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Secondary invariants in K-theory

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Introduction

Secondary invariants

Let $0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$ be an exact sequence of C*-algebras. One can associate to it a long exact sequence of K-groups

$$\cdots \longrightarrow K_*(J) \longrightarrow K_*(A) \longrightarrow K_*(A/J) \xrightarrow{\partial} K_{*+1}(J) \longrightarrow \cdots$$

and the boundary morphism ∂ is often called the index map. Let x a class in $K_*(A/J)$ such that its index $\partial(x)$ vanishes, then by exactness x lifts to a class $\rho(x)$ in $K_*(A)$. A lift of x to an element $\varrho(x)$ is called a K-theoretic secondary invariant, because the primary one, the index, vanishes.

Coarse algebras

Indeed we want to study K-theoretic invariants related to a particular exact sequence of K-groups, the so-called Analytic Surgery Exact Sequence:

$$\cdots \longrightarrow K_*(C^*(\widetilde{X})^\Gamma) \longrightarrow K_*(D^*(\widetilde{X})^\Gamma) \longrightarrow K_*(D^*(\widetilde{X})^\Gamma/C^*(\widetilde{X})^\Gamma) \longrightarrow \cdots$$

that is the long exact sequence in K-theory associated to the Roe extension of coarse algebras. Here $\widetilde{X}$ is a proper metric space and Γ is a discrete group that acts properly on $\widetilde{X}$. The algebras $C^*(\widetilde{X})^\Gamma$ and $D^*(\widetilde{X})^\Gamma$ are respectively the closure of the locally compact and the pseudolocal Γ-equivariant operators with finite propagation on a suitable Γ-equivariant $C_0(\widetilde{X})$-module.

By the Paschke duality one can prove that $K_*(D^*(\widetilde{X})^\Gamma/C^*(\widetilde{X})^\Gamma)$ is isomorphic to the Γ-equivariant analytic K-homology of $\widetilde{X}$ shifted by one: $K^\Gamma_{*+1}(\widetilde{X})$; moreover $K_*(C^*(\widetilde{X})^\Gamma) \cong K_*(C^*_r(\Gamma))$, that is the K-theory of the locally compact Γ-equivariant operators is isomorphic to the K-theory of the reduced C*-algebra of Γ. In particular it turns out that the boundary map of this exact sequence is the Baum-Connes assembly map $\mu^\Gamma_{\widetilde{X}}$. Then the group $K_*(D^*(\widetilde{X})^\Gamma)$ is the receptacle of K-theoretic invariants secondary with respect to the assembly map: in effect we are considering class of geometric operators that are homotopic to invertible ones and equivalently such that the index vanishes.

But the coarse algebras are not the only ones giving this K-groups exact sequence: we are going to see that in the case of a smooth Riemannian manifold $\widetilde{X}$ equipped with a proper and free action of Γ such that $X = \widetilde{X}/\Gamma$ is compact, there are several ways to realize the same exact sequence such that each ones generalize in further geometric contexts.

Localization algebras

The first one, the less different one, is given by Yu's localization algebras (see [50, 30]). The starting point is always the coarse algebra $C^*(\widetilde{X}^\Gamma)$, but the quotient $D^*(\widetilde{X})^\Gamma/C^*(\widetilde{X})^\Gamma$ is substituted by the localization algebra $C^*_L(\widetilde{X})^\Gamma$ of all bounded and uniformly continuous functions $f\colon [0,\infty)\to C^*(\widetilde{X})^\Gamma$ such that the propagation of $f(t)$ goes to 0 as t goes to ∞. Furthermore Yu defined a map, the local index, that constitutes an isomorphism between $K_*(C^*_L(\widetilde{X})^\Gamma)$ and $K^\Gamma_*(\widetilde{X})$.

The evaluation of functions at 0 gives an exact sequence

$$\cdots \longrightarrow K_*(C^*_{L,0}(\widetilde{X})^\Gamma) \longrightarrow K_*(C^*_L(\widetilde{X})^\Gamma) \xrightarrow{\mathrm{ev}_0} K_*(C^*(\widetilde{X})^\Gamma) \longrightarrow \cdots$$

that is proved to be naturally isomorphic to the previous one. Notice that the evaluation at 0 corresponds to the assembly map and here it is more clear what a secondary invariant is: something in $K_*(C^*_L(\widetilde{X})^\Gamma)$ that is trivial at 0.

Adiabatic deformation groupoid

Let us turn to another interesting realization of our exact sequence. In a short section of his famous book about Noncommutative Geometry Connes sketches a proof of the Atiyah-Singer index theorem using the tangent groupoid and groupoid techniques. As he notes, his proof is closely related to the K-theory proof of Atiyah and Singer, but it has the advantage of extending easily to more elaborate settings, for example to foliations.

The idea is not much different from the previous one. Consider the Lie groupoid $\widetilde{X}\times_\Gamma\widetilde{X}$, and consider functions in $C^\infty_c(\widetilde{X}\times_\Gamma\widetilde{X})$ as kernels of Γ-equivariant smoothing operators on $\widetilde{X}$. The closure of this *-algebra nothing but $C^*(\widetilde{X})^\Gamma$ (when we use the Γ,$C_0(\widetilde{X})$-module $L^2(\widetilde{X})$ to define it).

The tangent groupoid $\mathbb{T}_\Gamma(\widetilde{X})$, as set, is equal to

$$TX\times\{0\}\cup\widetilde{X}\times_\Gamma\widetilde{X}\times(0,1]$$

but the smooth structure at 0 is rather different from that one expects, see Section 1.2.3.

In [6] the authors defined a *-algebra of functions on the tangent groupoid that are nothing else than $C^\infty_c(\widetilde{X}\times_\Gamma\widetilde{X})$ at $t\neq 0$ and Schwartz functions in a certain sense at $t=0$, in particular such a function f_t is such that the support of f_t concentrates around the diagonal when t goes to 0. This dense subalgebra of the C*-algebra of the tangent groupoid sits isometrically in the localization algebra of Yu and gives a natural isomorphism between the exact sequence of the localization algebras and the long exact sequence of K-groups associated to the following exac sequence

$$0\longrightarrow C^*_r\left(\widetilde{X}\times_\Gamma\widetilde{X}\times(0,1)\right)\longrightarrow C^*_r\left(\mathbb{T}^0_\Gamma(\widetilde{X})\right)\longrightarrow C^*_r\left(\mathbb{T}_\Gamma(\widetilde{X})\right)\longrightarrow 0$$

where $\mathbb{T}^0_\Gamma(\widetilde{X})=TX\times\{0\}\cup\widetilde{X}\times_\Gamma\widetilde{X}\times(0,1)$, that is the deformation is open at 1.

Notice that the boundary map associated to the previous exact sequence is given again by the evaluation at 1. Moreover the evaluation at 0 from $C^*_r(\mathbb{T}_\Gamma(\widetilde{X}))$ to $C^*_r(TX)$ is a KK-equivalence and $C^*_r(TX)$ is isomorphic to $C_0(T^*X)$. It is known that $C_0(T^*X)$ is the Poincaré dual of $C(X)$, namely the K-theory of $C_0(T^*X)$ is isomorphic to the K-homology of X.

More generally if G is a Lie groupoid and $\mathfrak{A}(G)$ is its Lie algebroid we can construct a deformation groupoid

$$G_{ad} := \mathfrak{A}(G) \times \{0\} \cup G \times (0,1]$$

called the adiabatic groupoid that is equal to the tangent groupoid when $G = \widetilde{X} \times_\Gamma \widetilde{X}$.

And we get the analogous exact sequence

$$\cdots \longrightarrow K_*\left(C_r^*(G \times (0,1))\right) \longrightarrow K_*\left(C_r^*(G_{ad}^0)\right) \xrightarrow{\mathrm{ev}_0} K_*\left(C_r^*(G_{ad})\right) \longrightarrow \cdots$$

where $K_*\left(C_r^*(G_{ad}^0)\right)$ is the home of our secondary invariants. As above the evaluation map $\mathrm{ev}_0\colon C_r^*(G_{ad}^0) \to C_r^*(\mathfrak{A}(G))$ induces a KK-equivalence and the boundary map is given by the composition of the homomorphisms $[\mathrm{ev}_0]^{-1}\colon K_*(C_r^*(\mathfrak{A}(G)))K_*\left(C_r^*(G_{ad})\right)$ and $[\mathrm{ev}_1]\colon K_*\left(C_r^*(G_{ad})\right) \to K_*\left(C_r^*(G)\right)$.

As we can see in the case of the tangent groupoid secondary invariants are essentially given by a symbol (that is a function on the cotangent bundle of X) and a homotopy of its Γ-equivariant index to zero. Indeed if we consider $C_r^*(\mathbb{T}_\Gamma^0(\widetilde{X}))$ as the mapping cone C*-algebra of $\mathrm{ev}_0\colon C_r^*(\mathbb{T}_\Gamma(\widetilde{X})) \to C_r^*(\widetilde{X} \times_\Gamma \widetilde{X})$, this leads to the following and last realization of our exact sequence.

The Grothendieck group of a functor

In [19, II.2] it is defined the Grothendieck group of a functor $\varphi\colon \mathcal{C} \to \mathcal{C}'$, as the set of triples (E, F, α), where E and F are objects in the category $\mathcal{C}$ and α is an isomorphism $\varphi(E) \to \varphi(F)$ in the category $\mathcal{C}'$, modulo the following equivalence relation: two triples (E, F, α) and (E', F', α') are equivalent if there exist two isomorphisms $f\colon E \to E'$ and $g\colon F \to F'$ such that the following diagram

$$\begin{tikzcd}
\varphi(E) \arrow[r, "\alpha"] \arrow[d, "\varphi(f)"] & \varphi(F) \arrow[d, "\varphi(g)"] \\
\varphi(E') \arrow[r, "\alpha'"] & \varphi(F')
\end{tikzcd}$$

commutes.

In [19, II.3.28] it is shown that, when φ is the restriction of vector bundles over a space X to a closed subspace Y, one get the relative K-group $K(X, Y)$ defined as the K-theory of the mapping cone of the inclusion $i\colon Y \hookrightarrow X$.

In [35], G. Skandalis used the same idea: thinking of an element x in $KK(A, B)$ as a functor from $K(A)$ to $K(B)$ through the Kasparov product, one can construct a realtive K-group $K(x)$ and one can also prove that it is isomorphic to the K-theory of a mapping cone C*-algebra. Moreover this relative K-group fits in a long exact sequence

$$\ldots \longrightarrow K(B \otimes C_0(0,1)) \longrightarrow K(x) \longrightarrow K(A) \longrightarrow \ldots$$

such that the boundary map is given by the Kasparov product with x.

More generally, after fixing a separable C*-algebra D, one can think of an element $x \in KK(A, B)$ as a functor from $KK(D, A)$ to $KK(D, B)$, through the Kasparov product with x on the right and still get a relative KK-group $K(D, x)$. The group $KK(D, x)$ only depends on the class of (E, F) in $KK(A, B)$ (with separability assumptions). If this class is given by an exact sequence (admitting a completely positive cross section):

$$0 \longrightarrow B(0,1) \otimes \mathbb{K} \longrightarrow C \longrightarrow A \longrightarrow 0\,,$$

we find $KK(D,x) = KK(D,C)$ and $KK(x,D) = KK(C, D(0,1))$. Indeed one can prove that there always exists an algebra A_1 that is KK-equivalent to A and two morphisms $\varphi\colon A_1 \to B$ and $\psi\colon A_1 \to A$, this latter giving the KK-equivalence, such that $\mathbf{x} = [\psi]^{-1} \otimes_{A_1} [\varphi]$ and in particular the above C*-algebra C is given by the mapping cone $C_\varphi(A_1, B)$.

So constructing relative KK-groups corresponds, in a philosophical way, to take the Grothendieck group of a functor or, in a more concrete way, to take the Grothendieck group associated to homotopy classes of elements in a mapping cone. We want to construct a long exact sequence of groups such that the boundary map is the assembly map. Notice that the difficulty is that it is not induced by a morphism nor by a Kasparov product on the right. But still it is possible to construct a group.

If we see the assembly map $\mu^\Gamma_{\widetilde{X}}$ (see Section 1.3.2) as a functor between the semigroups of Kasparov bimodules $\mathbb{E}^*_\Gamma(C(\widetilde{X}), \mathbb{C})$ and $\mathbb{E}(\mathbb{C}, C^*_r(\Gamma))$ we can define a group $K_*(\mu^\Gamma_{\widetilde{X}})$ as the Grothendieck group of homotopy classes of pairs (ξ, η_t), where

- $\xi \in \mathbb{E}^*_\Gamma(C(\widetilde{X}), \mathbb{C})$ gives an equivariant K-homology class,
- $\eta_t \in \mathbb{E}^*(\mathbb{C}, C^*_r(\Gamma) \otimes C_0[0,1))$ is a path of bimodules, such that $[\eta_0] = \mu^\Gamma_{\widetilde{X}}[\xi]$ is the index of ξ and $[\eta_1] = 0$.

This group fits in a long exact sequence

$$\cdots \longrightarrow KK_*(\mathbb{C}, C^*_r(\Gamma) \otimes C_0(0,1)) \longrightarrow K_*(\mu^\Gamma_X) \longrightarrow K^\Gamma_*(X) \longrightarrow \cdots$$

whose boundary map turns out to be the assembly map and that is naturally isomorphic to the exact sequence given by the coarse algebras.

It is worth to point out that this construction work in a larger generality, that is one can obtain an analogous exact sequence starting from the assembly map $\mu^{\Gamma,A}_Z$ relative to a groupoid Γ and with coefficient in any Γ-algebra A.

Application to Geometric Topology

Higson-Roe and Piazza-Schick construction

In [9, 10, 11] Nigel Higson and John Roe construct a natural transformation of exact sequences between the surgery exact sequence of Browder, Novikov, Sullivan and Wall and the analytic surgery exact sequence of Higson and Roe.

The first one belongs to the realm of geometric topology: it encodes the problem of determining whether a Poincaré complex is or not a manifold and whether the solution of this problem is unique or not.

The second one is a long exact sequence in K-theory associated to an extension of C*-algebras. These algebras are connected to the large scale geometry of metric space and are a fundamental tool in coarse index theory.

Following these seminal papers, Paolo Piazza and Thomas Schick handled in a different way the problem of mapping the surgery exact sequence to analysis as well as for the Stolz sequence. This last sequence is related to the study of metric with scalar curvature. Let us rapidly see what they do.

Let X be a smooth, closed, oriented n-dimensional manifold, $\Gamma = \pi_1(X)$ its fundamental group and $\widetilde{X}$ its universal cover, classified by $u\colon X \to B\Gamma$. The surgery exact sequence associated to X is

$$\cdots \longrightarrow L_{n+1}(\mathbb{Z}\Gamma) \longrightarrow \mathcal{S}(X) \longrightarrow \mathcal{N}(X) \longrightarrow L_n(\mathbb{Z}\Gamma) \longrightarrow \cdots$$

The central object of interest in this sequence is the structure set $S(X)$; elements in the set $S(X)$ are given by homotopy manifold structures on X , i.e. orientation preserving homotopy equivalences $f\colon M \to X$, with M a smooth oriented closed manifold, considered up to h-cobordism. $N(X)$ is the set of degree one normal maps $f\colon M \to X$ considered up to normal bordism. Finally, the abelian groups $L_*(\mathbb{Z}\Gamma)$, the L-groups of the integral group ring $\mathbb{Z}\Gamma$, are defined algebraically but have a geometric realization as cobordism groups of manifolds with boundary with additional structure on the boundary. The surgery exact sequence plays a fundamental role in the classification of high-dimensional smooth compact manifolds.

In [29] the authors build a commutative diagram

$$\begin{tikzcd}
L_{n+1}(\mathbb{Z}\Gamma) \arrow[r] \arrow[d, "\mathrm{Ind}_\Gamma"] & \mathcal{S}(X) \arrow[r] \arrow[d, "\varrho"] & \mathcal{N}(X) \arrow[r] \arrow[d, "\beta"] & L_n(\mathbb{Z}\Gamma) \arrow[d, "\mathrm{Ind}_\Gamma"] \\
K_{n+1}(C^*_r(\Gamma)) \arrow[r] & K_{n+1}(D^*(\widetilde{X})^\Gamma) \arrow[r] & K_n(X) \arrow[r] & K_n(C^*_r(\Gamma))
\end{tikzcd}$$

where the vertical arrows are defined in the following way. The map ϱ is defined applying the following idea: if $f\colon M \to X$ is a homotopy equivalence then $\varrho[f\colon M \to X]$ is defined in terms of the projection onto the positive part of the spectrum of the self-adjoint invertible operator $\mathcal{D} + C_f$, with $\mathcal{D}$ the signature operator on the Galois covering defined by $(f \circ u) \cup (-u)\colon M \cup (-X) \to B\Gamma$ and C_f the Hilsum-Skandalis perturbation defined by f.

A geometrically given cycle for $L_n(Z\Gamma)$ consists in particular of a manifold with boundary (made up of two components), and the extra datum of a homotopy equivalence of the boundaries of the pieces. The Hilsum-Skandalis perturbation for this homotopy equivalence can then be used to perturb the signature operator to be invertible at the boundary. This allows for the definition of a generalized Atiyah-Patodi-Singer index class in $K_{n+1}(C^*_r\Gamma)$, which in the end defines the map Ind_Γ. Finally, β is defined as in Higson and Roe paper: if $f\colon M \to X$ defines a class in $\mathcal{N}(X)$ then its image through β is obtained as $f_*[\mathcal{D}_M] - [\mathcal{D}_X] \in K_n(X)$, with $[\mathcal{D}_M]$ and $[\mathcal{D}_X]$ the fundamental classes associated to the signature operators on the smooth compact manifolds M and X.

The main technical result, crucial for proving the commutativity of the diagram above, it is the so-called delocalized Atiyah-Patodi-Singer index theorem. If we have a manifold W with boundary $M_1 \sqcup M_2$, such that there exists a homotopy equivalence $f\colon M_1 \to M_2$, the theorem states a formula that relates the ϱ invariant of the boundary to the tiyah-Patodi-Singer index class $\mathrm{Ind}_\Gamma(\mathcal{D}, f)$ in $K_{n+1}(C^*_r\Gamma)$:

$$\iota_*(\mathrm{Ind}_\Gamma(\mathcal{D}, f)) = j_*(\varrho(\mathcal{D}_{\partial W} + C_f)) \in K_{n+1}(D^*(\widetilde{W})^\Gamma),$$

where here, $\mathcal{D}$ is the signature operator on W, and we use $j\colon D^*(\partial\widetilde{W})^\Gamma \to D^*(\widetilde{W})^\Gamma$induced by the inclusion $\partial\widetilde{W} \to \widetilde{W}$ and $i\colon C^*(\widetilde{W})^\Gamma \to D^*(\widetilde{W})^\Gamma$ the obvious inclusion.

Generalization to Lipschitz manifold

In [38] Sullivan proves that there always exists a Lipschitz manifold structure on M and that it is unique up to bi-Lipschitz homeomorphism isotopic to the identity. In [39, 40] Teleman studies index theory in the Lipschitz context and in [13] Hilsum develops it in the framework of unbounded Kasparov theory. In particular there is a signature operator arising from the Lipschitz structure and this operator determines a well defined class in the K-homology of M.

Thanks to these results it is possible to extend the work by Piazza and Schick [29] from the smooth to the topological category. We will obtain a natural transformation

$$\begin{tikzcd} L_{n+1}(\mathbb{Z}\Gamma) \arrow[r] \arrow[d, "\mathrm{Ind}_\Gamma"] & \mathcal{S}^{TOP}(M) \arrow[r] \arrow[d, "\varrho"] & \mathcal{N}^{TOP}(M) \arrow[r] \arrow[d, "\beta"] & L_n(\mathbb{Z}\Gamma) \arrow[d, "\mathrm{Ind}_\Gamma"] \\ K_{n+1}(C^*_r(\Gamma)) \arrow[r] & K_{n+1}(D^*(\widetilde{M})^\Gamma) \arrow[r] & K_n(M) \arrow[r] & K_n(C^*_r(\Gamma)) \end{tikzcd}$$

from the surgery exact sequence for topological manifolds to the analytic exact sequence of Higson and Roe, using tools and methods in coarse index theory. To this aim we will use as key tool the Lipschitz structure given by Sullivan theorem [38]. In particular we prove that the key results proven by Piazza and Schick, have a true abstract and K-theoretical meaning, that does not depend on the smooth structure and the pseudodifferential calculus.

One significant difference between the smooth SES and the topological SES is that the second one is an exact sequence of groups, whereas the first one is not. In this paper we deal with the mapping at the set level: to prove that diagram is commutative as a diagram of groups, the main difficulty is that the group structure of the topological structure set is rather hard to handle. The following question is wide open:

- is the map $\varrho\colon \mathcal{S}(M) \to K_{n+1}(D^*(\widetilde{M})^\Gamma)$ a homomorphism of groups?

A positive answer to this question would have direct consequences to Conjecture 3.8 in [47], using the methods in [48].

Generalization to Lie groupoids

As we saw above, there are several realizations of the Analytic Surgery Exact Sequence for a smooth manifold. One of them is the long exact sequence in K-theory

$$\cdots \longrightarrow K_*\left(C^*_r(\widetilde{X}\times_\Gamma \widetilde{X})(0,1)\right) \longrightarrow K_*\left(C^*_r(\mathbb{T}^0_\Gamma(\widetilde{X}))\right) \longrightarrow K_*\left(C_0(T^*X)\right) \longrightarrow \cdots$$

associated to the tangent groupoid.

In this framework secondary invariants sit in the K-theory of the C*-algebra of the tangent groupoid and, in general, in the one of the adiabatic deformation of a Lie groupoid. So it is extremely natural generalizing the definition of these K-theoretic invariants for geometrical situation that are more involved and well encoded by a Lie groupoid: the main example we have in mind is given by the holonomy (or the monodromy) groupoid of a foliation.

All that we did in the case of a smooth manifold gives a program to expand in the context of a Lie groupoid G:

- understanding which is the good definition of a homotopy equivalence between two Lie groupoids;
- defining a signature operator on a Lie groupoid and its index in the groupoid C*-algebra, that we call the analytic G-signature; the same for the Dirac operator;
- generalizing the Hilsum-Skandalis Theorem about homotopy invariance of the analytic signature of a groupoid: Hilsum and Skandalis already did it for the monodromy groupoid of a foliation and the generalization to a general Lie groupoid is completely analogous;

- defining the ϱ-invariant associated to a homotopy equivalence of Lie groupoids or to a metric with positive scalar curvature on the Lie algebroid, as a class in the K-theory of the C*-algebra of the adiabatic deformation;
- giving a formula as in the delocalized Atiyah-Patodi-Singer index theorem that relates the index of the operator on a groupoid over a manifold with boundary (here we assume that the operator on the boundary has zero index, that is it is homotopic to an invertible operator) and the ϱ invariant of the groupoid restricted to the boundary of the base;
- thanks to the above formula, proving that the ϱ invariants are well-defined on bordism classes of Lie groupoids.

The benefit of working with groupoid C*-algebras rather than with coarse C*-algebras is triple: they are smaller and separable algebras (as well as the localization algebras); the proof of theorems are more conceptual as we will see for the delocalized Atiyah-Patodi-Singer index theorem; finaly of course they easily generalize to more complex geometrical contexts. On the contrary definitions of basic objects (as for instance wrong-way functoriality) could seem more involved and, as we saw before, coarse algebras allows to handle the Lipschitz and the topological category.

Products

A further part of this thesis concerns product formulas for secondary invariants in both the Lipschitz context and the groupoid one.

We want to define the following exterior products:

$$K_j(\mu_{X_1}^{\Gamma_1}) \times K_i^{\Gamma_2}(X_2) \to K_{i+j}(\mu_{X_1\times X_2}^{\Gamma_1\times\Gamma_2})$$

in the case of a Lipschitz manifold;

$$K_j\left(C_r^*(G_{ad}^0)\right) \times K_i\left(C_r^*(\mathfrak{A}(H))\right) \to K_{i+j}\left(C_r^*((G\times H)_{ad}^0)\right)$$

in the context of a Lie groupoid.

The general idea is the following: in these two frameworks the home of secondary invariants are given by Grothendieck group of mapping cones of the assembly map $\mu_{X_1}^{\Gamma_1}$ and the evaluation map $\mathrm{ev}_0\colon C_r^*(G_{ad}) \to C_r^*(\mathfrak{A}(G))$ respectively. Whereas $K_i(X_2)$ and $K_i\left(C_r^*(\mathfrak{A}(H)\right)$ are isomorphic to the mapping cylinders of $\mu_{X_2}^{\Gamma_2}$ and $\mathrm{ev}_0\colon C_r^*(H_{ad}) \to C_r^*(\mathfrak{A}(H))$ respectively.

If we see mapping cones and mapping cylinders as deformations (with parameter $t\in[0,1)$ and $s\in[0,1]$ respectively) the cartesian product of them is a deformation of the square $\{(t,s)\,|\,t\in[0,1)\,,\,s\in[0,1]\}$. One can restrict this last deformation to the diagonal and obtain the mapping cylinder of $\mu_{X_1\times X_2}^{\Gamma_1\times\Gamma_2}$ and $\mathrm{ev}_0\colon C_r^*((G\times H)_{ad}) \to C_r^*(\mathfrak{A}(G\times H))$, namely the receptacle of the secondary invariants for the product manifold and the product groupoid.

Moreover it turns out that if we multiply the ϱ invariant associated to a homotopy equivalence $f\colon X_1\to Y$ with the K-homology class given by the signature operator on X_2, we obtain the ϱ class of the homotopy equivalence $f\times\mathrm{id}\colon X_1\times X_2 \to Y\times X_2$ and the same is true for Lie groupoids.

This allows to study the stability of these secondary invariants with respect to cartesian products with other manifolds or Lie groupoids. In particular it is worth to determine when the exterior product with a fixed K-homology class $\xi \in K_i^{\Gamma_2}(X_2)$ (resp. a class $\lambda \in K_i(C_r^*(\mathfrak{A}(H))))$ is injective or at least rational injective.

It turns out that the existence of a K-homology class $\zeta \in KK^{-i}(C_r^*(\Gamma_2), \mathbb{C})$ (resp. in $KK^{-i}(C_r^*(H), \mathbb{C})$) such that the Kasparov product of ζ and $\mu_{X_2}^{\Gamma_2}(\xi) \in KK^i(\mathbb{C}, C_r^*(\Gamma_2))$ (resp. with $\mathrm{Ind}_H(\lambda) \in KK^i(\mathbb{C}, C_r^*(H))$ is equal to $1 \in \mathbb{Z} \cong KK(\mathbb{C}, \mathbb{C})$ (resp. different from 0) implies injectivity (resp. rational injectivity).

Contents

Chapter 1

Exact sequences of K-groups

1.1 Coarse geometry

In this section we are going to recall the fundamental results and constructions of coarse geometry, coarse C*-algebras and coarse index theory. We shall base the exposition on the more complete and well presented discussion in [34].

Let X any set, if $A \subset X \times X$ and $B \subset X \times X$, we will use the following notation:

$$A^{-1} = \{(y,x) \,|\, (x,y) \in A\}$$

and

$$A \circ B = \{(x,z) \,|\, \exists\, y \in X \;: (x,y) \in A \,\text{and}\, (y,z) \in B\}.$$

Definition 1.1.1. A coarse structure on X is a collection of subsets of $X \times X$, called entourages, that have the following properties:

- For any entourages A and B, A^{-1}, $A \circ B$, and $A \cup B$ are entourages;
- Every finite subset of $X \times X$ is an entourage;
- Any subset of an entourage is an entourage.

If $\{(x,x)|x \in X\}$ is an entourage, then the coarse structure is said to be unital.

Definition 1.1.2. Let (X,d) a metric space and let S any set. Two function $f_1, f_2 \colon S \to X$ are said close if $\{d(f_1(s), f_2(s)) \,:\, s \in S\}$ is a bounded set of $\mathbb{R}$.

Definition 1.1.3. Let (X,d) a metric space. A subset $E \in X \times X$ is said to be controlled if the projection maps $\pi_1, \pi_2 \colon X \times X \to X$ are close.

The controlled sets are the ones that are contained in a uniformly bounded neighbourhood of the diagonal. The *metric coarse structure* on (X,d) is given by the collection of all controlled subset of X.

Lemma 1.1.4. *Two maps $f_1, f_2 \colon S \to X$ are close if and only if the image of $(f_1, f_2) \colon S \to X \times X$ is a controlled set.*

There is also a notion of morphism between coarse spaces.

Definition 1.1.5. Let X and Y two metric spaces. A map $f \colon X \to Y$ is said to be coarse if

- whenever $g_1 g_{,2} : S \to X$ are close, $f \circ g_1$ and $f \circ g_2$ are close;
- whenever B is a bounded set of Y, $f^{-1}(B)$ is a bounded set in X.

Let X be a proper metric space equipped with a free and proper action of a countable discrete group Γ of isometries of X.

Definition 1.1.6. Let H be a Hilbert space equipped with a representation

$$\rho \colon C_0(X) \to \mathbb{B}(H)$$

and a unitary representation

$$U \colon \Gamma \to \mathbb{B}(H)$$

such that $U(\gamma)\rho(f) = \rho(\gamma^{-1} f)U(\gamma)$ for every $\gamma \in \Gamma$ and $f \in C_0(X)$. We will call such a triple (H, U, ρ) a Γ-equivariant X-module.

Example 1.1.7. Let us set $H = L^2(X, \mu)$, where μ is a Γ-invariant Borel measure on X. Put

- $\rho \colon C_0(X) \to \mathbb{B}(H)$ the representation given by multipliation operators;
- $U \colon \Gamma \to \mathbb{B}(H)$ the representation given by traslation $U_\gamma \varphi(x) = \varphi(\gamma^{-1} x)$ for every $x \in X$.

Then (H, U, ρ) is a Γ-equivariant X-module.

Definition 1.1.8. Let A be a C*-algebra and let H be a Hilbert space. A rep- resentation $\rho \colon A \to \mathbb{B}(H)$ is said to be ample if it extends to a representation $\widetilde{\rho} \colon \widetilde{A} \to \mathbb{B}(H)$ of the unitalization of A which has the following properties:

- $\widetilde{\rho}$ is nondegenerate, meaning $\widetilde{\rho}(\widetilde{A})H$ is dense in H
- $\widetilde{\rho}(a)$ is compact for $a \in \widetilde{A}$ if and only if $a = 0$.

Moreover we will say that a representation $\rho \colon A \to \mathbb{B}(H)$ is very ample if it is the countable direct sum of a fixed ample representation.

If H is equipped with a unitary representation U of Γ, then we say that an operator $T \in \mathbb{B}(H)$ is Γ-equivariant if $U_\gamma T U_{\gamma^{-1}} = T$ for all $\gamma \in \Gamma$.

1.1.1 Controlled operators

Now we are going to see which kind of analysis arises from a metric coarse structure.

Definition 1.1.9. Let X and Y be two proper metric spaces; let $\rho_X \colon C_0(X) \to \mathbb{B}(H_X)$ and $\rho_Y \colon C_0(Y) \to \mathbb{B}(H_Y)$ be two representations on separable Hilbert spaces.

- The support of an element $\xi \in H_X$ is the set $\mathrm{supp}(\xi)$ of all $x \in X$ such that for every open neighbourhood U of x there is a function $f \in C_0(U)$ with $\rho_X(f)\xi \neq 0$.
- The support of an operator $T \in \mathbb{B}(H_X, H_Y)$ is the set $\mathrm{supp}(T)$ of all $(y, x) \in Y \times X$ such that for all open neighbourhoods $U \ni y$ and $V \ni x$ there exist $f \in C_0(U)$ and $g \in C_0(Y)$ such that $\rho_Y(f) T \rho_X(g) \neq 0$.
- An operator $T \in \mathbb{B}(H_X, H_Y)$ is properly supported if the slices $\{y \in Y : (y, x) \in \mathrm{supp}(T)\}$ and $\{x \in X : (y, x) \in \mathrm{supp}(T)\}$ are closed sets.

Definition 1.1.10. Let X as in the previous definition. An operator $T \in \mathbb{B}(H_X)$ is said to be controlled if its support is a controlled subset of $X \times X$.

This means that an operator is controlled if it is supported in a uniformly bounded neigbourhood of the diagonal of $X \times X$.

Proposition 1.1.11. *The set of all controlled operators for $\rho_X \colon C_0(X) \to H_X$ is a unital *-algebra of $\mathbb{B}(H_X)$.*

1.1.2 The algebra $C^*(X)^\Gamma$

Let (H_X, U, ρ) be an ample Γ-equivariant X-module.

Definition 1.1.12. We define the C*-algebra $C^*(X)^\Gamma$ as the closure in $\mathbb{B}(H_X)$ of the *-algebra of all Γ-equivariant operators T such tat

- T has finite propagation, i.e. there is a $R > 0$ such that $\rho(\varphi)T\rho(\psi) = 0$ for all $\varphi, \psi \in C_0(X)$ with $d(\mathrm{supp}(\varphi), \mathrm{supp}(\psi)) > R$;
- T is locally compact, i.e. $T\rho(\varphi)$ and $\rho(\varphi)T$ are compact operators.

If Γ is the trivial group, then we will suppress it from the notation and write $C^*(X)$.

The algebra $C^*(X)^\Gamma$ of course depends on the X-module used to construct it, but we are going to see that its K-theory does not.

Let (H_X, U^X, ρ_X) and (H_Y, U^Y, ρ_Y) be an X-module and a Y-module respectively, both of them Γ-equivariant.

Definition 1.1.13. An isometry $V \colon H_X \to H_Y$ coarsely covers a coarse map

$$f \colon X \to Y$$

if V is the norm-limit of linear maps V satisfying the following condition:

$$\exists R > 0 \text{ such that } \varphi V \psi = 0 \text{ if } d(\mathrm{supp}(\varphi), f(\mathrm{supp}(\psi))) > R$$

for $\varphi \in C_0(X)$ and $\psi \in C_0(Y)$.

Proposition 1.1.14. *Let (H_X, U^X, ρ_X) and (H_Y, U^Y, ρ_Y) be as above, with H_Y an ample Γ-equivariant representation. Then every Γ-equivariant coarse map $\varphi \colon X \to Y$ is coarsely covered by a Γ-equivariant isometry $V \colon H_X \to H_Y$. Moreover one can choose V such that its support is contained in the ε-neighbourhood of $\{(x, y) \in X \times Y \;:\; d(f(x), y) \leq \varepsilon\}$, the graph of f, for any $\varepsilon > 0$.*

Lemma 1.1.15. *If $V \colon H_X \to H_Y$ is an equivariant isometry that coarsely covers a coarse equivariant map $f \colon X \to Y$, then the application*

$$\mathrm{Ad}_V \colon \mathbb{B}(H_X) \to \mathbb{B}(H_Y),$$

given by $T \mapsto VTV^*$, maps $C^*(X)^\Gamma$ to $C^*(Y)^\Gamma$.

Lemma 1.1.16. *If V_1 and V_2 are two equivariant isometries that coarsely cover the same coarse equivariant map $f \colon X \to Y$, then Ad_{V_1} and Ad_{V_2} induce the same map $K_*(C^*(X)^\Gamma) \to K_*(C^*(Y)^\Gamma)$.*

It follows that if $f \colon X \to Y$ is an equivariant coarse map, then there is no ambiguity in the definition of the induced map $f_* \colon K_*(C^*(X)^\Gamma) \to K_*(C^*(Y)^\Gamma)$ as Ad_V, where V is any equivariant isometry which coarsely covers f.

Moreover if H_X and H'_X are two Γ-equivariant X-module, the previous results applied to the identity map $X \to X$ imply that $K_*(C^*(X)^\Gamma)$ is independent of the ample representation used to define $C^*(X)^\Gamma$.

1.1.3 The algebra $D^*(X)^\Gamma$

Let us still consider the geometrical setting of the previous section.

Definition 1.1.17. The algebra $D^*(X)^\Gamma$ is the norm closure in $\mathbb{B}(H_X)$ of the *-algebra of all Γ-equivariant operators T such that

- T has finite propagation, i.e. there is a $R > 0$ such that $\rho(\varphi)T\rho(\psi) = 0$ for all $\varphi, \psi \in C_0(X)$ with $d(\mathrm{supp}(\varphi), \mathrm{supp}(\psi)) > R$;
- T is pseudolocal, i.e. $[T, \rho(\varphi)]$ is a compact operator for any $\varphi \in C_0(X)$.

Here H_X has the structure of a very ample Γ-equivariant X-module.

Let us study the functoriality of this algebra with respect to coarse maps.

Definition 1.1.18. Let H_X be a X-module and let H_Y be a Y-module, both of them Γ-equivariant.

- A map $f\colon X \to Y$ is uniform if it is continuous and coarse.
- An isometry $V\colon H_X \to H_Y$ uniformly covers a uniform map f if it coarsely covers f and $\rho_Y(\varphi)V - V\rho(\varphi \circ f)$ is compact for any $\varphi \in C_0(Y)$.

One can prove the following results, analogous to the ones in the previous section.

Proposition 1.1.19. *Let (H_X, U^X, ρ_X) and (H_Y, U^Y, ρ_Y) be Γ-equivariant modules, with H_Y a very ample Γ-equivariant representation. Then every Γ-equivariant uniform map $\varphi\colon X \to Y$ is uniformly covered by a Γ-equivariant isometry $H_X \to H_Y$.*

Lemma 1.1.20. *If $V\colon H_X \to H_Y$ is an equivariant isometry that uniformly covers a coarse equivariant map $f\colon X \to Y$, then $\mathrm{Ad}_V\colon \mathbb{B}(H_X) \to \mathbb{B}(H_Y)$ maps $D^*(X)^\Gamma$ to $D^*(Y)^\Gamma$.*

Lemma 1.1.21. *If V_1 and V_2 are two equivariant isometries that uniformly cover the same coarse equivariant map $f\colon X \to Y$, then Ad_{V_1} and Ad_{V_2} induce the same map $K_*(D^*(X)^\Gamma) \to D_*(C^*(Y)^\Gamma)$.*

It turns out that the construction of $K_*(D^*(X)^\Gamma)$ is functorial as for $K_*(C^*(X)^\Gamma)$.

Remark 1.1.22. It turns out that $D^*(X)^\Gamma$ is a subalgebra of the multiplier algebra of $C^*(X)^\Gamma$.

1.1.4 Localization algebras

Yu [50] has introduced the localization algebra $C^*_L(X)^\Gamma$ to provide an alternative model for the K-homology of a proper metric space X. In fact, there is an isomorphism $K^\Gamma_n(X) \cong K_n(C^*_L(X)^\Gamma)$, called the local index map. This can be proved for simplicial complexes using a Mayer-Vietoris argument [50], or alternatively, the general case can be reduced to Paschke duality [30].

Let (H_X, U, ρ) be an ample Γ-equivariant X-module.

Definition 1.1.23. We define

- $C^*_L(X)^\Gamma$ as the C*-algebra generated by all bounded and uniformly continuous functions $f : [0, \infty) \to C^*(X)^\Gamma$ such that propagation of $f(t)$ goes to 0, as t goes to ∞;

- $D_L^*(X)^\Gamma$ as the C*-algebra generated by all bounded and uniformly continuous functions $f : [0, \infty) \to D^*(X)^\Gamma$ such that propagation of $f(t)$ goes to 0, as t goes to ∞;

- $C_{L,0}^*(X)^\Gamma$ as the kernel of the evaluation map $\mathrm{ev}_0 \colon C_L^*(X)^\Gamma \to C^*(X)^\Gamma$, such that $\mathrm{ev}_0(f) = f(0)$. In particular, $C_{L,0}^*(X)^\Gamma$ is an ideal of $C_L^*(X)^\Gamma$;

- $D_{L,0}^*(X)^\Gamma$ as the kernel of the evaluation map $\mathrm{ev}_0 \colon D_L^*(X)^\Gamma \to D^*(X)^\Gamma$, such that $\mathrm{ev}_0(f) = f(0)$. In particular, $D_{L,0}^*(X)^\Gamma$ is an ideal of $D_L^*(X)^\Gamma$;

As for $D^*(X)^\Gamma$, the C*-algebra $D_L^*(X)^\Gamma$ (resp. $D_{L,0}^*(X)^\Gamma$) is a subalgebra of the multiplier algebra of $C_L^*(X)^\Gamma$ (resp. $C_{L,0}^*(X)^\Gamma$).

Remark 1.1.24. Note that we can define the maximal version of all these C*-algebras.

There is a good functoriality for these algebras too with respect to uniformly continuous and coarse maps.

Let X and Y be proper metric spaces endowed with free and proper Γ-actions. A proper map $f \colon X \to Y$ is uniformly continuous and coarse (briefly u.c.c.) if and only if there exists a non-decreasing function $S \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which is continuous at 0 with S(0) = 0 such that $d_Y(f(x_1), f(x_2)) < S(d_X(x_1, x_2))$ for all $x_1, x_2 \in X$.

Definition 1.1.25. Let $f \colon X \to Y$ be a Γ-equivariant u.c.c. map and fix a Γ-equivariant X-module module H_X and a Γ-equivariant Y-module module H_Y . We say that a uniformly continuous family of Γ-equivariant isometries $\{V_t \colon H_X \to H_Y\ , t \in [0, \infty)\}$ covers f if $\sup\{d(y, f(x)) | (y, x) \in \mathrm{supp}(V_t)\} \to 0$ as $t \to \infty$.

Lemma 1.1.26. *Let $f \colon X \to Y$ be a Γ-equivariant u.c.c. map. Then there exists a family of isometries $V_t \colon H_X \to H_Y, , t \in [0, \infty)$, that covers f as in the previous definition.*

Lemma 1.1.27. *Let $f \colon X \to Y$ be a Γ-equivariant u.c.c. map and $V_t \colon H_X \to H_Y$ be a family of isometries that covers f. Then conjugation by V_t induces a *-homomorphism*

$$\mathrm{Ad}_{V_t} \colon C_L^*(X)^\Gamma \to C_L^*(Y)^\Gamma.$$

The induced map $(\mathrm{Ad}_{V_t})_* \colon K_*(C_L^*(X)^\Gamma) \to K_*(C_L^*(Y)^\Gamma)$ does not depend on the choice of the family of isometries V_t covering f. In particular, the K-theory of $C_L^*(X)^\Gamma$ does not depend on the choice of the ample Γ-equivariant X-module up to canonical isomorphism.

Hence the map f induces a well-defined map on K-theory of the localization algebras, and, as for coarse algebras, our notation will be $f_* = (\mathrm{Ad}_{V_t})_*$.

Now recall how the local index map $\mathrm{Ind}_L \colon K_*^\Gamma(X) \to K_*(C_L^*(X)^\Gamma)$ is defined. Consider an even cycle for $K_*^\Gamma(X)$ given by (H_X, F). Let $\{U_i\}$ be a Γ-invariant locally finite open cover of X with diameter$(U_i) < c$ for some fixed $c > 0$. Let $\{\phi_i\}$ be a Γ-invariant continuous partition of unity subordinate to $\{U_i\}$. We define

$$G = \sum_i \phi_i^{\frac{1}{2}} F \phi_i^{\frac{1}{2}},$$

where the sum converges in strong topology. It is not difficult to see that (H_X, F) is equivalent to (H_X, G) in $K_*^\Gamma(X)$. By using the fact that G has finite propagation, we see that G is a multiplier of $C^*(X)^\Gamma$ and G is a unitary modulo $C^*(X)^\Gamma$. Hence G produces a class $[G] \in K_0(C^*(X)^\Gamma)$. We define the index of (H_X, F) to be $[G]$. We denote this index class of (H_X, F) by $\mathrm{Ind}(H_X, F)$. For each $n \in \mathbb{N}$, let $\{U_n, j\}$ be a Γ-invariant locally finite open cover of X with diameter$(U_n, j) < 1/n$ and $\{\phi_n, j\}$ be a Γ-invariant continuous partition of unity subordinate to $\{U_n, j\}$. We define

$$G(t) = \sum_j (1-(t-n))\phi_{n,j}^{\frac{1}{2}} G \phi_{n,j}^{\frac{1}{2}} + (t-n)\phi_{n+1,j}^{\frac{1}{2}} G \phi_{n+1,j}^{\frac{1}{2}}$$

for $t \in [n, n+1]$. Here by convention, we assume that the open cover U_0, j is the trivial cover $\{X\}$ when $n = 0$. Then $G(t)$, $0 \leq t < \infty$, is a multiplier of $C_L^*(X)^\Gamma$ and a unitary modulo $C_L^*(X)^\Gamma$, hence defines a class in $K_0(C_L^*(X)^\Gamma)$. Hence it define a class $[G(t)]$ in $K_0(C_L^*(X)^\Gamma)$, that we call the local index of (H_X, F). We denote this local index class of (H_X, F) by $\mathrm{Ind}_L(H_X, F)$.

1.1.5 The analytic surgery exact sequence

Let X be a proper metric space such that a countable discrete group Γ acts properly, freely and isometrically on it.

The algebras so far defined fit in interesting exact sequences. We have the Higson-Roe analytic surgery exact sequence

$$\cdots \longrightarrow K_*(C^*(X)^\Gamma) \longrightarrow K_*(D^*(X)^\Gamma) \longrightarrow K_*(D^*(X)^\Gamma/C^*(X)^\Gamma) \longrightarrow \cdots$$

and the Yu's sequence

$$\cdots \longrightarrow K_*(C_{L,0}^*(X)^\Gamma) \longrightarrow K_*(C_L^*(X)^\Gamma) \xrightarrow{\mathrm{ev}_0} K_*(C^*(X)^\Gamma) \longrightarrow 0$$

and we know by [49, Section 6] that there is a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & K_*(C^*(X)^\Gamma) \arrow[r] \arrow[d] & K_*(D^*(X)^\Gamma) \arrow[r] \arrow[d] & K_*(D^*(X)^\Gamma/C^*(X)^\Gamma) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & K_*(C^*(X)^\Gamma) \arrow[r, "\partial"] & K_{*+1}(C_{L,0}^*(X)^\Gamma) \arrow[r] & K_{*+1}(C_L^*(X)^\Gamma) \arrow[r, "\mathrm{ev}_0"] & 0
\end{tikzcd}$$

where all vertical arrows are isomorphisms.

1.2 Groupoids

1.2.1 Basics

We refer the reader to [5] and bibliography inside it for the notations and a detailed overview about groupoids and index theory.

Definition 1.2.1. Let G and $G^{(0)}$ be two sets. A groupoid structure on G over $G^{(0)}$ is given by the following morphisms:

- An injective map $u : G^{(0)} \to G$ called the unit map. We can identify $G^{(0)}$ with its image in G.
- Two surjective maps: $r, s : G \to G^{(0)}$, which are respectively the range and source map.
- An involution: $i : G \to G$, $\gamma \mapsto \gamma^{-1}$ called the inverse map. It satisfies: $s \circ i = r$.

- A map $p : G^{(2)} \to G$, $(\gamma_1, \gamma_2) \mapsto \gamma_1 \cdot \gamma_2$ called the product, where the set

$$G^{(2)} := \{(\gamma_1, \gamma_2) \in G \times G \mid s(\gamma_1) = r(\gamma_2)\}$$

is the set of composable pair. Moreover for $(\gamma_1, \gamma_2) \in G^{(2)}$ we have $r(\gamma_1 \cdot \gamma_2) = r(\gamma_1)$ and $s(\gamma_1 \cdot \gamma_2) = s(\gamma_2)$.

The following properties must be fulfilled:

- The product is associative: for any $\gamma_1,\ \gamma_2,\ \gamma_3$ in G such that $s(\gamma_1) = r(\gamma_2)$ and $s(\gamma_2) = r(\gamma_3)$ the following equality holds

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 = \gamma_1 \cdot (\gamma_2 \cdot \gamma_3)\ .$$

- For any γ in G: $r(\gamma) \cdot \gamma = \gamma \cdot s(\gamma) = \gamma$ and $\gamma \cdot \gamma^{-1} = r(\gamma)$.

We denote a groupoid structure on G over $G^{(0)}$ by $G \rightrightarrows G^{(0)}$, where the arrows stand for the source and target maps.

We will adopt the following notations:

$$G_A := s^{-1}(A)\ ,\ G^B = r^{-1}(B) \ \text{ and } G_A^B = G_A \cap G^B$$

in particular if $x \in G^{(0)}$, the s-fiber (resp. r-fiber) of G over x is $G_x = s^{-1}(x)$ (resp. $G^x = r^{-1}(x)$).

Definition 1.2.2. We call G a Lie groupoid when G and $G^{(0)}$ are second-countable smooth manifolds with $G^{(0)}$ Hausdorff, the structural homomorphisms are smooth, u is an embedding, r and s are submersions, and i is a diffeomorphism.

1.2.2 Groupoid C*-algebras

We can associate to a Lie groupoid G the *-algebra $C_c^\infty(G, \Omega^{\frac{1}{2}}(\ker ds \oplus \ker dr))$ of the compactly supported sections of the half densities bundle associated to $\ker ds \oplus \ker dr$, with:

- the involution given by $f^*(\gamma) = \overline{f(\gamma^{-1})}$;
- and the product by $f * g(\gamma) = \int_{G_{s(\gamma)}} f(\gamma\eta^{-1}) g(\eta) d\eta$.

For all $x \in G^{(0)}$ the algebra $C_c^\infty(G, \Omega^{\frac{1}{2}}(\ker ds \oplus \ker dr))$ can be represented on $L^2(G_x, \Omega^{\frac{1}{2}}(G_x))$ by

$$\lambda_x(f)\xi(\gamma) = \int_{G_x} f(\gamma\eta^{-1}) g(\eta) d\eta,$$

where $f \in C_c^\infty(G, \Omega^{\frac{1}{2}}(\ker ds \oplus \ker dr))$ and $\xi \in L^2(G_x, \Omega^{\frac{1}{2}}(G_x))$.

Definition 1.2.3. The reduced C*-algebra of a Lie groupoid G, denoted by $C_r^*(G)$, is the completion of $C_c^\infty(G, \Omega^{\frac{1}{2}}(\ker ds \oplus \ker dr))$ with respect to the norm

$$||f||_r = \sup_{x \in G^{(0)}} ||\lambda_x(f)||.$$

The full C*-algebra of G is the completion of $C_c^\infty(G, \Omega^{\frac{1}{2}}(\ker ds \oplus \ker dr))$ with respect to all continuous representations.

Remark 1.2.4. Let A be a C*-algebra. If A is a $C(X)$-algebra, then we will use the following notation:

- if O is an open set in X, then $A_O := C_0(O) \cdot A$;
- if F is a closed set in X, then $A_F := A/\left(C_0(X \setminus F) \cdot A\right)$;
- if $Z = O \cap F$ is a locally closed set in X (with O open and F closed), then $A_Z := (A_O)_F = (A_F)_O$.

If $Y \subset Z$ is a closed set, then denote with e_Y^Z the restricion homomorphism from A_Z to A_Y.

Obviously the previously defined groupoid C*-algebras are $C_0(G^{(0)})$-algebras. From now on, if X is a locally closed subset of $G^{(0)}$ we will call $e_X \colon C_r^*(G) \to C_r^*(G_{|X})$ the evaluation map at X.

1.2.3 Lie algebroids and the adiabatic groupoid

Definition 1.2.5. A Lie algebroid $\mathfrak{A} = (p : \mathfrak{A} \to TM, [\ ,\ ]_{\mathfrak{A}})$ on a smooth manifold M is a vector bundle $\mathfrak{A} \to M$ equipped with a bracket $[\ ,\ ]_{\mathfrak{A}} : \Gamma(\mathfrak{A}) \times \Gamma(\mathfrak{A}) \to \Gamma(\mathfrak{A})$ on the module of sections of $\mathfrak{A}$, together with a homomorphism of fiber bundle $p : \mathfrak{A} \to TM$ from $\mathfrak{A}$ to the tangent bundle TM of M, called the anchor map, fulfilling the following conditions:

- the bracket $[\ ,\ ]_{\mathfrak{A}}$ is $\mathbb{R}$-bilinear, antisymmetric and satisfies the Jacobi identity,
- $[X, fY]_{\mathfrak{A}} = f[X,Y]_{\mathfrak{A}} + p(X)(f)Y$ for all $X,\ Y \in \Gamma(\mathfrak{A})$ and f a smooth function of M,
- $p([X,Y]_{\mathfrak{A}}) = [p(X), p(Y)]$ for all $X,\ Y \in \Gamma(\mathfrak{A})$.

The tangent space to s-fibers, that is $T_sG := \ker ds = \bigcup_{x \in G^{(0)}} TG_x$ has the structure of a Lie algebroid on $G^{(0)}$, with anchor map given by dr. It is denoted by $\mathfrak{A}(G)$ and we call it the Lie algebroid of G. We can also think of it as the normal bundle of the inclusion $G^{(0)} \hookrightarrow G$.

Let M_0 be a smooth compact submanifold of a smooth manifold M with normal bundle $\mathcal{N}$. As a set, the deformation to the normal cone is $D(M_0, M) = \mathcal{N} \times \{0\} \sqcup M \times (0,1]$.

In order to recall its smooth structure, we fix an exponential map, which is a diffeomorphism θ from a neighbourhood V' of the zero section M_0 in N to a neighbourhood V of M_0 in M. We may cover $D(M_0, M)$ with two open sets $M \times (0,1]$, with the product structure, and $W = \mathcal{N} \times 0 \sqcup V \times (0,1]$, endowed with the smooth structure for which the map

$$\Psi\{(m,\xi,t) \in \mathcal{N} \times [0,1] \,|\, (m, t\xi) \in V'\} \to W \tag{1.2.1}$$

given by $(m,\xi,t) \mapsto (\theta(m,t\xi),t)$, for $t \neq 0$, and by $(m,\xi,0) \mapsto (m,\xi,0)$, for $t = 0$, is a diffeomorphism. One can verify that the transition map on the overlap of thes two charts is smooth, see...

The adiabatic groupoid G_{ad} is given by the following one

$$\mathfrak{A}(G) \times \{0\} \cup G \times (0,1] \rightrightarrows G^{(0)} \times [0,1],$$

with the smooth structure given by the deformation to the normal cone associated to $G^{(0)} \hookrightarrow G$.

Definition 1.2.6. We will use the notation G^0_{ad} for the restriction adiabatic groupoid to the interval open on the right side, given by

$$\mathfrak{A}(G) \times \{0\} \cup G \times (0,1) \rightrightarrows G^{(0)} \times [0,1).$$

Definition 1.2.7. Let G be a Lie groupoid, then we associate to it a short exact sequence of C*-algebras

$$0 \longrightarrow C^*_r(G \times (0,1)) \longrightarrow C^*_r(G^0_{ad}) \xrightarrow{\mathrm{ev}_0} C^*(\mathfrak{A}(G)) \longrightarrow 0 \tag{1.2.2}$$

that we call the (reduced) adiabatic extension of G.

Notice that there is an analogous extension given by the full groupoid C*-algebras.

1.2.4 Groupoids over manifolds with boundary

For this section we refer the reader to [24] and [33, 3.1], where we take some notations from.

Let X be a manifold with boundary ∂X. We can think of X as a closed subspace of an open manifold $\overline{X}$. Let $\rho\colon \overline{X} \to \mathbb{R}$ be a defining function of the boundary, namely a function that is zero on ∂X and only there, with nowhere vanishing differential on it.

Definition 1.2.8. The *b*-calculus groupoid of X, denoted by $\Gamma(X, \partial X)$, is given by

$$\{(x, y, \alpha) \in X \times X \times \mathbb{R} \mid \rho(x) = e^{\alpha}\rho(y)\}.$$

It turns out that this groupoid is nothing but $\mathring{X} \times \mathring{X} \cup \partial X \times \partial X \times \mathbb{R} \rightrightarrows X$.

A topological groupoid is said to be longitudinally smooth if the s-fibers are smooth manifolds. If we consider G, the restriction to $(X, \partial X)$ of a Lie groupoid $\overline{G}$ on $\overline{X}$, this condition is not often fulfilled, because of the boundary. To get rid of this, we desingularize the groupoid on the boundary thanks to the following construction.

Definition 1.2.9. Let $G \rightrightarrows X$ be as above. Let the boundary be transverse with respect to $\overline{G}$ (this means that the range map and the inclusion of ∂X in $\overline{X}$ are transverse). Define $G(X, \partial X)$ as the following fibered product

$$\begin{tikzcd} G(X,\partial X) \arrow[r] \arrow[d] & G \arrow[d, "r\times s"] \\ \Gamma(X,\partial X) \arrow[r, "r\times s"] & X \times X \end{tikzcd}$$

where $\Gamma(X, \partial X)$ is the b-calculus groupoid of $(X, \partial X)$. Then $G(X, \partial X) \rightrightarrows X$ is a longitudinally smooth groupoid. As set, it turns out to be $G_{|\mathring{X}} \cup G_{|\partial X} \times \mathbb{R}$. See [24, Section 3] for a detailed construction.

Remark 1.2.10. For now on, when we say "Lie groupoid on a manifold with boundary", we will assume understood that we are dealing with the restriction of a true Lie groupoid a smooth manifold containing our manifold with boundary, such that it is transvere to the boundary.

In this context it is convenient to use a slight variation of the adiabatic groupoid.

Definition 1.2.11. Let $G(X, \partial X)$ be as in Definition1.2.9:

- let $G(X,\partial X)_{ad}^{\mathcal{F}}$ be the restriction of $G(X,\partial X)_{ad}$ to $X_{\mathcal{F}} := X \times [0,1] \setminus \partial X \times \{1\}$. One can see that it is the union $(G_{|\mathring{X}})_{ad} \cup (G_{|\partial X})_{ad}^{0} \times \mathbb{R}$;

- let $\mathcal{T}_{nc}G(X,\partial X)$ be the restriction of $G(X,\partial X)_{ad}^{\mathcal{F}}$ to $X_{\partial} := \mathring{X} \times \{0\} \cup \partial X \times [0,1)$. One can see that it is the union $\mathcal{A}(G_{|\mathring{X}}) \cup (G_{|\partial X})_{ad}^{0} \times \mathbb{R}$;

- finally let $G(X,\partial X)_{ad}^{0}$ be the restriction of $G(X,\partial X)_{ad}^{\mathcal{F}}$ to $X \times [0,1)$.

The superscript $\mathcal{F}$ is about a Fredholmness condition that will be clear later.

Remark 1.2.12. Consider the restriction morphism

$$e_{X_\partial} \colon C_r^*\left(G(X,\partial X)_{ad}^{\mathcal{F}}\right) \to C_r^*\left(\mathcal{T}_{nc}G(X,\partial X)\right).$$

Since $\ker(e_{X_\partial}) = C_r^*(G_{|\mathring{X}} \times (0,1])$ is KK-contractible, e_{X_∂} induces a KK-equivalence.

Lemma 1.2.13. *The C^*-algebra $C_r^*(\Gamma(\mathbb{R}_+,\{0\}))$ is K-contractible.*

Proof. We have that $\Gamma(\mathbb{R}_+,\{0\}) = \mathbb{R}_+^* \times \mathbb{R}_+^* \cup \{0\} \times \{0\} \times \mathbb{R} \rightrightarrows \mathbb{R}_+$. It is isomorphic to the groupoid $\mathbb{R}_+ \rtimes \mathbb{R}_+$ thanks to the morphism $\phi\colon \mathbb{R}_+^* \times \mathbb{R}_+^* \cup \{0\} \times \{0\} \times \mathbb{R} \to \mathbb{R}_+ \rtimes \mathbb{R}_+$ such that

- $(y_1,y_2) \mapsto (y_2, \frac{y_1}{y_2})$ if $y_1, y_2 \neq 0$;

- $(0,0,\lambda) \mapsto (0, e^{\lambda})$.

Hence $C_r^*(\Gamma(\mathbb{R}_+,\{0\})) \simeq C_r^*(\mathbb{R}_+ \rtimes \mathbb{R}_+) \simeq C_0(\mathbb{R}_+) \rtimes \mathbb{R}_+$ and, by the Connes-Thom isomorphism, $K_*\left(C_0(\mathbb{R}_+) \rtimes \mathbb{R}\right) \simeq K_{*-1}\left(C_0(\mathbb{R}_+)\right) = 0$. ☐

Corollary 1.2.14. *Let X be any smooth manifold and let $H \rightrightarrows X$ be a Lie groupoid. Consider the groupoid $G = H \times (-1,1) \times (-1,1) \rightrightarrows X \times (-1,1)$, the b-calculus groupoid of the retriction to $X \times [0,1)$ of G, denoted by $G(X \times [0,1), X \times \{0\})$, is given by $H \times \Gamma([0,1(,\{0\})$. Then the boundary map of the long exact sequence of K-groups associated to the following exact sequence of C^*-algebras*

$$0 \longrightarrow C_r^*\left(H \times (0,1) \times (0,1)\right) \longrightarrow C_r^*\left(G(X \times [0,1), X \times \{0\})\right) \longrightarrow C_r^*\left(H \times \mathbb{R}\right) \longrightarrow 0$$

is an isomorphism.

Proof. The b-calculus groupoid associated to G is given by $H \times \Gamma([0,1),\{0\})$. Then Lemma 1.2.13 implies that $K_*(C_r^*\left(G(X \times [0,1), X \times \{0\})\right)) = 0$, hence the result follows. ☐

Remark 1.2.15. The boundary map of the previous Corollary is nothing but the inverse of the Bott element.

1.2.5 Pull-back groupoid

Here we recall the pull-back construction for Lie groupoids. Let $G \rightrightarrows X$ be a Lie groupoid and let $\varphi\colon Y \to X$ be a transverse map with respect to G. This means that $d\varphi(T_yY) + q(\mathfrak{A}_{\varphi(y)}(G)) = T_{\varphi(y)}X$, where $q\colon \mathfrak{A}(G) \to TX$ is the anchor map of the Lie algebroid.

Definition 1.2.16. Let us fix the following notations:

- $G_\varphi = \{(\gamma, y) \in G \times Y \mid \varphi(y) = s(\gamma)\}$;

- $G^{\varphi} = \{(y, \gamma) \in Y \times G \,|\, \varphi(y) = r(\gamma)\}$;
- $G_{\varphi}^{\varphi} = \{(y_1, \gamma, y_2) \in Y \times G \times Y \,|\, \varphi(y_1) = r(\gamma)\,,\, \varphi(y_2) = s(\gamma)\}$.

Remark 1.2.17. The source and the target map for G_{φ}^{φ} are given by $s(y_1, \gamma, y_2) = y_2$ and $r(y_1, \gamma, y_2) = y_1$ respectively. Moreover $(y_1, \gamma, y_2)^{-1} = (y_2, \gamma^{-1}, y_1)$ and $(y_1, \gamma, y_2) \cdot (y_2, \gamma', y_3) = (y_1, \gamma \cdot \gamma', y_3)$.

Since $r, s \colon G \to X$ are submersions, G_{φ} and G^{φ} are submanifolds of $G \times Y$ and $Y \times G$ respectively. We are going to prove that G_{φ}^{φ} is a smooth manifold. The space G_{φ} in Definition 1.2.16 is given by the following pull-back

$$\begin{tikzcd} G_{\varphi} \arrow[r, "\tilde{\varphi}"] \arrow[d, "p"] & G \arrow[d, "s"] \\ Y \arrow[r, "\varphi"] & X \end{tikzcd}$$

and one can see that p is a surjective submersion, because s is so.

We want to prove that $k = r \circ \tilde{\psi}$ is a smooth submersion. Let $(\gamma_0, y_0) \in G_{\varphi}$ such that γ_0 is a unit of G. We define the following inclusion

- $i \colon G_{s(\gamma_0)} \to G_{\varphi}$, $i \colon \gamma \mapsto (\gamma, y_0)$ and put $\delta = k \circ i$;
- $j \colon Y \to G_{\varphi}$ is such that $j \colon y \mapsto (id_{\psi(y)}, y)$ and put $\varepsilon = k \circ j$.

Notice that $s(\gamma_0) = \alpha(\gamma_0) = \beta(y_0) = \psi(y_0)$ and then, by trasversality, it turns out that

$$\begin{aligned} dk_{(\gamma_0, y_0)}(di(T_{\gamma_0} G_{s(\gamma_0)}) + dj(T_{y_0} Y)) &= d\delta(T_{\gamma_0} G_{s(\gamma_0)}) + d\varepsilon(T_{y_0} Y) = \\ &= q(\mathfrak{A}_{\psi(y_0)}(G)) + d\psi(T_{y_0} Y) = T_{\psi(y_0)} X, \end{aligned}$$

hence that k is a submersion at (γ_0, y_0).

Now let us consider $(\gamma_1, y_1) \in G_{\varphi}$, with γ_1 not necessarily a unit. Construct the following pull-back

$$\begin{tikzcd} G_{\varphi}^{(2)} \arrow[r, "p_2"] \arrow[d, "p_1"] & G \arrow[d, "s"] \\ G_{\varphi} \arrow[r, "k"] & X \end{tikzcd}$$

where $G_{\varphi}^{(2)} = \{(\gamma, \gamma', y) \in G \times G_{\varphi} \,|\, s(\gamma) = r(\gamma')\}$. We have that p_1 is a submersion, because s is so. Moreover, at the point $(\gamma_1, \psi(y_1), y_1)$, p_2 is a submersion, since k is so at $(\varphi(y_1), y_1)$.

Let $(m, id) \colon G_{\varphi}^{(2)} \to U_{\varphi}$ be the map such that $(m, id) \colon (\gamma, \gamma', y) \mapsto (\gamma\gamma', y)$. Then we have that $r \circ p_2 = k \circ (m, id)$. But at $(\gamma_1, \psi(y_1), y_1)$ $r \circ p_2$ is a submersion, hence so is k at $(m, id)(\gamma_1, \psi(y_1), y_1) = (\gamma_1, y_1)$.

We have proven that the map $k \colon (\gamma, y) \mapsto r(\gamma)$ is a submersion because of transnversality. Then by the following pull-back diagram

$$\begin{tikzcd} G_{\varphi}^{\varphi} \arrow[r] \arrow[d] & G_{\varphi} \arrow[d, "k"] \\ Y \arrow[r, "\varphi"] & X \end{tikzcd}$$

it follows that G_{φ}^{φ} is a manifold. Moreover $G_{\varphi}^{\varphi} \rightrightarrows Y$ is a Lie groupoid that we will call the pull-back groupoid of G by φ.

One can easily show that

$$\mathfrak{A}(G_\varphi^\varphi) \simeq \{(\xi,\eta) \in TY \times \mathfrak{A}(G) \,|\, d\varphi(\xi) = q(\eta)\},$$

where q is the anchor map of $\mathfrak{A}(G)$. On the other hand the anchor map of $\mathfrak{A}(\,G_\varphi^\varphi)$ is the projection on TY.

There is a canonical way to construct a $C_r^*(G_\varphi^\varphi)$-$C_r^*(G)$-bimodule. Let us consider the groupoid

$$G_{\varphi\cup id_X}^{\varphi\cup id_X} \rightrightarrows Y \sqcup X$$

and let us denote it with L. We have that $L_X^X = G$, $L_Y^Y = G_\varphi^\varphi$, $L_X^Y = G^\varphi$ and $L_Y^X = G_\varphi$.

Remark 1.2.18. Because of this decomposition of $G_{\varphi\cup id_X}^{\varphi\cup id_X}$ we will keep the source and target notations for G^φ and G_φ too, though they are not groupoids in themeselves.

Let p_Y be the projection given by the restriction to L_Y^Y and let p_X be the projection given by the restriction to L_X^X, they are in the multipliers algebra of $C_r^*(L)$. Then $E_\varphi = p_Y C_r^*(L) p_X = C_r^*(G^\varphi)$ is the $C_r^*(G_\varphi^\varphi)$-$C_r^*(G)$bimodule we were searching for.

Definition 1.2.19. Denote by μ_φ the class of E_φ in $KK\left(C_r^*(G_\varphi^\varphi), C_r^*(G)\right)$.

Remark 1.2.20. The $C_r^*(G_\varphi^\varphi)$-valued inner product on E is given by $\langle x,y\rangle_{C_r^*(G_\varphi^\varphi)} = xy^*$ and the $C_r^*(G)$-valued one is given by $\langle x,y\rangle_{C_r^*(G)} = x^*y$. It is clear that, if φ is a surjective map, then E_φ is full with respect to the $C_r^*(G)$-valued inner product. Then μ_φ is a Morita equivalence, whose inverse μ_φ^{-1} is given by the bimodule $F = p_X C_r^*(L) p_Y$.

Proposition 1.2.21. *If $\varphi\colon Y \to X$ is transverse with respect to $G \rightrightarrows X$ and $\psi\colon Z \to Y$ is transverse with respect to $G_\varphi^\varphi \rightrightarrows Y$, then*

$$E_{\psi\circ\varphi} = E_\psi \otimes_{C_r^*(G_\varphi^\varphi)} E_\varphi.$$

Proof. Let the groupoid $H \rightrightarrows Z \sqcup Y \sqcup X$ be the pull-back of $G \rightrightarrows X$ by the map $\varphi\circ\psi \sqcup \varphi \sqcup \mathrm{id}_X\colon Z \sqcup Y \sqcup X \to X$. We can see $C_r^*(H)$ as 3×3 matrices of the following sort

$$\begin{pmatrix} C_r^*\left(G_{\varphi\circ\psi}^{\varphi\circ\psi}\right) & C_r^*\left(G_\varphi^{\varphi\circ\psi}\right) & C_r^*\left(G^{\varphi\circ\psi}\right) \\ C_r^*\left(G_{\varphi\circ\psi}^{\varphi}\right) & C_r^*\left(G_\varphi^\varphi\right) & C_r^*\left(G^\varphi\right) \\ C_r^*\left(G_{\varphi\circ\psi}\right) & C_r^*\left(G_{\varphi\circ\psi}^{\varphi}\right) & C_r^*\left(G\right) \end{pmatrix}$$

and that $E_{\varphi\circ\psi} = C_r^*\left(G^{\varphi\circ\psi}\right)$, $E_\psi = C_r^*\left((G_\varphi^\varphi)^\psi\right) \cong C_r^*\left(G_\varphi^{\varphi\circ\psi}\right)$ and $E_\varphi = C_r^*\left(G^\varphi\right)$ sit concretely in $C_r^*(H)$.

Now one can see that $E_\psi \otimes_{C_r^*(G_\varphi^\varphi)} E_\varphi$ is nothing else than $C_r^*\left(G_\varphi^{\varphi\circ\psi}\right) * C_r^*\left(G^\varphi\right)$, that is obviously equal to $E_{\varphi\circ\psi}$. □

Proposition 1.2.22. *If $\varphi\colon Y \to X$ is transverse with respect to $G \rightrightarrows X$ and Z is a saturated and locally compact subset of X and $\varphi' = \varphi_{|\varphi^{-1}(Z)}$, then the following equality holds*

$$e_{\varphi^{-1}(Z)} \otimes \mu_{\varphi'} = \mu_\varphi \otimes e_Z$$

in $KK\left(C_r^*\left((G_\varphi^\varphi)\right), C_r^*\left(G_{|Z}\right)\right)$.

Proof. First observe that $(G_\varphi^\varphi)_{|\varphi^{-1}(Z)} = (G_{|Z})_{\varphi'}^{\varphi'}$. Then, considering the C*-algebra of $L = G_{\varphi\cup id_X}^{\varphi\cup id_X} \rightrightarrows Y \sqcup X$ a 2×2 matrix algebra, the equality means just that restricting L to $W = \varphi^{-1}(Z) \sqcup Z$ and then taking the top left corner of $C_r^*(L_W)$ is the same of taking the top left corner of $C_r^*(L)$ an then restricting to W. □

1.2.6 Wrong-way functoriality

Submersions

Let $G \rightrightarrows G^{(0)}$ a Lie groupoid. Put $X = G^{(0)}$ and let $\varphi\colon Y \to X$ be a smooth submersion between smooth manifolds.

Consider the adiabatic groupoid of G

$$G_{ad} = \mathfrak{A}G \times \{0\} \sqcup G \times (0,1] \rightrightarrows X \times [0,1].$$

Setting $\bar{\varphi} = \varphi \times id_{[0,1]} \colon Y \times [0,1] \to X \times [0,1]$, we can construct the pull-back groupoid

$$(G_{ad})^{\bar{\varphi}}_{\bar{\varphi}} = \{((y_1,t),\gamma,(y_2,t)) \in (Y\times[0,1])\times G_{ad}\times(Y\times[0,1]) \mid r(\gamma) = \bar{\varphi}(y_1,t), s(\gamma) = \bar{\varphi}(y_2,t)\}.$$

Remark 1.2.23. The anchor map of the adiabatic algebroid $\mathfrak{A}(G_{ad}) \simeq \mathfrak{A}(G) \times [0,1]$ is given by $q_{ad}\colon (\eta,t) \mapsto (t\cdot q(\eta),t)$.

Hence we have

$$\mathfrak{A}\left((G_{ad})^{\bar{\varphi}}_{\bar{\varphi}}\right) = \{((\xi,t),(\eta,t)) \in (TY \times [0,1]) \times \mathfrak{A}(G_{ad}) \mid d\varphi(\xi) = t\cdot q(\eta)\}.$$

We deduce that, for $t \neq 0$, the fiber of $\mathfrak{A}\left((G_{ad})^{\bar{\varphi}}_{\bar{\varphi}}\right)$ on (x,t) is given by the following pull back

$$\begin{tikzcd} T_yY \times_{T_{\varphi(y)}X} \mathfrak{A}_{\varphi(y)}(G) \arrow[r] \arrow[d] & \mathfrak{A}_{\varphi(y)}(G) \arrow[d, "t\cdot q"] \\ T_yY \arrow[r, "d\varphi"'] & T_{\varphi(y)}X \end{tikzcd}$$

and for $t = 0$, because of the following exact sequence

$$0 \longrightarrow \ker(d\varphi)_y \longrightarrow T_yY \times_{T_{\varphi(y)}X} \mathfrak{A}_{\varphi(y)}(G) \longrightarrow \mathfrak{A}_{\varphi(y)}(G) \longrightarrow 0\,, \tag{1.2.3}$$

the fiber on $(y,0)$ is $\ker(d\varphi)_y \oplus \mathfrak{A}_{\varphi(y)}(G)$ that is (non canonically) isomorphic to $\mathfrak{A}\left((G_{ad})^{\bar{\varphi}}_{\bar{\varphi}}\right)_y$, because of the following exact sequence.

Remark 1.2.24. We will consider the restriction of the adiabatic groupoid to the interval $[0,1)$, open on the right hand side, instead of $[0,1]$ and we will denote it with G^0_{ad}.

Now we want to give an explicit picture of the adiabatic groupoid $\left((G_{ad})^{\bar{\varphi}}_{\bar{\varphi}}\right)_{ad}$. We fix the variables of the two deformations:

- on the horizontal axis we do the first adiabatic deformation with parameter t;
- on the vertical axis we do the second one, after the pull-back construction, with parameter u.

Then we obtain a groupoid, call it H, with objects set $Y \times [0,1]_t \times [0,1]_u$, such that

- H restricted to $\{u = c\}$, for any $c \in (0,1]$, is equal to $(G_{ad})^{\bar{\varphi}}_{\bar{\varphi}}$;
- H restricted to the t-axis, i.e. to $\{u = 0\}$, is the algebroid of $(G_{ad})^{\bar{\varphi}}_{\bar{\varphi}}$, that we have calculated above;
- H restricted to $\{t = 1\}$ is equal to the adiabatic groupoid $(G^{\varphi}_{\varphi})_{ad}$;

- H restricted to $\{t=0\}$ is equal to the adiabatic deformation of the Lie algebroid $(\mathfrak{A}(G))^{\varphi}_{\varphi}$.

Let $\mathcal{L}$ denote the reduced groupoid C*-algebra $C^*_r\left(((G_{ad})^{\bar{\varphi}}_{\bar{\varphi}})_{ad}\right)$. It is a $C(Y\times[0,1]\times[0,1])$-algebra.

Lemma 1.2.25. *The restriction morphism at $t=1$, $e^t_1\colon \mathcal{L}\to\mathcal{L}_{\{t=1\}}$, induces a KK-equivalence.*

Proof. Consider the following exact sequence

$$0\longrightarrow \mathcal{L}_{\{t\neq1\}}\longrightarrow\mathcal{L}\longrightarrow\mathcal{L}_{\{t=1\}}\longrightarrow 0\,,$$

the evaluation at $t=1$ has a completely positive section: since $\mathcal{L}_{(0,1]}\simeq\mathcal{L}_{\{t=1\}}\times(0,1]$, the map $\xi\mapsto t\cdot\xi$ does the job. Hence this exact sequence is semi-split and it is sufficient to prove the K-contractibility of $\mathcal{L}_{\{t\neq1\}}$.

But let us point out that the evaluation map at $u=0$ from $\mathcal{L}_{\{t\neq1\}}$ to $\mathcal{L}_{\{u=0\,,t\neq1\}}$ is a KK-equivalence: this follows from the KK-equivalence between $C^*_r(\mathcal{G}_{ad})$ and $C^*_r(\mathfrak{A}(\mathcal{G}))$ in the particular case $\mathcal{G}=(G^0_{ad})^{\bar{\varphi}}_{\bar{\varphi}}$.

Hence we have to prove the K-contractibility of $\mathcal{L}_{\{u=0\,,t\neq1\}}$. But, by (1.2.3), $\mathcal{L}_{\{u=0\,,t\neq1\}}$ is non canonically isomorphic to $C^*_r(\mathfrak{A}\,(G^{\varphi}_{\varphi}))\otimes C[0,1)$, that is clearly K-contractible. $\square$

Let $\mathcal{L}^0$ denote the C*-algebra $\mathcal{L}_{\{(t,u)\neq(1,1)\}}$. Then the above proof works for the following lemma too.

Lemma 1.2.26. *The evaluation map at $t=1$, $e^t_1\colon\mathcal{L}^0\to\mathcal{L}^0_{\{t=1\}}$, is a KK-equivalence.*

Let $[e^t_0]\in KK(\mathcal{L}^0,\mathcal{L}^0_{\{t=1\}})$ denote the KK-equivalence stated in Lemma 1.2.26 and let $e^u_1\colon\mathcal{L}^0\to\mathcal{L}^0_{\{u=1\}}$ be the evaluation map at $u=1$.

Definition 1.2.27. Let $G\rightrightarrows X$ be a Lie groupoid and let $\varphi\colon Y\to X$ be a smooth submersion between smooth manifolds. Hence we can define the shriek element $\varphi^{ad}_!$ as

$$\varphi^{ad}_!=[e^t_1]^{-1}\otimes_{\mathcal{L}^0}[e^u_1]\otimes_{\mathcal{L}^0_{\{u=1\}}}\mu_\varphi\in KK(C^*_r((G^{\varphi}_{\varphi}))^0_{ad}),C^*_r(G^0_{ad})),$$

where μ_φ is as in Definition 1.2.19.

Now we want to check that this definition behaves well with respect to the composition of submersions.

Proposition 1.2.28. *Let $G\rightrightarrows Z$ be a Lie groupoid. Let $f\colon Y\to X$ and $g\colon X\to Z$ be two smooth submersions between smooth manifolds. Then we have that*

$$(g\circ f)^{ad}_!=f^{ad}_!\otimes g^{ad}_!\in KK(C^*_r(G^{r\circ f}_{g\circ f})^0_{ad}),C^*_r(G^0_{ad})).$$

Proof. Consider the Lie groupoid K given by

$$\left(\left(((G_{ad})^{\bar{g}}_{\bar{g}})_{ad}\right)^{\bar{f}}_{\bar{f}}\right)_{ad}\rightrightarrows Y\times[0,1]\times[0,1]\times[0,1],$$

where we set t,u,v as the parameters respectively of the first, the second and the third adiabatic deformation in the construction of the groupoid. For sake of clarity let us set some notations:

- $H = Y \times ([0,1] \times [0,1] \times [0,1] \setminus \{(1,1,1)\}$ is a cube without a point;
- $T = \{t = 1\} \subset H$, $U = \{u = 1\} \subset H$ and $V = \{v = 1\} \subset H$ are the right, the posterior and the top faces of the cube, respectively;

Restricting K to the previous faces and their shared edges, we recognize the following groupoids:

- $K_T = \left(((G_g^g)_{ad})_{\bar{f}}^{\bar{f}}\right)_{ad}$, $K_U = \left(\left((G_{ad})_{\bar{g}}^{\bar{g}}\right)_{\bar{f}}^{f}\right)_{ad}$, $K_V = \left(\left((G_{ad})_{\bar{g}}^{\bar{g}}\right)_{ad}\right)_{\bar{f}}^{\bar{f}}$;
- $K_{T\cap U} = \left((G_g^g)_f^f\right)_{ad}$, $K_{U\cap V} = \left((G_{ad})_{\bar{g}}^{\bar{g}}\right)_{\bar{f}}^{\bar{f}}$, $K_{T\cap V} = \left((G_g^g)_{ad}\right)_{\bar{f}}^{\bar{f}}$.

Using Lemma 1.2.26, we get the following KK-equivalences: $e_{V\cap T}^V$, $e_{T\cap U}^T$, $e_{T\cap U}^U$, e_T^H and e_U^H.

We have that $f_!^{ad}$ is constructed through the groupoid K_T and it is equal to

$$(e_{T\cap U}^T)^{-1} \otimes e_{T\cap V}^T \otimes \mu_f.$$

Instea d $g_!^{ad}$ is constructed by means of the groupoid $L = \left((G_{ad})_{\bar{g}}^{\bar{g}}\right)_{ad}$ and it is equal to

$$(e_M)^{-1} \otimes e_N \otimes \mu_g,$$

where $L_M = (G_g^g)_{ad}$, $L_N = (G_{ad})_{\bar{g}}^{\bar{g}}$. Thus we have that

$$f_!^{ad} \otimes g_!^{ad} = (e_{T\cap U}^T)^{-1} \otimes e_{T\cap V}^T \otimes \mu_f \otimes (e_M)^{-1} \otimes e_N \otimes \mu_g.$$

Applying propositions 1.2.21 and 1.2.22, it is easy to get the following equality

$$\mu_f \otimes (e_M)^{-1} \otimes e_N \otimes \mu_g = (e_{T\cap V}^V)^{-1} \otimes e_{V\cap U}^V \otimes \mu_{g\circ f}.$$

Then

$$\begin{aligned}
f_!^{ad} \otimes g_!^{ad} &= (e_{T\cap U}^T)^{-1} \otimes e_{T\cap V}^T \otimes (e_{T\cap V}^V)^{-1} \otimes e_{V\cap U}^V \otimes \mu_{g\circ f} = \\
&= (e_{T\cap U}^T)^{-1} \otimes (e_T^H)^{-1} \otimes e_V^H \otimes e_{V\cap U}^V \otimes \mu_{g\circ f} = \\
&= (e_{T\cap U}^H)^{-1} \otimes e_{V\cap U}^H \otimes \mu_{g\circ f} = \\
&= (e_{T\cap U}^U)^{-1} \otimes (e_U^H)^{-1} \otimes e_U^H \otimes e_{V\cap U}^U \otimes \mu_{g\circ f} = \\
&= (e_{T\cap U}^U)^{-1} \otimes e_{V\cap U}^U \otimes \mu_{g\circ f} = \\
&= (g\circ f)_!^{ad}.
\end{aligned}$$

□

Proposition 1.2.29. *Let $p\colon E \to Y$ be a vector bundle and let $G \rightrightarrows Y$ be a Lie groupoid. Then $p_!^{ad} \in KK(C_r^*((G_p^p)_{ad}^0), C_r^*(G_{ad}^0))$ is a KK-equivalence.*

Proof. Since p is surjective μ_p is a Morita equivlence between $C_r^*\left((G_{ad})_{\bar{p}}^{\bar{p}}\right)$ and $C_r^*(G_{ad})$. Because of the definition of $p_!^{ad}$, it is sufficient to show that the evaluation at $u = 1$ gives a KK-equivalence in $KK(\mathcal{L}, \mathcal{L}_{\{u=1\}})$. But this is equivalent to prove that the kernel of the evaluation at $u = 1$, $\mathcal{L}_{\{u\neq 1\}}$, is KK-contractible. This turns to be equivalent to KK-contractibility of $\mathcal{L}_{\{t=0,u\neq 1\}}$ because of the following exact sequence

$$0 \longrightarrow \mathcal{L}_{\{t\neq 0,u\neq 1\}} \longrightarrow \mathcal{L}_{\{u\neq 1\}} \longrightarrow \mathcal{L}_{\{t=0,u\neq 1\}} \longrightarrow 0 ,$$

and KK-contractibility of $\mathcal{L}_{\{t\neq 0, u\neq 1\}} \simeq (G_p^p)^0_{ad} \otimes C_0(0,1]$.

But $\mathcal{L}_{\{t=0,u\neq 1\}}$ is equal to

$$(\mathfrak{A}(G))_p^p = \{(\eta_1, \xi, \eta_2) \in E \times \mathfrak{A}(G) \times E \,|,\ p(\eta_1) = r(\xi), p(\eta_2) = s(\xi)\}$$

for $u \neq 0$ and

$$\ker(dp) \oplus \mathfrak{A}(G)$$

for $u = 0$, see (1.2.3). But in both case we have something that is isomorphic to $E \oplus E \oplus \mathfrak{A}(G)$, because $\ker(dp)$ is the vertical bundle of p^*E. Then

$$\mathcal{L}_{\{t=0,u\neq 1\}} \simeq C_0(E \oplus E \oplus \mathfrak{A}(G)) \otimes C_0[0,1)$$

is KK-contractible.

□

Transverse maps deformable to a submersion

Definition 1.2.30. Let $G \rightrightarrows X$ be a Lie groupoid. And let $\psi\colon Y \to X$ a transverse map with respect to G. Such a map is said to be deformable to a submersion if there exists a map $T\colon Y \to G$ such that $s \circ T$ is equal to ψ and $r \circ T$ is a smooth submersion.

Remark 1.2.31. Observe that if $G \rightrightarrows X$ is a groupoid, then $G_r^r \rightrightarrows G$ is isomorphic to $G_s^s \rightrightarrows G$. Indeed

$$G_r^r = \{(g_1, \gamma, g_2) \in G \times G \times G \,|\, r(\gamma) = r(g_1) \text{ and } s(\gamma) = r(g_2)\},$$

while

$$G_s^s = \{(g_1, \gamma, g_2) \in G \times G \times G \,|\, r(\gamma) = s(g_1) \text{ and } s(\gamma) = s(g_2)\}.$$

Then the map $G_s^s \to G_r^r$ defined by $(g_1, \gamma, g_2) \mapsto (g_1, g_1\gamma g_2^{-1}, g_2)$ gives the isomorphism we were looking for.

Moreover, if $\psi\colon Y \to X$ is a transverse map deformable to a submersion through $T\colon Y \to G$, then there is an isomorphism

$$\alpha\colon G_\psi^\psi \to G_{r\circ T}^{r\circ T}$$

given by $\alpha\colon (y_1, \gamma, y_2) \mapsto (y_1, T(y_1)^{-1}\gamma T(y_2), y_2)$.

Definition 1.2.32. Let $\psi\colon Y \to X$ a map deformable to a submersion through $T\colon Y \to G$. Then define $\psi_!^{ad}$ as the element

$$[\alpha] \otimes (r \circ T)_!^{ad} \in KK(C_r^*(G_\psi^\psi), C_r^*(G)).$$

Proposition 1.2.33. *Let $\psi\colon Y \to X$ be deformable to a submersion with respect to G. The class $\psi_!^{ad}$ does not depend on the maps T and α.*

Proof. Let ψ be deformable to a submersion through another map $S\colon Y \to G$ and let $\beta\colon G_\psi^\psi \to G_{r\circ S}^{r\circ S}$ as in Remark 1.2.23.

Let us observe that α and β easily extend to isomorphisms A and B from $\left((G_{ad})_{\bar\psi}^{\bar\psi}\right)_{ad}$ to $\left((G_{ad})_{\overline{r\circ T}}^{\overline{r\circ T}}\right)_{ad}$ and $\left((G_{ad})_{\overline{r\circ S}}^{\overline{r\circ S}}\right)_{ad}$ respectively. It follows that $\alpha \circ \beta^{-1}\colon G_{r\circ S}^{r\circ S} \to G_{r\circ T}^{r\circ T}$ is an isomorphism and it extends to an isomorphism $A \circ B^{-1}$ as above.

Let us consider the following computation

$$\begin{aligned}[\beta^{-1}] \otimes [\alpha] \otimes (r \circ T)_!^{ad} &= [\alpha \circ \beta^{-1}] \otimes [e_1^t]^{-1} \otimes [e_1^u] \otimes \mu_{r \circ T} = \\ &= [e_1^t]^{-1} \otimes [A \circ B^{-1}] \otimes [e_1^u] \otimes \mu_{r \circ T} = \\ &= [e_1^t]^{-1} \otimes [e_1^u] \otimes [\alpha \circ \beta^{-1}] \otimes \mu_{r \circ T} = \\ &= [e_1^t]^{-1} \otimes [e_1^u] \otimes \mu_{r \circ S} = (r \circ S)_!^{ad},\end{aligned}$$

where it is understood the groupoids the evaluation maps refer to. Consequently we get the equality $[\alpha] \otimes (r \circ T)_!^{ad} = [\beta] \otimes (r \circ S)_!^{ad}$. □

Remark 1.2.34.
- If $\psi\colon Y \to X$ is a submersion, then the class $\psi_!^{ad}$ in Definition 1.2.27 and the one in Definition 1.2.32 coincide. Indeed

$$\psi_!^{ad} = (s \circ T)_!^{ad} = [\alpha] \otimes (r \circ T)_!^{ad}.$$

- If $\varphi\colon Z \to Y$ is a submersion and $\psi\colon Y \to X$ is a map deformable to a submersion through $T\colon Y \to G$, then $\psi \circ \varphi\colon Z \to X$ is deformable to a submersion through $T \circ \varphi\colon Z \to G$ and $\varphi_!^{ad} \otimes \psi_!^{ad} = (\psi \circ \varphi)_!^{ad}$ in the sense of Definition 1.2.32. Sure enough we have that

$$\begin{aligned}(\psi \circ \varphi)_!^{ad} &= [\beta] \otimes (r \circ T \circ \varphi)_!^{ad} = \\ &= [\beta] \otimes \varphi_!^{ad} \otimes (r \circ T)_!^{ad} = \\ &= [\beta] \otimes [e_1^t]^{-1} \otimes [e_1^u] \otimes \nu_\varphi \otimes (r \circ T)_!^{ad} = \\ &= [e_1^t]^{-1} \otimes [e_1^u] \otimes \mu_\varphi \otimes [\alpha] \otimes (r \circ T)_!^{ad} = \\ &= \varphi_!^{ad} \otimes \psi_!^{ad}.\end{aligned}$$

  Here $\mu_\varphi = \left[C_r^*\left((G_\psi^\psi)^\varphi\right)\right] \in KK\left(C_r^*\left((G_\psi^\psi)_\varphi^\varphi\right), C_r^*\left((G_\psi^\psi)\right)\right)$, $\nu_\varphi = \left[C_r^*\left((G_{r\circ T}^{r\circ T})^\varphi\right)\right] \in KK\left(C_r^*\left((G_{r\circ T}^{r\circ T})_\varphi^\varphi\right), C_r^*\left(G_{r\circ T}^{r\circ T}\right)\right)$, $\alpha\colon G_\psi^\psi \to G_{r\circ T}^{r\circ T}$ and $\beta\colon (G_\psi^\psi)_\varphi^\varphi \to (G_{r\circ T}^{r\circ T})_\varphi^\varphi$ are the isomorphisms defined above. The only point that is not obvious is the fact that

$$\begin{aligned}\mu_\varphi \otimes [\alpha] &= \left[C_r^*\left((G_\psi^\psi)^\varphi\right) \otimes_\alpha C_r^*\left(G_{r\circ T}^{r\circ T}\right), \lambda \otimes_\alpha 1, 0\right] = \\ &= \left[C_r^*\left((G_{r\circ T}^{r\circ T})^\varphi\right), \kappa \circ \beta, 0\right] = \\ &= [\beta] \otimes \nu_\varphi,\end{aligned}$$

  where we use the isomorphism $C_r^*\left((G_\psi^\psi)^\varphi\right) \otimes_\alpha C_r^*\left(G_{r\circ T}^{r\circ T}\right) \to C_r^*\left((G_{r\circ T}^{r\circ T})^\varphi\right)$ such that $a \otimes_\alpha b \mapsto ((\alpha^{-1})^* a) \cdot b$ (with inverse $c \mapsto \alpha^* c \otimes_\alpha 1$). We have dropped the adiabatic deformation subscripts to lighten the notation.

Transverse maps

Let $G \rightrightarrows X$ be a Lie groupoid. And let $\psi\colon Y \to X$ a transverse map with respect to G.

Definition 1.2.35. Let $p\colon E \to Y$ be a smooth submersion such that:

- $p_!^{ad}$ is invertible;
- $k = \psi \circ p$ is deformable to a submersion through $\tilde{\psi}\colon Y \to G$.

Let k be the composition $r \circ \tilde{\psi}$, then we can define the class

$$\psi_!^{ad} = (p_!^{ad})^{-1} \otimes k_!^{ad} \in KK(C_r^*((G_\psi^\psi)^0_{ad}), C_r^*(G^0_{ad})),$$

Proposition 1.2.36. *For any map $\psi\colon Y \to X$ tranvers with respect to $G \rightrightarrows X$, data as in Definition 1.2.35 always exist and the class $\psi_!^{ad}$ does not depend on them.*

Proof. Let $U \subset G$ a tubular neighbourhood of X in G. Consider the space $E = U_\psi$ as in Definition 1.2.16, given by the following pull-back

$$\begin{tikzcd}
U_\psi \arrow[r, "\tilde{\psi}"] \arrow[d, "p"] & U \arrow[d, "s"] \\
Y \arrow[r, "\psi"] & X
\end{tikzcd}$$

it is diffeomorphic to the normal bundle induced by $u \circ \psi\colon Y \to G$.

Now notice that p is a surjective submersion, because s is so. Moreover, up to diffeomorphism, it is a vector bundle projection, hence by Lemma 1.2.29 $p_!^{ad}$ is invertible. Furthermore by definition we have that $\psi \circ p = \tilde{\psi} \circ s$.

And one can prove that $r \circ \tilde{\psi}$ is a smooth submersion as in Remark 1.2.17. □

Proposition 1.2.37. *The element $\psi_!^{ad}$ does not depend on the choise of the map $p\colon E \to Y$, such that it satisfies the Definition 1.2.35 requirements.*

Proof. Let $p'\colon E' \to Y$ be an other smooth map that respects the conditions in Definition 1.2.35, E' being not necessarly a vector bundle.

Take the fibered product $E \times_Y E'$: it is just $(p')^*E$, the pull-back vector bundle on E'. Moreover $\pi\colon E \times_Y E' \to E$ and $\pi'\colon E \times_Y E' \to E'$ are smooth surjective submersions. We can define $P\colon E \times_Y E' \to Y$ as $p \circ \pi$ or $p' \circ \pi'$ equivalently. Notice that, since π' is a vector bundle projection, $(\pi')_!^{ad}$ is invertible and, because $p \circ \pi = p' \circ \pi'$, $\pi_!^{ad}$ is invertible too.

The following diagram

$$\begin{tikzcd}
 & E \times_Y E' \arrow[dl, dotted, "\pi"'] \arrow[dr, dotted, "\pi'"] \arrow[ddl, dotted, "P"] \arrow[ddr, "K"] & \\
E \arrow[d, dotted, "p"'] \arrow[drr, "k"] & & E' \arrow[d, "k'"] \arrow[dll, dotted, "p'"'] \\
Y \arrow[rr, "\psi"'] & & X
\end{tikzcd}$$

is commutative. Thanks to Remark 1.2.34, it is an easy task to check the well-definedness of $\psi_!^{ad}$ by the following computation:

$$\begin{aligned}
(p_!^{ad})^{-1} \otimes k_!^{ad} =& (p_!^{ad})^{-1} \otimes (\pi_!^{ad})^{-1} \otimes \pi_!^{ad} \otimes k_!^{ad} = \\
&= (P_!^{ad})^{-1} \otimes K_!^{ad} = \\
&= ((p')_!^{ad})^{-1} \otimes ((\pi')_!^{ad})^{-1} \otimes (\pi')_!^{ad} \otimes (k')_!^{ad} = \\
&= ((p')_!^{ad})^{-1} \otimes (k')_!^{ad}.
\end{aligned}$$

□

Remark 1.2.38. If $\psi\colon Y \to X$ is a submersion, the class $\psi_!^{ad}$ defined in 1.2.35 coincide with the one defined in 1.2.27.

Now we can verify that our definition behaves well with respect to composition of transverse maps.

Proposition 1.2.39. *Let $G \rightrightarrows Z$ be a Lie groupoid. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two smooth maps between smooth manifolds such that g is transverse w.r.t. G and f is transverse w.r.t. G_g^g. Then we have that*

$$(g \circ f)_!^{ad} = g_!^{ad} \otimes f_!^{ad} \in KK\left(C_r^*\left((G_{g\circ f}^{g\circ f})_{ad}^0\right), C_r^*\left(G_{ad}^0\right)\right).$$

Proof. Let $(p\colon E \to X, \tilde{f})$ and $(p'\colon E' \to Y, \tilde{g})$ be the input data for $f_!^{ad}$ and $g_!^{ad}$, where we can choose that E and E' are vector bundles. Then $(p \circ \pi\colon E \oplus E \to X, \tilde{g} \circ \pi')$ are input data for $(g \circ f)_!^{ad}$. Because of the commutativity of the following diagram

$$\begin{tikzcd} E \oplus E' \arrow[r, "\pi'"] \arrow[d, "\pi"] & E' \arrow[d, "p'"] \\ E \arrow[r, "k"] & Y \end{tikzcd}$$

we easily get the equality. In fact we have

$$\begin{aligned} g_!^{ad} \otimes f_!^{ad} &= (p_!^{ad})^{-1} \otimes k_!^{ad} \otimes ((p')_!^{ad})^{-1} \otimes (k')_!^{ad} = \\ &= (p_!^{ad})^{-1} \otimes ((\pi)_!^{ad})^{-1} \otimes (\pi')_!^{ad} \otimes (k')_!^{ad} = \\ &= (g \circ f)_!^{ad}. \end{aligned}$$

□

Let us state an other important property of this construction, that is its functoriality with respect to the restriction to open or closed sets.

Proposition 1.2.40. *Let $G \rightrightarrows X$ be a Lie groupoid. And let $\psi\colon Y \to X$ a transverse map with respect to G. Let $Y_1 \subset Y$ and $X_1 \subset X$ closed and saturated submanifolds such that $Y_1 = \psi^{-1}(X_1)$. Then we have the following commutative diagram*

$$\begin{tikzcd} \cdots \arrow[r] & K_*\left(C_r^*((G_\psi^\psi)_{ad}^0)_{Y\setminus Y_1}\right) \arrow[r, "i"] \arrow[d, "(\psi')_!^{ad}"] & K_*\left(C_r^*((G_\psi^\psi)_{ad}^0)\right) \arrow[r, "j"] \arrow[d, "\psi_!^{ad}"] & K_*\left(C_r^*((G_\psi^\psi)_{ad}^0)_{Y_1}\right) \arrow[r] \arrow[d, "(\psi'')_!^{ad}"] & \cdots \\ \cdots \arrow[r] & K_*\left(C_r^*(G_{ad}^0)_{X\setminus X_1}\right) \arrow[r, "i"] & K_*\left(C_r^*(G_{ad}^0)\right) \arrow[r, "j"] & K_*\left(C_r^*(G_{ad}^0)_{X_1}\right) \arrow[r] & \cdots \end{tikzcd}$$

where ψ' and ψ'' are the restriction of ψ to $Y \setminus Y_1$ and Y_1 respectively.

Proof. Let us prove the commutativity of the second square: we have to prove the equality of

$$[e_1^t]^{-1} \otimes [e_1^u] \otimes \mu_\psi \otimes [ev_{X_1}]$$

and

$$[ev_{Y_1}] \otimes [\bar{e}_1^t]^{-1} \otimes [\bar{e}_1^u] \otimes \mu_{\psi''}$$

in $KK\left(C_r^*((G_\psi^\psi)_{ad}^0, C_r^*(G_{ad}^0)_{X_1}\right)$. Here $\bar{e}$ are evaluation maps in the setting of the groupoids restricted to Y_1, whereas we use e for evaluation maps for groupoids over Y. But noticing that $[ev_{Y_1}] \otimes [\bar{e}_1^t]^{-1} \otimes [\bar{e}_1^u] = \times[\bar{e}_1^t]^{-1} \otimes [\bar{e}_1^u] \otimes [ev_{Y_1}]$ and applying Proposition 1.2.22, we obtain the commutativity of the second square. For the commutativity of the first one we use a similar argument. □

Finally it is worth to notice that if $\psi\colon Y \to X$ is transverse with respect to $G \rightrightarrows X$, then one has a natural transformation from the K-theory sequence associated to the adiabatic extension of G_ψ^ψ to the one of G:

$$
\begin{tikzcd}
\ldots \arrow[r] & K_*\left(C_r^*(G_\psi^\psi)\right) \arrow[r] \arrow[d, "\mu_\psi"] & K_*\left(C_r^*((G_\psi^\psi)_{ad}^0)\right) \arrow[r, "{[\mathrm{ev}_0]_*}"] \arrow[d, "\psi_!^{ad}"] & K_*\left(C_r^*(\mathfrak{A}(G_\psi^\psi))\right) \arrow[r] \arrow[d, "dd\psi_!"] & \ldots \\
\ldots \arrow[r] & K_*\left(C_r^*(G)\right) \arrow[r] & K_*\left(C_r^*((G)_{ad}^0)\right) \arrow[r, "{[\mathrm{ev}_0]_*}"] & K_*\left(C_r^*(\mathfrak{A}(G))\right) \arrow[r] & \ldots
\end{tikzcd}
\tag{1.2.4}
$$

where μ_ψ is the KK-element given in Definition 1.2.19 and $d\psi_! \in KK\big(C_r^*(\mathfrak{A}(G_\psi^\psi)), C_r^*(\mathfrak{A}(G))\big)$ is the KK-class obtained in the obvious way, as for $\psi_!^{ad}$, but restricting the process to the Lie algebroids.

Remark 1.2.41. Notice that one also has a wrong-way functoriality for the adiabatic deformation up to $t = 1$ included. It is given by the same construction and it enjoys the same properties. Moreover there is a commutative diagram analogous to (1.2.4) for the exact sequence

$$0 \longrightarrow C_r^*(G \times (0,1]) \longrightarrow C_r^*(G_{ad}) \xrightarrow{\mathrm{ev}_0} C_r^*(\mathfrak{A}(G)) \longrightarrow 0\,.$$

1.2.7 Pseudodifferential operators on groupoids

The material below is taken from [25, 26, 44], following closely the presentation in [5]. Let $G \rightrightarrows X$ be a Lie groupoid. We assume that X is a compact manifold and that the s-fibers G_x, $x \in X$, have no boundary. Let U_γ be the map induced on functions by right multiplication by γ:

$$U_\gamma\colon C^\infty(G_{s(\gamma)}, \Omega^{\frac{1}{2}}) \to C^\infty(G_{r(\gamma)}, \Omega^{\frac{1}{2}})$$

such that $U_\gamma f(\gamma') = f(\gamma'\gamma)$.

Definition 1.2.42. A linear G-operator is a continuous linear map $P : C_c^\infty(G, \Omega^{\frac{1}{2}}) \longrightarrow C^\infty(G, \Omega^{\frac{1}{2}})$ such that:

(i) P is given by a family $(P_x)_{x\in X}$ of Linear operators $P_x\colon C_c^\infty(G_x, \Omega^{\frac{1}{2}}) \to C^\infty(G_x)$ such that

$$P(f)(\gamma) = P_{s(\gamma)} f_{s(\gamma)}(\gamma), \quad \forall f \in C_c^\infty(G, \Omega^{\frac{1}{2}})$$

where f_x denotes the restriction $f|_{G_x}$.

(ii) The following invariance property holds:

$$U_\gamma P_{s(\gamma)} = P_{r(\gamma)} U_\gamma.$$

Let P be a G-operator and denote by k_x the Schwartz kernel of P_x, for each $x \in X$, as obtained from the Schwartz kernel theorem applied to the manifold G_x.

Thus, using the property (i):

$$Pf(\gamma) = \int_{\gamma' \in G_x} k_x(\gamma, \gamma') f(\gamma'), \quad \forall \gamma \in G_x\,,\; f \in C^\infty(G, \Omega^{\frac{1}{2}}).$$

Next:

$$U_\gamma Pf(\gamma') = Pf(\gamma'\gamma) = \int_{\gamma''\in G_x} k_x(\gamma'\gamma, \gamma'')f(\gamma''), \quad \forall\, \gamma \in G_x\,,\, \gamma' \in G^{r(\gamma)}$$

and

$$\begin{aligned} P(U_\gamma f)(\gamma') &= \int_{\gamma''\in G_{r(\gamma)}} k_y(\gamma', \gamma'')f(\gamma''\gamma) \\ &\overset{\eta=\gamma''\gamma}{=} \int_{\eta\in G_x} k_y(\gamma', \eta\gamma^{-1})f(\eta). \end{aligned}$$

Condition (ii) is equivalent to the following equalities of distributions on $G_x \times G_x$, for all $x \in V$:

$$k_{s(\gamma)}(\gamma'\gamma, \gamma'') = k_r(\gamma)(\gamma', \gamma''\gamma^{-1}), \quad \forall\, \gamma \in G.$$

Setting $k_P(\gamma) := k_{s(\gamma)}(\gamma, s(\gamma))$, it follows that $k_x(\gamma, \gamma') = k_P(\gamma\gamma'^{-1})$ and that the operator $P\colon C_c^\infty(G, \Omega^{\frac{1}{2}}) \to C^\infty(G, \Omega^{\frac{1}{2}})$ is given by:

$$P(f)(\gamma) = \int_{\gamma'\in G_x} k_P(\gamma\gamma'^{-1}).$$

We may consider k_P as a single distribution on G acting on smooth functions on G by convolution. With a slight abuse of terminology, we will refer to k_P as the Schwartz (or convolution) kernel of P. We define the support of P to be

$$\operatorname{supp}(P) = \bigcup_{x\in X} \operatorname{supp}(P_x)$$

Definition 1.2.43. The family $P = (P_x, x \in G^{(0)})$ is *properly supported* if $p_i^{-1}(K) \cap \operatorname{supp}(P)$ is a compact set for any compact subset $K \subset G$, where $p_1, p_2\colon G \times G \to G$ are the two projections; it is *compactly supported* or *uniformly supported* if k_P is compactly supported (which implies that each P_x is properly supported). Finally we say that P is smoothing if k_p lies in $C^\infty(G)$.

We now turn to the definition of pseudodifferential operators on a Lie groupoid G.

Definition 1.2.44. A G-operator P is a G-pseudodifferential operator of order m if:

1. The Schwartz kernel k_P is smooth outside $G^{(0)}$.

2. For every distinguished chart $\psi : U \subset G \to \Omega \times s(U) \subset \mathbb{R}^{n-p} \times \mathbb{R}^p$ of G:

$$\begin{tikzcd} U \arrow[rr, "\psi"] \arrow[dr, "s"'] & & \Omega \times s(U) \arrow[dl, "p_2"] \\ & s(U) & \end{tikzcd}$$

   the operator $(\psi^{-1})^* P \psi^* : C_c^\infty(\Omega \times s(U)) \to C_c^\infty(\Omega \times s(U))$ is a smooth family parametrized by $s(U)$ of pseudodifferential operators of order m on Ω.

It turns out that the space $\Psi_c^*(G)$ of compactly supported G-pseudodifferential operators is an involutive algebra.

The principal symbol of a G-pseudodifferential operator P of order m is defined as a function $\sigma_m(P)$ on $\mathfrak{A}^*(G) \setminus G^{(0)}$ by:

$$\sigma_m(P)(x,\xi) = \sigma_{pr}(P_x)(x,\xi)$$

where $\sigma_{pr}(P_x)$ is the principal symbol of the pseudodifferential operator P_x on the manifold G_x. Conversely, given a symbol f of order m on $\mathfrak{A}^*(G)$ together with the following data:

1. A smooth embedding $\theta : U \to \mathfrak{A}G$, where U is a open set in G containing $G^{(0)}$, such that $\theta(G^{(0)}) = G^{(0)}$, $(d\theta)|_{G^0} = \mathrm{Id}$ and $\theta(\gamma) \in \mathfrak{A}_{s(\gamma)}G$ for all $\gamma \in U$;

2. A smooth compactly supported map $\phi : G \to \mathbb{R}_+$ such that $\phi^{-1}(1) = G^{(0)}$;

then a G-pseudodifferential operator $P_{f,\theta,\phi}$ is obtained by the formula:

$$P_{f,\theta,\phi}u(\gamma) = \int_{\gamma' \in G_{s(\gamma)},\, \xi \in \mathfrak{A}^*_{r(\gamma)}(G)} e^{-i\theta(\gamma'\gamma^{-1})\cdot\xi} f(r(\gamma),\xi)\phi(\gamma'\gamma^{-1})u(\gamma'))$$

with $u \in C_c^\infty(G, \Omega^{\frac{1}{2}})$. The principal symbol of $P_{f,\theta,\phi}$ is just the leading part of f.

The principal symbol map respects pointwise product while the product law for total symbols is much more involved. An operator is *elliptic* when its principal symbol never vanishes and in that case, as in the classical situation, it has a parametrix inverting it modulo $\Psi_c^{-\infty}(G) = C_c^\infty(G)$.

Operators of zero order $\Psi_c^0(G)$ are a subalgebra of the multiplier algebra $M(C^*(G))$ and operators of negative order actually in $C^*(G)$.

All these definitions and properties immediately extend to the case of operators acting between sections of bundles on $G^{(0)}$ pulled back to G with the range map r. The space of compactly supported pseudodifferential operators on G acting on sections of r^*E and taking values in sections of r^*F will be noted $\Psi_c^*(G,E,F)$. If $F = E$ we get an algebra denoted by $\Psi_c^*(G,E)$.

Let us provide some examples of (classical) pseudodifferential G-operators.

Examples 1.2.45. *1. $G = X \times X \rightrightarrows X$ the pair groupoid, where X is a compact smooth manifold. It turns out that for any compactly supported pseudodifferential G-operator P:*

$$P(g)(x,y) = \int_{X\times\{x\}} k_P(x,z)g(z,y) = \int_X k_P(x,z)g(z,y)$$

which immediately implies that $P_x = P_{x'}$ for all elements $x, x' \in X$. Namely the family is constant and reduces to one single compactly supported pseudodifferential operator P_0 on X. In this case one obtain that $\Psi_c^0(G) = \Psi^0(X)$.

2. Let $p : X \to Z$ a submersion, and $G = X \times_Z X = \{(x,y) \in X \times X \,|\, p(x) = p(y)\}$ the associated subgroupoid of the pair groupoid $X \times X$. The manifold G_x can be identified with the fiber $p^{-1}(p(x))$. The invariance condition implies that for any pseudodifferential G-operator P, we have $P_x = P_y$ on $p^{-1}(p(x))$ as soon as $y \in p^{-1}(p(x))$. Thus, P is actually given by a family $\tilde{P}_z$, $z \in Z$ of operators on $p^{-1}(z)$, with the relation $P_x = \tilde{P}_{p(x)}$.

3. *If $G = \mathcal{G} \rightrightarrows *$ is a Lie group, then $\Psi^0_c(G)$ is equal to $\Psi^0_{prop}(\mathcal{G})^{\mathcal{G}}$, the algebra of properly supported pseudodifferential operators on G, invariant with respect to right translations. In this example, every invariant properly supported operator is also uniformly supported.*

4. *Let $G = E \rightrightarrows X$ be the total space of a (euclidean, hermitian) vector bundle $p \colon E \to X$ over a compact smooth manifold X, with $r = s = p$. If P is a pseudodifferential G-operator:*
$$Pf(v) = \int_{w \in E_x} k_P(v - w) f(w)$$
Thus, for all $x \in X$, P_x is a translation-invariant convolution operator on the linear space E_x such that the underlying distribution k_P identifies with the Fourier transform of a symbol on E. Consequently we have that $\Psi^0_c(G) = C(BE)$, the algebra of continuous functions on the unit ball bundle of E.

5. *Let G be the fundamental groupoid of a compact smooth manifold M with fundamental group $\pi_1(M) = \Gamma$. Recall that if we denote by $\widetilde{M}$ a universal covering of M and let Γ act by covering transformations, then $G^(0) = \widetilde{M}/\Gamma = M$, $G = \widetilde{M} \times_\Gamma \widetilde{M}$ and the source and the range maps are the two projections. Each fiber G_x can be identified with $\widetilde{M}$, uniquely up to the action of an element in Γ. Let $P = (P_x, x \in M)$ be an invariant, uniformly supported, pseudodifferential G-operator. Then each P_x is a pseudodifferential operator on $\widetilde{M}$. The invariance condition applied to the elements $\gamma \in G$ such that $x = s(g) = r(g)$ implies that each operator P_x is invariant with respect to the action of Γ. This means that we can identify P_x with an operator on $\widetilde{M}$ and that the resulting operator does not depend on the identification of G_x with $\widetilde{M}$. Then the invariance condition applied to an arbitrary element $\gamma \in G$ gives that all operators P_x acting on $\widetilde{M}$ coincide. We obtain that $\Psi^0_c(G) = \Psi^0_{prop}(\widetilde{M})^\Gamma$, the algebra of properly supported Γ-invariant pseudodifferential operators on the universal covering $\widetilde{M}$ of M.*

6. *If G is the holonomy groupoid of a foliation $\mathcal{F}$ on a smooth manifold X, then $\Psi^0_c(G)$ is the algebra of pseudodifferential operators along the leaves of the foliation.*

7. *If we consider the adiabatic deformation G_{ad} of a Lie groupoid $G \rightrightarrows X$, a pseudodifferential G_{ad}-operator P is given by a family $(P_{x,t}\,,\,(x,t) \in X \times [0,1])$ such that P_t is a family of pseudodifferential G-operator for $t > 0$ and P_0 is a family of operators as in Exemple 4, where $P_{0,x}$ is given by the complete symbol of $P_{1,x}$.*

8. *If $G \rightrightarrows X$ is the groupoid of Definition 1.2.8, then $\Psi^0_c(G)$ is the b-calculus of R. Melrose.*

1.2.8 Index as deformation

In this section we are going to recall the result of [25], that compares the analytic index map defined by Atiyah and Singer in [1, Section 6] and the analytic index defined by Connes by means of the tangent groupoid, let us call the last one the adiabatic index.

The KK-element of a deformation groupoid

Let G be a smooth deformation groupoid, namely a Lie groupoid of the following kind:
$$G = G_1 \times \{0\} \cup G_2 \times]0,1] \rightrightarrows G^{(0)} = M \times [0,1].$$

One can consider the saturated open subset $M\times]0,1]$ of $G^{(0)}$. Using the isomorphisms $C_r^*(G|_{M\times]0,1]}) \simeq C_r^*(G_2)\otimes C_0(]0,1])$ and $C_r^*(G|_{M\times\{0\}}) \simeq C_r^*(G_1)$, we obtain the following exact sequence of C^*-algebras:

$$0 \longrightarrow C_r^*(G_2)\otimes C_0(]0,1]) \xrightarrow{i} C_r^*(G) \xrightarrow{\mathrm{ev}_0} C_r^*(G_1) \longrightarrow 0$$

where i is the inclusion map and ev_0 is the evaluation map at 0.

We assume now that $C_r^*(G_1)$ is nuclear. Since the C^*-algebra $C_r^*(G_2)\otimes C_0(]0,1])$ is contractible, the long exact sequence in KK-theory shows that the group homomorphism $KK\left(A, C_r^*(G)\right) \to KK\left(A, C_r^*(G_1)\right)$, given by the Kasparov product with the element $[\mathrm{ev}_0]$, is an isomorphism for each C^*-algebra A.

In particular with $A = C_r^*(G)$ we get that $[\mathrm{ev}_0]$ is invertible in KK-theory: there is an element $[ev_0]^{-1}$ in $KK\left(C_r^*(G_1), C_r^*(G)\right)$ such that $[ev_0]\otimes[\mathrm{ev}_0]^{-1} = 1_{C_r^*(G)}$ and $[\mathrm{ev}_0]^{-1}\otimes[\mathrm{ev}_0] = 1_{C^*(G_1)}$.

Let $\mathrm{ev}_1 : C_r^*(G) \to C_r^*(G_2)$ be the evaluation map at 1 and $[\mathrm{ev}_1]$ the corresponding element of $KK\left(C_r^*(G), C_r^*(G_2)\right)$.

Definition 1.2.46. The KK-element associated to the deformation groupoid G is defined by:

$$\partial_G = [\mathrm{ev}_0]^{-1}\otimes[\mathrm{ev}_1] \in KK\left(C_r^*(G_1), C_r^*(G_2)\right) \ .$$

The analytical index

Let $G \rightrightarrows X$ be a Lie groupoid and consider its adiabatic deformation $G_{ad} \rightrightarrows X\times[0,1]$. Recall that it is of the form

$$\mathfrak{A}(G)\times\{0\}\sqcup G\times(0,1]$$

and that $C_0^*(\mathfrak{A}(G)) \cong C_0(\mathfrak{A}^*(G))$. This is a particular case of a smooth deformation groupoid. Therefore we can associate to it a KK-element as in definition 1.2.46.

Definition 1.2.47. We will denote by

$$\mathrm{Ind_G} \in KK\left(C_r^*(\mathfrak{A}(G)), C_r^*(G)\right)$$

the KK-element $\partial_{G_{ad}}$ and we will call the *adiabatic G-index* the homomorphism

$$K_*\left(C_0(\mathfrak{A}^*(G))\right) \to K_*\left(C_r^*(G)\right),$$

given by the Kasparov product with Ind_G.

Let X be a closed smooth manifold. We are going to prove that, when G is the pair groupoid $X\times X$ (and then $\mathfrak{A}^*(G) = T^*X$), the adiabatic G-index is nothing but the classical analytic index defined by Atiyah and Singer in [1].

Let $\pi\colon V \to X$ be a real vector bundle over X. Let us recall the realization of $K_0\left(C_0(V)\right)$ given by Atiyah and Singer in [1, Section 2], but in the slightly different formulation of [4].

Definition 1.2.48. A Clifford symbol is a pair (E,σ), where E is a $\mathbb{Z}_2$-graded smooth Hermitian vector bundle and $\sigma \in C(S^*X, \mathrm{End}(E))$ satisfies the following conditions:

- $\sigma(x,\xi)$ is of degree 1 for the $\mathbb{Z}_2$-grading of $\mathrm{End}(E)$ for all $(x,\xi)\in S^*(X)$;
- $\sigma^* = \sigma$ and $\sigma^2 = 1$.

Such a symbol is called *trivial* is does not depend on the variable ξ.

The group of stable homotopy classes of Clifford symbols, modulo the trivial ones, is canonically isomorphic to $K_0\left(C_0(V)\right)$.

Now observe that any Clifford symbol on the vector bundle (E, σ), up to add a trivial one, is equivalent to a Clifford symbol with E a trivial bundle $\mathbb{C}^n$. Consider the following commutative diagram

$$\begin{tikzcd} C(X) \arrow[r, "\pi^*"] \arrow[d, "j"] & C(S^*X) \arrow[d, "="] \\ \Psi^0(X) \arrow[r, "\mathrm{symb}_{pr}"] & C(S^*X) \end{tikzcd} \tag{1.2.5}$$

where $\Psi^0(X)$ is the C*-algebra of the 0-order classical pseudodifferential operator on X and $\mathrm{symb}_{pr}\colon \Psi^0(X) \to C(S^*X)$ is the principal symbol map. Thus we can associate to the Clifford symbol $(\mathbb{C}^n, \sigma)$ the relative K-cycle (p_n, p_n, σ) for the pair $\big(\Psi^0(X), C(S^*X)\big)$, see Section A for the definition. Therefore we have a well-defined map

$$K_0\left(C_0(T^*X)\right) \to K_0\left(\Psi^0(X), C(S^*X)\right).$$

Composing this application with the inverse of the excision map A.0.3

$$K_0\left(\Psi^0(X), C(S^*X)\right) \to K_0\left(\mathbb{K}\right) \cong \mathbb{Z}$$

and applying this composition to the Clifford symbol $(\mathbb{C}^n, \sigma)$, by Example A.0.8 we obtain the analytical index of $(\mathbb{C}^n, \sigma)$ defined by Atiyah and Singer in [1].

Definition 1.2.49. The analytical index is the map

$$\mathrm{Ind}_a\colon\ K_0\left(C_0(T^*X)\right) \to \mathbb{Z}$$

given by the composition

$$K_0\left(C_0(T^*X)\right) \to K_0\left(\Psi^0(X), C(S^*X)\right) \to K_0\left(\mathbb{K}\right) \cong \mathbb{Z}$$

as above.

Of course, if instead of the cotangent bundle we consider the dual Lie algebroid $\mathfrak{A}^*(G)$ of a Lie groupoid $G \rightrightarrows X$, after doing cosmetic modification to the construction, we obtain an analytical G-index in the following way.

Definition 1.2.50. The analytical G-index is the map

$$\mathrm{Ind}_a^G\colon\ K_0\left(C_0(\mathfrak{A}^*(G))\right) \to C_r^*(G)$$

given by the composition

$$K_0\left(C_0(A^*(G))\right) \to K_0\left(\Psi_c^0(G), C(\mathfrak{A}^*(G))\right) \to K_0\left(C_r^*(G)\right)$$

as above.

Proposition 1.2.51. *Let $G = X \times X \rightrightarrows X$ the pair groupoid associated to a smooth closed manifold X, then we have the following equality*

$$\mathrm{Ind}_G = \mathrm{Ind}_a\colon K_0\left(C_0(T^*X)\right) \to \mathbb{Z}$$

between the adiabatic and the analytical index.

Proof. Let $G_{ad} \rightrightarrows X \times [0,1]$ the tangent groupoid of X. Notice that $\mathfrak{A}^*(G) \cong T^*X$ and that $\mathfrak{A}^*(G) \cong T^*X \times [0,1]$. Recall that the evaluation map $\mathrm{ev}_0 \colon C_r^*(G_{ad}) \to C_r^*(\mathfrak{A}(G)) \cong C_0(\mathfrak{A}^*(G))$ induces a KK-equivalence. Then we have the following commutative diagram

$$\begin{tikzcd}
K_0\left(C_0(T^*X)\right) \arrow[r, "{[\mathrm{ev}_0]^{-1}}"] \arrow[d] & K_0\left(C_0(T^*X \times [0,1])\right) \arrow[r, "{[\mathrm{ev}_1]}"] \arrow[d] & K_0\left(C_0(T^*X)\right) \arrow[d] \\
K_0\left(\Psi_c^0(T^*X), C(S^*X)\right) \arrow[r, "{[\mathrm{ev}_0]^{-1}}"] \arrow[d] & K_0\left(\Psi_c^0(G_{ad}), C(S^*X)\right) \arrow[r, "{[\mathrm{ev}_1]}"] \arrow[d] & K_0\left(\Psi^0(X), C(S^*X)\right) \arrow[d] \\
K_0\left(C_0(T^*X)\right) \arrow[r, "{[\mathrm{ev}_0]^{-1}}"] & K_0\left(C_r^*(G_{ad})\right) \arrow[r, "{[\mathrm{ev}_1]}"] & \mathbb{Z}
\end{tikzcd}$$

where the third vertical composition is the analytical index Ind_a and the bottom line composition is the adiabatic index Ind_G.

Both the top line and the left-hand side compositions are obviously the identity maps. Then considering the exterior big square we immediately the wished equality.

□

1.3 The relative K-theory of the assembly map

In [19, II.2] it is defined the Grothendieck group of a functor $\varphi \colon \mathcal{C} \to \mathcal{C}'$, as the set of triples (E, F, α), where E and F are objects in the category $\mathcal{C}$ and α is an isomorphism $\varphi(E) \to \varphi(F)$ in the category $\mathcal{C}'$, modulo the following equivalence relation: two triples (E, F, α) and (E', F', α') are equivalent if there exist two isomorphisms $f \colon E \to E'$ and $g \colon F \to F'$ such that the following diagram

$$\begin{tikzcd}
\varphi(E) \arrow[r, "\alpha"] \arrow[d, "\varphi(f)"] & \varphi(F) \arrow[d, "\varphi(g)"] \\
\varphi(E') \arrow[r, "\alpha'"] & \varphi(F')
\end{tikzcd}$$

commutes.

In [19, II.3.28] it is shown that, when φ is the restriction of vector bundles over a space X to a closed subspace Y, one get the relative K-group $K(X, Y)$ defined as the K-theory of the mapping cone of the inclusion $i \colon Y \hookrightarrow X$.

1.3.1 The relative KK-theory

In [35] G. Skandalis used the same idea, as we are going to explain, to define the K-theory of the functor given by the Kasparov product with a KK-element.

Let A, B, D be C*-algebras. Let $x = (E, F) \in \mathbb{E}(A, B)$. Define $\mathbb{E}(x, D)$ as the set of pairs $\left((E_0, F_0), (\widetilde{E}, \widetilde{F})\right)$ where $(E_0, F_0) \in \mathbb{E}(B, D)$ and $(\widetilde{E}, \widetilde{F}) \in \mathbb{E}(A, D[0,1))$ such that $(\widetilde{E}_0, \widetilde{F}_0)$ is a Kasparov product of (E, F) by (E_0, F_0). Define also $\mathbb{E}(D, x)$ as the set of pairs $\left((E_0, F_0), (\widetilde{E}, \widetilde{F})\right)$ where $(E_0, F_0) \in \mathbb{E}(D, A)$ and $(\widetilde{E}, \widetilde{F}) \in \mathbb{E}(D, B[0,1))$ such that $(\widetilde{E}_0, \widetilde{F}_0)$ is a Kasparov product of (E_0, F_0) by (E, F). Define $KK(x, D)$ as the group of homotopy classes of elements in $\mathbb{E}(x, D)$ and $KK(D, x)$ as the group of homotopy classes of elements in $\mathbb{E}(D, x)$. One then proves easily the following facts.

If A is separable we have the exact sequence:

$$KK(B, D(0,1)) \xrightarrow{x\otimes\cdot} KK(A, D(0,1)) \longrightarrow KK(x, D) \longrightarrow KK(B, D) \xrightarrow{x\otimes\cdot} KK(A, D)\ .$$

If D is separable we have the exact sequence:

$$KK(D, A(0,1)) \xrightarrow{\cdot\otimes x} KK(D, B(0,1)) \longrightarrow KK(D, x) \longrightarrow KK(D, A) \xrightarrow{\cdot\otimes x} KK(D, B)\ .$$

The groups $KK(x, D)$ and $KK(D, x)$ only depend on the class of (E, F) in $KK(A, B)$ (with separability assumptions). If this class is given by an exact sequence (admitting a completely positive cross section):

$$0 \longrightarrow B(0,1) \otimes \mathbb{K} \longrightarrow C \longrightarrow A \longrightarrow 0\ ,$$

we find $KK(D, x) = KK(D, C)$ and $KK(x, D) = KK(C, D(0, 1))$. Indeed one can prove that there always exists an algebra A_1 that is KK-equivalent to A and two morphisms $\varphi\colon A_1 \to B$ and $\psi\colon A_1 \to A$, this latter giving the KK-equivalence, such that $\mathbf{x} = [\psi]^{-1} \otimes_{A_1} [\varphi]$ and in particular the above C*-algebra C is given by the mapping cone $C_\varphi(A_1, B)$.

Remark 1.3.1. Notice that if x is the adiabatic G-index $\mathrm{Ind}_G \in KK(C_0(\mathfrak{A}^*(G)), C_r^*(G))$, since $\mathrm{Ind}_G = [\mathrm{ev}_0]^{-1} \otimes_{C_r^*(G_{ad})} [\mathrm{ev}_1]$, the previous reasoning tell us that the relative KK-theory of x is nothing but $KK(D, C_r^*(G_{ad}^0))$, where $C_r^*(G_{ad}^0) \cong C_{\mathrm{ev}_1}(C_r^*(G_{ad}), C_r^*(G))$.

We also have a notion of functoriality with respect to pairs of *-homomorphism: let $\alpha\colon A \to A'$ and $\beta\colon B \to B'$ be *-homomorphisms and let $x \in \mathbb{E}(A, B)$ and $x' \in \mathbb{E}(A', B')$ be such that $x \otimes [\beta] = [\alpha] \otimes x'$. Then one can check that the morphism $K(x) \to K(x')$ given by

$$\Big((E_0, F_0), (\widetilde{E}, \widetilde{F})\Big) \mapsto \Big(\alpha_*(E_0, F_0), (\beta \times \mathrm{id}_{[0,1)})_*(\widetilde{E}, \widetilde{F})\Big) \tag{1.3.1}$$

is well defined.

So we have seen that constructing relative KK-groups corresponds, in a philosophical way, to take the Grothendieck group of a functor or, in a more concrete way, to take the Grothendieck group of a mapping cone. We want to construct a long exact sequence of groups such that the boundary map is the assembly map. Notice that the difficulty is that it is not induced by a morphism nor by a Kasparov product on the right. But still it is possible to construct a group.

1.3.2 The assembly map

We refer the reader to [20] for precise definitions of the objects in this section. Let Γ be a discrete group. For any Γ-C*-algebras A and B, there exists a descent homomorphism

$$j^\Gamma\colon KK_\Gamma(A, B) \to KK(A \rtimes \Gamma, B \rtimes \Gamma)$$

which is functorial and compatible with respect to Kasparov products. It associates to an equivariant KK-cycle $[H, \phi, F]$ the Kasparov bimodule $[H \rtimes \Gamma, \widetilde{\phi}, \widetilde{F}]$, where

- $H \rtimes G$ is the $A \rtimes \Gamma$-$B \rtimes \Gamma$-bimodule given by the completion of $C_c(G, H)$, with the usual $C_c(\Gamma, B)$-valued inner product and left $C_c(\Gamma, A)$-action;
- $\widetilde{\phi}$ is the extension to $A \rtimes \Gamma$ of the representation induced by ϕ of $C_c(\Gamma, A)$;

- $\widetilde{F}$ is extension the operator on $C_c(\Gamma, H)$ to an operator on $H \rtimes \Gamma$, such that it associate to $\gamma \mapsto \alpha(\gamma)$ the element $\gamma \mapsto F(\alpha(\gamma))$

Moreover we know that for any proper and cocompact Γ-space X one can construct an imprimitivity $C(X/\Gamma)$-$C_0(X) \rtimes \Gamma$-bimodules E_X. Since $C(X/\Gamma)$ is unital E_X defines an element in $KK(\mathbb{C}, C_0(X) \rtimes \Gamma)$.

Definition 1.3.2. The assembly map $\mu_X^\Gamma \colon KK_\Gamma^*(C_0(X), \mathbb{C}) \to KK(\mathbb{C}, C_r^*(\Gamma))$ is given by the composition

$$KK_\Gamma^*(C_0(X), \mathbb{C}) \xrightarrow{j^\Gamma} KK^*(C_0(X) \rtimes \Gamma, C_r^*(\Gamma)) \xrightarrow{[E_X]\otimes -} K_*(C_r^*(\Gamma)) \; .$$

1.3.3 The group $K_*(\mu_X^\Gamma)$

Definition 1.3.3. Let X be as above. An even-μ_X^Γ relative cycle on X consists of the following data:

- a Γ-equivariant selfadjoint Kasparov bimodule $(H, \phi, T) \in \mathbb{E}_\Gamma(C_0(X), \mathbb{C})$;
- a Kasparov bimodule $(\mathcal{E}(t), \psi(t), S(t)) \in \mathbb{E}(\mathbb{C}, C_r^*(\Gamma)[0,1))$, such that $\mathcal{E}(0) = E_X \otimes_{C_0(X)\rtimes\Gamma} H \rtimes \Gamma$, $\psi(0) = 1 \otimes \phi$, $S(0)$ is a $\widetilde{T}$-connection and $\tilde{S}(1)$ is invertible. Here $\widetilde{\phi}$ and $\widetilde{T}$ are like in the definition of the descent homomorphism. That is the class of $(\mathcal{E}(0), \psi(0), S(0))$ is equal to $\mu_X^\Gamma(H, \phi, T)$.

For the odd case the definition is analogous. Such a cycle is said to be degenerate if both (H, ϕ, T) and $(\mathcal{E}(t), \psi(t), S(t))$ are degenerate Kasparov bimodules.

Definition 1.3.4. Let $(H_i, \phi_i, T_i, \mathcal{E}(t)_i, \psi(t)_i, S_i(t))$, $i = 0, 1$, be two μ_X^Γ-relative cycles.

We will say that they are homotopic if there exists a path $(H_s, \phi_s, T_s, \mathcal{E}_s(t), \psi(t)_s, S_s(t))$ of μ_X^Γ-relative cycles that joins them. Then we denote by $K_j(\mu_X^\Gamma)$, $i = 0, 1$, the Grothendieck group generated by all homotopy classes of μ_X^Γ-relative cycles on X.

We can define in an analogous way a group $K_j(\mu_{X,A}^\Gamma)$, where A is any Γ-C*-algebra, using the assembly map with coefficient $\mu_{X,A}^\Gamma \colon KK^\Gamma(C_0(X), A) \to KK(\mathbb{C}, A \rtimes \Gamma)$.

Remark 1.3.5. Let $\varphi \colon A \to B$ a C*-algebras morphism and C_φ its mapping cone. Then we obtain naturally the long exact sequence of K-groups

$$\cdots \longrightarrow K_*(SB) \longrightarrow K_*(C_\varphi) \longrightarrow K_*(A) \xrightarrow{\varphi_*} K_*(B) \longrightarrow \cdots$$

whose boundary morphism is induced by φ. Conversely, if we start from a homomorphism $K_*(A) \to K_*(B)$ induced by morphism $\varphi \colon A \to B$, then this homomorphism fits into a sequence as above.

As explained before, the idea behind the construction of $K_j(\mu_X^\Gamma)$ is considering the assembly map as a functor. But instead of a Kasparov product on the right, we have the assembly map, that is the composition of the descent morphism and a Kasparov product on the left, and instead of the K-theory of a mapping cone C*-algebra we obtain the Grothendieck group of a "mapping cone" of Kasparov bimodules.

Definition 1.3.6. Let Y, X be two spaces and assume that a group Γ acts on Y and X in a proper and cocompact way. Let $f \colon Y \to X$ be a Γ-equivariant continuous map. We can define a homomorphism

$$f_* \colon K_*(\mu_Y^\Gamma) \to K_*(\mu_X^\Gamma)$$

such that

$$f_*[H, \phi, T, \mathcal{E}(t), \psi(t), S(t)] = [H, \phi \circ f, T, \mathcal{E}'(t), \psi'(t), S'(t))].$$

Here $(\mathcal{E}'(t), \psi'(t), S'(t))$ is the concatenation of the path we are going to describe in a moment and the path $(\mathcal{E}(t), \psi(t), S(t))$. The first one is the path connecting the Kasparov bimodules $(E_X \otimes_{C_0(X)\rtimes\Gamma} H \rtimes \Gamma, 1 \otimes \widetilde{(\phi \circ f)}, S')$ and $(E_Y \otimes_{C_0(Y)\rtimes\Gamma} H \rtimes \Gamma, 1 \otimes \widetilde{\phi}, S)$, where S' is a $\widetilde{T}$-connection on $E_Y \otimes_{C_0(Y)\rtimes\Gamma} H \rtimes \Gamma$. This path always exists, because of the functoriality of the assembly map: since $\mu_Y^\Gamma = \mu_X^\Gamma \circ f_* \colon KK_\Gamma(C_0(Y), \mathbb{C}) \to K_0(C_r^*(\Gamma))$, it turns out that $(E_X \otimes_{C_0(X)\rtimes\Gamma} H \rtimes \Gamma, 1 \otimes \widetilde{(\phi \circ f)}, S')$ and $(E_Y \otimes_{C_0(Y)\rtimes\Gamma} H \rtimes \Gamma, 1 \otimes \widetilde{\phi}, S)$ define the same class.

Moreover the class obtained does not depends on the choice of this path.

Definition 1.3.7. Define the group $\hat{K}_0^\Gamma(X)$ as the Grothendieck group of homotopy classes of cycles given by the following data:

- a Γ-equivariant selfadjoint Kasparov bimodule $(H, \phi, T) \in \mathbb{E}_\Gamma(C_0(X), \mathbb{C})$;
- a Kasparov bimodule $(\mathcal{E}(t), \psi(t), S(t)) \in \mathbb{E}(\mathbb{C}, C_r^*(\Gamma)[0, 1))$, such that $\mathcal{E}(0) = E_X \otimes_{C_0(X)\rtimes\Gamma} H \rtimes \Gamma$, $\psi(0) = 1 \otimes \phi$, $S(0)$ is a $\widetilde{T}$-connection. Here $\widetilde{\phi}$ and $\widetilde{T}$ are like in the definition of the descent homomorphism. That is the class of $(\mathcal{E}(0), \psi(0), S(0))$ is equal to $\mu_X^\Gamma(H, \phi, T)$.
- such a cycle is said to be degenerate if both (H, ϕ, T) and $(\mathcal{E}(t), \psi(t), S(t))$ are degenerate Kasparov bimodules.

For the odd case $\hat{K}_1^\Gamma(X)$ the definition is analogous.

Remark 1.3.8. The group $\hat{K}_j^\Gamma(X)$ is as in definition 1.3.3, but dropping the condition of $S(1)$ being invertible, we get that the map

$$\hat{K}_j^\Gamma(X) \ni [H, \phi, T, \mathcal{E}(t), \psi(t), S(t)] \mapsto [H, \phi, T] \in KK_\Gamma^j(C_0(X), \mathbb{C})$$

is a group isomorphism. Indeed one can easily check the kernel of this map is isomorphic to $KK_j(\mathbb{C}, C_r^*(\Gamma) \otimes C_0(0, 1])$, that is trivial since $C_r^*(\Gamma) \otimes C_0(0, 1]$ is a cone. The inverse map is obviously given by

$$[H, \phi, T] \mapsto [H, \phi, T, \mathcal{E}, 1 \otimes \widetilde{\phi}, S]$$

where S is the path constantly equal to any $\widetilde{T}$-connection.

Definition 1.3.9. Let Γ a discrete group, we can define the following exact sequence of groups

$$\cdots \longrightarrow KK_*(\mathbb{C}, C_r^*(\Gamma) \otimes C_0(0, 1)) \longrightarrow K_*(\mu^\Gamma) \longrightarrow \hat{K}_*^\Gamma \longrightarrow \cdots$$

as the direct limit of

$$\cdots \longrightarrow KK_*(\mathbb{C}, C_r^*(\Gamma) \otimes C_0(0, 1)) \longrightarrow K_*(\mu_X^\Gamma) \longrightarrow \hat{K}_*^\Gamma(X) \longrightarrow \cdots$$

over all cocompact Γ-subspace X of $E\Gamma$.

1.4 Comparing exact sequences

Here we want to compare the three different realization of the Analytic Surgery Exact Sequence and prove that they are naturally isomorphic when we consider the case of a proper and free action of a group Γ on a smooth Riemannian manifold $\widetilde{X}$.

1.4.1 Coarse geometry vs Relative KK-groups

In [32] the J. Roe shows that the following diagram

$$\begin{tikzcd} K_{*+1}(D^*(X)^\Gamma/C^*(X)^\Gamma) \arrow[r, "\mathrm{Ind}"] \arrow[d, "P"] & K_*(C^*(X)^\Gamma) \arrow[d, "\cong"] \\ KK^*_\Gamma(C_0(X),\mathbb{C}) \arrow[r, "\mu_X^\Gamma"] & K_*(C^*_r(\Gamma)) \end{tikzcd}$$

is commutative. Here P is given by the Pashcke duality and μ_X^Γ is the assembly map defined by Kasparov in [20]. In particular if $F \in \mathbb{B}(L^2(X))$ is an element of $D^*(X)^\Gamma$ such that $F^2 = 1$ and $F^* = F$ modulo $C^*(X)^\Gamma$, it gives a representative for

$$\mu_X^\Gamma \left[L^2(X), \varphi\colon C_0(X) \to \mathbb{B}(L^2(X)), F\right]$$

in the following way. Since we can assume that F is exactly of finite propagation, it define an operator F_c on the pre-Hilbert space $L^2_c(X)$ of the compactly supported L^2-functions of X. One can endow $L^2_c(X)$ with the following $\mathbb{C}\Gamma$-valued inner product

$$\langle f, g\rangle_{\mathbb{C}\Gamma}(\gamma) = \langle f^\gamma, g\rangle_{\mathbb{C}}$$

where $\langle f^\Gamma, g\rangle_{\mathbb{C}}$ is the standard inner product between g and the function f traslated by γ. With a standard double completion of the pair $(L^2_c(X), \mathbb{C}\Gamma)$ we obtain an Hilbert module over $C^*_r(\Gamma)$ that we denote by $L^2_\Gamma(X)$. Now F_c extends to an adjointable operator $\widetilde{F}$ on $L^2_\Gamma(X)$ and the class $[L^2_\Gamma(X), 1\otimes\widetilde{\varphi}, \widetilde{F}] \in KK(\mathbb{C}, C^*_r(\Gamma))$ is equal to $\mu_X^\Gamma\left[L^2(X), \varphi, F\right]$.

Remark 1.4.1. Notice that if F is invertible, then $\widetilde{F}$ is so.

Moreover one can proved that $L^2_\Gamma(X)$ is a complemented sub-Hilbert module of $L^2(X)\rtimes\Gamma$. In fact if $\phi\colon X \to [0,1]$ is a compactly supported function such that

$$\sum_{\gamma\in\Gamma} (\phi^2)^\gamma = 1,$$

then the projection

$$p = \sum_{\gamma\in\Gamma} \phi\cdot\phi^{\gamma^{-1}}[\gamma] \in C_0(X)\rtimes\Gamma$$

has as range the $C^*_r(\Gamma)$-module $L^2_\Gamma(X)$.

Remark 1.4.2. Indeed the projection p gives the class $[E_X] \in KK(\mathbb{C}, C_0(X)\rtimes\Gamma)$ used in the Definition 1.3.2 of the assembly map.

Proposition 1.4.3. *There is a commutative diagram*

$$\begin{tikzcd} \cdots \arrow[r] & K_j(C^*(X)^\Gamma) \arrow[r] \arrow[d, "\beta"] & K_j(D^*(X)^\Gamma) \arrow[r] \arrow[d, "\alpha"] & K_j(D^*(X)^\Gamma/C^*(X)^\Gamma) \arrow[r] \arrow[d, "P"] & \cdots \\ \cdots \arrow[r] & KK^{j-1}(\mathbb{C}, C^*_r(\Gamma)\otimes C_0(0,1)) \arrow[r] & K_{j-1}(\mu_X^\Gamma) \arrow[r] & KK^{j-1}_\Gamma(C_0(X),\mathbb{C}) \arrow[r] & \cdots \end{tikzcd}$$

whose vertical arrows are isomorphisms.

Proof. Let $(L^2(X), \phi)$ be the Γ-equivariant $C_0(X)$-module used to construct the algebra $D^*(X)^\Gamma$. The map P is given by the Pashcke duality. The homomorphism β is given by the composition of the isomorphism between $C^*(X)^\Gamma$ and $C_r^*(\Gamma)$, and the Bott periodicity. Let us describe the homomorphism $\alpha\colon K_0(D^*(X)^\Gamma) \to K_1(\mu_X^\Gamma)$. It associates to a projection p over $D^*(X)^\Gamma$ the cycle $(H, \varphi, T, \mathcal{E}(t), \psi(t), S(t))$, where

- $(H, \varphi, T) = (L^2(X), \varphi, 2p - 1)$;
- $(\mathcal{E}(t), \psi(t), S(t))$ is given by the path constatly equal to $(L^2_\Gamma(X), 1 \otimes \widetilde{\varphi}, \widetilde{T})$, that is the triple built in the discussion at the beginning of the present section.

Observe that $L^2_\Gamma(X)$ is nothing but $E_X \otimes_{C_0(X)\rtimes\Gamma} L^2(X) \rtimes \Gamma$ and that $\widetilde{T}$ is invertible by construction (see Remark 1.4.1).

The homomorphism β associate to a projection p over $C^*(X)^\Gamma$ the Kasparov bimodule $[L^2_\Gamma(X), 1 \otimes \widetilde{\phi}, G(t)]$, where $G(t)$ is the loop of invertible elements $\widetilde{T}(1 - e^{2i\pi t}) - 1$ over $C^*(X)^\Gamma$, given by the Bott periodicity.

The second square is obviously commutative. Concerning the first one, since $(L^2(X), \phi, T)$ is degenerate as Kasparov bimodule, it is easy to produce a homotopy of cycles between $(0, 0, 0, L^2_\Gamma(X), 1 \otimes \widetilde{\varphi}, G(t))$ and $\alpha(i_*[p]) = [H, \varphi, T, L^2_\Gamma(X), 1 \otimes \widetilde{\varphi}, S(t)]$, where $S(t)$ is the constant path equal to $\widetilde{T}$. To do it, observe that $G\left(\frac{1}{2}\right) = \widetilde{T}$ and that $G_s(t) = G\left((1-s)t + \frac{1}{2}s\right)$ does the job. □

Now let us compare the functoriality of these two construction with respect to uniform maps.

Lemma 1.4.4. *Let Y, X be two Riemannian manifolds and assume that a group Γ acts on Y and X freely, isometrically and such that Y/Γ is compact. Let $f\colon Y \to X$ be a Γ-equivariant continuous coarse map and let $V\colon H_Y \to H_X$ be an isometry that covers f in the D^*-sense. Then the following diagram*

$$\begin{array}{ccc}
K_j(D^*(Y)^\Gamma) & \xrightarrow{f_*} & K_j((D^*(X)^\Gamma) \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\
K_{j-1}(\mu_Y^\Gamma) & \xrightarrow{f_*} & K_{j-1}(\mu_X^\Gamma)
\end{array}$$

commutes.

Proof. Let (H_Y, ϕ_Y) and (H_X, ϕ_X) the representations of $C_0(Y)$ and $C_0(X)$ respectively, used to construct the algebras $D^*(Y)^\Gamma$ and $D^*(X)^\Gamma$. Let p be a projection over $D^*(Y)^\Gamma$. Recall that $f_*[p] = [\mathrm{Ad}_V(p)] \in K_0(D^*(X)^\Gamma)$. Then we get two element of $K_1(\mu_X^\Gamma)$:

- the first one is $\alpha([\mathrm{Ad}_V(p)]) = [H_X, \phi_X, T, \mathcal{E}_X, 1 \otimes \widetilde{\phi}_X, S]$. Here $T = 2\mathrm{Ad}_V(p) - 1$, $\mathcal{E}_X = E_X \otimes_{C_0(X)\rtimes\Gamma} H_X \rtimes \Gamma$ and S is the path constantly equal to a $\widetilde{T}$-connection;
- the second one is $f_*(\alpha[p]) = [H_Y, \phi_Y \circ f^*, U, \mathcal{E}'(t), \psi'(t), S'(t)]$. Here $U = 2p - 1$ and $(\mathcal{E}'(t), \psi'(t), S'(t))$ is the path connecting $(E_X \otimes_{C_0(X)\rtimes\Gamma} H \rtimes \Gamma, 1 \otimes \widetilde{(\phi \circ f)}, S')$ and $(E_Y \otimes_{C_0(Y)\rtimes\Gamma} H \rtimes \Gamma, 1 \otimes \widetilde{\phi}, S)$, where S' is a $\widetilde{U}$-connection on $E_Y \otimes_{C_0(Y)\rtimes\Gamma} H \rtimes \Gamma$.

We have to prove that these two classes are the same.

Consider the projection $Q = VV^*$, then we can decompose $\alpha([\mathrm{Ad}_V(p)])$ in two direct summand:

$$\alpha([\mathrm{Ad}_V(p)]) = [QH_X, \phi_X, T_1, R\mathcal{E}_X, 1\otimes\widetilde{\phi}_X, S_1]\oplus[(1-Q)H_X, \phi_X, T_2, (1-R)\mathcal{E}_X^2, 1\otimes\widetilde{\phi}_X, S_2],$$

where $T_1 = QTQ$, $T_2 = (1-Q)T(1-Q)$, R is a $\widetilde{Q}$-connection and S_1 and S_2 are defined similarly.

The second summand is clearly degenerate; since the following diagram

$$\begin{tikzcd}
C_0(X) \arrow[r, "f^*"] \arrow[drr, "\varphi_X"'] & C_0(Y) \arrow[r, "\varphi_Y"] & \mathbb{B}(H_Y) \arrow[d, "\simeq"', "\mathrm{Ad}_V"] \\
 & & \mathbb{B}(QH_X)
\end{tikzcd}$$

commutes modulo compacts operators, the first one is equal to $f_*(\alpha[p])$. □

1.4.2 Localization algebras vs Lie groupoids C*-algebras

Let X be a smooth manifold with fundamental group $\pi_1(X) = \Gamma$ and let us fix $G(X) \rightrightarrows X$ to be the Lie groupoid $\widetilde{X} \times_\Gamma \widetilde{X}$. In this section we want to state the following natural isomorphism of exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & K_*\left(C_r^*(G(X)\times(0,1))\right) \arrow[r] \arrow[d, "\alpha"] & K_*\left(C_r^*(G(X)^0_{ad})\right) \arrow[r] \arrow[d, "\beta"] & K_*\left(C_0(T^*X)\right) \arrow[r] \arrow[d, "\gamma"] & \cdots \\
\cdots \arrow[r] & K_{*+1}(C^*(\widetilde{X})^\Gamma) \arrow[r, "\partial"] & K_*(C^*_{L,0}(\widetilde{X})^\Gamma) \arrow[r] & K_*(C^*_L(\widetilde{X})^\Gamma) \arrow[r, "\mathrm{ev}_0"] & 0
\end{tikzcd} \tag{1.4.1}$$

in a rather direct way. From now on in this section we are fixing the ample Γ-equivariant $C_0(\widetilde{X})$-representation used to define the localization algebras to be $H = L^2(\widetilde{X}) \otimes l^2$.

First let us recall that, for a general Lie groupoid $G \rightrightarrows X$, $\mathrm{ev}_0\colon C_r^*(G_{ad}) \to C_r^*(\mathfrak{A}(G))$ induces a KK-equivalence, then we can replace $C_r^*(\mathfrak{A}(G(X)))$ with $C_r^*(G(X)_{ad})$ at the right-upper place in the diagram.

Since $C^*(\widetilde{X})^\Gamma$ is isomorphic to $C_r^*(G(X)) \otimes \mathbb{K}$, the first vertical map α is just the composition of the map induced by $f \mapsto f \otimes e_{11}$ and the Bott periodicity.

In order to define β, it is useful to recall what a compact subset of $G(X)_{ad}$ is. We have to understand what it happen for t near 0. Since the topology of $G(X)_{ad}$ is given by the map Ψ in (1.2.1), (it turns out that a compact is the image of a compact $K \subset \mathcal{N} \times [0,1]$ through Ψ, where $\mathcal{N}$ in this case is the Lie algebroid $\mathfrak{A}(G(X))$. Then it is easy to check that the image of any bounded neighbourhood of the zero section is a set that concentrates around $G^{(0)} = X$ when t goes to 0. This in particular means that if $f \in C_c^\infty(G(X)_{ad})$, then the f is given by a family $(f_t)_{t\in[0,1]}$ such that the support of f_t concentrates around the diagonal when t goes to 0. Moreover, since $(0,1] \times G$ is dense in $G(X)_{ad}$, f is determined in a unambiguous way by $(f_t)_{t\neq 0}$. Now we can immediately define a map

$$C_c^\infty(G(X)_{ad}) \to C_L^*(\widetilde{X})^\Gamma$$

sending a family $(f_t)_{t\in(0,1]}$ to the paths $T_s \otimes e_{11}$ of integral operators with smooth kernel $k_s = f_{\exp(-s)}$. By the previous discussion, one has that the propagation of T_s goes to 0 as s goes to infinity.

By continuity we extend this map to a *-homomorphism $\gamma'\colon C^*_r(G(X)_{ad}) \to C^*_L(\widetilde{X})^\Gamma$. Then we finally obtain the group morphism

$$\gamma\colon K_*(C_0(T^*X)) \to K_*\left(C^*_L(\widetilde{X})^\Gamma\right)$$

as KK-element $[\mathrm{ev}_0]^{-1} \otimes [\gamma']$.

Clearly γ' restricts to a *-homomorphism $\gamma''\colon C^*_r(G(X)^0_{ad}) \to C^*_{L,0}(\widetilde{X})^\Gamma$ that induces the group morphism

$$\beta\colon K_*\left(C^*_r(G(X)^0_{ad})\right) \to K_*\left(C^*_{L,0}(\widetilde{X})^\Gamma\right).$$

With this definition of the vertical maps, the diagram (1.4.1) is obviously commutative.

Furthermore α is given by a Morita equivalence, hence it is an isomorphism. The map γ is also an isomorphism: indeed if we compose γ on the left with the Poincaré duality $KK(C(X),\mathbb{C}) \to KK(\mathbb{C}, C_0(T^*X))$ (see [4, Corollary 3.8]), we obtain nothing but the Yu's local index Ind_L, defined in Section 1.1.4.

Finally, by the Five Lemma, it follows that β is an isomorphism too.

Lemma 1.4.5. *The following diagram*

$$\begin{tikzcd} K_*\left(C^*_r(G(X))\right) \arrow[r, "\mu_f"] \arrow[d, "\alpha_X"] & K_*\left(C^*_r(G(Y))\right) \arrow[d, "\alpha_Y"] \\ K_*\left(C^*(\widetilde{X})^\Gamma\right) \arrow[r, "f_*"] & K_*\left(C^*(\widetilde{Y})^\Gamma\right) \end{tikzcd} \tag{1.4.2}$$

is commutative. Here μ_f is as in Definition 1.2.19.

Proof. It is obvious. □

Lemma 1.4.6. *The following diagram*

$$\begin{tikzcd} K_*\left(C_0(T^*X)\right) \arrow[r, "df_!"] \arrow[d, "\gamma_X"] & K_*\left(C_0(T^*Y)\right) \arrow[d, "\gamma_Y"] \\ K_*\left(C^*_L(\widetilde{X})^\Gamma\right) \arrow[r, "f_*"] & K_*\left(C^*_L(\widetilde{Y})^\Gamma\right) \end{tikzcd} \tag{1.4.3}$$

is commutative. Here $df_!$ is as in (1.2.4).

Proof. Let us prove that $df_! \in KK(C_0(T^*X), C_0(T^*Y)$, defined as in (1.2.4), coincides with the class $df_! \in KK(C_0(T^*X), C_0(T^*Y)$ defined as in [4, Corollary 3.8], in this proof we will denote that class by $df_!^{CS}$. It is sufficient to prove it when f is a submersion.

Let $k\colon X \to \mathbb{R}^N$ be any embedding, for N big enough. Define the smooth embedding $\iota\colon X \to \mathbb{R}^N \times Y$ as the map $x \mapsto (k(x), f(x))$. Let U be a sufficiently small tubular neighbourhood of $\iota(X)$ in $\mathbb{R}^N \times Y$. Then we have the following commutative diagram:

$$\begin{tikzcd} K_*(C_0(T^*U)) \arrow[r, "di_!"] \arrow[d, "d\pi_!"] & K_*(C_0(T^*\mathbb{R}^N \times T^*Y)) \arrow[d, "dp!"] \\ K_*(C_0(T^*X)) \arrow[r, "df_!"] & K_*(C_0(T^*Y)) \end{tikzcd}$$

where $p\colon \mathbb{R}^N \times Y \to Y$ is the projection, $\pi\colon U \to X$ is the bundle projection given by the fact that U is isomorphic to the normal bundle of $\iota(X)$ in $\mathbb{R}^N \times Y$ and i is the natural inclusion of U into $\mathbb{R}^N \times Y$.

Notice that both $di_!$ and $di_!^{CS}$ are induced by the obvious inclusion of C*-algebras $C_0(T^*U) \hookrightarrow C_0(T^*\mathbb{R}^N \times T^*Y)$.

Moreover notice that by Proposition 1.2.29 $d\pi_!$ and $dp_!$ are invertible and by [5, Propositon 7.2] they are the inverses of the Thom isomorphism for $\pi\colon U \to X$ and $p\colon \mathbb{R}^N \times Y \to Y$ respectively.

Since $\mathbb{R}^N \times Y$ is a trivial vector bundle, $dp_!$ is given by the Bott element $bott^{-1} \in KK(C_0(\mathbb{R}^N), \mathbb{C})$. Let $j\colon X \to U$ the inclusion of X as the zero section, then $dj_!^{CS}$ coincides with the Thom isomorphism (see [4]) and then it is equal to $(d\pi_!)^{-1}$.

So we have that

$$\begin{aligned} df_! &= (dp_!)^{-1} \otimes di_! \otimes d\pi_! = \\ &= dj_!^{CS} \otimes di_!^{CS} \otimes bott^{-1} = \\ &= df_!^{CS} \end{aligned}$$

where the last equality is given by [4, Theorem 4.4].

Now consider the following commutative diagram

$$\begin{tikzcd} KK(C(X),\mathbb{C}) \arrow[r, "{[f]}"] \arrow[d, "\sigma_X"] & KK(C(Y),\mathbb{C}) \arrow[d, "\sigma_Y"] \\ K_*\left(C_0(T^*X)\right) \arrow[r, "df_!"] \arrow[d, "\gamma_X"] & K_*\left(C_0(T^*Y)\right) \arrow[d, "\gamma_Y"] \\ K_*\left(C_L^*(\widetilde{X})^\Gamma\right) \arrow[r, "f_*"] & K_*\left(C_L^*(\widetilde{Y})^\Gamma\right) \end{tikzcd}$$

where σ_X and σ_Y the classes defined in [4, Section 3] that give the Poincaré duality. The upper square is commutative by [4, Corollary 3.8] and the big exterior square is obviously commutative, hence the diagram (1.4.3) is commutative too.

□

Lemma 1.4.7. *The following diagram*

$$\begin{tikzcd} K_*\left(C_r^*(G(X)^0_{ad})\right) \arrow[r, "f_!^{ad}"] \arrow[d, "\beta_X"] & K_*\left(C_r^*(G(Y)^0_{ad})\right) \arrow[d, "\beta_Y"] \\ K_*\left(C_{L,0}^*(\widetilde{X})^\Gamma\right) \arrow[r, "f_*"] & K_*\left(C_{L,0}^*(\widetilde{Y})^\Gamma\right) \end{tikzcd} \tag{1.4.4}$$

is commutative.

Proof. Since

$$\begin{tikzcd} K_*\left(C_r^*(G(X)_{ad})\right) \arrow[r, "f_!^{ad}"] \arrow[d, "{[\mathrm{ev}_0]}"] & K_*\left(C_r^*(G(Y)_{ad})\right) \arrow[d, "{[\mathrm{ev}_0]}"] \\ K_*\left(C_0(T^*X)\right) \arrow[r, "df_!"] & K_*\left(C_0(T^*Y)\right) \end{tikzcd}$$

is commutative, then

$$\begin{tikzcd}
K_*\left(C_r^*(G(X)_{ad})\right) \arrow[r, "f_!^{ad}"] \arrow[d, "{[\gamma']}"] & K_*\left(C_r^*(G(Y)_{ad})\right) \arrow[d, "{[\gamma']}"] \\
K_*\left(C_L^*(\widetilde{X})^\Gamma\right) \arrow[r, "f_*"] & K_*\left(C_L^*(\widetilde{Y})^\Gamma\right)
\end{tikzcd}$$

is commutative too by the previous lemma. Hence it follows that diagram (1.4.4) is also commutative.

□

Chapter 2

Secondary invariants in K-theory

2.1 Lipschitz manifolds, the signature operator and the Surgery Exact Sequence

2.1.1 Lipschitz manifolds

We start recalling fundamental results on Lipschitz manifolds. For further details we refer to [39, 40, 38, 43]. Here we are following the presentation in [13].

Definition 2.1.1. Let $U \in \mathbb{R}^m$ be an open set. We recall that a map $\varphi\colon U \to \mathbb{R}^k$ is called Lipschitz if one of the following equivalent two conditions are satisfied:

1. There exists $C > O$ such that for all $x, y \in U$,
$$||\varphi(x) - \varphi(y)|| \leq C||x - y||.$$
2. φ possesses partial derivatives almost everywhere and the maps $x \mapsto \frac{\partial \varphi_i}{\partial x_i}(x)$ belongs to $L^\infty = (U)$.

We can define the Jacobian matrix of φ as the function $d\varphi \in L^\infty(U, M_{m,k}(\mathbb{R}))$ given by $d\varphi(x) = \left(\frac{\partial \varphi_i}{\partial x_i}(x)\right)$.

Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^k$, $W \subset \mathbb{R}^h$ be open subsets and $\varphi\colon U \to V$, $\psi\colon V \to W$ Lipschitz maps, then $\psi \circ \varphi$ is Lipschitz and we have for almost every $x \in U$ that

$$d(\psi \circ \varphi)(x) = d\psi(\varphi(x)) \circ d\varphi(x). \tag{2.1.1}$$

If φ, φ^{-1} are Lipschitz homeomorphism between two open subsets $U, V \subset \mathbb{R}^m$, then the class of Lebesgue measure is conserved by φ (this follows from equation (2.1.1)).

Let $\varphi\colon U \to \mathbb{R}^k$ be a Lipschitz map and let $\omega\colon \mathbb{R}^k \to \bigwedge(\mathbb{R}^k)$ be a measurable map . By condition 2) in Definition 2.1.1 we can form the pull-back $\varphi * (\omega)$ on U as follows : we can suppose that $\omega(y) = a(y)dy_{i_1} \wedge dy_{i_2} \wedge \cdots \wedge dy_{i_p}$ and put

$$\varphi * (w)(x) = a(\varphi(x)) \left(\sum_j \frac{\partial \varphi_i}{\partial x_j} dx_j\right) \wedge \ldots \left(\sum_j \frac{\partial \varphi_p}{\partial x_j} dx_j\right).$$

In particular we get a continuous linear map :

$$\varphi^*\colon L^2(V, \Lambda(\mathbb{R}^k)) \to L^2(U, \Lambda(\mathbb{R}^m)).$$

Let $\omega \in L^2(U, \Lambda(\mathbb{R}^m))$ considered by a current on U by the formula

$$\langle \omega, \alpha \rangle = \int \omega \wedge \alpha,$$

where $\alpha \in C_c^\infty(U, \bigwedge(\mathbb{R}^m))$. The exterior derivative of ω is the current defined by the following equality

$$\langle d\omega, \alpha \rangle = \pm 1 \int \omega \wedge d\alpha,$$

where $\alpha \in C_c^\infty(U, \bigwedge(\mathbb{R}^m))$ is homogenous.

Definition 2.1.2. We define $\Omega_d(U)$ to be the subspace of $L^2(U, \bigwedge(\mathbb{R}^m))$ of the ω for which the current $d\omega$ is again a square-integrable differential form; $\Omega_d(U)$ is the maximal domain of d.

In particular one can prove that, for any Lipschitz map $\varphi\colon U \to V$ we have that

$$\varphi^*(\Omega_d(V)) \subset \Omega_d(U)$$

and that

$$d\varphi^*(\omega) = \varphi^*(d\omega).$$

Definition 2.1.3. A Lipschitz manifold M is a topological manifold provided with an atlas $(U_i, \varphi_i)_{i \in I}$ such that for any $i, j \in I$, the homeomorphism

$$\varphi_j \circ \varphi_i^{-1} \colon U_i \cap U_j \to U_i \cap U_j$$

is a Lipschitz map.

It follows from (2.1.1) that a Lipschitz manifold possesses a well defined Lebesgue class of measure: this is the class of measure μ on M such that $\varphi_i(\mu)$ is equivalent to the Lebesgue measure on $\varphi_i(U_i) \subset \mathbb{R}^{\dim(M)}$. The Lipschitz structure on M determines the dense *-subalgebra $\mathcal{L}$ of $C(M)$ of Lipschitz functions: namely $f \in \mathcal{L}$ if and only if $\varphi_i^*(f)$ is a Lipschitz function on $\varphi_i(U_i)$, for all $i \in I$. Conversely a sub-algebra of $C(M)$ satisfying this type of conditions will determine a unique Lipschitz structure.

In what follows will be crucial the following result of Sullivan.

Theorem 2.1.4 ([38])**.** *Any topological manifold of dimension $n \neq 4$ has a Lipschitz atlas of coordinates. For any two such structures $\mathcal{L}_1$ and L_2, there exists a Lipschitz homeomorphism $h\colon \mathcal{L}_1 \to \mathcal{L}_2$ isotopic to the identity.*

2.1.2 Signature operator on Lipschitz manifolds

Let M be a Lipschitz manifold, oriented, of dimension m and $(U_i, \varphi_i)_{i \in I}$ an atlas of M. We shall denote by $\mathcal{M}(M, T^*M)$ and $\mathcal{M}(M, \Lambda(M))$ the measurable sections over M obtained by patching together the local trivial measurable sections of $T^*U_i = U_i \times \mathbb{R}^m$ and $T^*U_i = U_i \times \Lambda(\mathbb{R}^m)$. The sections of $\mathcal{M}(M, \Lambda(M))$ are families $(\omega_i)_{i \in I}$ where $\omega_i \colon \varphi(U_i) \to \Lambda(\mathbb{R}^\cdot)$ is measurable and $(\varphi_j \circ \varphi_i^{-1})^* \omega_i = \omega_j$ on $U_i \cap U_j$. Let $L^2(M, \Lambda(M))$ be thespace of square integrable differential forms, i.e. $\omega_i \in L^2(U_i, \Lambda(\mathbb{R}^m))$ for every $i \in I$. If $\xi, \eta \in L^2(M, \Lambda(M))$ and $\deg(\xi) + \deg(\eta) = m$, then the m-form $\xi \wedge \eta$ is integrable (M being orientable) and we get a bilinear pairing :

$$(\xi, \eta) = \int_M \xi \wedge \eta$$

on $L^2(M,\Lambda^p(M)) \times L^2(M,\Lambda^{m-p}(M))$ for all integer $0 \leq p \leq m$, which extends to a bilinear pairing on $L^2(M,\Lambda(M))$.

Now let us recall some elementary facts about Riemannian metrics on Lipschitz manifolds.

Let $U \in \mathbb{R}^m$ an open set and $Q(U)$ the space of all measurable Riemannian metrics on U equivalent to the standard one: then $Q(U)$ is the space of measurable map $x \mapsto g_x$ where g_x is a positive definite quadratic form on $\mathbb{R}^m$ for which there exist $\alpha, \beta > 0$ such that for almost all $\xi \in T^*U$ we have :

$$-\alpha||\xi||^2 \leq g_x(\xi,\eta) \leq \beta||\xi||^2$$

A Riemannian metric on M is a collection $(g_i)_{i\in I}$, where $g_i \in Q(\varphi(U_i))$ and and

$$(\varphi_j \circ \varphi_i^{-1})^* g_i = g_j$$

on $U_i \cap U_j$.

As M is oriented, a Riemannian metric g determines a $*$-operator of Hodge in the following way: in any local chart U_i we define the measurable field of Hodge operators

$$* \in \mathcal{M}(U_i, \mathrm{End}(\Lambda(\mathbb{R}^m)).$$

These operators patch together to give an inversible map of which satisfy

$$*^2 = (-1)^p$$

on $L^2(M,\Lambda^p(M))$. In particular the operator τ of $L^2(M,\Lambda(M))$ defined by

$$\tau = i^{p(p-1)+m} *$$

on $L^2(M,\Lambda^p(M))$ is an involution and we get a grading on

$$L^2(M,\Lambda(M))^i = \ker(\tau + (-1)^{i+1}),$$

for $i = 0, 1 \bmod 2$.

The metric g on M gives a hermitian bilinear form on $L^2(M,\Lambda(M))$:

$$(\alpha,\beta)_g = \int_M \alpha \wedge *\bar{\beta}.$$

One can prove that the Hilbert space structure on $L^2(M,\Lambda(M))$ induced by that hermitian bilinear form does not depends on the metric g, up to isomorphism.

Let $\Omega_d(M) \subset L^2(M,\Lambda(M))$ be the dense linear space of differential forms ω such that ω is locally in $\Omega_d(U_i)$, for any local chart U_i.

The operator $\omega \mapsto d\omega$ of $L^2(M,\Lambda(M))$ with domain $\Omega_d(M)$ is closed and satisfies:

$$d^2 = 0.$$

Definition 2.1.5. The signature operator on a Lipschitz manifold M is the operator

$$D_Z = d - *d* .$$

Theorem 2.1.6 ([39, 13])**.** *Let M be a closed oriented Lipschitz manifold of even dimension. Then from the complex of L^2-differential forms on M (with respect to some choice of a Lipschitz Riemannian metric g) one obtains a signature operator D_g which is closed and self-adjoint. Therefore D_g determines a class $[D]$ in $K_0(M) \simeq KK(C(M),\mathbb{C})$ which is independent of the choice of the metric g. The image of $[D]$ in $K_0(pt) \simeq KK(\mathbb{C},\mathbb{C})$ (i.e., the index of D_g) is the usual signature of the manifold.*

In [14] Hilsum proves that the signature operator gives a KK-class as above in the case of non compact manifolds too, provided the manifold M is endowed with a metric g such that it is metrically complete with respect to the induced structure of metric space. Moreover he showed a result on the finite propagation speed for solutions of the wave equation.

Theorem 2.1.7 (Hilsum). *Let M be an oriented Lipschitz manifold with a Riemannian structure, such that the manifold is complete as metric space. Let d be the associated distance function and let D be the associated signature operator. For all $t \in \mathbb{R}$, we have that:*

$$\operatorname{supp}(e^{itD}) \subset \{(x, y) \in M \times M \mid d(x, y) \leq t\}.$$

For $f \in \mathcal{S}(\mathbb{R})$ such that $\operatorname{supp}(\hat{f}) \subset [-a, a]$, with $a > 0$, we have that:

$$\operatorname{supp}(f(D)) \subset \{(x, y) \in M \times M \mid d(x, y) \leq a\}.$$

This theorem will be key in the coarse geometrical setting.

2.1.3 ϱ classes

We refer the reader to [11, sect.1] and [28, sect.1] for notations about coarse geometry and coarse algebras.

Let N be an oriented topological manifold of dimension $n \geq 5$; an element of the topological structure set N is given by an orientation preserving homotopy equivalence $f\colon M \to N$. Two different homotopy equivalences $f\colon M \to N$ and $f'\colon M' \to N$, are equivalent if there is a h-cobordism W between them and a homotopy equivalence $F\colon W \to N \times [0, 1]$, such that $F_{|M} = f$ and $F_{|M'} = f'$.

Definition 2.1.8. We define the topological structure set $\mathcal{S}^{TOP}(N)$ of N as the set of the h-cobordism classes of oriented homotopy equivalences.

Given a class $[f\colon M \to N]$, we set $Z = M \cup -N$. Let Γ be the fundamental group of N. The universal covering $\widetilde{N} \to N$ is induced by a map $u\colon N \to B\Gamma$: $\widetilde{N} = u^*(E\Gamma)$, where $B\Gamma$ is the classifying space of Γ and $E\Gamma$ its universal covering. Let $\widetilde{M}$ be the Γ-Galois covering induced by $u_M := u \circ f$, then we get a Γ-Galois covering $\widetilde{Z} = \widetilde{M} \cup -\widetilde{N}$ on Z. let $\mathcal{F} = \widetilde{Z} \times_\Gamma C^*_r(\Gamma)$ be the associated Mischenko bundle.

Now, starting from a Lipschitz structure on Z given by Theorem 2.1.4, consider the L^2-forms complex $L^2(Z, \Lambda_{\mathbb{C}}(Z))$.

Like in [13], we have that $\big(L^2(Z, \Lambda_{\mathbb{C}}(Z)), \mu, D_Z\big)$ defines an unbounded class $[D_Z] \in KK(C(Z), \mathbb{C})$, where μ is the representation that associates the multiplication operator μ_f to a function f.

Perturbed signature operator

We want to associate a class in the K-group $K_*(D^*(\tilde{M})^\Gamma)$ to a homotopy equivalence $f\colon M \to N$ and show that this mapping is well defined on the h-cobordism classes.

The key result for what follows is the homotopy invariance of the index class of the signature operator for compact oriented smooth manifolds, proved by M. Hilsum and G. Skandalis in [17]. Remember that, in the equivariant setting, this class is given by

$$\operatorname{Ind}_\Gamma(D_Z) = [\mathcal{F}] \otimes_{C(Z) \otimes C^*_r(\Gamma)} [\mathcal{D}_Z] \in KK(\mathbb{C}, C^*_r(\Gamma)),$$

where $[\mathcal{F}] \in KK(\mathbb{C}, C(Z) \otimes C_r^*(\Gamma))$ is the class given by the Mischenko bundle and $[\mathcal{D}_Z] \in KK(C(Z) \otimes C_r^*(\Gamma), C_r^*(\Gamma)$ is the class of the Signature operator twisted by the Mischenko bundle.

Theorem 2.1.9 (Hilsum-Skandalis)**.** *Let $f\colon M \to N$ be a homotopy equivalence. Then the class $\mathrm{Ind}_\Gamma(D_Z) \in KK(\mathbb{C}, C_r^*(\Gamma))$ vanishes.*

In remark [17, p.95] the authors observe that all arguments can be applied to the Lipschitz case: we can easily check that the smoothness of the objects is not necessary.

Remark 2.1.10. Of particular interest to us is a byproduct of the proof of Theorem 3.1, namely the construction of a homotopy $\mathcal{D}_\alpha$ between the signature operator $\mathcal{D}_Z = \mathcal{D}_0$ and an invertible operator $\mathcal{D}_1$. Here $\mathcal{D}_Z$ is the signature operator twisted by the Mishchenko bundle. Moreover we point out that the perturbation creates a gap in its spectrum near 0. This is the reason for the vanishing of the index class $\mathrm{Ind}_\Gamma(D_Z)$.

Proposition 2.1.11. *The difference $\mathcal{D}_0 - \mathcal{D}_1$ is a compact operator on the Hilbert module $L^2(Z, \Lambda_{\mathbb{C}}(Z) \otimes \mathcal{F})$ both in the smooth and in the Lipschitz case.*

Proof.

The proof of [17, Theorem 3.3] is based on the construction of an operator $T_{p,v}$, that plays the role of the pull-back of forms.

Let us take the following data:

- a submersion $p\colon M \times B^k \to N$, where B^k is the unit open disk of $\mathbb{R}^k$;
- a smooth k-form v with compact support on B^k, such that $\int_{B^k} v = 1$. Put then $\omega = p_{B^k}^*(v)$.

Then $p^*\colon L^2(N, \Lambda_{\mathbb{C}}(N) \otimes \mathcal{F}_N) \to L^2(M \times B^k, \Lambda_{\mathbb{C}}(M \times B^k) \otimes p^*\mathcal{F}_N)$ is a bounded operator and $T_{p,v}$ is defined as the operator $\xi \mapsto q_*(\omega \wedge p^*(\xi))$. Consider the following commutative diagram

$$\begin{tikzcd} & M \times B^k \arrow[dl, "q"'] \arrow[d, "t"] \arrow[dr, "p"] & \\ M & M \times N \arrow[l, "p_M"] \arrow[r, "p_N"'] & N \end{tikzcd} \tag{2.1.2}$$

where $t = id_M \times p$. We get that, for $\xi \in L^2(N, \Lambda_{\mathbb{C}}(N) \otimes \mathcal{F}_N)$

$$\begin{aligned} T_{p,v}(\xi) &= q_*(\omega \wedge p^*(\xi)) = \\ &= (p_M)_* t_* \left(\omega \wedge (t)^* p_N^*(\xi)\right) = \\ &= (p_M)_* \left(t_*\omega \wedge p_N^*(\xi)\right). \end{aligned}$$

Notice that $(p_M)_*$ is nothing else than the integration over N. Assume tht k and p are chosen so that t is a submersion. If we denote the form $t_*\omega$ on $M \times N$ with $k(y, x)$, it turns out that $T_{p,v}(\xi) = \int_N k(x, y)\xi(x)$ is an integral operator with smooth kernel and consequently a smoothing operator.

The operator Y in [17, Lemma 2.1(c)], such that $1 + T'_{p,v} \circ T_{p,v} = d_N \circ Y + Y \circ d_N$, is bounded of order -1 (see the proof of [46, Lemma 2.2] for an explicit expression of Y).

Now we can follow word by word the proof of [27, Lemma 9.14], using the conventions in [46, Section 3]. For simplicity let us consider the odd case . The perturbed signature operator is then given by

$$\mathcal{D}_t = -iU_t(d_t \circ S_t + S_t \circ d_t) \circ U_t^{-1}$$

where

$$d_t = \begin{pmatrix} d_M & tT'_{p,v} \\ 0 & -d_N \end{pmatrix}, \quad S_t = \operatorname{sign}(\tau_Z \circ L_t) \text{ and } U_t = |\tau_Z \circ L_t|^{\frac{1}{2}},$$

with

$$L_t = \begin{pmatrix} 1 + T'_{p,v} \circ T_{p,v} & (1 - it\gamma \circ Y) \circ T'_{p,v} \\ T_{p,v} \circ (1 + it\gamma \circ Y) & 1 \end{pmatrix}$$

One can easily see that $L_t = 1 + H_t$, with H_t smoothing. Moreover one has that $|\tau_Z \circ L_t| = \sqrt{L_t^* L_t} = \sqrt{1 + R_t}$, with R_t smoothing. Observe that $0 < L_t^* L_t = 1 + R_t$ implies that $R_t > 1$. It follows that $\sqrt{L_t^* L_t} - 1 = f(R_t)$, where $f(x) = \sqrt{1+x} - 1$ is holomorphic on the spectrum of R_t (-1 is a branch point for f). Since $f(0) = 0$, we have that $f(z) = az + zh(z)z$, where h is a holomorphic function.

Let us point out that if S_0 and S_1 are smoothing operators and T is a bounded operator, then $S_0 \circ T \circ S_1$ is smoothing. Then we immediately get that $F_t := |\tau_Z \circ L_t| - 1 = f(R_t)$ is smoothing.

With the same argument one can prove that $U_t = 1 + H_t$ with H_t smoothing.

By [27, Lemma A.12], $|\tau_Z \circ L_t|^{-1} = 1 + F'_t$ and $U_t = 1 + H'_t$ with F'_t and H'_t smoothing. Then one obtains that, $S_t = \tau_Z + G_t$ and $d_t = d_Z + E_t$, where G_t and E_t are smoothing operators.

Consequently one has that

$$\mathcal{D}_t = -i(1 + H_t)\left((d + E_t) \circ (\tau_Z + G_t) + (\tau_Z + G_t) \circ (d + E_t)\right) \circ (1 + H'_t)$$

is equal to $\mathcal{D} + C_f$ with C_f a compact operator.

Now we have to prove that the *Lipschitz* Hilsum-Skandalis perturbation is bounded. In the smooth case we tackled the problem geometrically, here we try with a more analytical approach. An operator of order $-n$ is a bounded operator between $H^s(Z, E)$ and $H^{s+n}(Z, E)$, the Sobolev sections of E of order s and $s + n$, for any s. An operator is regularizing if it is of order $-\infty$. Equivalently one can say that an operator T is regularizing (of order $-\infty$) if $D^n \circ T \circ D^m$ is a bounded operator on L^2-section for any $m, n \in \mathbb{Z}$.

By [13, Proposition 5.6] we know that the signature operator has compact resolvent, therefore his spectrum is a countable and discrete subset $\{\lambda_n\}_{n\in\mathbb{N}}$ of $\mathbb{R}$ such that $\lim_{n\longrightarrow\infty} \lambda_n^2 = +\infty$.

Now let $\psi \in C_0^\infty(\mathbb{R})$ be a rapidly decreasing even function such that $\psi(1) = 1$. Since ψ is even, it turns out that $\psi(d_N + d_N^*)$ preserves the degree of forms and it is a Hilbert-Schmidt operator: the proof of the first statement of [31, Prop. 5.31] works putting 'Hilbert-Schmidt' instead of 'smoothing'. Let us denote its kernel by $k(x, y) \in L^2(N \times N, \operatorname{End}(\Lambda_{\mathbb{C}}(N) \otimes \mathcal{F}_N))$.

Define the compact operator $T_f \colon L^2(N, \Lambda_{\mathbb{C}}(N) \otimes \mathcal{F}_N) \to L^2(M, \Lambda_{\mathbb{C}}(M) \otimes f^*\mathcal{F}_N)$ as the integral operator with kernel $(q \times id_N)_*(p \times id_N)^* k \in L^2(M \times N, \operatorname{Hom}(\Lambda_{\mathbb{C}}(N) \otimes \mathcal{F}_N, \Lambda_{\mathbb{C}}(M) \otimes f^*\mathcal{F}_N))$, where p and q are as in the diagram (2.1.2).

This operator satisfies the hypothesis of [17, Lemma 2.1]. Indeed, because of our choice of ψ, we have that $1 - \psi(x) = x \cdot \varphi(x)$, where φ is a rapidly decreasing odd function. Moreover $d_N^* \circ \varphi(d_N + d_N^*) = \varphi(d_N + d_N^*) \circ d_N$, since φ is odd. Then we get the following formula

$$1 - \psi(d_N + d_N^*) = d_N \circ \varphi(d_N + d_N^*) + \varphi(d_N + d_N^*) \circ d_N$$

and by construction $T'_f T_f = \psi(d_N + d_N^*)' \psi(d_N + d_N^*)$. Now it's easy to check that there exists an operator $Y \in \mathbb{B}(L^2(N, \Lambda_{\mathbb{C}}(N) \otimes \mathcal{F}_N))$ such that $Y(\operatorname{dom}(d_N)) \subset \operatorname{dom}(d_N)$ and that $1 - T_f' \circ T_f = d_N \circ Y + Y \circ d_N$:

$$\begin{aligned}
1 - T_f' \circ T_f &= 1 - \psi(d_N + d_N^*)' \circ \psi(d_N + d_N^*) = \\
&= 1 - (1 - d_N \circ \varphi(d_N + d_N^*) + \varphi(d_N + d_N^*) \circ d_N)' \cdot \\
&\cdot (1 - d_N \circ \varphi(d_N + d_N^*) + \varphi(d_N + d_N^*) \circ d_N) = \\
&= -\varphi(d_N + d_N^*)' \circ d_N \varphi(d_N + d_N^*) + \varphi(d_N + d_N^*) \circ d_N + \\
&+ \varphi(d_N + d_N^*)' \circ d_N \circ d_N \circ \varphi(d_N + d_N^*) + \varphi(d_N + d_N^*)' \circ \varphi(d_N + d_N^*) \circ d_N + \\
&+ d_N \circ \varphi(d_N + d_N^*) \circ d_N \circ \varphi(d_N + d_N^*) + d_N \circ \varphi(d_N + d_N^*)' \circ \varphi(d_N + d_N^*) \circ d_N = \\
&= d_N \circ Y + d_N \circ Y,
\end{aligned}$$

with $Y = \varphi(d_N + d_N^*) - \varphi(d_N + d_N^*)' + \varphi(d_N + d_N^*)' \circ d_N \circ \varphi(d_N + d_N^*)$.

It is easy to check that the operator T_f is a regularizing operator (and hence a compact operator), therefore the image of T_f is in the domain of the Lipschitz signature operator.

Then the boundedness of the Lipschitz Hilsum-Skandalis perturbation follows as in the smooth case.

The only thing we have to care about is the dependence of this construction on the choice of the metric on N. In particular we have to check that $\psi(d_N + d_N^*)$ is Hilbert-Schmidt no matter which metric we use to take the adjoint.

If we have two different metrics g_0 and g_1 on N, then by [13, Lemma 5.1] we can complete the $Lip(N)$-module $Lip(N, \Lambda_{\mathbb{C}}(N) \otimes \mathcal{F}_N)$ with respect the two metrics and we obtain two isomorphic $C(N)$-Hilbert modules with compatible metrics:

$$K^{-1}||\cdot||_1 \leq ||\cdot||_0 \leq K||\cdot||_1 \ \exists K \in \mathbb{R}^+ \setminus \{0\}.$$

Then by the Minmax Theorem $|\lambda_n^0| \leq K^2|\lambda_n^1|$, where for any $n \in \mathbb{N}$, λ_n^i is the n-th eigenvalue of $d + d_i^*$.

So it is easy to check that if ψ is a rapidly decreasing function on the spectrum of $d + d_0^*$, it is rapidly decreasing on the spectrum of $d + d_1^*$ too. Therefore $\psi(d + d^*)$ is Hilbert-Schmidt independently of the metric we choice.

$\square$

Definition 2.1.12. Let $f\colon M \to N$ be a homotopy equivalence and $Z = M \cup -N$. Denote by $\mathcal{C}_f$ the perturbation of $\mathcal{D}_Z$ arising in the previous remark and call it a trivializing perturbation. Note that it depends on the homotopy equivalence f.

We recall from [29] that $L^2(Z, \Lambda_{\mathbb{C}}(Z) \otimes \mathcal{F})$ is a $C_r^*(\Gamma)$-Hilbert module and that there is an isomorphism of C*-algebras

$$\mathbb{K}(L^2(Z, \Lambda_{\mathbb{C}}(Z) \otimes \mathcal{F})) \simeq C^*(\widetilde{Z})^\Gamma.$$

By [21, Proposition 2.1], the above isomorphism gives an isomorphism at level of multiplier algebras

$$\mathbb{B}(L^2(Z, \Lambda_{\mathbb{C}}(Z) \otimes \mathcal{F})) \simeq \mathcal{M}(C^*(\widetilde{Z})^\Gamma). \tag{2.1.3}$$

This isomorphism is given by the map L_π defined in [29, Section 2.2.1]. Hence we can go from the Mishchenko bundle setting to the covering one. From now on C_f will be the element in $\mathcal{M}(C^*(\widetilde{Z})^\Gamma)$ associated to $\mathcal{C}_f \in \mathbb{B}(L^2(Z, \Lambda_{\mathbb{C}}(Z) \otimes \mathcal{F}))$ through the map L_π. Moreover $\widetilde{D}_Z$ will indicate the operator on the covering induced by the signature without coefficients in the Mishchenko bundle.

Remark 2.1.13. Consider a chopping function $\psi \in C_b(\mathbb{R})$ with compactly supported Fourier transform. Thanks to Theorem 2.1.7 we can prove that the functional calculus through ψ of the operator $\widetilde{D}_Z$ is an operator of finite propagation. The pseudolocality of $\widetilde{D}_Z$ comes from [13, 6.1]. Hence $\psi(\widetilde{D}_Z) \in D^*(\widetilde{Z})^\Gamma$.

Proposition 2.1.14. *The difference between $\psi(\widetilde{D}_Z)$ and $\psi(\widetilde{D}_Z + C_f)$ belongs to $C^*(\widetilde{Z})^\Gamma$.*

Proof. Moving to the Mishchenko bundle setting through (1.2.23), we should prove that the difference $\psi(\mathcal{D}_Z) - \psi(\mathcal{D}_Z + \mathcal{C}_f)$ belongs to $\mathbb{K}(L^2(Z, \Lambda_{\mathbb{C}}(Z) \otimes \mathcal{F}))$. If $\psi_1(t) = t(1+t^2)^{-\frac{1}{2}}$, by [2, Proposition 2.2] we have that $[\psi_1(\mathcal{D}_Z), a]$ belongs to the algebra of compact C*-module operators. So if we consider the matrices $\begin{bmatrix} \mathcal{D}_Z & 0 \\ 0 & \mathcal{D}_Z + \mathcal{C}_f \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, their bracket consists in $\begin{bmatrix} 0 & -\mathcal{C}_f \\ \mathcal{C}_f & 0 \end{bmatrix}$, that is known to be bounded. Then, after applying the functional calculus through ψ_1, we deduce that the matrix components in the bracket

$$\pm(\psi_1(\mathcal{D}_Z) - \psi_1(\mathcal{D}_Z + \mathcal{C}_f))$$

are compact.

Now notice that two different chopping functions differ by a function in $C_0(\mathbb{R})$. Taking into account the correspondence stated in (1.2.23), we have that the resolvent of $\mathcal{D}_Z$, given by $(i + \mathcal{D}_Z)^{-1}$, is compact (see [13, Proposition 5.6]) and since $\phi(t) = (i+t)^{-1}$ generates $C_0(\mathbb{R})$, the functional calculus of D_Z through a function in $C_0(\mathbb{R})$ gives a compact operator. Then if ψ' is any chopping function, it turns out that

$$\psi(\mathcal{D}_Z) - \psi(\mathcal{D}_Z + \mathcal{C}_f) = \psi_1(\mathcal{D}_Z) - \psi_1(\mathcal{D}_Z + \mathcal{C}_f) + \text{compacts operators}$$

and the right-hand side term is compact.

□

Corollary 2.1.15. *The operator $\chi(\tilde{D}_Z + C_f)$, with $\chi(x) = \frac{x}{|x|}$, is a bounded involution in $D^*(\widetilde{Z})^\Gamma$.*

Thanks to Corollary 2.1.15 we can define a class by setting

$$\varrho(\widetilde{D}_Z + C_f) = \left[\frac{1}{2}(1 + \chi(\widetilde{D}_Z + C_f))\right] \in K_0(D^*(\widetilde{Z})^\Gamma).$$

Now consider the map $\varphi\colon Z \to N$ such that $\varphi_{|M} = f$ and $\varphi_{|-N} = -\mathrm{Id}_N$; we can clearly see that φ is covered by a Γ-equivariant map $\widetilde{\varphi}\colon \widetilde{Z} \to \widetilde{N}$.

Definition 2.1.16. Let $f\colon M \to N$ be a homotopy equivalence between two compact oriented Lipschitz manifolds. Consider $Z = M \cup -N$ and its covering $\widetilde{Z}$ associated, as above, to a classifying map $u\colon Z \to B\Gamma$. Let $\widetilde{D}_Z$ be the Lipschitz signature operator and let C_f be the trivializing perturbation associated to f. We define

$$\varrho(f\colon M \to N) := \widetilde{\varphi}_* \left[\frac{1}{2}(1 + \chi(\widetilde{D}_Z + C_f))\right] \in K_0(D^*(\widetilde{N})^\Gamma)$$

and

$$\varrho_\Gamma(f\colon M \to N) = u_* \varrho(f\colon M \to N) \in K_0(D^*_\Gamma).$$

Proposition 2.1.17. *The ϱ-class does not depend on the choice of the Lipschitz structure.*

Proof. The second part of Theorem 2.1.4 can be formulated as follows: let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two different Lipschitz structures on Z, then there exists a bi-Lipschitz homeomorphism $\phi\colon Z \to Z$, isotopic to the identity through a path ϕ^t and such that $\phi^*(\mathcal{L}_2) = \mathcal{L}_1$, where $\phi^*\colon C(Z) \to C(Z)$ is the induced *-homomorphism. Because of the functoriality of the Teleman's construction we know that $\phi_*(\varrho_1) = \varrho_2$, where ϱ_1 and ϱ_2 are the ϱ-invariants associated to two different Lipschitz structures. The isotopy ϕ^t induces a paths of *-isomorphisms $\phi^t_*\colon D^*(\tilde{Z})^\Gamma \to D^*(\tilde{Z})^\Gamma$. Then $\phi^t_*(\varrho_1)$ gives a homotopy between ϱ_2 and ϱ_1. □

Perturbed signature operator on manifolds with cylindrical ends

In this section we check that the construction we made of ϱ and ϱ_Γ are well defined on the structure set $\mathcal{S}^{TOP}(N)$.

For this purpose we will use results in [46, 28, 29], where the authors have developed the theory in the smooth setting. Their methods are rather abstract and they also hold in the Lipschitz context.

In order to develop the theory for manifolds with cylindrical ends, we are going to use the same notations as [29, 2.19].

The geometrical setting is the following: let $f\colon M \to N$ and $f'\colon M' \to N$ be two topological structures for N; let W be a cobordism between $\partial_0 W = M$ and $\partial_1 W = M'$ and let W_∞ be the manifold with the infinite semi-cylinder $\partial W \times \mathbb{R}_{\leq 0}$ attached to the boundary; let $V = N \times [0,1]$ and V_∞ be the complete cylinders with base $\partial V = N$; there is a homotopy equivalence $F\colon W_\infty \to V_\infty$ which has the product form $F_\partial \times \mathrm{id}_{\mathbb{R}_{\leq 0}}$ on the cylindrical ends, where $F_{\partial_0} = f\colon M \to N$ and $F_{\partial_1} = f'\colon M' \to N$, both of them being homotopy equivalences.

Thanks to the results in [14] we have a well defined Lipschitz signature complex on $X = W_\infty \cup -V_\infty$. Notice that $\partial_0 X = Z$ and $\partial_1 X = Z'$.

First of all we need a generalization of Theorem 2.1.9 for manifolds with cylindrical ends. This result is given by [46, Proposition 8.1], where a perturbation is associated of the signature operator to the homotopy equivalence F. Such a perturbation makes the operator invertible, like in the usual case.

Remark 2.1.18. Like the case presented in Theorem 2.1.9, the generalization developed in [46, Proposition 8.1] is still valid in the Lipschitz setting.

The goal of this section is to check that the ϱ-class is well defined on the h-cobordism classes: as pointed out in [29, Proposition 4.7], this is obtained from the combination of [46, Theorem 8.4] and [29, Corollary 3.3].

In [46, Theorem 8.4] all constructions work in the Lipschitz framework, where we do not consider the parameter ε. Wahl builds a perturbation of the signature operator $\mathcal{C}^{\mathrm{cyl}}_F$, supported on the cylindrical ends, from the perturbations on Z and Z'; hence she constructs a homotopy of operators between $\mathcal{D}_X + \mathcal{C}^{\mathrm{cyl}}_F$ and an other operator that, thanks to the Bunke's relative index theorem, has vanishing index.

For the proof of the equality we just mentioned, the only point that is not obvious in the Lipschitz case is the one concerning the use of the relative index theorem proved in [3], since what remains of the proof uses abstract theory of unbounded operators and spectral flow methods.

It is worth formulating Bunke's Theorem in the Lipschitz case and giving a sketch of its proof.

Bunke's relative index theorem for Lipschitz manifolds

The idea of the theorem is the following: let X be a manifold, let $E \to X$ be a bundle and D a Fredholm operator on the sections of this bundle; if there exists a hypersurface Y in X such that the operator is invertible near Y, we can cut the manifold (and the bundle) along Y and we can paste a semicylinder to the boundary of both parts obtained after cutting, extending the bundle and the operator along the semicylinder. Then we obtain an operator whose index equals the index of the original operator.

More precisely we are considering the following data: the Lipschitz manifold X we have defined in the previous subsection, the Hilbert module $L^2(X, \Lambda_{\mathbb{C}}(X) \otimes \mathcal{F})$ of L^2-forms on X twisted by the Mishchenko bundle, that we are going to denote by $\mathcal{H}^0$; a regular operator G that is the twisted Lipschitz signature operator, possibly perturbed by a bounded operator; we suppose that there is a Lipschitz function with compact support $f \geq 0$ and $(G^2 + f)^{-1} \in \mathbb{B}(\mathcal{H}^0, \mathcal{H}^2)$ (here $\mathcal{H}^2$ is the maximal domain of the square of the signature operator).

Definition 2.1.19. Let $\mathrm{Lip}_K(X)$ be the set of bounded Lipschitz functions h such that for all $\varepsilon > 0$ there exists a compact $C \subset X$, with $||dh_{|X \setminus C}||_{L^\infty} < \varepsilon$. Let us call $C_K(X)$ the closure of $\mathrm{Lip}_K(X)$ in the sup-norm.

For the benefit of the reader, we recall the theorem stated in the Lipschitz setting. Let $E_i \to X_i$, $i = 1, 2$, be the two $C^*(\Gamma)$-C* bundles $\Lambda_{\mathbb{C}}(X_i) \otimes \mathcal{F}_i$, with operator G_i, associated to them as above. Let $W_i \cup_{Y_i} V_i$ be partitions of X_i where Y_i are compact hypersurfaces. Assume that there is a commutative diagram of isomorphisms of all structures

$$\begin{tikzcd}
\Psi\colon & E_{1|U(Y_1)} \arrow[r] \arrow[d] & E_{2|U(Y_2)} \arrow[d] \\
\psi\colon & U(Y_1) \arrow[r] & U(Y_2) \\
\psi_{|Y_1} & :Y_1 \arrow[r] \arrow[u] & Y_2 \arrow[u]
\end{tikzcd}$$

where the $U(Y_i)$ are tubular neighbourhoods of Y_i, $i = 1, 2$. We cut X_i at Y_i, glue the pieces together interchanging the boundary components and obtain $X_3 := W_1 \cup_Y V_2$ and $X_4 := W_2 \cup_Y V_1$. Moreover, we glue the bundles using Ψ, which yields A-C* bundles $E_3 \to X_3$ and $E_4 \to X_4$ and we assume that G_i, $i = 3, 4$ are again invertible at infinity. We define $[X_i]$ as the class $[\mathcal{H}_i^0, \frac{G_i}{G_i^2+f}] \in KK(C_K(X_i), C^*(\Gamma))$. The algebra $C_K(X)$ is unital. Hence, there is an embedding $i\colon \mathbb{C} \to C_K(X)$ and an induced map

$$i^*\colon KK(C_K(X), C_r^*(\Gamma)) \to KK(\mathbb{C}), C_r^*(\Gamma)).$$

Set $\{X_i\} := i^*[X_i] \in KK(\mathbb{C}, C_r^*(\Gamma))$ for $i = 1, \ldots, 4$.

Theorem 2.1.20 ([3])**.**

$$\{X_l\} + \{X_2\} - \{X_3\} - \{X_4\} = 0.$$

Here are two facts:

- thanks to [39, Theorem 7.1] we have the following Rellich-type result: the inclusion $\mathcal{H}^2 \hookrightarrow \mathcal{H}^0$ is compact;

- for any f Lipschitz function compactly supported on X, the multiplication operator $f\colon \mathcal{H}^2 \to \mathcal{H}^0$ is compact. And this also holds for the Clifford multiplication by $\mathrm{grad}(f)$, the gradient of f.

Let $R(\lambda)$ be the bounded operator $(G^2+f+\lambda)^{-1} \in \mathbb{B}(\mathcal{H}^0, \mathcal{H}^2)$, for $\lambda \geq 0$; because of the Rellich-type result, we know that $R(\lambda)$ defines a compact operator in $\mathbb{B}(\mathcal{H}^0)$ and that there is a positive constant C such that $||R(\lambda)|| \leq (C+\lambda)^{-1}$.

Lemma 2.1.21. *The integral*

$$F = \frac{G}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda$$

is convergent and defines an operator in $\mathbb{B}(\mathcal{H}^0)$.

Lemma 2.1.22. *The operator* $[D, R(\lambda)]$ *extends to a bounded operator that coincides with*

$$-R(\lambda)\mathrm{grad}(f)R(\lambda).$$

Moreover such an operator is compact.

Proof. See [3, Lemma 1.7 and Lemma 1.8]. □

Lemma 2.1.23. *For any* $h \in C_K(X)$, $h(F^2 - I) \in \mathbb{K}(\mathcal{H}^0)$.

Proof. We have

$$\begin{aligned}
&\left(\frac{G}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda\right)\left(\frac{G}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda\right) = \\
&\frac{G^2}{\pi^2}\left(\int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda\right)^2 + \frac{G}{\pi}\left[\int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda, \frac{G}{\pi}\right]\int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda = \\
&\frac{G^2}{\pi^2}\left(G^2+f\right)^{-1} - \frac{G}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda)\mathrm{grad}(f)R(\lambda) d\lambda \int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda \sim \\
&\frac{G^2}{\pi^2}\left(G^2+f\right)^{-1},
\end{aligned}$$

where in the third step we have used Lemma 2.1.22. Here $\sim$ means "equal modulo compacts". Hence

$$h(F^2 - I) \sim h\frac{f}{G^2+f}$$

that is compact, since multiplication by f is. □

Lemma 2.1.24. *For any* $h \in C_K(X)$, $[F, h] \in \mathbb{K}(\mathcal{H}^0)$.

Proof. Since we chose as G as a perturbation of the signature operator D and since the perturbation becomes compact under bounded transform, we have that

$$\begin{aligned}
[F,h] \sim &\left[\frac{D}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda, h\right] = \\
&\frac{D}{\pi}\left[\int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda, h\right] + \left[\frac{D}{\pi}, h\right]\int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda = \\
&\frac{D}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}} \left[R(\lambda), h\right] d\lambda + \mathrm{grad}(h)\int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda \sim \\
&\frac{D}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}} \left[R(\lambda), h\right] d\lambda.
\end{aligned}$$

The term in the last line is compact as in the proof of [3, Lemma 1.12]. □

Lemma 2.1.25. *Let f, f_1 be two positive and compactly supported Lipschitz functions such that $(G^2+f)^{-1}, (G^2+f_1)^{-1} \in \mathbb{B}(\mathcal{H}^0, \mathcal{H}^2)$. Then the two associated operators F, F_1 differ each other by a compact operator.*

Proof. See [3, Lemma 1.10]. □

The lemmas we presented yield to the following

Proposition 2.1.26. *The pair $(\mathcal{H}^0, F)$ defines a Kasparov $(C_K(X), C_r^*(\Gamma))$-module and its class in $KK(C_K(X), C_r^*(\Gamma))$ does not depend on the choice of f.*

After checking this technical part, the proof of Theorem 2.1.20 is completely abstract and it follows in the Lipschitz case as in the smooth one.

Now we treat another fundamental result proved by Piazza ans Schick: the delocalized Atiyah-Patodi-Singer index theorem. As noticed in [29, Section 5.2], the proof of the delocalized APS index theorem is based on abstract functional analysis for unbounded operators on Hilbert spaces. The reader can check that it works almost completely in the same way in the Lipschitz context and we will not give all details again.

The only proof to be modified is [29, Prop 5.33]. Being the context and the notation understood, we state the following Proposition.

Proposition 2.1.27. *Given a Dirac type operator D, the operator $(1+D^2)^{-1} \colon L^2 \to H^2$ is a norm limit of finite propagation operators $G_n \colon L^2 \to H^2$ with the property that $[\varphi, G_n] \colon L^2 \to H^2$ is compact for any compactly supported continuous function on M.*

Proof. It is an easy computation to show that

$$\frac{1}{1+x^2} = \int_{-\infty}^{+\infty} \frac{e^{-|t|}}{2} e^{-itx} dt.$$

Let $f \colon \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that

- $0 \le f \le 1$,
- $f = 1$ on a neighbourhood of 0,
- f has compact support.

Define $G_n = \int_{-\infty}^{+\infty} f\left(\frac{t}{n}\right) \frac{e^{-|t|}}{2} e^{-itD} dt$.

Finite propagation: since $f\left(\frac{t}{n}\right) \frac{e^{-|t|}}{2}$ has compact support, G_n has finite propagation speed.

Pseudolocality: thanks to the above formula, $(1+D^2)^{-1} - G_n = \int_{-\infty}^{+\infty} (1-f\left(\frac{t}{n}\right)) \frac{e^{-|t|}}{2} e^{-itD} dt$. Notice that $(1-f\left(\frac{t}{n}\right)) \frac{e^{-|t|}}{2}$ is C^∞ and moreover it is a rapidly decreasing function on the spectrum of D. By [31, Prop 5.31], $(1+D^2)^{-1} - G_n$ is a bounded operator from H^m to H^k for any $m, k \in \mathbb{N}$, hence G_n is pseudolocal because so is $(1+D^2)^{-1}$. Indeed, using Jacobi identities for commutators and the fact that $[\varphi, D] = c(d\varphi)$, $[\varphi, (1+D^2)^{-1}] = (1+D^2)^{-1} c(d\phi) D (1+D^2)^{-1} + (1+D^2)^{-1} D c(d\phi)(1+D^2)^{-1})$ is compact, because the Clifford multiplication $c(d\varphi)$ is compact too.

In fact we need less: it is sufficient to show that $(1+D^2)^{-1} - G_n$ is a bounded operator from L^2 to H^3 and then, by Rellich Theorem, the commutator $[\varphi, (1+D^2)^{-1} - G_n]$ turns out to be a compact operator from L^2 to H^2. To prove this, we only need that the third derivative of $(1-f\left(\frac{t}{n}\right))\frac{e^{-|t|}}{2}$ has a bounded supremum norm (less than being rapidly decreasing).

In fact, under these hypotheses and by the properties of the Fourier transform, we get that

$$||(1+D^2)^{-1}-G_n||_{L^2\to H^3}=\left|\left|\left((1-f\left(\frac{t}{n}\right))\frac{e^{-|t|}}{2}\right)'''\right|\right|_\infty \tag{2.1.4}$$

is bounded. Moreover $\left((1-f\left(\frac{t}{n}\right))\frac{e^{-|t|}}{2}\right)'''$ is equal to

$$-\frac{1}{n^3}f'''\left(\frac{t}{n}\right)e^{-|t|}+\frac{3}{n^2}f''\left(\frac{t}{n}\right)|t|e^{-|t|}-\frac{3}{n}f'\left(\frac{t}{n}\right)e^{-|t|}-(1-f\left(\frac{t}{n}\right))|t|^3e^{-|t|}$$

that clearly goes to zero as n goes to infinity. This also holds in the Lipschitz case. □

Now we can state the delocalized Atiyah-Patodi-Singer index theorem, that holds in the Lipschitz context too.

Theorem 2.1.28 ([29])**.** *If $i\colon C^*(\widetilde{X})^\Gamma \hookrightarrow D^*(\widetilde{X})^\Gamma$ is the inclusion and $j_*\colon D^*(\partial\widetilde{X})^\Gamma \to D^*(\widetilde{X})^\Gamma$ is the map induced by the inclusion $j\colon \partial\widetilde{X}\hookrightarrow\widetilde{X}$, we have*

$$i_*(\mathrm{Ind}_\Gamma(D_X+C_F^{\mathrm{cyl}}))=j_*(\varrho(D_{\partial X}+C_{F_\partial}))\in K_0(D^*(\widetilde{X})^\Gamma).$$

Using the functoriality of the classifying map $u\circ F\cup u\colon \widetilde{X}\to E\Gamma$ and the map $\Phi:=\pi_1\circ(F\cup -\mathrm{id}_{V\times[0,1]})$ we obtain

$$i_*\widetilde{\Phi}_*(\mathrm{Ind}_\Gamma(D_X+C_F^{\mathrm{cyl}}))=\varrho(F_\partial)\in K_0(D^*(\widetilde{V})^\Gamma)$$

$$i_*u_*\widetilde{\Phi}_*(\mathrm{Ind}_\Gamma(D_X+C_F^{\mathrm{cyl}}))=\varrho_\Gamma(F_\partial)\in K_0(D^*_\Gamma).$$

Observe that ϱ_Γ is additive on disjoint unions like $\partial X=Z\cup -Z'$ and in particular that

$$\varrho(F_\partial)=\varrho(f)-\varrho(f').$$

Combining this with [46, Theorem 8.4], we finally have that

$$\varrho(f)=\varrho(f'),$$

and similarly for ϱ_Γ, hence they are well defined on $\mathcal{S}^{TOP}(N)$.

2.1.4 Mapping surgery to analysis: the odd dimensional case

Here we refer the reader to [29, Section 4] for definitions. We state the main theorem.

Theorem 2.1.29. *Let N be an n-dimensional closed oriented topological manifold with fundamental group Γ. Assume that $n\geq 5$ is odd. Then there is a commutative diagram with exact rows*

$$\begin{tikzcd}
L_{n+1}(\mathbb{Z}\Gamma) \arrow[r] \arrow[d, "\mathrm{Ind}_\Gamma"] & \mathcal{S}^{TOP}(N) \arrow[r] \arrow[d, "\varrho"] & \mathcal{N}^{TOP}(N) \arrow[r] \arrow[d, "\beta"] & L_n(\mathbb{Z}\Gamma) \arrow[d, "\mathrm{Ind}_\Gamma"] \\
K_{n+1}(C^*_r(\Gamma)) \arrow[r] & K_{n+1}(D^*(\widetilde{N})^\Gamma) \arrow[r] & K_n(N) \arrow[r] & K_n(C^*_r(\Gamma))
\end{tikzcd}$$

and through the classifying map $u\colon N \to B\Gamma$ of the universal cover $\widetilde{N}$ of N, we have the analogous commutative diagram that involves the universal Higson-Roe exact sequence

$$\begin{tikzcd}
L_{n+1}(\mathbb{Z}\Gamma) \arrow[r] \arrow[d, "\mathrm{Ind}_\Gamma"] & \mathcal{S}^{TOP}(N) \arrow[r] \arrow[d, "\varrho_\Gamma"] & \mathcal{N}^{TOP}(N) \arrow[r] \arrow[d, "\beta_\Gamma"] & L_n(\mathbb{Z}\Gamma) \arrow[d, "\mathrm{Ind}_\Gamma"] \\
K_{n+1}(C^*_r(\Gamma)) \arrow[r] & K_{n+1}(D^*_\Gamma) \arrow[r] & K_n(B\Gamma) \arrow[r] & K_n(C^*_r(\Gamma))
\end{tikzcd}$$

Remark 2.1.30. Let us recall the fact that, despite $\mathcal{S}^{TOP}(N)$ has a group structure, we do not deal with it and the top row is considered just as a sequence of sets, as in the smooth case.

Thanks to the results in the previous section we can check that the results in [29, Sections 4.2 and 4.3] hold in the category of topological manifolds instead of the one of smooth manifolds: all proofs are still valid in the Lipschitz context. Thanks to the work by C.Whal [46, Theorem 9.1], that can be combined with Theorem 2.1.7, the first vertical arrow is well defined in the Lipschitz setting. The second one is also well defined for the previous section. Concerning the third one there are no significant problems.

The same method in Proposition2.1.17 applies to the class of the signature and its index class, then all vertical arrows do not depend on the chosen Lipschitz structure.

One has to check the commutativity of the three squares.

- The third square is obviously commutative: let $(f\colon M \to N)$ be a normal map in $\mathcal{N}^{TOP}(N)$, it is sent horizontally to the same map forgetting the fact that it is normal and then through Ind_Γ to the difference $\mathrm{Ind}_\Gamma(D_M) - \mathrm{Ind}_\Gamma(D_N)$; on the other hand $\beta(f\colon M \to N) = f_*[D_M] - [D_N]$, that gives, through the analytic assembly map, the index class just founded.

- Let us study the second square: let $(f\colon M \to N)$ be a structure in $\mathcal{S}^{TOP}(N)$, it goes to the same map forgetting the fact that f is a homotopy equivalence; the ϱ-class $\varrho(f)$, as in Definition 2.1.16, is the push-forward through $\tilde{\varphi}$ of the class $\left[\frac{1}{2}(1 + \chi(\widetilde{D}_Z + C_f))\right] \in K_0(D^*(\widetilde{Z})^\Gamma)$; this goes, in the horizontal direction, to the class in $K_0(D^*(\widetilde{Z})^\Gamma)/C^*(\widetilde{Z})^\Gamma$ that represents, by Pashcke duality, the K-homology class of the signature operator of Z; then by functoriality of $\tilde{\varphi}_*$ and the fact that $\beta(f\colon M \to N) = f_*[D_M] - [D_N]$, we obtain the commutativity of the second square.

- For the first square commutativity we refer the reader to [29, 4.10]. Let $a \in L_{n+1}(\mathbb{Z}\Gamma)$ and let $(f\colon M \to N)$ be a structure in $\mathcal{S}^{TOP}(N)$. The commutativity of the first square means that the following equation holds:

$$i_*(\mathrm{Ind}_\Gamma(a)) = \varrho(a[f\colon M \to N]) - \varrho([f\colon M \to N]) \in K_0(D^*(\widetilde{Z})^\Gamma);$$

  this is proved identifying the right hand side with the class predicted by the APS delocalized index theorem, that, as we know, holds in the Lipschitz case too. The proof is based on an addition formula, as in [46, 7.1], and algebraic identifications of ϱ-classes, that the reader can check still holding, word-for-word, in the Lipschitz case.

2.1.5 Products

Let M and N be two Cartesian products with a common factor, namely $M = M_1 \times M_2$ and $N = N_1 \times M_2$, and let $f_1 \colon M_1 \to N_1$ be a homotopy equivalence. Therefore $f = f_1 \times \mathrm{id} \colon M \to N$ is a homotopy equivalence.

Observe that the signature operator on $Z = M \cup (-N)$ has this form: $D_Z = D_1 \hat{\otimes} 1 + 1 \hat{\otimes} D_2$, i.e. the graded tensor product of the signature operator D_1 on $M_1 \cup (-N_1)$ and the signature operator D_2 on M_2.

As before we construct from f a bounded operator C_f that produces an invertible perturbation $D_Z + C_f$. Notice that, from the construction in [17] and as it has been pointed out in [46, (6.1)], the operator C_f has the form $C_{f_1} \hat{\otimes} 1$, where all grading operators are understood in the graded tensor product. We have

$$D_Z + C_f = (D_1 + C_{f_1}) \hat{\otimes} 1 + 1 \hat{\otimes} D_2$$

so we can associate an invertible perturbation of D_Z to an invertible perturbation of D_1.

We would like to state a product formula involving the ϱ-class invariant of the first factor and the K-homology class of the second one. To this end it will useful to pass from the group $K_*(D^*(\widetilde{X})^\Gamma)$ to the realization $K_*(\mu_X^\Gamma)$ and define a product in the fol lowing way.

Let

$$\xi = [H_1, \phi_1, T_1, \mathcal{E}_1(t), \psi_1(t), S_1(t)] \in K_*(\mu_{X_1}^{\Gamma_1})$$

and let

$$\lambda = [H_2, \phi_2, T_2, \mathcal{E}_2(t), \psi_2(t), S_2(t)] \in \hat{K}_i^{\Gamma_2}(X_2),$$

where X_1 and X_2 are two proper and cocompact spaces with respect to Γ_1 and Γ_2 respectively. Let $(H_1 \hat{\otimes} H_2, \phi_1 \hat{\otimes} \phi_2, T)$ be an exterior Kasparov product of (H_1, ϕ_1, T_1) and (H_2, ϕ_2, T_2). Let $(\mathcal{E}(t), \psi(t), S(t))$ be the restriction to the diagonal of the a Kasparov product of $(\mathcal{E}_1(t), \psi_1(t), S_1(t))$ and $(\mathcal{E}_2(t), \psi_2(t), S_2(t))$ (that is a Kasparov $\mathbb{C}$-A-bimodule, where A is equal to the algebra $C_r^*(\Gamma_1) \otimes C_r^*(\Gamma_2) \otimes C_0([0,1]^2 \setminus \{1\} \times [0,1])$).

Definition 2.1.31. We define a product

$$K_j(\mu_{X_1}^{\Gamma_1}) \times \hat{K}_i^{\Gamma_2}(X_2) \to K_{i+j}(\mu_{X_1 \times X_2}^{\Gamma_1 \times \Gamma_2})$$

that associates to $\xi \times \lambda$ the class

$$\xi \boxtimes \lambda := [H_1 \hat{\otimes} H_2, \phi_1 \hat{\otimes} \phi_2, T, \mathcal{E}(t), \psi(t), S(t)]$$

where the entries are as described above. The product is compatible with homotopies in both factors and so it is well defined.

Remark 2.1.32. A similar product is defined in an obvious way on $KK^{j-1}(\mathbb{C}, C_r^*(\Gamma_1) \otimes C_0(0,1))$ and $\hat{K}_j^{\Gamma_1}(C(X_1), \mathbb{C})$. It is natural in the sense that the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & KK^j(\mathbb{C}, A) \times \hat{K}_i^{\Gamma_2}(X_2) \arrow[r] \arrow[d] & K_j(\mu_{X_1}^{\Gamma_1}) \times \hat{K}_i^{\Gamma_2}(X_2) \arrow[r] \arrow[d] & \hat{K}_j^{\Gamma}(X_1) \times \hat{K}_i^{\Gamma_2}(X_2) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & KK^{j+i}(\mathbb{C}, B) \arrow[r] & K_{i+j}(\mu_{X_1 \times X_2}^{\Gamma_1 \times \Gamma_2}) \arrow[r] & \hat{K}_{j+i}^{\Gamma_1 \times \Gamma_2}(X_1 \times X_2)) \arrow[r] & \cdots
\end{tikzcd}$$

is commutative. Here $A = C^*(\widetilde{X}_1)^{\Gamma_1} \otimes C_0(0,1)$ and $B = C^*(\widetilde{X}_1 \times \widetilde{X}_2)^{\Gamma_1 \times \Gamma_2} \otimes C_0(0,1)$.

Lemma 2.1.33. *Let Y, X, Z be three space and assume that a group Γ_1 acts properly and cocompactly on Y and X and Γ_2 acts properly and cocompactly on Z. Let $f\colon Y \to X$ be a Γ-equivariant continuous map. Then the following diagram*

$$\begin{tikzcd} K_j(\mu_Y^{\Gamma_1}) \times \hat{K}_i^{\Gamma_2}(Z) \arrow[r, "f_*\times id"] \arrow[d] & K_j(\mu_X^{\Gamma_1}) \times \hat{K}_i^{\Gamma_2}(Z) \arrow[d] \\ K_{i+j}(\mu_{Y\times Z}^{\Gamma_1\times\Gamma_2}) \arrow[r, "(f\times id_Z)_*"] & K_{i+j}(\mu_{X\times Z}^{\Gamma_1\times\Gamma_2}) \end{tikzcd}$$

where the vertical arrows are given by 2.1.31, is commutative.

Proof. This is straightforward since $(\phi_1 \otimes \phi_2) \circ (f \times id_Z)^* = (\phi_1 \circ f^*) \otimes \phi_2$. □

2.1.6 Stability of ϱ classes

The signature operator

Let $f\colon M \to N$ be a structure in $\mathcal{S}^{TOP}(N)$ and $\varrho(f)$ be the associated ϱ-class in $K_*(D^*(\widetilde{Z})^\Gamma)$. Let us see the different realisations of this class with respect to the different models of the analytical structure set.

- In $K_0(D^*(\widetilde{Z})^\Gamma)$ we have the element $\left[\frac{1}{2}(1+\chi(\widetilde{D}_Z + C_f))\right]$.
- In $K_1(\mu_{\widetilde{Z}}^\Gamma)$ this element turns into

$$\left[H, \phi, F, \mathcal{E}, 1 \otimes \widetilde{\phi}, G\right],$$

  where $F = \left(\chi(\widetilde{D}_Z + C_f)\right)$, $\mathcal{E} = E_Z \otimes_{C_0(Z)\rtimes\Gamma} H \rtimes \Gamma$ and G is the path constantly equal to any $\widetilde{F}$-connection.
- Finally observe that the image of the last element through the natural map $K_1(\mu_{\widetilde{Z}}^\Gamma) \to K_1(\mu^\Gamma)$ is the image of $\varrho_\Gamma \in K_0(D_\Gamma^*)$ by means of the obvious isomorphism.

Proposition 2.1.34. *Let M_1 and N_1 be two n-dimensional Lipschitz manifolds with n odd and let M_2 be an m-dimensional Lipschitz manifold with m even. Let M be $M_1 \times M_2$, let N be $N_1 \times M_2$ and let $f_1\colon N_1 \to M_1$ be a homotopy equivalence. Let Γ_i be the fundamental groups of M_i, with $i = 1, 2$. We get that*

$$\varrho(f_1 \times \mathrm{id}_{M_2}) = \varrho(f_1) \boxtimes [D_2] \in K_1(\mu_{\widetilde{M}_1\times\widetilde{M}_2}^{\Gamma_1\times\Gamma_2})$$

and the same holds for ϱ_Γ.

Proof. Let $Z_1 = M_1 \cup N_1$ and $Z_2 = M_1 \times M_2 \cup N_1 \times M_2$.

The class $\varrho(f_1)$ is represented in $S_1^{\Gamma_1}(Z_1)$ by the cycle

$$\left[H_1, \phi_1, F_1, \mathcal{E}_1, 1 \otimes \widetilde{\phi}_1, G_1\right],$$

where $F_1 = \chi(\widetilde{D}_{Z_1} + C_{f_1})$.

The class $[D_2] \in \hat{K}_1^{\Gamma_2}(\widetilde{M}_2)$ is represented by

$$\left[H_2, \phi_2, F_2, \mathcal{E}_2, 1 \otimes \widetilde{\phi}_2, G_2\right],$$

where $F_2 = \psi(\widetilde{D}_{M_2})$.

Finally the class $\varrho(f_1 \times \mathrm{id}_{M_2}) \in K_1(\mu^{\Gamma_1\times\Gamma_2}_{\widetilde{Z}\times\widetilde{M}_2})$ is represented by

$$\left[H_1 \otimes H_2, \phi_1 \otimes \phi_2, F, \mathcal{E}_1 \otimes \mathcal{E}_2, 1 \otimes \widetilde{\phi}_1 \otimes \widetilde{\phi}_2, G\right],$$

where $F = \chi(\widetilde{D}_{Z_1} \otimes 1 + 1 \otimes \widetilde{D}_{M_1} + C_{f_1\times \mathrm{id}_{M_2}})$.

We have to prove the identity of the last class mentioned with the product $\varrho(f)\boxtimes[D_2] \in S_1^{\Gamma_1\times\Gamma_2}(Z_1 \times M_2)$ given by

$$\left[H_1 \otimes H_2, \phi_1 \otimes \phi_2, F', \mathcal{E}_1 \otimes \mathcal{E}_2, 1 \otimes \widetilde{\phi}_1 \otimes \widetilde{\phi}_2, G'\right],$$

where $F' = \chi(\widetilde{D}_{Z_1} + C_f) \otimes 1 + 1 \otimes \psi(\widetilde{D}_{M_2})$.

Since $\widetilde{D}_{Z_1} \otimes 1 + 1 \otimes \widetilde{D}_{M_1} + C_{f_1\times\mathrm{id}_{M_2}} = (\widetilde{D}_{Z_1} + C_f) \otimes 1 + 1 \otimes \widetilde{D}_{M_2}$ and that χF and ψF differ by a compact operator, the identity follows from [2, Theorem 3.2].

Trivially this holds for ϱ_Γ too. □

We would like that, after fixing a non zero K-homology class λ, under suitable assumptions, the product with this element behaves injectively.

To prove such a result we need to define a new group we will denote by $T_*^{\Gamma_1,\Gamma_2}(X_1, X_2)$ (notice that the order of X_1 and X_2 is not irrelevant).

Definition 2.1.35. A cycle of $T_j^{\Gamma_1,\Gamma_2}(X_1, X_2)$ consists of the following data:

- a Kasparov bimodule $(H, \phi, T) \in \mathbb{E}^{\Gamma_1\times\Gamma_2}(C_0(X_1) \otimes C_0(X_2), \mathbb{C})$;
- a Kasparov bimodule $(\mathcal{E}_s, \psi_s, S_s) \in \mathbb{E}^{\Gamma_1}(C_0(X_1), C_r^*(\Gamma_2) \otimes C[0,1])$, where $\mathcal{E}_0 = E_{X_2} \otimes_{C_0(X_2)\rtimes\Gamma_2} H \rtimes \Gamma_2$, $\psi_0 = 1 \otimes \widetilde{\phi}$ and S_0 is any $\widetilde{T}$-connection;
- a Kasparov bimodule $(\mathcal{E}'_{t,s}, \psi'_{t,s}, S'_{t,s}) \in \mathbb{E}(\mathbb{C}, C_r^*(\Gamma_1) \otimes C_r^*(\Gamma_2) \otimes C_0(\mathcal{T}))$, where $\mathcal{T}$ is the triangle $\{(s,t) \in [0,1]^2 \setminus \{1,1\} \mid s \geq t\}$, $\mathcal{E}'_{0,s} = E_{X_1} \otimes_{C_0(X_1)\rtimes\Gamma_1} \mathcal{E}_1 \rtimes \Gamma_1$, $\psi'_{0,s} = 1 \otimes \widetilde{\psi}$ and $S'_{s,0}$ is any S_s-connection;

  modulo homotopies of cycles, defined in a obvious way.

Remark 2.1.36. To have an intuition of what this group is, accordingly with the idea in Remark 1.3.5, one can think of it as the restriction to the triangle $\mathcal{T} = \{(s,t) \in [0,1]^2 \setminus \{1,1\} \mid s \geq t\}$ of the product of the "mapping cone" $\mu_{X_1}^{\Gamma_1}$ and the "mapping cylinder" of $\mu_{X_2}^{\Gamma_2}$. This idea was used in Definition 2.1.31 too.

It could be useful to think to the naturalness of these constructions in the case of true mapping cones and mapping cylinders.

Lemma 2.1.37. *The group $T_*^{\Gamma_1,\Gamma_2}(X_1, X_2)$ is isomorphic to $K_1(\mu^{\Gamma_1\times\Gamma_2}_{X_1\times X_2})$.*

Proof. Define the homomorphism $\Phi\colon T_*^{\Gamma_1,\Gamma_2}(X_1, X_2) \to S^{\Gamma_1\times\Gamma_2}(X_1 \times X_2)$ given by

$$\big((H, \phi, T), (\mathcal{E}_s, \psi_s, S_s), (\mathcal{E}'_{t,s}, \psi'_{t,s}, S'_{t,s})\big) \mapsto \big(H, \phi, T, \mathcal{E}'_{t,t}, \psi'_{t,t}, S'_{t,t}\big).$$

Define the following homomorphism

$$\Psi\colon (H, \phi, T, \mathcal{H}_t, \alpha_t, U_t) \mapsto \big((H, \phi, T), (\mathcal{E}_s, \psi_s, S_s), (\mathcal{E}'_{t,s}, \psi'_{t,s}, S'_{t,s})\big)$$

where

- $(\mathcal{E}_s, \psi_s, S_s)$ is the path constantly equal to $(E_{X_2} \otimes_{C_0(X_2)\rtimes\Gamma_2} H \rtimes \Gamma_2, 1 \otimes \widetilde{\phi}, S)$, with S any $\widetilde{T}$-connection;

- for all fixed $t \in [0,1)$, $(\mathcal{E}'_{t,s}, \psi'_{t,s}, S'_{t,s})$ is the paths constantly equal to $(\mathcal{H}_t, \alpha_t, U_t)$.

It is easy to check that the third condition in Definition 2.1.35 is satisfied and that Φ and Ψ are inverse to each other.

$\square$

Lemma 2.1.38. *Let λ be a class in $\hat{K}_i^{\Gamma_2}(X_2)$. If there exists a class $\zeta \in KK^{-i}(C_r^*(\Gamma_2), \mathbb{C})$ such that $\mu_{X_2}^{\Gamma_2}(\lambda) \otimes_{C_r^*(\Gamma_2)} \zeta = n$ with $n \neq 0$, then*

$$\boxtimes\lambda \colon K_*(\mu_{X_1}^{\Gamma_1}) \otimes \mathbb{Z}\left[\frac{1}{n}\right] \to K_{i+j}(\mu_{X_1\times X_2}^{\Gamma_1\times\Gamma_2}) \otimes \mathbb{Z}\left[\frac{1}{n}\right]$$

is injective. In particular if $\mu_{X_2}^{\Gamma_2}(\lambda) \otimes_{C_r^*(\Gamma_2)} \zeta = 1$, then the product with λ is honestly injective.

Proof. To prove the Lemma we are going to build a left inverse for $\boxtimes\lambda$. Define the map c_ζ as the composition of the following ones:

- the isomorphism $\Psi \colon K_*(\mu_{X_1\times X_2}^{\Gamma_1\times\Gamma_2}) \to T_*^{\Gamma_1,\Gamma_2}(X_1, X_2)$,

- the evaluation at $s = 0$, $\mathrm{ev}_{\mathrm{s=0}} \colon T_*^{\Gamma_1,\Gamma_2}(X_1, X_2) \to K_*(\mu_{X_1}^{\Gamma_1, C_r^*(\Gamma_2)})$,

- the morphism $K_*(\mu_{X_1}^{\Gamma_1, C_r^*(\Gamma_2)}) \to K_{*-i}(\mu_{X_1}^{\Gamma_1})$ given by

  $$(H, \phi, T, \mathcal{E}(t), \psi(t), S(t)) \mapsto (H', \phi', T', \mathcal{E}'(t), \psi'(t), S'(t)),$$

  where (H', ϕ', T') is any Kasparov product of $(H, \phi, T, \mathcal{E}(t))$ and ζ and $(\mathcal{E}'(t), \psi'(t), S'(t))$ is any Kasparov product of $(\mathcal{E}(t), \psi(t), S(t))$ and ζ.

It is easy to check that

$$\mathrm{ev}_{s=0} \circ \Psi \circ \boxtimes\lambda \colon K_i(\mu_{X_1}^{\Gamma_1}) \to K_{i+j}(\mu_{X_1}^{\Gamma_1, C_r^*(\Gamma_2)})$$

is just the exterior product with $\mu_{X_2}^{\Gamma_2}(\lambda)$. Then, by hypothesis, $c_\zeta(x \boxtimes \lambda) = n \cdot x$ for any $x \in K_i(\mu_{X_1}^{\Gamma_1})$. After inverting n, we get an inverse for $\boxtimes\lambda$. $\square$

Remark 2.1.39. The same argument fits to prove that if we fix an element $x \in \hat{K}_i^{\Gamma_2}(X_2)$ satisfying the above condition, then the vertical arrows of the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & KK^j(\mathbb{C}, A) \arrow[r] \arrow[d, "\boxtimes x"] & K_j(\mu_{X_1}^{\Gamma_1}) \arrow[r] \arrow[d, "\boxtimes x"] & \hat{K}_1^{\Gamma_1}(X_1) \arrow[r] \arrow[d, "\boxtimes x"] & \cdots \\
\cdots \arrow[r] & KK^{j+i}(\mathbb{C}, B) \otimes \mathbb{Z}\left[\frac{1}{n}\right] \arrow[r] & K_{i+j}(\mu_{X_1\times X_2}^{\Gamma_1\times\Gamma_2}) \otimes \mathbb{Z}\left[\frac{1}{n}\right] \arrow[r] & \hat{K}_1^{\Gamma_1\times\Gamma_2}(X_1 \times X_2) \otimes \mathbb{Z}\left[\frac{1}{n}\right] \arrow[r] & \cdots
\end{tikzcd}$$

are rationally injective. Here $A = C^*(X_1)^{\Gamma_1} \otimes C_0(0,1)$ and $B = C^*(X_1 \times X_2)^{\Gamma_1\times\Gamma_2} \otimes C_0(0,1)$.

We can obtain the condition of Lemma 2.1.38 under certain hypotheses on Γ_2: we impose that the group has a γ element, this means that there exists a C*-algebra on which Γ acts properly and elements

$$\eta \in KK_\Gamma(\mathbb{C}, A) \quad \text{and} \quad d \in KK_\Gamma(A, \mathbb{C}),$$

such that $\gamma = \eta \otimes_A d \in KK_\Gamma(\mathbb{C}, \mathbb{C})$ satisfies $p^*\gamma = 1 \in KK_{\underline{E}\Gamma \rtimes \Gamma}(C_0(\underline{E}\Gamma), C_0(\underline{E}\Gamma))$, where $\underline{E}\Gamma$ is the classifying space for proper actions of Γ and $p\colon \underline{E}\Gamma \rtimes \Gamma \to \Gamma$ is the homomorphism defined by $p(z, g) = g$. We refer the reader to [41, 42].

The existence of the γ element implies that the Baum-Connes assembly map (with coefficients) is split injective and that the group is K-amenable: this last property gives the existence of a non trivial element $\zeta \in KK(C^*_r(\Gamma_2), \mathbb{C})$.

Corollary 2.1.40. *Let M_2 be an even dimensional Lipschitz manifold with fundamental group Γ_2 such that it has a γ element and the index of $[D_2] \in K_*(M_2)$ is different from 0. If $f_1\colon N_1 \to M_1$ and $f_1'\colon N_1' \mapsto M_1$ are homotopy equivalences between odd dimensional Lipschitz manifolds, with different ϱ-class invariants, then*

$$[f_1 \times \mathrm{id}_{M_2}] \neq [f_1' \times \mathrm{id}_{M_2}] \in \mathcal{S}^{TOP}(M_1 \times M_2).$$

Dirac operators and positive scalar curvature

We would like to apply the methods of the previous sections to get similar results about the secondary invariants described in [28].

Let us recall [28, Definition 1.6]: let (M, g) be a Riemannian spin manifold of dimension $n > 0$, with fundamental group Γ. If g has uniformly positive scalar curvature then the Dirac operator $\widetilde{\not{D}}_M$ is invertible and $\chi(\widetilde{\not{D}}_M)$, the bounded transform of the lift of $\not{D}_M$ to the universal covering of M, defines a class $\varrho_g \in D^*(\widetilde{M})^\Gamma$.

Thanks to that and the APS-delocalized Theorem, for n odd, one obtains the following commutative diagram

$$\begin{tikzcd}
\Omega^{\mathrm{spin}}_{n+1}(M) \arrow[r] \arrow[d, "\beta"] & \mathrm{R}^{\mathrm{spin}}_{n+1}(M) \arrow[r] \arrow[d, "\mathrm{Ind}_\Gamma"] & \mathrm{Pos}^{\mathrm{spin}}_n(M) \arrow[r] \arrow[d, "\varrho"] & \Omega^{\mathrm{spin}}_n(M) \arrow[d, "\beta"] \\
K_{n+1}(M) \arrow[r] & K_{n+1}(C^*_r(\Gamma)) \arrow[r] & K_{n+1}(D^*(\widetilde{M})^\Gamma) \arrow[r] & K_n(M)
\end{tikzcd}$$

where M is a compact space with fundamental group Γ and universal covering $\widetilde{M}$. The first row in the diagram is the Stolz exact sequence, see for instance [28, Definition 1.39].

In the $S^\Gamma_*(M)$ picture of the analytic structure set, the class ϱ_g is given by the quadruple

$$[L^2(M, \not{S}), C(M), \not{D}_M, \widetilde{\chi(\not{D}_M)}].$$

Here the last term is the constant path $\widetilde{\chi(\not{D}_M)}$ because the operator is invertible and there is no need to perturb it.

Remark 2.1.41. If (M, g) has positive scalar curvature and (N, h) is another Riemannian manifold, then for $\varepsilon > 0$ small enough, $(M \times N, g \times \varepsilon h)$ has positive scalar curvature. Hence if M admits a metric with positive scalar curvature, so does $M \times N$.

Proposition 2.1.42. *Let M be a spin manifold of dimension n and let g be a Riemannian metric with positive scalar curvature on M. Let N be a spin manifold of dimension m and h a Riemannian metric such that $(M \times N, g \times h)$ has positive scalar curvature. Then*

$$\varrho_g \cdot [\not{D}_h] = \varrho_{g\times h} \in S^{\Gamma_1\times\Gamma_2}_{n+m}(M \times N),$$

where Γ_1 and Γ_2 are the fundamental groups of M and N respectively and $[\not{D}_h]$ is the class of the Dirac operator on N in $K_m(N)$.

Proof. We can prove the result as we did 2.1.34. Moreover since the class ϱ_g is represented by a quadruple whose last term is the constant path $\widetilde{\chi(\not{D}_h)}$, it turns out that we can prove it in an easier way (see for instance [34, Proposition 6.2.13]). □

Proposition 2.1.43. *Let M be a spin manifold of odd dimension n with fundamental group Γ_1 and let g_1 and g_2 be two Riemannian metrics with positive scalar curvature on M such that $\varrho_{g_1} \neq \varrho_{g_2} \in S^{\Gamma}_n(M)$. Let (N, h) be a Riemannian spin manifold of even dimension m with fundamental group Γ_2, such that the index of $[\not{D}_N] \in K_m(N)$ is $n \neq 0$, Γ_2 has a γ element and $g_i \times h$ has positive scalar curvature on $M_i \times N$.*

Then

$$\varrho_{g_1\times h} \neq \varrho_{g_2\times h} \in K_*(\mu^{\Gamma_1\times\Gamma_2}_{M\times N}) \otimes \mathbb{Z}\left[\frac{1}{n}\right].$$

Proof. We can use the arguments we used for Lemma 2.1.38 to obtain immediately the result. □

2.1.7 The delocalized APS index Theorem in the odd-dimensional case

Another application of the product formula is the proof of the delocalized APS index theorem for odd dimensional cobordisms.

We will do it for the perturbed signature operator, the theorem for the Dirac operator on a spin manifold with positive scalar curvature is completely analogous.

Because of motivations well explained in [29, Remark 4.6], we will do prove the theorem at the cost of inverting 2. We recall that here and in [29] the signature operator on an odd dimensional manifold is not the odd signature operator of Atiyah, Patodi and Singer, but the direct sum of two (unitarily equivalent) versions of this operator.

Since in the statement of the delocalized APS index theorem in the odd dimensional case we will compare the ϱ invariant of the boundary with the index of the APS odd signature operator on the cobordism, it is worth to specify the notation we shall follow: on an odd dimensional manifold we denote by D^{APS} the odd signature operator of Atiyah, Patodi and Singer and we denote by D the odd signature operator that we used so far.

The strategy of the proof is to reduce the odd dimensional case to the even dimensional one through the product by the K-homology class of the signature operator on the circle. Then it is useful to review the behaving of the signature operator with respect to cartesian products of manifolds. For a detailed treatment we refer the reader to sections 5 and 6 of [45].

Let W be an n-dimensional manifold with boundary ∂W endowed with a cocompact free Γ-action. We assume that n odd and that the boundary of W is composed by a pairs of homotopy equivalent manifolds. Let $j\colon \partial W \hookrightarrow W$ and $j'\colon \partial W \times \mathbb{R} \hookrightarrow W \times \mathbb{R}$ be the obvious inclusions. Let us recall some useful facts:

- the even signature operator $D_{W\times S^1}$ is equivalent to the direct sum of two copies of the exterior product $D_W^{APS}\hat{\otimes}1 + 1\hat{\otimes}D_{S^1}^{APS}$, see [45, Section 6.3]. Since D_{S^1} is the sum of two equivalent versions of $D_{S^1}^{APS}$, one has that $D_{W\times S^1}$ is equivalent to $D_W^{APS}\hat{\otimes}1 + 1\hat{\otimes}D_{S^1}$. Consequently the higher index of $(D_{W\times S^1} + C_{F\times\mathrm{id}}^{\mathrm{cyl}})^+$ is equal to the class given by the product $\frac{1}{2}[\mathrm{Ind}_\Gamma(D_W^{APS} + C_F^{\mathrm{cyl}})] \boxtimes [D_{S^1}]$, where here $\boxtimes\colon K_i(C_r^*(\Gamma)) \times K_j(S^1) \to K_{i+j}(C_r^*(\Gamma\times\mathbb{Z}))$;
- the operator $D_{\partial W\times S^1}^{APS}$ is equivalent to the exterior product of the even dimensional signature operator $D_{\partial W}$ and the odd dimensional signature operator $D_{S^1}^{APS}$, see [45, Section 6.1]. Thus we obtain that the odd dimensional operator $D_{\partial W\times S^1}$ is equivalent to the exterior product of the even dimensional signature operator $D_{\partial W}$ and the odd dimensional signature operator D_{S^1}. In particular this means that $\varrho(D_{\partial W} + C_{F_\partial}) \boxtimes [D_{S^1}]$ is equal to $\varrho(D_{\partial W\times S^1} + C_{F_\partial\times\mathrm{id}})$, where here $\boxtimes\colon K_i(\mu_{\widetilde{W}}^{\Gamma_1}) \times K_j(S^1) \to K_{i+j}(\mu_{\widetilde{W}\times\mathbb{R}}^{\Gamma\times\mathbb{Z}})$.

Remark 2.1.44. Notice that, since $D_{S^1}^{APS}$ is nothing else than the Dirac operator on the circle and since D_{S^1} is unitarily equivalent to two copies of $D_{S^1}^{APS}$, its index is two times the generator of $C_r^*(\mathbb{Z})$. Now $KK(C_r^*(\mathbb{Z}),\mathbb{C}) \cong KK(C(S^1),\mathbb{C})$ by Fourier transform and $KK(C(S^1),\mathbb{C}) \cong \mathrm{Hom}(K_0(C(S^1)),\mathbb{Z})$, by [8, Theorem 7.5.5] for instance. So choosing any homomorphism from $K_0(C(S^1))$ to $\mathbb{Z}$ that sends the index of $D_{S^1}^{APS}$ to 1, we obtain a class $\zeta \in KK(C_r^*(\mathbb{Z}),\mathbb{C})$ that satisfies the assumptions of Lemma 2.1.38, with $n = 2$.

Theorem 2.1.45. *If $i\colon C^*(\widetilde{W})^\Gamma \hookrightarrow D^*(\widetilde{W})^\Gamma$ is the inclusion and $j_*\colon D^*(\partial\widetilde{W})^\Gamma \to D^*(\widetilde{W})^\Gamma$ is the map induced by the inclusion $j\colon \partial\widetilde{W} \hookrightarrow \widetilde{W}$, we have*

$$i_*\left(\frac{1}{2}\mathrm{Ind}_\Gamma(D_W^{\mathrm{APS}} + C_F^{\mathrm{cyl}})\right) = j_*(\varrho(D_{\partial W} + C_{F_\partial})) \in K_0(D^*(\widetilde{W})^\Gamma) \otimes \mathbb{Z}\left[\frac{1}{2}\right],$$

where $\frac{1}{2}\mathrm{Ind}_\Gamma(D_W^{APS} + C_F^{\mathrm{cyl}}) \in K_0(C^*(\widetilde{W})^\Gamma) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$.

Proof. Let W be as above. Because of Proposition 1.4.3 and Lemma 1.4.4 we will prove the theorem in the $K_*(\mu^\Gamma_\cdot)$ setting.

Let

$$\Pi_D\colon K_0(\mu_{\widetilde{W}}^\Gamma) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \to K_0(\mu_{\widetilde{W}\times\mathbb{R}}^{\Gamma\times\mathbb{Z}}) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$$

and

$$\Pi_C\colon K_1(C^*(\widetilde{W})^\Gamma) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \to K_1(C^*(\widetilde{W}\times\mathbb{R})^{\mathbb{Z}\times\Gamma}) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$$

be the morphism induced by the product with the of class of the signature operator D_{S^1} in $K_1(S^1)$. By Lemma 2.1.38, we have that

$$i_*\left(\frac{1}{2}\mathrm{Ind}_\Gamma(D_W^{APS} + C_F^{\mathrm{cyl}})\right) = j_S(\varrho(D_{\partial W} + C_{F_\partial})) \tag{2.1.5}$$

holds if and only if

$$\Pi_D\left(i_*\left(\frac{1}{2}\mathrm{Ind}_\Gamma(D_W^{APS} + C_F^{\mathrm{cyl}})\right)\right) = \Pi_D(j_S(\varrho(D_{\partial W} + C_{F_\partial})))$$

holds.

But by Remark 2.1.32 it turns out that

$$\Pi_D\left(i_*\left(\frac{1}{2}\mathrm{Ind}_\Gamma(D_W^{APS}+C_F^{\mathrm{cyl}})\right)\right)=i_*\left(\Pi_C\left(\frac{1}{2}\mathrm{Ind}_\Gamma(D_W^{APS}+C_F^{\mathrm{cyl}})\right)\right)$$

and, and by [45, Section 6.3], that

$$\Pi_C\left(\frac{1}{2}\mathrm{Ind}_\Gamma(D_W^{APS}+C_F^{\mathrm{cyl}})\right)=\mathrm{Ind}_\Gamma(D_{W\times S^1}+C_{F\times\mathrm{id}}^{\mathrm{cyl}}).$$

Moreover by Lemma 2.1.33 it follows that

$$\Pi_D(j_S(\varrho(D_{\partial W}+C_{F_\partial})))=j'_S(\Pi_D(\varrho(D_{\partial W}+C_{F_\partial})))$$

and, by Proposition 2.1.42, that

$$\Pi_D\left(\varrho(D_{\partial W}+C_{F_\partial})\right)=\varrho(D_{\partial W\times S^1}+C_{F_\partial\times\mathrm{id}}).$$

Thus we have that (2.1.5) holds if and only if

$$i_*\left(\mathrm{Ind}_\Gamma(D_{W\times S^1}^{APS}+C_{F\times\mathrm{id}}^{\mathrm{cyl}})\right)=j'_S\left(\varrho(D_{\partial W\times S^1}+C_{F_\partial\times\mathrm{id}})\right)$$

holds. But, since $W\times S^1$ is even dimensional, the equality on the right-hand side holds by 2.1.28 and the Theorem is proved. □

If W is a Spin Riemannian manifold with boundary, such that the metric on the boundary has positive scalar curvature, then we can state the analogous theorem for the ϱ invariants associated to Dirac operators.

Theorem 2.1.46. *If $i\colon C^*(\widetilde{W})^\Gamma\hookrightarrow D^*(\widetilde{W})^\Gamma$ is the inclusion and $j_*\colon D^*(\partial\widetilde{W})^\Gamma\to D^*(\widetilde{W})^\Gamma$ is the map induced by the inclusion $j\colon\partial\widetilde{W}\hookrightarrow\widetilde{W}$, we have*

$$i_*(\mathrm{Ind}_\Gamma(\not{D}_W))=j_*(\varrho(\not{D}_{\partial W}))\in K_0(D^*(\widetilde{W})^\Gamma).$$

Notice that in this case it is not necessary to invert 2. Moreover the proof of the theorem is very similar to the case of the signature operator, but easier because we have not to perturb the Dirac operator to obtain an invertible operator.

2.2 Lie groupoids and secondary invariants

2.2.1 ϱ classes for Lie groupoids

Let $G\rightrightarrows X$ be a Lie groupoid. Let us quikly recall this construction of the adiabatic index: let $G\rightrightarrows X$ be a Lie groupoid. let P be an elliptic pseudodifferential G-operator, see [44] for definitions and properties. Its principal symbol $\sigma(P)$ defines a class in the K-group $K_*(C_0(\mathfrak{A}^*(G)))$ that is nothing else than $K_*(C^*_r(\mathfrak{A}(G)))$, by means of the Fourier transform.

We know that $ev_0\colon C^*_r(G_{ad})\to C^*_r(\mathfrak{A}(G))$ is a KK-equivalence: concretely, to the class of the symbol $\sigma(P)\in C_0(\mathfrak{A}^*(G))$, we can associate the class in $K_*(C^*_r(G_{ad}))$ defined by the unbounded regular operator P_{ad} on the $C^*_r(G_{ad})$-module $C^*_r(G_{ad})$ defined by

$$(P_{ad}f)(\gamma,t)=\int_{\xi\in\mathfrak{A}^*(G)_{r(\gamma)}}\int_{\gamma'\in G_{s(\gamma)}}e^{\frac{i\langle\exp^{-1}(\gamma'\gamma^{-1})|\xi\rangle}{t}}\chi(\gamma'\gamma^{-1})\sigma(P)(r(\gamma),\xi)f(\gamma',t)\frac{d\xi d\gamma'}{t}\tag{2.2.1}$$

for $t \neq 0$ and

$$(P_{ad}f)(x,V,0) = \int_{\xi\in\mathfrak{A}^*(G)_x}\int_{V'\in\mathfrak{A}(G)_x} e^{i\langle\exp^{-1}(V')|\xi\rangle}\chi(\exp(V))\sigma(P)(x,\xi)f(x,V-V',0)d\xi dV' \tag{2.2.2}$$

for $t = 0$, with $f \in C_c^\infty(G_{ad}, \Omega^{\frac{1}{2}})$. Here we have chosen an exponential map $\exp\colon U \to W$, from a neighbourood of the zero section in the algebroid $\mathfrak{A}(G)$ to a tubular neighbourhood W of X in G, and a cut-off function χ with support in W. Of course we can also do it with coefficients in any vector bundle E over X.

From Proposition 1.2.51, evaluating the class $[P_{ad}]$ at 1, we get $\mathrm{Ind}(P) = [\sigma(P)] \otimes [ev_0]^{-1} \otimes [ev_1]$, the analytic index of P in the group $K_*(C_r^*(G))$.

Notice that $C_r^*(G_{ad}^0)$ is isomorphic to the mapping cone $C_{ev_1}(C_r^*(G_{ad}), C_r^*(G))$ of the evaluation at 1. Hence, if $\mathrm{Ind}(P)$ is the zero class, the pair given by the operator P and a homotopy P_t of P to an invertible operator defines, through the above isomorphism, a class in $K_*(C_r^*(G_{ad}^0))$. This will be the home of the secondary invariants that we will study in section. They are called secondary because they arise when the index, the primary invariant, vanishes.

Definition 2.2.1. In the situation described above, let us denote the ϱ-invariant associated to P and P_t as

$$\varrho(P_t) \in K_*(C_r^*(G_{ad}^0)).$$

2.2.2 Cobordism relations

Let W be a smooth manifold with boundary ∂W and let $G(W,\partial W) \rightrightarrows W$ be the b-calculus groupoid of a Lie groupoid G transverse to the boundary, as in Definition 1.2.9.

Let P be an elliptic pseudodifferential $G(W,\partial W)$ operator and denote its restriction to the boundary by P^∂: this is a $G_{|\partial W} \times \mathbb{R}$-opertor.

Assume that there exists a homotopy P_t^∂ from $P_0^\partial = P^\partial$ to an invertible operator P_1^∂. Then we get the following classes:

- a secondary invariant $\rho(P_t^\partial) \in K_*\left((G_{|\partial W})_{ad}^0 \times \mathbb{R}\right) \simeq K_{*+1}\left((G_{|\partial W})_{ad}^0\right)$;
- a class $[P_{ad}^{\mathcal{T}}] \in K_*\left(\mathcal{T}_{nc}G(W,\partial W)\right)$ (see Definition 1.2.11), naturally defined by means of the symbol of P and the homotopy P_t^∂. In fact we can extend it to a class $[P_{ad}^{\mathcal{F}}] \in K_*\left(G(W,\partial W)_{ad}^{\mathcal{F}}\right)$ through the KK-equivalence in the Remark 1.2.12;
- finally we get a class $\mathrm{Ind}(P, P_t^\partial) \in K_*\left(G_{|\mathring{W}}\right)$. This is the generalized Fredholm index of P associated to the perturbation on the boundary P_t^∂, obtained as the Kasparov product $[P_{ad}^{\mathcal{F}}] \otimes [ev_1]$.

Let us point out that $\rho(P^\partial)$ is equal to $[P_{ad}^{\mathcal{F}}] \otimes [ev_{\partial W\times[0,1)}]$.

Lemma 2.2.2. *Let A, B, C, J_B, J_C be C^*-algebras such that one has the following exact sequences:*

1) $0 \longrightarrow J_B \longrightarrow A \xrightarrow{\beta} B \longrightarrow 0$,

2) $0 \longrightarrow J_C \longrightarrow A \xrightarrow{\gamma} C \longrightarrow 0$,

then we have the following ones:

3) $0 \longrightarrow J \longrightarrow J_C \longrightarrow B \longrightarrow 0$,

4) $0 \longrightarrow J \longrightarrow J_B \longrightarrow C \longrightarrow 0$,

5) $0 \longrightarrow J \longrightarrow A \xrightarrow{\beta\oplus-\gamma} B \oplus C \longrightarrow 0$,

where $J = J_B \cap J_C$. Let ∂_B and ∂_C be the boundary homomorphisms associated to the exact sequences 3) and 4) respectively. Then ∂, the boundary homomorphism associated to the exact sequences 5), is such that

$$\partial\colon x \oplus y \mapsto \partial_B(x) - \partial_C(y)$$

where $x \in K_n(B)$, $y \in K_n(C)$ and $\partial_B(x) - \partial_C(y) \in K_{n+1}(J)$.

Proof. Let $\lambda\colon J \oplus J \to J$ be the homomorphism

$$\lambda\colon j_1 \oplus j_2 \mapsto j_1 + j_2$$

and let $\mu\colon J_C \oplus J_B \to A$ be the homomorphism

$$\mu\colon j_3 \oplus j_4 \mapsto j_3 + j_4.$$

We have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & J \oplus J \arrow[r] \arrow[d, "\lambda"] & J_C \oplus J_B \arrow[r, "\beta\oplus\gamma"] \arrow[d, "\mu"] & B \oplus C \arrow[r] \arrow[d, "id_B\oplus-id_C"] & 0\,. \\
0 \arrow[r] & J \arrow[r] & A \arrow[r, "\beta\oplus-\gamma"] & B \oplus C \arrow[r] & 0
\end{tikzcd}$$

The commutativity of the first square is obviuos, for the second one we have that

$$\begin{aligned}\beta \oplus -\gamma(\mu(h \oplus k)) = \beta \oplus -\gamma(h + k) = \beta(h + k) \oplus -\gamma(h + k) = \\ = \beta(h) \oplus -\gamma(k) = (id_B \oplus -id_C)(\beta \oplus \gamma(h \oplus k)).\end{aligned}$$

Here we have used the fact that $\beta(k) = 0$ and $\gamma(k) = 0$ because of exactness of sequences 1) and 2). Then by the commutative of the following square

$$\begin{tikzcd}
K_n(B \oplus C) \arrow[r, "\partial_B\oplus\partial_C"] \arrow[d, "id_B\oplus-id_C"] & K_{n+1}(J \oplus J) \arrow[d, "\lambda_*"] \\
K_n(B \oplus C) \arrow[r, "\partial"] & K_{n+1}(J)
\end{tikzcd}$$

we obtain that $\partial\colon x \oplus y \mapsto \partial_B(x) - \partial_C(y)$. $\square$

We want to apply this Lemma to the following case. The C*-algebras we are interested in are:

$$A = C^*_r\left(G(W, \partial W)^{\mathcal{F}}_{ad}\right)\,,\; B = C^*_r\left(G_{|\mathring{W}}\right)\,,\; C = C^*_r\left((G_{|\partial W})^0_{ad} \times \mathbb{R}\right).$$

Consequently we get the following diagram of exact sequences:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & C_r^*\left((G_{|\mathring{W}})^0_{ad}\right) \arrow[r] \arrow[d] & C_r^*\left((G_{|\mathring{W}})_{ad}\right) \arrow[d] \arrow[dr, bend left] & & \\
0 \arrow[r] & C_r^*\left(G(W,\partial W)^0_{ad}\right) \arrow[r] \arrow[dr, bend right] & C_r^*\left(G(W,\partial W)^{\mathcal{F}}_{ad}\right) \arrow[r] \arrow[d] & C_r^*\left(G_{|\mathring{W}}\right) \arrow[r] \arrow[d] & 0 \\
 & & C_r^*\left((G_{|\partial W})^0_{ad} \times \mathbb{R}\right) \arrow[r] \arrow[d] & 0 & \\
 & & 0 & &
\end{tikzcd}$$

that is fit for the situation in Lemma 2.2.2. Using the notations in the Lemma one obtains that

- $\partial_B \colon K_{*+1}\left(C_r^*\left(G_{|\mathring{W}}\right)\right) \to K_*\left(C_r^*\left((G_{|\mathring{W}})^0_{ad}\right)\right)$ is given by $b \mapsto b \otimes \mathrm{Bott} \otimes [i]$. Where $i \colon C_r^*\left(G_{|\mathring{W}} \times (0,1)\right) \to C_r^*\left((G_{|\mathring{W}})^0_{ad}\right)$ is the obvious inclusion;
- $\partial_C \colon K_{*+1}\left(C_r^*\left((G_{|\partial W})^0_{ad} \times \mathbb{R}\right)\right) \to K_*\left(C_r^*\left((G_{|\mathring{W}})^0_{ad}\right)\right)$ is given by $c \mapsto c \otimes \mathrm{Bott}^{-1} \otimes [j]$.

  Here Bott^{-1} is given by the boundary map in Corollary 1.2.14 and j is the inclusion of $C_r^*\left(G_{|\partial W} \times (0,1) \times (0,1)\right)$ in $C_r^*\left(G_{|\mathring{W}}\right)$. We have this inclusion because we can always assume that W is of the form $\partial W \times [0,1)$ and that $G(W,\partial W)$ is of the form $G_{|\partial W} \times \Gamma([0,1),\{0\})$ near the boundary.

Then we get the following result.

Theorem 2.2.3 (Delocalized APS index Theorem for Lie groupoids)**.** *In the situation above, we have the following equality*

$$\partial_C\left(\rho(P_t^{\partial})\right) = \partial_B\left(Ind(P, P_t^{\partial})\right) \in K_{*+1}\left((G_{|\mathring{W}})^0_{ad}\right).$$

Proof. It is an easy consequence of Lemma 2.2.2. ☐

Proposition 2.2.4. *If $W = X \times [0,1]$ and $G(W,\partial W) = G \times \Gamma([0,1],\{0,1\})$, then the boundary map associated to the following exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & C_r^*\left(G \times (0,1) \times (0,1)\right) \arrow[d] & & \\
 & C_r^*\left(G \times \Gamma([0,1],\{0,1\})\right) \arrow[r] & C_r^*\left(G \times \mathbb{R} \times \{0\}\right) \oplus C_r^*\left(G \times \mathbb{R} \times \{1\}\right) \arrow[r] & 0
\end{tikzcd}$$

is given by

$$\partial(x_0 \oplus x_1) = x_0 \otimes Bott_0^{-1} - x_1 \otimes Bott_1^{-1}, \tag{2.2.3}$$

where $Bott_i$ is the Bott element for $C_r^*\left(G \times \mathbb{R} \times \{i\}\right)$, defined as the boundary map in Corollary 1.2.14.

Proof. Put $A = C_r^*(G \times \Gamma([0,1],\{0,1\}))$, $B = C_r^*(G \times \mathbb{R} \times \{0\})$ and $C = C_r^*(G \times \mathbb{R} \times \{1\})$. Then the equality (2.2.3) follows from Lemma 2.2.2 and Corollary 1.2.14.

□

2.2.3 Products

Let $G \rightrightarrows X$ and $H \rightrightarrows Y$ be two Lie groupoids. In this section we will define an external product between the C*-algebra of the adiabatic deformation of G and the C*-algebra of the Lie algebroid of H, valued in the C*-algebra of the adiabatic deformation of $G \times H$.

Let us build a KK-class $\alpha \in KK\left(C_r^*(G_{ad}^0) \otimes C_r^*(\mathfrak{A}(H)), C_r^*((G \times H)_{ad}^0)\right)$ in the following way: notice that

$$\mathrm{id} \otimes \mathrm{ev}_0 \colon C_r^*(G_{ad}^0) \otimes C_r^*(H_{ad}) \to C_r^*(G_{ad}^0) \otimes C_r^*(\mathfrak{A}(H))$$

induces a KK-equivalence; moreover, since $C_r^*(G_{ad}^0) \otimes C_r^*(H_{ad}) = C_r^*(G_{ad}^0 \times H_{ad})$, we have a $C_0([0,1]^2 \setminus \{1\} \times [0,1])$-algebra and the restriction the diagonal of the square (open on the right side) $[0,1]^2 \setminus \{1\} \times [0,1]$ induces a KK-element $\Delta \in KK\left(C_r^*(G_{ad}^0 \times H_{ad}), C_r^*((G \times H)_{ad}^0)\right)$.

Hence let us define α as the Kasparov product

$$[\mathrm{id} \otimes \mathrm{ev}_0]^{-1} \otimes_{C_r^*(G_{ad}^0 \times H_{ad})} \Delta \in KK\left(C_r^*(G_{ad}^0) \otimes C_r^*(\mathfrak{A}(H)), C_r^*((G \times H)_{ad}^0)\right).$$

Definition 2.2.5. The external product

$$\boxtimes \colon K_i\left(C_r^*(G_{ad}^0)\right) \times K_j\left(C_r^*(\mathfrak{A}(H))\right) \to K_{i+j}\left(C_r^*((G \times H)_{ad}^0)\right)$$

is defined as the map

$$x \times y \to (x \otimes_{\mathbb{C}} y) \otimes_D \alpha,$$

where $D = C_r^*(G_{ad}^0) \otimes C_r^*(\mathfrak{A}(H))$.

Now we want to investigate the injectivity of this product, after fixing an element $y \in K_*(C_r^*(\mathfrak{A}(H)))$. To do it, let us construct an element

$$\beta \in KK\left(C_r^*((G \times H)_{ad}^0), C_r^*(G_{ad}^0) \otimes C_r^*(H)\right).$$

Let T be the restriction of $G_{ad}^0 \times H_{ad}$ to the triangle

$$\{s \geq t \,|\, (t,s) \in [0,1]^2 \setminus \{1\} \times [0,1]\}.$$

Observe that the restriction of T to the diagonal induces a KK-equivalence

$$\Delta' \in KK(C_r^*(T), C_r^*((G \times H)_{ad}^0)),$$

indeed the kernel of the restiction morphism turns out to be isomorphic to the C*-algebra $C_r^*(G) \otimes C_r^*(H_{ad}^0) \otimes C[0,1)$, that is K-contractible.

Then let us define β as the Kasparov product

$$(\Delta')^{-1} \otimes_{C_r^*(T)} [\mathrm{ev}_{\{s=1\}}] \in KK\left(C_r^*((G \times H)_{ad}^0), C_r^*(G_{ad}^0) \otimes C_r^*(H)\right).$$

It is easy to verify that

$$\alpha \otimes_{C_r^*\left((G \times H)_{ad}^0\right)} \beta \in KK\left(C_r^*(G_{ad}^0) \otimes C_r^*(\mathfrak{A}(H)), C_r^*(G_{ad}^0) \otimes C_r^*(H)\right)$$

is nothing but the class $\mathrm{id}_{C_r^*(G_{ad}^0)} \otimes \mathrm{Ind}_H$, where $\mathrm{Ind}_H \in KK\left(C_r^*(\mathfrak{A}(H), C_r^*(H))\right)$ is the index KK-class as in the Remark 1.2.47.

Lemma 2.2.6. *Let y be a class in $K_i(\mathfrak{A}(H))$. Assume that there extists a K-homology class $\eta \in KK(H, pt)$ such that*

$$y \otimes_{C_r^*(\mathfrak{A}(H))} \mathrm{Ind}_H \otimes_{C_r^*(H)} \eta = n \in \mathbb{Z},$$

with $n \neq 0$, then the map $K_j\left(G^0_{ad}\right) \to K_{i+j}\left((G \times H)^0_{ad}\right)$ given by

$$x \mapsto x \boxtimes y$$

is rationally injective. If $n = 1$, then the map is honestly injective.

Proof. From the previous discussion it turns out that the map

$$x \mapsto (x \boxtimes y) \otimes_{C_r^*((G\times H)^0_{ad})} \beta \otimes_{C_r^*(H)} \eta$$

is equal to the multiplication by n. So up to invert n we have that the exterior product with y is rationally injective and that if $n = 1$ it is injective. $\square$

2.2.4 The Signature operator

In [17] the authors prove that the K-theory classes of the higher signatures are homotopy invariant. They also prove it in the case of foliations, using a method that can be easely presented in a more abstract way for any Lie groupoid.

Let $G \rightrightarrows X$ be a Lie groupoid of dimension n and let $\Lambda_{\mathbb{C}}\mathfrak{A}^*(G)$ be the exterior algebra of $\mathfrak{A}^*(G)$. We can construct a right $C_r^*(G)$-module $\mathcal{E}(G)$ as the completion of $C_c^\infty(G, r^*\Lambda_{\mathbb{C}}\mathfrak{A}^*(G) \otimes s^*\Omega^{\frac{1}{2}}(\mathfrak{A}(G)))$. Furthermore we can define the following $C_r^*(G)$-valued quadratic form putting

$$Q(\xi, \zeta)(\gamma) = m_*\left(\overline{p_1^*(i^*\xi)} \wedge p_2^*\zeta\right),$$

where $m, p_k \colon G^{(2)} \to G$ are such that $m \colon (\gamma_1, \gamma_2) \mapsto \gamma_1 \cdot \gamma_2$ and $p_k \colon (\gamma_1, \gamma_2) \mapsto \gamma_k$ (and i is the inversion map of the groupoid). If $T \in \mathcal{L}(\mathcal{E}(G))$, let us denote T' its adjoint with respect to the quadratic form Q (i.e. $Q(T\xi, \zeta) = Q(\xi, T'\zeta)$ for any $\xi, \zeta \in \mathcal{E}(G)$).

The quadratic form Q is regular in the sense of [17, Definition 1.3], by means of the operator T, given by

$$T\alpha = i^{-\partial(\alpha)(n-\partial(\alpha))}(*_{\mathfrak{A}^*(G)} \otimes id_{C_r^*(G)})(\alpha),$$

where $*_{\mathfrak{A}^*(G)}$ is the Hodge operator of the Lie algebroid.

Consider the exterior derivation operator on $C_c^\infty(X, \Lambda_{\mathbb{C}}\mathfrak{A}^*(G))$. Its pull-back through the submersion $r \colon G \to X$ induces an operator d_0 on $C_c^\infty\left(G, r^*\Lambda_{\mathbb{C}}\mathfrak{A}^*(G) \otimes s^*\Omega^{\frac{1}{2}}(\mathfrak{A}(G))\right)$ that is closable; let us still denote with d_0 its closure: it defines a regular operator on $\mathcal{E}(G)$. We have that $\mathrm{Im} d_0 \subset \mathrm{dom} d_0$ and that $d_0^2 = 0$.

Now put $d\xi = i^{\partial\xi} d_0 \xi$. By Theorem B.0.13, we have that $D_G = d + d^*$ is an elliptic, regular, self-adjoint, G-pseudodifferential operator on $\mathcal{E}(G)$.

Definition 2.2.7. The class $[\mathcal{E}(G), D_G] \in K_*(C_r^*(G))$, defined as in [2], is the analytic G-signature of X.

Definition 2.2.8. Let $G \rightrightarrows X$ and $H \rightrightarrows Y$ be two Lie groupoids. A morphism $\varphi \colon H \to G$ is a homotopy equivalence if there are a morphism $\psi \colon G \to H$ and maps $T \colon X \to G$ and $S \colon Y \to H$, such that

- for any $x \in X$ we have that $s(T(x)) = \varphi \circ \psi(x)$ and $r(T(x)) = x$,

- for any $\gamma \in G^x_{\varphi\circ\psi(x)}$ we have that $T(x)\cdot(\varphi\circ\psi(\gamma)) = \gamma\cdot T(\varphi\circ\psi(x))$,
- T is homotopic to id_X through r-fiber;

and similarly for S and $\psi\circ\varphi$.

Remark 2.2.9. This is nothing else than a strong equivalence of groupoids, see [23, 5.4], with natural transformations homotopic to identities. This implies that $H = G^{\varphi}_{\varphi}$, see [23, Proposition 5.11].

Of course a homotopy equivalence φ between two groupoids gives a Morita equivalence, whose imprimitivity bimodule is given by μ_φ as in the subsection 1.2.5.

Theorem 2.2.10 (Hilsum-Skandalis)**.** *Let $H \rightrightarrows Y$ and $G \rightrightarrows X$ be two Lie groupoids, with X and Y compact manifolds, and let $\varphi\colon H \to G$ be a homotopy equivalence of groupoids. Then*

$$[\mathcal{E}(H), D_H]\otimes \mu_\varphi = [\mathcal{E}(G), D_G] \in K_*(C^*_r(G)). \tag{2.2.4}$$

Proof. Let us notice that by hypoteses $H \simeq G^{\varphi}_{\varphi}$. Consider the Lie groupoid $L = G^{\varphi\cup\mathrm{id}_X}_{\varphi\cup\mathrm{id}_X} \rightrightarrows Y\cup X$ and the $C^*_r(L)$-module $\mathcal{E}(L)$, that is the completion of $C^\infty_c(L, r^*\Lambda\mathfrak{A}(L)\otimes s^*\Omega^{\frac{1}{2}}(\mathfrak{A}(L)))$.

We can see an element in $\mathcal{E}(L)$ as a 2×2 matrix in $\begin{pmatrix}\mathcal{E}(G) & \mathcal{E}(G_\varphi)\\ \mathcal{E}(G^\varphi) & \mathcal{E}(G^\varphi_\varphi)\end{pmatrix}$, where the notation is self-explenatory. Then the L-operator d_L given by the exterior derivation is a matrix $\begin{pmatrix} d_G & 0\\ 0 & -d_{G^\varphi_\varphi}\end{pmatrix}$. Put $(\mathcal{E}_1, Q_1, D_1) = (\mathcal{E}(G_\varphi)\oplus\mathcal{E}(G), Q, d)$ and $(\mathcal{E}_2, Q_2, D_2) = (\mathcal{E}(G^\varphi_\varphi)\oplus\mathcal{E}(G^\varphi), Q, d_{G^\varphi_\varphi})$. We want to construct a operator $\mathcal{T}\in\mathcal{L}(\mathcal{E}_1,\mathcal{E}_2)$ that satisfiess the hypoteses of [17, Lemma 2.1].

- Let us point out that there is an embedding of Γ_φ, the graph of φ in $Y\times X$, in G_φ, through the map $j\colon (y,\varphi(y))\mapsto (id_{\varphi(y)}, y)$.

  Consider an n-form α that is in the class of the Poincaré dual of Γ_φ in G_φ: namely $\alpha \in C^\infty_c(G_\varphi, \Lambda^n(T^*G_\varphi))$ is a closed form such that $\int_{j(\Gamma_\varphi)}\omega = \int_{G_\varphi}\alpha\wedge\omega$ for any $\omega\in C^\infty_c(G_\varphi,\Lambda^*(T^*G_\varphi))$. One can see it as the Thom class of the normal bundle associated to the inclusion $\Gamma_\varphi \hookrightarrow G_\varphi$.

  Now, because the exterior algebra of a direct sum is the tensor product of the exterior algebras, we get a map $C^\infty_c(G_\varphi,\Lambda^*(T^*G_\varphi)) \to C^\infty_c(G_\varphi, \Lambda^*(\ker dr)^*\otimes\Lambda^*(\ker ds)^*)$, the last being $C^\infty_c\left(G_\varphi, r^*\Lambda^*\mathfrak{A}^*(L)\otimes s^*\Lambda^*\mathfrak{A}^*(L)\right)$, and we can take the image of α, extend it to L putting it equal to zero outside G_φ and still denote it by α.

  We can define an integral operator with integral kernel given by α as follows

  $$\mathcal{T}_\alpha(\xi) = m_*(p_1^*(\alpha)\wedge p_2^*(\xi)),$$

  where $\xi\in C^\infty_c(L, r^*\Lambda^*\mathfrak{A}^*(L)\otimes\Omega^{\frac{1}{2}}(\mathfrak{A}(L)))$. Since m, p_1 and p_2 are submersions it is easy to check that $\mathcal{T}_\alpha$ extends to a bounded operator on $\mathcal{E}(L)$ and that as matrix it is of the form $\begin{pmatrix}0 & T_\alpha\\ 0 & 0\end{pmatrix}$.

  Since α is closed and the pull-back and the push-forward of forms are chain maps, $\mathcal{T}_\alpha$ commutes with d_L.

  Moreover, because the integration kernel is a smooth form, the image of $\mathcal{T}_\alpha$ is contained in the domain of d^{φ}_{φ}.

- Let $m^{123}, q_k\colon H^{(3)} \to H$ and $m^{23}\colon H^{(3)} \to H^{(2)}$ be the obvious maps. Then we can calculate the Q-adjoint of T_α:

$$\begin{aligned}
Q(T_\alpha\xi,\zeta) &= m_*\left(\overline{p_1^*i^*(m_*(p_1^*\alpha\wedge p_2^*\xi)}\wedge p_2^*\zeta\right) = \\
&= m_*\left(\overline{p_1^*(m_*(p_1^*i^*\xi\wedge p_2^*i^*\alpha)}\wedge p_2^*\zeta\right) = \\
&= m_*^{123}\left(\overline{q_1^*i^*\xi}\wedge q_2^*i^*\alpha\wedge q_3^*\zeta\right) = \\
&= m_*\left(m_*^{23}(\overline{q_1^*i^*\xi}\wedge q_2^*i^*\alpha\wedge q_3^*\zeta)\right) = \\
&= m_*\left(\overline{p_1^*i^*\xi}\wedge p_2^*(m_*(p_1^*i^*\alpha\wedge p_2^*\zeta))\right).
\end{aligned}$$

  Then it turns out that $T'_\alpha\zeta = m_*(p_1^*i^*\alpha\wedge p_2^*\zeta)$. Let us notice that $i^*\alpha$ is in the class of the Poincaré dual of $i\left(j(\Gamma_\varphi)\right)$.

- Now we will prove that the cohomological inverse of $\mathcal{T}_\alpha$ is given by its Q-adjoint $\mathcal{T}'_\alpha$ and that there exists a smoothing operator $\mathcal{Y}\in\mathcal{L}(\mathcal{E}_1)$ such that $1-\mathcal{T}'_\alpha\mathcal{T}_\alpha = d_G\mathcal{Y}+\mathcal{Y}d_G$. Let us make the following computation

$$\begin{aligned}
T'_\alpha T_\alpha\xi &= m_*\left(p_1^*i^*\alpha\wedge p_2^*(m_*(p_1^*\alpha\wedge p_2^*\xi))\right) = \\
&= m^{123}\left(q_1^*i^*\alpha\wedge q_2^*\alpha\wedge q_3^*\alpha\right) = \\
&= m_*\left(m_*^{12}(q_1^*i^*\alpha\wedge q_2^*\alpha\wedge q_3^*\alpha)\right) = \\
&= m_*\left(p_1^*(m_*(p_1^*i^*\alpha\wedge p_2^*\alpha))\wedge p_2^*\xi\right).
\end{aligned}$$

  Recall that the Poincaré dual of X is the Thom class of the normal bundle of $X\hookrightarrow G$. Then using the exponential map $\exp\colon N_X^G\to G$, we can produce a homotopy between $\beta := m_*(p_1^*i^*\alpha\wedge p_2^*\alpha)$ and a current supported only on X. The operator associated to it is nothing but the identity. Consequently, by standard methods, we can produce a chain homotopy $\mathcal{Y}$ in the following way:

$$\mathcal{Y}(\xi) = \int_0^1 i_{\partial/\partial t}(m_*(p_1^*(\exp_t^*\beta)\wedge p_2^*(\xi))dt,$$

  where $\exp_t(V) = \exp(tV)$ for any vector in N_X^G.

Now we are able to apply [17, Lemma 2.1] and hence we get a homotopy D_L^{HS} from D_L to an invertible operator. This in particular means that $[\mathcal{E}(L), D_L] = 0$ and the following equality

$$[\mathcal{E}(L), D_L]\otimes\mu_{\varphi\cup\mathrm{id}_X} = [\mathcal{E}(H), D_H]\otimes\mu_\varphi - [\mathcal{E}(G), D_G]$$

gives the equality (2.1.9). □

Definition 2.2.11. Let $G\rightrightarrows X$ be a Lie groupoid on a compact smooth manifold. We define the G-structure set $\mathcal{S}_G(X)$ of X as the set

$$\{\varphi\colon H\to G\,|\,\varphi \text{ is a homotopy equivalence of groupoids }\}/\sim,$$

where $(H\rightrightarrows Y,\varphi)\sim(H'\rightrightarrows Y',\varphi')$ if there exist

- a cobordism W with boundary $Y\cup Y'$,
- a Lie groupoid $K\rightrightarrows W$, transverse to the boundary

- a morphism $\Phi\colon K(W,\partial W)\to G(X\times[0,1],X\times\{0,1\})$ such that Φ is a groupoid homotopy equivalence and if we restrict it to the boundary we have that $\Phi_{|Y}=\varphi\times\mathrm{id}\colon H\times\mathbb{R}\to G\times\mathbb{R}$ and $\Phi_{|Y'}=\varphi'\times\mathrm{id}\colon H'\times\mathbb{R}\to G\times\mathbb{R}$.

If $\varphi\colon H\to G$ is a homotopy equivalence of groupoids, and let n be the rank of the Lie algebroid of G. Then we can define a secondary invariant as in Definition 2.2.1.

Definition 2.2.12. Let us define the secondary invariant $\rho(\varphi)$ as the class

$$[D_L^{HS}]\otimes(\varphi\cup\mathrm{id}_X)_!^{ad}\in K_n\left(C_r^*(G_{ad}^0)\right),$$

where $L=G_{\varphi\cup\mathrm{id}_X}^{\varphi\cup\mathrm{id}_X}$ and we are using Theorem 2.2.10 to produce the homotopy D_L^{HS} from D_L to an invertible operator.

Proposition 2.2.13. *The map*

$$\rho\colon\mathcal{S}_G(X)\to K_n\left(C_r^*(G_{ad}^0)\right)$$

is well defined with respect to the cobordism equivalence relation.

Proof. Let $\Phi\colon K(W,\partial W)\to G(X\times[0,1],X\times\{0,1\})$ be a groupoid homotopy equivalence with $\Phi_{|Y}=\varphi\colon H\to G$ and $\Phi_{|Y'}=\varphi'\colon H'\to G$. We have to show that

$$[D_L^{HS}]\otimes(\varphi\cup\mathrm{id}_X)_!^{ad}=[D_{L'}^{HS}]\otimes(\varphi'\cup\mathrm{id}_X)_!^{ad}\in K_n\left(C_r^*(G_{ad}^0)\right).$$

Since Φ is a homotopy equivalence of groupoids, $K(W,\partial W)=G(X\times[0,1],X\times\{0,1\})_\Phi^\Phi$. Let $\mathcal{L}\rightrightarrows W\cup X\times[0,1]$ be the pull-back of $G(X\times[0,1],X\times\{0,1\})$ through $\Phi\cup\mathrm{id}_{X\times[0,1]}$.

Thanks to Theorem 2.2.10, we get a class $[D_{\mathcal{L}}^{HS}]\in K_{n+1}\left(C_r^*(\mathcal{L}_{ad}^0)\right)$. The formula in Proposition 2.2.3 tell us that

$$\partial_C\left([D_{\mathcal{L}}^{HS}]\otimes[ev_\partial]\right)=0\in K_{n+1}\left((\mathcal{L}_{\mathring{W}\cup X\times(0,1)})_{ad}^0\right).$$

Let $\pi_X\colon X\times[0,1]\to X$ the projection onto the first factor. If we prove that

$$\partial_C\left([D_{\mathcal{L}}^{HS}]\otimes[ev_\partial]\right)\otimes(\Phi_{|\partial W}\cup\mathrm{id}_{X\times\{0,1\}})_!^{ad}\otimes(\pi_X)_!^{ad}$$

and

$$[D_L^{HS}]\otimes(\varphi\cup\mathrm{id}_{X\times\{0\}})_!^{ad}-[D_{L'}^{HS}]\otimes(\varphi'\cup\mathrm{id}_{X\times\{1\}})_!^{ad}$$

are the same class, we have done.

By Proposition 1.2.40, we get the following equality

$$\partial_C\left([D_{\mathcal{L}}^{HS}]\otimes[ev_\partial]\right)\otimes(\Phi\cup\mathrm{id}_{X\times(0,1)})_!^{ad}=\partial_C'\left([D_{\mathcal{L}}^{HS}]\otimes[ev_\partial]\otimes(\Phi_{|\partial W}\cup\mathrm{id}_{X\times\{0,1\}})_!^{ad}\right)$$

where ∂_C' is the boundary map associated to the restriction from the cylinder $X\times[0,1]$ to the boundary $X\times\{0,1\}$.

Now, since $[D_{\mathcal{L}}^{HS}]\otimes[ev_\partial]=\left([D_L^{HS}]\otimes\mathrm{Bott}_0\right)\oplus\left([D_{L'}^{HS}]\otimes\mathrm{Bott}_1\right)$, we get that

$$\begin{aligned}
&\partial_C\left([D_{\mathcal{L}}^{HS}]\otimes[ev_\partial]\right)\otimes(\Phi_{|\partial W}\cup\mathrm{id}_{X\times\{0,1\}})_!^{ad}\otimes(\pi_X)_!^{ad}=\\
&\quad=\partial_C'\left([D_{\mathcal{L}}^{HS}]\otimes[ev_\partial]\otimes(\Phi_{|\partial W}\cup\mathrm{id}_{X\times\{0,1\}})_!^{ad}\right)\otimes(\pi_X)_!^{ad}\\
&\quad=\partial_C'\left(\left([D_L^{HS}]\otimes\mathrm{Bott}_0\oplus[D_{L'}^{HS}]\otimes\mathrm{Bott}_1\right)\otimes(\Phi_{|\partial W}\cup\mathrm{id}_{X\times\{0,1\}})_!^{ad}\right)\otimes(\pi_X)_!^{ad}=\\
&\quad=\partial_C'\left(\left([D_L^{HS}]\otimes(\varphi\cup\mathrm{id}_{X\times\{0\}})_!^{ad}\otimes\mathrm{Bott}_0\right)\oplus\left([D_{L'}^{HS}]\otimes(\varphi'\cup\mathrm{id}_{X\times\{1\}})_!^{ad}\otimes\mathrm{Bott}_1\right)\right)\otimes(\pi_X)_!^{ad}=\\
&\quad=\left([D_L^{HS}]\otimes(\varphi\cup\mathrm{id}_{X\times\{0\}})_!^{ad}\oplus-[(D_{L'}^{HS}]\otimes(\varphi'\cup\mathrm{id}_{X\times\{1\}})_!^{ad}\right)\otimes(\pi_X)_!^{ad}=\\
&\quad=[D_L^{HS}]\otimes(\varphi\cup\mathrm{id}_X)_!^{ad}-[D_{L'}^{HS}]\otimes(\varphi'\cup\mathrm{id}_X)_!^{ad},
\end{aligned}$$

where we have used Proposition 1.2.40 and Proposition 2.2.4.

□

2.2.5 The Dirac operator

Let $G \rightrightarrows X$ be a Lie groupoid over a compact manifold X, with Lie algebroid $\mathfrak{A}(G) \to X$. Let g be a metric on $\mathfrak{A}(G)$, by pull-back it defines a G-invariant metric on $\ker ds$ along the s-fibers of G. Let ∇ be the fiber-wise Levi-Civita connection associated to this metric.

Definition 2.2.14. Let $\mathrm{Cliff}\,(\mathfrak{A}(G))$ be the Clifford algebra bundle over X associated to the metric g. Let S be a bundle of Clifford modules over $\mathrm{Cliff}\,(\mathfrak{A}(G))$ and let $c(X)$ denotes the Clifford multiplication by $X \in \mathrm{Cliff}\,(\mathfrak{A}(G))$. Assume that S is equipped with a metric g_S and a compatible connection ∇^S such that:

- Clifford multiplication is skew-symmetric, that is

$$\langle c(X)s_1, s_2\rangle + \langle s_1, c(X)s_2\rangle = 0$$

  for all $X \in C^\infty(X, \mathfrak{A}(G))$ and $s_1, s_2 \in C^\infty(X, S)$;

- ∇^S is compatible with the Levi-Civita connection ∇, namely

$$\nabla^S_X(c(Y)s) = c(\nabla_X Y)s + c(Y)\nabla^S_X(s)$$

  for all $X, Y \in C^\infty(X, \mathfrak{A}(G))$ and $s \in C^\infty(X, S)$.

The Dirac operator associated to these data is defined as

$$\not{D}_S \colon s \mapsto \sum_\alpha c(e_\alpha)\nabla^S_\alpha(s)$$

for $s \in C^\infty(X, S)$ and $\{e_\alpha\}_{\alpha\in A}$ a local orthonormal frame.

With this local expression one can easily prove the analogue of the Weitzenbock formula:

$$\not{D}_S^2 = (\nabla^S)^*\nabla^S + \sum_{\alpha<\beta} c(e_\alpha)c(e_\beta)R(\nabla^S)_{\alpha\beta}, \tag{2.2.5}$$

where $R(\nabla^S)_{\alpha\beta}$ denote the terms of the curvature of ∇^S. Assume that the Lie algebroid $\mathfrak{A}(G)$ is Spin, namely it is orientable and its structure group $SO(n)$ can be lifted to the double cover $\mathrm{Spin}(n)$. Moreover we can consider the *spinors* bundle $\not{S}$ and denote the associated Dirac operator just by $\not{D}$. In this case the second term in (2.2.5) is equal to $\frac{1}{4}$ of the scalar curvature of $\nabla^{\not{S}}$.

Remark 2.2.15. The above discussion implies that if the scalar curvature of $\nabla^{\not{S}}$ is positive everywhere, then the Dirac operator $\not{D}$ is invertible.

Thanks to the previous remark the operator $\not{D}_{ad}$, defined as in (2.2.1) and (2.2.2), is an unbounded multiplier of G_{ad} that defines directly an element of $K_*\left(C^*_r(G^0_{ad})\right)$.

Remark 2.2.16. Remeber that for the Signature operator we need to perform a homotopy of the operator to an invertible one, whereas in the case of the Dirac operator we already have the invertiblity condition at 1 in the adiabatic deformation, thanks to the positivity of the scalar curvature.

From now on we will assume that BG, the classifying space of G, is a manifold and $\mathcal{BG} \rightrightarrows BG$ is the Lie groupoid associated to the universal 1-cocycle ξ (see Appendix C for definitions). The general case will be the subject of future works.

We want to define a groupoid version of the Stolz sequence

$$\Omega^{\mathrm{spin}}_{n+1}(BG) \longrightarrow \mathrm{R}^{\mathrm{spin}}_{n+1}(BG) \longrightarrow \mathrm{Pos}^{\mathrm{spin}}_{n}(BG) \longrightarrow \Omega^{\mathrm{spin}}_{n}(BG)$$

(see for instance [36] for the definition in the case where G is a group).

Definition 2.2.17. Let $G \rightrightarrows X$ be a Lie groupoid.

- Let $\mathrm{Pos}^{spin}_n(BG)$ be the set of bordism classes of triples $(M, f\colon M \to BG, g)$. Here $f\colon M \to BG$ is a smooth transverse map from a n-dimensional smooth closed manifold M such that: f is transverse with respect to $\mathcal{BG}$; $\mathfrak{A}(\mathcal{BG}^f_f)$ is spin and it is equipped with a metric g with positive scalar curvature.

  A bordism between $(M, f\colon M \to BG, g)$ and $(M', f'\colon M' \to BG, g')$ is a triple
  $$(W, F\colon W \to BG \times [0,1], h),$$
  where W is a compact smooth manifold with boundary $\partial W = M \sqcup -M'$, a reference map F (which sends the boundary to the boundary) that restricts to f and f' on the boundary and such that $\mathfrak{A}(\mathcal{BG}^F_F)$ is spin equipped with a metric h with positive scalar curvature, which as a product structure near the boundary and restricts to g and g' on the boundary.

- Let $R^{spin}_{n+1}(BG)$ be the set of bordism classes $(W, f\colon W \to BG, g)$. Here W is a compact $n+1$-dimensional smooth manifold, possibly with boundary; $f\colon W \to BG$ is a smooth transverse map (sending boundary to boundary) that is transverse with respect to $\mathcal{BG}$ and such that $\mathfrak{A}(\mathcal{BG}^f_f)$ is spin and equipped with the metric g; the metric g has positive scalar curvature on the boundary.

  Two triples (W, f, g) and (W', f', g') are bordant if there exists a bordism
  $$(N, \varphi\colon N \to BG, h)$$
  between ∂W and $\partial W'$ such that $(\mathfrak{A}(\mathcal{BG}^\varphi_\varphi), h)$ is spin with positive scalar curvature and
  $$Y := W \sqcup_{\partial W} N \sqcup_{\partial W'} W'$$
  is the boundary of a manifold Z such that the reference map $F = f \sqcup \varphi \sqcup f'$ extends to reference map $F'\colon Z \to BG \times [0,1]$ and the Lie algebroid of the associated Lie algebroid is spin.

- Let $\Omega^{spin}_n(BG)$ be the set of bordims classes $(M, f\colon M \to BG)$. Here M is a closed n-dimensional smooth manifold; $f\colon M \to BG$ is a smooth transverse map that is transverse with respect to $\mathcal{BG}$ and such that $(\mathfrak{A}(\mathcal{BG}^f_f), h)$ is spin. The bordism equivalence between triples is as for $\mathrm{Pos}^{spin}_n(BG)$, without conditions about the metric.

Indeed we obtain a groupoid version of Stolz sequence, as in the classical case, and we want to build a diagram

$$\begin{tikzcd}
\Omega^{\mathrm{spin}}_{n+1}(BG) \arrow[r] \arrow[d, "\beta"] & \mathrm{R}^{\mathrm{spin}}_{n+1}(BG) \arrow[r] \arrow[d, "\mathrm{Ind}_{\mathcal{BG}}"] & \mathrm{Pos}^{\mathrm{spin}}_{n}(BG) \arrow[r] \arrow[d, "\varrho"] & \Omega^{\mathrm{spin}}_{n}(BG) \arrow[d, "\beta"] \\
K_{n+1}\left(\mathfrak{A}(\mathcal{BG})\right) \arrow[r] & K_n\left(\mathcal{BG} \times (0,1)\right) \arrow[r] & K_n\left(\mathcal{BG}^0_{ad}\right) \arrow[r] & K_n\left(\mathfrak{A}(\mathcal{BG})\right)
\end{tikzcd} \tag{2.2.6}$$

such that all the squares are commutative. Let us give the definition of the vertical homomorphisms.

Definition of $\beta\colon \Omega_n^{\mathrm{spin}}(BG) \to K_n\left(\mathfrak{A}(\mathcal{BG})\right)$

Let $(M, f\colon M \to BG)$ an element of $\Omega_n^{spin}(BG)$. Then the Lie algebroid $\mathcal{BG}_f^f$ is spin and, as in Definition 2.2.14, we can define a Dirac operator associated to this spin structure. We will denote it by $\not{D}_f$ and its symbol $\sigma(\not{D}_f) \in C_0(\mathfrak{A}^*(\mathcal{BG}_f^f))$ defines a class

$$\beta(M, f) \in K_n\left(\mathfrak{A}(\mathcal{BG}_f^f)\right)$$

by Fourier transform. Using Corollary 1.2.14 it is easy to prove that β is well defined: indeed if (M, f) and (M', f') are bordant through (W, F) then the Dirac operator $\not{D}_F$ defines a x class in the K-theory of the C*-algebra of the Monthubert groupoid $H := \mathfrak{A}(\mathcal{BG}_F^F)(W, \partial W)$. Consider the following commutative diagram:

$$\begin{tikzcd}
K_{n+1}\left(H\right) \arrow[r, "\mathrm{ev}_\partial"] \arrow[d, "dF_!"] & K_{n+1}\left(\mathfrak{A}(\mathcal{BG}_f^f) \times \mathbb{R}\right) \oplus K_{n+1}\left(\mathfrak{A}(\mathcal{BG}_{f'}^{f'}) \times \mathbb{R}\right) \arrow[r, "\alpha"] \arrow[d, "df_! \oplus df'_!"] & K_n\left(\mathfrak{A}(\mathcal{BG}_f^f)\right) \oplus K_n\left(\mathfrak{A}(\mathcal{BG}_{f'}^{f'})\right) \arrow[d, "df_! + df'_!"] \\
K_{n+1}\left(\mathcal{BG}([0,1], \{0,1\})\right) \arrow[r, "\mathrm{ev}_\partial"] & K_{n+1}\left(\mathfrak{A}(\mathcal{BG} \times \{0\}) \times \mathbb{R}\right) \oplus K_{n+1}\left(\mathfrak{A}(\mathcal{BG} \times \{1\}) \times \mathbb{R}\right) \arrow[r, "\alpha"] & K_n((\mathfrak{A}(\mathcal{BG})))
\end{tikzcd}$$

where α is given by the invers of the Bott periodicity on each summand. We have that

$$(df_! + df'_!) \circ \alpha \circ \mathrm{ev}_\partial(x) = \beta(M, f) - \beta(M', f').$$

By Proposition 2.2.4 the bottom line is exact and by the commutativity of the diagram we have that

$$\beta(M, F) = \beta(M', f').$$

Definition of $\mathrm{Ind}_{\mathcal{BG}}\colon R_{n+1}^{spin}(BG) \to K_n\left(\mathcal{BG} \times (0,1)\right)$

Now let consider an element $(W, f\colon W \to BG, g) \in R_{n+1}^{spin}(BG)$. Consider the Dirac operator $\not{D}_f$, since we have positive scalar curvature on the boundary, using (**??**) we obtain a class

$$y \in K_{n+1}(\mathcal{BG}_f^f(W, \partial W)_{ad}^{\mathcal{F}})$$

(see Definition 1.2.11). Hence we have definition of the map $\mathrm{Ind}_{\mathcal{BG}}$ in the following way

$$(W, f\colon W \to BG, g) \mapsto Bott \circ \mu_f \circ \mathrm{ev}_1(y) \in K_n\left(\mathcal{BG} \times (0,1)\right),$$

where

- $\mathrm{ev}_1\colon \mathcal{BG}_f^f(W, \partial W)_{ad}^{\mathcal{F}} \to \mathcal{BG}_f^f(\mathring{W})$ is the evaluation at $t = 1$ in the adiabatic deformation;
- μ_f is the Morita equivalence associated to the pull-back construction;
- $Bott\colon K_{n+1}\left(\mathcal{BG} \times (0,1) \times (0,1)\right) \to K_n\left(\mathcal{BG} \times \mathbb{R}\right)$ is the Bott periodicity given by Corollary 1.2.14.

This map is well-defined on bordism classes: let (W, f, g) and (W', f', g') be two triples in $R^{spin}_{n+1}(BG)$; let $(\mathcal{BG}^{\varphi}_{\varphi}, h)$, Y and (Z, F') be as in Definition 2.2.17.

Since Y is a boundary, the $\mathcal{BG}$-index $z \in K_{n+1}\left(\mathcal{BG}^F_F\right)$ of $\not{D}_F$ is zero. Consider the following exact sequence:

$$\ldots \longrightarrow K_{n+1}\left(\mathcal{BG}^f_f(\mathring{W})\right) \oplus K_{n+1}\left(\mathcal{BG}^{f'}_{f'}(\mathring{W}')\right) \xrightarrow{i} K_{n+1}\left(\mathcal{BG}^F_F\right) \xrightarrow{\mathrm{ev}_\partial} K_{n+1}\left(\mathcal{BG}^{\varphi}_{\varphi}(N)\right) \longrightarrow \ldots$$

where $i(a \oplus b) = \iota(a) + \iota'(b)$, with ι and ι' induced by the natural inclusions of $\mathcal{BG}^f_f(\mathring{W})$ and $\mathcal{BG}^{f'}_{f'}(\mathring{W}')$ in $\mathcal{BG}^F_F$. As the scalar curvature on N is positive, we have that $\mathrm{ev}_\partial(z) = 0$ and then z lifts to an element of $K_{n+1}\left(\mathcal{BG}^f_f(\mathring{W})\right) \oplus K_{n+1}\left(\mathcal{BG}^{f'}_{f'}(\mathring{W})\right)$ that is of course the direct sum of $\mathrm{ev}_1(y)$ and $-\mathrm{ev}_1(y')$ (the sign $-$ is given by the orientation in the pasting process), the index of $\not{D}_f$ and $\not{D}_{f'}$ respectively. By the definition of F, it follows that $\mu_F \circ \iota = \mu_f$ and $\mu_F \circ \iota' = \mu_{f'}$. Hence

$$\begin{aligned}
\mathrm{Ind}_{\mathcal{BG}}(W, f, g) - \mathrm{Ind}_{\mathcal{BG}}(W', f', g') &= Bott^{-1} \circ \mu_f \circ \mathrm{ev}_1(y) - Bott^{-1} \circ \mu_{f'} \circ \mathrm{ev}_1(y') = \\
&= Bott^{-1} \circ \mu_F \circ \iota \circ \mathrm{ev}_1(y) - Bott^{-1} \circ \mu_F \circ \iota' \circ \mathrm{ev}_1(y') = \\
&= Bott^{-1} \circ \mu_F \circ i(\mathrm{ev}_1(y) \oplus -\mathrm{ev}_1(y')) = \\
&= Bott^{-1} \circ \mu_F(z) = 0
\end{aligned}$$

Definition of ϱ: $\mathrm{Pos}^{spin}_n(BG) \to K_n\left(\mathcal{BG}^0_{ad}\right)$

Let (M, f, g) be a triple in $\mathrm{Pos}^{spin}_n(BG)$. In this case, since the algeborid is spin and the scalar curvature is positive, the Dirac operator $\not{D}_f$ defines directly a class $[\not{D}_f] \in K_n\left((\mathcal{BG}^f_f)^0_{ad}\right)$. Then we can give the following definition of ρ-class:

$$\varrho(M, f, g) := f^{ad}_!([\not{D}_f]) \in K_n(\mathcal{BG}^0_{ad}). \tag{2.2.7}$$

We should check that this map is well-defined, but the poof of this fact is completely analogous to the one of Proposition 2.2.13.

Finally, using the Theorem 2.2.3, the commutativity of the diagram (2.2.6) is obvious.

Remark 2.2.18. Of course, if we consider the Poincaré groupoid $\widetilde{X} \times_\Gamma \widetilde{X}$ of a smooth manifold X with fundamental group Γ, Section 1.4.2 tell us that the Piazza-Schick ϱ-invariant defined in [28, 29] and the ϱ-invariant defined through Lie groupoids are the same.

2.2.6 Product formulas for secondary invariants

Now we would like to apply the product in Definition 2.2.5 to the ρ invariant of Definition 2.2.12 and (2.2.7).

Proposition 2.2.19. *Let $G \rightrightarrows X$ and $H \rightrightarrows Y$ be two Lie groupoid homotopy equivalent by means of the groupoid morphsm $\varphi\colon H \to G$. Let $J \rightrightarrows Z$ be another Lie groupoid. Consider the secondary invariant $\rho(\varphi) \in K_i(C^*_r(G^0_{ad}))$ and the symbol class of the J-signature operator on Z, given by $[\sigma_J] \in K_j(\mathfrak{A}(J))$.*

Then we have the following product formula

$$\rho(\varphi) \boxtimes [\sigma_J] = \rho(\varphi \times \mathrm{id}_Z) \in K_{i+j}\left((G \times J)^0_{ad}\right),$$

where $\varphi \times \mathrm{id}_J$ is a homotopy equivalence between $H \times J$ and $G \times J$.

Proof. If $L = G^{\varphi\cup \mathrm{id}_X}_{\varphi\cup \mathrm{id}_X}$, then $\rho(\varphi) = [D^{HS}_L] \otimes (\varphi \cup \mathrm{id}_X)^{ad}_!$. Consequently, following the notations of Definition 2.2.5, one has that $\rho(\varphi) \boxtimes [\sigma_J]$ is equal to

$$\Big(\Big([D^{HS}_L] \otimes (\varphi \cup \mathrm{id}_X)^{ad}_! \Big) \otimes_{\mathbb{C}} [\sigma_J] \Big) \otimes_D [\mathrm{id} \otimes \mathrm{ev}_0]^{-1} \otimes_{D'} \Delta,$$

where $D = C^*_r(G^0_{ad}) \otimes C^*_r(\mathfrak{A}(J))$ and $D' = C^*_r(G^0_{ad}) \otimes C^*_r(J_{ad})$.

That is equal to

$$\Big(([D^{HS}_L] \otimes_{\mathbb{C}} [\sigma_J]) \otimes \Big((\varphi \cup \mathrm{id}_X)^{ad}_! \otimes (\mathrm{id}_Z)_! \Big) \Big) \otimes_D [\mathrm{id} \otimes \mathrm{ev}_0]^{-1} \otimes_{D'} \Delta,$$

Notice that the following equalities holds:

- by Remark 1.2.41 we have that

$$\Big((\varphi \cup \mathrm{id}_X)^{ad}_! \otimes (\mathrm{id}_Z)_! \Big) \otimes_D [\mathrm{id} \otimes \mathrm{ev}_0]^{-1} = [\mathrm{id} \otimes \mathrm{ev}_0]^{-1} \otimes \Big((\varphi \cup \mathrm{id}_X)^{ad}_! \otimes (\mathrm{id}_Z)^{ad}_! \Big)\,;$$

- moreover it is easy to verify that

$$\Big((\varphi \cup \mathrm{id}_X)^{ad}_! \otimes (\mathrm{id}_Z)^{ad}_! \Big) \otimes \Delta = \Delta \otimes (\varphi \times \mathrm{id}_Z \cup \mathrm{id}_{X\times Z})^{ad}_!.$$

Then it turns out that

$$\rho(\varphi) \boxtimes [\sigma_J] = [D^{HS}_L] \boxtimes [\sigma_J] \otimes (\varphi \times \mathrm{id}_Z \cup \mathrm{id}_{X\times Z})^{ad}_!.$$

So it only remains to notice that $[D^{HS}_L] \boxtimes [\sigma_J] = [D^{HS}_{L\times J}] \in K_{i+j}\big(C^*_r((L\times J)^0_{ad})\big)$. And the proposition is proved.

□

One can similarly prove the analogous result for Dirac operators.

Proposition 2.2.20. *Let $G \rightrightarrows X$ and $H \rightrightarrows Y$ be two Lie groupoids such that both BG and BH are smooth manifolds. Let (M, f, g) be a triple in $\mathrm{Pos}^{spin}_n(BG)$ and let (N, f') be an element in $\Omega^{spin}_m(BH)$. Then we have that*

$$\varrho(M, f, g) \boxtimes \beta(N, f') = \varrho(M \times N, f \times f', g \times h) \in K_{n+m}\left((\mathcal{BG} \times \mathcal{BH})^0_{ad}\right)$$

where h any metric on $\mathfrak{A}(\mathcal{BH}^{f'}_{f'})$ such that $g \times h$ on $\mathfrak{A}(\mathcal{BG}^f_f) \boxtimes \mathfrak{A}(\mathcal{BH}^{f'}_{f'})$ has positive scalar curvature.

Appendix A

Relative K-theory and Excision

The following section follows closely the exposition in [8, Section 4.3], with some slight modifications. Let J an ideal in a unital C*-algebra A and let $\pi\colon A \to A/J$ be the quotient homomorphism.

Definition A.0.1. A relative K-cycle for the pair $(A, A/J)$ is a triple (p, q, σ), where

- p and q are projections over A;
- σ is an invertible element over A/J such that $\pi(q) = \sigma\pi(p)\sigma^{-1}$.

A triple (p, q, σ) is degenerate if there exists an invertible element S over A such that $\pi(S) = \sigma$ and $q = SpS^{-1}$.

Remark A.0.2. In [8, Definition 4.3.1] σ is a partial isometry implementing a Murray-von Neumann equivalence between $\pi(p)$ and $\pi(q)$, but this property and the one used in the previous definition are equivalent by elementary arguments of K-theory for C*-algebras. Equivalently, other texts define a relative K-cycles as triple (E, F, σ), where E and F are finitely generated projective A-modules such that $\sigma\colon \pi_*E \to \pi_*F$ is an isomorphism.

Example A.0.3. Let T a Fredholm operator on an Hilbert space H, then $(\mathrm{Id}, \mathrm{Id}, \pi(T))$ is a relative cycle for the pair $(\mathbb{B}(H), \mathbb{Q}(H))$, where $\mathbb{Q}(H)$ is the Calkin algebra.

The set of relative K-cycles as a natural structure of semigroup given by the following operation

$$(p, q, \sigma) \oplus (p', q', \sigma') = (p \oplus p', q \oplus q', \sigma \oplus \sigma').$$

Two relative K-cycles (p_0, q_0, σ_0) and (p_1, q_1, σ_1) are homotopic if there is a continuous path of relative K-cycles (p_t, q_t, σ_t) connecting them.

Definition A.0.4. The relative K-group $(K_0(A, A/J), \oplus)$ is the abelian group with one generator $[p, q, \sigma]$ for each relative K-cycle (p, q, σ), modulo the following equivalence relation: $(p, q, \sigma) \sim (p', q', \sigma')$ if there exist two degenerate relative K-cycle τ and τ' such that $(p, q, \sigma) \oplus \tau$ and $(p', q', \sigma') \oplus \tau'$ are homotopic.

Let I and J be ideals in A and B respectively and let $\theta\colon A \to B$ be a *-homomorphism such that $\theta(I) \subset J$, then we can define a morphism

$$\theta_*\colon K_0(A, A/I) \to K_0(B, B/J) \tag{A.0.1}$$

such that

$$\theta_*[p, q, \sigma] = [\theta(p), \theta(q), \bar{\theta}(\sigma)],$$

where $\bar{\theta}\colon A/I \to B/J$ is the *-homomorphism induced by θ between the quotients.

There exists a natural homomorphism $K_0(A, A/J) \to K_0(A)$ given by

$$[p, q, \sigma] \mapsto [p] - [q] \tag{A.0.2}$$

and one can prove the following result.

Proposition A.0.5. *The sequence of groups*

$$K_0(A, A/J) \longrightarrow K_0(A) \longrightarrow K_0(A/J)$$

is exact in the middle.

Let J a possibly non-unital C*-algebra and $\widetilde{J}$ its unitalization. Then the following exact sequence

$$0 \longrightarrow J \longrightarrow \widetilde{J} \longrightarrow \mathbb{C} \longrightarrow 0$$

is split and we define the K-theory of J is defined as $K_0(J) := \ker\{K_0(\widetilde{J}) \to K_0(\mathbb{C}) \cong \mathbb{Z}\}$. Notice that the image of homomorphism A.0.2 $\lambda\colon K_0(\widetilde{J}, \mathbb{C}) \to K_0(\widetilde{J})$ is contained in $K_0(J)$, by definition. Thus from the following commutative diagram diagram

$$\begin{tikzcd}
K_0(\widetilde{J}, \mathbb{C}) \arrow[r] \arrow[d, "\lambda"] & K_0(\widetilde{J}) \arrow[r] \arrow[d, "="] & K_0(\mathbb{C}) \arrow[r] \arrow[d, "="] & 0 \\
K_0(J) \arrow[r] & K_0(\widetilde{J}) \arrow[r] & K_0(\mathbb{C}) \arrow[r] & 0
\end{tikzcd}$$

we deduce directly that λ is surjective. The injectivity of λ is obvious.

Definition A.0.6. If J is an ideal in a C*-algebra A, then $\widetilde{J}$ may be regarded as a subalgebra of A and we have a natural map

$$K_0(J) \cong K_0(\widetilde{J}, \mathbb{C}) \to K_0(A, A/J) \tag{A.0.3}$$

called the *excision map.*

Theorem A.0.7. *[8, Theorem 4.3.8] The excision map A.0.3 is an isomorphism.*

Example A.0.8. Let $D \in \mathbb{M}_n(A)$ be such that $\sigma = \pi(D) \in \mathbb{M}_n(A/J)$ is invertible. Then the triple (p_n, p_n, σ), where p_n is the identity matrix in $\mathbb{M}_n(A)$, is a relative K-cycle for the pair $(A, A/J)$.

Now we would like to find explicitly the image in $K_0(J)$ of $[p_n, p_n, \sigma]$ under the inverse of the excision map. First form the direct sum of $[p_n, p_n, \sigma]$ with the degenerate element $[0, 0, p_n]$. Let $Q \in \mathbb{M}_n(A)$ be any lift of $\sigma^{-1} \in \mathbb{M}_n(A/J)$: we have that $DQ - p_n$ and $QD - p_n$ are elements in $\mathbb{M}_n(J)$. Now observe that the invertible element $\sigma \oplus \sigma^{-1}$ has an invertible lift in $\mathbb{M}_{2n}(A)$ given by

$$T = \begin{bmatrix} (p_n - DQ)D + D & DQ - p_n \\ p_n - QD & Q \end{bmatrix}$$

with inverse

$$T^{-1} = \begin{bmatrix} Q & p_n - QD \\ DQ - p_n & (p_n - DQ)D + D \end{bmatrix}$$

such that T is homotopic to the identity through a path T_t. Then

$$[T_t(p_n \oplus 0)T_t^{-1}, p_n \oplus 0, \pi(T_t)(\sigma \oplus p_n)\pi(T_t^{-1})]$$

is a path from $[p_n \oplus 0, p_n \oplus 0, \sigma \oplus p_n]$ to $[T(p_n \oplus 0)T^{-1}, p_n \oplus 0, p_{2_n}]$. After doing a simple calculation, we obtain that $[T(p_n \oplus 0)T^{-1}, p_n \oplus 0, p_{2_n}]$ is a relative K-cycle for the pair $(\widetilde{J}, \mathbb{C})$.

Finally observe that the image of this last class in $K_0(J)$ is equal to

$$[T(p_n \oplus 0)T^{-1}] - [p_n \oplus 0],$$

that is nothing but the image of the class $[\sigma] \in K_1(A/J)$ through the index map

$$\partial\colon K_1(A/J) \to K_0(J).$$

Appendix B

A criterion for the regularity of operators on Hilbert modules

Let A be a C*-algebra and let E, F, H be Hilbert A-modules. We will use the term morphism to indicate adjointable operators.

Lemma B.0.1. *Let $T \in \mathcal{L}(E, F)$ be a surjective morphism, then there exists a morphism $S \in \mathcal{L}(F, E)$ such that $TS = id_F$.*

Proof. We have that $E = \ker T \oplus \mathrm{im} T^*$. Then $T_{|\mathrm{im}T^*}$ is a bijective morphism in $\mathcal{L}(\mathrm{im}T^*, F)$.

It follows that there exists $S' \colon F \to \mathrm{im}T^*$ such that $T_{|\mathrm{im}T^*}S' = id_F$. It is obviously adjointable, because $G(S)$ is just the image of $G(T_{|\mathrm{im}T^*})$ through the flip map $u \colon \mathrm{im}T^* \oplus F \to F \oplus \mathrm{im}T^*$, defined as $u(x, y) = (y, -x)$. The lemma follows setting $S = i \circ S$, where $i \colon imT^* \to E$ is the inclusion. $\square$

Lemma B.0.2. *Let $T \in \mathcal{L}(E, F)$ be a morphism and let $S \colon H \to F$ be a densely defined operator such that $imT \subset domS^*$. Then $S^*T \in \mathcal{L}(E, H)$.*

Proof. The adjoint of a densely defined operator is closed by definition, then S^*T is closed.

Let $x \in \mathrm{dom}S$, then we have

$$\langle x, S^*Tz\rangle = \langle Sx, Tz\rangle = \langle T^*Sx, z\rangle \quad \forall z \in E.$$

This imply that $\mathrm{dom}S \subset \mathrm{dom}(S^*T)^*$ and then $(S^*T)^*$ is densely defined. Now setting $R = id_E$, we have that S^*T is regular. Finally we obtain that $S^*T \in \mathcal{L}(E, H)$. $\square$

Definition B.0.3. Let $T \colon E \to F$ and $S \colon F \to H$ be regular operators such that $\mathrm{im}T + \mathrm{dom}S = F$. We will say that T and S are transverse (with respect to F).

Lemma B.0.4. *Let $T \colon E \to F$ and $S \colon F \to H$ be transverse and regular operators, then there exist $Q \in \mathcal{L}(F)$ an $R \in \mathcal{L}(F, E)$ such that $imQ \subset domS$, $imR \subset domT$ and $Q + TR = id_F$.*

Proof. Consider the operator from $E \oplus F \to F$ defined by

$$(x, y) \mapsto \frac{T}{\sqrt{1 + T^*T}}x + \frac{1}{\sqrt{1 + S^*S}}y,$$

it is surjective because $\mathrm{im}\frac{1}{\sqrt{1+S^*S}} = \mathrm{dom}S$ and $\mathrm{im}\frac{T}{\sqrt{1+T^*T}} = \mathrm{im}T$.

Hence by B.0.1 there exists $V \colon F \to E \oplus F$ such that $(\frac{T}{\sqrt{1+T^*T}} \oplus 0 + 0 \oplus \frac{1}{\sqrt{1+S^*S}})V = id_F$. If we set

- $Q = \left(0 \oplus \frac{1}{\sqrt{1+S^*S}}\right) V$,
- $R = \left(\frac{1}{\sqrt{1+T^*T}} \oplus 0\right) V$,

then the lemma follows. $\square$

Lemma B.0.5. *Let $T\colon E \to F$ and $S\colon F \to H$ be transverse and regular operators, then $domST = RdomS + (1-RT)domT$, where R is given by B.0.4.*

Proof. Let's prove the inclusion $\mathrm{dom}ST \supset R\mathrm{dom}S + (1-RT)\mathrm{dom}T$.

Let $y \in \mathrm{dom}S$. By B.0.4 we have that $Ry \in \mathrm{dom}T$ and $TRy = y - Qy \in \mathrm{dom}S$. Then $R\mathrm{dom}S \subset \mathrm{dom}ST$.

Let $y \in \mathrm{dom}T$. By B.0.4 we have that $y - RTy \in \mathrm{dom}T$ and $Ty - TRTy = (1 - TR)(Ty) = QTy \in \mathrm{dom}S$. Then $(1-RT)\mathrm{dom}T \subset \mathrm{dom}ST$.

Let's prove the opposite inclusion $\mathrm{dom}ST \subset R\mathrm{dom}S + (1-RT)\mathrm{dom}T$. Let $x \in \mathrm{dom}ST \subset \mathrm{dom}T$. Let $z \in \mathrm{dom}T$ such that $x - z \in \mathrm{im}R$, such a z always exists. Then $x - z + RTz \in \mathrm{im}R$ and hence there exists $w \in F$ such that $Rw = x - z + RTz$. By B.0.4 $w = Qw + TRw$, hence $w - TRw \in \mathrm{dom}S$.

We have that

$$\begin{aligned} R(w - TRw) &= x - z + RTz - RT(x - z + RTz) = \\ &= x - z + RTz - RTx + RTz - RTRTz, \end{aligned}$$

that is

$$x = R(TRw - w + Tx) + (1-RT)(z + RTz),$$

where $TRw - w + Tx \in \mathrm{dom}S$ and $z + RTz \in \mathrm{dom}T$.

Hence we have that $x \in R\mathrm{dom}S + (1-RT)\mathrm{dom}T$. $\square$

Lemma B.0.6. *Let $T\colon E \to F$ and $S\colon F \to H$ be transverse and regular operators, then $domST$ is a core for T.*

Proof. Let $(x, Tx) \in E \oplus F$ be an element in the graph of T. Let $y_n \in F$ such that $y_n \to Tx$. By B.0.5 we have that $Ry_n + (1-RT)x$ is a sequence in $\mathrm{dom}ST$ and it converge to $x \in \mathrm{dom}T$. Moreover, by B.0.4 and by continuity of $1-Q$, we have that $TRy_n = y_n - Qy_n \to (1-Q)Tx = TRTx$. Then $T(Ry_n + (1-RT)x) \to Tx$ and the lemma is proved. $\square$

Lemma B.0.7. *Let $T\colon E \to F$ and $S\colon F \to H$ be transverse and regular operators, then $dom(ST)^* \subset domS^*$. Moreover, if $x \in dom(ST)^*$, then $S^*x = (SQ)^*x + R^*(ST)^*x$.*

Proof. Recall that

$$\mathrm{dom}(ST)^* = \{x \in H \,|\, \exists z \in E\,,\, \langle z, y\rangle = \langle x, STy\rangle \,\forall y \in \mathrm{dom}ST\}$$

and similarly

$$\mathrm{dom}(S)^* = \{x \in H \,|\, \exists w \in F\,,\, \langle w, v\rangle = \langle x, Sv\rangle \,\forall v \in \mathrm{dom}S\}.$$

Let $x \in \mathrm{dom}(ST)^*$ and $v \in \mathrm{dom}S$. We have that

$$\begin{aligned} \langle x, Sv\rangle &= \langle x, S(Q+TR)v\rangle = \\ &= \langle x, SQv\rangle + \langle (ST)^*x, Ru\rangle \\ &= \langle (SQ)^*x, v\rangle + \langle R^*(ST)^*x, v\rangle = \\ &= \langle (SQ)^*x + R^*(ST)^*x, v\rangle, \end{aligned}$$

where in the second row we use the fact that $Ru \in \mathrm{dom} ST$, in the third one the facts that $(SQ)^*$ and R are morphisms. This show that $x \in \mathrm{dom} S$ and that $S^*x = (SQ)^*x + R^*(ST^*)x$. $\square$

Lemma B.0.8. *Let $T\colon E \to F$ and $S\colon F \to H$ be transverse and regular operators, then $T^*S^* = (ST)^*$.*

Proof. By and B.0.6, we know that $T^*S^* \subset (ST)^*$. We have to show the converse $T^*S^* \supset (ST)^*$. Recall that

$$\mathrm{dom} T^*S^* = \{x \in \mathrm{dom} S^* \mid S^*x \in \mathrm{dom} T^*\}.$$

By B.0.7 it is sufficient to show that if $x \in \mathrm{dom}(ST)^*$, then $S^*x \in \mathrm{dom} T^*$.

We have that

$$\begin{aligned}
\langle S^*x, Tv\rangle &= \langle (SQ)^*x, v\rangle + \langle R^*(ST^*)x, v\rangle = \\
&= \langle x, SQTv\rangle + \langle (ST)^*x, RTv\rangle = \\
&= \langle S^*x, QTv\rangle + \langle (ST)^*x, RTv\rangle = \\
&= \langle S^*x, Tv\rangle - \langle S^*x, TRTv\rangle + \langle (ST)^*x, RTv\rangle = \\
&= \langle S^*x, T(1-RT)v\rangle + \langle (ST)^*x, RTv\rangle = \\
&= \langle x, ST(1-RT)v\rangle + \langle (ST)^*x, RTv\rangle = \\
&= \langle (ST)^*x, (1-RT)v\rangle + \langle (ST)^*x, RTv\rangle = \\
&= \langle (ST)^*x, v\rangle.
\end{aligned}$$

where in the third row we use the fact that $QTv \in \mathrm{dom} S$, in the sixth one that $(1-RT)v \in \mathrm{dom} ST$ and then $T(1-RT)v \in \mathrm{dom} S$.

So we obtain that there exists $w = (ST)^*x$ such that $\langle S^*x, Tv\rangle = \langle (ST)^*x, v\rangle$ for any $v \in \mathrm{dom} T$, that is $S^*x \in \mathrm{dom} T$ and the lemma is prooved. $\square$

Lemma B.0.9. *Let $T\colon E \to F$ and $S\colon F \to H$ be transverse and regular operators, such that S^* and T^* are transverse too. Then ST is closed and dom$(ST)^*$ is dense in H.*

Proof. By B.0.8 we have that $(ST)^{**} = (T^*S^*)^*$ and, by B.0.6 applied to T^*S^*, we have that $(T^*S^*)^* = ST$. Hence ST is closed.

Moreover by B.0.6 applied to T^* and S^* we have that $\mathrm{dom}(ST)^* = \mathrm{dom} T^*S^*$ is dense in $\mathrm{dom} S^*$, that is dense in H. Hence $\mathrm{dom}(ST)^*$ is dense in H. $\square$

Proposition B.0.10. *Let $T\colon E \to F$ and $S\colon F \to H$ be transverse and regular operators, such that S^* and T^* are transverse too. Then ST is regular.*

Proof. By B.0.6 and B.0.9, we know that ST and $(ST)^*$ are densely defined and that ST is closed.

We know that there exist $\Phi \in \mathcal{L}(M, E)$ and $\Psi \in \mathcal{L}(N, F)$ such that $\mathrm{im}\Phi = \mathrm{dom} T$ and $\mathrm{im}\Psi = \mathrm{dom} S$, with $T\Phi \in \mathcal{L}(M, F)$ and $S\Psi \in \mathcal{L}(N, F)$, for some Hibert A-modules M and N.

Consider $\Xi = R\Psi + (1-RT)\Phi \in \mathcal{L}(N \oplus M, E)$, then $\mathrm{im}\Xi = \mathrm{dom} ST$. Since $ST = (T^*S^*)^*$, we can apply B.0.2 to Ξ and T^*S^*: we obtain that $ST\Xi$ is a morphism and ST is regular.

$\square$

Lemma B.0.11. *Let $T\colon E \to F$ be a regular operator. Then T and T^* are transverse with respect to both E and F.*

Proof. It is sufficient give the proof with respect to E, then inverting the roles of T and T^* we obtain the transversality with respect to F.

Let $U\mathcal{L}(F \oplus E, E \oplus F)$ the flip morphism $(y, x) \mapsto (x - y)$. Since T is regular, then it is densely defined as T^*, by definition of regularity. Then by $\mathrm{im}(1+T^*T)$ is $\mathrm{dom}T + \mathrm{im}T^*$ and by $(1 + T^*T)$ is surjective. Then $E = \mathrm{dom}T + \mathrm{im}T^*$. □

Lemma B.0.12. *Let T and S be regular operators from E to F such that $domT + domS = E$ and $domT^* + domS^* = F$. Then $S + T$ is regular and $(S + T)^* = (S^* + T^*)$.*

Proof. We can write $S + T$ as the composition of the following operators:

- A: $x \mapsto (x, Tx)$ from E to $E \oplus F$, with $\mathrm{dom}A = \mathrm{dom}T$ and $\mathrm{im}A = \mathrm{dom}T \oplus \mathrm{im}T$;
- B: $(y, z) \mapsto Sy + z$ from F to $E \oplus F$, with $\mathrm{dom}B = \mathrm{dom}S \oplus F$ and $\mathrm{im}B = F$.

We want apply B.0.4 and for this aim we have to verify the transversality of A and B and the trasversality of their adojint operators.

It's easy to see that

$$\mathrm{im}A + \mathrm{dom}B = \mathrm{dom}T \oplus \mathrm{im}T + \mathrm{dom}S \oplus F = (\mathrm{dom}T + \mathrm{dom}S) \oplus F = E \oplus F.$$

Let's study $\mathrm{im}B^* + \mathrm{dom}A^*$. We have that

$$\mathrm{dom}A^* = \{(x, y) \in E \oplus F \,|\, \exists z \in E \text{ s.t. } \langle z, w\rangle = \langle x, w\rangle + \langle y, Tw\rangle \,\forall w \in \mathrm{dom}T\},$$

then one can verify that for any $x \in E$ we have trivially that $(x, 0) \in \mathrm{dom}A^*$; similarly for any $y \in \mathrm{dom}T^*$ we have that $(0, y) \in \mathrm{dom}A^*$. Then $\mathrm{dom}A^* \supset E \oplus \mathrm{dom}T^*$. The opposite inclusion is also true: let $(x, y) \in \mathrm{dom}A^*$, then there exists $z \in E$ such that $\langle z - x, w\rangle = \langle y, Tw\rangle$ for all $w \in \mathrm{dom}T$, that is $y \in \mathrm{dom}T^*$. This yields to the equality $\mathrm{dom}A^* = E \oplus \mathrm{dom}T^*$ On the other side

$$\mathrm{dom}B^* = \{x \in F \,|\, \exists (y, z) \in E \oplus F \text{ s.t. } \langle (y, z), (v, w)\rangle = \langle x, Sv\rangle + \langle x, w\rangle \,\forall (v, w) \in \mathrm{dom}S \oplus F\}.$$

Let $y \in \mathrm{dom}S^*$, then we have trivially that $(y, 0) \in \mathrm{dom}B^*$, that is $\mathrm{dom}S^* \subset \mathrm{dom}B^*$. The converse is also true: infact let $x \in \mathrm{dom}B^*$ then there exists $(y, z) \in E \oplus F$ such that $\langle (y, z), (v, 0)\rangle = \langle x, Sv\rangle$ for all $v \in \mathrm{dom}S$; in particular $y \in E$ is such that $\langle y, v\rangle = \langle x, Sv\rangle$ for all $v \in \mathrm{dom}S$, that is $x \in \mathrm{dom}S^*$. Then $\mathrm{dom}S^* = \mathrm{dom}B^*$ It's easy to verify that, if $x \in \mathrm{dom}S^*$, $B^*x = S^*x \oplus x$, in fact

$$\begin{aligned}\langle x, B(v, w)\rangle &= \langle x, Sv\rangle + \langle x, w\rangle = \\ &= \langle S^*x, v\rangle + \langle x, w\rangle = \\ &= \langle (S^*x, x), (v, w)\rangle.\end{aligned}$$

So $\mathrm{im}B^* = \mathrm{im}S^* \oplus F$. Finally we can conclude that

$$\mathrm{im}B^* + \mathrm{dom}A^* = E \oplus \mathrm{dom}T^* + \mathrm{im}S^* \oplus F = E \oplus F.$$

Now we can apply B.0.10 and conclude that $BA = S + T$ is regular.

To prove the last part of Lemma we apply B.0.8 to A and B. In fact it is sufficient to observe that $S * +T*$ is nothing else A^*B^*. Then we have that $(S + T)^* = (BA)^* = A^*B^* = S^* + T^*$. □

Theorem B.0.13. *Let T a regular operator from E to E such that $imT \subset domT$. Then $T + T^*$ is regular and self-adjoint.*

Proof. The proof is immediate as consequence of the previous lemmas. In fact by B.0.11 we have that $E = \mathrm{dom}T^* + \mathrm{im}T \subset \mathrm{dom}T^* + \mathrm{dom}T$. Then we can apply B.0.12 and we get that $T + T^*$ is regular and $(T + T^*)^* = T^* + T$, that is $T + T^*$ is self-adjoint. □

Appendix C

Classifying spaces and 1-cocycles

In this section we are going to recall some basic construction from [7, 16]. Let $G \rightrightarrows X$ be a topological groupoid.

Definition C.0.1. Let Y a topological space and $\{U_i\}_{i\in I}$ an open cover of Y.

- A 0-cocycle with values in G, defined on the cover $\{U_i\}_{i\in I}$ is the data of a continuous application $\mu_i \colon U_i \to G$ for each $i \in I$, such that $\mu_i(y) = \mu_j(y)$ for any pair $(i,j) \in I$ and for any $y \in U_i \cap U_j$. It is actually a global continuous function $\mu \colon Y \to G$.
- A 1-cocycle witb values in G, defined on the cover $\{U_i\}_{i\in I}$ is the data of a continuous application
$$\lambda_{ij} \colon U_i \cap U_j \to G$$
for any pair (i,j), in a such way that, if $y \in U_i \cap U_j \cap U_k$, then $\lambda_{ij}(y)$ is compasable with $\lambda_{jk}(y)$ and
$$\lambda_{ik}(y) = \lambda_{ij}(y)\lambda_{jk}(y).$$

A 1-cocycle witb values in G, defined on the cover $\{U_i\}_{i\in I}$ is the data of a continuous application
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$$\lambda_{ik}(y) = \lambda_{ij}(y)\lambda_{jk}(y).$$
We will say that two 1-cocycles λ and λ' are cohomologous if there exists a function $\mu \colon Y \to G$ such that $\lambda_{ij}(y) = \mu(y)\lambda'_{ij}(y)\mu(y)^{-1}$. Let $H^1(Y, \{U_i\}_{i\in I}, G)$ be the set of the cohomology classes of G-valued 1-cocycles associated to the covering $\{U_i\}_{i\in I}$. Finally define $H^1(Y,G)$ as the limit $\lim H^1(Y, \{U_i\}_{i\in I}, G)$, where $\{U_i\}_{i\in I}$ runs over all open covers of Y.

If $\varphi \colon Y' \to Y$ is a continuous map, then we have a natural map $\varphi^* \colon H^1(Y,G) \to H^1(Y',G)$ that associates to a 1-cocycle $(\lambda_{ij}, \{U_i\}_{i\in I})$ the pull-back $(\lambda_{ij} \circ \varphi, \{\varphi^{-1}(U_i)\}_{i\in I})$.

Remark C.0.2. If $(\lambda_{ij}, \{U_i\}_{i\in I})$ is a G-valued 1-cocycle on Y, then λ_{ii} takes values in $G^{(0)} = X$ for any $i \in I$ and $\lambda_{ij}(x) = \lambda_{ji}^{-1}(x)$ for any $i,j \in I$ and $x \in U_i \cap U_j$.

Let $(\lambda_{ij}, \{U_i\}_{i\in I})$ be a G-valued 1-cocycle on Y, then one can canonically construct a groupoid G_λ^λ over Y in the following way:

- take the disjoint union $\bigsqcup_i U_i$ of all the open sets of the cover;

- consider the map $\Lambda\colon \bigsqcup_i U_i \to X$ given by λ_{ii} on each U_i;
- build the pull-back groupoid $G_\Lambda^\Lambda = \bigsqcup_{i,j} U_i \times_X G \times_X U_j$;
- finally define G_λ^λ as the quotient of G_λ^λ by the following equivalence relation: $(y_i, \gamma, y_j) \sim (y_k, \gamma', y_h)$, with $(y_i, \gamma, y_j) \in U_i \times_X G \times_X U_j$ and $(y_k, \gamma', y_h) \in U_k \times_X G \times_X U_h$, if $y_i = y_k \in U_i \cap U_k$, $y_j = y_h \in U_j \cap U_h$ and $\gamma' = \lambda_{ki}(y_i)\gamma\lambda_{jh}(y_j)$.

Of course the isomorphism class of the groupoid $G_\lambda^\lambda \rightrightarrows Y$ depends only on the cohomology class of λ. Notice that the groupoid $G \rightrightarrows X$ itself is associated to the cocycle $\lambda \in H^1(X, G)$ given by the identity on X.

In the literature there are many equivalent definition of the classifying space BG of G. In this thesis we will take as definition the following proposition.

Proposition C.0.3. *There exists a unique space BG up to homotopy, equipped with a universal 1-cocycle $\xi \in H^1(BG, G)$ such that for any 1-cocycle $\lambda \in H^1(Y, G)$ on a topological space Y, there exists a unique function $f\colon Y \to G$, up to conjugation by 0-cocycles, such that*

$$\lambda = f^*\xi \in H^1(Y, G).$$

Let $\mathcal{BG} \rightrightarrows BG$ be groupoid associated to the 1-cocycle $\xi \in H^1(BG, G)$. One can easily check that for any $f\colon Y \to BG$, the groupoid $G_{f^*\xi}^{f^*\xi} \rightrightarrows Y$ is isomorphic to $\mathcal{BG}_f^f \rightrightarrows Y$, the pull-back of $\mathcal{BG}$ along f.

Remark C.0.4. Let $\lambda \in H^1(Y, G)$ be represented by $(\lambda_{ij}, \{U_i\}_{i\in I})$ and let $\{\alpha_i\}_{i\in I}$ a partition of the unity associated to a locally finite cover $\{U_i\}_{i\in I}$. Then the function $f\colon Y \to BG$ such that $G_\lambda^\lambda \cong \mathcal{BG}_f^f$ is given by

$$\sum_{i\in I} \alpha_i \lambda_{ii}\colon Y \to BG.$$

Now let us assume that G is a Lie groupoid. A smooth 1-cocycle $(\lambda_{ij}, \{U_i\}_{i\in I})$ is transverse if λ_{ii} is transverse for any $i \in I$. It is clear that if λ is a 1-cocycle smooth and transverse, then G_λ^λ is a Lie groupoid. In particular if BG is a smooth manifold, then the function $f\colon Y \to BG$ is a smooth function tranverse with respect to $\mathcal{BG}$.

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