Fluctuations of Stochastic Heat Equation and KPZ equation

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Noise

- White noise – De-correlation in space and in time– is a standard assumption.
- Yet, time series data indicates other wise; common random source also indicates spatial correlations.
- Mollified white noise are typically spatial de-correlation for any two points with distance greater or equal to a given number.
- Take the scenario of a finite grid. Why would we expect the noise at one corner pf the grid being totally de-correlated from that sitting on the opposite corner? As the grid size increased, it make more sense to assume the decay of correlation in proportion to the distance.
- For this reason and for mathematical curiosity, we study long range correlated noise.

Long Range Dependence in Time

In the statistical mechanics word, LRD has already been take noticed by e.g. Sinai.

- According to J. Jona-Lasino, ‘the critical point of a second order phase transitions so far represents in physics the most important instance where the central limit theorem breaks down’. By this he refers to the convergence of the rescaled sum of an infinite number of mean zero random variables $\frac{1}{n^{-\alpha}} \sum_{i=1}^{n} X_i$ to a Gaussian random variable.
- This breaks down precisely when there is a strong correlation of the said random variables.
- According to Rosenblatt, Y_k is a mean zero Gaussian sequence with covariance $E(Y_0 Y_n) \sim n^{-a}$, $a \in (0, \frac{1}{2})$, then

$$n^{a-} \sum_{i=1}^{n} H_2(Y_i) \to \text{ non -Gaussian.}$$

Functional Limit Theorem

In fact even when the limits are Gausssian distributed, Donsker's invariance principle may fail:

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{i=1}^{[nt]} X_i$$

may not be a Brownian motion.

The scale α would actually yield the self-similarity exponent of the process in the **domain of attraction**. In today's language the domain of attraction $=$ Universality class.
One of these processes are fractional Brownian motions, the others are Hermit processes $Z^{H,m}$.

Self-similarity

- Lamberti 62: if $X(t)$ is continuous at 0 with non-degenerate laws, $\frac{1}{h(\lambda)}Y(\lambda t) \to X(t)$, then

$$X(ta) = a^H X(t).$$

- Consider a mean zero stationary increments process, with self-similarity ($Y(0) = 0$, and $\sigma^2 = \mathbf{E}(Y(1)^2) < \infty$). Then

$$\mathbf{E}(Y(t)Y(s)) = \frac{1}{2}\sigma^2(t^{2H} + s^{2H} - |t-s|^{2H})^2,$$

where $H \in (0,1)$. Lamperti, Embrechts and Maejima.
- A Gaussian process with the above properties is a fBM.

Correlated noise

The 'derivative' of a fractional Brownian motion is the simplest noise with correlation. A fBM is a Gaussian process with stationary increment and $\mathbf{E}(B_{t+s} - B_s)^2 = t^{2H}$.
For t large, $H \neq \frac{1}{2}$:

$$\mathbf{E}(B_{t+s+1} - B_{t+s})(B_{s+1} - B_s) \sim t^{2H-2}.$$

- The dynamics of an equation driven by a long range fractional Brownian motion ($H > \frac{1}{2}$) is quite different from that of a Brownian motion. Very little is know of its invariant measure (except the fractional Ornstein-Uhlenbeck equation and its generalizations) . Even little is know of its densities and tails.

The Nile

Each summer, the river Nile overflows and floods the surrounding areas, leaving behind rich fertile silt for agriculture. If the inundation was inadequate, only a small area would be covered with the life-giving silt, famine follows.

During the Pharaonic Period, forecast for the water flow was used to compute taxes.

Time series data

Records on the height of the annual flow has been kept for 3 millennia, with numerous Nilometers.

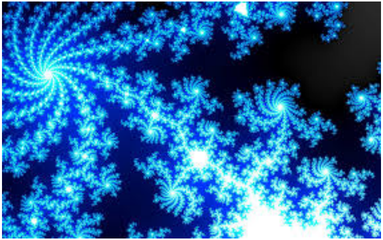
In 1906, Harold Hurst started to work in the Survey Department of Egypt in October 1906, which was responsible for collecting data throughout the Nile basin.

Harold Hurst

Hurst worked in Egypt from 1906-1968, studying the annual Nile overflow series data he discovered a heavier flood year is followed by a heavier than average flood year and a draught year flow is followed by a lighter than average river flow.

The Hurst phenomenon is modelled by Mandelbrot and van Ness with fractional BM (1968).

Mandelbrot studied fractals to capture the roughness persistent at all levels.



$$B_t = \int_0^t (t-u)^{H-\frac{1}{2}}\, dW_u + \int_{-\infty}^0 \left[(t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}\right] dW_u.$$

Self-similar: $B_{at} = a^H B_t$.

Multi-scale

A dynamical system such as from engineering, science, economics, and ecology consists of many different variables interacting with each other. Given an object of study, the interacting elements are classified as into two categories: influential or negligible, the negligible is either neglected or included in the model through the CLT theorem and modelled by randomness.

Some influential interacting variables evolve at the same time scale as our objects, some evolves at a slower scale (so can be treated to be constant in time), others at faster scales. If we model the evolution of the slow variables by a random differential equations, the fast variable entered into the expression for the vector fields.

On the time scale of the object of interest, the precise positions of fast variables are not tractable but often not needed. Instead, one focuses on the persistent effects of the

Two time scale stochastic equations

Stochastic equations with slow and fast variables already separated:

$$\dot{x}_t^\varepsilon = F_0(x_t^\varepsilon, y_t^\varepsilon) + F(x_t^\varepsilon, y_t^\varepsilon)\dot{\xi}_t$$

where $y_t^\varepsilon = y_{\frac{t}{\varepsilon}}$ for a suitable process y or

$$\dot{y}_t^\varepsilon = \frac{1}{\varepsilon}\sigma_0(x_t^\varepsilon, y_t) + \varepsilon^{-\alpha}\sigma(x_t^\varepsilon, y_t)\dot{\eta}_t$$

As the time separation parameter $\varepsilon \to 0$, the position of y_t^ε is not tractable and irrelevant. The aim is to track down its persistent effect and deduce an autonomous equation for the variable of interests.
Examples of such fast motions are for example periodic functions or ergodic (stationary) Markovian process.

Functional LLN and Averaging Principle

$$\dot{x}_t^\varepsilon = F_0(x_t^\varepsilon, y_{t/\varepsilon}) + F(x_t^\varepsilon, y_{t/\varepsilon})\dot{\xi}_t$$

- y_t is said to satisfy a **Functional LLN** if for any f regular
$$\left|\varepsilon \int_0^t f(y_{r/\varepsilon})dr - t\bar{f}\right| = o(\varepsilon)$$
- Let Y_s be an independent stationary ergodic Markov process with generator $\mathcal{L}$ which one assumes nice. Functional LLN implies that
$$\int_0^t f(Y_{s/\varepsilon})dW_s \to \hat{W}_t$$
a Wiener process with covariance $(\overline{f \otimes f})^{\frac{1}{2}}$.
$$dx_t^\varepsilon = f(x_t^\varepsilon, Y_{s/\varepsilon})dW_s$$

Diffusion Creation /Homogenesation

The functional CLT for the Markov process
If $\bar f = 0$ and regular

$$\frac{1}{\sqrt{\varepsilon}} \int_0^t f(Y_{s/\varepsilon}) ds \to (\overline{f \mathcal{L}^{-1} f})^{\frac{1}{2}} W_t.$$

Diffusion creation problem:

$$\dot{x}_t^\varepsilon = \sqrt{\varepsilon}\, f(x_t^\varepsilon, y_{\frac{t}{\varepsilon}}).$$

Integration and enhanced processes

$$\int_0^t f(B_s)dB_s \sim \sum \int_u^v f(B_u) + f'(B_u)\delta B_{ur} + \dots)dB_r$$

$$\sim \overbrace{\sum f(B_u)\delta B_{uv} + f'(B_u)\int_u^v \delta B_{ur} dB_r + \dots}^{A_{uv}}.$$

For $H > \frac{1}{2}$, this is Riemann-Stieljes, for $H \in (\frac{1}{3}, \frac{1}{2})$, we need the second order term, the iterated integrals of B.
Enhanced process $(B, \mathbb{B})$ where

$$\mathbb{B}_{uv} = \int_u^v (B_u - B_r)dB_r.$$

For $H \in (\frac{1}{4}, \frac{1}{3})$, need to Taylor expand to order 3.

To define $\int f(x_s)dB_s$ we assume that x_s is controlled by B: it is similar to B plus smooth order terms.

Rough functional limit theorem

Y_s is a fractional Ornstein-Uhlenbeck process,
$H^*(m) = m(H-1)+1$, G function with Hermite rank m.

- $$\frac{1}{\sqrt{\varepsilon}}\int_0^t G(Y_{s/\varepsilon})ds \to cW_t, \qquad \text{if} \quad H^*(m) < \frac{1}{2},$$
- $$\frac{1}{\sqrt{\varepsilon}\sqrt{|\ln \varepsilon|}}\int_0^t G(Y_{s/\varepsilon})ds \to cW_t, \qquad \text{if} \quad H^*(m) = \frac{1}{2},$$
- $$\varepsilon^{H^*(m)-1}\int_0^t G(y_{s/\varepsilon})ds \to c\bar{Z}_t^{H*(m),m}, \qquad \text{if} \quad H^*(m) > \frac{1}{2}, .$$

Essentially known: weak convergence.
Theorem (Gehringer-L.) Functional limit theorem holds in rough path topology, so limit of $\dot{x}_t^\varepsilon = f(x_t^\varepsilon)G(y_{\frac{t}{\varepsilon}})$ converge.

Fractional Averaging

$$dx_t^\varepsilon = f(x_t^\varepsilon, y_t^\varepsilon)\, dB_t + g(x_t^\varepsilon, y_t^\varepsilon)\, dt \ ,$$

Proposition. Let $H > \frac{1}{2}$ and y_t is a uniformly elliptic Markov process on a compact manifold. Then $x_t^\varepsilon \to \bar{x}_t$ in probability

$$d\bar{x}_t = \bar{f}(\bar{x}_t)\, dB_t + \bar{g}(\bar{x}_t)\, dt \ .$$

For any $\beta < H$, there exists $\kappa > 0$ such that

$$\lim_{\varepsilon \to 0} \mathbf{P}\Big(\Big| \int_s^t f(y_{\frac{r}{\varepsilon}}) dB_r - \bar{f}(B_t - B_s) \Big|_\beta > \varepsilon^\kappa \Big) = 0 \ .$$

Notes

Annealed limit. $x_t^\varepsilon \to \bar{x}$ in probability.

Technical difficulty: Tightness

Quenched Problem. If we fix a path $h = B(\omega)$, does the convergence hold for each fixed fBM path?

Feedback model.

Ingredients for the proof

Let $H > \frac{1}{2}$. Let $f_n, \bar{f} : \mathbf{R}_+ \times \mathbf{R}^d \to L(\mathbf{R}^m, \mathbf{R}^d)$ be in $\mathcal{C}^{\zeta,2}$.

- **Lemma 1** [Hairer-L'20] $x_0^n = x_0$

$$dx_t^n = f_n(t, x_t^n)\, dB_t, + f_0(t, x_t^n) dt, \quad dx_t = \bar{f}(x_t)\, dB_t + \bar{f}_0(x_t) dt\ ,$$

  Suppose that $\kappa, \gamma \geq 0$,

$$\lim_{n\to\infty} |f_n - \bar{f}|_{-\kappa,\gamma} = 0\ .$$

  Then, $x^n \to x$ in probability in $\mathcal{C}^\alpha$, for $\alpha \in (\frac{1}{2}, H - \kappa)$, $\zeta + \alpha > 1$ and $H - \kappa + \gamma\alpha > 1$.

- **Lemma 2.** If y is strong mixing with rate δ, $F : \mathbf{R}^d \times \mathcal{Y} \to \mathbf{R}$ bounded measurable, uniformly Lipschitz continuous in the first variable ($\mathcal{Y}$ compact), then $\|F(x, y_{\cdot/\varepsilon}) - \bar{F}\|_{-\kappa,\gamma} \to 0$.

Ingredients for the proof

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Unusual Limit Theorems

Y_s stationary ergodic Markov process with all nice properties and generator $\mathcal{L}$.

Lemma (Hairer+Li 21)

- If $H > \frac{1}{2}$, the following converges in probability

$$\int_0^t f(Y_{s/\varepsilon})dB_s \to \bar{f} B_t.$$

- If $H \in (\frac{1}{3}, \frac{1}{2})$ (and $\bar{f} = 0$ incase $H > \frac{1}{2}$)

$$\varepsilon^{\frac{1}{2}-H} \int_0^t f(Y_{s/\varepsilon})dB_s \to \Sigma W_t.$$

$$\Sigma = \frac{1}{2}\Gamma(2H+1)\overline{F \otimes \mathcal{L}^{1-2H} F}.$$

Spatial Dependence

SPDEs with spatial long range dependent noise

Now consider a Gaussian family ξ, white in time, spatially correlated noise. Keep space-time homogeneity. Formally

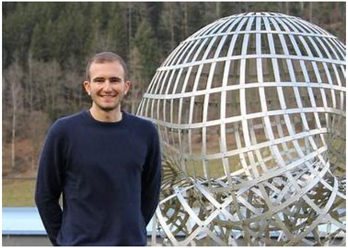
$$\mathbf{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)R(x-y).$$

Not concerned with the well-poseness problem, we consider a regular SPDE, assuming that R is smooth.

- $$\partial_t h = \tfrac{1}{2}\Delta h + \tfrac{1}{2}|\nabla h|^2 + \beta\xi \ .$$
- $$\begin{aligned} \partial_t u(t,x) &= \tfrac{1}{2}\Delta u(t,x) + \beta u(t,x)\xi(t,x) \ , \\ u(0,\cdot) &= 1 \ , \end{aligned}$$

Collaborators

Martin Hairer



Luca Gerolla

Shape at infinity

$$|R(x)| \leq \frac{c_R}{1+|x|^\kappa}, \quad \kappa \in (2,d), d \geq 3.$$

Shape of noise at infinity:

$$\lim_{\varepsilon \to 0} \varepsilon^{-\kappa} R(\frac{x}{\varepsilon}) = x^{-\kappa}.$$

- For every ε, there is a function valued solution u_t to the SHE and and a solution h_t to the KPZ. Take $u_0 \equiv 1$ for simplicity.
- As $t \to \infty$, the solutions converge to stationary random fields. To see this, we note that for any s, u_{t-s} equals in law to the pull back solution $u^{(s)}(t,x)$, which solves the SHE with $u^{(s)}(s,\cdot) \equiv 1$.

Large time behaviour

Set $s' < s < 0$,

$$u_t^{(s)}(x) = 1 + \beta \int_s^t \int_{\mathbf{R}^d} P_{t-r}(x-y) u_r^{(s)}(y) \xi(dr, dy).$$

By heat kernel estimates, decay condition on R,

$$\begin{aligned} & \|u^{(s')}(t,x) - u^{(s)}(t,x)\|_p \\ \leq \; & M_p \, (1+t-s)^{\frac{2-\kappa}{4}} + c(\beta_0)(1+t-s)^{\frac{2-\kappa}{4}} \\ M_p := \; & \sup_{x \in \mathbf{R}^d} \sup_{t>s} (t-s+1)^{\frac{\kappa-2}{4}} \|u^{(s')}(t,x) - u^{(s)}(t,x)\|_p \, . \end{aligned}$$

$$\sup_{x \in \mathbf{R}^d} \|u^{s'}(t,x) - u^{s}(t,x)\|_p \lesssim M_p \beta (1 + t + (-s) \wedge (-s'))^{\frac{2-\kappa}{4}}.$$

Theorem: large time

Interpolate with

$$\|u^{(K)}(t,x_1) - u^{(K)}(t,x_2)\|_p \lesssim |x_1 - x_2|^\delta \ ,$$

Theorem [GLH] Let $p \geq 1$, there exists a space-time stationary random field $\vec{Z}$ such that, for $\beta \leq \beta_1(p,d,R)$, any $t \in \mathbf{R}$,

$$\lim_{s \to -\infty} \mathbf{E} \sup_{x \in \mathfrak{K}} |u^{(s)}(t,x) - \vec{Z}(t,x)|^p \to 0.$$

- $\mathbf{E}[u(t,x)] = 1$, by

$$u_t(x) = 1 + \beta \int_0^t \int_{\mathbf{R}^d} P_{t-s}(x-y)\sigma(u_s(y))\xi(ds,dy).$$

  If $\beta \sim 0$ is small, $u \sim P_t u(0,\cdot) = 1$.
- We study the large scale fluctuations: $u^\varepsilon(t,x) = u(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$.

Good coupling

$$u^\varepsilon(t,x) = u(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}), \qquad \xi^\varepsilon = \varepsilon^{\kappa/2-1}\xi(t/\varepsilon^2, x/\varepsilon)$$

$$\mathbf{E}[\xi^\varepsilon(t,x)\xi^\varepsilon(s,y)] = \delta(t-s)\varepsilon^{-\kappa}R(\tfrac{x-y}{\varepsilon}) \ .$$

$\xi^\varepsilon \to \xi^0$ in law, as $\varepsilon \to 0$, where

$$\mathbf{E}[\xi^0(t,x)\xi^0(s,y)] = \delta(t-s)|x-y|^{-\kappa} \ .$$

- can choose a good coupling such that for every $\psi \in \mathcal{C}_c^\infty(\mathbf{R}^{d+1})$, there exists $\xi^0(\psi)$ with $\xi^\varepsilon(\psi) \overset{(P)}{\Rightarrow} \xi^0(\psi)$.
- *Example.* $\xi(t,x) = \int_{\mathbf{R}^d} \varphi(x-y)\eta(t,y)\,dy$, η white noise. We have a natural good coupling:

$$\xi^\varepsilon(t,x) = \varepsilon^{-(d+\kappa)/2}\int_{\mathbf{R}^d} \varphi((x-y)/\varepsilon)\eta(t,y)\,dy \ .$$

Further choose $\varphi(x) \sim |x|^{-\frac{d+\kappa}{2}}$.

Role played by nonlinear interaction

- $v_\varepsilon = \varepsilon^{1-\frac{\kappa}{2}}(u_\varepsilon - 1)$ solves:

$$\partial_t v_\varepsilon(t,x) = \frac{1}{2}\Delta v_\varepsilon(t,x) + \beta\sigma(\varepsilon^{\frac{\kappa}{2}-1}v_\varepsilon(t,x)+1)\xi_\varepsilon(t,x).$$

- We expect the solution of the equation below to v:

$$\lim_{\varepsilon\to 0}\varepsilon^{1-\frac{\kappa}{2}}\int_{\mathbf{R}^d}(u_\varepsilon(t,x)-1)g(x)dx$$

$$\partial_t v = \frac{1}{2}\Delta v + \sigma(1)\dot{W}^\kappa.$$

The guess for this limiting variance is actually wrong.

Homogenization lemma

$$\partial_t w_{s,y}(t,x) = \tfrac{1}{2}\Delta w_{s,y}(t,x) + \beta w_{s,y}(t,x)\xi(t,x) \;,$$
$$w_{s,y}(s,\cdot) = \delta_y \;,$$

$$\vec{Z}(t,x) = 1 + \beta \int_{-\infty}^{t} \int_{\mathbf{R}^d} P_{t-r}(x-z)\vec{Z}(r,z)\,\xi(dr,dz);$$
$$\overleftarrow{Z}(s,y) = 1 + \beta \int_{s}^{\infty} \int_{\mathbf{R}^d} P_{r-s}(y-z)\overleftarrow{Z}(r,z)\,\xi(dr,dz).$$

Lemma:

$$\left\| \frac{w_{s,y}(t,x)}{P_{t-s}(y-x)} - \vec{Z}(t,x)\overleftarrow{Z}(s,y) \right\|_p$$
$$\lesssim (1+t-s)^{-\frac{1}{2}+\frac{1}{\kappa}}\left(1 + (1+t-s)^{-\frac{1}{2}}|x-y|\right) \;.$$

Fluctuation Theorem for SHE

$$X_t^{\varepsilon,g} = \varepsilon^{1-\frac{\kappa}{2}} \int_{\mathbf{R}^d} \big(u_\varepsilon(t,x) - \mathbf{E} u_\varepsilon(t,x)\big) g(x) dx.$$

Theorem. [GHL] Let $\nu_{\text{eff}}^2 = |\mathbf{E}[\sigma(\vec{Z}(0,x))]|^2$, $d \geq 3$. For $\beta < \beta_0$.

$$(X_{t_1}^{\varepsilon,g_1}, \dots, X_{t_n}^{\varepsilon,g_n}) \Rightarrow \left(\int_{\mathbf{R}^d} \mathcal{U}(t_1,x) g_1(x)\, dx, \dots, \int_{\mathbf{R}^d} \mathcal{U}(t_n,x) g_n(x)\, dx \right)$$

$$\partial_t \mathcal{U}(t,x) = \frac{1}{2} \Delta \mathcal{U}(t,x) + \beta\, \nu_{\text{eff}}^2 \dot{W}^\kappa(t,x), \qquad \mathcal{U}(0,x) = 0,$$

$$\mathbf{E}[\, \dot{W}^\kappa(t,x) \dot{W}^\kappa(s,y)\,] = \delta(t-s)|x-y|^{-\kappa}.$$

Unlike, for the compactly supported correlation case, where

$$\bar{\nu}^2 = \int R(x) \mathbf{E}_B \left(e^{\frac{1}{2}\beta^2 \int_0^\infty R(x+B_s) ds} \right) dx.$$

Compare to compactly supported case

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \beta u(t,x)\eta * \varphi(x)$$

Limit solves Edwards-Wilkinsons Equation.

$$\partial_t \mathcal{U}(t,x) = \frac{1}{2}\Delta \mathcal{U}(t,x) + \beta\bar{\nu}\,\eta(t,x),$$

$$\bar{\nu}^2 = \int R(x)\mathbf{E}_B\left(e^{\frac{1}{2}\beta^2\int_0^\infty R(x+B_s)ds}\right)dx.$$

J. Magnen, J. Unterberger (2008), C. Mukherjee, Shamov, Zeitouni (2016)
Yu Gu, L. Ryzhik, O. Zeitouni, (2018)- C. Cosco, S. Nakajim, M. Najashima 2020– D. Lygkonis, N. Zygouras (2022)– A. Dunlap, Y. Gu, L. Ryzhik, O. Zeitouni (2020).

KPZ

$$h_\varepsilon(t,x) := \varepsilon^{1-\frac{\kappa}{2}} h\Big(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\Big) - \tfrac{t}{2}\beta^2 R(0)\varepsilon^{-1-\frac{\kappa}{2}} ,$$

Allow a slowly varying initial condition:

$$\partial_t h_\varepsilon = \tfrac{1}{2}\Delta h_\varepsilon + \tfrac{1}{2}\varepsilon^{\frac{\kappa}{2}-1}|\nabla h_\varepsilon|^2 - \tfrac{1}{2}\beta^2 R(0)\varepsilon^{-1-\frac{\kappa}{2}} + \beta\xi^\varepsilon$$

$$\partial_t \mathcal{U}^0 = \frac{1}{2}\Delta\mathcal{U}^0 + \beta\xi^0 , \qquad \mathcal{U}^0(0,\cdot) = h(0) .$$

Theorem (GHL)

Let $p \geq 1$, $\alpha < 1 - \frac{\kappa}{2}$, and $\sigma < -1 - \frac{\kappa}{2}$. For β small,

$$(h_\varepsilon(t,x) - \varepsilon^{1-\frac{\kappa}{2}} c, \xi^\varepsilon) \Rightarrow (\mathcal{U}, \xi^0)$$

in probability in $L^p([T_0,T], \mathbf{C}^\alpha(E))$.

- Fast variable non-Markovian, driven by fBM. Time homogenisation problem (multiple scaling constants, the effect dynamics is much richer than that given by the Markovian dynamics), Johann Gehringer +L.
- Slow variable driven by fBM. The analysis is a puzzle. Hairer+L.
- Fractional averaging with fast fractional dynamics, Sieber+L.
- non-product form for Volterra kernels, Gehringer +L+Sieber
  - for SPDE

  $$dX_t^\varepsilon = AX_t^\varepsilon \, dt + f\big(X_t^\varepsilon, Y_{\frac{t}{\varepsilon}}\big) \, dt + g\big(X_t^\varepsilon, Y_{\frac{t}{\varepsilon}}\big) \, dB_t.$$

  L.+Sieber
- L. + Planloup+Sieber: smooth dependence of invariant measures for SDEs driven by fBMs depending on a parameter.