Probability ACSAI 2024-25
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WEEK 11

Exercise 1. Let X_i, $i = 1, 2$ be independent uniform random variables in $[0, 1]$.

1) Compute the distribution (i.e. the probability density function) of $X_1 + X_2$.
2) Compute the distribution (i.e. the probability density function) of $\max\{X_1, X_2\}$.
3) Compute the distribution (i.e. the probability density function) of $\min\{X_1, X_2\}$.

Exercise 2. Let U be a uniform random variable in $[0, 1]$, and let V be a random variable independent of U, uniformly distributed in $[-1, 1]$.

1) Compute the distribution (i.e. the probability density function) of V^2.
2) Compute the distribution (i.e. the probability density function) of $\log(1/U)$.
3) Compute $\mathbb{P}(U \leq V)$.

Exercise 3. Let X be a Gaussian random variable with mean 2 and variance 25. Provide answers to the following questions by using the Gaussian integral tables.

1) Compute $\mathbb{P}(|X - 2| \geq 7)$.
2) Compute $\mathbb{P}(0 \leq X \leq 7)$.
3) Determine α such that $\mathbb{P}(X \geq \alpha) \leq 0.1$.

Exercise 4. In order to transmit a bit from a source A to a receiver B via a pair of electrical wires, one applies a potential difference of $+2\,\mathrm{V}$ for the value 1 and $-2\,\mathrm{V}$ for the value 0. Due to electromagnetic disturbance, if A applies $\mu = \pm 2\,\mathrm{V}$, B reads $X = \mu + Z$, where Z represents the noise, described by a Gaussian random variable of mean 0 and variance $1\,\mathrm{V}^2$. After reading X, B registers the message with the following rule: if $X \geq 0.5\,\mathrm{V}$ then B registers 1, while if $X < 0.5\,\mathrm{V}$ then B registers 0.

1) If A sends 0, compute the probability that B registers 1.
2) If A sends 1, compute the probability that B registers 0.

Suppose now that A sends 0 or 1 with equal probability.

3) Compute the probability that B registers 1.
4) If B has registered 1, compute the probability that the message registered coincides with the message sent.

Exercise 5. Two fair dice are rolled 300 times. Let X denote the number of rolls at which the pair $(1, 1)$ is obtained.

1) Compute $\mathbb{E}(X)$ and $\mathbb{V}(X)$.
2) Using the Gaussian approximation, compute the probability of obtaining $(1, 1)$ more than 10 times.

Now consider the case where the two dice are rolled n times.

3) Using the Gaussian approximation, determine how large should n be so that the probability of obtaining $(1, 1)$ at least 10 times exceeds $1/2$.