asymptotic, as ε goes to 0, of the corresponding value functions V^ε, depending on slow, fast variable and time, in view of proving convergence, in the sense of weak semilimits, to some functions independent of y, related to a limit control problem where y does not appear any more, at least as state variable.

More precisely, we exploit that the V^ε are solutions, in the viscosity sense, to a time–dependent Hamilton–Jacobi–Bellman equation of the form

$$u_t^\varepsilon + H\left(x, y, D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}\right) = 0$$

and show that the upper/lower weak semilimit is sub/supersolution to a limit equation

$$u_t + \overline{H}(x, Du) = 0$$

containing the so–called effective Hamiltonian $\overline{H}$, obtained via a canonical procedure we describe below from the Hamiltonian of the approximating equations. We also show that initial conditions, i.e. terminal costs, are transferred, with suitable adaptations, to the limit. See Theorems 4.3, 4.4, which are the main results of the paper.

We tackle the subject through a PDE approach first proposed in this context by Alvarez–Bardi, see [1], [2] and the survey booklet [3], in turn inspired by techniques developed in the framework of homogenization of Hamilton–Jacobi equations by Lions–Papanicolau–Varadhan and Evans, see [17], [11], [12]. The singular perturbation can be actually viewed as a relative homogenization of slow with respect fast variable. In the original formulation, homogenization was obtained assuming periodicity in the underlying space plus coercivity of the Hamiltonian in the momentum variable.

Alvarez–Bardi keep periodicity in y, but do without coercivity, and assume instead bounded time controllability in the fast variable. A condition of this kind is indeed unavoidable, otherwise it cannot be expected to get rid of y at the limit, or even to get any limit. Another noncoercive homogenization problem, arising from turbulent combustion models, has been recently investigated with similar techniques in [18].

The novelty of our contribution is that we remove any compactness condition on the fast variable, and this requires major adaptations in perturbed test function method, which is the core of the asymptotic procedure. We further comment on it later on.

Following a more classical control–theoretic approach, namely directly working on the trajectory of the dynamics, Arstein–Gaitsgory, see [7] and [5], [6], have studied a similar model replacing in a sense periodicity by a coercivity condition in the cost, and allowing y to vary in the whole of $\mathbb{R}^M$, for some dimension M. Beside proving convergence, they also provide a thorough description of the limit control problem, in terms of occupational measures, see [6]. This is clearly a relevant aspect of the topic, but we do not treat it here.

Our aim is to recover their results adapting Alvarez–Bardi techniques. We assume, as in [7] and [5], coercivity of running cost, see **(H4)**, and a controllability condition, see **(H3)**, stronger than the one used in [1], [2], [3] and implying, see Lemma 2.9, coercivity of the corresponding Hamiltonian, at least in the fast variable. We do believe that our methods can also work under bounded time controllability, and so without any coercivity on H, but this requires more work, and the details have still to be fully checked and written down.

The focus of our analysis is on the associate cell problem, namely the one–parameter family of stationary equations, posed in the space of fast variable, obtained by freezing in H slow variable and momentum, say at a value (x_0, p_0). Its role, at least in the periodic case, is twofold: it provides a definition of the effective Hamiltonian $\overline{H}$ at (x_0, p_0) as the minimum value of the parameter for which there is a subsolution (then also supersolutions or solutions do exist), the corresponding equation will be called critical in what follows, and critical sub/supersolutions play the crucial role of correctors in the perturbed test function method.

The absence of compactness calls into questions the very status of the critical value $\overline{H}(x_0, p_0)$ since, in contrast to what happens when periodicity is assumed, the existence of solutions does not characterize any more the critical equation, see Appendix A. Moreover critical sub/supersolutions must enjoy suitable additional properties, as explained below, to be effective in the asymptotic procedure.

The two issues are intertwined. By performing a rather accurate qualitative analysis of the cell problems, we show that (sub/super) solutions usable as correctors can be obtained only for the critical equation. We make essentially use for that of tools issued from weak KAM theory, and in particular of the capital notion of Aubry set. As far as we know, it is the first time that this methodology finds a specific application in singular perturbation or homogenization problems.

The geometric counterpart of coercivity in the cost functional is that the critical equation has a nonempty compact Aubry set for every fixed (x_0, p_0), see Lemma 3.8, which in turn implies existence of coercive solutions possessing a simple representation formula in terms of a related intrinsic metric, and bounded subsolutions as well, see Propositions 3.7, 3.9. Coercive solutions, up to modification depending on ε (see Subsection 3.3), are used in the upper semilimit part of the asymptotic, which is the most demanding point of the analysis.

The paper is organized as follows. In Section 2 we give some preliminary material and standing assumptions, we then study some relevant property of controlled dynamics and how they affect value functions. Approximating Hamilton–Jacobi–Bellman equations and limit problem are also defined. Section 3 is about cell problems and construction of distinguished critical sub/supersolutions to be used as correctors. Sections 4 contain the main results. The appendix is devoted to review some basic facts of metric approach and Weak KAM theory for general Hamilton–Jacobi equations.

2. Setting of the problem

2.1. **Notations and terminology.** Given an Euclidean space, say to fix ideas $\mathbb{R}^N$, for some $N \in \mathbb{N}$, $x \in \mathbb{R}^N$ and $R > 0$ we denote by $B(x, R)$ the open ball centered at x with radius R. Given $B \subset \mathbb{R}^N$, we indicate by $\overline{B}$, $\operatorname{int} B$, its closure and interior, respectively. Given subsets B, C, and a scalar λ, we set

$$\begin{aligned} B + C &= \{x + y \mid x \in B,\, y \in C\} \\ \lambda B &= \{\lambda x \mid x \in B\}. \end{aligned}$$

We make precise that in all Hamilton–Jacobi equations we will consider throughout the paper the term (sub/super) solution must be understood in the viscosity sense.

Given an upper semicontinuous (resp. lower semicontinuous) $u : \mathbb{R}^N \to \mathbb{R}$, we say that a function ψ is supertangent (resp. subtangent) to a u at some point x_0 if it is of class C^1, $u = v$ at x_0 and

$$\psi \geq u \ \text{ (resp. } \psi \leq u), \quad \text{locally at } x_0.$$

If strict inequalities hold in the above formula then ψ will be called strict supertangent (resp. subtangent).

Given a sequence of locally equibounded functions $u_n : \mathbb{R}^M \to \mathbb{R}$, the upper weak semilimit (resp lower weak semilimit) is defined via the formula

$$\begin{aligned} (\limsup{}^{\#} u_n)(x) &= \sup\{\limsup_n u_n(x_n) \mid x_n \to x\} \\ (\text{resp.} \quad (\liminf{}_{\#} u_n)(x) &= \inf\{\liminf_n u_n(x_n) \mid x_n \to x\}). \end{aligned}$$

If u is a locally bounded function and we take in the above formula the sequence u_n constantly equal to u then we get through upper (resp. lower) weak semilimit the upper (resp. lower) semicontinuous envelope of u, denoted by $u^{\#}$ (resp. $u_{\#}$). It is minimal (resp. maximal) upper (resp. lower) semicontinuous function greater (resp. less) than or equal to u.

2.2. **Assumptions.** We assume that the slow variable, usually denoted by x, lives in $\mathbb{R}^N$ and the fast variable y in $\mathbb{R}^M$, for given positive integers N, M. We denote by A the control set, by $f : \mathbb{R}^N \times \mathbb{R}^M \times A \to \mathbb{R}^N$, $g : \mathbb{R}^N \times \mathbb{R}^M \times A \to \mathbb{R}^M$ the controlled vector fields related to slow and fast dynamics, respectively. We also have a running cost $\ell : \mathbb{R}^N \times \mathbb{R}^M \times A \to \mathbb{R}$ and a terminal cost $u_0 : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$. We call, as usual, control a measurable trajectory defined in $[0, +\infty)$ taking values in A. We require:

- **(H1)** Control set: A is a compact subset of some Euclidean space;
- **(H2)** Controlled dynamics: There is a constant $L_0 > 0$ with

$$\begin{aligned} |f(x_1, y_1, a) - f(x_2, y_2, a)| &\leq L_0 \left(|x_1 - x_2| + |y_1 - y_2|\right) \\ |g(x_1, y_1, a) - g(x_2, y_2, a)| &\leq L_0 \left(|x_1 - x_2| + |y_1 - y_2|\right) \end{aligned}$$

  for any (x_i, y_i), $i = 1, 2$ in $\mathbb{R}^N \times \mathbb{R}^M$ and $a \in A$; we assume in addition that $|f|$ is bounded with upper bound denoted by Q_0;
- **(H3)** Total controllability: For any compact set $K \subset \mathbb{R}^N \times \mathbb{R}^M$ there exists $r = r(K) > 0$ such that

$$B(0, r) \subset \overline{\text{co}}\, g(x, y, A) \qquad \text{for } (x, y) \in K,$$

  where $g(x, y, A) = \{g(x, y, a) \mid a \in A\}$;
- **(H4)** Running cost: ℓ is continuous in $\mathbb{R}^N \times \mathbb{R}^M \times A$, and for any compact set $B \subset \mathbb{R}^N$

$$\lim_{|y| \to +\infty} \min_{(x,a) \in B \times A} \ell(x, y, a) = +\infty; \tag{1}$$

- **(H5)** Terminal cost: u_0 is continuous and bounded from below in $\mathbb{R}^N \times \mathbb{R}^M$. To simplify notations, $-Q_0$, see **(H2)**, is also taken as lower bound of u_0 in $\mathbb{R}^N \times \mathbb{R}^M$.

Taking into account Assumption **(H5)**, we define

$$\overline{u}_0(x) = \inf_{y \in \mathbb{R}^M} u_0(x, y) \qquad \text{for any } x \in \mathbb{R}^N. \tag{2}$$

This function is apparently upper semicontinuous, and will play the role of initial condition in the limit equation we get in the asymptotic procedure.

Remark 2.1. Due to Relaxation Theorem plus Filippov Implicit Function Lemma, see for instance [4], [10], the integral trajectories of the differential inclusion

$$\dot{\zeta} \in \overline{\mathrm{co}}\, g(x, \zeta, A) \qquad \text{for } x \text{ fixed in } \mathbb{R}^N,$$

are locally uniformly approximated in time by solutions to

$$\dot{\eta} = g(x, \eta, \alpha) \qquad \text{for some control } \alpha. \tag{3}$$

By iteratively applying this property to a concatenation of a sequence of curves of (3) for infinitesimal times, we derive local bounded time controllability for fast dynamics, namely, given R_1, R_2 positive, there is $T_0 = T_0(R_1, R_2)$ such that for any y_1, y_2 in $B(0, R_1)$, $x \in B(0, R_2)$, we can find a trajectory η of (3) joining y_1 to y_2 in a time $T \leq T_0$.

2.3. **Controlled dynamics.** For any $\varepsilon > 0$, any control α, the controlled dynamics is defined as

$$\begin{cases} \dot{\xi}(t) = & \varepsilon\, f(\xi(t), \eta(t), \alpha(t)) \\ \dot{\eta}(t) = & g(\xi(t), \eta(t), \alpha(t)) \end{cases} \tag{CD_ε}$$

Notice that if ξ, η are solutions to (CD_ε) with initial data (x, y) then the trajectories

$$t \mapsto \xi(t/\varepsilon), \qquad t \mapsto \eta(t/\varepsilon)$$

are solutions to

$$\begin{cases} \dot{\xi}_0(t) = & f(\xi_0(t), \eta_0(t), \alpha(t/\varepsilon)) \\ \varepsilon\, \dot{\eta}_0(t) = & g(\xi_0(t), \eta_0(t), \alpha(t/\varepsilon)) \end{cases} \tag{$\overline{CD}_\varepsilon$}$$

with the same initial data.

Given a trajectory ξ, η of (CD_ε) with initial data (x, y) and control α, for some $\varepsilon > 0$, and $T > 0$, we deduce from standing assumptions and Grönwall Lemma, the following basic estimates:

$$|\xi(t) - x| \leq Q_0\, T \qquad \text{for } t \in [0, T/\varepsilon]. \tag{4}$$

If ζ satisfies

$$\dot{\zeta} = g(x, \zeta, \alpha) \quad \zeta(0) = y,$$

then

$$\begin{aligned} |\eta(T) - \zeta(T)| \ &\leq \ \int_0^T |g(\xi, \eta, \alpha) - g(x, \zeta, \alpha)|\, \mathrm{d}s \\ &\leq \ L_0 \int_0^T \big(|\xi - x| + |\eta - \zeta|\big)\, \mathrm{d}s \leq L_0\, \varepsilon\, Q_0\, T^2\, e^{L_0 T}. \end{aligned} \tag{5}$$

Finally

$$
\begin{aligned}
|\eta(T) - y| &\leq \int_0^T |g(\xi,\eta,\alpha) - g(\xi,y,\alpha)|\,\mathrm{d}s + \int_0^T |g(\xi,y,\alpha)|\,\mathrm{d}s \\
&\leq L_0\, R\, T\, e^{L_0 T},
\end{aligned} \tag{6}
$$

where R is an upper bound of $|g|$ in $B(x,\varepsilon\, T) \times \{y\} \times A$, and similarly

$$
\begin{aligned}
|\eta(T) - y| &\leq \int_0^T |g(\xi,\eta,\alpha) - g(\xi,\eta(T),\alpha)|\,\mathrm{d}s + \int_0^T |g(\xi,\eta(T),\alpha)|\,\mathrm{d}s \\
&\leq L_0\, R'\, T\, e^{L_0 T},
\end{aligned} \tag{7}
$$

where R' is an upper bound of $|g|$ in $B(x,\varepsilon\, T) \times \{\eta(T)\} \times A$.

By using bounded time controllability condition, we further get:

Lemma 2.2. *Given R_1, R_2 positive , $x \in B(0,R_1)$, y, z in $B(0,R_2)$, there is, for any ε, a trajectory $(\xi_\varepsilon,\eta_\varepsilon)$ of (CD_ε), starting at (x,y) and a time T_ε with*

$$
T_0(R_1,R_2) < T_\varepsilon < 3\, T_0(R_1,R_2) \tag{8}
$$

such that

$$
|\eta_\varepsilon(T_\varepsilon) - z| = \mathrm{O}(\varepsilon).
$$

The quantity $T_0(\cdot,\cdot)$ is as in Remark 2.1.

Proof: By controllability condition, see Remark 2.1, there is a control α and a trajectory ζ with

$$
\dot\zeta = g(x,\zeta,\alpha) \qquad \text{for a suitable } \alpha \tag{9}
$$

starting at y and reaching z in a time $T_\varepsilon \leq T_0(R_1,R_2)$. Up to adding a cycle based on z and satisfying (9) for some control, we can assume T_ε to satisfy (8). Note that such a cycle does exist again in force of the controllability condition. We then take, for any ε, the trajectories $(\xi_\varepsilon,\eta_\varepsilon)$ of (CD_ε) starting at (x,y) corresponding to the same control α, and invoke (5) to get the assertion. □

We derive:

Proposition 2.3. *Given a bounded set B of $\mathbb{R}^N \times \mathbb{R}^M$ and $S > 0$, there exists a bounded subset $B_0 \supset B$ such that for any initial data in B and any ε, we can find a trajectory of (CD_ε) lying in B_0 as $t \in [0, S/\varepsilon]$.*

Proof: We fix $(x,y) \in B$. By (4), we can find R_1, R_2 such that $B \subset B(0,R_1) \times B(0,R_2)$, and the first component ξ of any trajectory (ξ,η) of (CD_ε), for any ε, starting at (x,y) is contained in $B(0,R_1)$. We write T_0 for $T_0(R_1,R_2)$. Clearly, it is enough to establish the assertion for ε small.

By applying Lemma 2.2 with ε suitably small and $z = 0$, we find a time T_ε and a trajectory $(\xi_\varepsilon, \eta_\varepsilon)$ of (CD_ε) such that $(\xi_\varepsilon(T_\varepsilon), \eta_\varepsilon(T_\varepsilon)) \in B(0, R_1) \times B(0, R_2)$. Taking into account that the time T_ε is estimated from above and below by a positive quantity, see (8), we can iterate the procedure and get by concatenation of the curves so obtained, a trajectory (ξ_0, η_0) in $[0, t_0/\varepsilon]$, starting at (x, y), with the crucial property that there are times $\{t_i\}$, $i = 1, \cdots k$, for some index k, in $[0, S/\varepsilon]$ such that

for any $t \in [0, S/\varepsilon]$, there is t_i with $|t - t_i| \leq 3\,T_0$;
$\eta_\varepsilon(t_i) \in B(0, R_2)$ for any i.

We derive as $t \in \left[0, \frac{S}{\varepsilon}\right]$

$$|\xi_\varepsilon(t) - x_0| < Q_0\, S \tag{10}$$

$$|\eta_\varepsilon(t)| \leq R_2 + 3\, P\, T_0 \tag{11}$$

with constant P solely depending, see (6), upon R_1, R_2, $T_0(R_1, R_2)$. This proves the assertion.

□

The next result is a strengthened version of Lemma 2.2 stating that the approximation of a value of the fast variable by a trajectory of the fast dynamics can be realized in any predetermined suitably large time. To establish it, we need exploiting total controllability assumption **(H3)** in its full extent. The lemma will be used in the proof of Theorem 4.4.

Lemma 2.4. *Given $x \in \mathbb{R}^N$, y, z in $\mathbb{R}^M$, and $S > 0$ suitably large, there is, for any ε, a trajectory $(\overline{\xi}_\varepsilon, \overline{\eta}_\varepsilon)$ of (CD_ε), starting at (x, y) such that*

$$|\overline{\eta}_\varepsilon(S) - z| = \mathrm{O}(\varepsilon).$$

Proof: We fix R_1, R_2 such that $x \in B(0, R_1)$, and y, z are in $B(0, R_2)$. We take S with $S > 3\,T_0(R_1, R_2)$. By applying Lemma 2.2, we find $T_\varepsilon < 3\,T_0(R_1, R_2) < S$ and, for any ε, a curve $(\xi_\varepsilon, \eta_\varepsilon)$ of (CD_ε) starting at (x, y) with

$$|\eta_\varepsilon(T_\varepsilon) - z| = \mathrm{O}(\varepsilon).$$

By iterating the procedure, if necessary, as in the proof of Lemma 2.2, we can extend it to an interval $[0, S_\varepsilon]$, with $S - S_\varepsilon < T_\varepsilon$, still getting

$$|\eta_\varepsilon(S_\varepsilon) - z| = \mathrm{O}(\varepsilon). \tag{12}$$

By **(H3)** and Relaxation Theorem, see Remark 2.1, we find a control β and a trajectory ζ_ε satisfying

$$\dot{\zeta}_\varepsilon = g(\xi_\varepsilon(S_\varepsilon), \zeta_\varepsilon, \beta) \quad \zeta_\varepsilon(0) = \eta_\varepsilon(S_\varepsilon)$$

with

$$|\zeta_\varepsilon(t) - \eta_\varepsilon(S_\varepsilon)| = \mathrm{O}(\varepsilon) \qquad \text{for } t \in [0, S - S_\varepsilon]. \tag{13}$$

Owing to (5), the trajectory $(\xi_\varepsilon^0, \eta_\varepsilon^0)$ of (CD_ε) starting at $(\xi_\varepsilon(S_\varepsilon), \eta_\varepsilon(S_\varepsilon))$, with control β satisfies

$$|\eta_\varepsilon^0(S - S_\varepsilon) - \zeta_\varepsilon(S - S_\varepsilon)| = \mathrm{O}(\varepsilon). \tag{14}$$

By concatenation of η_ε and η_ε^0, we finally get, in force of (12), (13), (14), a trajectory satisfying the assertion. □

2.4. **Minimization problems and value functions.** We consider for any $(x, y) \in \mathbb{R}^N \times \mathbb{R}^M$, $t > 0$, $\varepsilon > 0$, the optimization problems

$$\inf_\alpha \varepsilon \int_0^{\frac{t}{\varepsilon}} \ell\big(\xi_\varepsilon, \eta_\varepsilon, \alpha\big) \, \mathrm{d}s + u_0 \left(\xi_\varepsilon \left(\frac{t}{\varepsilon} \right), \eta_\varepsilon \left(\frac{t}{\varepsilon} \right) \right) \tag{15}$$

with ξ_ε, η_ε are solutions to (CD_ε) in $[0, +\infty)$, issued from the initial datum (x, y). Or equivalently with the change of variables $r = \varepsilon\, s$

$$\inf_\alpha \int_0^t \ell\big(\xi_\varepsilon^0, \eta_\varepsilon^0, \alpha\big) \, \mathrm{d}r + u_0(\xi_\varepsilon^0(t), \eta_\varepsilon^0(t))) \tag{16}$$

with ξ_ε^0, η_ε^0 are solutions to $(\overline{CD}_\varepsilon)$ in $[0, +\infty)$, issued from (x, y). We denote by V^ε the corresponding value functions, namely the functions associating to any initial datum (x, y) and time t the infimum of the functional in (15)/ (16). They are apparently continuous with respect to all arguments.

Remark 2.5. Looking at the form of the above minimization problem, we understand that coercivity assumption **(H4)** plus **(H5)** plays the role of a compactness condition for the fast variable, inasmuch as it implies that the trajectories of the fast dynamics realizing the value function, up to some small constant, lie in a compact subset of $\mathbb{R}^M$. This fact will be crucial in the asymptotic analysis.

We derive from Proposition 2.3:

Proposition 2.6. *The value functions V^ε are locally equibounded.*

Proof: Let C be a bounded set of $\mathbb{R}^N \times \mathbb{R}^M \times [0, +\infty)$, and $(x_0, y_0, t_0) \in C$. Thanks to Proposition 2.3, there are for any ε trajectories (ξ_0, η_0), we drop the dependence on ε to ease notations, of (CD_ε) starting at (x_0, y_0), and contained in a bounded set of $\mathbb{R}^N \times \mathbb{R}^M$ solely depending on C. By using the formulation (15) of the minimization problem, we get

$$V^\varepsilon(x_0, y_0, t_0) \le \varepsilon \int_0^{\frac{t_0}{\varepsilon}} \ell(\xi_0(s), \eta_0(s), \alpha(s)) \, \mathrm{d}s + u_0(\xi_0(t_0/\varepsilon), \eta_0(t_0/\varepsilon)).$$

Since the integrand in the above formula and u_0 are bounded independently of ε, we obtain the equiboundedness from above of the V^ε.

We now consider any trajectory (ξ, η) of (CD_ε) starting (x_0, y_0) and corresponding to a control β. By (4), $\xi(t)$ lies in a compact subset K of $\mathbb{R}^N$, only depending on C, for $t \in [0, t_0/\varepsilon]$, and by coercivity assumption **(H4)**, there is a constant P_0 with

$$\ell(x,y,a) \geq P_0 \qquad \text{for any } (x,y,a) \in K \times \mathbb{R}^M \times A. \tag{17}$$

Since $-Q_0$ is a lower bound of u_0 in $\mathbb{R}^N \times \mathbb{R}^M$, see **(H5)**, this implies

$$\begin{aligned} &\varepsilon \int_0^{\frac{t_0}{\varepsilon}} \ell(\xi(s), \eta(s), \beta(s))\, \mathrm{d}s + u_0(\xi(t_0/\varepsilon), \eta(t_0/\varepsilon)) \geq \\ &\varepsilon \frac{t_0}{\varepsilon} P_0 + u_0(\xi(t_0/\varepsilon), \eta(t_0/\varepsilon)) \geq P_0\, t_0 - Q_0. \end{aligned} \tag{18}$$

Being (ξ, η) an arbitrary trajectory with initial point (x_0, y_0), the above inequality shows the claimed local equiboundedness from below of value functions.

□

The previous result allows us to define $\limsup^\# V^\varepsilon$, $\liminf_\# V^\varepsilon$, these functions will be denoted by $\overline{V}$, $\underline{V}$, respectively, in what follows. The next proposition shows that they only depend on time and slow variable, at least for positive times.

Proposition 2.7. *We have*

$$\begin{aligned} (\liminf{}_\# V^\varepsilon)(x_0, y_0, t_0) &= (\liminf{}_\# V^\varepsilon)(x_0, z_0, t_0) =: \underline{V}(x_0, t_0) \\ (\limsup{}_\# V^\varepsilon)(x_0, y_0, t_0) &= (\limsup{}_\# V^\varepsilon)(x_0, z_0, t_0) =: \overline{V}(x_0, t_0) \end{aligned}$$

for any $x_0 \in \mathbb{R}^N$, y_0, z_0 in $\mathbb{R}^M$ and $t_0 > 0$.

Proof: We start by

Claim: *Given positive constants R_1, R_2, S we can determine $P = P(R_1, R_2, S) > 0$ such that for any $\varepsilon > 0$, $x \in B(0, R_1)$, y, z in $B(0, R_2)$, $t \in [0, S]$ there exist x', x'', z', z'', t', t'', depending on ε, with*

$$|x - x'| < \varepsilon P, \quad |z - z'| < \varepsilon P, \quad |t - t'| < \varepsilon P,$$

$$|x - x''| < \varepsilon P, \quad |z - z''| < \varepsilon P, \quad |t - t''| < \varepsilon P$$

such that

$$\begin{aligned} V^\varepsilon(x', z', t') &< V^\varepsilon(x, y, t) + \varepsilon P \\ V^\varepsilon(x'', z'', t'') &> V^\varepsilon(x, y, t) - \varepsilon P. \end{aligned}$$

We fix ε. By controllability assumption (see Remark 2.1) z and y can be joined in a time T less than or equal to $T_0 = T_0(R_1, R_2)$ by a curve ζ satisfying

$$\dot{\zeta} = g(x, \zeta, \alpha) \qquad \text{for a suitable control } \alpha.$$

We consider the trajectory (ξ, η) of (CD_ε) with the same control α satisfying

$$\xi(T) = x \quad \text{and} \quad \eta(T) = y,$$

and set

$$x' = \xi(0) \quad \text{and} \quad z' = \eta(0).$$

By (4), (5), we get

$$|x' - x| < \varepsilon P_0 \tag{19}$$

$$|z' - z| < \varepsilon P_0 \tag{20}$$

for a suitable $P_0 > 0$. We select a trajectory (ξ_0, η_0) of (CD_ε) with initial datum (x, y), corresponding to a control β, such that

$$V^\varepsilon(x, y, t) \geq \varepsilon \int_0^{\frac{t}{\varepsilon}} \ell(\xi_0, \eta_0, \beta) \, \mathrm{d}s + u_0 \left(\xi_0 \left(\frac{t}{\varepsilon} \right), \eta_0 \left(\frac{t}{\varepsilon} \right) \right) - \varepsilon. \tag{21}$$

We set

$$t' = t + \varepsilon T, \tag{22}$$

by concatenation of α and β, ξ and ξ_0, η and η_0, we get a control γ and trajectory $(\overline{\xi}, \overline{\eta})$ of (CD_ε) starting at (x', z'), defined in $\left[0, \frac{t'}{\varepsilon}\right]$. We consequently have

$$\begin{aligned}
&V^\varepsilon(x', z', t') \leq \\
&\varepsilon \int_0^{\frac{t'}{\varepsilon}} \ell(\overline{\xi}, \overline{\eta}, \gamma) \, \mathrm{d}s + u_0 \left(\overline{\xi} \left(\frac{t'}{\varepsilon} \right), \overline{\eta} \left(\frac{t'}{\varepsilon} \right) \right) = \\
&\varepsilon \int_0^{T} \ell(\xi, \eta, \alpha) \, \mathrm{d}s + \varepsilon \int_T^{\frac{t'}{\varepsilon}} \ell(\xi_0(s-T), \eta_0(s-T), \beta(s-T)) \, \mathrm{d}s + \\
&u_0 \left(\xi_0 \left(\frac{t'}{\varepsilon} \right), \eta_0 \left(\frac{t'}{\varepsilon} \right) \right).
\end{aligned}$$

By taking into account (5) and (21), we derive

$$V^\varepsilon(x', z', t') \leq \varepsilon \, Q \, T_0 + V^\varepsilon(x, y, t) + \varepsilon \quad \text{for a suitable } Q > 0. \tag{23}$$

The first part of the claim is therefore proved taking into account (19), (20), (22), (23), and defining

$$P = \max\{P_0, T_0, Q \, T_0 + 1\}.$$

The estimates for x'', y'', z'', t'' can be obtained slightly modifying the above argument.

We sketch the proof for reader's convenience. We denote by ζ' a curve joining y to z in a time $T' \leq T_0$ and satisfying

$$\dot{\zeta}' = g(x, \zeta', \alpha) \qquad \text{for a suitable control } \alpha'.$$

We consider the trajectory (ξ', η') of (CD_ε) with the same control α' satisfying

$$\xi'(0) = x \quad \text{and} \quad \eta'(0) = y,$$

and set

$$x'' = \xi'(T') \quad \text{and} \quad z'' = \eta'(T').$$

As in the first part of the proof we get

$$\begin{aligned} |x'' - x| &\leq P_0\, \varepsilon \\ |z'' - z| &\leq P_0\, \varepsilon, \end{aligned}$$

for a suitable P_0. We select a trajectory (ξ_0', η_0') of (CD_ε) with initial datum (x'', z''), corresponding to a control β', which is optimal for $V^\varepsilon(x'', z'', t - \varepsilon\, T')$ up to ε, namely

$$V^\varepsilon(x'', z'', t'') \geq \varepsilon \int_0^{\frac{t''}{\varepsilon}} \ell(\xi_0', \eta_0', \beta')\, \mathrm{d}s + u_0 \left(\xi_0' \left(\frac{t''}{\varepsilon} \right), \eta_0' \left(\frac{t''}{\varepsilon} \right) \right) - \varepsilon.$$

Here we are assuming ε so small that $t'' := t - \varepsilon\, T'$ is positive, this does not entail any limitation to the argument since we are interested to ε infinitesimal. From this point we go on as in the previous part.

We exploit the first part of the claim to show that for any pair of values y_0, z_0 of the fast variable, any $x_0 \in \mathbb{R}^N$, $t_0 > 0$

$$(\liminf{}_\# V^\varepsilon)(x_0, z_0, t_0) \leq (\liminf{}_\# V^\varepsilon)(x_0, y_0, t_0), \tag{24}$$

which in turn implies by the arbitrariness of y_0, z_0, that $\liminf_\# V^\varepsilon$ independent of the fast variable. We consider ε_n, x_n, y_n, t_n converging to 0, x_0, y_0, t_0, respectively, with

$$\lim_n V^{\varepsilon_n}(x_n, y_n, t_n) = (\liminf{}_\# V^\varepsilon)(x_0, z_0, t_0).$$

Since all the x_n , y_n, and z_0, t_n are contained in compact subsets of $\mathbb{R}^N$, $\mathbb{R}^M$, $[0, +\infty)$, respectively, we can apply, for any given $n \in \mathbb{N}$, the claim to $\varepsilon = \varepsilon_n$, $x = x_n$, $y = y_n$, $z = z_0$, $t = t_n$ and get of x_n', z_n', t_n' with

$$|x_n - x_n'| < \varepsilon_n\, P, \quad |z_0 - z_n'| < \varepsilon_n\, P, \quad |t_n - t_n'| < \varepsilon_n\, P$$

and

$$V^{\varepsilon_n}(x_n', z_n', t_n') < V^{\varepsilon_n}(x_n, y_n, t_n) + \varepsilon_n\, P$$

for a suitable P. Sending n to infinity we deduce

$$\liminf V^{\varepsilon_n}(x_n', z_n', t_n') \leq \lim V^{\varepsilon_n}(x_n, y_n, t_n) = (\liminf{}_\# V^\varepsilon)(x_0, z_0, t_0),$$

which implies (24) since $x_n' \to x_0$, $z_n' \to z_0$ and $t_n' \to t_0$.

The assertion relative to $\limsup^{\#} V^\varepsilon$ is obtained using the second part of the claim and slightly adapting the above argument.

□

As a consequence of coercivity of running cost assumed in **(H4)** we deduce:

Proposition 2.8. *The value function V^ε satisfy for any ε, any compact subset K of $\mathbb{R}^N \times (0,+\infty)$*

$$\lim_{|y|\to+\infty} \min_{(x,t)\in K} V^\varepsilon(x,y,t) = +\infty.$$

Proof: We fix ε, we assume, without loosing any generality, that K is of the form $\widetilde{K} \times [S,T]$, where $\widetilde{K}$ is a compact subset of $\mathbb{R}^N$ and S, T are positive times. Given any $P > 0$, we can determine by **(H4)** a constant R such that the ball $B(0,R)$ of $\mathbb{R}^M$ satisfies

$$\ell(x,y,a) > P \qquad \text{for any } (x,a) \in \widetilde{K} \times A,\ y \in \mathbb{R}^M \setminus B(0,R). \tag{25}$$

Taking into account the estimate (7), we see that there exists $R_0 > R$ such that

$$\eta(t) \notin B(0,R) \qquad \text{for } t \in [0,T] \tag{26}$$

for any trajectory of (CD_ε) starting in $K_0 \times \big(\mathbb{R}^M \setminus B(0,R_0)\big)$. Given $\delta > 0$, we find, for any

$$(x,y,t) \in K_0 \times \big(\mathbb{R}^M \setminus B(0,R_0)\big) \times [S,T]$$

a trajectory (ξ_0, η_0) of (CD_ε), corresponding to a control α, starting at (x,y) with

$$V^\varepsilon(x,y,t) \geq \varepsilon \int_0^{\frac{t}{\varepsilon}} \ell(\xi_0,\eta_0,\alpha)\,\mathrm{d}s + u_0\left(\xi_0\left(\frac{t}{\varepsilon}\right), \eta_0\left(\frac{t}{\varepsilon}\right)\right) - \delta.$$

We deduce by (25), (26), **(H5)**

$$V^\varepsilon(x,y,t) \geq P\,S - Q_0 - \delta,$$

which gives the assertion, since P can be chosen as large as desired, and δ as small as desired. □

2.5. **HJB equations.** We define the Hamiltonian

$$H(x,y,p,q) = \max_{a\in A}\{-p\cdot f(x,y,a) - q\cdot g(x,y,a) - \ell(x,y,a)\}$$

The main contribution of Assumption **(H3)** is the following coercivity property on H:

Lemma 2.9. *For any given bounded set $C \subset \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^N$, we have*

$$\lim_{|q|\to+\infty} \min_{(x,y,p)\in C} H(x,y,p,q) = +\infty.$$

Proof: We denote by r the positive constant provided by **(H3)** in correspondence to the projection of C on the state variables space $\mathbb{R}^N \times \mathbb{R}^M$. We consequently have for (x,y) in such projection and $q \in \mathbb{R}^M$

$$\max\{q \cdot v \mid v \in g(x,y,A)\} = \max\{q \cdot v \mid v \in \overline{\text{co}}\, g(x,y,A)\} \geq r\,|q|. \tag{27}$$

We take $(x,y,p) \in C$, and denote by a_0 an element in the control set such that $g(x,y,a_0)$ realizes the maximum in (27). We get from the very definition of H and (27)

$$H(x,y,p,q) \geq -|p|\,|f(x,y,a_0)| + r\,|q| - |\ell(x,y,a)| \qquad \text{for any } q.$$

When we send $|q|$ to infinity, all the terms in the right hand–side of the above formula stay bounded except $r\,|q|$. This gives the assertion.

□

Given a bounded set B in $\mathbb{R}^N \times \mathbb{R}^M$, one can check by direct calculation that H satisfies

$$\begin{aligned} &|H(x_1,y_1,p,q) - H(x_2,y_2,p,q)| \leq \\ &L_0\,(|x_1-x_2| + |y_1-y_2|)(|p|+|q|) + \omega(|x_1-x_2| + |y_1-y_2|) \end{aligned} \tag{28}$$

for any (x_1,y_1), (x_2,y_2) in B and $(p,q) \in \mathbb{R}^N \times \mathbb{R}^M$, where ω is an uniform continuity modulus of ℓ in $B \times A$ and L_0 is as in **(H2)**. We also have

$$\begin{aligned} &|H(x,y,p_1,q_1) - H(x,y,p_2,q_2)| \leq \\ &|f(x,y,a_0)|\,|p_1-p_2| + |g(x,y,a_0)|\,|q_1-q_2| \end{aligned} \tag{29}$$

for any $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$, (p_1,q_1), (p_2,q_2) in $\mathbb{R}^N \times \mathbb{R}^M$, a suitable $a_0 \in A$.

We write, for any $\varepsilon > 0$, the family of Hamilton–Jacobi–Bellman problems

$$\begin{cases} u^\varepsilon_t + H\left(x,y,D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}\right) & = 0 \\ u^\varepsilon(x,y,0) & = u_0(x,y) \end{cases} \tag{HJ$_\varepsilon$}$$

It is well known that the value functions V^ε are solutions to (HJ$_\varepsilon$), even if not necessarily unique in our setting. However, due to the estimate (28), we have the following local comparison result (see for instance [9]):

Proposition 2.10. *Given a bounded open set B of $\mathbb{R}^N \times \mathbb{R}^M$ and times $t_2 > t_1$, let u, v be continuous subsolution and supersolution, respectively, of the equation in* (HJ$_\varepsilon$). *If $u \leq v$ in $\partial_p\big(B \times (t_1,t_2)\big)$ then $u \leq v$ in $B \times (t_1,t_2)$, where ∂_p stands for the parabolic boundary.*

We define the *effective Hamiltonian*

$$\overline{H}(x,p) = \inf\{b \in \mathbb{R} \mid H(x,y,p,Du) = b \text{ admits a subsolution in } \mathbb{R}^M\} \tag{30}$$

for any fixed $(x,p) \in \mathbb{R}^N \times \mathbb{R}^N$, where the equation appearing in the formula is solely in the fast variable y with slow variable x and corresponding momentum p frozen. This quantity can be in principle infinite, however we will show in what follows that not only it is finite for any (x, p), but also that the infimum is actually a minimum.

We write the limit equation

$$(\overline{\text{HJ}}) \qquad u_t + \overline{H}(x, Du) = 0.$$

3. Cell problems

The section is devoted to the analysis of the stationary Hamilton–Jacobi equations in $\mathbb{R}^M$ appearing in the definition of effective Hamiltonian, namely with slow variable and corresponding momentum frozen.

3.1. **Basic analysis.** We fix $(x_0, p_0) \in \mathbb{R}^N \times \mathbb{R}^N$, and set to ease notations

$$\begin{array}{rcll} H_0(y, q) & = & H(x_0, y, p_0, q) & \text{for any } (y, q) \in \mathbb{R}^M \times \mathbb{R}^M \\ \ell_0(y, a) & = & \ell(x_0, y, a) + p_0 \cdot f(x_0, y, a) & \text{for any } (y, a) \in \mathbb{R}^M \times A \\ g_0(y, a) & = & g(x_0, y, a) & \text{for any } (y, a) \in \mathbb{R}^M \times A \end{array}$$

Given a control $\alpha(t)$, we consider the controlled differential equation in $\mathbb{R}^M$

$$(31) \qquad \dot{\eta}(t) = g_0(\eta(t), \alpha(t)).$$

We directly derive from Lemma 2.9:

Lemma 3.1. *We have*

$$\lim_{|q| \to +\infty} \min_{y \in K} H_0(y, q) = +\infty$$

for any compact subset K of $\mathbb{R}^M$.

This result implies, according to Lemma A.1, that all subsolutions are locally Lipschitz–continuous, and allows adopting the metric method, see Appendix A, in the analysis of the cell equations. To ease notation, we set $c_0 = \overline{H}(x_0, p_0)$, also called the critical value of H_0, see (86). We will prove in Proposition 3.3 that c_0 is finite. We denote by Z, σ, S the corresponding sublevels, support function and intrinsic distance, see Appendix A for the corresponding definitions. Same objects for a supercritical value b will be denoted by Z_b, σ_b, S_b.

To compare the metric and control–theoretic viewpoint, we notice

$$Z_b(y) = \{q \in \mathbb{R}^M \mid q \cdot (-g_0(y, a)) \le \ell_0(y, a) + b \text{ for any } a \in A\}.$$

for any given supercritical $b \in \mathbb{R}$, namely $b \geq c_0$, and $y \in \mathbb{R}^M$. This implies that the support function $\sigma_b(y, \cdot)$ is the maximal subadditive positively homogeneous function $\rho : \mathbb{R}^M \to \mathbb{R}$ with

$$\rho(-g_0(y,a)) \leq \ell_0(y,a) + b \qquad \text{for any } a \in A, \tag{32}$$

which somehow justifies the next equivalences.

Proposition 3.2. *Given a supercritical value b, the following conditions are equivalent:*

- **(i)** *u is a subsolution to $H_0 = b$;*
- **(ii)** *$u(y_2) - u(y_1) \leq S_b(y_1, y_2)$ for any y_1, y_2;*
- **(iii)** *$u(y_1) - u(y_2) \leq \int_0^T (\ell_0(\eta(t), \alpha(t)) + b)\, \mathrm{d}t$ for any y_1, y_2, time T, control α, any trajectory η of (31) with $\eta(0) = y_1$, $\eta(T) = y_2$.*

Proof: The equivalence **(i)** $\iff$ **(ii)** is given in Proposition A.3 **(i)**, the equivalence **(i)** $\iff$ **(iii)** is the usual characterization of subsolutions to Hamilton–Jacobi–Bellman equations in terms of suboptimality, see [8]. □

One advantage of metric method is that any curve is endowed of a length, while integral cost functional is only defined on trajectories of the controlled dynamics. Also notice that there is a change of orientation between length and cost functional, that can detected from (32) and comparison between items **(ii)** and **(iii)** in Proposition 3.2. This just depends on u_0 being terminal cost and initial condition in (HJ_ε), the discrepancy should be eliminated if (HJ_ε) were posed in $(-\infty, 0)$ and u_0 should consequently play the role of terminal condition and initial cost.

Proposition 3.3. *The critical value c_0 is finite.*

Proof: Owing to coercivity of ℓ and boundedness of f

$$H_0(y,0) = \max_{a \in A}\{-\ell_0(y,a)\} \to -\infty \quad \text{as } |y| \to +\infty$$

and consequently

$$H_0(y,0) < 0 \quad \text{outside some compact subset } K \text{ of } \mathbb{R}^M.$$

We set

$$b_0 \quad = \quad \max\big\{0, \max\{H_0(x,0) \mid x \in K\}\big\},$$

then the null function is subsolution to $H_0 = b_0$ in $\mathbb{R}^M$, and so $c_0 < +\infty$.

By controllability condition **(H3)**, we find a cycle η defined in $[0,T]$, for a positive T, solution to (31) for some control α. We put

$$R = \int_0^T \ell_0(\eta, \alpha)\, \mathrm{d}t,$$

and for $b < -\frac{R}{T}$ we get

$$\int_0^T (\ell_0(\eta, \alpha) + b)\, \mathrm{d}t < R - \frac{R}{T} T = 0.$$

The above cycle, repeated infinite times, gives a trajectory of (31) in $[0, +\infty)$, still denoted by η, such that

$$\int_0^\infty (\ell_0(\eta, \alpha) + b)\, \mathrm{d}t = -\infty. \tag{33}$$

If there were a subsolution u to $H_0 = b$ then

$$u(\eta(0)) - u(\eta(t_0)) \leq \int_0^{t_0} (\ell_0(\eta(t), \alpha(t)) + b)\, \mathrm{d}t \quad \text{for any } t_0 > 0. \tag{34}$$

But the support of η is equal to $\eta([0, T])$ which is a compact subset of $\mathbb{R}^M$, so that the oscillation of u (which is locally Lipschitz continuous) on it is bounded. This shows that (33) and (34) are in contradiction. We then deduce that the equation $H_0 = b$ cannot have any subsolution, showing in the end that $c_0 > -\infty$. □

We deduce from standing assumptions a sign and a coercivity condition on the critical distances. To do that, we start selecting a compact set C of $\mathbb{R}^M$ with

$$H_0(y, 0) = -\min_{a \in A} \ell_0(y, a) < c_0 - Q_0 \qquad \text{for any } y \in \mathbb{R}^M \setminus C, \tag{35}$$

where Q_0 is as in **(H2)**. This is possible since ℓ_0 is coercive. Further we set

$$K_0 = \left\{ y \mid d(y, C) \leq \max_{C \times C} |S| \right\}. \tag{36}$$

Proposition 3.4. *The following properties hold true:*

- **(i)** $\lim_{|y| \to +\infty} \inf_{y_0 \in K} S(y_0, y) = +\infty$ *for any compact set* $K \subset \mathbb{R}^M$*;*
- **(ii)** $Z(y) \supset B(0, 1)$ *for any* y *outside the compact set* K_0 *defined as in* (36)*;*
- **(iii)** $S(y_1, y_2) > 0$ *for any pair* y_1*,* y_2 *outside* K_0*.*

Proof: If $q \in \mathbb{R}^M$ satisfies

$$H_0(y, q) = c_0 \qquad \text{for some } y \text{ in } \mathbb{R}^M \setminus C, \tag{37}$$

where C is defined as in (35), then

$$c_0 = H_0(y, q) = \max_{a \in A} \{-g_0(y, a) \cdot q - \ell_0(y, a)\} \leq Q_0\, |q| - \min_{a \in A} \ell_0(y, a)$$

and by the very definition of C

$$|q| \geq \frac{c_0 + \min_{a \in A} \ell_0(y, a)}{Q_0} > \frac{Q_0}{Q_0} = 1, \tag{38}$$

Since 0 in the interior of $Z(y)$ by (35), we derive a stronger version of item **(ii)**, with C in place of K_0, which in turn implies

$$\frac{v}{|v|} \in Z(y) \qquad \text{for any } y \in \mathbb{R}^M \setminus C,\ v \in \mathbb{R}^M \text{ with } v \neq 0$$

and consequently

$$\sigma(y,v) \geq v \cdot \left(\frac{v}{|v|}\right) = |v| \qquad \text{for any } y \in \mathbb{R}^M \setminus C,\ v \in \mathbb{R}^M \text{ with } v \neq 0. \tag{39}$$

Next, we fix a compact set K and consider two points $y_1 \in K$, $y_2 \notin C$ and any curve ζ, defined in $[0,1]$, linking them. We distinguish two cases according on whether the intersection of ζ with C is nonempty or empty. In the first instance we set

$$t_1 = \min\{t \in [0,1] \mid \zeta(t) \in C\} \tag{40}$$

$$t_2 = \max\{t \in [0,1] \mid \zeta(t) \in C\}. \tag{41}$$

We denote by R an upper bound of $|S|$ in $C \times C$ and exploit (39) to get

$$\begin{aligned} \int_0^1 \sigma(\zeta,\dot\zeta)\,\mathrm{d}t &= \int_0^{t_1} \sigma(\zeta,\dot\zeta)\,\mathrm{d}t + \int_{t_1}^{t_2} \sigma(\zeta,\dot\zeta)\,\mathrm{d}t + \int_{t_2}^1 \sigma(\zeta,\dot\zeta)\,\mathrm{d}t \\ &\geq |y_1 - \zeta(t_1)| + S(\zeta(t_1),\zeta(t_2)) + |y_2 - \zeta(t_2)| \\ &\geq -R + d(y_1,C) + d(y_2,C). \end{aligned} \tag{42}$$

If instead the curve ζ entirely lies outside C, we have by (39)

$$\int_0^1 \sigma(\zeta,\dot\zeta)\,\mathrm{d}t \geq |y_1 - y_2|. \tag{43}$$

In both cases we get item **(i)** sending y_2 to infinity and taking into account that y_1 has been arbitrarily chosen in K.

We finally see, looking at (42), (43), and slightly adapting the above argument that K_0, defined as in (36), satisfies item **(iii)**. □

Remark 3.5. Given a compact set $K \subset \mathbb{R}^M$, the same argument of Proposition 3.4 allows also proving

$$\lim_{|y|\to+\infty} \inf_{y_0 \in K} S(y,y_0) = +\infty \tag{44}$$

Corollary 3.6. *For any bounded open set B there exists $R > 0$ such that if y_1, y_2 belong to B then all 1–optimal curves for $S(y_1,y_2)$ are contained in $B(0,R)$.*

Proof: We can assume without loosing generality that $B \supset K_0$, where K_0 is the set defined in (36). We set

$$P = \sup_{B \times B} |S|.$$

By Proposition 3.4 **(i)** there is R such that

$$\inf_{y_0 \in B} S(y_0, y) > 2\,P + 2 \qquad \text{for } y \text{ with } |y| > R.$$

We claim that such an R satisfies the claim. In fact, assume by contradiction that there are y_1, y_2 in B and an 1–optimal curve ζ, defined in $[0,1]$, for $S(y_1, y_2)$ not contained in $B(0,R)$. Let t_1 be a time in $(0,1)$ with $\zeta(t_1) \notin B(0,R)$ and set

$$t_2 = \min\{t \in (t_1, 1) \mid \zeta(t) \in K_0 \subset B\}$$

then, taking into account Proposition 3.4

$$\begin{aligned} S(y_1, y_2) &\geq \int_0^1 \sigma(\zeta, \dot\zeta)\,\mathrm{d}t - 1 \\ &= \int_0^{t_1} \sigma(\zeta, \dot\zeta)\,\mathrm{d}t + \int_{t_1}^{t_2} \sigma(\zeta, \dot\zeta)\,\mathrm{d}t + \int_{t_2}^1 \sigma(\zeta, \dot\zeta)\,\mathrm{d}t - 1 \\ &\geq S(y_1, \zeta(t_1)) + S(\zeta(t_1), \zeta(t_2)) + S(\zeta(t_2), y_2) - 1 \\ &\geq 2\,P + 2 - P - 1 = P + 1, \end{aligned}$$

which is in contrast with the very definition of P.

□

3.2. **Existence of special subsolutions and solutions.** Here we show the existence of bounded critical subsolutions, and of coercive critical solutions.

Proposition 3.7. *There exists a bounded Lipschitz–continuous critical subsolution u, vanishing and strict outside the compact set K_0 defined as in* (36)*.*

Proof: By Proposition 3.4, item **(iii)**

$$S(y_1, y_2) \geq 0 \qquad \text{for any } y_1,\, y_2 \text{ in } \overline{\mathbb{R}^M \setminus K_0}, \tag{45}$$

and consequently the null function is an admissible trace for subsolutions to $H_0 = c_0$ on $\overline{\mathbb{R}^M \setminus K_0}$ in the sense of Proposition A.3 **(iii)**, so that owing to Proposition A.3 **(iii)**

$$u(y) := \inf\{S(z, y) \mid z \in \overline{\mathbb{R}^M \setminus K_0}\}$$

is a subsolution to $H_0 = c_0$ in $\mathbb{R}^M$ vanishing on $\overline{\mathbb{R}^M \setminus K_0}$, in addition

$$H_0(y, Du) = H_0(y, 0) < c_0 - Q_0 \qquad \text{for } y \in \mathbb{R}^M \setminus K_0 \subset \mathbb{R}^M \setminus C$$

by the very definition of C in (35). Since u is locally Lipschitz–continuous by Lemma 3.1 and vanishes outside a compact set, it is actually globally Lipschitz–continuous in $\mathbb{R}^M$. This fully shows the assertion.

□

We denote by $\mathcal{A}_0$ the Aubry set of H_0, see Proposition A.4 for the definition. We have:

Lemma 3.8. *The Aubry set $\mathcal{A}_0$ is nonempty and contained in K_0, where K_0 is defined as in* (36)*.*

Proof: We know from Proposition 3.7 that there is a critical subsolution which is strict outside K_0, so that by Proposition A.4 **(iii)** $\mathcal{A}_0 \subset K_0$. The point is then to show that the Aubry set is nonempty.

We argue by contradiction using a covering argument. If $\mathcal{A}_0 = \emptyset$, then we can associate by Proposition A.4 **(iii)** to any point $y \in K_0$ an open neighborhood B_y, a value $d_y < c_0$, and a critical subsolution w_y with

$$H_0(\cdot, Dw_y) \leq d_y < c_0 \quad \text{in } B_y.$$

We extract a finite subcovering $\{B_1, \cdots, B_m\}$ corresponding to points $y_1, \cdots, y_m$ of K_0, and set

$$\begin{aligned} w_j &= w_{y_j} \\ d_j &= d_{y_j} \qquad for \quad j = 1, \cdots, m. \end{aligned}$$

Then

$$\{B_0, B_1, \cdots, B_m\},$$

where $B_0 = \mathbb{R}^M \setminus K_0$, is an finite open cover of $\mathbb{R}^M$. We denote by u the critical subsolution constructed in Proposition 3.7 and set $d_0 = c_0 - Q_0$, so that

$$H_0(y, Du(y)) \leq d_0 < c_0 \qquad \text{for any } y \in B_0.$$

We define

$$w = \lambda_0 \, u + \sum_{i=1}^{m} \lambda_j \, w_j,$$

where $\lambda_0, \lambda_1, \cdots, \lambda_m$ are positive coefficients summing to 1. We have by convexity of H_0

$$H_0(y, Dw(y)) \leq \lambda_0 \, H_0(y, Du(y)) + \sum_{j=1}^{m} \lambda_j \, H_0(y, Dw_j(y)),$$

for a.e. $y \in \mathbb{R}^M$, and we derive

$$H_0(y, Dw(y)) \leq \sum_{i \neq j} \lambda_i \, c_0 + \lambda_j \, d_j = (1 - \lambda_j) \, c_0 + \lambda_j \, d_j = c_0 + \lambda_j \, (d_j - c_0)$$

for a.e. $y \in B_j$, $j = 0, \cdots, m$. We set $\widetilde{d} = \max_j \lambda_j (d_j - c_0) < 0$ and conclude

$$H_0(y, Dw(y)) \leq c_0 + \widetilde{d} < c_0 \quad \text{for a.e. } y \in \mathbb{R}^M,$$

which is impossible by the very definition of c_0. This gives by contradiction $\emptyset \neq \mathcal{A}_0 \subset K_0$, as desired.

□

From the previous lemma and Proposition 3.4, item **(i)** we get:

Proposition 3.9. *All the functions $y \mapsto S(y_0, y)$, for $y_0 \in \mathcal{A}_0$, are coercive critical solutions.*

The previous line of reasoning can be somehow reversed. We proceed showing that the existence of coercive solutions, plus the coercivity of intrinsic distance, characterizes the critical equation and also directly implies that the Aubry set is nonempty, as made precise by the following result:

Proposition 3.10. *Assume that the equation*

$$H_0(y, Du) = b$$

admits a coercive solution in $\mathbb{R}^M$ and limit relation (44) holds true with S_b in place of S, then $b = c_0$ and the corresponding Aubry set is nonempty.

Proof: The argument is by contradiction. Let w be a coercive solution of the equation in object. If $b \neq c_0$ or $\mathcal{A}_0 = \emptyset$ then by Corollary A.5, Proposition A.6, there is, for any $R > 0$, an unique solution of the Dirichlet problem

$$\begin{cases} H_0(y, Du) = & b \quad \text{in } B(0,R) \\ u = & w \quad \text{on } \partial B(0,R) \end{cases}$$

which therefore must coincide with w, and

$$w(0) = w(z) + S_b(z, 0) \qquad \text{for any } R > 0, \text{ some } z \in \partial B(0,R). \tag{46}$$

Since we have assumed (44), with S_b in place of S, we have

$$\lim_{|z| \to +\infty} S_b(z, 0) = +\infty$$

and by assumption w is coercive. This shows that (46) is impossible, and concludes the proof. □

We derive:

Proposition 3.11. *The effective Hamiltonian $\overline{H} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is continuous in both components and convex in p.*

Proof: It is easy to see using the continuity of H and the argument in the proof of Proposition 3.3 that $\overline{H}$ is locally bounded. We consider a sequence (x_n, p_n) converging to some (x, p), and assume that $\overline{H}(x_n, p_n)$ admits limit. We consider a sequence v_n of solutions to

$$H(x_n, y, p_n, Du) = \overline{H}(x_n, p_n)$$

of the form as in Proposition 3.9. By exploiting the continuity of H we see that the v_n are locally equiLipschitz–continuous, locally equibounded and equicoercive. They are consequently locally uniformly convergent, up to a subsequence, by Ascoli Theorem, with limit function, say w, locally Lipschitz– continuous and coercive. In addition, by basic stability properties of viscosity solutions theory, w satisfies

$$H(x, y, p, Dw) = \lim_n \overline{H}(x_n, p_n),$$

which implies by Proposition 3.10 that $\lim_n \overline{H}(x_n, p_n) = \overline{H}(x, p)$. This shows the claimed continuity of $\overline{H}$.

We see by the very definition of H that

$$H(x, y, \lambda\, p_1 + (1-\lambda)\, p_2, \lambda\, q_1 + (1-\lambda)\, q_2) \leq \lambda\, H(x, y, p_1, q_1) + (1-\lambda)\, H(x, y, p_2, q_2).$$

We derive from this that if u_i, $i = 1, 2$, satisfy $H(x, y, p_i, Du_i) \leq \overline{H}(x, p_i)$ in the viscosity sense, then

$$H(x, y, \lambda\, p_1 + (1-\lambda)\, p_2, \lambda\, Du_1 + (1-\lambda)\, Du_2) \leq \lambda\, \overline{H}(x, p_1) + (1-\lambda)\, \overline{H}(x, p_2),$$

which in turn implies

$$\overline{H}(x, \lambda\, p_1 + (1-\lambda)\, p_2) \leq \lambda\, \overline{H}(x, p_1) + (1-\lambda)\, \overline{H}(x, p_2)$$

as desired.

□

3.3. **Construction of a supersolution.** We sill keep (x_0, p_0) fixed. Starting from Proposition 3.9, we construct a supersolution of the cell problem which will play the role of corrector in Theorem 4.3. We denote by K_0 the set defined in (36). We fix $y_0 \in \mathcal{A}_0$; by the coercivity of $S(y_0, \cdot)$, see Proposition 3.9, there is a constant d such that

$$d + S(y_0, y) > 0 \qquad \text{for any } y \in \mathbb{R}^M. \tag{47}$$

We select a constant R_0 satisfying

$$B(0, R_0 - 3) \supset K_0 \tag{48}$$

(49) $R_0 - 3$ satisfies Corollary 3.6 for a neighborhood of y_0.

We aim at proving:

Theorem 3.12. *Let $U : \mathbb{R}^M \to \mathbb{R}$ be a function bounded from above in $\overline{B}(0, R_0)$ with*

$$U \leq 0 \qquad \text{in } B(0, R_0 - 1), \tag{50}$$

then there exists for any $\lambda > 0$, a locally Lipschitz–continuous supersolution w_λ of $H_0 = c_0$ in $\mathbb{R}^M$ with

$$U \leq \lambda\, w_\lambda \qquad \text{in } \overline{B}(0, R_0) \tag{51}$$

$$w_\lambda = d + S(y_0, \cdot) \qquad \text{in a neighborhood of } y_0. \tag{52}$$

To construct the supersolutions w_λ some preliminary steps are needed.

We define

$$M_0 = \max\left\{ \sup_{\overline{B}(0,R_0)} \frac{1}{\lambda}\, U, 1 \right\}.$$

We denote by $h_\lambda : [0, +\infty) \to [0, +\infty)$ a nondecreasing continuous function with

$$h_\lambda \equiv 1 \qquad \text{in } [0, R_0 - 3] \tag{53}$$

$$h_\lambda \equiv M_0 \qquad \text{in } [R_0 - 2, +\infty). \tag{54}$$

We introduce the length functional

$$\int_0^1 h_\lambda(\xi)\, \sigma(\xi, \dot{\xi})\, ds$$

for any curve ξ defined in $[0, 1]$, and denote by S^h the distance obtained by minimization of it among curves linking two given points, we drop dependence on λ to ease notations.

Lemma 3.13. *The function $S^h(y_0, \cdot)$ is a locally Lipschitz–continuous supersolution to $H_0 = c_0$ in $\mathbb{R}^M$, and coincides with $S(y_0, \cdot)$ in a neighborhood of y_0.*

Proof: The function h_λ, defined in (53), (54), satisfies $h_\lambda \geq 1$ and if $h_\lambda(|y|) > 1$ then by (53)

$$y \notin B(0, R_0 - 3) \supset K_0$$

so that by Proposition 3.4 **(ii)** $H_0(y, 0) < c_0$. We are thus in position to apply Proposition A.7, which directly gives the asserted supersolution property outside y_0, as well as the Lipschitz continuity. We also know by (49) and $h_\lambda \equiv 1$ in $B(0, R_0 - 3)$ that

$$S^h(y_0, \cdot) = S(y_0, \cdot) \qquad \text{in a neighborhood of } y_0,$$

and $S^h(y_0, \cdot)$ is solution to $H_0 = c_0$ on the whole space, by Proposition 3.9. This concludes the proof. □

By the very definition of S^h, we have:

$$S^h \geq S \qquad \text{in } \mathbb{R}^M \times \mathbb{R}^M. \tag{55}$$

We define

$$w_\lambda = d + S^h(y_0, \cdot) \tag{56}$$

where d, y_0 are as in (47).

Lemma 3.14. *The following inequalities hold true:*

$$\begin{aligned} w_\lambda &> 0 && \text{in } \mathbb{R}^M \\ w_\lambda &\geq M_0 && \text{in } \mathbb{R}^M \setminus B(0, R_0 - 1). \end{aligned}$$

Proof: From (55) and the definition of w_λ we derive

$$w_\lambda \geq d + S(y_0, \cdot)$$

and this in turn yields $w_\lambda > 0$ in $\mathbb{R}^M$ because of (47).

We fix $y \notin B(0, R_0 - 1)$, $\delta > 0$, and consider a δ–optimal curve ζ defined in $[0,1]$ for $S^h(y_0, y)$. We set

$$t_1 = \max\{t \in [0,1] \mid \zeta(t) \in B(0, R_0 - 2)\},$$

notice that

$$|\zeta(t_1) - y| > 1.$$

Owing to the above inequality, $w_\lambda > 0$, Proposition 3.4 item **(ii)**, the definition of h_λ, we have

$$\begin{aligned} d + \int_0^1 h_\lambda(|\zeta|)\, \sigma(\zeta, \dot\zeta)\, \mathrm{d}t &= d + \int_0^{t_1} h_\lambda(|\zeta|)\, \sigma(\zeta, \dot\zeta)\, \mathrm{d}t + \int_{t_1}^1 h_\lambda(|\zeta|)\, \sigma(\zeta, \dot\zeta)\, \mathrm{d}t \\ &\geq w_\lambda(\zeta(t_1)) + \int_{t_1}^1 h_\lambda(|\zeta|)\, |\dot\zeta|\, \mathrm{d}t + \delta \\ &\geq w_\lambda(\zeta(t_1)) + M_0\, |y - \zeta(t_1)| + \delta > M_0 + \delta. \end{aligned}$$

Taking into account the definition of w_λ and the fact that the curve ζ joining y_0 to $y \notin B(0, R_0 - 1)$ and δ are arbitrary, we deduce from the above computation the desired inequality. □

Proof: (of Theorem 3.12) In view of Lemma 3.13, it is just left to show (51). It indeed holds true in $B(0, R_0 - 1)$ because of (50) and $w_\lambda > 0$. If $y \in \overline{B(0, R_0)} \setminus B(0, R_0 - 1)$, then by Lemma 3.14, we have

$$w_\lambda(y) \geq M_0 \geq \sup_{\overline{B}(0,R_0)} \frac{1}{\lambda} U \geq \frac{1}{\lambda} U(y).$$

□

4. Asymptotic analysis

We summarize the relevant output of the previous section in the following

Theorem 4.1. *We consider* $(x_0, p_0) \in \mathbb{R}^N \times \mathbb{R}^N$*, a constant* R_0 *satisfying* (48), (49)*, a function* U *bounded from above in* $\overline{B}(0, R_0)$ *and less than or equal zero in* $B(0, R_0 - 1)$*, any positive constant* λ*. Then the equation*

$$H(x_0, y, p_0, Du) = \overline{H}(x_0, p_0) \qquad \text{in } \mathbb{R}^M$$

admits a bounded Lipschitz–continuous subsolution and a locally Lipschitz–continuous supersolution, say w_λ*, satisfying* (51), (52)

We recall the notations $\overline{V} = \limsup^{\#} V^\varepsilon$, $\underline{V} = \liminf_{\#} V^\varepsilon$, where the V^ε are the value functions of problems (15)/ (16). We consider a point $(x_0, t_0) \in \mathbb{R}^N \times (0, +\infty)$, and set

$$K_\delta = B(x_0, \delta) \times (t_0 - \delta, t_0 + \delta) \qquad \text{for } \delta < t_0. \tag{57}$$

We further consider a constant $R_0 > 0$ satisfying (48), (49). The next lemma, based on Theorem 3.12, will be of crucial importance. The entities $y_0 \in \mathcal{A}_0$ and d appearing in the statement are defined as in (47) :

Lemma 4.2. *Let* ψ *be a strict supertangent to* $\overline{V}$ *at* (x_0, t_0) *such that* (x_0, t_0) *is the unique maximizer of* $\overline{V} - \psi$ *in* K_{δ_0}*, for some* $\delta_0 < t_0$*. Then, given any infinitesimal sequence* ε_j*, and* $\delta < \delta_0$*, we find a constant* ρ_δ *and a family* w^j *of supersolutions to* $H(x_0, y, D\psi(x_0, t_0), Du) = \overline{H}(x_0, D\psi(x_0, t_0))$ *in* $\mathbb{R}^M$ *satisfying for* j *suitably large*

$$\varepsilon_j \, w^j \;\geq\; V^{\varepsilon_j} - \psi + \rho_\delta \qquad \text{in } \partial\big(K_\delta \times B(0, R_0)\big) \tag{58}$$

$$w^j \;=\; d + S(y_0, \cdot) \qquad \text{in a neighborhood } A_0 \text{ of } y_0, \tag{59}$$

where S *is the intrinsic critical distance, see Subsection 3.1, related to* $(x_0, D\psi(x_0, t_0))$*.*

Proof: By supertangency properties of ψ at (x_0, t_0), we find, for any $\delta < \delta_0$, a $\rho_\delta > 0$ with

$$\max_{\partial K_\delta} \overline{V} - \psi < -3\,\rho_\delta. \tag{60}$$

We fix a δ and define

$$U^\varepsilon(y) = \begin{cases} \max_{(x,t) \in \partial K_\delta} \{V^\varepsilon(x, y, t) - \psi(x, t) + \rho_\delta\} & \text{for } y \in B(0, R_0 - 1/2) \\ \max_{(x,t) \in K_\delta} \{V^\varepsilon(x, y, t) - \psi(x, t) + \rho_\delta\} & \text{for } y \in \mathbb{R}^M \setminus B(0, R_0 - 1/2). \end{cases}$$

Notice that the U^ε are continuous for any ε and locally equibounded, since the V^ε are locally equibounded in force of Proposition 2.6. To ease notations we set

$$U^j = U^{\varepsilon_j}.$$

Claim : *There is* $j_0 = j_0(R_0)$ *such that*

$$U^j \leq -\rho_\delta \qquad \text{in } B(0, R_0 - 1), \text{ for } j > j_0.$$

Were the claim false, there should be a subsequence y_j contained in $B(0, R_0 - 1)$ with

$$U^j(y_j) > -\rho_\delta.$$

The y_j converge, up to further extracting a subsequence, to some $\overline{y}$, and, being ε_j infinitesimal, we get

$$(\limsup{}^{\#} U^\varepsilon)(\overline{y}) \geq -\rho_\delta. \tag{61}$$

Moreover, there exists an infinitesimal sequence ε_i and elements z_i converging to $\overline{y}$ with

$$\lim_i U^{\varepsilon_i}(z_i) = (\limsup{}^{\#} U^\varepsilon)(\overline{y}),$$

at least for i large $z_i \in B(0, R_0 - 1/2)$, and by the very definition of U^ε in $B(0, R_0 - 1/2)$, we get

$$U^{\varepsilon_i}(z_i) = V^{\varepsilon_i}(x_i, z_i, t_i) - \psi(x_i, t_i) + \rho_\delta \qquad \text{for some } (x_i, t_i) \in \partial K_\delta,$$

up to extracting a subsequence, (x_i, t_i) converges to some $(\overline{x}, \overline{t}) \in \partial K_\delta$ so that by (60)

$$\begin{aligned} (\limsup{}^{\#} U^\varepsilon)(\overline{y}) &= \lim U^{\varepsilon_i}(z_i) = \lim V^{\varepsilon_i}(x_i, z_i, t_i) - \psi(x_i, t_i) + \rho_\delta \\ &\leq \overline{V}(\overline{x}, \overline{t}) - \psi(\overline{x}, \overline{t}) + \rho_\delta \leq -2\,\rho_\delta. \end{aligned}$$

which is in contradiction with (61). This ends the proof of the claim.

We are then in the position to apply Theorem 3.12 to any U^j, and get a supersolution w^j to $H(x_0, \cdot, D\psi(x_0, t_0), \cdot) = \overline{H}(x_0, D\psi(x_0, t_0))$, which satisfies, for $j > j_0$, the condition (59) and

$$\varepsilon_j\, w^j \geq U^j \qquad \text{in } \overline{B}(0, R_0).$$

Owing to the very definition of U^j, we derive from the latter inequality that

$$\varepsilon_j\, w^j(y) \geq V^{\varepsilon_j}(x, y, t) - \psi(x, t) + \rho_\delta$$

holds in

$$\partial K_\delta \times B(0, R_0) \cup K_\delta \times \partial B(0, R_0) = \partial\big(K_\delta \times B(0, R_0)\big).$$

This proves (58) and conclude the proof.

□

We proceed establishing the asymptotic result for upper weak semilimit of the V^ε. The first part of the proof is a version, adapted to our setting, of perturbed test function method. We are going to use as correctors, depending on ε, the special supersolutions to cell equations constructed in Subsection 3.3 in the frame of Lemma 4.2. The argument of the second half about behavior of limit function at $t = 0$ makes a direct use of the material of Subsections 2.3, 2.4.

Theorem 4.3. *The function* $\overline{V} = \limsup^{\#} V^\varepsilon$ *is a subsolution to* ($\overline{\mathrm{HJ}}$) *satisfying*

$$\limsup_{\substack{(x,t)\to(x_0,0)\\ t>0}} \overline{V}(x,t) \le \overline{u}_0(x_0) \qquad \text{for any } x_0 \in \mathbb{R}^N. \tag{62}$$

Proof: Let (x_0, t_0) be a point in $\mathbb{R}^N \times (0, +\infty)$, and ψ a strict supertangent to $\overline{V}$ at (x_0, t_0) such that (x_0, t_0) is the unique maximizer of $\overline{V} - \psi$ in K_{δ_0}, for some $\delta_0 > 0$ (see (57) for the definition of K_δ).

By Proposition 2.7, we can find an infinitesimal sequence ε_j and (x_j, y_j, t_j) converging to (x_0, y_0, t_0), where y_0 is as in (47), with

$$\lim_j V^{\varepsilon_j}(x_j, y_j, t_j) = \overline{V}(x_0, t_0) = \psi(x_0, t_0). \tag{63}$$

We assume by contradiction

$$\psi_t(x_0, t_0) + \overline{H}(x_0, D\psi(x_0, t_0)) > 2\,\eta \tag{64}$$

for some positive η. We apply Lemma 2.9, about coercivity of H, to the bounded set

$$C := B(x_0, \delta_0) \times B(0, R_0) \times D\psi(K_{\delta_0}),$$

where R_0 satisfies (48), (49), and exploit that $\overline{H}$ is locally bounded to find $P > 0$ with

$$H(x, y, p, q) > \overline{H}(x, p) \qquad \text{for } (x, y, p) \in C,\ q \text{ with } |q| \ge P. \tag{65}$$

Applying the estimates (28) to $B(x_0, \delta_0) \times B(0, R_0)$ and (29), we find

$$\begin{aligned} &|H(x_0, y, D\psi(x_0, t_0), q) - H(x, y, p, q)| \le \\ &L_0\,(|x - x_0|)(|D\psi(x_0, t_0)| + |q|) + \\ &\omega(|x - x_0|) + Q\,|D\psi(x_0, t_0) - p| \end{aligned} \tag{66}$$

for any $(x, y) \in B(x_0, \delta_0) \times B(0, R_0)$ and $(p, q) \in \mathbb{R}^N \times \mathbb{R}^M$, where ω is an uniform continuity modulus of ℓ in $B(x_0, \delta_0) \times B(0, R_0) \times A$, L_0 is as in **(H2)** and Q is an upper bound of $|f|$ in $B(x_0, \delta_0) \times B(0, R_0) \times A$.

Exploiting the continuity of $D\psi$, ψ_t, $\overline{H}$, we can determine, $\delta_0 > \delta > 0$ such that using (64), (66) with $q \in B(0, P)$ and p of the form $D\psi(x, t)$, we get

$$|H(x_0, y, D\psi(x_0, t_0), q) - H(x, y, D\psi(x, t), q)| < \eta \tag{67}$$

$$|D\psi(x, t) - D\psi(x_0, t_0)| < \eta \tag{68}$$

$$\psi_t(x, t) + \overline{H}(x, D\psi(x, t)) > 0 \tag{69}$$

for $(y,q) \in B(0,R_0) \times B(0,P)$, $(x,t) \in K_\delta$. By applying Lemma 4.2 to such a δ, we find a constant $\rho_\delta > 0$ and a family w^j of supersolutions to

$$H(x_0, y, p_0, D\psi(x_0,t_0), Du) = \overline{H}(x_0, D\psi(x_0,t_0)) \qquad \text{in } \mathbb{R}^M$$

with

$$\varepsilon_j\, w^j \;\geq\; V^{\varepsilon_j} - \psi + \rho_\delta \qquad \text{in } \partial\big(K_\delta \times B(0,R_0)\big) \tag{70}$$

$$w^j \;=\; d + S_0(y_0,\cdot) \qquad \text{in a neighborhood } A_0 \text{ of } y_0, \tag{71}$$

for j large enough, see (47) for the definition of d. We claim that the corrected test function $\psi + w^j$ satisfies

$$\psi_t(x,t) + H(x,y,D\psi(x,t),Dw^j) \geq 0$$

in $K_\delta \times B(0,R_0)$ in the viscosity sense. In fact, let ϕ be a subtangent to $\psi + w^j$ at some point $(x,y,t) \in K_\delta \times B(0,R_0)$, then

$$\begin{aligned} \phi_t(x,y,t) &= \psi_t(x,t) \\ D_x\phi(x,y,t) &= D\psi(x,t) \end{aligned}$$

and so, to prove the claim, we have to show the inequality

$$\psi_t(x,t) + H(x,y,D\psi(x,t),D_y\phi(x,y,t)) \geq 0.$$

We have that

$$z \mapsto \phi(x,z,t)$$

is supertangent to w^j at y, which implies by the supersolution property of w^j

$$H(x_0,y,D\psi(x_0,t_0),D_y\phi(x,y,t)) \geq \overline{H}(x_0,D\psi(x_0,t_0))$$

If $|D_y\phi(x,y,t)| < P$ then by (64), (67) and (68)

$$\begin{gathered} \psi_t(x,t) + H(x,y,D\psi(x,t),D_y\phi(x,y,t)) \geq \\ \psi_t(x_0,t_0) - \eta + H(x_0,y,D\psi(x_0,t_0),D_y\phi(x,y,t)) - \eta \geq \\ \psi_t(x_0,t_0) + \overline{H}(x_0,D\psi(x_0,t_0)) - 2\,\eta \geq 0. \end{gathered}$$

If instead $|D_y\phi(x,y,t)| \geq P$ then by (65), (69)

$$\begin{gathered} \psi_t(x,t) + H(x,y,D\psi(x,t),D_y\phi(x,y,t)) \geq \\ \psi_t(x,t) + \overline{H}(x,D\psi(x,t)) \geq 0. \end{gathered}$$

The claim is then proved. For j large enough, the functions V^{ε_j}, $\psi + \varepsilon_j\, w^j - \rho_\delta$ are then subsolutions and supersolutions, respectively, to

$$u_t + H\left(x,y,D_xu,\frac{D_yu}{\varepsilon_j}\right) = 0$$

in $K_\delta \times B(0,R_0)$, then taking into account the boundary inequality (70), we can apply the comparison principle of Proposition 2.10 to the above equation to deduce

$$V^{\varepsilon_j} \leq \psi + \varepsilon_j\, w^j - \rho_\delta \qquad \text{in } K_\delta \times B(0,R_0). \tag{72}$$

On the other side, let (x_j, y_j, t_j) be the sequence converging to (x_0, y_0, t_0) introduced in (63), then for j large $(x_j, y_j, t_j) \in K_\delta \times B(0,R_0)$, and $w^j(y_j) = d + S(y_0, y_j)$ by (71), so that

$$\lim_j \varepsilon_j\, w^j(y_j) = 0.$$

We therefore get

$$\lim_j \left[V^{\varepsilon_j}(x_j,y_j,t_j) - \psi(x_j,t_j) - \varepsilon_j\, w^j(y_j)\right] = \overline{V}(x_0,t_0) - \psi(x_0,t_0) = 0$$

which contradicts (72).

We proceed proving (62). We consider (x_n, t_n) converging to $(x_0, 0)$ such that $\overline{V}(x_n,t_n)$ admits limit. Our task is then to show

$$\lim_n \overline{V}(x_n,t_n) \leq \overline{u}_0(x_0).$$

We find for any n an infinitesimal sequence ε_j^n and (x_j^n, y_j^n, t_j^n) converging to $(x_n, 0, t_n)$ with

$$\lim_j V^{\varepsilon_j^n}(x_j^n, y_j^n, t_j^n) = \overline{V}(x_n,t_n),$$

$0 \in \mathbb{R}^M$ is clearly an arbitrary choice, in view of Proposition 2.7. By applying a diagonal argument we find ε_n converging to 0 and (z_n, y_n, s_n) converging to $(x_0, 0, 0)$ with

$$\lim_n V^{\varepsilon_n}(z_n,y_n,s_n) = \lim_n \overline{V}(x_n,t_n) \tag{73}$$

$$\lim_n \frac{s_n}{\varepsilon_n} = +\infty. \tag{74}$$

Given $\delta > 0$, we denote by $\widetilde{y}$ a δ–minimizer of $y \mapsto u_0(x_0, y)$ in $\mathbb{R}^M$, see assumption **(H5)**. By applying Proposition 2.3, Lemma 2.4 and taking into account (74), we find for any n sufficiently large a trajectory (ξ_n, η_n) of (CD_ε), with $\varepsilon = \varepsilon_n$, corresponding to controls α_n and starting at (z_n, y_n), such that

(75) (ξ_n, η_n) is contained in a compact subset independent of n as $t \in [0, s_n/\varepsilon_n]$

$$|\eta_n(s_n/\varepsilon_n) - \widetilde{y}| = \mathrm{O}(\varepsilon_n) \tag{76}$$

By using formulation (15) of minimization problem, we discover

$$V^{\varepsilon_n}(z_n,y_n,s_n) \leq \varepsilon_n \int_0^{\frac{s_n}{\varepsilon_n}} \ell(\xi_n(t), \eta_n(t), \alpha_n(t))\, \mathrm{d}t + u_0(\xi_n(s_n/\varepsilon_n), \eta_n(s_n/\varepsilon_n)),$$

where the integrand is estimated from above by a constant, say Q, independent of n, because of (75), therefore

$$V^{\varepsilon_n}(x_n, y_n, s_n) \le Q\, s_n + u_0(\xi_n(s_n/\varepsilon_n), \eta_n(s_n/\varepsilon_n))$$

Owing to (4), (76), (73), and the fact that s_n is infinitesimal, we then get

$$\lim_n \overline{V}(x_n, t_n) = \lim_n V^{\varepsilon_n}(z_n, y_n, s_n) \le u_0(x_0, \widetilde{y}) \le \overline{u}_0(x_0) + \delta.$$

This concludes the proof because δ is arbitrary. □

The second main result concerns lower weak semilimit. Here we essentially exploit the existence of bounded Lipschitz–continuous subsolutions to cell equations established in Proposition 3.7 plus the coercivity of the V^ε proved in Proposition 2.8. The part of the proof about behavior of limit function at $t = 0$ is direct and not based on a PDE approach. We recall that $(\overline{u}_0)_\#$ stands for the lower semicontinuous envelope of $\overline{u}_0$, see Subsection 2.1 for definition.

Theorem 4.4. *The function* $\underline{V} = \liminf_\# V^\varepsilon$ *is a supersolution to* ($\overline{\text{HJ}}$) *satisfying*

$$\liminf_{\substack{(x,t)\to(x_0,0)\\ t>0}} \underline{V}(x,t) \ge (\overline{u}_0)_\#(x_0) \qquad \text{for any } x_0 \in \mathbb{R}^N. \tag{77}$$

Proof: Let (x_0, t_0) be a point in $\mathbb{R}^N \times (0, +\infty)$, and φ a strict subtangent to $\underline{V}$ at (x_0, t_0) such that (x_0, t_0) is the unique minimizer of $\underline{V} - \varphi$ in K_{δ_0}, for some $\delta_0 > 0$ (see (57) for the definition of K_δ). We assume by contradiction

$$\varphi_t(x_0, t_0) + \overline{H}(x_0, D\varphi(x_0, t_0)) < 0. \tag{78}$$

Given $\varepsilon > 0$, we can find by Proposition 2.8 about coercivity of value functions, $R_\varepsilon > 1$ satisfying

$$V^\varepsilon(x, y, t) > \sup_{K_{\delta_0}} \varphi + 1 \qquad \text{for } (x, y) \in K_{\delta_0},\ y \in \mathbb{R}^M \setminus B(0, R_\varepsilon). \tag{79}$$

We can also find, exploiting Proposition 3.7, a Lipschitz–continuous subsolution u to the cell problem

$$H(x_0, y, D\varphi(x_0, t_0), Du) = \overline{H}(x_0, D\varphi(x_0, t_0)) \qquad \text{in } \mathbb{R}^M \tag{80}$$

with

$$u(y) < 0 \qquad \text{for any } y \in \mathbb{R}^M. \tag{81}$$

By using estimate (28) on H, Lipschitz continuity of u, continuity of $\overline{H}$, $D\varphi$, φ_t and (78), (80) we can determine $0 < \delta < \delta_0$ such that $u + \varphi$ is subsolution to

$$w_t + H(x, y, D\varphi(x,t), Dw) = 0 \qquad \text{in } K_\delta \times \mathbb{R}^M.$$

Owing to strict subtangency property of φ, there is $1 > \rho > 0$ with

$$\underline{V} - \varphi > 2\,\rho \qquad \text{in } \partial K_\delta,$$

and, taking into account that $\underline{V}$ is the lower semilimit of the V^ε, we derive

$$V^\varepsilon - \varphi > \rho \qquad \text{in } \partial K_\delta \times B(0, R_\varepsilon)$$

for ε sufficiently small, which in turn implies by (81)

$$V^\varepsilon - \varphi - u > \rho \qquad \text{in } \partial K_\delta \times B(0, R_\varepsilon). \tag{82}$$

Owing to (79), (81), we also have

$$V^\varepsilon - \varphi - u > \rho \qquad \text{in } K_\delta \times \partial B(0, R_\varepsilon). \tag{83}$$

Since V^ε, $\varphi + \varepsilon\, u + \rho$ are supersolution and subsolution, respectively, to

$$w_t + H\left(x, y, D_x w, \frac{D_y w}{\varepsilon}\right) = 0$$

in $K_\delta \times B(0, R_0)$, the boundary conditions (82), (83) plus the comparison principle in Proposition 2.10 implies

$$V^\varepsilon \geq \varphi + \varepsilon\, u + \rho \qquad \text{in } K_\delta \times B(0, R_\varepsilon), \text{ for } \varepsilon \text{ small.} \tag{84}$$

On the other side, there is by Proposition 2.7 an infinitesimal sequence ε_j and a sequence (x_j, y_j, t_j) converging to $(x_0, 0, t_0)$ with

$$\lim_j V^{\varepsilon_j}(x_j, y_j, t_j) = \underline{V}(x_0, t_0)$$

and consequently

$$\lim_j \left[V^{\varepsilon_j}(x_j, y_j, t_j) - \varphi(x_j, t_j) - \varepsilon_j\, u(y_j)\right] = \overline{V}(x_0, t_0) - \varphi(x_0, t_0) = 0.$$

Taking into account that $R_\varepsilon > 1$ for any ε, and (x_j, y_j, t_j) are in $K_\delta \times B(0, 1)$ for j large, the last limit relation contradicts (84).

We proceed proving (77). We consider (x_n, t_n) converging to $(x_0, 0)$ such that $\underline{V}(x_n, t_n)$ admits limit, with the aim of showing

$$\lim_n \underline{V}(x_n, t_n) \geq (\overline{u}_0)_\#(x_0).$$

Arguing as in the final part of Theorem 4.3, we find an infinitesimal sequence ε_n and (z_n, y_n, s_n) converging to $(x_0, \widetilde{y}, 0)$, for some $\widetilde{y} \in \mathbb{R}^M$, with

$$\lim_n V^{\varepsilon_n}(z_n, y_n, s_n) = \lim_n \underline{V}(x_n, t_n).$$

We fix $\delta > 0$. Arguing as in second half of Proposition 2.6, see estimate (18), we determine a constant P_0 independent of n and trajectories (ξ_n, η_n) of the controlled dynamics starting at (z_n, y_n) with

$$V^{\varepsilon_n}(z_n, y_n, s_n) \geq P_0\, s_n + u_0(\xi_n(s_n/\varepsilon_n), \eta_j(s_j/\varepsilon_j)) - \delta \geq P_0\, s_n + \overline{u}_0(\xi_n(s_n/\varepsilon_n)) - \delta.$$

Since by the boundedness assumption on f

$$|\xi_n(s_n/\varepsilon_n) - z_n| \leq Q_0\, s_n,$$

we get at the limit

$$\lim_n \underline{V}(x_n, t_n) = \lim_n V^{\varepsilon_n}(z_n, y_n, s_n) \geq \liminf_n \overline{u}_0(\xi_n(s_n/\varepsilon_n)) - \delta \geq (\overline{u}_0)_\#(x_0) - \delta,$$

which gives the assertion since δ is arbitrary.

□

Appendix A. Facts from weak KAM theory

Here we consider an Hamiltonian $F(y,q)$ defined in $\mathbb{R}^M \times \mathbb{R}^M$ and the family of equations

$$F(y, Du) = b \qquad \text{in } \mathbb{R}^M, \text{ for } b \in \mathbb{R} \tag{85}$$

We assume F to satisfy

F is continuous in both variables;
F is convex in q;
$\lim_{|q|\to+\infty} \min_{y\in K} F(y,q) = +\infty$ for any compact subset K of $\mathbb{R}^M$.

Our aim is to recall some basic facts of weak KAM theory, which will be exposed here through the so–called metric method for equation (85), see [13], [15], [16], [14]. We define the critical value of F as

$$c = \inf\{b \mid (85) \text{ has subsolutions in } \mathbb{R}^M\}. \tag{86}$$

Being the ambient space non compact c can also be infinite. We assume in what follows

The critical value of F is finite.

We call supercritical a value b with $b \geq c$. By stability properties of viscosity (sub)solutions, subsolutions for the critical equation do exist. We derive from coercivity of F:

Lemma A.1. *Let b a supercritical value. The subsolutions to $F = b$ are locally equiLipschitz–continuous.*

We adopt the so–called metric method which is based on the definition of an intrinsic distance starting from the sublevels of the Hamiltonian for any supercritical value. For any $b \geq c$ we set

$$Z_b(y) = \{q \mid F(y,q) \leq b\} \qquad y \in \mathbb{R}^M.$$

Owing to continuity, convexity and coercivity of F, we have:

Lemma A.2. *For any $b \geq c$, the multifunction $y \mapsto Z_b(y)$ takes convex compact values, it is in addition Hausdorff–continuous at any point y_0 where $\operatorname{int} Z_b(y_0) \neq \emptyset$ and upper semicontinuous elsewhere.*

We further set

$$\sigma_b(y,v) = \max\{q \cdot v \mid q \in Z_b(y)\} \qquad \text{for any } y,\ v \text{ in } \mathbb{R}^M,$$

namely the support function of $Z_b(y)$ at q, and define for any curve ξ defined in $[0,1]$ the associated intrinsic length via

$$\int_0^1 \sigma_b(\xi,\dot\xi)\,\mathrm{d}s.$$

Notice that the above integral is invariant for orientation-preserving change of parameter, being the support function positively homogeneous and subadditive, as a length functional should be. Also notice that because of this invariance the choice of the interval $[0,1]$ is not restrictive. For any pair y_1, y_2 we define the intrinsic distance as

$$S_b(y_1,y_2) = \inf\left\{\int_0^1 \sigma_b(\xi,\dot\xi)\,\mathrm{d}s \mid \xi \ \text{ with } \xi(0)=y_1, \quad \xi(1)=y_2\right\}.$$

The intrinsic distance is finite for any supercritical value b.

Proposition A.3. *Given $b \geq c$, we have*

- **(i)** *a function u is a subsolution to $F=b$ if and only if*

$$u(y_2) - u(y_1) \leq S_b(y_1,y_2) \qquad \text{for any } y_1,\ y_2;$$

- **(ii)** *for any fixed y_0, the function $y \mapsto S_b(y_0,y)$ is subsolution to $F=b$ in $\mathbb{R}^M$ and solution in $\mathbb{R}^M \setminus \{y_0\}$;*
- **(iii)** *Let C, w be a closed set of $\mathbb{R}^M$ and a function defined in C satisfying*

$$w(y_2) - w(y_1) \leq S_b(y_1,y_2) \qquad \text{for any } y_1,\ y_2 \text{ in } C$$

then the function

$$y \mapsto \inf\{w(z) + S_b(z,y) \mid z \in C\}$$

is subsolution to $F=b$ in $\mathbb{R}^M$, solution in $\mathbb{R}^M \setminus C$ and equal to w in C.

In contrast to what happens when the ambient space is compact, namely $F=b$ admits solutions in the whole space if and only if $b=c$, in the noncompact case instead there are solutions for any supercritical equation. It is in fact enough that the intrinsic length is finite, as always is the case for supercritical values, to get a solution.

The construction of such a solution is in fact quite simple. One considers a sequence y_n with $|y_n|$ diverging and the functions

$$u_n = S_b(y_n,\cdot) - S_b(y_n,0).$$

By Lemma A.1 and Proposition A.3 the u_n are solutions except at y_n, are locally equiLipschitz–continuous, and also equibounded, since they vanish at 0. They therefore converge, up to a subsequence, by Ascoli Theorem. Having swept away the bad (in the sense of Proposition A.3 **(ii)**) points y_n to infinity, but kept the solution property by stability properties of viscosity solutions under uniform convergence, we see that the limit function is indeed the sought solution of $F=b$.

We say that a function u is a strict subsolution to $F=b$ in some open set B if

$$F(x,Du) \leq b - \delta \qquad \text{for some } \delta > 0, \text{ in the viscosity sense in } B.$$

The points satisfying the equivalent properties stated in the following proposition, make up the so–called Aubry set, denoted by $\mathcal{A}$.

Proposition A.4. *Given $y_0 \in \mathbb{R}^M$, the following three properties are equivalent:*

- **(i)** *there exists a sequence of cycles ξ_n based on y_0 and defined in $[0,1]$ with*

$$\inf_n \int_0^1 \sigma_c(\xi_n, \dot{\xi}_n)\,\mathrm{d}s = 0 \quad \text{and} \quad \inf_n \int_0^1 |\dot{\xi}_n|\,\mathrm{d}s > 0;$$

- **(ii)** *$y \mapsto S_c(y_0, y)$ is solution to $F = c$ in the whole of $\mathbb{R}^M$;*
- **(iii)** *if a function u is a strict critical subsolution in a neighborhood of y_0, then u cannot be subsolution to $F = c$ in $\mathbb{R}^M$.*

Notice that, in contrast with the compact case, even if the critical value is finite, the Aubry set can be empty for Hamiltonian defined in $\mathbb{R}^M \times \mathbb{R}^M$. We derive from Proposition A.4 **(iii)** adapting the same argument of Lemma 3.8:

Corollary A.5. *Assume that the Aubry set is empty, then for any bounded open set B of $\mathbb{R}^M$, there is a critical subsolution which is strict in B.*

We record for later use:

Proposition A.6. *Let B, b be an open bounded set of $\mathbb{R}^M$, and a critical value, respectively. Assume that the equation $F = b$ admits a strict subsolution in B, and denote by w a subsolution of $F = b$ in $\mathbb{R}^M$. Then the Dirichlet problem*

$$\begin{cases} F(y, Du) = & b \quad \text{in } B \\ u = & w \quad \text{on } \partial B \end{cases}$$

admits an unique solution u given by the formula

$$u(y) = \inf\{w(z) + S_b(z, y) \mid z \in \partial B\}.$$

We now consider a supercritical value b and a function $h : \mathbb{R}^M \to \mathbb{R}$ with

$$h \geq 1 \quad \text{in } \mathbb{R}^M \quad \text{and} \quad h(y) > 1 \Rightarrow F(y, 0) \leq b \tag{87}$$

We define for any curve ξ in $[0,1]$ the length functional

$$\int_0^1 h(\xi)\,\sigma_b(\xi, \dot{\xi})\,ds$$

and denote by S_b^h the corresponding distance obtained as the infimum of lengths of curves joining two given points of $\mathbb{R}^M$. We have

Proposition A.7. *Let b, h be a supercritical value for F and a function satisfying (87), respectively, then $S_b^h(z_0, \cdot)$ is a locally Lipschitz–continuous supersolution to (85) in $\mathbb{R}^M \setminus \{z_0\}$, for any $z_0 \in \mathbb{R}^M$.*

Proof: We fix z_0. For any $(y,v) \in \mathbb{R}^M \times \mathbb{R}^M$, $h(y)\,\sigma_b(y,v)$ is the support function of the b–sublevel of the Hamiltonian

$$(y,q) \mapsto F\left(y, \frac{q}{h(y)}\right) \tag{88}$$

and S_b^h is the corresponding intrinsic distance. According to Proposition A.3 **(ii)**, $w := S_b^h(z_0,\cdot)$ is subsolution to (85) in $\mathbb{R}^M$, and supersolution in $\mathbb{R}^M \setminus \{z_0\}$, with F replaced by the Hamiltonian in (88). Since the Hamiltonian in (88) keeps the coercivity property of F, this implies that w is locally Lipschitz–continuous in force of Lemma A.1.

Taking into account the supersolution information on w, we consider a subtangent ψ to w at a point y. If $h(y)=1$ then

$$F(y, D\psi(y)) = F\left(y, \frac{D\psi(y)}{h(y)}\right) \geq b. \tag{89}$$

If instead $h(y) > 1$ then by (87) and convex character of F

$$\begin{aligned} F\left(y, \frac{D\psi(y)}{h(y)}\right) &= F\left(y, \left(1 - \frac{1}{h(y)}\right) 0 + \frac{D\psi(y)}{h(y)}\right) \\ &\leq \frac{1}{h(y)}\, F(y, D\psi(y)) + \left(1 - \frac{1}{h(y)}\right) b \end{aligned}$$

and consequently

$$\frac{1}{h(y)}\, F(y, D\psi(y)) \geq b - \left(1 - \frac{1}{h(y)}\right) b = \frac{1}{h(y)}\, b. \tag{90}$$

Formulas (89), (90) provide the assertion. □

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