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ASYMPTOTIC PROBABILITY OF ENERGY INCREASING SOLUTIONS TO THE HOMOGENEOUS BOLTZMANN EQUATION

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Weak solutions to the homogeneous Boltzmann equation with increasing energy have been constructed by Lu and Wennberg. We consider an underlying microscopic stochastic model with binary collisions (Kac's model) and show that these solutions are atypical. More precisely, we prove that the probability of observing these paths is exponentially small in the number of particles and compute the exponential rate. This result is obtained by improving the established large deviation estimates in the canonical setting. Key ingredients are the extension of Sanov's theorem to the microcanonical ensemble and large deviations for the Kac's model in the microcanonical setting.

1. Introduction. The derivation of the Boltzmann equation from an underlying microscopic dynamics of N interacting particles is a paradigmatic problem in nonequilibrium statistical mechanics. It is based on the validity of the *Stosszahlansatz* with probability one in the limit $N \to +\infty$. At a more refined level, it is possible to analyze the corresponding large deviations, whose derivation is related to the validity of the *Stosszahlansatz* with probability super-exponentially close to one for N large.

In this perspective, the most challenging case of Newtonian dynamics of hard spheres in the Boltzmann–Grad limit has been recently discussed in [3]. Nevertheless, the case of stochastic dynamics presents interesting features. The first result in this setting has been obtained in [13], where a large deviation upper bound is derived in the space homogeneous case. A complete large deviation principle has been obtained in [20] for a space inhomogeneous model with a finite set of velocities. In [1] a large deviation upper bound is achieved for a homogeneous model which conserves momentum but not energy, while the matching lower bound is obtained for a restricted class of paths. A similar result, in the case of energy and momentum conservation, has been proven in [9]. In this case the upper and lower bound match for a subset of paths for which energy is conserved.

For energy preserving microscopic dynamics with unbounded velocities, a main obstacle to a complete proof of large deviations is the occurrence of macroscopic paths with finite rate function that violate the conservation of the energy. In particular, as discussed in [9], a class of such paths is given by the solutions to the homogeneous Boltzmann equations constructed by Lu and Wennberg in [15], for which the energy is increasing. Another example of large deviation asymptotic for nonconserving energy path has been constructed in [2], for a Kac-like microscopic dynamics with discrete energies. More precisely, as proven in [9], the upper bound rate function derived in [13] vanishes on Lu and Wennberg solutions, while their asymptotic probability is e^{-cN}, which implies the upper bound rate function in [13] is not optimal.

The homogeneous Boltzmann equation with hard sphere cross-section reads as

$$\partial_t f_t(v) = \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{d}v_* \int_{S_{d-1}} \mathrm{d}\omega \, \big|(v - v_*)\cdot\omega\big| \big(f_t(v') f_t(v'_*) - f_t(v) f_t(v_*)\big), \tag{1.1}$$

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where S_{d-1} is the sphere in $\mathbb{R}^d$ and $\mathrm{d}\omega$ is the Haar measure on S_{d-1}. The associate Cauchy problem has a unique solution in the class of functions with constant energy [17]. Let us discuss how the Lu and Wennberg solutions, with increasing energy, can be constructed in the special case in which the energy has a unique jump at time zero. Consider a sequence of initial densities f_0^n such that f_0^n converges weakly to f_0 but $e := \lim_n \int f_0^n(v)|v|^2\,\mathrm{d}v > \int f_0(v)|v|^2\,\mathrm{d}v$, namely a fraction of energy evaporates at infinity. Denoting by f_t^n the unique energy conserving solution of the homogeneous Boltzmann equation with initial datum f_0^n, we then have that f_t^n, $t \geq 0$, converges to a solution to the homogeneous Boltzmann equation, with initial datum f_0, but the energy has a positive jump at time 0. Observe that this construction does not yield a jump in the energy if the total cross section is bounded, in fact in this case there is uniqueness of the solution without the requirement of energy conservation. A model with this feature has been analyzed in [8].

A main result of this paper is the proposal of a rate function that improves the one in [13], being strictly positive on Lu and Wennberg solutions. In particular we consider a Kac walk with the hard sphere cross section, and prove the large deviation upper bound with such rate function. The matching large deviation lower bound is achieved for both Lu and Wennberg solutions and the same restricted class of path as in [1, 9].

As it is clear from the previous construction, Lu and Wennberg solutions can be produced from a microscopic model only if there exists a fluctuation of the initial energy and then following the typical behavior. To introduce the improved rate function we consider first the case in which the initial velocities are sampled from the microcanonical ensembles, namely the total energy and momentum are not random, and given by (eN, uN). After [1, 2, 9], we consider as empirical observable the pair (π^N, Q^N) where π^N is the empirical distribution of velocities, while the empirical flux Q^N records the collision times together with the incoming and outgoing velocities. The microcanonical rate function reads

$$I_{e,u}(\pi, Q) = H_{e,u}(\pi_0) + J_{e,u}(\pi, Q), \tag{1.2}$$

where $H_{e,u}$ takes into account the fluctuation of the initial data, while $J_{e,u}$ is the dynamical contribution, that is defined as follows. Set $\mathrm{d}Q^\pi := \frac{1}{2}\,\mathrm{d}\pi \otimes \mathrm{d}\pi\, B\,\mathrm{d}\omega\,\mathrm{d}t$, with $B = B(v - v_*, \omega) = \frac{1}{2}|(v - v_*) \cdot \omega|$, and let $J(\pi, Q)$ be the relative entropy of Q with respect to Q^π, namely

$$J(\pi, Q) = \int \left\{ \mathrm{d}Q \log \frac{\mathrm{d}Q}{\mathrm{d}Q^\pi} - \mathrm{d}Q + \mathrm{d}Q^\pi \right\}. \tag{1.3}$$

Then, by the microcanonical constraint, $J_{e,u}(\pi, Q)$ is equal to $J(\pi, Q)$ if the energy of π does not exceed e and its momentum is equal to u, while $J_{e,u}(\pi, Q) = +\infty$ otherwise. The functional $H_{e,u}$ will be derived by extending Sanov's theorem to the microcanonical ensemble. In particular, $H_{e,u}(\pi_0)$ is infinite when the energy of π_0 exceed e, but it can be finite when the energy is below e. Namely, loss of energy at time 0 occurs with exponentially small probability. According to (1.2), the asymptotic probability of Lu and Wennberg solutions is then $\exp(-N H_{e,u}(\pi_0))$.

We then analyze the case in which the initial velocities are sampled from the canonical ensemble, namely are i.i.d. m-distributed random variables. The canonical rate function can then be obtained from (1.2) as follows

$$I(\pi, Q) = \inf_{e,u} \big(A(e, u) + I_{e,u}(\pi, Q) \big), \tag{1.4}$$

where A is the rate function for the energy and momentum of the sum of i.i.d. m-distributed random variables, given by Cramér's theorem. The rate function introduced in [13] and further analyzed in [9] is given by

$$\mathcal{I}(\pi, Q) = \mathrm{Ent}(\pi_0|m) + J(\pi, Q),$$

where $\mathrm{Ent}(\pi_0|m)$ is the relative entropy. In particular, $\mathcal{I}$ vanishes on Lu and Wennberg solutions. We show that I defined in (1.4) is larger than $\mathcal{I}$ and vanishes only on the unique energy conserving solution to (1.1). Moreover, we compute explicitly its value on the Lu and Wennberg solutions, which is given by $c\Delta\mathcal{E}$, where c is a strictly positive constant depending on the tail of initial distribution m and $\Delta\mathcal{E}$ is the total gain of the energy. Hence, the asymptotic probability of Lu and Wennberg solutions is $\mathrm{e}^{-cN\Delta\mathcal{E}}$.

The present work is organized as follows. In Section 2 we consider the static case, by analyzing the large deviations of the empirical measure when the velocities are sampled from the microcanonical ensemble. As discussed before, we show that the large deviation functional is finite on probability measures with energy evaporation. In Section 3 we state the large deviation principle for the Kac model with hard sphere cross section and microcanonical initial data. The corresponding proof is carried out in Sections 4, 5. In Section 6 we derive the large deviation asymptotic for the Kac model with canonical initial distribution. Section 7 is finally devoted to the asymptotic probability of Lu and Wennberg solutions.

2. Sanov theorem for microcanonical ensemble.

Sanov's theorem, that describes the asymptotic behavior of the empirical measures associated to a sequence of N i.i.d. random variables, is a basic result in the theory of large deviations. A natural question is to replace the independence assumption by some dependency structure. For instance, the case of the empirical measure associated to Markov chains is the content of the classical Donsker–Varadhan theorem. We here analyze the case in which the underlying sequence of random variables is sampled according to a microcanonical ensemble, that can be realized by conditioning i.i.d. random variables to the sum of their squares, that is, to the total kinetic energy in the physical interpretation. A particular case of this situation has been previously discussed in [11]; there it is in fact analyzed the case where N real random variables are sampled according to the uniform measure on the sphere of radius $\sqrt{N}$ on $\mathbb{R}^N$ and corresponds to the microcanonical ensemble associated to i.i.d. Gaussians. A peculiar feature of this setting is the possibility of observing—at the large deviations level—probabilities that violate the microcanonical constraint. More precisely, while for each N the law of the empirical measure is supported by the probabilities with fixed second moment, the large deviations rate function is finite also for probabilities with second moment strictly smaller than the prescribed value. In view of the application to homogeneous Boltzmann equations, we shall next consider microcanonical ensembles that are obtained by conditioning both to the total energy and to the total momentum.

Fix hereafter $d \geq 2$ and denote by $\mathcal{P}(\mathbb{R}^d)$ the set of probability measures on $\mathbb{R}^d$ equipped with the topology induced by the weak convergence and the associated Borel σ-algebra. Let $\boldsymbol{\zeta}: \mathbb{R}^d \mapsto [0, +\infty) \times \mathbb{R}^d$ be the map given by $\boldsymbol{\zeta} = (\zeta_0, \zeta)(v) = (|v|^2/2, v)$. For $\boldsymbol{\gamma} = (\gamma_0, \gamma) \in \mathbb{R} \times \mathbb{R}^d$ let $\boldsymbol{\gamma} \cdot \boldsymbol{\zeta}$ be the function on $\mathbb{R}^d$ given by $\gamma_0\zeta_0 + \gamma \cdot \zeta$. We shall consider probabilities $m \in \mathcal{P}(\mathbb{R}^d)$ satisfying the following conditions.

ASSUMPTION 2.1. There exists $\gamma_0^* \in (0, +\infty]$ such that:

(i) m is absolutely continuous with respect to the Lebesgue measure and m is strictly positive on open sets;

(ii) $m(\mathrm{e}^{\gamma_0\zeta_0}) < +\infty$ for any $\gamma_0 \in (-\infty, \gamma_0^*)$, and $\lim_{\gamma_0 \uparrow \gamma_0^*} m(\mathrm{e}^{\gamma_0\zeta_0}) = +\infty$;

(iii) for each $\boldsymbol{\gamma} = (\gamma_0, \gamma) \in (-\infty, \gamma_0^*) \times \mathbb{R}^d$ the Fourier transform of $\frac{\mathrm{d}m}{\mathrm{d}v}\mathrm{e}^{\boldsymbol{\gamma}\cdot\boldsymbol{\zeta}}$ belongs to $L^1(\mathbb{R}^d)$;

(iv) there exists $c > 0$ such that $\frac{\mathrm{d}m}{\mathrm{d}v} \geq \frac{1}{c}\exp\{-c|v|^2\}$.

Condition (iv) is mainly technical, and will be used only to derive the lower bound for the dynamical rate function.

We observe that, as follows from Hölder inequality, the map $(-\infty, \gamma_0^*) \times \mathbb{R}^d \ni \boldsymbol{\gamma} \mapsto \log m(\mathrm{e}^{\boldsymbol{\gamma}\cdot\boldsymbol{\zeta}})$ is strictly convex. Set $Z = \{(e,u) \in (0,+\infty) \times \mathbb{R}^d : e > |u|^2/2\}$, then $\nabla \log m(\mathrm{e}^{\boldsymbol{\gamma}\cdot\boldsymbol{\zeta}})$ is a bijection from $(-\infty, \gamma_0^*) \times \mathbb{R}^d$ to Z. We denote by $(e,u) \mapsto \boldsymbol{\gamma}(e,u)$ the inverse map and by $m_{e,u}$ the probability on $\mathbb{R}^d$ defined by

$$m_{e,u}(\mathrm{d}v) := \frac{\mathrm{e}^{\boldsymbol{\gamma}(e,u)\cdot\boldsymbol{\zeta}(v)}}{m(\mathrm{e}^{\boldsymbol{\gamma}(e,u)\cdot\boldsymbol{\zeta}})} m(\mathrm{d}v). \tag{2.1}$$

In words, $m_{e,u}$ is the *exponential tilt* of m such that $m_{e,u}(\boldsymbol{\zeta}) = (e,u)$. Namely, u and e are the average values of velocity and total energy, respectively. Note that $m_{e,u}$ satisfies the conditions in Assumption 2.1 with γ_0^* replaced by $\gamma_0^* - \gamma_0(e,u) > 0$. We denote by U the internal energy defined by the relation $e = U + |u|^2/2$, so that U is the expected value of $|v-u|^2/2$.

Let $\Sigma^N := (\mathbb{R}^d)^N$ be the configuration space for N velocities in $\mathbb{R}^d$. Given $(e,u) \in Z$, we denote by

$$\Sigma_{e,u}^N := \left\{ \boldsymbol{v} \in (\mathbb{R}^d)^N : \frac{1}{N}\sum_{i=1}^N \boldsymbol{\zeta}(v_i) = (e,u) \right\} \tag{2.2}$$

the set of configurations with total momentum Nu and total energy Ne.

Let μ^N be the probability on Σ^N given by $m^{\otimes N}$, interpreted as the canonical ensemble. Let also $(e,u) \mapsto \nu_{e,u}^N$ be a regular version of the probability μ^N conditioned to $\frac{1}{N}\sum_{i=1}^N \boldsymbol{\zeta}(v_i)$. In particular, $\nu_{e,u}^N$, interpreted as the microcanonical ensemble, is the probability supported by $\Sigma_{e,u}^N$ informally given by $\nu_{e,u}^N = \mu^N(\,\cdot\,|\Sigma_{e,u}^N)$. As $N \to \infty$ the one-marginal of $\{\nu_{e,u}^N\}$ converge to $m_{e,u}$ (equivalence of ensembles in the thermodynamic limit) [5, 6, 18]. Our aim is to describe the corresponding large deviations asymptotic. In order to apply this result to Kac's walk with a canonical initial distribution of the velocities, the large deviation principle will be proven uniformly for (e,u) in compact subsets of Z.

We define the *empirical measure* as the map $\pi^N : \Sigma^N \to \mathcal{P}(\mathbb{R}^d)$ given by

$$\pi^N(\boldsymbol{v}) = \frac{1}{N}\sum_{i=1}^N \delta_{v_i}. \tag{2.3}$$

Given two probabilities m_1, m_2, recall that the relative entropy $\mathrm{Ent}(m_2|m_1)$ is defined as $\mathrm{Ent}(m_2|m_1) = \int \mathrm{d}m_1 \rho \log \rho$, where $\mathrm{d}m_2 = \rho\, \mathrm{d}m_1$, understanding that $\mathrm{Ent}(m_2|m_1) = +\infty$ if m_2 is not absolutely continuous with respect to m_1.

Given $(e,u) \in Z$ set

$$C_{e,u} := \{\pi \in \mathcal{P}(\mathbb{R}^d) : \pi(\zeta) = u,\ \pi(\zeta_0) \le e\} \tag{2.4}$$

that is, a compact and convex subset of $\mathcal{P}(\mathbb{R}^d)$. Note that $C_{e,u}$ is the closure in $\mathcal{P}(\mathbb{R}^d)$ of the set of probabilities π satisfying the microcanonical constraint $\pi(\boldsymbol{\zeta}) = (e,u)$.

THEOREM 2.2. *Fix $(e,u) \in Z$ and a sequence $(e_N, u_N) \to (e,u)$. If m satisfies item (i)–(iii) in Assumption 2.1 then the family of probabilities $\{\nu_{e_N,u_N}^N \circ (\pi^N)^{-1}\}_N$ on $\mathcal{P}(\mathbb{R}^d)$ satisfies a large deviation principle with speed N and good convex rate function $H_{e,u} : \mathcal{P}(\mathbb{R}^d) \to [0,+\infty]$ given by*

$$H_{e,u}(\pi) = \begin{cases} \mathrm{Ent}(\pi|m_{e,u}) + [\gamma_0^* - \gamma_0(e,u)][e - \pi(\zeta_0)] & \text{if } \pi \in C_{e,u}, \\ +\infty & \text{otherwise}. \end{cases} \tag{2.5}$$

The rate function $H_{e,u}$ can be seen as the canonical rate function $\mathrm{Ent}(\,\cdot\,|m_{e,u})$ with an extra penalization for violations of the energy constraint. When m is the standard Gaussian on $\mathbb{R}$

and the momentum constraint is dropped, this result reduces to the one obtained in [11]. If $\gamma_0^* = +\infty$ we understand that $H_{e,u}(\pi) = +\infty$ when $\pi(\zeta_0) < e$. In particular, when the tails of m are sub-Gaussian, then the energy constraint holds also at the large deviation level. In the terminology of the calculus of variations, as shown in Lemma 7.4 below, the functional $H_{e,u}$ can be obtained by taking the lower semicontinuous envelope of the functional given by $\mathrm{Ent}(\cdot|m_{e,u})$ when $\pi(\zeta_0) = e$ and infinite otherwise.

While the arguments in [11] rely on the representation of the uniform measure on the spheres in terms of i.i.d. Gaussian, the proof of the above theorem will be achieved by applying the Gärtner–Ellis theorem, which provides the large deviation rate function as the Legendre transform of the log-moment generating function. To this end, for $\phi \in C_{\mathrm{b}}(\mathbb{R}^d)$ set

$$\mathrm{d}m_{e,u}^{\phi} := \frac{\mathrm{d}m_{e,u}\mathrm{e}^{\phi}}{m_{e,u}(\mathrm{e}^{\phi})}, \tag{2.6}$$

and

$$\Lambda_{e,u}(\phi) := -\boldsymbol{\gamma} \cdot (e,u) + \log m_{e,u}(\mathrm{e}^{\phi+\boldsymbol{\gamma}\cdot\zeta}), \tag{2.7}$$

where $\boldsymbol{\gamma} = \boldsymbol{\gamma}(\phi)$ is chosen so that

$$\frac{m_{e,u}^{\phi}(\exp\{\boldsymbol{\gamma}\cdot\boldsymbol{\zeta}\}\boldsymbol{\zeta})}{m_{e,u}^{\phi}(\exp\{\boldsymbol{\gamma}\cdot\boldsymbol{\zeta}\})} = (e,u), \tag{2.8}$$

namely, it is chosen in order that the exponential tilt of m^{ϕ} has average energy and momentum (e,u).

LEMMA 2.3. *For each $\phi \in C_{\mathrm{b}}(\mathbb{R}^d)$,*

$$\lim_{N\to\infty} \frac{1}{N} \log \nu_{e_N,u_N}^N(\mathrm{e}^{N\pi^N(\phi)}) = \Lambda_{e,u}(\phi). \tag{2.9}$$

PROOF. As simple to check,

$$\left|\frac{1}{N}\log \nu_{e_N,u_N}^N(\mathrm{e}^{N\pi^N(\phi_2)}) - \frac{1}{N}\log \nu_{e_N,u_N}^N(\mathrm{e}^{N\pi^N(\phi_1)})\right| \le \sup_{v\in\mathbb{R}^d}|\phi_2(v)-\phi_1(v)|.$$

By a density argument, it is therefore enough to prove the statement for smooth ϕ.

Observing that $m^{\otimes N}(\cdot|\Sigma_{e,u}^N) = m_{e,u}^{\otimes N}(\cdot|\Sigma_{e,u}^N)$, for $\boldsymbol{\gamma} \in (-\infty, \gamma_0^*) \times \mathbb{R}^d$ we write

$$\frac{1}{N}\log \nu_{e_N,u_N}^N(\mathrm{e}^{N\pi^N(\phi)}) = -\boldsymbol{\gamma}\cdot(e_N,u_N) + \frac{1}{N}\log m_{e_N,u_N}^{\otimes N}(\mathrm{e}^{N\pi^N(\phi+\boldsymbol{\gamma}\cdot\boldsymbol{\zeta})}|\Sigma_{e_N,u_N}^N).$$

By a direct computation (cf. Lemma 3.5 in [1]) for any $\psi \in C_{\mathrm{b}}(\mathbb{R}^d)$

$$m_{e_N,u_N}^{\otimes N}(\mathrm{e}^{N\pi^N(\psi)}|\Sigma_{e_N,u_N}^N) = \left(m_{e_N,u_N}(\mathrm{e}^{\psi})\right)^N \frac{f_N^{\psi}(e_N,u_N)}{f_N(e_N,u_N)},$$

where f_N^{ψ}, f_N are the densities of the random vector $\frac{1}{N}\sum_{i=1}^N \boldsymbol{\zeta}(v_i)$, in which $\{v_i\}$ are i.i.d. with law m_{e_N,u_N}^{ψ}, m_{e_N,u_N} respectively. Observe that, as we assumed that m is strictly positive on open sets, the law of $\frac{1}{N}\sum_i \zeta(v_i)$ is absolutely continuous for $N > 2$. Choosing $\psi = \phi + \boldsymbol{\gamma}\cdot\boldsymbol{\zeta}$, with $\boldsymbol{\gamma} = \boldsymbol{\gamma}_N(\phi)$ such that (2.8) holds with (e,u) replaced by (e_N,u_N), by the local central limit theorem (see, e.g., [19]) we deduce

$$\lim_{N\to\infty}\frac{1}{N}\log\frac{f_N^{\phi+\boldsymbol{\gamma}\cdot\boldsymbol{\zeta}}(e_N,u_N)}{f_N(e_N,u_N)} = 0.$$

Note indeed that the local central limit holds in view of Assumption 2.1 and the smoothness of ϕ. Gathering the above computations, the statement follows. □

LEMMA 2.4. *Let $\Lambda^*_{e,u}(\pi) := \sup_\phi\{\pi(\phi) - \Lambda_{e,u}(\phi)\}$ be the Legendre transform of $\Lambda_{e,u}$. Then $\Lambda^*_{e,u} = H_{e,u}$.*

PROOF. Let $\{\phi_n\}$ be a sequence of bounded functions monotonically convergent to $\varepsilon\zeta_0$, for some $\varepsilon \in (0, \gamma_0^* - \gamma_0(e,u))$. In view of Assumption 2.1, item (ii), $\overline{\lim}_n \Lambda_{e,u}(\phi_n) < +\infty$. Therefore $\Lambda^*_{e,u}(\pi) < +\infty$ implies $\pi(\zeta_0) < +\infty$. It thus suffices to prove $\Lambda^*_{e,u}(\pi) = H_{e,u}(\pi)$ for π with bounded energy.

Observe now that if $\boldsymbol{\gamma}(\phi)$ is such that (2.8) holds then, by the strict convexity of $\boldsymbol{\gamma} \mapsto \log m_{e,u}(\mathrm{e}^{\phi+\boldsymbol{\gamma}\cdot\zeta})$,

$$\boldsymbol{\gamma}(\phi)\cdot(e,u) - \log m_{e,u}\big(\mathrm{e}^{\phi+\boldsymbol{\gamma}(\phi)\cdot\zeta}\big) = \sup_{\boldsymbol{\gamma}:\gamma_0<\gamma_0^*-\gamma_0(e,u)} \big\{\boldsymbol{\gamma}\cdot(e,u) - \log m_{e,u}\big(\mathrm{e}^{\phi+\boldsymbol{\gamma}\cdot\zeta}\big)\big\}.$$

For π with bounded energy we thus have

$$\begin{aligned}
\Lambda^*_{e,u}(\pi) &= \sup_\phi\big\{\pi(\phi) + \boldsymbol{\gamma}(\phi)\cdot(e,u) - \log m_{e,u}\big(\mathrm{e}^{\phi+\boldsymbol{\gamma}(\phi)\cdot\zeta}\big)\big\}\\
&= \sup_{\boldsymbol{\gamma}:\gamma_0<\gamma_0^*-\gamma_0(e,u)} \sup_\phi\big\{\pi(\phi) + \boldsymbol{\gamma}\cdot(e,u) - \log m_{e,u}\big(\mathrm{e}^{\phi+\boldsymbol{\gamma}\cdot\zeta}\big)\big\}\\
&= \sup_{\boldsymbol{\gamma}:\gamma_0<\gamma_0^*-\gamma_0(e,u)} \sup_\phi\big\{\pi(\phi+\boldsymbol{\gamma}\cdot\boldsymbol{\zeta}) + \boldsymbol{\gamma}\cdot\big(e-\pi(\zeta_0), u-\pi(\zeta)\big) - \log m_{e,u}\big(\mathrm{e}^{\phi+\boldsymbol{\gamma}\cdot\zeta}\big)\big\}\\
&= \mathrm{Ent}(\pi|m_{e,u}) + \sup_{\boldsymbol{\gamma}:\gamma_0<\gamma_0^*-\gamma_0(e,u)} \big\{\boldsymbol{\gamma}\cdot\big(e-\pi(\zeta_0), u-\pi(\zeta)\big)\big\} = H_{e,u}(\pi)
\end{aligned}$$

where we used the variational representation of the relative entropy. □

PROOF OF THEOREM 2.2. For $\delta > 0$ let $C^\delta_{e,u}$ be the compact subset of $\mathcal{P}(\mathbb{R}^d)$ given by $C^\delta_{e,u} := \{\pi \in \mathcal{P}(\mathbb{R}^d) \colon \pi(\zeta_0) \le e + \delta, |\pi(\zeta) - u| \le \delta\}$. By the very definition of $\nu^N_{e,u}$, eventually in N we have $\nu^N_{e_N,u_N}(\pi^N \in C^\delta_{e,u}) = 1$, which implies the exponential tightness of the sequence $\nu^N_{e_N,u_N} \circ (\pi^N)^{-1}$.

In view of Lemmata 2.3 and 2.4, to conclude the proof it is enough to apply the abstract Gärtner–Ellis theorem as stated in [7], Thm. 4.5.20. In order to verify its hypotheses, we note that the strict convexity of the map $\pi \mapsto H_{e,u}(\pi)$ implies, in the terminology of convex analysis used in [7], that every $\pi \in C_{e,u}$ is an *exposed* point of $H_{e,u}$. □

Large deviations from total probability formula. We next show how the Sanov's theorem for i.i.d. random variables can be recovered from Theorem 2.2. While this route is over-complicated in the present context, it will be crucial to deduce the large deviations for Kac's walks with canonical initial distribution of the velocities.

We first state a general argument to deduce the large deviation principle from the total probability formula. Let $\mathcal{X}$ be a Hausdorff topological space and $\{\mu^n\}$ be a sequence of probabilities on $\mathcal{X}$. Let also $\mathcal{Y}$ be a locally compact Polish space, Y be a $\mathcal{Y}$-valued random variable on $\mathcal{X}$ and denote by $\{p_n\}$ the corresponding laws. Fix n and let $y \mapsto \nu^n_y$ be a regular version of the conditional probability of μ^n given Y. We have the disintegration

$$\mu^n = \int p_n(\mathrm{d}y)\, \nu^n_y. \tag{2.10}$$

We will deduce the large deviation of $\{\mu^n\}$ from the large deviations of $\{p_n\}$ and the large deviations on $\{\nu^n_y\}$, that will be assumed to hold uniformly for y in compact subsets of $\mathcal{Y}$.

PROPOSITION 2.5. *Assume*:

(i) *the family* $\{p_n\}$ *is exponentially tight and satisfies a large deviation principle with good rate function* $A\colon \mathcal{Y} \to [0, +\infty]$;

(ii) *for each compact* $K \subset\subset \mathcal{Y}$ *there exists a sequence of compacts* $H_\ell \subset\subset \mathcal{X}$ *such that* $\sup_{y\in K} \nu_y^n(H_\ell^c) \le e^{-n\ell}$;

(iii) *for each* $y \in \mathcal{Y}$ *and each sequence* $y_n \to y$ *the family* $\{\nu_{y_n}^n\}$ *satisfies a large deviation principle with good rate function* $F_y\colon \mathcal{X} \to [0, +\infty]$.

Then the family $\{\mu^n\}$ *is exponentially tight and satisfies a large deviation principle with good rate function* $I\colon \mathcal{X} \to [0, +\infty]$ *given by*

$$I(x) = \inf_{y\in\mathcal{Y}} \{A(y) + F_y(x)\}. \tag{2.11}$$

PROOF. *Step 1. Exponential tightness*. As follows from (2.10), for each compact $K \subset\subset \mathcal{Y}$ and each compact $H \subset\subset \mathcal{X}$

$$\mu^n(H^c) \le \sup_{y\in K} \nu_y^n(H^c) + p_n(K^c).$$

The assumptions on $\{p_n\}$ and $\{\nu_y^n\}$ thus yield the exponential tightness of $\{\mu^n\}$.

Step 2. Lower semicontinuity of the rate function. Since A is lower semicontinuous, the lower semicontinuity of I in (2.11) is implied by the (joint) lower semicontinuity of the map $\mathcal{X} \times \mathcal{Y} \ni (x, y) \mapsto F_y(x)$ that we next deduce. Since $\mathcal{Y}$ is Polish, the joint lower semicontinuity of F is in fact equivalent to the following statement. For each $(x, y) \in \mathcal{X} \times \mathcal{Y}$, each sequence $y_k \to y$, and each $\delta > 0$ there exists an open neighborhood $\mathcal{N} \ni x$ such that

$$\varliminf_k \inf_{x'\in\mathcal{N}} F_{y_k}(x') \ge F_y(x) - \delta. \tag{2.12}$$

Fix $(x, y) \in \mathcal{X} \times \mathcal{Y}$, a sequence $y_k \to y$, and $\delta > 0$. By the lower semicontinuity of $\mathcal{X} \ni x \mapsto F_y(x)$, there exists an open neighborhood $\mathcal{N}' \ni x$ such that

$$\inf_{x'\in\mathcal{N}'} F_y(x') \ge F_y(x) - \delta. \tag{2.13}$$

Denoting by an over-line the closure, let now $\mathcal{N}$ be a open neighborhood such that $x \in \mathcal{N} \subset \overline{\mathcal{N}} \subset \mathcal{N}'$. We then claim that the bound (2.12) holds. In order to show it, by passing to a not relabeled subsequence, we may assume that $\varliminf_k \inf_{x'\in\mathcal{N}} F_{y_k}(x') = \lim_k \inf_{x'\in\mathcal{N}} F_{y_k}(x')$. For k fixed, by the lower bound for the sequence $\{\nu_{y_k}^n\}$,

$$\varliminf_n \frac{1}{n} \log \nu_{y_k}^n(\mathcal{N}) \ge - \inf_{x'\in\mathcal{N}} F_{y_k}(x')$$

which, by taking the inferior limit in k, implies

$$\varliminf_k \varliminf_n \frac{1}{n} \log \nu_{y_k}^n(\mathcal{N}) \ge - \lim_k \inf_{x'\in\mathcal{N}} F_{y_k}(x').$$

By a diagonal argument, there exists a sequence $n_k \uparrow +\infty$ such that

$$\begin{aligned}\varliminf_k \varliminf_n \frac{1}{n} \log \nu_{y_k}^n(\mathcal{N}) &= \varliminf_k \frac{1}{n_k} \log \nu_{y_k}^{n_k}(\mathcal{N}) \le \varlimsup_k \frac{1}{n_k} \log \nu_{y_k}^{n_k}(\overline{\mathcal{N}}) \\ &\le - \inf_{x'\in\overline{\mathcal{N}}} F_y(x') \le - \inf_{x'\in\mathcal{N}'} F_y(x') \le -[F_y(x) - \delta],\end{aligned}$$

where we used the large deviations upper bound for the sequence $\{\nu_{y_k}^{n_k}\}$ and (2.13) in the last step. Comparing the two last displayed equations the bound (2.12) follows.

Step 3. Lower bound. It is enough to show that for each $x \in \mathcal{X}$ and each open neighborhood $\mathcal{N} \ni x$

$$\varliminf_n \frac{1}{n} \log \mu^n(\mathcal{N}) \geq -I(x). \tag{2.14}$$

Fix a metric inducing the topology of $\mathcal{Y}$ and, given $y \in \mathcal{Y}$ and $\delta > 0$, let $B_\delta(y)$ the corresponding open ball of radius δ centered in y. In order to show (2.14), fix $y \in \mathcal{Y}$. By the large deviations lower bound of the sequence $\{p_n\}$, for each $\delta > 0$ we then have

$$\varliminf_n \frac{1}{n} \log p_n\big(B_\delta(y)\big) \geq -A(y).$$

Therefore, by a diagonal argument, there exists a sequence $\delta_n \downarrow 0$ such that

$$\varliminf_n \frac{1}{n} \log p_n\big(B_{\delta_n}(y)\big) \geq -A(y).$$

From the disintegration (2.10) we then obtain

$$\mu^n(\mathcal{N}) \geq \int_{B_{\delta_n}(y)} p_n\big(\mathrm{d}y'\big)\, \nu^n_{y'}(\mathcal{N}) \geq p_n\big(B_{\delta_n}(y)\big) \inf_{y' \in B_{\delta_n}(y)} \nu^n_{y'}(\mathcal{N}).$$

Whence, for a suitable sequence $y'_n \to y$,

$$\varliminf_n \frac{1}{n} \log \mu^n(\mathcal{N}) \geq \varliminf_n \frac{1}{n} \log p_n\big(B_{\delta_n}(y)\big) + \varliminf_n \frac{1}{n} \log \nu^n_{y'_n}(\mathcal{N}) \geq -\big[A(y) + F_y(x)\big],$$

where we used the large deviations lower bound for the family $\{\nu^n_{y'_n}\}$. By optimizing over $y \in \mathcal{Y}$ and recalling (2.11) we then deduce (2.14).

Step 4. Upper bound for compacts. Fix a compact set $H \subset\subset \mathcal{X}$, $\ell > 0$, $\varepsilon > 0$, and observe that, by the joint lower semicontinuity of F proven in Step 2 above, the map $\mathcal{Y} \ni y \mapsto \inf_{x \in H} F_y(x)$ is lower semicontinuous. By the exponential tightness of $\{p_n\}$, there exists a compact $K_\ell \subset\subset \mathcal{Y}$ such that $p_n(K_\ell^{\mathrm{c}}) \leq \mathrm{e}^{-n\ell}$. For each $y \in K_\ell$, by the lower semicontinuity of A and the previous observation, there exists $\delta > 0$ such that $A(y') \geq A(y) - \varepsilon/2$ and $\inf_{x \in H} F_{y'}(x) \geq \inf_{x \in H} F_y(x) - \varepsilon/2$ for any $y' \in B_{2\delta}(y)$. By the local compactness of $\mathcal{Y}$, possibly by decreasing δ, we can assume that $B_{2\delta}(y)$ is relatively compact. Furthermore, by the compactness of K_ℓ, there exists a finite family $\{B_{\delta_i}(y_i)\}_{i=1,\dots,r}$ such that $K_\ell \subset \bigcup_i B_{\delta_i}(y_i)$. In view of (2.10),

$$\begin{aligned} \mu^n(H) &\leq \sum_{i=1}^r \int_{B_{\delta_i}(y_i)} p_n\big(\mathrm{d}y'\big)\, \nu^n_{y'}(H) + p_n\big(K_\ell^{\mathrm{c}}\big) \\ &\leq \sum_{i=1}^r p_n\big(\overline{B}_{\delta_i}(y_i)\big) \sup_{y' \in B_{\delta_i}(y_i)} \nu^n_{y'}(H) + \mathrm{e}^{-n\ell}. \end{aligned} \tag{2.15}$$

Since the sets $B_{\delta_i}(y_i)$ are relatively compact, by passing if necessary to a not relabeled subsequence, for each $i = 1, \dots, r$ there exist $\bar{y}_i \in \overline{B}_{\delta_i}(y_i)$ and a sequence $y_i^n \to \bar{y}_i$ such that

$$\varlimsup_n \sup_{y' \in B_{\delta_i}(y_i)} \frac{1}{n} \log \nu^n_{y'}(H) = \varlimsup_n \frac{1}{n} \log \nu^n_{y_i^n}(H).$$

Letting $a \vee b := \max\{a, b\}$ and using the large deviation upper bound both for $\{p_n\}$ and $\{\nu^n_{y_i^n}\}$ in (2.15) we thus get

$$\begin{aligned}\overline{\lim_n} \frac{1}{n} \log \mu^n(H) &\le \max_{i=1,\dots,r} \Big\{- \inf_{y' \in \overline{B}_{\delta_i}(y_i)} A(y') - \inf_{x \in H} F_{\bar{y}_i}(x)\Big\} \vee (-\ell) \\ &\le - \min_{i=1,\dots,r} \Big\{A(y_i) + \inf_{x\in H} F_{y_i}(x) - \varepsilon\Big\} \vee (-\ell) \\ &\le - \inf_{x \in H} \inf_{y \in \mathcal{Y}} \{A(y) + F_y(x) - \varepsilon\} \vee (-\ell).\end{aligned}$$

Recalling (2.11), we conclude by taking the limits $\varepsilon \downarrow 0$ and $\ell \uparrow +\infty$. $\square$

Let $\boldsymbol{v} \in \Sigma^N$ be sampled according to the product probability $\mu^N = m^{\otimes N}$ and denote by p_N the law of $\frac{1}{N}\sum_i \boldsymbol{\zeta}(v_i)$. We then have the disintegration

$$\mu^N = \int p_N\big(\mathrm{d}(e,u)\big)\, \nu^N_{e,u}.$$

Moreover the sequence $\{p_N\}$ satisfies a large deviations principle with rate function $A \colon (0,+\infty) \times \mathbb{R}^d \to [0,+\infty]$ given by

$$A(e,u) = \sup_{\boldsymbol{\gamma}} \{\boldsymbol{\gamma} \cdot (e,u) - \log m(\mathrm{e}^{\boldsymbol{\gamma}\cdot\boldsymbol{\zeta}})\}. \tag{2.16}$$

This follows from the multidimensional Cramér's theorem in [7], Thm. 2.3.6. Indeed, in the terminology of convex analysis used in [7], the function $\boldsymbol{\gamma} \mapsto \log m(\mathrm{e}^{\boldsymbol{\gamma}\cdot\boldsymbol{\zeta}})$ is *steep*. Namely $|\nabla \log m(\mathrm{e}^{\boldsymbol{\gamma}\cdot\boldsymbol{\zeta}})|$ diverges when $\gamma_0 \to \gamma_0^*$. This follows from item (ii) in Assumption 2.1.

In view of Proposition 2.5 and the following remark, Sanov's theorem for i.i.d. random variables can be deduced from Theorem 2.2.

REMARK 2.6. We have

$$\mathrm{Ent}(\pi|m) = \inf_{(e,u)} \{A(e,u) + H_{e,u}(\pi)\}.$$

In fact, by a direct computation, the infimum is achieved for $(e,u) = \pi(\boldsymbol{\zeta})$.

3. Large deviations for Kac model with microcanonical initial data.

The model. Recall that $\Sigma^N = (\mathbb{R}^d)^N$. We consider the Kac walk given by the Markov process on the configuration space Σ^N, whose generator acts on bounded continuous functions $f \colon \Sigma^N \to \mathbb{R}$ as

$$\mathcal{L}_N f(\boldsymbol{v}) = \frac{1}{N} \sum_{\{i,j\}} L_{i,j} f(\boldsymbol{v}),$$

where the sum is carried over the unordered pairs $\{i,j\} \subset \{1,\dots,N\}$, $i \neq j$, and

$$L_{i,j} f(\boldsymbol{v}) = \int_{\mathbb{S}_{d-1}} \mathrm{d}\omega\, B(v_i - v_j, \omega)\big[f\big(T^{\omega}_{i,j}\boldsymbol{v}\big) - f(\boldsymbol{v})\big].$$

Here $\mathbb{S}_{d-1}$ is the sphere in $\mathbb{R}^d$, $\mathrm{d}\omega$ is the Haar measure on $\mathbb{S}_{d-1}$, and

$$\big(T^{\omega}_{i,j}\boldsymbol{v}\big)_k = \begin{cases} v_i + \big(\omega\cdot(v_j - v_i)\big)\omega & \text{if } k = i, \\ v_j - \big(\omega\cdot(v_j - v_i)\big)\omega & \text{if } k = j, \\ v_k & \text{otherwise,}\end{cases} \tag{3.1}$$

and the collision kernel B is given by

$$B(v-v_*,\omega)=\frac{1}{2}\big|(v-v_*)\cdot\omega\big|. \tag{3.2}$$

Observe that the dynamics preserves energy and momentum, that is, can be restricted to the set $\Sigma^N_{e,u}$ as defined in (2.2). We denote by $(\boldsymbol{v}(t))_{t\ge 0}$ the Markov process generated by $\mathcal{L}_N$.

Fix hereafter $T>0$. Given a probability ν on Σ^N we denote by $\mathbb{P}^N_\nu$ the law of this process on the time interval $[0,T]$. Observe that $\mathbb{P}^N_\nu$ is a probability on the Skorokhod space $D([0,T];\Sigma^N)$. As usual if $\nu=\delta_{\boldsymbol{v}}$ for some $\boldsymbol{v}\in\Sigma^N$, the corresponding law is simply denoted by $\mathbb{P}^N_{\boldsymbol{v}}$.

Empirical observables. Recall that $\mathcal{P}(\mathbb{R}^d)$ is the set of probability measures π on $\mathbb{R}^d$ equipped with the weak topology and the corresponding Borel σ-algebra. Let $D([0,T];\mathcal{P}(\mathbb{R}^d))$ the set of $\mathcal{P}(\mathbb{R}^d)$-valued cádlág paths endowed with the Skorokhod topology and the corresponding Borel σ-algebra. Recalling the empirical measure π^N defined in (2.3), with a slight abuse of notation we denote also by π^N the map from $D([0,T];\Sigma^N)$ to $D([0,T];\mathcal{P}(\mathbb{R}^d))$ defined by $\pi^N_t(\boldsymbol{v}):=\pi^N(\boldsymbol{v}(t))$, $t\in[0,T]$.

We denote by $\mathcal{M}$ the subset of the finite measures Q on $[0,T]\times\mathbb{R}^{2d}\times\mathbb{R}^{2d}$ that satisfy $Q(\mathrm{d}t;\mathrm{d}v,\mathrm{d}v_*,\mathrm{d}v',\mathrm{d}v'_*)=Q(\mathrm{d}t;\mathrm{d}v_*,\mathrm{d}v,\mathrm{d}v',\mathrm{d}v'_*)=Q(\mathrm{d}t;\mathrm{d}v,\mathrm{d}v_*,\mathrm{d}v'_*,\mathrm{d}v')$. We consider $\mathcal{M}$ endowed with the weak topology and the corresponding Borel σ-algebra. By definition, the weak topology is the weakest topology such that the map $Q\mapsto Q(F)$ is continuous for each F in $C_{\mathrm{b}}([0,T]\times\mathbb{R}^{2d}\times\mathbb{R}^{2d})$.

For paths $\boldsymbol{v}(\cdot)$ sampled according to $\mathbb{P}^N_{\boldsymbol{v}}$, the *empirical flow* is the map Q^N taking values on $\mathcal{M}$ defined by

$$Q^N(\boldsymbol{v})(F):=\frac{1}{N}\sum_{\{i,j\}}\sum_{k\ge 1}F\big(\tau^{i,j}_k;v_i\big(\tau^{i,j}_k-\big),v_j\big(\tau^{i,j}_k-\big),v_i\big(\tau^{i,j}_k\big),v_j\big(\tau^{i,j}_k\big)\big), \tag{3.3}$$

where $F\colon[0,T]\times\mathbb{R}^{2d}\times\mathbb{R}^{2d}\to\mathbb{R}$ is a continuous and bounded function that satisfies $F(t;v,v_*,v',v'_*)=F(t;v_*,v,v',v'_*)=F(t;v,v_*,v'_*,v')$, while $(\tau^{i,j}_k)_{k\ge1}$ are the jump times of the pair (v_i,v_j). Here, $v_i(t-)=\lim_{s\uparrow t}v_i(s)$. In view of the conservation of the energy and momentum, the measure $Q^N(\mathrm{d}t;\cdot)$ is supported on $\{\boldsymbol{\zeta}(v)+\boldsymbol{\zeta}(v_*)=\boldsymbol{\zeta}(v')+\boldsymbol{\zeta}(v_*)\}\subset\mathbb{R}^{2d}\times\mathbb{R}^{2d}$.

Let $\mathcal{S}$ be the subset of $D([0,T];\mathcal{P}(\mathbb{R}^d))\times\mathcal{M}$ given by elements (π,Q) that satisfies the balance equation

$$\begin{aligned}&\pi_T(\phi_T)-\pi_0(\phi_0)-\int_0^T\mathrm{d}t\,\pi_t(\partial_t\phi_t)\\&\quad+\int Q\big(\mathrm{d}t;\mathrm{d}v,\mathrm{d}v_*,\mathrm{d}v',\mathrm{d}v'_*\big)\big[\phi_t(v)+\phi_t(v_*)-\phi_t\big(v'\big)-\phi_t\big(v'_*\big)\big]=0\end{aligned} \tag{3.4}$$

for each $\phi\in C_{\mathrm{b}}([0,T]\times\mathbb{R}^d)$ continuously differentiable in t, with bounded derivative. For each $\boldsymbol{v}\in\Sigma^N$, with $\mathbb{P}^N_{\boldsymbol{v}}$ probability one, the pair (π^N,Q^N) belongs to $\mathcal{S}$.

The rate function. Given $(e,u)\in Z$, recall $C_{e,u}:=\{\mu\in\mathcal{P}(\mathbb{R}^d):\mu(\zeta_0)\le e,\ \mu(\zeta)=u\}$, and set

$$\mathcal{C}_{e,u}:=\big\{\pi\in C\big([0,T],\mathcal{P}\big(\mathbb{R}^d\big)\big)\ \pi_t\in C_{e,u},t\in[0,T]\big\}, \tag{3.5}$$

that is a closed subset of $C([0,T],\mathcal{P}(\mathbb{R}^d)$.

For notation convenience, let $r(v,v^*,\cdot)$ be the measure on $\mathbb{R}^{2d}$ supported on $\{\boldsymbol{\zeta}(v)+\boldsymbol{\zeta}(v_*)=\boldsymbol{\zeta}(v')+\boldsymbol{\zeta}(v'_*)\}$ such that

$$r\big(v,v_*,\mathrm{d}v',\mathrm{d}v'_*\big)=\mathrm{d}\omega\,B(v-v_*,\omega),$$

where v' and v'_* are related to ω by the collision rules, as in (3.1). For $\pi \in D([0,T]; \mathcal{P}(\mathbb{R}^d))$ let Q^π be the measure defined by

$$Q^\pi\big(\mathrm{d}t;\, \mathrm{d}v,\, \mathrm{d}v_*,\, \mathrm{d}v',\, \mathrm{d}v'_*\big) := \frac{1}{2}\,\mathrm{d}t\, \pi_t(\mathrm{d}v)\pi_t(\mathrm{d}v_*)\, r\big(v, v_*;\, \mathrm{d}v',\, \mathrm{d}v'_*\big) \tag{3.6}$$

and observe that $Q^\pi(\mathrm{d}t, \cdot)$ is supported on $\{\boldsymbol{\zeta}(v) + \boldsymbol{\zeta}(v_*) = \boldsymbol{\zeta}(v') + \boldsymbol{\zeta}(v_*)\} \subset \mathbb{R}^{2d} \times \mathbb{R}^{2d}$.

DEFINITION 3.1. Let $\mathcal{S}^{\mathrm{ac}}_{e,u}$ be the subset of $\mathcal{S}$ given by the elements (π, Q) that satisfy the following conditions:

(i) $\pi_t \in C_{e,u}$ for each $t \in [0,T]$;
(ii) $Q \ll Q^\pi$.

Observe that if $(\pi, Q) \in \mathcal{S}^{\mathrm{ac}}_{e,u}$ then Q^π is a finite measure. Moreover, from the balance equation (3.4), as $Q \ll Q^\pi$ we deduce that $\pi \in C([0,T]; \mathcal{P}(\mathbb{R}^d))$ so that $\pi \in \mathcal{C}_{e,u}$.

The dynamical rate function $J_{e,u} : \mathcal{S} \to [0, +\infty]$ is defined by

$$J_{e,u}(\pi, Q) := \begin{cases} \displaystyle\int \mathrm{d}Q^\pi \left[\frac{\mathrm{d}Q}{\mathrm{d}Q^\pi} \log \frac{\mathrm{d}Q}{\mathrm{d}Q^\pi} - \left(\frac{\mathrm{d}Q}{\mathrm{d}Q^\pi} - 1 \right) \right] & \text{if } (\pi, Q) \in \mathcal{S}^{\mathrm{ac}}_{e,u}, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.7}$$

By the remark following Definition 3.1, $J_{e,u}(\pi, Q) < +\infty$ implies $\pi \in \mathcal{C}_{e,u}$.

Recalling $H_{e,u}$ has been defined in (2.5), the microcanonical rate function is

$$I_{e,u}(\pi, Q) := H_{e,u}(\pi_0) + J_{e,u}(\pi, Q). \tag{3.8}$$

Let also $\hat{\mathcal{S}}$ be the subset of $\mathcal{S}$ given by the pairs (π, Q) such that

$$\int_{[0,T]\times\mathbb{R}^{4d}} \mathrm{d}Q \left[\zeta_0(v) + \zeta_0(v_*) + \zeta_0(v') + \zeta_0(v'_*)\right] < +\infty.$$

If $(\pi, Q) \in \hat{\mathcal{S}}$, by a standard truncation procedure, we can test the balance equation (3.4) against $\phi(t, v) = a(t)\zeta_0(v)$ with $a \in C^1(0,T) \cap C[0,T]$ deducing that $\pi_t(\zeta_0) = \pi_0(\zeta_0)$ for every $t \in [0,T]$.

THEOREM 3.2. *Assume m satisfies conditions* (i)–(iii) *in Assumption* 2.1. *Fix $(e,u) \in Z$, a sequence $(e_N, u_N) \to (e,u)$, and let $\nu^N_{e_N,u_N}$ be the microcanonical probabilities as in Section* 2. *The family $\mathbb{P}^N_{\nu^N_{e_N,u_N}} \circ (\pi^N, Q^N)^{-1}$ satisfies a large deviation upper bound with good rate function $I_{e,u} : \mathcal{S} \to [0, +\infty]$, namely $I_{e,u}$ has compact level sets and for each closed $C \subset \mathcal{S}$*

$$\varlimsup_{N\to+\infty} \frac{1}{N} \log \mathbb{P}^N_{\nu^N_{e_N,u_N}}\big((\pi^N, Q^N) \in C\big) \le -\inf_C I_{e,u}. \tag{3.9}$$

Moreover, if m satisfies also condition (iv) *in Assumption* 2.1, *then for each open $O \subset \mathcal{S}$*

$$\varliminf_{N\to+\infty} \frac{1}{N} \log \mathbb{P}^N_{\nu^N_{e_N,u_N}}\big((\pi^N, Q^N) \in O\big) \ge -\inf_{O\cap\hat{\mathcal{S}}} I_{e,u}. \tag{3.10}$$

4. Proof of the upper bound. The proof follows the same strategy as in [1] and in [9]. For the reader's convenience we here provide the details. The upper bound is achieved by an established pattern in large deviation theory. We first prove the exponential tightness, which allows us to reduce to compacts. By an exponential tilting of the measure, we prove an upper bound for open balls and finally we use a mini-max argument to conclude.

The basic observation is the following. Let $F\colon [0,T]\times\mathbb{R}^{4d}\to\mathbb{R}$ be bounded, measurable and such that $F(t;v,v_*,v',v'_*)=F(t;v_*,v,v',v'_*)=F(t;v,v_*,v'_*,v')$. Set

$$\lambda^F(t;v,v_*)=\int r(v,v_*,\mathrm{d}v',\mathrm{d}v'_*)\mathrm{e}^{F(t;v,v_*,v',v'_*)}. \tag{4.1}$$

If $F=0$ we drop it from the notation. Denoting by $Q^N_{[0,t]}$ the restriction of the measure Q^N on $[0,t]$, $t\in(0,T]$, and using that $\lambda(v,v)=\lambda^F(t,v,v)=0$ the process

$$\mathbb{M}^F_t=\exp\biggl\{N\biggl(Q^N_{[0,t]}(F)-\frac{1}{2}\int_0^t \mathrm{d}s\,\pi^N_s\otimes\pi^N_s(\lambda^F-\lambda)\biggr)\biggr\} \tag{4.2}$$

is a $\mathbb{P}^N_{\boldsymbol{v}}$ positive martingale for each $\boldsymbol{v}\in\Sigma^N$, see for example, [12], App. 1, Prop. 2.6.

For any $\delta>0$ we define the compact set $C^\delta_{e,u}:=\{\mu\in\mathcal{P}(\mathbb{R}^d):\mu(\zeta_0)\le e+\delta,\ |\mu(\zeta)-u|\le\delta\}$. By the conservation of the energy and the momentum

$$\mathbb{P}^N_{\nu^N_{e_N,u_N}}\bigl(\pi^N_t\in C^\delta_{e,u},t\in[0,T]\bigr)=1, \tag{4.3}$$

eventually in N. To prove the exponential tightness of the empirical measure it is therefore enough to obtain an estimate on the continuity modulus of the map $t\mapsto\pi_t(\phi)$ which is stated in the next lemma.

LEMMA 4.1. *For each $\varepsilon>0$ and $\phi\in C_{\mathrm{b}}(\mathbb{R}^d)$*

$$\lim_{\eta\downarrow 0}\varlimsup_{N\to+\infty}\frac{1}{N}\log\mathbb{P}^N_{\nu^N_{e_N,u_N}}\Bigl(\sup_{t,s\in[0,T]:|t-s|<\eta}\bigl|\pi^N_t(\phi)-\pi^N_s(\phi)\bigr|>\varepsilon\Bigr)=-\infty. \tag{4.4}$$

Set

$$\bar F(v,v_*,v',v'_*):=\log\bigl(\mathrm{e}+\zeta_0(v)+\zeta_0(v_*)+\zeta_0(v')+\zeta_0(v'_*)\bigr). \tag{4.5}$$

Since $\bar F\ge 1$ and $\bar F$ has compact level sets, the Chebyshev inequality and the Prohorov theorem [4], Thm.8.6.2, imply that, given $\ell>0$, the set $\{Q\in\mathcal{M}\colon Q(\bar F)\le\ell\}$ is compact. The exponential tightness of the empirical flow thus follows from the next lemma.

LEMMA 4.2.

$$\lim_{\ell\to+\infty}\varlimsup_{N\to+\infty}\frac{1}{N}\log\mathbb{P}^N_{\nu^N_{e_N,u_N}}\bigl(Q^N(\bar F)>\ell\bigr)=-\infty. \tag{4.6}$$

It is convenient to prove Lemmata 4.1 and 4.2 in the reverse order.

PROOF OF LEMMA 4.2. Observe that by the conservation of the energy, there exists a constant $c>0$, depending on e, such that for any N the bound $Q^{\pi^N}(\mathrm{e}^{\frac{1}{2}\bar F})\le c$ holds with $\mathbb{P}^N_{\nu^N_{e_N,u_N}}$ probability one.

Let $\mathbb{M}_T$ be the exponential martingale in (4.2) with $F=\frac{1}{2}\bar F$. Then, for each $\ell>0$,

$$\mathbb{P}^N_{\nu^N_{e_N,u_N}}\bigl(Q^N(\bar F)>\ell\bigr)=\mathbb{E}^N_{\nu^N_{e_N,u_N}}\bigl(\mathbb{M}_T\,(\mathbb{M}_T)^{-1}\mathbb{1}_{Q^N(\bar F)>\ell}\bigr)\le\exp\{-N\ell/2+cN\}. \qquad\square$$

PROOF OF LEMMA 4.1. In view of the balance equation (3.4) it is enough to show that there exists a function $c\colon(0,1)\to\mathbb{R}_+$ with $c(\eta)\uparrow+\infty$ as $\eta\downarrow 0$ such that, for any $\varepsilon>0$

$$\mathbb{P}^N_{\nu^N_{e_N,u_N}}\Bigl(\sup_{t\in[0,T-\eta]}Q^N_{[t,t+\eta]}(1)>\varepsilon\Bigr)\le\mathrm{e}^{-Nc(\eta)}.$$

By a straightforward inclusion of events, the previous bound follows from

$$\frac{1}{\eta}\sup_{t\in[0,T-\eta]}\mathbb{P}^N_{\nu^N_{e_N,u_N}}\big(Q^N_{[t,t+\eta]}(1)>\varepsilon\big)\le \mathrm{e}^{-Nc(\eta)}.$$

Consider the super-martingale (4.2) with $F=\gamma\,\mathbb{1}_{[t,t+\eta]}$, $\gamma>0$. Using the same argument of the previous lemma we deduce

$$\mathbb{P}^N_{\nu^N_{e_N,u_N}}\big(Q^N_{[t,t+\eta]}(1)>\varepsilon\big)\le \exp\{-N[\gamma\varepsilon-\eta(\mathrm{e}^\gamma-1)C(1+e)]\}.$$

The proof is concluded by choosing $\gamma=\log(1/\eta)$. □

Upper bound on compacts. Recalling the set $C^\delta_{e,u}$ defined above (4.3), let $\mathcal{C}^\delta_{e,u}$ be the closed subset of $C([0,T];\mathcal{P}(\mathbb{R}^d))$ defined as

$$\mathcal{C}^\delta_{e,u}:=\bigcap_{t\in[0,T]}\{\pi:\pi_t\in C^\delta_{e,u}\}. \tag{4.7}$$

By Urysohn's lemma, for each $\eta>0$ there exists $\psi^{\delta,\eta}_{e,u}: C([0,T];\mathcal{P}(\mathbb{R}^d))\to[0,1]$ continuous such that

$$\psi^{\delta,\eta}_{e,u}(\pi)=\begin{cases}0 & \text{if } \pi\in\mathcal{C}^\delta_{e,u},\\ 1 & \text{if } \operatorname{dist}(\pi,C^\delta_{e,u})\ge\eta,\end{cases}$$

where dist is the uniform distance. Moreover, for $\pi\in D([0,T],\mathcal{P}(\mathbb{R}^d))$, we extend it to a function defined on $\mathbb{R}$ by setting $\pi_t=\pi_0$ if $t<0$, $\pi_t=\pi_T$ if $t>T$. Let $\imath_\varepsilon$ be the a smooth approximation of the δ function, and denote by $\imath_\varepsilon*\pi$ the time convolution of π.

LEMMA 4.3. *Fix a measurable subset $B\subset\mathcal{S}$. For any $(\phi,F)\in C_{\mathrm{b}}(\mathbb{R}^d)\times C_{\mathrm{b}}(\mathbb{R}^+\times(\mathbb{R}^d)^2\times(\mathbb{R}^d)^2)$ such that and $F(t;v,v_*,v',v'_*)=F(t;v_*,v,v',v'_*)=F(t;v,v_*,v'_*,v')$, and any $\delta,\eta,\varepsilon,\alpha>0$,*

$$\varlimsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}^N_{\nu^N}\big((\pi^N,Q^N)\in B\big)\le-\inf_{(\pi,Q)\in B}\{I_{\phi,F}(\pi,Q)+\alpha\psi^{\delta,\eta}_{e,u}(\imath_\varepsilon*\pi)\}, \tag{4.8}$$

where

$$I_{\phi,F}(\pi,Q):=\pi_0(\phi)-\Lambda_{e,u}(\phi)+Q(F)-\frac{1}{2}\int_0^T \mathrm{d}t\,\pi_t\otimes\pi_t\big(\lambda^F-\lambda\big). \tag{4.9}$$

PROOF. Let $\tilde{\nu}^N_{e_N,u_N}$ be the probability on Σ^N defined by

$$\mathrm{d}\tilde{\nu}^N_{e_N,u_N}=\mathrm{d}\nu^N_{e_N,u_N}\exp\{N\pi^N(\phi)-\log\nu^N_{e_N,u_N}(\mathrm{e}^{N\pi^N(\phi)})\},$$

where $\pi^N:\Sigma^N\to\mathcal{P}(\mathbb{R}^d)$ is the map defined in (2.3).

Let now $\pi^N: D([0,T];\Sigma^N)\to D([0,T];\mathcal{P}(\mathbb{R}^d))$ be the map introduced before equation (3.3). Recalling (4.3) and the definition of the martingale $\mathbb{M}^F_t$ in (4.2), we write

$$\begin{aligned}
&\mathbb{P}^N_{\nu^N_{e_N,u_N}}\big((\pi^N,Q^N)\in B\big)\\
&\quad=\int\nu^N_{e_N,u_N}(\mathrm{d}\mathbf{v})\mathbb{E}^N_{\mathbf{v}}\big(\mathrm{e}^{-N\alpha\psi^{\delta,\eta}_{e,u}(\imath_\varepsilon*\pi^N)}\mathbb{1}_B(\pi^N,Q^N)\big)\\
&\quad=\int\mathrm{d}\tilde{\nu}^N_{e_N,u_N}\frac{\mathrm{d}\nu^N_{e_N,u_N}}{\mathrm{d}\tilde{\nu}^N_{e_N,u_N}}\mathbb{E}^N_{\mathbf{v}}\big(\mathrm{e}^{-N\alpha\psi^{\delta,\eta}_{e,u}(\imath_\varepsilon*\pi^N)}\mathbb{M}^F_T(\mathbb{M}^F_T)^{-1}\mathbb{1}_B(\pi^N,Q^N)\big).
\end{aligned}$$

We get

$$\begin{aligned}&\mathbb{P}^N_{\nu^N_{e_N,u_N}}\big((\pi^N, Q^N) \in B\big)\\&\quad\le \sup_{(\pi,Q)\in B} \exp\bigg\{-N\bigg[\pi_0(\phi) - \frac{1}{N}\log \nu^N_{e_N,u_N}\big(\mathrm{e}^{N\pi^N(\phi)}\big) + \alpha\psi^{\delta,\eta}_{e,u}(\iota_\varepsilon * \pi)\\&\qquad + Q(F) - \frac{1}{2}\int_0^T \mathrm{d}t\, \pi_t \otimes \pi_t\big(\lambda^F - \lambda\big)\bigg]\bigg\},\end{aligned}$$

where we used that $\mathbb{E}^N_{\tilde{\nu}^N_{e_N,u_N}}(\mathbb{M}^F_T) = 1$. The statement follows from Lemma 2.3. □

LEMMA 4.4 (Variational characterization of the dynamical rate functional). *For any* $(\pi, Q) \in \mathcal{S}$ *such that* $\pi \in C([0,T]; \mathcal{P}(\mathbb{R}^d))$

$$J_{e,u}(\pi, Q) = \sup_{F,\alpha,\delta,\eta,\varepsilon}\bigg\{Q(F) - \frac{1}{2}\int_0^T \mathrm{d}t\, \pi_t \otimes \pi_t\big(\lambda^F - \lambda\big) + \alpha\psi^{\delta,\eta}_{e,u}(\iota_\varepsilon * \pi)\bigg\}, \tag{4.10}$$

where the supremum is carried out over all continuous and bounded $F\colon [0,T] \times (\mathbb{R}^d)^2 \times (\mathbb{R}^d)^2 \to \mathbb{R}$ *such that* $F(t; v, v_*, v', v'_*) = F(t; v_*, v, v', v'_*) = F(t; v, v_*, v'_*, v')$, *and* $\alpha, \delta, \eta, \varepsilon > 0$.

PROOF. By monotonicity

$$\sup_{\alpha,\delta,\eta,\varepsilon} \alpha\psi^{\delta,\eta}_{e,u}(\iota_\varepsilon * \pi) = \sup_{\varepsilon} \lim_{\alpha\uparrow+\infty} \lim_{\delta\downarrow 0} \lim_{\eta\downarrow 0} \alpha\psi^{\delta,\eta}_{e,u}(\iota_\varepsilon * \pi) = \begin{cases} 0 & \text{if } \pi \in \mathcal{C}_{e,u},\\ +\infty & \text{otherwise,}\end{cases}$$

where we have used that if $(\iota_\varepsilon * \pi)_t \in C_{e,u}$, for any $t \in [0,T]$ and $\varepsilon > 0$, then $\pi_t \in C_{e,u}$ for any $t \in [0,T]$. To complete the proof, it remains to show that for $\pi \in \mathcal{C}_{e,u}$

$$J_{e,u}(\pi, Q) = \sup_F \bigg\{Q(F) - \frac{1}{2}\int_0^T \mathrm{d}t\, \pi_t \otimes \pi_t\big(\lambda^F - \lambda\big)\bigg\}. \tag{4.11}$$

Recall the definition of Q^π in (3.6) and observe that

$$\frac{1}{2}\int_0^T \mathrm{d}t\, \pi_t \otimes \pi_t\big(\lambda^F - \lambda\big) = Q^\pi\big(\mathrm{e}^F - 1\big).$$

This implies that if $\sup_F[Q(F) - \frac{1}{2}\int_0^T \mathrm{d}t\, \pi_t \otimes \pi_t(\lambda^F - \lambda)]$ is finite, then Q is absolutely continuous with respect to Q^π. The proof is now completed by a direct computation. □

PROOF OF THEOREM 3.2, UPPER BOUND. In view of (4.3), Lemma 4.2 and Lemma 4.1 imply the exponential tightness of the family $\{\mathbb{P}^N_{\nu^N_{e_N,u_N}} \circ (\pi^N, Q^N)^{-1}\}$. Therefore it is enough to show the statement for compacts. In view of Lemma 4.3 and the mini-max argument in [12], App.2, Lemma 3.2, the statement follows from Lemmata 2.4 and 4.4. □

5. Proof of the lower bound.

We premise a general lemma concerning the large deviations lower bound which is proven in [10], Prop. 4.1, see also [16].

LEMMA 5.1. *Let* $\{P_n\}$ *be a sequence of probability measures on a Polish space* $\mathcal{X}$. *Assume that for each* $x \in \mathcal{X}$ *there exists a sequence of probability measures* $\{\tilde{P}^x_n\}$ *weakly convergent to* δ_x *and such that*

$$\varlimsup_{n\to\infty} \frac{1}{n}\,\mathrm{Ent}\big(\tilde{P}^x_n | P_n\big) \le \mathcal{J}(x) \tag{5.1}$$

for some $\mathcal{J}\colon \mathcal{X} \to [0, +\infty]$. *Then the sequence* $\{P_n\}$ *satisfies the large deviation lower bound with rate function given by* $\mathrm{sc}^{-}\mathcal{J}$, *the lower semi-continuous envelope of* $\mathcal{J}$, *that is*,

$$(\mathrm{sc}^{-}\mathcal{J})(x) := \sup_{U \in \mathcal{N}_x} \inf_{y \in U} \mathcal{J}(y),$$

where $\mathcal{N}_x$ *denotes the collection of the open neighborhoods of* x.

We shall first prove the entropy bound (5.1) when (π, Q) belongs to a "nice" set, understanding that the functional $\mathcal{J}$ is infinite otherwise. This will be achieved by adapting the strategy introduced in [1], see also [9]. By a density argument we then identify the lower-semicontinuous envelope $\mathrm{sc}^{-}\mathcal{J}$ with the rate function $I_{e,u}$ on the set $\hat{S}$. In particular, as in [1] and [9], the large deviations upper and lower bound are here proven to match only when Q has bounded second moment. On the other hand, we do not require, as in [9], that $B \geq c > 0$.

Perturbed Kac walks. We start by the following law of large numbers for a class of perturbed Kac's walks. Consider perturbed time dependent collision kernels $\tilde{B}$ that are continuous and satisfy

$$\sup_{t,v,v_*} \tilde{\lambda}_t(v, v_*) = \sup_{t,v,v_*} \int \tilde{B}_t(v, v_*, \omega)\, \mathrm{d}\omega \leq C, \tag{5.2}$$

for some $C < +\infty$. Fix $(e, u) \in Z$, a sequence $(e_N, u_N) \to (e, u)$, and let $\nu^N_{e_N,u_N}$ be the family of probabilities on Σ^N as in Section 2, and denote by $\tilde{\mathbb{P}}^N_{\nu^N_{e_N,u_N}}$ the law of the perturbed Kac walk with initial datum $\nu^N_{e_N,u_N}$.

LEMMA 5.2. *As* $N \to +\infty$, *the pair* (π^N, Q^N) *converges, in* $\tilde{\mathbb{P}}^N_{\nu^N_{e_N,u_N}}$ *probability, to* $(f\, \mathrm{d}v, q\, \mathrm{d}t\, \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\omega)$, *where* $q_t(v, v_*, \omega) = \frac{1}{2} f_t(v) f_t(v_*) \tilde{B}_t(v, v_*, \omega)$ *and* $f \in C([0, T]; L^1(\mathbb{R}^d))$ *is the unique solution to the perturbed Kac's equation*

$$\begin{cases} \partial_t f_t(v) = \iint \mathrm{d}v_*\, \mathrm{d}\omega \big[\tilde{B}_t(v', v'_*, \omega) f_t(v') f_t(v'_*) - \tilde{B}_t(v, v_*, \omega) f_t(v) f_t(v_*)\big], \\ f_0(\cdot) = \dfrac{\mathrm{d}m_{e,u}}{\mathrm{d}v}. \end{cases} \tag{5.3}$$

Here we understand that (5.3) *holds by integrating against continuous, bounded test functions which are continuous differentiable in time.*

The proof follows from the fact the large deviation upper bound holds also for the perturbed Kac's walk, and from the uniqueness of the solution to (5.3). The latter holds in view of the condition (5.2), see the proof of Lemma 4.1 in [1] for the details.

The following specifies the collection of "nice" (π, Q). Recall $\mathcal{S}^{\mathrm{ac}}_{e,u}$ in Definition 3.1.

DEFINITION 5.3. Let $\tilde{\mathcal{S}}_{e,u}$ be the collection of elements $(\pi, Q) \in \mathcal{S}^{\mathrm{ac}}_{e,u}$ whose densities (f, q) are continuous and such that

$$\sup_{t,v,v_*,\omega} \frac{q_t(v,\ v_*, \omega)}{f_t(v) f_t(v_*)} < +\infty, \tag{5.4}$$

and

$$\sup_{t,v,v_*,\omega} \frac{q_t(v,\ v_*, \omega)}{f_t(v) f_t(v_*) B(v, v_*, \omega)} < +\infty. \tag{5.5}$$

Given $(\pi, Q) \in \tilde{\mathbb{S}}_{e,u}$, denote by $\tilde{B}_t$ the time dependent perturbed kernel defined by

$$\tilde{B}_t(v, v_*, \omega) = 2\frac{q_t(v,\, v_*, \omega)}{f_t(v) f_t(v_*)}, \tag{5.6}$$

that meets (5.2).

The next statement provides the large deviation lower bound for neighborhood of elements in $\tilde{\mathbb{S}}_{e,u}$.

PROPOSITION 5.4. *Let $(\pi, Q) \in \tilde{\mathbb{S}}_{e,u}$. Assume that m satisfies items (i)–(iii) in Assumption 2.1, and suppose $\pi_0(\mathrm{d}v) = \mathrm{e}^{\phi} m(\mathrm{d}v)/m(\mathrm{e}^{\phi})$ for some ϕ continuous and bounded. Fix a sequence $(e_N, u_N) \to (e, u)$, and denote by $\tilde{\nu}^N_{e_N,u_N}$ the regular version of the probability $\pi_0^{\otimes N}$ conditioned to $(\frac{1}{N}\sum_{i=1}^N \frac{1}{2}|v_i|^2,\ \frac{1}{N}\sum_{i=1}^N v_i)$ evaluated at (e_N, u_N). Then*

$$\varlimsup_{N\to\infty} \frac{1}{N} \operatorname{Ent}\big(\tilde{\mathbb{P}}^N_{\tilde{\nu}^N_{e_N,u_N}} | \mathbb{P}^N_{\nu^N_{e_N,u_N}}\big) \le I_{e,u}(\pi, Q).$$

We premise the following Lemma.

LEMMA 5.5. *If $F \in C_{\mathrm{b}}([0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times S_{d-1})$, then*

$$\limsup_{N\to\infty} \tilde{\mathbb{E}}^N_{\tilde{\nu}^N_{e_N,u_N}} \big(Q^N(F)^2\big) < +\infty.$$

PROOF. Set

$$\tilde{M}^N_t := Q^N_{[0,t]}(F) - \frac{1}{N^2} \sum_{\{i,j\}} \int_0^t \mathrm{d}s \int \mathrm{d}\omega\, \tilde{B}_s(v_i, v_j, \omega) F_s(v_i, v_j, \omega),$$

and note that it is a $\tilde{\mathbb{P}}_{\tilde{\nu}^N_{e_N,u_N}}$ martingale with predictable quadratic variation

$$\langle \tilde{M}^N \rangle_t = \frac{1}{N^2} \sum_{\{i,j\}} \int_0^t \mathrm{d}s \int \mathrm{d}\omega\, \tilde{B}_s(v_i, v_j, \omega) F_s(v_i, v_j, \omega)^2.$$

In view of (5.2), the random variable $\langle \tilde{M}^N \rangle_T$ is uniformly bounded in N, which implies the statement. □

PROOF OF PROPOSITION 5.4. By using Theorem 2.2, it is enough to show that

$$\varlimsup_{N\to\infty} \frac{1}{N} \operatorname{Ent}\big(\tilde{\mathbb{P}}^N_{\tilde{\nu}^N_{e_N,u_N}} | \mathbb{P}^N_{\tilde{\nu}^N_{e_N,u_N}}\big) \le J_{e,u}(\pi, Q). \tag{5.7}$$

In view of the assumptions on $\tilde{B}$, the value at time T of the martingale defined in (4.2) with $F_t = \log(\tilde{B}/B)$ is the Radon–Nykodim derivative of $\tilde{\mathbb{P}}^N_{\tilde{\nu}^N_{e_N,u_N}}$ with respect to $\mathbb{P}^N_{\tilde{\nu}^N_{e_N,u_N}}$. Since $\lambda^F_t = \tilde{\lambda}_t$,

$$\begin{aligned} &\frac{1}{N} \operatorname{Ent}\big(\tilde{\mathbb{P}}^N_{\tilde{\nu}^N_{e_N,u_N}} | \mathbb{P}^N_{\tilde{\nu}^N_{e_N,u_N}}\big) \\ &\quad = \tilde{\mathbb{E}}_{\tilde{\nu}^N_{e_N,u_N}} \bigg(Q^N_{[0,T]}(F) - \frac{1}{2}\int_0^T \mathrm{d}s\, \pi^N_s \otimes \pi^N_s(\tilde{\lambda}_s - \lambda)\bigg). \end{aligned}$$

Now observe that, by Lemma 5.2, (π^N, Q^N) converges to (π, Q) in $\tilde{\mathbb{P}}^N_{\tilde{\nu}^N_{e_N,u_N}}$ probability. By the definition of $\tilde{\mathbb{S}}_{e,u}$, F satisfies the assumption of Lemma 5.5, then the sequence $Q^N_{[0,T]}(F)$ is uniformly summable with respect to $\tilde{\mathbb{P}}^N_{\tilde{\nu}^N_{e_N,u_N}}$. By (5.2), $\pi^N_s \otimes \pi^N_s(\tilde{\lambda}_s)$ converges to $\pi_s \otimes \pi_s(\tilde{\lambda}_s)$ for almost all $s \in [0, T]$. Moreover, by energy conservation, λ is uniformly summable with respect to $\mathrm{d}s\, \pi^N_s \otimes \pi^N_s$. Therefore (5.7) follows. □

Approximating paths. Recall that the set $\hat{\mathcal{S}}$ has been defined above in Theorem 3.2.

THEOREM 5.6. *Assume that m satisfies items* (i)–(iv) *in Assumption 2.1. For each $(\pi, Q) \in \hat{\mathcal{S}}$ such that $I_{e,u}(\pi, Q) < +\infty$ there exists a sequence $\{(\pi_n, Q_n)\} \subset \tilde{\mathcal{S}}_{e,u} \cap \hat{\mathcal{S}}$ satisfying $(\pi_n, Q_n) \to (\pi, Q)$ and $I_{e,u}(\pi_n, Q_n) \to I(\pi, Q)$.*

This result, together with Proposition 5.4 and Lemma 5.1, concludes the proof of the large deviations lower bound as stated in (3.10).

PROOF. The proof is achieved by combining the following three steps and a standard diagonal argument. In particular, in Step 1 we construct positive regular approximating probability paths, in Step 2 we regularize in time, in Step 3 we perform a truncation argument as in [1], adapted to the hard-sphere kernel.

Step 1. Velocity convolution. Since $I_{e,u}(\pi, Q) < +\infty$ and $(\pi, Q) \in \hat{\mathcal{S}}$, $\pi_t(\zeta) = u$, $\pi_t(\zeta_0) = \pi_0(\zeta_0) = U + |u|^2/2 \in (0, e]$, where $U = \frac{1}{2}\int \pi_t(\mathrm{d}v)|v-u|^2$ is the internal energy.

Let (f, q) be the densities of (π, Q). Given $0 < \delta < 1$, let g_δ be the Gaussian kernel on $\mathbb{R}^d$ with variance δ and define

$$
\begin{aligned}
f_t^\delta(v) &= \alpha(g_\delta * f_t)\big(\alpha(v-u)+u\big), \\
q_t^\delta(v, v_*, \omega) &= \alpha^2(g_\delta \otimes g_\delta \otimes \mathrm{id} * q)\big(\alpha(v-u)+u, \alpha(v_*-u)+u, \omega\big),
\end{aligned}
\tag{5.8}
$$

where id is the identity function and $\alpha = \alpha(\delta) > 0$ is chosen such that $\int \mathrm{d}v f_t^\delta(v)|v-u|^2/2 = U$. Observe that for any $\alpha > 0$, $\int \mathrm{d}v f_t^\delta(v) v = u$.

Let (π^δ, Q^δ) be the pair with densities (f_t^δ, q_t^δ), which satisfies the balance equation. In order to prove the convergence of the rate function, we first observe that, by item (ii) in Assumption 2.1, we can write

$$
\mathrm{Ent}(\pi_0^\delta | m_{e,u}) = \int f_0^\delta \log f_0^\delta + \int f_0^\delta \log \frac{1}{m_{e,u}}.
$$

Since $\alpha(\delta) \to 1$ as $\delta \to 0$, by Jensen's inequality and item (ii) in Assumption 2.1,

$$
\varlimsup_{\delta \to 0} \mathrm{Ent}(\pi_0^\delta | m_{e,u}) \le \mathrm{Ent}(\pi_0 | m_{e,u}).
$$

By the choice of α, f_0^δ has the same energy as f_0. Therefore

$$
\varlimsup_{\delta \to 0} H_{e,u}(\pi_0^\delta) \le H_{e,u}(\pi_0).
$$

We will conclude the proof showing that $\varlimsup J_{e,u}(f^\delta, q^\delta) \le J_{e,u}(f, q)$. We first observe that by a straightforward approximation argument we can choose $F = \log 1/B$ in (4.11), and deduce

$$
Q\left(\log \frac{1}{B}\right) \le J_{e,u}(f, q) + \frac{1}{2}\int_0^T \mathrm{d}t \int \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\omega f f_* B\left(\frac{1}{B} - 1\right) < \infty. \tag{5.9}
$$

We prove in the Appendix that $Q^\delta(\log 1/B)$ is bounded and converges to $Q(\log 1/B)$ as $\delta \to 0$. Therefore

$$
J_{e,u}(\pi^\delta, Q^\delta) = \int_0^T \mathrm{d}t \int \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\omega\, q^\delta \log \frac{2q^\delta}{f^\delta f_*^\delta} + Q^\delta\left(\log \frac{1}{B}\right) - Q^\delta(1) + Q^{\pi^\delta}(1).
$$

Since the map $[0, +\infty)^2 \ni (a, b) \mapsto a \log(a/b)$ is one-homogeneous and convex, by (5.8) and Jensen's inequality the first term on the r.h.s. is bounded by $Q(\log \frac{2q}{f f_*})$. Moreover, $Q^\delta(1) = Q(1)$, while, since $B = \frac{1}{2}|(v - v_*) \cdot \omega|$, $Q^{\pi^\delta}(1) = \frac{1}{\alpha} Q^\pi(1)$.

Step 2. Time convolution. Consider $(\pi, Q) \in \hat{\mathcal{S}}$ such that $I_{e,u}(\pi, Q) < +\infty$, and denote with (f, q) their densities. Assume that f and q are smooth in the velocities, and $f > 0$. Observe that the approximating path constructed in Step 1 meets these requirements.

Extend $[0, T] \ni t \mapsto (f_t, q_t)$ to a function defined on $(-\infty, T]$ by setting $(f_t, q_t) = (f_0, 0)$ if $t < 0$. Let $\imath_\varepsilon$ be the a smooth approximation of the δ function, with support in $(-\varepsilon, 0)$, and denote by $(\pi^\varepsilon, Q^\varepsilon)$ the path with densities $(f^\varepsilon, q^\varepsilon) = \imath_\varepsilon * (f, q)$; here we understand the convolution in time. The pair $(\pi^\varepsilon, Q^\varepsilon)$ converges to (π, Q) and satisfies the balance equation (3.4). Observe that $f_0^\varepsilon = f_0$ and, since $(\pi, Q) \in \hat{\mathcal{S}}$, $\pi_t(\zeta) = \pi_0(\zeta)$ for any $t \in [0, T]$, so that $\pi_t^\varepsilon(\zeta) = \pi_0(\zeta)$ for any $t \in [0, T]$,

We claim that $\lim_{\varepsilon\to 0} I_{e,u}(\pi^\varepsilon, Q^\varepsilon) = I_{e,u}(\pi, Q)$. To this end, as $H_{e,u}(\pi_0^\varepsilon) = H_{e,u}(\pi_0)$, by lower semi-continuity it is enough to show that $\overline{\lim}_{\varepsilon\to 0} J_{e,u}(\pi^\varepsilon, Q^\varepsilon) \le J_{e,u}(\pi, Q)$.

Let g_1 be the standard Gaussian density on $\mathbb{R}^d$. We observe that, by a standard approximation argument, we can choose $F = \log g_1/f$ in the variational formula (4.11), and deduce that $\int q \log \frac{1}{f} < +\infty$ is finite. Since $J_{e,u}(\pi, Q)$ is bounded, using (5.9), we then deduce that $\int q \log q < +\infty$.

By Jensen's inequality $\int q^\varepsilon \log q^\varepsilon \le \int q \log q < +\infty$. On the other hand, by convexity, the maps $q \mapsto \int q \log q$ is lower semi-continuous, therefore we conclude that

$$\lim_{\varepsilon\to 0} \int q^\varepsilon \log q^\varepsilon = \int q \log q.$$

We write

$$J_{e,u}(\pi^\varepsilon, Q^\varepsilon) = -\int q^\varepsilon \log 2q^\varepsilon + \int q^\varepsilon \log \frac{2q^\varepsilon}{f^\varepsilon} + \int q^\varepsilon \log \frac{2q^\varepsilon}{f_*^\varepsilon} + \int q^\varepsilon\Big(\log \frac{1}{B} - 1\Big) + \int f^\varepsilon f_*^\varepsilon B.$$

As already stated, the first term on the right-hand-side converges. By Jensen's inequality the second term is bounded by $\int q \log(2q/f)$ and the third by $\int q \log(2q/f_*)$. Moreover, the fourth does not depend on ε. The convergence of the last term follows from the fact that, since the energy is uniformly bounded and $\pi \in C([0, T], \mathcal{P}(\mathbb{R}^d))$, the map $[0, T]^2 \ni (s, s') \mapsto \int \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\omega f_s(v) f_s(v_*) B(v - v_*, \omega)$ is continuous.

Step 3. Truncation. Consider $(\pi, Q) \in \hat{\mathcal{S}}$ with $I_{e,u}(\pi, Q) < +\infty$, with densities (f, q). We denote by $q_t^{(i)}$, $i = 1, \dots, 4$ the marginal of q_t respectively on v, v_*, v', v'_*. Then $q_t^{(1)} = q_t^{(2)}$, $q_t^{(3)} = q_t^{(4)}$, and the balance equation is the weak version of the identity

$$\partial_t f_t = 2\big(q_t^{(3)} - q_t^{(1)}\big).$$

In the sequel we assume (f, q) smooth, f strictly positive, and $q_t^{(3)} \in L^2([0, T] \times \mathbb{R}^d)$. Observe that the approximating path defined by applying sequentially Step 1 and 2 meets the above conditions. Indeed, the last condition above follows by Young's inequality for convolutions.

Given $\ell > 0$, let $\chi^\ell(v, v_*, \omega) \in [0, 1]$ be a continuous function such that

$$\chi^\ell(v, v_*, \omega) = \begin{cases} 1 & \text{if } |v|^2 + |v|_*^2 < \ell \text{ and } |(v - v_*) \cdot \omega| > 1/\ell, \\ 0 & \text{if } |v|^2 + |v|_*^2 \ge (\ell + 1) \text{ or } |(v - v_*) \cdot \omega| \le 1/(\ell + 1). \end{cases}$$

We define $(\tilde{f}^\ell, \tilde{q}^\ell)$ by

$$\begin{aligned} &\tilde{q}^\ell(v, v_*, \omega) = q(v, v_*, \omega)\chi^\ell(v, v_*, \omega), \\ &\tilde{f}_t^\ell = f_0 + 2\int_0^t \mathrm{d}s\big(\tilde{q}_s^{\ell,(3)} - \tilde{q}_s^{\ell,(1)}\big) + 2\int_0^T \mathrm{d}s\big(q_s^{(3)} - \tilde{q}_s^{\ell,(3)}\big). \end{aligned} \tag{5.10}$$

Observe that $\tilde{q}_t^\ell \le q_t$. Moreover $\tilde{f}_t^\ell \ge f_t$, since

$$\int_0^t \mathrm{d}s\big(\tilde{q}_s^{\ell,(3)} - \tilde{q}_s^{\ell,(1)}\big) + \int_0^T \mathrm{d}s\big(q_s^{(3)} - \tilde{q}_s^{\ell,(3)}\big) = \int_0^t \mathrm{d}s\big(q_s^{(3)} - \tilde{q}_s^{\ell,(1)}\big) + \int_t^T \mathrm{d}s\big(q_s^{(3)} - \tilde{q}_s^{\ell,(3)}\big). \tag{5.11}$$

Set

$$c_\ell^{-1} = 1 + 2\int_0^T \mathrm{d}s \int \mathrm{d}v\big(q_s^{(3)} - \tilde{q}_s^{\ell,(3)}\big),$$

and denote by (e_ℓ, u_ℓ) the energy and momentum of the probability $c_\ell \tilde{f}_t^\ell \,\mathrm{d}v$. Note that (e_ℓ, u_ℓ) does not depend on t since $(\pi, Q) \in \hat{\mathbb{S}}$. We define (f^ℓ, q^ℓ) by:

$$f^\ell(v) = \alpha c_\ell \tilde{f}^\ell\big(\alpha(v-u) + u_\ell\big),$$

$$q^\ell(v, v_*, \omega) = \alpha^2 c_\ell \tilde{q}^\ell\big(\alpha(v-u) + u_\ell, \alpha(v-u) + u_\ell\big),$$

where $\alpha = \alpha_\ell > 0$ is chosen such that $\int \mathrm{d}v f_0^\ell(v)\boldsymbol{\zeta}(v) = \int \mathrm{d}v f_0(v)\boldsymbol{\zeta}(v)$. Observe that the pair (f^ℓ, q^ℓ) satisfies the balance equation. As $\ell \to +\infty$, $c_\ell \to 1$, $u_\ell \to u$, $\alpha_\ell \to 1$, therefore (f^ℓ, q^ℓ) converges to (f, q).

We claim that

$$\varlimsup_{\ell \to +\infty} I_{e,u}(\pi^\ell, Q^\ell) \le I_{e,u}(\pi, Q).$$

We start by proving that

$$\varlimsup_{\ell \to +\infty} H_{e,u}(\pi_0^\ell) \le H_{e,u}(\pi_0). \tag{5.12}$$

Let m^ℓ be the probability measure satisfying

$$\int m^\ell(\mathrm{d}v)\varphi(v) = \int m_{e,u}(\mathrm{d}v)\alpha\varphi\big(\alpha(v-u) + u_\ell\big),$$

for any $\varphi \in C_\mathrm{b}(\mathbb{R}^d)$, and let ρ^ℓ be its density. By a change of variable

$$\mathrm{Ent}\big(\pi_0^\ell | m_{e,u}\big) = \mathrm{Ent}\big(c_\ell \tilde{f}_0^\ell \,\mathrm{d}v | m^\ell\big). \tag{5.13}$$

By (5.10),

$$c_\ell \tilde{f}_0^\ell = c_\ell f_0 + (1 - c_\ell)\bar{h}^\ell,$$

where $h^\ell = 2\int_0^T \mathrm{d}s(q_s^{(3)} - \tilde{q}_s^{\ell,(3)})$ and $\bar{h}^\ell = h^\ell / \int h^\ell$. By convexity

$$\mathrm{Ent}\big(c_\ell \tilde{f}_0^\ell \,\mathrm{d}v | m^\ell\big) \le c_\ell \,\mathrm{Ent}\big(\pi_0 | m^\ell\big) + (1 - c_\ell)\,\mathrm{Ent}\big(\bar{h}^\ell \,\mathrm{d}v | m^\ell\big).$$

Since $c_\ell \to 1$, $\alpha_\ell \to 1$, $u_\ell \to u$, in view of item (iv) in Assumption 2.1, by dominated convergence the first term on the right-hand-side of (5.13) converges to $\mathrm{Ent}(\pi_0 | m)$.

We now show that the second term vanishes. Observe that

$$(1 - c_\ell)\,\mathrm{Ent}\big(\bar{h}^\ell \,\mathrm{d}v | m^\ell\big) = c_\ell \int h^\ell \log h^\ell + (1 - c_\ell)\log \frac{c_\ell}{1 - c_\ell} - c_\ell \int h^\ell \log \rho^\ell.$$

Since, by assumption on $q^{(3)}$, $h^\ell \in L^2$ and it converges to zero pointwise, the first term vanishes. The second term vanishes since $c_\ell \to 1$. Finally, using item (iv) of Assumption 2.1, the last term vanishes by dominated convergence. Since $\pi_0^\ell(\boldsymbol{\zeta}) = \pi_0(\boldsymbol{\zeta})$, (5.12) follows.

We conclude the proof by showing that

$$\lim_{\ell \to +\infty} J_{e,u}(\pi^\ell, Q^\ell) = J_{e,u}(\pi, Q).$$

By a change of variables,

$$J_{e,u}(\pi^\ell, Q^\ell) = c_\ell \int \tilde{q}^\ell \log \frac{2\tilde{q}^\ell}{c_\ell \tilde{f}^\ell \tilde{f}^\ell_* B} + c_\ell \log\alpha \int \tilde{q}^\ell - c_\ell \int \tilde{q}^\ell + \frac{c_\ell^2}{\alpha} \int \tilde{f}^\ell \tilde{f}^\ell_* B.$$

Since $\tilde{q}^\ell \le q$, $\tilde{f}^\ell \ge f$, and $c_\ell \to 1$, by dominated convergence the first term on the right-hand-side converges to $\int q \log(2q/ff_*B)$. Since $\int q^\ell \to \int q$ and $\alpha \to 1$, the second term tends to 0, and the third converges to $Q(1)$. Finally, since $\int q^{(3)}\zeta_0 < +\infty$, B is uniformly summable with respect to $\tilde{f}^\ell \tilde{f}^\ell_*$, therefore the last term converges to $Q^\pi(1)$. $\square$

6. Large deviations for Kac model with canonical initial data. In this section we consider the Kac model with canonical initial data, namely when the initial velocities are i.i.d. sampled from a given probability m. In view of the abstract Proposition 2.5, the large deviation principle for the pair empirical measure and flow can be deduced from the large deviation principle of the Kac model with microcanonical initial data.

The canonical rate function is given by

$$I(\pi, Q) = \inf_{(e,u)\in Z} \big(A(e,u) + I_{e,u}(\pi, Q)\big), \tag{6.1}$$

where A, as defined in (2.16), is the rate function relative to the sum of i.i.d. random variables given by Cramér's theorem.

In order to compare this rate function with the one in [9, 13], consider the dynamical function as in (3.7), but without the microcanonical constraint, namely

$$J(\pi, Q) := \int \mathrm{d}Q^\pi \left[\frac{\mathrm{d}Q}{\mathrm{d}Q^\pi} \log \frac{\mathrm{d}Q}{\mathrm{d}Q^\pi} - \left(\frac{\mathrm{d}Q}{\mathrm{d}Q^\pi} - 1 \right) \right]. \tag{6.2}$$

Then functional in [9, 13] reads

$$\mathcal{I}(\pi, Q) = \mathrm{Ent}(\pi_0|m) + J(\pi, Q).$$

By Remark 2.6, for any $(\pi, Q) \in \mathcal{S}$ we have $\mathcal{I}(\pi, Q) \le I(\pi, Q)$. For some path (π, Q) this inequality is strict because, as discussed in detail in the next section, $\mathcal{I}$ vanishes on Lu and Wennberg solutions, while I is strictly positive.

THEOREM 6.1. *Let m by a probability measure in $\mathbb{R}^d$ and set $\mu^N = m^{\otimes N}$. If m satisfies item* (i)–(iii) *in Assumption 2.1 then the family $\mathbb{P}^N_{\mu^N} \circ (\pi^N, Q^N)^{-1}$ satisfies a large deviation upper bound with good rate function $I : \mathcal{S} \to [0, +\infty]$, namely I has compact level sets and for each closed $C \subset \mathcal{S}$*

$$\varlimsup_{N\to+\infty} \frac{1}{N} \log \mathbb{P}^N_{\mu^N}\big((\pi^N, Q^N) \in C\big) \le -\inf_C I. \tag{6.3}$$

Moreover, if m satisfies also condition (iv) *in Assumption 2.1, then for each open $O \subset \mathcal{S}$*

$$\varliminf_{N\to+\infty} \frac{1}{N} \log \mathbb{P}^N_{\mu^N}\big((\pi^N, Q^N) \in O\big) \ge -\inf_{O\cap\hat{\mathcal{S}}} I. \tag{6.4}$$

PROOF. By the definition of the microcanonical ensemble $\nu^N_{e,u}$ given below equation (2.2), we have

$$\mathbb{P}^N_{\mu^N} = \int p_N\big(\mathrm{d}(e,u)\big)\mathbb{P}^N_{\nu^N_{e,u}},$$

where p_N is the law of $\frac{1}{N}\sum_i \boldsymbol{\zeta}(v_i)$ with $\boldsymbol{v}$ sampled according to μ^N. By Cramér's theorem, as discussed before in Remark 2.6, p_N satisfies a large deviation principle with rate function A. The proof is thus essentially achieved by combining Theorem 3.2 with the abstract

Proposition 2.5. However, since in the large deviation result with microcanonical initial data the upper and lower bound rate function may differ, we need a replacement for Step 2 in the the proof of Proposition 2.5.

Upper bound. The argument in Step 4 in the proof of Proposition 2.5 applies, provided we show that the map $Z \times \mathcal{S} \ni (e, u, \pi, Q) \mapsto I_{e,u}(\pi, Q)$ is lower semicontinuous.

Recall the set $\mathcal{C}_{e,u}$ defined in (3.5), and let $\mathfrak{C}$ be the subset of $Z \times \mathcal{S}$ defined by

$$\mathfrak{C} := \{(e, u, \pi, Q) : \pi \in \mathcal{C}_{e,u}\}.$$

By the lower semicontinuity of the map $\pi \mapsto \pi(\zeta_0)$, and the continuity of the map $\pi \mapsto \pi(\zeta)$ when the energy of π is uniformly bounded, we deduce that $\mathfrak{C}$ is closed. By the variational representation (4.11), this implies the joint lower semicontinuity of $J_{e,u}(\pi, Q)$.

By Theorem 2.2 and Step 2 in the proof of Proposition 2.5, we also deduce the joint lower semicontinuity of $H_{e,u}(\pi_0)$, that conclude the proof.

Lower bound. Fix $(\pi, Q) \in \hat{\mathcal{S}}$. By Step 3 in the proof of proposition 2.5, we deduce that for any open neighborhood $\mathcal{N}$ of (π, Q) we have

$$\varliminf_{N \to +\infty} \frac{1}{N} \log \mathbb{P}^N_{\mu^N}\big((\pi^N, Q^N) \in \mathcal{N}\big) \geq -I(\pi, Q),$$

that implies the statement. □

7. Asymptotic probability of Lu and Wennberg solutions. We start by observing that the balance equation (3.4) for a pair (π, Q) with $Q = Q^\pi$ is equivalent to the statement that π is a weak solution (1.1). Recalling that the functional J, as defined in (6.2), vanishes if and only if $Q = Q^\pi$, then we deduce that the zero level set of J are the weak solutions to the homogeneous Boltzmann equation (1.1). As we next state, the zero level set of both the functional $I_{e,u}$ and I respectively defined in (3.8), (6.1) is a singleton. As a consequence the large deviation upper bound stated in Theorems (3.2) and (6.1) implies the convergence of the empirical measure to the unique energy solution to the homogeneous Boltzmann equation (1.1) with an exponential bound on the error.

THEOREM 7.1.

(i) $I_{e,u}(\pi, Q) = 0$ *if and only if* $\pi = f\,\mathrm{d}v$, $Q = Q^\pi$ *and* f *is the unique energy conserving solution to the Cauchy problem associated to* (1.1) *with initial datum* $\frac{\mathrm{d}m_{e,u}}{\mathrm{d}v}$ *as defined in* (2.1).

(ii) $I(\pi, Q) = 0$ *if and only if* $\pi = f\,\mathrm{d}v$, $Q = Q^\pi$ *and* f *is the unique energy conserving solution to the Cauchy problem associated to* (1.1) *with initial datum* $\frac{\mathrm{d}m}{\mathrm{d}v}$.

PROOF. We prove only the first statement. By the definition of $I_{e,u}$ if f is an energy conserving solution to the Cauchy problem associated to (1.1) with initial datum $\frac{\mathrm{d}m_{e,u}}{\mathrm{d}v}$, then $\pi = f\,\mathrm{d}v$ and $Q = Q^\pi$ belong to the zero level set of $I_{e,u}$. To prove the converse, we observe that, by the very definition (3.7), $J_{e,u}(\pi, Q) = 0$ implies that $Q = Q^\pi$ and $\pi_t(\zeta_0) \leq e$ for any $t \in [0, T]$. Since $H_{e,u}(\pi_0) = 0$ implies that $\pi_0 = m_{e,u}$ we deduce $\pi_t = f_t\,\mathrm{d}v$ where f is a weak solution to the Cauchy problem associated to (1.1) with initial datum $\frac{\mathrm{d}m_{e,u}}{\mathrm{d}v}$ and nonincreasing energy. Since for any weak solution to (1.1) the energy can not decrease in time (see [14, 17]), f_t is the unique energy conserving solution. □

Fix a nondecreasing piecewise constant, left-continuous profile $\mathcal{E} : [0, T] \to \mathbb{R}_+$, with finite, nonzero, number of jumps.

DEFINITION 7.2. A Lu and Wennberg solution to the Cauchy problem associated to the homogeneous Boltzmann equation with initial datum f_0 and *energy profile* $\mathcal{E}$ is a measurable function $f : [0, T] \times \mathbb{R}^d \to [0, +\infty)$ such that:

(i) the map $t \mapsto f_t(v)\,\mathrm{d}v =: \pi_t$ in $C([0, T]; \mathcal{P}(\mathbb{R}^d))$;
(ii) f is a weak solution to the homogeneous Boltzmann equation;
(iii) $\pi_t(\zeta_0) = \mathcal{E}(t)$, $t \in [0, T]$.

Observe that for any $e \geq \mathcal{E}(T)$, for $\pi = f\,\mathrm{d}v$, with f a Lu and Wennberg solution, $J_{e,u}(\pi, Q^\pi) = 0$. Hence

$$I_{e,u}(\pi, Q^\pi) = \mathrm{Ent}(\pi_0 | m_{e,u}) + [\gamma_0^* - \gamma_0][e - \mathcal{E}(0)],$$

namely the Lu and Wennberg solutions contribute to the rate function only at time zero. We remark that these pairs (π, Q^π) do not belong to the set $\hat{\mathcal{S}}$ for which the upper and lower bound in Theorem 3.2 is proven to match. In the next theorem we will show they actually match also for a suitable class of Lu and Wennberg solutions.

THEOREM 7.3. *Fix $(e, u) \in Z$ and a sequence $(e_N, u_N) \to (e, u)$. For each energy profile $\mathcal{E}$ with $\mathcal{E}(T) < e$ and each f_0 with energy $\mathcal{E}(0)$, there exists a Lu and Wennberg solution f with energy profile $\mathcal{E}$ such that for every open neighborhood A of (π, Q^π), $\pi = f\,\mathrm{d}v$,*

$$\varliminf_{N \to +\infty} \frac{1}{N} \log \mathbb{P}^N_{\nu^N_{e_N, u_N}} \big((\pi^N, Q^N) \in A \big) \geq -I_{e,u}(\pi, Q^\pi). \tag{7.1}$$

Observe that, by the upper bound in Theorem 3.2

$$\varlimsup_{N \to +\infty} \frac{1}{N} \log \mathbb{P}^N_{\nu^N_{e_N, u_N}} \big((\pi^N, Q^N) \in \bar{A} \big) \leq -\inf_{\bar{A}} I_{e,u},$$

which, together with (7.1), identifies the asymptotic probability of Lu and Wennberg solutions.

As in [15], the Lu and Wennberg solutions will be constructed as a limit of a suitable sequence. In particular we will consider a sequence f^n which conserve the energy and such that $t \mapsto f^n_t(v)\,\mathrm{d}v \in \mathcal{P}(\mathbb{R}^d)$ is continuous. We start with the static result.

LEMMA 7.4. *Consider $\rho \in \mathcal{P}(\mathbb{R}^d)$ such that $H_{e,u}(\rho)$ is finite and $e_0 := \rho(\zeta_0) < e$. Given $e_1 \in (e_0, e]$ and $n \in \mathbb{N}$, let $g_n = m_{n(e_1 - e_0) + e_0, u}$ be the exponential tilt of m with energy $n(e_1 - e_0) + e_0$ and momentum u. Set $\rho_n = (1 - \frac{1}{n})\rho + \frac{1}{n} g_n$, so that $\rho_n(\zeta_0) = e_1$, then*

$$\lim_{n \to \infty} H_{e,u}(\rho_n) = H_{e,u}(\rho). \tag{7.2}$$

PROOF. By the lower semicontinuity of $H_{e,u}$, it is enough to show that $\varlimsup H_{e,u}(\rho_n) \leq H_{e,u}(\rho)$. By the convexity of $H_{e,u}$ and Jensen's inequality

$$H_{e,u}(\rho_n) \leq \left(1 - \frac{1}{n}\right) H_{e,u}(\rho) + \frac{1}{n} H_{e,u}(g_n).$$

Let $\boldsymbol{\lambda}^n$ such that

$$g_n(\mathrm{d}v) = \frac{\mathrm{e}^{\boldsymbol{\lambda}_n \cdot \boldsymbol{\zeta}} m(\mathrm{d}v)}{m(\mathrm{e}^{\boldsymbol{\lambda}_n \cdot \boldsymbol{\zeta}})} = \frac{\mathrm{e}^{(\boldsymbol{\lambda}_n - \boldsymbol{\gamma}(e,u)) \cdot \boldsymbol{\zeta}}}{m_{e,u}(\mathrm{e}^{(\boldsymbol{\lambda}_n - \boldsymbol{\gamma}(e,u)) \cdot \boldsymbol{\zeta}})} m_{e,u}(\mathrm{d}v),$$

where we used (2.1). Observe that $\lambda_0^n \uparrow \gamma_0^*$ as $n \to +\infty$. Since g_n has energy $n(e_1 - e_0) + e_0$ we get

$$\varlimsup_{n \to +\infty} \frac{1}{n} \mathrm{Ent}(g_n | m_{e,u}) \leq \varlimsup_{n \to +\infty} (\lambda_0^n - \gamma_0(e, u))(e_1 - e_0) = (\gamma_0^* - \gamma_0(e, u))(e_1 - e_0),$$

which concludes the proof. □

For any probability density h with finite energy let $\mathcal{U}_t(h), t \geq 0$, be the unique energy conserving solution to the Cauchy problem associated to the homogeneous Boltzmann equation with initial datum h. In the following statement we collect the result on the moments estimates in [17, 21].

LEMMA 7.5. *Let h be a probability density on $\mathbb{R}^d$ with finite energy and entropy. Then*:

(i) *For each $p > 2$ and $t > 0$ there exists a real $C > 0$ depending only on p, t and the initial energy, such that*

$$\int \mathrm{d}v\, \mathcal{U}_t(h)(v)|v|^p \leq C.$$

(ii) *For each $p > 2$, if $\int \mathrm{d}v\; h(v)|v|^p < +\infty$, then*

$$\sup_{t\in[0,T]} \int \mathrm{d}v\; \mathcal{U}_t(h)(v)|v|^p < +\infty.$$

Fix an energy profile $\mathcal{E} : [0, T] \to \mathbb{R}_+$ and denote by $0 \leq t_1 < \cdots < t_k < T$ the discontinuity set of $\mathcal{E}$. Given f_0 with finite entropy and energy $\mathcal{E}(0)$, let h_0^n be a sequence weakly convergent to f_0 satisfying the following requirements. The energy of h_0^n is independent on n and equal to $\mathcal{E}(0)$, its entropy converges the entropy of f_0, and it has finite (n-dependent) p-moment for some $p \geq 3$. For $n \geq 1$ and $i = 1, \dots, k$, set $e_{n,i} = nk[\mathcal{E}(t_i^+) - \mathcal{E}(t_i)] + \mathcal{E}(0)$ and define g_i^n as the density of the tilted probability $m_{e_{n,i},u}$. Define

$$f_t^n = \begin{cases} \Bigl(1 - \dfrac{1}{n}\Bigr)\mathcal{U}_t(h_0^n) + \dfrac{1}{nk}\displaystyle\sum_{i=1}^k g_i^n & t \in [0, t_1], \\ \Bigl(1 - \dfrac{k-1}{nk}\Bigr)\mathcal{U}_{t-t_1}(h_1^n) + \dfrac{1}{nk}\displaystyle\sum_{i=2}^k g_i^n & t \in (t_1, t_2], \\ \cdots & \cdots, \\ \Bigl(1 - \dfrac{1}{nk}\Bigr)\mathcal{U}_{t-t_{k-1}}(h_{k-1}^n) + \dfrac{1}{nk} g_k^n & t \in (t_{k-1}, t_k], \\ \mathcal{U}_{t-t_k}(h_k^n) & t \in (t_k, T], \end{cases} \tag{7.3}$$

where h_i^n are recursively defined so that $t \mapsto f_t^n(v)\,\mathrm{d}v$ is continuous, namely

$$h_i^n = \frac{1}{1 - \frac{k-i}{nk}}\Biggl[f_{t_i}^n - \frac{1}{nk}\sum_{j=i+1}^k g_j^n\Biggr].$$

Let also $q_t^n(v, v_*, \omega)$ be such that, for $t \in (t_i, t_{i+1}]$,

$$q_t^n(v, v_*, \omega) = \Bigl(1 - \frac{k-i}{nk}\Bigr)\mathcal{U}_{t-t_i}(h_i^n)(v)\mathcal{U}_{t-t_i}(h_i^n)(v_*)B(v, v_*, \omega).$$

Here $i = 0, \dots, k$, with $t_0 = 0$ and $t_{k+1} = T$. Observe that, by construction, the pair (π^n, Q^n) with densities (f^n, q^n) satisfies the balance equation (3.4). Furthermore, by definition of h_0^n and item (ii) in Lemma 7.5, for each n the pair $(\pi^n, Q^n) \in \hat{\mathcal{S}}$.

LEMMA 7.6. *The sequence $\{(\pi^n, Q^n)\}$ is relatively compact in $\mathcal{S}$. Any cluster point (π, Q) is such that $Q = Q^\pi$, $\pi = f\,\mathrm{d}v$, where f is a Lu and Wennberg solution with initial datum f_0 and energy profile $\mathcal{E}$. Moreover*

$$\lim_{n\to\infty} I_{e,u}(\pi^n, Q^n) = H_{e,u}(f_0\,\mathrm{d}v). \tag{7.4}$$

PROOF. We start by proving (7.4). Observe that $\int \mathrm{d}v\, h_i^n \zeta_0 = \mathcal{E}(t_i^+)$, for $i = 1, \dots, k$. Then by Lemma 7.4 and Jensen's inequality,

$$\lim_{n\to\infty} H_{e,u}(\pi_0^n) = \mathrm{Ent}(f_0\,\mathrm{d}v|m_{e,u}) + (\gamma_0^* - \gamma(e,u))\left[e - \mathcal{E}(T) + \sum_{i=1}^k (\mathcal{E}(t_i^+) - \mathcal{E}(t_i))\right]$$
$$= H_{e,u}(f_0\,\mathrm{d}v).$$

We now show that

$$\lim_{n\to\infty} J_{e,u}(\pi^n, Q^n) = 0. \tag{7.5}$$

By definition, for $t \in (t_i, t_{i+1}]$, $i = 0, \dots, n$, we have

$$f_t^n \geq \left(1 - \frac{k-i}{n}\right)\mathcal{U}_{t-t_i}(h_i^n).$$

Hence the the contribution to $J_{e,u}(\pi^n, Q^n)$ in the time window $(t_i, t_{i+1}]$ is bounded by

$$\int_{t_i}^{t_{i+1}} \mathrm{d}t \int \mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega \left\{ q_t^n(v, v_*, \omega) \log\left(1 - \frac{k-i}{n}\right)^{-1} \right.$$
$$\left. - q_t^n(v, v_*, \omega) + f_t^n(v) f_t^n(v_*) B(v, v_*, \omega) \right\}.$$

Since the energy of f_t^n is $\mathcal{E}(T)$, the mass of q^n is bounded uniformly in n, therefore the first term vanishes as $n \to \infty$. The same bound, together with the fact that the energy of $\frac{1}{n} g_i^n$ is bounded uniformly in n and $B \leq C(1 + |v| + |v_*|)$, implies that the second line also vanishes. Hence (7.5) follows.

Equation (7.4) and the goodness of $I_{e,u}$ imply that the sequence (π^n, Q^n) is relatively compact. Let (π, Q) be a cluster point. By the lower semicontinuity of $J_{e,u}$ and (7.5) we deduce that $J_{e,u}(\pi, Q) = 0$, hence $Q = Q^\pi$, $\pi = f\,\mathrm{d}v$, where f is a solution to the Cauchy problem associated to (1.1) and initial datum f_0. It remains to show that f has energy profile $\mathcal{E}$. For any $i = 0, \dots, k$, the energy of h_i^n is uniformly bounded. Fix i and $t \in (t_i, t_{i+1}]$. By item (i) in 7.5, the p-moment of $\mathcal{U}_{t-t_i}(h_i^n)$, $p > 2$, is bounded uniformly in n, therefore ζ_0 is uniformly summable with respect to $\mathcal{U}_{t-t_i}(h_i^n)$, then

$$\int \mathrm{d}v\, f_t \zeta_0 = \lim_{n\to+\infty} \int \mathrm{d}v\, \mathcal{U}_{t-t_i}(h_i^n)\,\zeta_0 = \mathcal{E}(T) - \sum_{j=i+1}^k (\mathcal{E}(t_j^+) - \mathcal{E}(t_j)) = \mathcal{E}(t). \qquad \square$$

THEOREM 7.7. *Let m be a probability measure satisfying Assumption 2.1, and set $\mu^N := m^{\otimes N}$. For each energy profile $\mathcal{E}$ with $\mathcal{E}(0) = m(\zeta_0)$ there exists a Lu and Wennberg solution f with $f_0 = m$ and energy profile $\mathcal{E}$ such that for every open neighborhood A of (π, Q^π), $\pi = f\,\mathrm{d}v$,*

$$\varliminf_{N\to+\infty} \frac{1}{N} \log \mathbb{P}_{\mu^N}^N((\pi^N, Q^N) \in A) \geq -I(\pi, Q^\pi) = \gamma_0^*(\mathcal{E}(T) - \mathcal{E}(0)). \tag{7.6}$$

PROOF. The proof of the inequality in (7.6) follows the same arguments of the proof of Theorem 7.3. We here discuss the equality. Since $\pi = f\,\mathrm{d}v$ is a weak solution to (1.1), $J_{e,u}(\pi, Q) = 0$ if $e \geq \mathcal{E}(T)$, otherwise is infinity. Then, by definition (6.1) and Theorem 2.2 we have that

$$I(\pi, Q) = \inf_{e \geq \mathcal{E}(T)} (A(e,u) + \mathrm{Ent}(m|m_{e,u}) + (\gamma_0^* - \gamma_0(e,u)(e - \mathcal{E}(0)),$$

where $u = m(\zeta)$. The supremum in the definition (2.16) of $A_{e,u}$ is achieved in $\boldsymbol{\gamma} = \boldsymbol{\gamma}(e,u)$. By definition of the relative entropy

$$\mathrm{Ent}(m|m_{e,u}) = -\boldsymbol{\gamma}(e,u)\cdot m(\boldsymbol{\zeta}) + \log m\big(\mathrm{e}^{\boldsymbol{\gamma}(e,u)\cdot\boldsymbol{\zeta}}\big).$$

Then, by direct computation, $I(\pi, Q) = \inf_{e\ge\mathcal{E}(T)} \gamma_0^*(e-\mathcal{E}(0)) = \gamma_0^*(\mathcal{E}(T)-\mathcal{E}(0))$. □

APPENDIX

It is sufficient to prove that $Q^\delta([\log 1/B]^+)$ converges to $Q([\log 1/B]^+)$ as $\delta \to 0$, since the result for the negative part easily follows from the fact that $|[\log 1/B]^-|$ is sublinear in $|v - v_*|$, and $(\pi, Q) \in \hat{\mathbb{S}}$. We indicate with g_δ^1 the Gaussian kernel in one dimension, and note that

$$(g_\delta \otimes g_\delta \otimes \mathrm{id}) * \left[\log\frac{1}{2B}\right]^+(v, v_*, \omega) = \int_{\mathbb{R}} g_\delta^1(w\cdot\omega - y)\left[\log\frac{1}{|y|}\right]^+ \mathrm{d}y,$$

where $w = (v - v_*)/\sqrt{2}$. We now prove that there exist some constants $c_1, c_2 > 0$ such that

$$\int_{\mathbb{R}} g_\delta^1(x-y)\left[\log\frac{1}{|y|}\right]^+ \mathrm{d}y \le c_1\left[\log\frac{1}{|x|}\right]^+ + c_2,$$

which implies that

$$(g_\delta \otimes g_\delta \otimes \mathrm{id}) * \left[\log\frac{1}{B}\right]^+(v, v_*, \omega) \le c_1\left[\log\frac{1}{B}\right]^+(v, v_*, \omega) + c_2.$$

Using this fact and that $Q([\log 1/B]^+) < +\infty$, we achieve the convergence result by using the Fubini–Tonelli theorem and dominate convergence.

We denote by z a standard Gaussian stochastic variable and note that

$$\int_{\mathbb{R}} g_\delta^1(x-y)\left[\log\frac{1}{|y|}\right]^+ \mathrm{d}y = \mathbb{E}\left(\left[\log\frac{1}{|x-\delta z|}\right]^+\right) \le \log\frac{1}{\delta} + \mathbb{E}\left(\left[\log\frac{1}{|x/\delta - z|}\right]^+\right).$$

Since $[\log 1/|y|]^+$ is summable, by Young's inequality the second term is uniformly bounded, so that, if $|x| \le \sqrt{\delta}$ we have

$$\mathbb{E}\left(\left[\log\frac{1}{|x-\delta z|}\right]^+\right) \le 2\log\frac{1}{|x|} + c.$$

To handle the case $|x| \ge \sqrt{\delta}$, we use the Jensen inequality:

$$\mathbb{E}\left(\left[\log\frac{1}{|x-\delta z|}\right]^+\right) = 2\log \mathrm{e}^{\mathbb{E}([\log 1/\sqrt{|x-\delta z|}]^+)} \le 2\log\mathbb{E}\left(\frac{1}{\sqrt{|x-\delta z|\wedge 1}}\right).$$

We estimate

$$\mathbb{E}\left(\frac{1}{\sqrt{|x-\delta z|\wedge 1}}\right) = \int_{\mathbb{R}} g_\delta^1(y)\frac{1}{\sqrt{|x-y|\wedge 1}}\,\mathrm{d}y$$

by noticing that in the region $|y| < |x|/2$ or $|y| > 2|x|$ we have $1/\sqrt{|x-y|\wedge 1} \le \sqrt{2}/\sqrt{|x|\wedge 1}$. Therefore

$$\mathbb{E}\left(\frac{1}{\sqrt{|x-\delta z|\wedge 1}}\right) \le c\frac{1}{\sqrt{|x|\wedge 1}} + g_\delta^1(|x|/2)\int_{|x|/2}^{2|x|}\frac{1}{\sqrt{|x-y|\wedge 1}}\,\mathrm{d}y.$$

We conclude the proof observing that the last term is estimated by $c\mathrm{e}^{-1/8\delta}(1 + 1/\delta)$, which is uniformly bounded in δ.

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REFERENCES

[1] BASILE, G., BENEDETTO, D., BERTINI, L. and ORRIERI, C. (2021). Large deviations for Kac-like walks. *J. Stat. Phys.* **184** Paper No. 10, 27. MR4281234 https://doi.org/10.1007/s10955-021-02794-2

[2] BASILE, G., BENEDETTO, D., CAGLIOTI, E. and BERTINI, L. (2023). Large deviations for a binary collision model: Energy evaporation. *Math. Eng.* **5** Paper No. 001, 12. MR4370323 https://doi.org/10.1007/bf02828297

[3] BODINEAU, T., GALLAGHER, I., SAINT-RAYMOND, L. and SIMONELLA, S. (2023). Statistical dynamics of a hard sphere gas: fluctuating Boltzmann equation and large deviations. *Ann. of Math.* (2) **198** 1047–1201. MR4660136 https://doi.org/10.4007/annals.2023.198.3.3

[4] BOGACHEV, V. I. (2007). *Measure Theory. Vol. I, II*. Springer, Berlin. MR2267655 https://doi.org/10.1007/978-3-540-34514-5

[5] CARLEN, E. A., CARVALHO, M. C., LE ROUX, J., LOSS, M. and VILLANI, C. (2010). Entropy and chaos in the Kac model. *Kinet. Relat. Models* **3** 85–122. MR2580955 https://doi.org/10.3934/krm.2010.3.85

[6] CHATTERJEE, S. (2017). A note about the uniform distribution on the intersection of a simplex and a sphere. *J. Topol. Anal.* **9** 717–738. MR3684622 https://doi.org/10.1142/S1793525317500224

[7] DEMBO, A. and ZEITOUNI, O. (1998). *Large Deviations Techniques and Applications*, 2nd ed. *Applications of Mathematics* (*New York*) **38**. Springer, New York. MR1619036 https://doi.org/10.1007/978-1-4612-5320-4

[8] ERBAR, M. (2023). A gradient flow approach to the Boltzmann equation. *J. Eur. Math. Soc.* published online first.

[9] HEYDECKER, D. (2023). Large deviations of Kac's conservative particle system and energy nonconserving solutions to the Boltzmann equation: A counterexample to the predicted rate function. *Ann. Appl. Probab.* **33** 1758–1826. MR4583658 https://doi.org/10.1214/22-aap1852

[10] JENSEN, L. H. (2000). *Large Deviations of the Asymmetric Simple Exclusion Process in One Dimension*. ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)—New York University. MR2700635

[11] KIM, S. S. and RAMANAN, K. (2018). A conditional limit theorem for high-dimensional ℓ^p-spheres. *J. Appl. Probab.* **55** 1060–1077. MR3899928 https://doi.org/10.1017/jpr.2018.71

[12] KIPNIS, C. and LANDIM, C. (1999). *Scaling Limits of Interacting Particle Systems*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **320**. Springer, Berlin. MR1707314 https://doi.org/10.1007/978-3-662-03752-2

[13] LÉONARD, C. (1995). On large deviations for particle systems associated with spatially homogeneous Boltzmann type equations. *Probab. Theory Related Fields* **101** 1–44. MR1314173 https://doi.org/10.1007/BF01192194

[14] LU, X. (1999). Conservation of energy, entropy identity, and local stability for the spatially homogeneous Boltzmann equation. *J. Stat. Phys.* **96** 765–796. MR1716814 https://doi.org/10.1023/A:1004606525200

[15] LU, X. and WENNBERG, B. (2002). Solutions with increasing energy for the spatially homogeneous Boltzmann equation. *Nonlinear Anal. Real World Appl.* **3** 243–258. MR1893976 https://doi.org/10.1016/S1468-1218(01)00026-8

[16] MARIANI, M. (2018). A Γ-convergence approach to large deviations. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **18** 951–976. MR3807592

[17] MISCHLER, S. and WENNBERG, B. (1999). On the spatially homogeneous Boltzmann equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **16** 467–501. MR1697562 https://doi.org/10.1016/S0294-1449(99)80025-0

[18] NAM, K. (2020). Large deviations and localization of the microcanonical ensembles given by multiple constraints. *Ann. Probab.* **48** 2525–2564. MR4152650 https://doi.org/10.1214/20-AOP1430

[19] PETROV, V. V. (1975). *Sums of Independent Random Variables*. *Ergebnisse der Mathematik und Ihrer Grenzgebiete* [*Results in Mathematics and Related Areas*], *Band* 82. Springer, New York. MR0388499

[20] REZAKHANLOU, F. (1998). Large deviations from a kinetic limit. *Ann. Probab.* **26** 1259–1340. MR1640346 https://doi.org/10.1214/aop/1022855753

[21] WENNBERG, B. (1997). Entropy dissipation and moment production for the Boltzmann equation. *J. Stat. Phys.* **86** 1053–1066. MR1450762 https://doi.org/10.1007/BF02183613