Stochastic equations and particle systems, La Sapienza, Roma

Renormalisation and sharp asymptotics for metastable transition times in stochastic Allen–Cahn equations

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Joint works with Barbara Gentz (Bielefeld), Giacomo Di Gesù (Rome), Hendrik Weber (Münster), Tom Klose (Warwick)




Allen–Cahn and Φ^4_d equations

- Allen–Cahn equation [Chafee & Infante 74, Allen & Cahn 75]

$$\partial_t \phi(t,x) = \Delta \phi(t,x) + \phi(t,x) - \phi(t,x)^3 + \sqrt{2\varepsilon}\xi$$

- Dynamic Φ^4_d model/stochastic quantization eq. [Parisi & Wu 81]

$$\partial_t \phi(t,x) = \Delta \phi(t,x) - \phi(t,x)^3 + \sqrt{2\varepsilon}\xi$$

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- Same solution theory, different dynamics:
  Allen–Cahn is metastable, Φ^4_d is not

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Deterministic Allen–Cahn PDE in $d = 1$

[Chafee & Infante 74, Allen & Cahn 75]

$$\partial_t \phi(t,x) = \Delta\phi(t,x) + f(\phi(t,x))$$

- $x \in [0, L]$, L: bifurcation parameter
- $\phi(t,x) \in \mathbb{R}$
- Either periodic or zero-flux Neumann boundary conditions
- In this talk: $f(\phi) = \phi - \phi^3$ (results more general)

Energy function:

$$V[\phi] = \int_0^L \left[\frac{1}{2}\phi'(x)^2 - \frac{1}{2}\phi(x)^2 + \frac{1}{4}\phi(x)^4\right] \mathrm{d}x \qquad \to \quad \min$$

Stationary solutions: $\phi_0''(x) = -\phi_0(x) + \phi_0(x)^3$ critical points of V

Stability: Sturm–Liouville problem $\partial_t \psi_t(x) = \psi_t''(x) + [1 - 3\phi_0(x)^2]\psi_t(x)$

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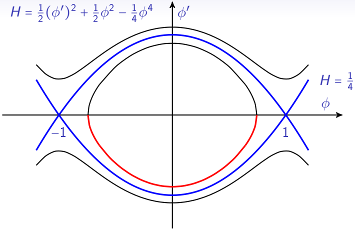
- $\phi_\pm(x) \equiv \pm 1$: stable
- $\phi_0(x) \equiv 0$: unstable
- Nonconstant solutions satisfying b.c.
  (expressible in terms of Jacobi elliptic fcts)



- Neumann b.c: $2k$ nonconstant solutions when $L > k\pi$

Number of positive eigenvalues
(= unstable directions)
Transition state

- Periodic b.c: k families when $L > 2k\pi$

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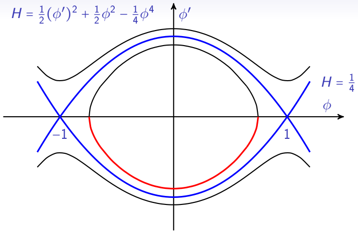

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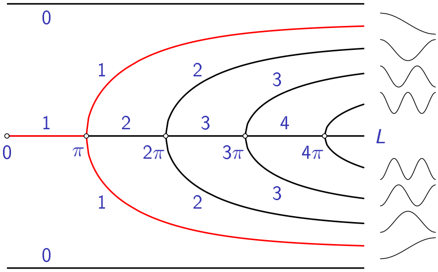

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Eyring–Kramers law for 1D SPDEs: heuristics

$$\partial_t \phi(t,x) = \Delta\phi(t,x) + f(\phi(t,x)) \quad + \sqrt{2\varepsilon}\,\xi(t,x) \qquad (f(\phi) = \phi - \phi^3)$$

Initial condition: ϕ_{in} near $\phi_- \equiv -1$ with eigenvalues $\nu_k = (\frac{\beta k\pi}{L})^2 + 2$

Target: $\phi_+ \equiv 1$, $\tau_+ = \inf\{t > 0 \colon \|\phi(t,\cdot) - \phi_+\|_{L^\infty} < \rho\}$

Transition state: ($\beta = 1$ for Neumann b.c., $\beta = 2$ for periodic b.c.)

$$\phi_{\text{ts}}(x) = \begin{cases} \phi_0(x) \equiv 0 & \text{if } L \leqslant \beta\pi \quad \text{with ev } \lambda_k = (\frac{\beta k\pi}{L})^2 - 1 \\ \phi_1(x)\ \beta\text{-kink stationary sol.} & \text{if } L > \beta\pi \quad \text{with ev } \lambda'_k \end{cases}$$

[Faris & Jona-Lasinio 82]: large-deviation principle

$\Rightarrow$ Arrhenius law: $\mathbb{E}^{\phi_{\text{in}}}[\tau_+] \simeq e^{(V[\phi_{\text{ts}}] - V[\phi_-])/\varepsilon}$

[Maier & Stein 01]: formal computation; for Neumann b.c. and $L < \pi$

$\Rightarrow \mathbb{E}^{\phi_{\text{in}}}[\tau_+] \simeq 2\pi\sqrt{\frac{1}{|\lambda_0|\nu_0}\prod_{k=1}^{\infty}\frac{\lambda_k}{\nu_k}}\, e^{(V[\phi_{\text{ts}}] - V[\phi_-])/\varepsilon}$

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Eyring–Kramers law for 1D SPDEs: main result

Theorem: Neumann b.c. [B & Gentz, 2013]

- If $L < \pi - c$ with $c > 0$, then

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- If $L > \pi + c$, then same formula with extra factor $\frac{1}{2}$ (since 2 saddles) and λ'_k instead of λ_k

- Results also for L near π and periodic b.c.
- Prefactor involves a Fredholm determinant: $\Delta_\perp$ Laplacian acting on mean zero functions

$$\prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k} = \det\big[(-\Delta_\perp - 1)(-\Delta_\perp + 2)^{-1}\big] = \det\big[\mathbb{1} - 3(-\Delta_\perp + 2)^{-1}\big]$$

converges because $\log\det = \operatorname{Tr}\log$ and $(-\Delta_\perp + 2)^{-1}$ is trace class (limit $= \frac{\sqrt{2}\sin(L)}{\sinh(\sqrt{2}L)}$)

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Ideas of proof

- ▷ Spectral Galerkin approximation: $\phi(t,x) = \sum_{|k| \leqslant N} z_k(t) e_k(x)$ (Fourier)

- ▷ Potential-theoretic approach:

$$\mathbb{E}^{\nu_{AB}}[\tau_B] := \int_{\partial A} \mathbb{E}^z[\tau_B] \nu_{AB}(\mathrm{d}z) = \frac{1}{\mathrm{cap}(A,B)} \int_{B^c} \mathrm{e}^{-V(z)/\varepsilon} h_{AB}(x) \, \mathrm{d}z$$

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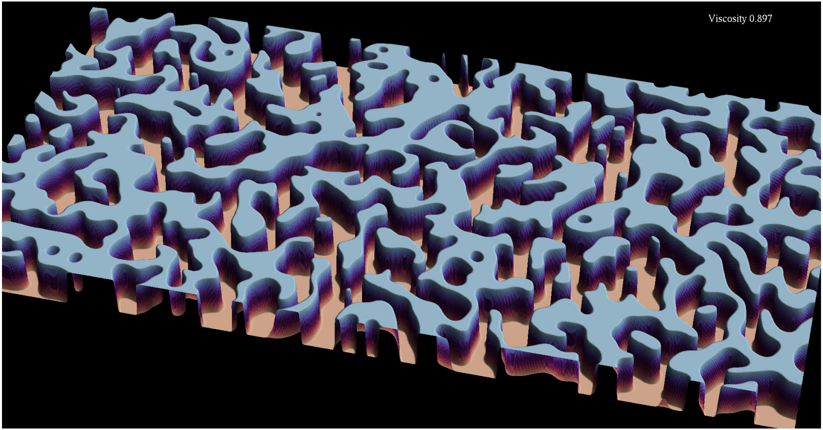
(Stochastic) Allen–Cahn equation on $\mathbb{T}^2$

$$d\phi(t,x) = \left[\nu(\varepsilon t)\Delta\phi(t,x) + \phi(t,x) - \phi(t,x)^3\right] dt + \sigma \, dW(t,x)$$



(Online: https://youtu.be/yX0EAxZHNCQ)

The two-dimensional case

- Large-deviation principle: [Hairer & Weber, 2015]
- Naive computation of prefactor fails:

$$\log \prod_{k\in(\mathbb{N}^2)^*} \frac{1-\bigl(\frac{L}{|k|\pi}\bigr)^2}{1+2\bigl(\frac{L}{|k|\pi}\bigr)^2} \simeq \sum_{k\in(\mathbb{N}^2)^*} \log\biggl(1-\frac{3L^2}{|k|^2\pi^2}\biggr)$$

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- In fact, the equation needs to be renormalised

Theorem: [Da Prato & Debussche 2003]

Let ξ^δ be a mollification on scale δ of white noise. Then

$$\partial_t\phi = \Delta\phi + \left[1 + 3\varepsilon C(\delta)\right]\phi - \phi^3 + \sqrt{2\varepsilon}\xi^\delta$$

with $C(\delta) \simeq \log(\delta^{-1})$ admits local solution converging as $\delta \to 0$
(Global version: [Mourrat & Weber 2015])
[Mourrat & Weber 2014]: Renormalised eq = scaling limit of Ising–Kac model

Renormalisation

Problem: Stoch. convolution $w_t(x) = \int_0^t e^{(t-s)\Delta} \xi(s,x)\, ds$ is distribution

- δ-mollification should be equivalent to Galerkin approx. $|k| \leqslant N = \delta^{-1}$:

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$$\mu_k = (\Omega|k|)^2 \qquad \Omega = \beta\pi/L$$

- $\lim_{t\to\infty} \int_0^t e^{(\Delta-1)(t-s)} \xi_N(s,x)\, ds = \phi_N$ is a Gaussian free field, s.t.

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Computation of the prefactor

- Consider for simplicity $L < \beta\pi \Rightarrow$ transition state in 0
- Galerkin-truncated renormalised potential

$$V_N = \frac{1}{2}\int_{\mathbb{T}^2}\left[\|\nabla\phi_N(x)\|^2 - \phi_N(x)^2\right]\mathrm{d}x + \frac{1}{4}\int_{\mathbb{T}^2} :\phi_N(x)^4:\mathrm{d}x$$

- Using Nelson estimate: $\operatorname{cap}(A,B) \simeq \sqrt{\frac{|\lambda_0|\varepsilon}{2\pi}} \prod_{0<|k|\leqslant N} \sqrt{\frac{2\pi\varepsilon}{\lambda_k}}$
- Symmetry argument:

$$\int_{B^c} h_{A,B}(z)\,\mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z = \frac{1}{2}\int \mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z = \frac{1}{2}\mathcal{Z}_N(\varepsilon)$$

- $\mathcal{Z}_N(\varepsilon) \simeq 2\prod_{|k|\leqslant N}\sqrt{\frac{2\pi\varepsilon}{\nu_k}}\,\mathrm{e}^{-V_N(L,0)/\varepsilon}$ where $-V_N(L,0) = \frac{1}{4}L^2 + \frac{3}{2}L^2 C_N\varepsilon$
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Main result in dimension 2

Theorem: [B, Di Gesù, Weber, 2017]

For $L < 2\pi$, appropriate $A \ni \phi_-$, $B \ni \phi_+$, $\exists \mu_N$ probability measures on ∂A:

$$\limsup_{N\to\infty} \mathbb{E}^{\mu_N}\big[\tau_B\big] \leqslant \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k}\, e^{\frac{\nu_k-\lambda_k}{|\lambda_k|}}}\; e^{(V[\phi_{ts}]-V[\phi_-])/\varepsilon}\big[1 + c_+\sqrt{\varepsilon}\,\big]$$

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with $T = 3(-\Delta_\perp - 1)^{-1}$

$\det_2$ defined whenever T is only Hilbert–Schmidt (true for $d \leqslant 3$)

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Allen–Cahn equation in dimension 3

- Equation needs two counterterms:

$$\partial_t \phi = \Delta\phi + \Big[1 + 3\varepsilon C_1(\delta) - 9\varepsilon^2 C_2(\delta)\Big]\phi - \phi^3 + \sqrt{2\varepsilon}\xi^\delta$$

  with $C_1(\delta) \sim \delta^{-1}$, $C_2(\delta) \sim \log(\delta^{-1})$
- Solution theories:
  - Regularity structures [Hairer 2014]
  - Paracontrolled distributions [Gubinelli–Imkeller-Perkowski 2015]
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Some literature on the static Φ^4_3 model

- Glimm & Jaffe (1968, 1973), Feldman (1974):
  Combinatorics of Feynman diagrams
- Benfatto, Cassandro, Gallavotti, Nicolò & Olivieri (1978, 1980):
  Renormalisation group (integrating out scales)
- Brydges, Fröhlich & Sokal (1983):
  Generating function and skeleton inequalities
- Brydges, Dimock & Hurd (1995):
  Polymer expansions
- Connes & Kreimer (2000, 2001):
  Hopf algebras
- ...
- Barashkov & Gubinelli (2020):
  Boué–Dupuis formula

Perturbative computation of partition function

- Wick powers: $X =$  $= \int_\Lambda :\phi(x)^4: \mathrm{d}x$, $Y =$  $= \int_\Lambda :\phi(x)^2: \mathrm{d}x$
- Parameters: $\alpha = \frac{g}{4}$, $\beta = \frac{1}{2} g^2 C_N^{(2)}$, $\gamma = g^2 C_N^{(3)} - g^3 C_N^{(4)}$

  Then $\dfrac{\mathcal{Z}_{N,g}}{\mathcal{Z}_{N,0}} = \mathbb{E}^{\nu_0}\left[\mathrm{e}^{-\alpha X - \beta Y - \gamma}\right] = \mathrm{e}^{-\gamma}\, \mathbb{E}^{\nu_0}\left[\mathrm{e}^{-\alpha X - \beta Y}\right]$
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$$\Pi_N(\Gamma) = \int_{\Lambda^{\mathscr{V}}} \prod_{e \in \mathscr{E}} G_N(x_{e_+} - x_{e_-})\, \mathrm{d}x$$

  For instance

$$C_N^{(1)} = \Pi_N$$

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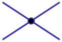
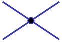
  $\mu_n = (-1)^n \mathbb{E}^{\nu_0}\Big[\Big(\alpha$  $+\beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\nu_0}\Big[$  ^{m}  $^{n-m}\Big]$

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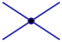
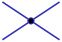
  $\mu_n = (-1)^n \mathbb{E}^{\nu_0}\Big[\Big(\alpha$  $+\beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\nu_0}\Big[$  ^{m}  $^{n-m}\Big]$

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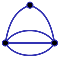
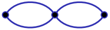
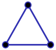
  $\mu_3 = -\alpha^3 \binom{4}{2}^3 2^3 \Pi_N$  $-3\alpha^2\beta(4^2\cdot 2\cdot 3!)\Pi_N$ 

  $-3\alpha\beta^2 4!\,\Pi_N$  $-8\beta^3\Pi_N$ 

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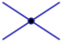
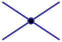
  $\mu_n = (-1)^n \mathbb{E}^{\nu_0}\Big[\Big(\alpha$  $+ \beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\nu_0}\Big[$  ^{m}  $^{n-m}\Big]$

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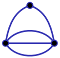
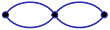
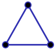
  $\mu_3 = -\alpha^3 \binom{4}{2}^3 2^3 \Pi_N$  $- 3\alpha^2\beta(4^2 \cdot 2 \cdot 3!)\Pi_N$ 

  $- 3\alpha\beta^2 4!\, \Pi_N$  $- 8\beta^3 \Pi_N$ 

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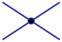
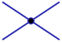
  $\mu_n = (-1)^n \mathbb{E}^{\nu_0}\Big[\Big(\alpha$  $+\beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\nu_0}\Big[$  ^{m}  $^{n-m}\Big]$

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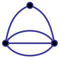
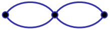
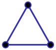
  $\mu_3 = -\alpha^3 \binom{4}{2}^3 2^3 \Pi_N$  $- 3\alpha^2\beta(4^2 \cdot 2 \cdot 3!)\Pi_N$ 

  $- 3\alpha\beta^2 4!\, \Pi_N$  $- 8\beta^3 \Pi_N$ 

- Cumulant expansion: (Leonov & Shiraev)

  $$-\log \mathbb{E}[e^{-\alpha X - \beta Y - \gamma}] = \gamma - \sum_{n=2}^{\infty} \frac{\kappa_n}{n!} \qquad \kappa_n = \mu_n - \sum_{m=2}^{n-2} \binom{n-1}{m} \kappa_m \mu_{n-m}$$

- Linked Cluster Theorem: κ_n projection of μ_n on connected graphs

  Proof: for instance Peccati & Taqqu (2011)

Commutative diagram

$$\begin{array}{ccccc}
e^{-\alpha X} & \xmapsto{\mathscr{P}} & \sum_p \frac{(-\alpha)^p}{p!} \mathscr{P}(X^p) & & \\
\downarrow W & & \downarrow M^{g^{\mathrm{BPHZ}}} & \searrow \Pi_N^{\mathrm{BPHZ}} & \\
e^{-\alpha X - \beta Y} & \xmapsto{\mathscr{P}} & \sum_{n,m} \frac{(-\alpha)^m (-\beta)^{n-m}}{m!(n-m)!} \mathscr{P}(X^m Y^{n-m}) & \xmapsto{\Pi_N} & \log \mathbb{E}[e^{-\alpha X - \beta Y}]
\end{array}$$

- $\mathscr{P} = \Pi_{\text{connected}}\left(\sum_{\text{pairings}}\right)$
- $e^{-\alpha X}, e^{-\alpha X - \beta Y} \in H = \operatorname{span}\{X^n Y^m\}$
- Construction of W inspired by [Ebrahimi-Fard, Patras, Tapia & Zambotti]

Lemma: [B, Klose & Tapia]

- $W(e^{-\alpha X}) = e^{-\alpha X - \beta Y}$
- $\mathscr{P} \circ W = M^{g^{\mathrm{BPHZ}}} \circ \mathscr{P}$

Proof of commutativity based on Zimmermann's forest formula for $\mathscr{A}$

Commutative diagram

$$
\begin{array}{ccccc}
e^{-\alpha X} & \overset{\mathscr{P}}{\longmapsto} & \displaystyle\sum_p \frac{(-\alpha)^p}{p!}\mathscr{P}(X^p) & & \\
W \big\downarrow & & \big\downarrow M^{g^{\text{BPHZ}}} & \overset{\Pi_N^{\text{BPHZ}}}{\searrow} & \\
e^{-\alpha X-\beta Y} & \overset{\mathscr{P}}{\longmapsto} & \displaystyle\sum_{n,m} \frac{(-\alpha)^m(-\beta)^{n-m}}{m!(n-m)!}\mathscr{P}(X^mY^{n-m}) & \overset{\Pi_N}{\longmapsto} & \log\mathbb{E}[e^{-\alpha X-\beta Y}]
\end{array}
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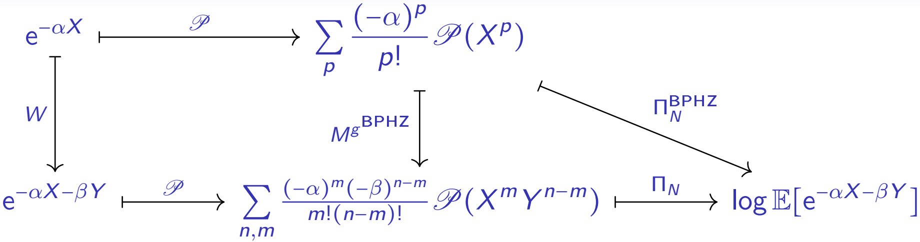

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Proof of commutativity based on Zimmermann's forest formula for $\mathscr{A}$

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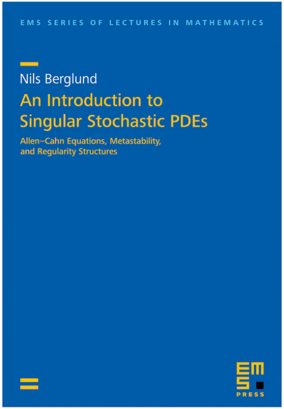

Thanks for your attention!

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