Probability ACSAI 2024-25
L. Bertini and V. Silvestri

WEEK 7

Exercise 1. Write a random word made of 10 characters by choosing a character uniformly at random 10 times independently, from an alphabet of 26 characters. Let X be the random variable that counts the number of A's in the resulting word. What is the distribution of X? Compute $E(X)$, that is the average number of A's in a random word of length 10. How about repeating the experiment with a word of length N?

Exercise 2. A fair 6-faced die is tossed, and let X denote the observed value.

1) Compute the probability distribution of X.
2) Compute the expected value of X.
3) Compute the variance of X.

Answer the above questions in the case of an n-faced die, $n \in \mathbb{N}$.

Exercise 3. Toss two fair 6-faced dice, and let X denote the minimum between the observed values.

1) Compute the probability distribution of X.
2) Compute the expected value of X.

Exercise 4. A box contains 10 transistors, of which 3 are broken. You check one transistor at a time (without replacement) until you find a broken one. Compute the expected value of the number of checked transistors.

Exercise 5. Consider a multiple choice exam with the following rules. There are a total of 10 questions, and for each question there are 4 possible answers, of which exactly one is correct. The evaluation algorithms is as follows: each correct answer gets $+3$ marks, and each wrong answer gets -1 mark. Alice did not study, so she answers all 10 questions at random.

1) Compute the probability that Alice passes the exam (i.e. she scores at least 18/30).
2) Compute Alice's expected final grade.
3) Compute the variance of Alice's final grade.

Exercise 6. (HYPERGEOMETRIC RANDOM VARIABLE) Consider an urn with a white balls and b black balls. You pick k balls sequentially without replacement ($k \leq a+b$). Let X_i, $i = 1, \dots, k$ be the random variable taking value 1 if the i-th ball is white and 0 if it is black. Let, moreover, X denote the total number of white balls picked.

1) Compute the distribution of X.
2) Compute the expected value of X.
(You should do both the direct calculation using the distribution of X, and the calculation using the expectations of the X_i's.)
3) Compute the covariance between X_i and X_j, $i, j = 1, \dots, k$.

4) Compute the variance of X.
(You should do both the direct calculation using the distribution of X, and the calculation using that $X = \sum_{i=1}^{k} X_i$ and the previous question.)

Exercise 7.(A LIMIT THEOREM FOR THE HYPERGEOMETRIC DISTRIBUTION) For $a, b, k \in \mathbb{N}$, consider the *hypergeometric distribution*

$$P_{a,b,k}(h) = \frac{\binom{a}{h}\binom{b}{k-h}}{\binom{a+b}{k}}, \qquad h = 0, \dots, k.$$

1) Compute the limit of $P_{a,b,k}$ as $a, b \to \infty$ with $a/(a+b) \to p \in (0,1)$ (k is fixed).

2) Discuss a probabilistic interpretation of the result.