Probability ACSAI 2024-25
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WEEK 8

Exercise 1. Let X_1, X_2 be independent random variables, uniformly distributed in $\{1, \ldots, n\}$.

1) Compute the probability distribution of $X_1 + X_2$.
2) Compute the expected value of $X_1 + X_2$.
3) Compute the variance of $X_1 + X_2$.

Exercise 2. A fair die is tossed repeatedly, until 5 or 6 is obtained. Let T denote the number of tosses, and let X denote the number seen on the die in the last toss.

1) Compute $\mathbb{P}(T = 3, X = 5)$.
2) Compute the probability distribution of T.
3) Compute the probability distribution of X.
4) Are T and X independent random variables? Explain.

Exercise 3. How many times should you roll – on average – a fair die in order to see all faces?
HINT: Use geometric random variables to avoid computations.

Exercise 4. The breaking time of component C_i is given by a random variable T_i, $i = 1, \ldots, k$. Assume that the random variables $T_1, \ldots, T_k$ are independent, and that $T_i \sim \text{Geom}(p)$ with $p \in (0, 1)$, $i = 1, \ldots, k$.

1) Find the probability distribution of the breaking time of the circuit C_{ser} obtained by organising in series all the components C_1, ..., C_k.
2) Find the probability distribution of the breaking time of the circuit C_{par} obtained by organising in parallel all the components C_1, ..., C_k.

Exercise 5. In a Bernoulli scheme with head probability $p \in (0, 1)$ let X denote the random variable counting the number of consecutive outcomes identical to the first one: that is, $X = 1$ if the first coin toss gives head and the second gives tail or the first gives tail and the second head, $X = 2$ if two heads followed by a tail or two tails followed by a head, and so on.

1) Find the probability distribution of X.
2) Compute the expected value of X.
3) Compute the variance of X.

Exercise 6. Assume that – on average – 2% of the population is left-handed. Given a sample of 100 individuals, use the Poisson approximation for Binomial random variables to compute the probability that at least 3 individuals are left-handed.

Exercise 7. A biased coin with head probability $p \in (0, 1)$ is tossed a random number of times (indipendently of the results of the coin tosses) with Poisson distribution of parameter $\lambda > 0$. Find the probability distribution of the total number of heads and tails obtained, and prove that these two random variables are independent.