Probability ACSAI 2024-25
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Week 10

Exercise 1. From a collection of 7 batteries, of which 3 are new, 2 are used but working and 2 are broken, 3 batteries are chosen at random. Let X and Y denote respectively the number of new batteries and used working batteries, among the ones chosen.

1) Determine the joint distribution of (X, Y) and the marginal distributions of X and Y.

2) Compute $\text{cov}(X, Y)$. Are the random variables X and Y independent?

3) The three chosen batteries are inserted in a machine which works if none of them is broken. Determine the probability that the machine works.

Exercise 2. A die has a blue face, two red faces and three green faces. The die is rolled twice. Let R denote the number of times a red face is obtained, and G denote the number of times a green face is obtained.

1) Construct the table of the joint distribution of (R, G).

2) Determine the distribution of $Z = \max\{R, G\}$ and compute $\mathbb{E}(Z)$ and $\mathbb{V}(Z)$.

Exercise 3. The electrical components produced in a factory are defective, independently from each other, with probability p, and they are functioning with probability $1 - p$, where $p \in (0, 1)$. The components are subject to a quality check with the following procedure: each component, independently of the others, is checked with probability α and it is not checked with probability $1-\alpha$, where $\alpha \in (0, 1)$. A component which is found defective is discarded, while the others are put on sale. Suppose that the factory produced n components.

1) Compute the distribution of the number of components which are discarded after the quality check procedure.

2) Knowing that k components have been discarded after the quality check procedure, for $k = 0, 1, \dots, n$, compute the distribution of the number of defective components among the $n - k$ components which are put on sale.

Exercise 4. Let X, Y be two independent Bernoulli random variables with parameter p. Define

$$Z = X(1 - Y) \quad \text{e} \quad W = 1 - XY.$$

1) What is the joint distribution of (Z, W)?

2) What are the marginal distributions of Z and W?

3) For which values of p are the random variables Z and W independent?

Exercise 5. Let X be a continuous random variable with $X \geq 0$. Show that

$$\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > x)\, dx.$$

Exercise 6. A and B play the following game: A writes 1 or 2 on a piece of paper, and B guesses the number written by A. If A wrote $i \in \{1, 2\}$ and B guesses right, then A pays i euros to B. If, on the other hand, B guesses wrong, then B pays 0.75 euros to A.

Assume that B uses a random strategy, guessing 1 with probability p and 2 with probability $1 - p$.

1) Knowing that A wrote 1, determine the expected gain of B.

2) Knowing that A wrote 2, determine the expected gain of B.

3) Determine the value of p which maximises the minimum between the two expected gains above.

Suppose that A uses a random strategy, writing 1 with probability q and 2 with probability $1 - q$.

4) Knowing that B guessed 1, determine the expected loss of A.

5) Knowing that B guessed 2, determine the expected loss of A.

6) Determine the value of q which minimises the maximum between the two expected gains above.

Compare the answers to 3) and 6).

Exercise 7. Let X, Y be two random variables. For $y \in \text{Im}(Y)$ let $\mathbb{E}(X|y)$ denote the expected value of X conditional on the event $\{Y = y\}$ (that is, the expected value with respect to the conditional distribution of X given the event $\{Y = y\}$). Prove that

$$\mathbb{E}(X) = \sum_{y \in \text{Im}(Y)} \mathbb{P}(Y = y)\, \mathbb{E}(X|y).$$

Exercise 8. Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{1}{6}(x + k) & \text{if } x \in [0, k] \\ 0 & \text{otherwise.} \end{cases}$$

1) Compute the value of $k \geq 0$.

2) Compute $\mathbb{P}(1 \leq X \leq 2)$.