Probability ACSAI 2024-25
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Week 4

Exercise 1. Let A, B be events with $\mathbb{P}(A) = 0.3$, $\mathbb{P}(A \cup B) = 0.5$ and $\mathbb{P}(B) = p$. Find the value of p in the following cases:

1) A and B are disjoint,
2) A and B are independent,
3) A is a subset of B.

Exercise 2. Let A, B, C be three independent events. Prove that the following events are independent.

1) $A^{\complement}, B, C$,
2) $A^{\complement}, B^{\complement}, C$,
3) $A^{\complement}, B^{\complement}, C^{\complement}$.

Exercise 3. Two standard dice are rolled.

1) Show that the event "the sum of the dice is 9" is not independent of the outcome of the first die.
2) Show that the event "the sum of the dice is 7" is independent of the outcome of the first die.
3) Give an intuitive explanation of why the above two cases are different.

Exercise 4. When 3 horses compete in a race, their winning probabilities are $0.3, 0.5, 0.2$ respectively. They compete in 3 consecutive races. Compute the probability of the following events:

1) the same horse wins all races,
2) each horse wins exactly one race.

Exercise 5. A rocket hits its target with probability $1/3$.

1) If 3 rockets are fired, what is the probability that at least one of them hits the target?
2) Find the minimum number of rockets which need to be fired in order to guarantee that the probability that at least one of them hits the target is at least 90%.

Exercise 6. Albert plays 10 rounds at roulette betting on red 1 euro each round. According to the casino rules, the probability of winning each round is $18/37$

1) Compute the probability that Albert wins for the first time at the fifth round.
2) Compute the probability that Albert wins at least two rounds.
3) Compute the probability that after the 10 rounds the net gain by Albert is 2 euro.

Exercise 7. For $n \in \mathbb{N}$ and $p \in (0,1)$ consider the binomial distribution (number of heads in n biased coin tosses)

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, \dots, n.$$

Prove that $P(k)$ is increasing for $k \leq \bar{k}$, with $\bar{k} = \bar{k}(n,p)$ to be determined, and decreasing for $k > \bar{k}$.

Exercise 8.

1) Let $B, N, n \in \mathbb{N}$ with $B, N \geq n$. Prove, via a probabilistic interpretation, the identity:

$$\sum_{k=0}^{n} \binom{B}{k} \binom{N}{n-k} = \binom{N+B}{n}.$$

2) Alice and Bob toss a fair coin n times each. Compute the probability that they obtain the same number of heads.