Stochastic transport models of turbulence

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Stochastic equations and particle systems

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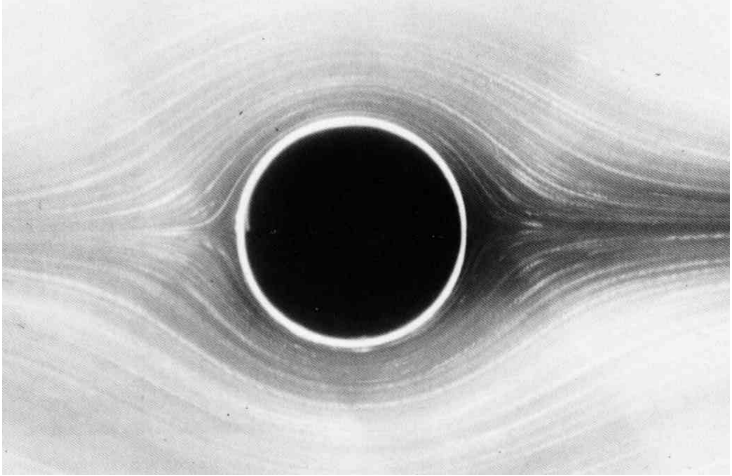
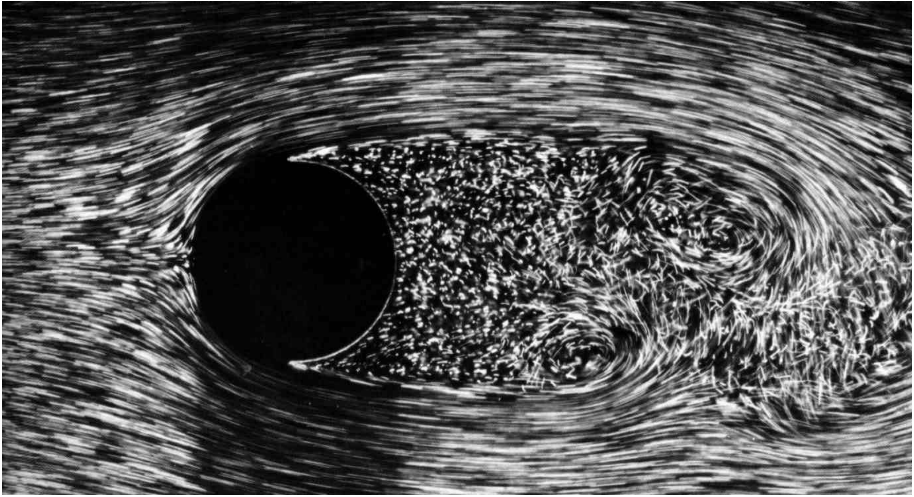
Joint work with Marco Bagnara, Lucio Galeati and Mario Maurelli

An Unassuming Laminar Flow



Shear flow past an obstacle at Re= 0.16
(aluminum dust in water, *An Album of Fluid Motion*, van Dyke 1982).

Turbulence emerges: Chaos ensues



Turbulence past an obstacle at Re$=2000$
(air bubbles in water, *An Album of Fluid Motion*, van Dyke 1982).

Upcoming

- A linear, stochastic PDE model for turbulence.
- Anomalous Regularization.
- Some integral asymptotics.

Advection of Passive Scalars

Features of (Nonlinear, 3D) Turbulence

Kolmogorov-Obukhov predictions on structure functions and energy cascades in isotropic turbulence:

$$E\left[|u(x+r)-u(x)|^2\right] \sim \varepsilon^{2/3} r^{2/3} \quad r \to 0,$$

$$E\left[E(k)\right] = E\left[\left|\int_{|\underline{k}|=k} |\hat{u}(\underline{k})|^2 d\underline{k}\right|\right] \sim \varepsilon^{2/3} k^{-5/3} \quad \text{(inertial range)}$$

(ε the energy dissipation rate, assumed scale-independent)
Anomalous dissipation of energy:

$$\lim_{\nu \to 0} \nu \int_0^T \|\nabla u_t^\nu\|_{L^2}^2 \, \mathrm{d}t > 0$$

These phenomena should be reproduced by sol. of 3D Navier-Stokes eqs.

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = \nu \Delta u + \nabla p \\ \nabla \cdot u = 0 \end{cases}$$

with $\nu \ll 1$, however simpler phenomenological models are sufficient.

Advection of Passive Scalars

Advection of a scalar field ρ by a (decoupled) random vector field $u : \mathbb{R}^d \to \mathbb{R}^d$ (and possibly small viscosity ν),

$$\partial_t \rho + u \cdot \nabla \rho = \nu \Delta \rho,$$

is able to replicate many features of turbulence.

(first proposals by Obukhov, Corrsin, Batchelor, see Sreenivasan '18 for a survey).

Kraichnan considered the case where u is a isotropic Gaussian field, delta-correlated in time, with power-law covariance spectrum.

(Kraichnan *Phys. Fluids* '68, '70, *J. Fluid Mech.* '74, '76, '76, *PRL* '89)

The model allows explicit (although sometimes only formal) computations and can replicate the energy cascade, anomalous dissipation, spontaneous stochasticity.

Kraichnan's Model

Consider $u = \frac{\partial}{\partial t} W(t,x)$ as a sample of a divergence-free, isotropic Gaussian random velocity field Gaussian velocity field, study the Stratonovich transport SPDE

$$d\rho + \circ dW \cdot \nabla \rho = 0, \quad \rho : [0,\infty) \times \mathbb{R}^d,\ d \geq 2.$$

We choose covariance with power law spectrum,

$$E[W(t,x) \otimes W(s,y)] = (t \wedge s) Q(x-y), \quad \hat{Q}(\xi) = \frac{c_d}{(1+|\xi|^2)^{d/2+\alpha}} P_\xi^\perp,$$

that is

$$Q(0) - Q(x) = |x|^{2\alpha} \left(I - \frac{2\alpha}{d-1} P_x^\perp \right) + o\left(|x|^{2\alpha}\right), \quad |x| \to 0.$$

Regular Kraichnan's Model

When $\alpha > 1, W(t,x)$ is continuously differentiable in x, and

- there exists a C^1-regular, measure preserving stochastic flow $X_t(x)$ for the underlying SDE $dX_t = W(\circ dt, X_t)$;
- $\rho_t(x) = \rho_0\left(X_t^{-1}(x)\right)$ and

$$\|\rho_t\|_{L^2} = \|\rho_0\|_{L^2} \quad \forall t \geq 0, \quad \mathbb{P}\text{-a.s.}$$

- Initial data δ_x evolve as $\delta_{X_t(x)}$.
- Explicitly computable Lyapunov exponents, strictly positive top Lyapunov $\lambda = \lim_{t\to\infty} \frac{1}{t} \log |DX_t(x)|$.

  cf. Le Jan '85, Baxendale-Harris '86
- On compact manifolds: exponential mixing of passive scalars.

  cf. Dolgopyat-Kaloshin-Koralov '04, Gess-Yaroslavtsev '21.

Rough Kraichnan's Model

When $\alpha \in (0,1)$, W is Hölder continuous and the SPDE is well-posed, and

(strong existence, pathwise uniqueness for $\rho_0 \in L^1$, cf. Le Jan-Raimond '02, Maurelli '11, Galeati-Luo '23)

- Spontaneous stochasticity: particle trajectories X_t, Y_t starting at the same position x depart istantaneously.
- SDE dynamics not described by a map $x \mapsto X_t(x)$ but a flow of Markovian kernels.
- Initial data δ_x of the SPDE become L^1-densities at $t > 0$.
- Solutions to SPDE are limit of vanishing viscosity/smooth approximations.
- Anomalous dissipation: $\|\rho_t\|_{L^2} < \|\rho_0\|_{L^2}$ with positive probability.

  (related to spontaneous stochasticity, cf. Drivas-Eyink '17)

Evolution of Multiscale Norms

Negative Sobolev norms

$$\|f\|^2_{\dot{H}^{-s}} = \int_{\mathbb{R}^d} |\xi|^{-2s} |\hat{f}(\xi)|^2 \, \mathrm{d}\xi$$

allow to gauge mixing behaviour and cascade mechanisms.
Previous results include:

- $\alpha > 1$: [Gess, Yaroslavtsev '21] show pathwise exponential decay of negative Sobolev norms (uniform-in-diffusivity estimates):

$$\|\rho_t\|_{\dot{H}^{-s}} \leq D e^{-\gamma t} \|\rho_0\|_{\dot{H}^s} \quad \forall t \geq 0, \quad E\left[|D|^p\right] < \infty.$$

- $\alpha = 1$, statistically self-similar case: [Coti Zelati, Gvalani, Drivas '24]

$$E\left[\|\rho_t\|^2_{\dot{H}_{-s}}\right] = e^{-\lambda_{d,s} t} \|\rho_0\|^2_{\dot{H}^{-s}}, \quad \lambda_{d,s} \sim 2s(d-2s)$$

- $\alpha < 1$, on $\mathbb{T}^d$ [Rowan '23] proves exponential decay of energy:

$$E\left[\|\rho_t\|^2_{L^2}\right] \leq e^{-Ct} \|\rho_0\|^2_{L^2}, \quad \forall t \geq \tau(\alpha)$$

Regularization in Rough Kraichnan's Model

Anomalous Regularization for $\alpha < 1$

For $\alpha > 1, \rho_t(x) = \rho_0\left(X_t^{-1}(x)\right)$, reversible dynamics and no regularization.

Theorem (Galeati-G.-Maurelli)

Let $d \geq 2, \alpha \in (0,1), s \in (0, d/2)$. There exist positive constants $C_1, C_2 > 0$ such that

$$\frac{d}{dt} E\left[\|\rho_t\|^2_{\dot{H}^{-s}}\right] + C_1 E\left[\|\rho_t\|^2_{\dot{H}^{1-\alpha-s}}\right] \leq C_2 E\left[\|\rho_t\|^2_{\dot{H}^{-s}}\right] \quad \forall t > 0.$$

In particular, for $T > 0$

$$\sup_{t\in[0,T]} E\left[\|\rho_t\|^2_{\dot{H}^{-s}}\right] + C_1 E\left[\int_0^T \|\rho_t\|^2_{\dot{H}^{1-\alpha-s}}\, \mathrm{d}t\right] \leq e^{2C_2 T} E\left[\|\rho_0\|^2_{\dot{H}^{-s}}\right].$$

As a consequence, the solution map extends uniquely to any initial data $\rho \in \dot{H}^{-s}$ with $s \in (0, d/2)$ and solutions become istantaneously L^2-regular.

About Regularization

- Result motivated by [Coghi, Maurelli '23] who obtained the result for $d = 2, s = 1$ and applied it to regularization by Kraichnan noise for 2D Euler.
- Solutions become regular: $\mathbb{P}$-a.s. $\rho \in L^2_t H^{1-\alpha-}$.
- Optimality of threshold $s < d/2$, up to equality: white-noise (which is supported on $H^{-d/2-}$) is formally invariant.
- Constant C_1 dictated by the local Hölder behaviour of W, while constant C_2 by how much W deviates from self-similarity. In the (extremely formal!) statistically self-similar case, the balance becomes

$$\frac{d}{dt} E\left[\|\rho_t\|^2_{\dot{H}^{-s}}\right] + C_1 E\left[\|\rho_t\|^2_{\dot{H}^{1-\alpha-s}}\right] = 0$$

Heuristics

Given a smoothing kernel G, ρ solution as above,

$$\frac{d}{dt} E\left[\langle G * \rho_t, \rho_t \rangle\right] + 2\nu E\left[\langle G * \nabla\rho_t, \nabla\rho_t \rangle\right]$$
$$= E\left[\int_{\mathbb{R}^d \times \mathbb{R}^d} \mathrm{Tr}\left((Q(0) - Q(x-y))D^2 G(x-y)\right) \rho_t(x)\rho_t(y) dx\, \mathrm{d}y\right]$$
$$=: E\left[\langle H * \rho_t, \rho_t \rangle\right].$$

Consider the (ill-defined) self-similar case $\hat{Q}(\xi) = c_d |\xi|^{-d-2\alpha} P_\xi^\perp$. Take $R_s(z) = |z|^{d-2s}$, so that $\langle R_s * \rho_t, \rho_t \rangle \sim \|\rho_t\|^2_{\mathcal{H}^{-s}}$. By an explicit computation:

$$H(z) = 2(2s-d)(s+\alpha-1)|z|^{2(s+\alpha-1)-d} = -2(d-2s)(s+\alpha-1)R_{s+\alpha-1}(z)$$

(consistent with $\alpha = 1$ examined in Coti Zelati-Gvalani-Drivas '24)

Flux Functions

Lemma

Let $d \geq 2, \alpha \in (0,1)$*; let* $\rho_0 \in L^1 \cap L^2 \cap \dot{H}^{-s}$, ρ *associated solution. Then*

$$\frac{d}{dt} E\left[\|\rho_t\|^2_{\dot{H}^{-s}}\right] = \int_{\mathbb{R}^d} F(\xi) E\left[|\hat{\rho}_t(\xi)|^2\right] d\xi - 2\nu E\left[\|\rho_t\|^2_{\dot{H}^{-s}}\right]$$

for the flux function $F = F(\alpha, s, d)$ *defined by*

$$F(\xi) := \int_{\mathbb{R}^d} \frac{1}{(1+|\xi-\eta|^2)^{\frac{d}{2}+\alpha}} \left|P^{\perp}_{\xi-\eta}\xi\right|^2 \left(\frac{1}{|\eta|^{2s}} - \frac{1}{|\xi|^{2s}}\right) d\eta$$

We are therefore reduced to study the high-frequency asymptotics of $F(\xi)$.

Beyond Regularization in Kraichnan's model

- Rigorous derivation of intermittency in the Kraichnan model.
- Study more realistic and complicated models (e.g. W not white in time, replaced by a solution to an SPDE).
- Rigorously derivation of anomalous dissipation and anomalous regularization in nonlinear SPDEs driven by "Kraichnan noise".

  (recent results in Coghi-Maurelli '23, Bagnara-Galeati-Maurelli '24)

  Results in the nonlinear case are restricted to $\alpha \in (0, 1/2)$, but a physically motivated choice would be $\alpha = 2/3$.
- Regularization by Kraichnan's transport noise for vector fields, aiming to an application to 3D Navier-Stokes.

Kazantsev-Kraichnan

Theorem (Bagnara-G.-Maurelli)

The induction equation

$$\partial_t B + (u \cdot \nabla) B = (B \cdot \nabla) u, \quad \nabla \cdot B = 0,$$

driven by the random field u as in Kraichnan's Model, has a unique solution in $L^2_{t,\omega}(H^{-s+(1-\alpha)})$ if the initial datum is in H^{-s}, for $d \geq 3$, small enough $\alpha \in (0,1)$ and an appropriate negative s.

Asymptotics

A rather tough integral

The asymptotics at $|\xi| \to \infty$ of the flux function

$$F(\xi) := \int_{\mathbb{R}^d} \frac{1}{(1+|\xi-\eta|^2)^{\frac{d}{2}+\alpha}} \left| P^{\perp}_{\xi-\eta}\xi \right|^2 \left(\frac{1}{|\eta|^{2s}} - \frac{1}{|\xi|^{2s}} \right) d\eta$$

can be reduced to that of

$$J(\lambda) = \int_0^\infty h(\lambda t) f(t) dt,$$

at $\lambda (= |\xi|) \to \infty$, with

$$f(r) = r^{d-1} \int_0^\pi \frac{\sin^d \theta d\theta}{|1 - 2r\cos\theta + r^2|^s}, \quad h(r) = \frac{1}{(1+r^2)^{d/2+\alpha}}.$$

Parseval Theorem for Mellin Transforms

The Mellin transform of f is defined by

$$M[f,z]=\int_0^\infty t^{z-1}f(t)dt$$

for $z\in\mathbb{C}$ for which the integral is absolutely convergent.

$$J(\lambda)=\int_0^\infty h(\lambda t)f(t)dt=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\frac{M[h,z]M[f,1-z]}{\lambda^z}dz,$$

if the line $r+i\mathbb{R}$ is entirely included in the intersection of the fundamental strips of the involved Mellin transforms.

Computing Coefficients

Taking the difference of Parseval formulas for two lines

$$J(\lambda) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{M[h,z]M[f,1-z]}{\lambda^z}dz$$
$$= \sum_{r<\mathrm{Re}\, z<r'} \mathrm{Res}\left\{-\lambda^{-z}M[h,z]M[f,1-z]\right\}$$
$$+ \frac{1}{2\pi i}\int_{r'-i\infty}^{r'+i\infty} \frac{M[h,z]M[f,1-z]}{\lambda^z}dz,$$

so we can compute an asymptotic expansion by evaluating residues. The relevant pole (producing the leading coefficient) is at $z = d + 2\alpha$, and we need the residue to be a **positive** real number.

Computing Coefficients

One therefore needs to evaluate

$$M[h,z] = \frac{\Gamma\left(z/2\right)\Gamma\left(d/2+\alpha-z/2\right)}{2\Gamma\left(d/2+\alpha\right)},$$

and the crucial

$$\begin{aligned} M[f,1-z] &= \int_0^\infty r^{d-1-z}\int_0^\pi \frac{\sin^d\theta d\theta}{|1-2r\cos\theta+r^2|^s} \\ &= \frac{\sqrt{\pi}\Gamma\left(\frac{d-2s+2}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{2\Gamma(s)}\cdot\frac{\Gamma\left(\frac{2s-d+z}{2}\right)\Gamma\left(\frac{d-z}{2}\right)}{\Gamma\left(\frac{z+2}{2}\right)\Gamma\left(\frac{2d-2s+2-z}{2}\right)}. \end{aligned}$$

Why the explicit computation? The relevant pole comes from $M[h,z]$, and it is located **outside of the analyticity region of** $M[f,1-z]$.

A representation with special functions

Theorem

Let $\alpha \in (0,1)$ and $s \in (0, d/2)$. There exists a finite constant $C_{d,\alpha,s} > 0$ such that

$$|F(\xi) + K_{d,\alpha,s}|\xi|^{2-2\alpha-2s}| \leq C_{d,\alpha,s}|\xi|^{-2s} \quad \forall \xi \in \mathbb{R}^d \setminus \{0\},$$

where

$$K_{d,\alpha,s} = -\frac{2^{d/2-1}(d+1)\Gamma(s+\alpha)\Gamma(-\alpha)\Gamma\left(\frac{d-2s+2}{2}\right)}{\Gamma(s)\Gamma\left(\frac{d+2\alpha+2}{2}\right)\Gamma\left(\frac{d-2s+2-2\alpha}{2}\right)} > 0.$$

Thank you!

(arXiv:2407.16668, arXiv:2411.09482)