

SOLVABLE STATIONARY NON ① EQUILIBRIUM STATES

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OVERVIEW

2

- Interacting stochastic particle systems
- Out of equilibrium by interaction with external sources
- Stationary non equilibrium states (SNS)
- Effective models for non equilibrium physics
- Sources of interesting mathematical problems

THE FRAMEWORK

3

- One dimensional lattices $\Lambda_N = \{1, 2, \dots, N\}$
- Configurations $\eta = (\eta_x)_{x=1}^N$; $\eta_x \in \mathbb{N}, \mathbb{R}^+$;
 $\eta_x = \text{number of particles or energy at } x \in \Lambda_N$
- Markovian evolution; interaction with external sources at $\{0\}$ and $\{N+1\}$
- Non reversible stochastic processes;
invariant measure (SNS) difficult to compute;
long range correlations

THE KIPNIS-MARCHIORO-PRESUTTI (KMP) MODEL

④

- Stochastic model for the evolution of the energies of a chain of harmonic oscillators

η_x = energy of the oscillator at $x \in \Lambda_N$

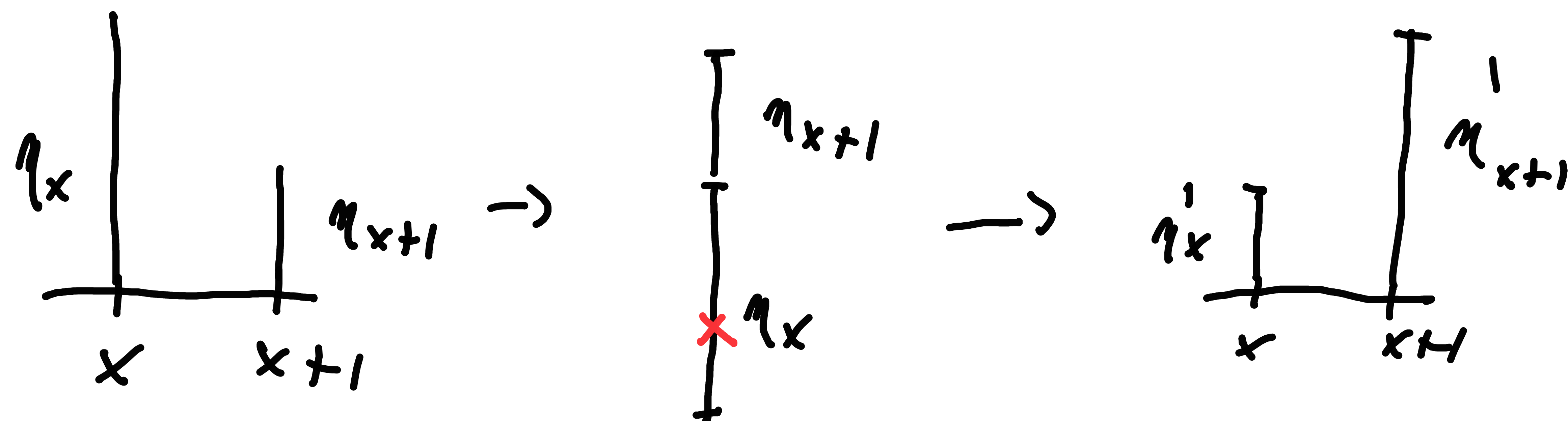
BULK DYNAMICS:

At time 1 $(\eta_x, \eta_{x+1}) \rightarrow (\eta'_x, \eta'_{x+1})$

$$\begin{cases} \eta'_x = U(\eta_x + \eta_{x+1}) \\ \eta'_{x+1} = (1-U)(\eta_x + \eta_{x+1}) \end{cases}$$

$$U \in [0, 1]$$

uniform



Conservative
dynamics

KMP: BOUNDARY DYNAMICS

5

- The left boundary is in contact with an external source at Temperature T_-
- At the one we have

$$\eta_1 \rightarrow \eta'_1 = U(\bar{E}_0 + \eta_1)$$

U uniform in $[0,1]$

E_0 exponential of parameter $\frac{1}{T_-}$

$$\boxed{\frac{1}{T_-} e^{-\frac{\eta}{T_-}} d\eta} \quad (\text{independent})$$

- At The right boundary The same mechanism but the external source has Temperature T_+
- When $T_- = T_+$, reversible, product invariant measure $\boxed{\frac{1}{x} \frac{1}{T} e^{-\eta_x/T} d\eta_x}$; when $T_- \neq T_+$?

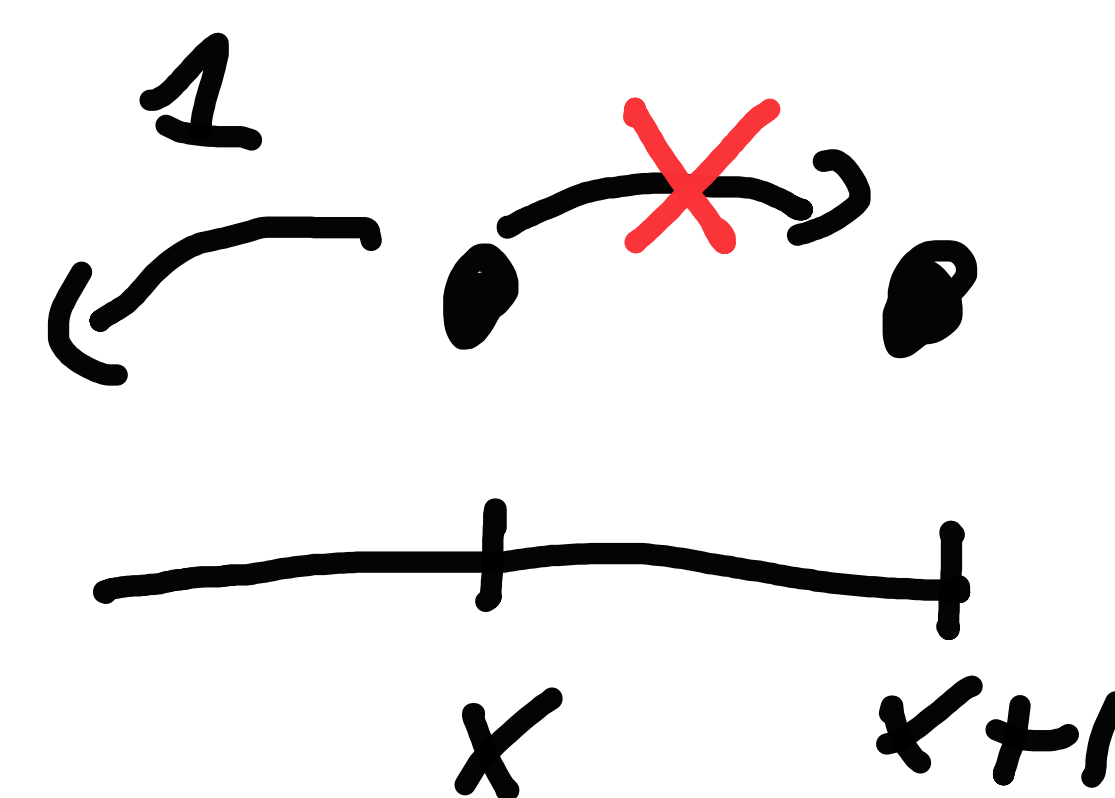
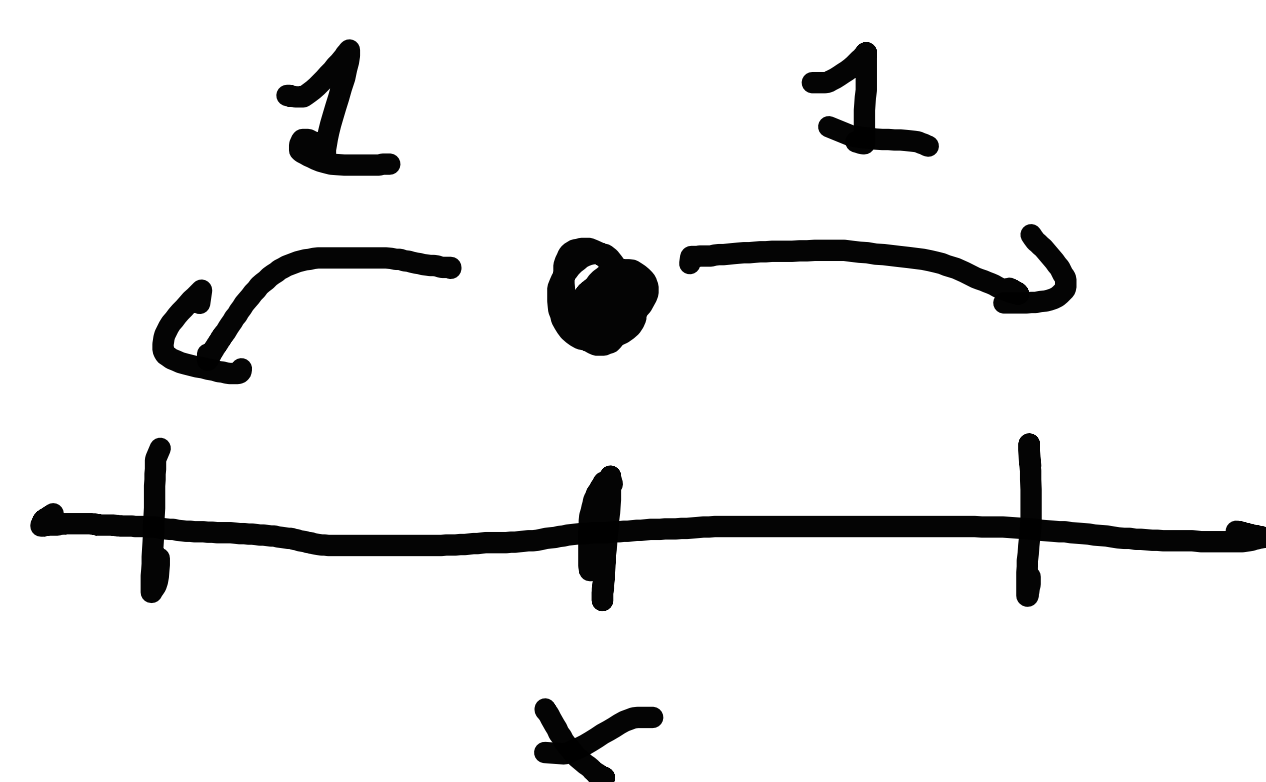
EXCLUSION

PROCESS

6

- $\eta_x = \begin{cases} 1 & \text{there is one particle at } x \\ 0 & \text{there are no particles at } x \end{cases}$

- In the bulk



- At the left boundary $\eta_1 \rightarrow 1 - \eta_1$ at rate

$$(1 - \alpha)\eta_1 + \alpha(1 - \eta_1)$$

- At the right boundary The same but with $\alpha \rightarrow \beta$

- If $\alpha = \beta$, reversible, product invariant measure

$$\prod_x B_\alpha(\eta_x)$$

- If $\alpha \neq \beta$ Combinatorial representations (Matrix product ansatz and many other)

"SOLVABLE" MODELS

7

- KMP and SEP are in a class of "solvable" SWS
- Simple representations of the large deviations rate functional for the invariant measure
- Obtained by a direct computation from the microscopic combinatorial representations or by a dynamic continuous variational approach.

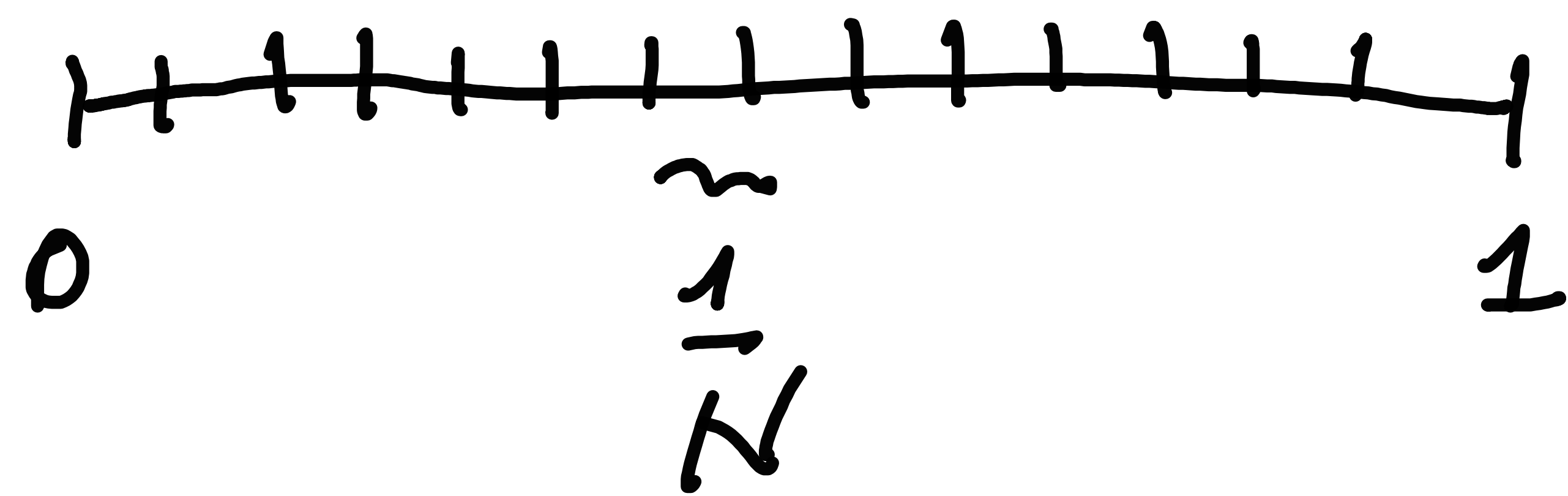
LARGE DEVIATIONS

8

- Configurations η distributed according to the invariant measure

$$\bar{\pi}_N(\eta) = \frac{1}{N} \sum_x \eta_x \delta_{x/N} \in \mathcal{M}^+(\bar{[0,1]})$$

Empirical measure



Rescaling of Λ_N

$$\mathbb{P}(\bar{\pi}_N(\eta) \simeq \rho(x) dx) \sim e^{-N I(\rho)}$$

I is rate functional

RATE FUNCTIONAL

9

- In The equilibrium cases ($T_- = T_+$, $\alpha = \beta$)

$$\boxed{I(p) = \int_0^1 f(p(x)) dx} \quad \text{local functional}$$

- In The non equilibrium cases ($T_- \neq T_+$, $\alpha \neq \beta$)

$$\text{SEP } I(p) = \sup_{\bar{F}} \int_0^1 \left(p(x) \log \frac{p(x)}{\bar{F}(x)} + (1-p(x)) \log \frac{1-p(x)}{1-\bar{F}(x)} + \log \frac{\bar{F}'(x)}{\Delta} \right) dx$$

$$\text{KMP } I(p) = \inf_{\bar{F}} \int_0^1 \left(\frac{p(x)}{\bar{F}(x)} - 1 - \log \frac{p(x)}{\bar{F}(x)} - \log \frac{\bar{F}'(x)}{\Delta} \right) dx$$

Inf and sup are over piecewise C^1 functions $\bar{F}$ that are monotone and satisfy boundary conditions $\bar{F}(0) = \bar{F}_- < \bar{F}_+ = \bar{F}(1)$.

A CONJECTURE

10

- Macroscopically $I(p) = \sup_{inf} G(p, F)$; adding 1 functional variable (The function F) The functional is simpler.
- Is This True also microscopically?
- Can The inf in The KMP case be interpreted as a contraction principle?
- Yes (Bellini, De Sole, G., Ione-Losinio, Gentile 2007)
→ a Toy model for The invariant measure.

ORDER STATISTICS

(17)

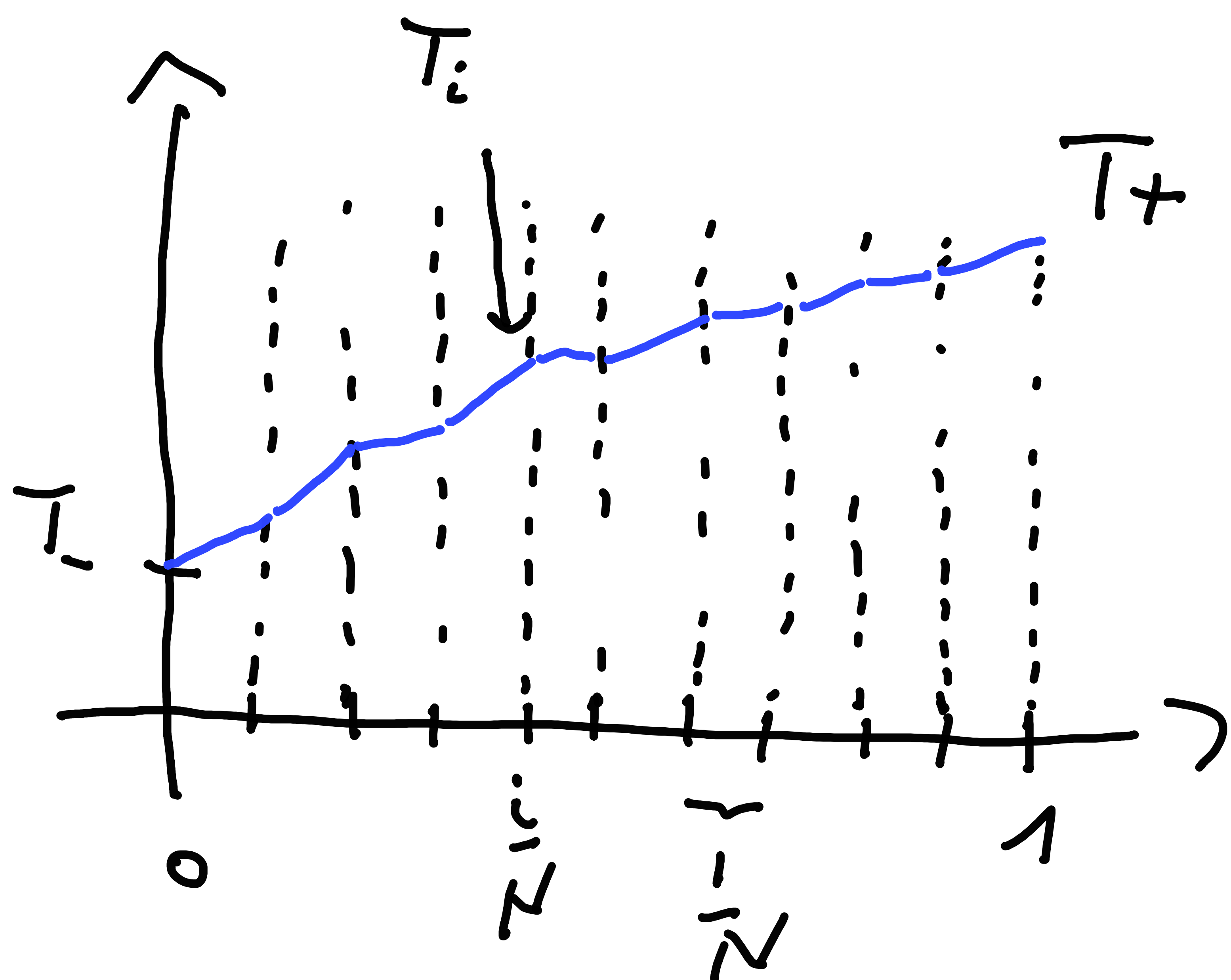
$$\bar{I}(P) = \inf_{\bar{T}} \int_0^1 \left[\underbrace{\left(\frac{P(x)}{F(x)} - 1 - \log \frac{P(x)}{F(x)} \right)}_{\text{I}} - \log \frac{F'(x)}{T_+ - T_-} \right] dx$$

LD rate for a product of inhomogeneous exponentials

II LD rate for the order statistics of uniform in $[T_-, T_+]$

$$\text{Let } T_- \leq T_1 \leq T_2 \leq \dots \leq T_N \leq T_+$$

order statistics of N uniform in $[T_-, T_+]$



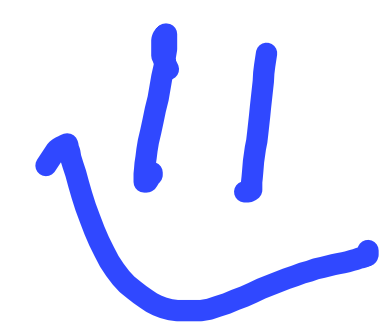
← F_N random interface has rate functional II

A TOY MEASURE

12

Consider $n_1, \dots, n_N$ random variables such that conditioned to $T_1, \dots, T_N$ they are independent exponentially distributed n_i with parameter $\frac{1}{T_i}$

$$\mathcal{V}_N^{\text{Toy}}(n, \bar{T}) = \frac{N!}{(T_+ - T_-)^N} \mathbb{1}(T_- \leq T_1 \leq T_2 \leq \dots \leq T_N \leq T_+) \prod_{i=1}^N \frac{1}{T_i} e^{-\frac{n_i}{T_i}} dn_i dT_i$$



$$\mu_N^{\text{Toy}}(n) = \int \mathcal{V}_N^{\text{Toy}}(n, d\bar{T})$$

it is not The invariant measure of KMP

$$N=1 \quad \mu_1(n_1) dn_1 = \int_{T_-}^{T_+} \left(\frac{dT_1}{\pi V(T_+ - T_-)(T_+ - T_1)} \frac{1}{T_1} e^{-n_1/T_1} \right) dn_1$$

BDJL-2007

Arcsine and not uniform

A COUPLING

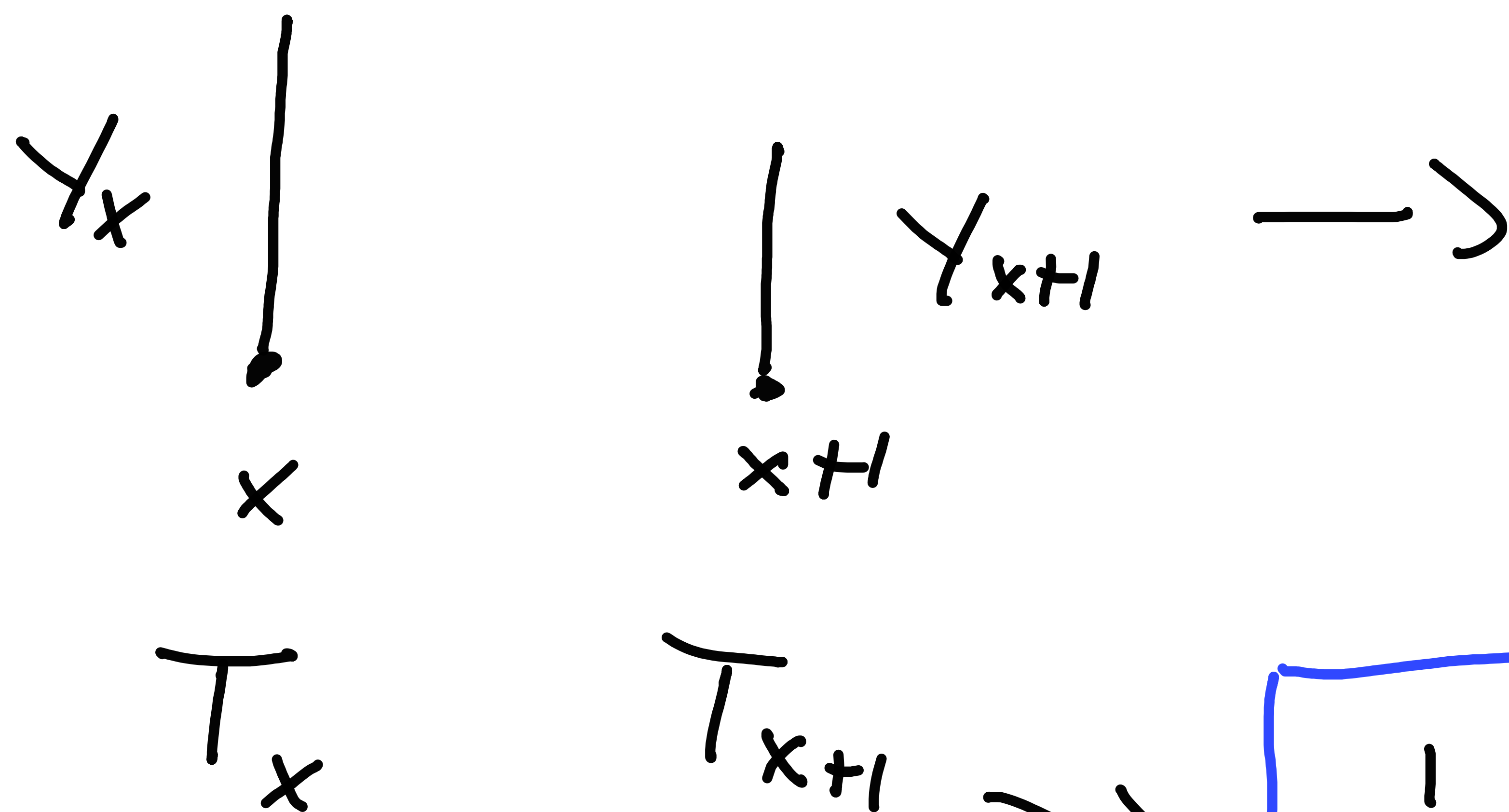
13

- We show that there exists a natural enlargement of the state space from energy configurations η to energy-Temperature configurations (η, T) .
- It works on any graph, not only on the 1-d lattice
- Basic idea: construct a coupling between $\gamma_1, \dots, \gamma_N$ The energy variables of an equilibrium KMP model (in contact with all external sources at the same temperature $\equiv 1$) and new variables $T_1, \dots, T_N$

THE GRAPHICAL CONSTRUCTION

14

In The bulk at wire 1



$$\gamma'_x = U(\gamma_x + \gamma_{x+1})$$

$$\gamma'_{x+1} = (1-U)(\gamma_x + \gamma_{x+1})$$

$$T'_x = T'_{x+1} = \frac{\gamma_x}{\gamma_x + \gamma_{x+1}} T_x + \frac{\gamma_{x+1}}{\gamma_x + \gamma_{x+1}} T_{x+1}$$

A similar mechanism at the boundary

$$\eta_x = \gamma_x T_x$$

is a non equilibrium boundary

when $K \neq 1$

SURPRISING FEATURES

15

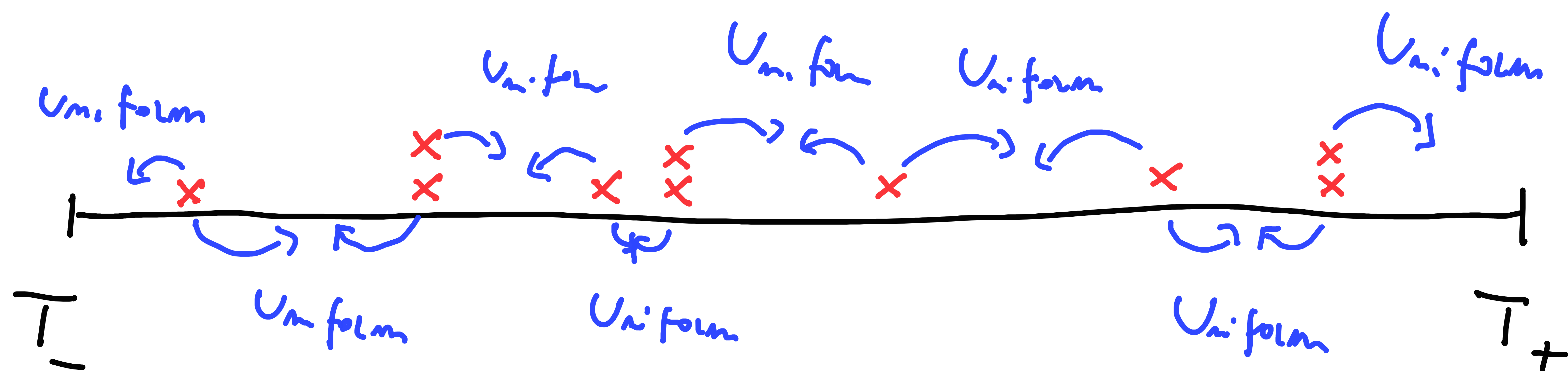
FACT 1: When $Y(0)$ is at equilibrium
(i.e. distributed) as a product of i.i.d. exponentials)
and $T(0)$ is independent from $Y(0)$, then
 $(Y(t), T(t))$ are independent $\forall t$.

FACT 2: In addition, under the above
assumptions, $T(t)$ is Markovian
and evolves as an auxiliary
Markov opinion process defined in the
following.

EFFECTIVE TEMPERATURE/OPINION MODEL

16

- Configurations $T_- \leq T_1 \leq T_2 \dots \leq T_N \leq T_+$
- There may be pairs (and only pairs) of coinciding Temperatures



THE INVARIANT MEASURE

17

- The invariant measure of $(Y(x), T(x))$

is therefore a product measure $\mu^Y(x) \mu^{\text{Eff}}(T)$

μ_N^Y = product measure with exponential of parameter 1 marginals

μ_N^{Eff} = invariant measure of the effective Temperature model

- Since $\eta_x = Y_x T_x$ Then The invariant measure of the boundary given KMP

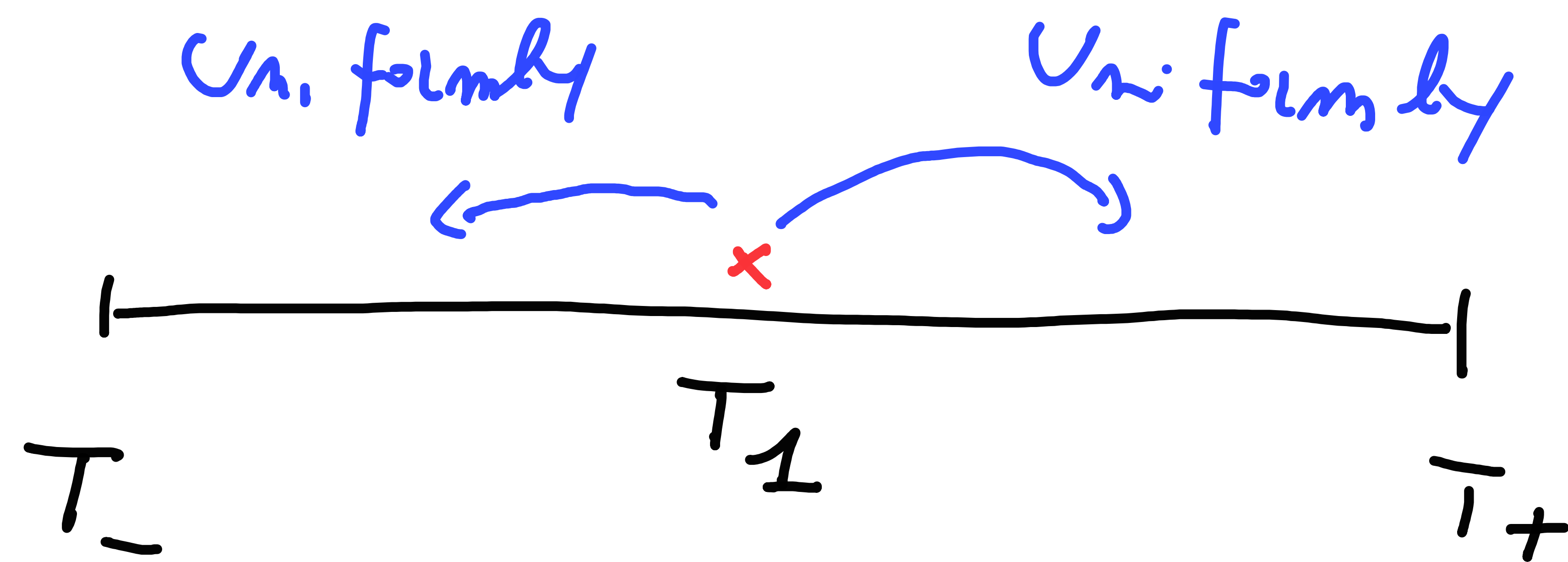
$$\mu(\eta) d\eta = \int \left(\mu_N^{\text{Eff}}(dT) \prod_x \frac{1}{T_x} e^{-\frac{\eta_x}{T_x}} \right) d\eta$$

Mixture of product of inhomogeneous exponential distributions

THE SINGLE OSCILLATOR CASE

18

The effective Temperature model with $N=1$ oscillators is



$\rightarrow$ invariant measure $\mu_1^{\text{Ett}}(T_1) d\bar{T}_1 = \frac{d\bar{T}_1}{\sqrt{(T_1 - T_-)(T_+ - T_1)}}$

A TRIPLE OF PROCESSES

19

- We have a Triple (γ, T, η) of processes
- Two of them determine the Third one
- (γ, η) is a coupling between an equilibrium and a non equilibrium KMP (evolving by the same graph col constitution)
- (η, T) is a coupled evolution of energies and Temperatures of the oscillators

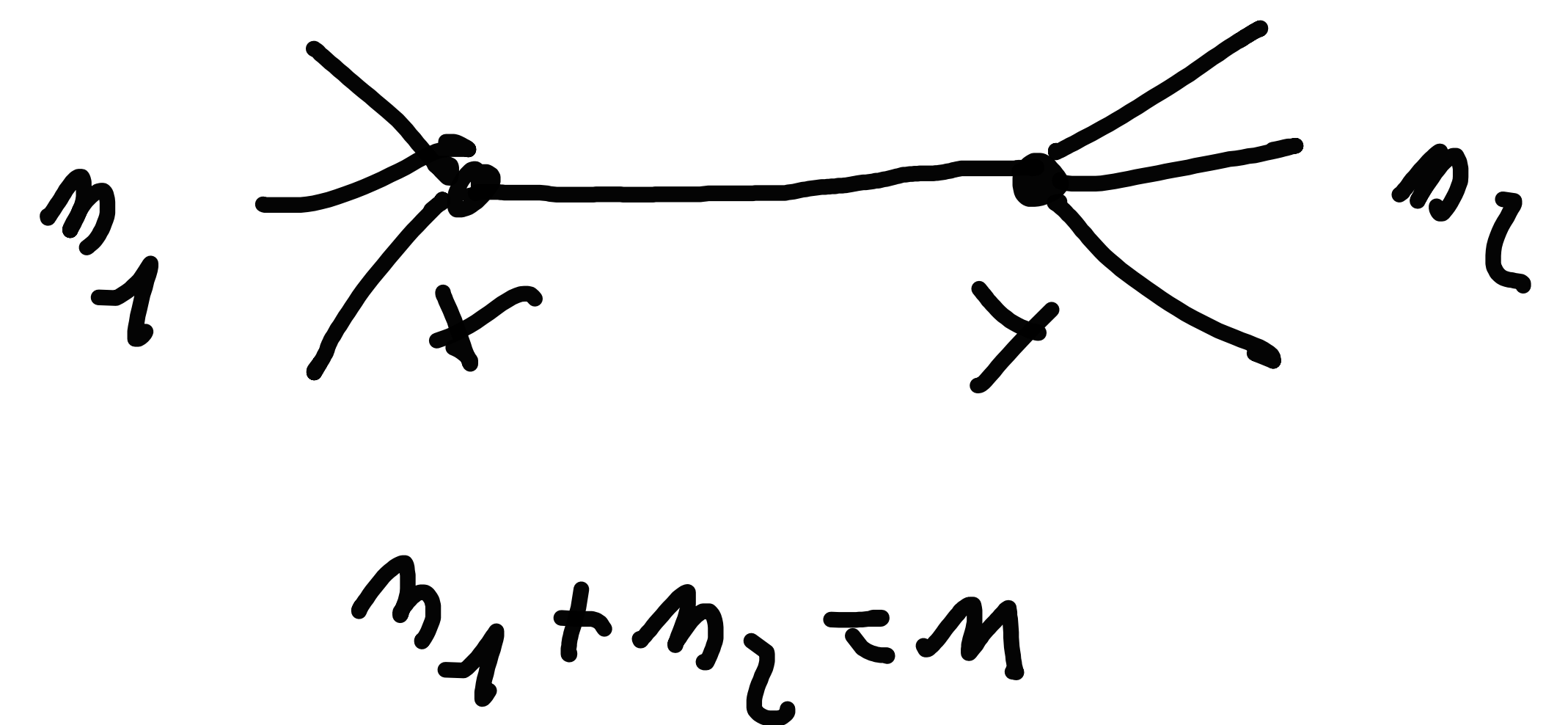
$(\eta_x, \eta_{x+1}) \rightarrow$ interaction $\rightarrow$ exchange of energy (η'_x, η'_{x+1})

$(T_x, T_{x+1}) \rightarrow$ Thermalization $\rightarrow T'_x = T'_{x+1} = \frac{\eta_x/T_x}{\frac{\eta_x}{T_x} + \frac{\eta_{x+1}}{T_{x+1}}}$

DISAGREEMENT PROCESS

20

- The invariant measure $\mu_N^{\text{Eff}}(\tau)$ is not known; partial informations.
- The disagreement process $D_{xy}(t) = \mathbb{1}(T_x(t) \neq T_y(t))$
- (xy) edge of the graph:
- τ is Markov when restricted to only $E' \subseteq E$
- If $E' \subset (xy)$ $1 \rightarrow 0$ rate 1 $0 \rightarrow 1$ rate n
 $\Rightarrow \mu_N^{\text{Eff}}(D_{xy}=1) = \frac{n}{n+1}$
- $\{(x,y) : D_{xy}(t)=0\}$ is an independent subset of E
- Perfect simulation algorithm.



EXCLUSION PROCESSES

21

- Matrix product ansatz
- Invariant measures of boundary driven exclusion processes may be written in terms of product of matrices
- $\eta \in \{0, 1\}^{\mathbb{N}}$ M^0, M^1 $A \times A$ matrices ($A = \mathbb{N}$)
 $v, w \in \mathbb{R}_+^A$ vectors; M^0, M^1 satisfy some commutation relations; we assume $M_{i,i}^e > 0$.

$$\mu_N(\eta) = \frac{1}{Z_N} v \cdot \prod_{x=1}^N M^{\eta(x)} w$$

ENLARGING THE STATE SPACE

(27)

- Introduce new variables $\{$

$$\hat{\mu}_N(n, \{z\}) = \frac{1}{Z_N} \prod_{z_1}^{\eta(1)} \prod_{z_1, z_2}^{\eta(1)} \dots \prod_{z_N, z_{N+1}}^{\eta(N)} \psi_{z_{N+1}}$$

$$\sum_{\{z\}} \hat{\mu}_N(n, \{z\}) = \mu_N(n)$$

- Call for simplicity $z = (n, \{z\})$ $z_x = (n_x, \{z_x\})$
and define the $(\{0,1\} \times A) \times (\{0,1\} \times A)$ met $T_2 \times$

$$\prod_{z_x, z_{x+1}}^{\eta_x} = \prod_{z_x, z_{x+1}}^{\eta_x}$$

$\Rightarrow$

$$\hat{\mu}_N(z) = \hat{\nu}_{z_1} \prod_{x=1}^N \hat{\mu}_{z_x, z_{x+1}} \psi_{z_{N+1}}$$

MARKOVIANITY

23

- $\hat{M}_{ep} \geq 0$

$$\hat{M} = \begin{pmatrix} M^0 & M^0 \\ M^1 & M^1 \end{pmatrix}$$

- $\hat{M}$ encodes a nearest neighbor pair interaction
→ Hammersley-Clifford → Markov

- GROUND STATE TRANSFORMATION

$$\hat{P}_{z_x, z_{x+1}} = \frac{1}{\lambda e_{z_x}} \hat{M}_{z_x, z_{x+1}} e_{z_{x+1}}$$

is stochastic; (λ, e) Perron eigenvalue and eigenvector.

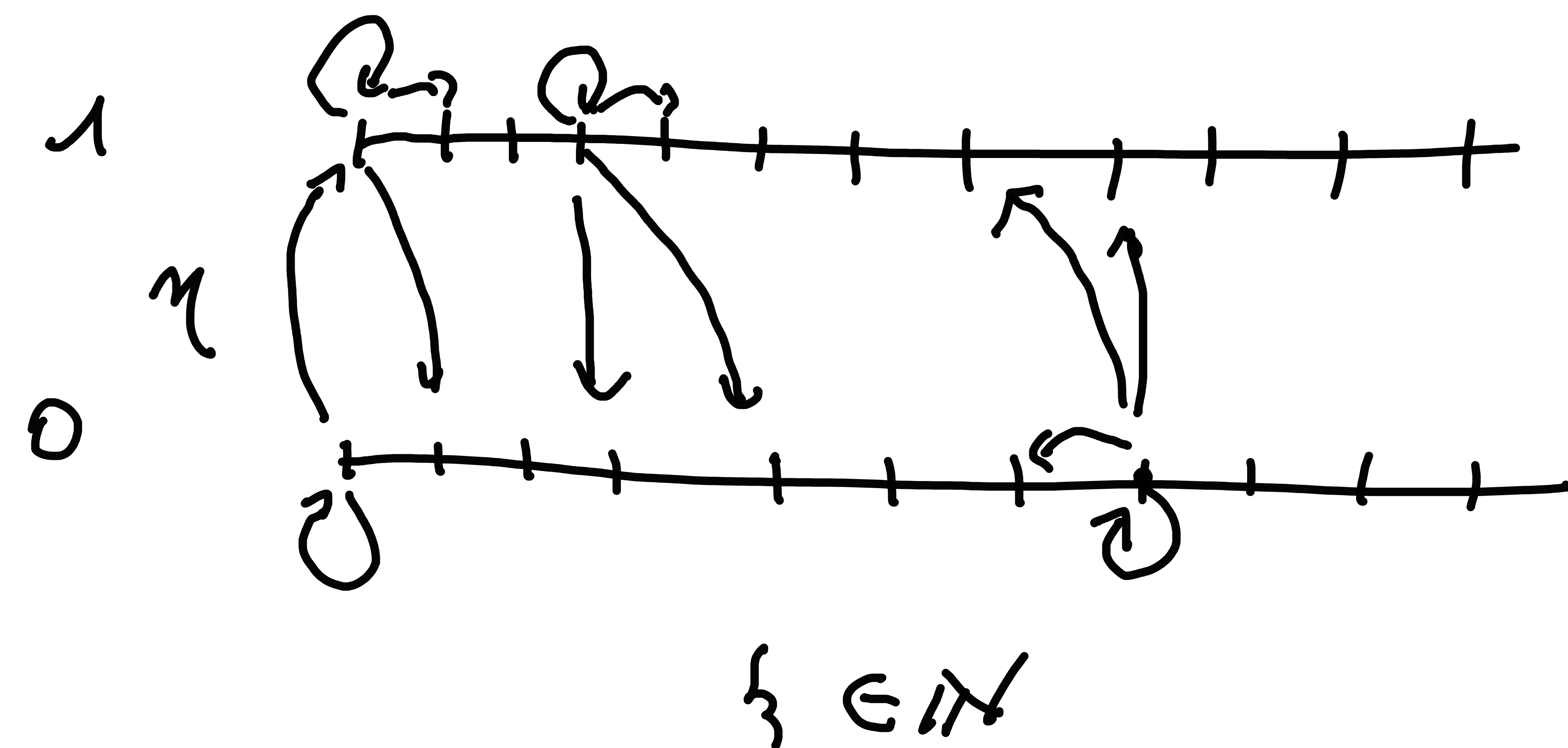
$$\hat{\mu}_N(z) = \frac{1}{Z_N} \prod_{x=1}^N \hat{P}_{z_x, z_{x+1}} \hat{\psi}_{z_{N+1}}$$

Is a Markov
bz. bzgl.

BOUNDARY DRIVEN TASEP

24

- The measure $\mu_N(\eta)$ is an hidden Markov model
- For boundary driven TASEP (on some region of the parameters) we can compute ell; $\hat{P}$ is



$$\bar{I}(\rho) = \inf_{\bar{F}} G(\rho, \bar{F})$$

