Probability ACSAI 2024-25
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Week 9

Exercise 1. (Independence of random variables) Let X and Y be random variables.

1) Prove that if X is a degenerate random variable, that is $X = c$ for some $c \in \mathbb{R}$, then X and Y are independent.

2) Prove that if X and Y are binary, that is $\big|\mathrm{Im}(X)\big| = \big|\mathrm{Im}(Y)\big| = 2$, then the random variables X and Y are independent if and only if $\mathrm{cov}(X, Y) = 0$.

3) Give an example of two random variables X and Y such that $\mathrm{cov}(X, Y) = 0$ but X and Y are not independent.

Exercise 2. (Alternative proof of the inclusion/exclusion principle) For an event A let $\mathbf{1}_A$ denote the random variable which takes value 1 if $\omega \in A$ and value 0 if $\omega \in A^c$.

1) Let A and B be events. Check that $\mathbf{1}_{A^c} = 1 - \mathbf{1}_A$ and that $\mathbf{1}_{A\cap B} = \mathbf{1}_A \mathbf{1}_B$.

2) Let $a_1, b_1, \dots, a_n, b_n \in \mathbb{R}$. Convince yourself of the binomial identity:

$$\prod_{i=1}^{n}(a_i + b_i) = \sum_{I\subset\{1,\dots,n\}} \prod_{i\in I} a_i \prod_{j\in I^c} b_j.$$

3) Use the previous points together with the properties of the expectation to prove the inclusion/exclusion principle.

Exercise 3. (Confidence intervals) Consider a biased coin with head probability p unknown. In order to estimate p, the coin is tossed n times and p is estimated via S_n/n, where S_n is the number of heads in the n coin tosses. Given $\delta > 0$, determine how large should n be in order for the probability that $|S_n/n - p| < \delta$ to be at least 95%.

Exercise 4. (Coupon collector) Consider an album with n coupons. In order to complete the album, you buy one coupon per day (uniformly chosen among all possible coupons, independently of the other days).

1) Show that the expected value of the number of days needed to complete the album is given by

$$K_n = n\Big[1 + \frac{1}{2} + \cdots + \frac{1}{n}\Big].$$

2) Prove the asymptotic relation

$$K_n = n\big[\log n + o(\log n)\big].$$

Hint: Formalise the problem in terms of geometric random variables.

Exercise 5. Show that if a random variable $X \geq 0$ takes integer values, then

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} \mathbb{P}(X \geq k).$$

Exercise 6. A server can process two type of requests: A and B. Assume that the total number of requests is described by a Poisson random variable with parameter

5 and each request, independently from the others, is of type A with probability 2/3 and of type B with probability 1/3.

1) Compute the probability that the server receives 5 requests of type A.

2) Compute the expectation valued of the requests of type A and B.

3) Knowing that the total number of requests is 9, compute the probbility that 6 are of type A.

4) Knowing that the total number of requests is at least three, compute the probability that the first three are of type A.

Exercise 7. Alice (A) and Bob (B) take place in a prize game. They have a fair die (common to A and B) and two independent coins (one for A and one for B) which return head (H) with probability $p \in (0, 1)$ and tail (T) with probability $1 - p$. The rule of the game are the following. First the die is tossed, if it returns 1 there are no prizes, otherwise both A and B toss their coin until it returns H and they win 1 euro for each toss they did. Let X and Y the euro won by A and B.

1) Are X and Y independent random variables?

2) Compute the distribution of X and Y.

3) Compute the expectation value of X and Y.

4) Compute the covariance between X and Y.