Probability ACSAI 2024-25
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WEEK 6

Exercise 1. (COUPON COLLECTOR) Your goal is to collect n coupons for your album. What is the probability that you will do so by buying k coupons, $k \geq n$? You may use the uniform probability measure on the k-ple of coupons you buy. [Hint: use the inclusion/exclusion principle]

Exercise 2. Two fair dice are rolled.

1) What is the (conditional) probability that one turns up two spots, given they show different numbers?

2) What is the (conditional) probability that the first turns up 6, given that the sum is k, for each k from 2 through 12?

3) What is the (conditional) probability that at least one turns up 6, given that the sum is k, for each k from 2 through 12?

Exercise 3. An urn contains a red ball and a green ball. One ball is picked at random from the urn, its colour is observed, and the ball is placed back in the urn together with a new ball of the same colour. This procedure is repeated two more times. Let R_i, for $i = 1, 2, 3$, denote the event "the i-th picked ball is red".

1) Compute $P(R_1|R_2)$.

2) Compute $P(R_3|R_2)$.

3) Compute $P(R_1|R_3)$.

Exercise 4. A biased coin is given, with bias p unknown and to be determined. The Maximum Likelihood Estimator $\hat{p}$ for p is defined by requiring that $\hat{p}$ maximises the probability of the observed event. Compute the Maximum Likelihood Estimator $\hat{p}$ for the following two experiments:

1) Toss the coin 200 times and obtain 67 heads.

2) The first head is observed on the fifth coin toss.

Determine the Maximum Likelihood Estimator $\hat{p}$ in the following more general cases: (i) toss the coin n times and obtain k heads; (ii) the first head is observed on the h-th coin toss.

Exercise 5. A biased coin, with probability of head $p \in (0, 1)$, is tossed repeatedly. Given $a, b \geq 1$, compute the probability that the coin gives a heads before b tails.

Exercise 6. A referendum is called in a population of n individuals, all of them having the right to vote. Each individual will vote with probability $1/2$, independently of the others. Moreover, if the individual votes, he/she votes YES with probability $1/2$, independently of the others.

1) Compute the probability that a given individual goes to vote and votes YES.

2) Compute the probability that the number of YES votes is k, for $k = 0 \ldots, n$.

3) Knowing that there have been exactly k YES votes, compute the probability that exactly m individuals voted, for $m = k, \ldots, n$.

Exercise 7. (Monty Hall problem) In a TV show a guest is asked to choose among three doors. Behind one door there is a car while behind the other two there are two goats, denoted goat A and goat B. The guest wins what there is behind the chosen door. After the guest has made his initial choice, that door is not opened. Instead, it is opened one of two unchosen doors which reveals a goat. The guest is then offered the possibility of exchanging the door he initially chose with the one still closed. Let p be the (conditional) probability that the car is behind the last offered door. Show that:

1) $p = 2/3$ if the opened door is choosen to always reveal a goat without bias between the two goats;

2) $p = 1/2$ if the opened door is choosen at random;

3) $p = 1/(1 + a)$ if the goat seen is the goat A and the opened door is selected so that a goat is always chosen and in the case there are two goats it is opened the door revealing goat A with probability a.